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Finite heating duration based-sensitivity analysis in transient heat conduction

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Abstract. In the parameter estimation the sensitivity analysis of the temperature to the unknown parameter (such as thermal conductivity, heat capacity) plays a fundamental role. In the experimental setup for thermal properties measurements of solid materials, the heater in contact with the specimen can be modelled through an high conductivity thin layer. In such a case the one dimensional finite rectangular body, representing the sample, is subject to a boundary condition of the 4th kind at the heated boundary. Also, a constant heat flux applied for a finite period of time is considered in the analysis and the sample backside is assumed insulated. The temperature field within the finite body is obtained by using the superposition principle. Then, the scaled sensitivity coefficients are computed analytically for two different locations: at the interface between the heater and the sample, and at the sample backside. The results show that the sensitivity coefficients with respect to the thermal conductivity and to the volumetric heat capacity of the sample are uncorrelated.

1. Introduction

Sensitivity coefficients have many applications such as parameter estimation, optimal experiment design [1] and uncertainty or error analysis [2]. In parameter estimation, valuable insight is provided by carrying out a sensitivity analysis of the temperature to the unknown parameters, such as thermal conductivity, volumetric heat capacity or thermal diffusivity. For instance, in an apparatus for which the sensitivity coefficients had not been investigated it is possible that additional materials in the experimental configuration (such as the thin heater giving up heat to the sample) can have a large impact on the temperature than the material of interest (sample), with direct consequences on the quality of the results [1].

Also, on the one hand the sensitivity coefficients make possible a preliminary evaluation of the goodness of the experimental results (at least from a qualitative point of view), on the other hand they are directly involved in the estimation of the parameters when minimizing the ordinary least square norm [3-5].

As described in [6], the parameter estimation technique requires measured temperature values. It is essential that the measured response be sensitive to the parameters of interest: the more sensitive the temperature (or large the sensitivity coefficient) is, the more valuable the temperature measurements are [1]. Also, in order to gain much insight and information from the results as possible, it must be that



the sensitivity coefficients with respect to the parameters of interest are uncorrelated and large in magnitudes [7].

The focus of the current paper is to provide sensitivity coefficients in the event that the effect of the thin heater in contact with the specimen is simulated through an high-conductivity thin layer to which a constant heat flux is applied for a finite period of time. Sensitivity coefficients for a similar case, but involving a constant heat flux applied for a unlimited period of time, are provided in a previous work of the same authors [7], where they were computed numerically using a two-points central difference approximation for large times. The addressed heat conduction problem concerns a one dimensional finite rectangular body, representing the sample, subject to a boundary condition of the fourth kind [8, Chap. 2] at the heated boundary $x=0$, and insulated at the backside ($x=L$). As the problem is linear, the thermal field is obtained by using the superposition principle. Then, the so-called “scaled” sensitivity coefficients are computed analytically by performing the partial derivatives of the temperature with respect to the parameters of interest. The coefficients are presented in a graphical form and they are calculated for two different location: 1) at the interface between the heater and the sample, and 2) at the sample backside. The results of the analysis show that the sensitivity coefficients with respect the thermal properties of the sample are uncorrelated and, therefore, the thermal conductivity and the heat capacity can be estimated simultaneously.

2. Mathematical formulation

The schematic of an experimental apparatus for the thermal properties measurements of solid materials is shown in figure 1a. In particular, a thin layer heater is located at the interface of two samples of the same material and thickness and gives heat up at surface of both samples. Also, all the external surfaces are insulated and the uniform volumetric heat generation within the heater, $g(t)$, is applied for a finite period of time (from $t=0$ to t_h) as follows.

$$g(t) = g_0 H(t) - g_0 H(t - t_h) \quad (\tilde{t} > 0) \quad (1)$$

where g_0 is the constant volumetric heat generation, while $H(\cdot)$ is the Heaviside function.

For the sake of thermal symmetry, the three-layer configuration (specimen-heater-specimen) reduces to a simplified two-layer configuration depicted in figure 1b. In particular, it is assumed that 1) the heater is in perfect thermal contact with the sample; 2) the heater can be modelled as a lumped body, and 3) the thermal properties of the sample (thermal conductivity k and volumetric heat capacity C) are temperature-independent as well as the volumetric heat capacity of the heater (C_f). Note that due to the second assumption the volumetric heat generation can be considered as a surface heat flux, $q''_{f,0} = g_0 L_f$, applied for a finite period of time as shown by figure 1b, where L_f is a half of the heater thickness. Also, the sample and the heater are initially at the same uniform temperature T_{in} .

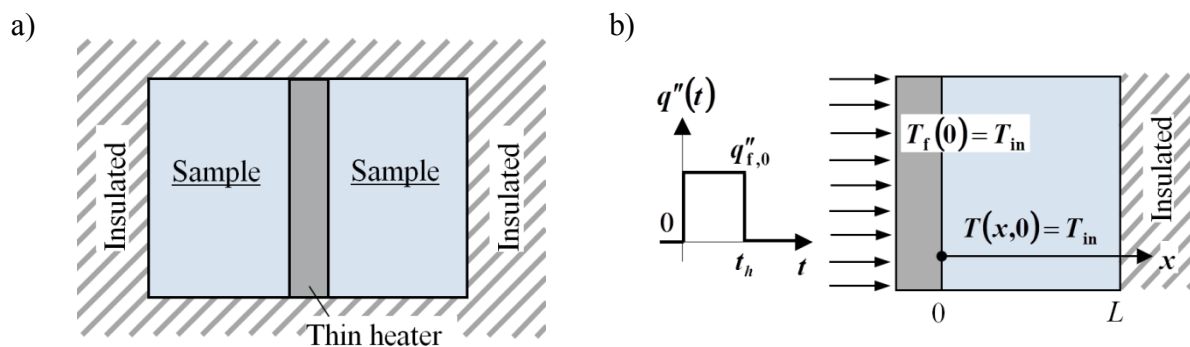


Figure 1. Schematic of the experimental apparatus for thermal properties measurements a) and simplified schematic for the addressed problem b).

The mathematical formulation of this transient, linear, rectangular heat conduction problem is defined in dimensionless form as follows.

$$\frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} = \frac{\partial \tilde{T}}{\partial \tilde{t}} \quad (0 < \tilde{x} < 1; \tilde{t} > 0) \quad (2a)$$

$$-\left(\frac{\partial \tilde{T}}{\partial \tilde{x}}\right)_{\tilde{x}=0} + P \frac{\partial \tilde{T}_f}{\partial \tilde{t}} = \underbrace{H(\tilde{t}) - H(\tilde{t} - \tilde{t}_h)}_{q_f''(\tilde{t})}; \quad \tilde{T}_f(\tilde{t}) = \tilde{T}(0, \tilde{t}) \quad (\tilde{t} > 0) \quad (2b)$$

$$-\left(\frac{\partial \tilde{T}}{\partial \tilde{x}}\right)_{\tilde{x}=1} = 0 \quad (\tilde{t} > 0) \quad (2c)$$

$$\tilde{T}_f(0) = 0; \quad \tilde{T}(\tilde{x}, 0) = 0 \quad (0 < \tilde{x} < 1) \quad (2d)$$

where equation (2b) represents the boundary condition of the fourth kind at $\tilde{x} = 0$ [8] and the dimensionless variables appearing in the above equations are defined as:

$$\tilde{T} = \frac{T - T_{in}}{q_{f,0}'' L/k}, \quad \tilde{T}_f = \frac{T_f - T_{in}}{q_{f,0}'' L/k}, \quad \tilde{q}'' = \frac{q''}{q_{f,0}''}, \quad \tilde{q}_f'' = \frac{q_f''}{q_{f,0}''}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{t} = \frac{kt}{CL^2}, \quad \tilde{t}_h = \frac{kt_h}{CL^2}, \quad P = \frac{C_f L_f}{CL}$$

In detail the P parameter appearing in equation (2) denotes the ratio between the heat capacity of the heater and the sample.

According to the numbering system devised in [9] this linear problem may be denoted by X42B50T00. In detail, this number denotes a transient heat conduction problem concerning a 1D rectangular finite body (by the “X”), subject to a boundary condition of the fourth kind at the surface $x=0$ (by the “4” in X42) where a finite duration heat flux is applied (by the B5), and having an insulated boundary at $x=L$ (heat flux boundary condition by the “2” in X42, which is homogeneous by the “0” in B50); also, T00 denotes a zero initial temperature for both layers.

3. Temperature solution

The solution to the current problem, equations (2a)-(2d), has to be calculated both when the heating is “on” ($0 < \tilde{t} \leq \tilde{t}_h$) and when it is “off” ($\tilde{t} > \tilde{t}_h$). By using the superposition principle it results in

$$\tilde{T} = \begin{cases} \tilde{T}_{X42}(\tilde{x}, \tilde{t}) & (0 \leq \tilde{t} \leq \tilde{t}_h) \\ \tilde{T}_{X42}(\tilde{x}, \tilde{t}) - \tilde{T}_{X42}(\tilde{x}, \tilde{t} - \tilde{t}_h) & (\tilde{t} > \tilde{t}_h) \end{cases} \quad (3)$$

where \tilde{T}_{X42} is the response to a constant heat flux applied at $\tilde{t} = 0$ for an infinite duration. For such a heat flux the problem is denoted by X42B10T00 whose solution is available in references [10, p. 128, # 5] and [11] (note that the “1” in B10 denotes a constant heat flux applied for any $\tilde{t} > 0$). It is

$$\tilde{T}_{X42}(\tilde{x}, \tilde{t}) = \underbrace{\frac{\tilde{t}}{\tilde{N}_0} + \frac{\tilde{x}^2}{2\tilde{N}_0} - \frac{\tilde{x}}{\tilde{N}_0} + \frac{1}{3\tilde{N}_0^2}}_{\tilde{T}_{X42,qs}(\tilde{x}, \tilde{t})} - \underbrace{\sum_{m=1}^{\infty} \frac{\cos(\beta_m \tilde{x}) - P\beta_m \sin(\beta_m \tilde{x})}{\tilde{N}_m \beta_m^2}}_{\tilde{T}_{X42,\sigma}(\tilde{x}, \tilde{t})} e^{-\beta_m^2 \tilde{t}} \quad (4a)$$

where the dimensionless norm \tilde{N}_m and the eigenvalues β_m are defined as, respectively,

$$\tilde{N}_m = \begin{cases} P + 1 & m = 0 \\ \frac{(P\beta_m)^2 + P + 1}{2} & m = 1, 2, \dots \end{cases} \quad (4b)$$

$$\beta_m \cot(\beta_m) = -P^{-1} \quad m = 0, 1, 2, \dots \quad (4c)$$

The $\beta_0 = 0$ eigenvalue is responsible of the term \tilde{t}/\tilde{N}_0 appearing in equation (4a). The computation of the remaining β_m eigenvalues as well as the number of terms m_{max} are discussed in the next subsections.

3.1. Computation of the eigenvalues

The eigenvalues appearing in equation (4a) are computed as roots of the eigencondition (transcendental equation) given by equation (4c).

Its roots may be computed by using explicit approximate relations based on the second-order modified Newton method [12]. These relations provide an approximate value of the exact eigenvalue with high accuracy (8-decimal place after one iteration, and 15-decimal place after two iterations) for the P range $[0, \infty)$.

3.2. Computational solution

The infinite series appearing in the temperature solution, equation (4a), exhibits an exponential convergence, that is, very fast. However, as we cannot take into account infinite terms a convergence criterion for this series is needed. Following the procedure given in [13], the maximum number of required terms m_{max} to get a truncation error of 10^{-A} ($A=2,3,..15$) in the series solution may be taken as:

$$m_{max} = \text{ceil} \left\{ \frac{1}{2} + \frac{1}{\pi} \left[\frac{A \ln(10)}{\tilde{t}} \right]^{1/2} \right\} \quad (5)$$

where $(m-1/2)\pi$ has been used as a conservative estimate for β_m and “ceil(z)” is a Matlab function that rounds the number z to the nearest integer greater than or equal to z .

3.2.1. Early time solution. In addition, according to equation (5) a large number of terms is required for early times. However, for times less than the so-called deviation time \tilde{t}_d , defined as [9, 14]

$$\tilde{t}_d = \frac{(2 - \tilde{x})^2}{10A} \quad (A = 2, 3, \dots, 15), \quad (6)$$

the temperature \tilde{T}_{X42} appearing in equation (3) can be replaced by the solution of the semi-infinite transient X40B1T00 problem with an error less than 10^{-A} .

However, as \tilde{t}_d can be less or greater than \tilde{t}_h , two cases has to be considered:

- $\tilde{t}_d \leq \tilde{t}_h$. In this case, the temperature solution is simply given by

$$\tilde{T}(\tilde{x}, \tilde{t}) \cong \tilde{T}_{X40}(\tilde{x}, \tilde{t}) \quad (0 \leq \tilde{t} \leq \tilde{t}_d \leq \tilde{t}_h) \quad (7a)$$

- $\tilde{t}_d > \tilde{t}_h$. In such a case, equation (3) holds the two parts and may be rewritten as

$$\tilde{T}(\tilde{x}, \tilde{t}) \cong \begin{cases} \tilde{T}_{X40}(\tilde{x}, \tilde{t}) & (0 \leq \tilde{t} \leq \tilde{t}_h < \tilde{t}_d) \\ \tilde{T}_{X40}(\tilde{x}, \tilde{t}) - \tilde{T}_{X40}(\tilde{x}, \tilde{t} - \tilde{t}_h) & (\tilde{t}_h < \tilde{t} \leq \tilde{t}_d) \end{cases} \quad (7b)$$

where $\tilde{T}_{X40}(\tilde{x}, \tilde{t})$ denotes the solution of the X40B1T00 problem. It is available in the heat conduction literature [10, p 306, #12] and it is also discussed in [15]. In dimensionless form it results in

$$\tilde{T}_{X40}(\tilde{x}, \tilde{t}) = \sqrt{\frac{4\tilde{t}}{\pi}} \exp\left(-\frac{\tilde{x}^2}{4\tilde{t}}\right) - \tilde{x} \operatorname{erfc}\left(\frac{\tilde{x}}{\sqrt{4\tilde{t}}}\right) - P \left[\operatorname{erfc}\left(\frac{\tilde{x}}{\sqrt{4\tilde{t}}}\right) - \operatorname{Am}(\tilde{x}, \tilde{t}, P) \right] \quad (0 \leq \tilde{t} \leq \tilde{t}_d) \quad (8a)$$

where $\operatorname{Am}(\tilde{x}, \tilde{t}, P)$ is the so-called Amos function defined as

$$\operatorname{Am}(\tilde{x}, \tilde{t}, P) = \exp\left(\frac{\tilde{x}}{P} + \frac{\tilde{t}}{P^2}\right) \operatorname{erfc}\left(\frac{\tilde{x}}{\sqrt{4\tilde{t}}} + \frac{\sqrt{\tilde{t}}}{P}\right) = \exp\left(-\frac{\tilde{x}^2}{4\tilde{t}}\right) \operatorname{erfcx}\left(\frac{\tilde{x}}{\sqrt{4\tilde{t}}} + \frac{\sqrt{\tilde{t}}}{P}\right) \quad (8b)$$

Moreover, $\tilde{T}_{X40}(\tilde{x}, \tilde{t} - \tilde{t}_h)$ may be derived from equation (8a) by simply replacing \tilde{t} with $\tilde{t} - \tilde{t}_h$.

A verified computer code in Matlab ambient for calculating the solution of the X42B10T00 problem is provided in [11]. Also, for the sake of completeness, a plot of the dimensionless temperature as a function of time with \tilde{t}_h as a parameter is given in figure 2.

4. Sensitivity coefficients

The scaled sensitivity coefficients are defined as partial derivatives of the temperature with respect to the model parameter p of interest (e.g. k, C, C_f) multiplied by the parameter itself:

$$X_p = p \frac{\partial T}{\partial p} \quad \text{with } p = k, C, C_f \quad (9)$$

Note that the scaled sensitivity coefficients have units of °C and their calculation requires the use of the chain rule [16].

When the parameter of interest is the thermal conductivity, the corresponding sensitivity coefficient can be expressed as follows

$$X_k = k \frac{\partial T}{\partial k} = k \frac{\partial}{\partial k} \left(\tilde{T} \frac{q_{\tilde{t}0}'' L}{k} \right) = \frac{q_{\tilde{t}0}'' L}{k} \left[k \frac{\partial \tilde{T}}{\partial k} - \tilde{T} \right] \quad (10)$$

where $\tilde{T} = \tilde{T}[\tilde{x}, \tilde{t}, P, \beta_m(P), \tilde{t} - \tilde{t}_h]$, with $\tilde{t}(k, C)$, $P(C, C_f)$, $\tilde{t} - \tilde{t}_h(k, C)$. Therefore, by using the chain

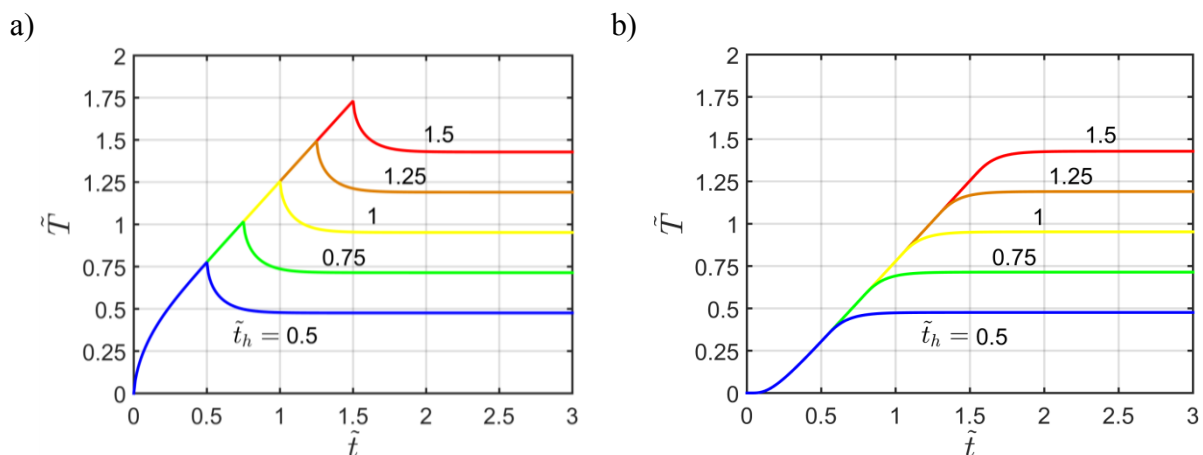


Figure 2. Dimensionless temperature as a function of the dimensionless time, with \tilde{t}_h as a parameter and $P=0.05$, at a) $\tilde{x}=0$ and at b) $\tilde{x}=1$.

rule it is found that

$$\frac{\partial \tilde{T}}{\partial k} = \frac{\partial \tilde{T}}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial k} + \frac{\partial \tilde{T}}{\partial (\tilde{t} - \tilde{t}_h)} \frac{\partial (\tilde{t} - \tilde{t}_h)}{\partial k} \quad (11)$$

By substituting equation (11) in equation (10), one can obtain the scaled sensitivity coefficient with respect to the thermal conductivity. In dimensionless form it results in:

$$\tilde{X}_k = \tilde{t} \frac{\partial \tilde{T}}{\partial \tilde{t}} + (\tilde{t} - \tilde{t}_h) \frac{\partial \tilde{T}}{\partial (\tilde{t} - \tilde{t}_h)} - \tilde{T} \quad (12)$$

where $\tilde{X}_k = X_k / (q_{t0}'' L / k)$.

By following the same procedure described above, the dimensionless scaled sensitivity coefficient with respect to the volumetric heat capacity C is obtained.

$$\tilde{X}_C = C \frac{\partial \tilde{T}}{\partial C} = -\tilde{t} \frac{\partial \tilde{T}}{\partial \tilde{t}} - P \frac{\partial \tilde{T}}{\partial P} - (\tilde{t} - \tilde{t}_h) \frac{\partial \tilde{T}}{\partial (\tilde{t} - \tilde{t}_h)} \quad (13)$$

In similar manner the scaled sensitivity coefficient with respect to the volumetric heat capacity of the heater C_f , in dimensionless form, results in:

$$\tilde{X}_{C_f} = C_f \frac{\partial \tilde{T}}{\partial C_f} = P \frac{\partial \tilde{T}}{\partial P} \quad (14)$$

Also, the following relationship involving the scaled sensitivity coefficients listed above is valid for all values of time and position.

$$\tilde{X}_k + \tilde{X}_C + \tilde{X}_{C_f} = -\tilde{T} \quad (15)$$

4.1. Computation of the scaled sensitivity coefficients \tilde{X}_k , \tilde{X}_C and \tilde{X}_{C_f}

The sensitivity coefficients discussed above may be computed both when the heating is “on” ($0 < \tilde{t} \leq \tilde{t}_h$) and when it is “off” ($\tilde{t} > \tilde{t}_h$). By using the superposition principle they may be given as

$$\tilde{X}_i = \begin{cases} \tilde{X}_{i,X42}(\tilde{x}, \tilde{t}, P) & (0 \leq \tilde{t} \leq \tilde{t}_h) \\ \tilde{X}_{i,X42}(\tilde{x}, \tilde{t}, P) - \tilde{X}_{i,X42}(\tilde{x}, \tilde{t} - \tilde{t}_h, P) & (\tilde{t} > \tilde{t}_h) \end{cases} \quad \text{with } i = k, C, C_f \quad (16)$$

where $\tilde{X}_{i,X42}$ (with $i=k, C, C_f$) denotes the sensitivity coefficients of the X42B10T00 case which, bearing in mind equations (12)-(14), can be evaluated using the following relations

$$\tilde{X}_k(\tilde{x}, \tilde{t}, P) = \tilde{t} \frac{\partial \tilde{T}}{\partial \tilde{t}} - \tilde{T}; \quad \tilde{X}_k(\tilde{x}, \tilde{t} - \tilde{t}_h, P) = (\tilde{t} - \tilde{t}_h) \frac{\partial \tilde{T}}{\partial (\tilde{t} - \tilde{t}_h)} - \tilde{T} \quad (17a)$$

$$\tilde{X}_C(\tilde{x}, \tilde{t}, P) = -\tilde{t} \frac{\partial \tilde{T}}{\partial \tilde{t}} - P \frac{\partial \tilde{T}}{\partial P}; \quad \tilde{X}_C(\tilde{x}, \tilde{t} - \tilde{t}_h, P) = -(\tilde{t} - \tilde{t}_h) \frac{\partial \tilde{T}}{\partial (\tilde{t} - \tilde{t}_h)} - P \frac{\partial \tilde{T}}{\partial P} \quad (17b)$$

$$\tilde{X}_{C_f}(\tilde{x}, \tilde{t}, P) = P \frac{\partial \tilde{T}}{\partial P}; \quad \tilde{X}_{C_f}(\tilde{x}, \tilde{t} - \tilde{t}_h, P) = P \frac{\partial \tilde{T}}{\partial P} \quad (17c)$$

Then, by applying the above equations to equation (4a) the sensitivities $\tilde{X}_{i,X42}(\tilde{x}, \tilde{t}, P)$ (with $i=k, C,$

C_t) result in

$$\tilde{X}_{k,X42} = \frac{\tilde{x}(2-\tilde{x})}{2\tilde{N}_0} - \frac{1}{3\tilde{N}_0^2} + \sum_{m=1}^{\infty} \frac{\cos(\beta_m \tilde{x}) - P\beta_m \sin(\beta_m \tilde{x})}{\tilde{N}_m \beta_m^2} (1 + \beta_m^2 \tilde{t}) e^{-\beta_m^2 \tilde{t}} \quad (18a)$$

$$\tilde{X}_{C,X42} = -\frac{\tilde{t}}{\tilde{N}_0} + \frac{P[2\tilde{t} + \tilde{x}(\tilde{x}-2)]}{2\tilde{N}_0^2} + \frac{2P}{3\tilde{N}_0^3} + \sum_{m=1}^{\infty} \frac{D_m \cos(\beta_m \tilde{x}) - E_m \sin(\beta_m \tilde{x})}{\tilde{N}_m^2 \beta_m^4} e^{-\beta_m^2 \tilde{t}} \quad (18b)$$

$$\tilde{X}_{C_t,X42} = -\frac{P[2\tilde{t} + \tilde{x}(\tilde{x}-2)]}{2\tilde{N}_0^2} - \frac{2P}{3\tilde{N}_0^3} + P \sum_{m=1}^{\infty} \frac{F_m \sin(\beta_m \tilde{x}) + G_m \cos(\beta_m \tilde{x})}{\tilde{N}_m^2 \beta_m^4} e^{-\beta_m^2 \tilde{t}} \quad (18c)$$

where

$$D_m = \tilde{N}_m \beta_m^4 \tilde{t} - P G_m, \quad E_m = P(\tilde{N}_m \beta_m^5 \tilde{t} + F_m) \quad (18d)$$

Also,

$$F_m = -P\beta_m^2 \frac{d\beta_m}{dP} (2P^2 \beta_m^2 + \tilde{N}_0) + \tilde{N}_m \beta_m^2 \left[\frac{d\beta_m}{dP} (\tilde{x} - 2P\beta_m^2 \tilde{t} - P) + \beta_m \right] - P\beta_m^3 \left(P\beta_m^2 + \frac{1}{2} \right) \quad (18e)$$

$$G_m = \beta_m \frac{d\beta_m}{dP} [2P^2 \beta_m^2 + \tilde{N}_m \beta_m^2 (P\tilde{x} + 2\tilde{t}) + \tilde{N}_0] + \beta_m^2 \left(P\beta_m^2 + \frac{1}{2} \right) \quad (18f)$$

It is worth noting that the sensitivity coefficients given by equations (18a)-(18c) can be computed analytically as the derivative $d\beta_m/dP$ appearing in equations (18e) and (18f) may be given in an algebraic form. In fact, the eigencondition defined by equation (4c) can be rewritten as

$$P = -\frac{\tan(\beta_m)}{\beta_m} \quad (19)$$

Then, by differentiating equation (19) with respect to the eigenvalue β_m and by taking the corresponding reciprocals, it is found that

$$\frac{d\beta_m}{dP} = \frac{\tan(\beta_m)[1 - \beta_m \tan(\beta_m)] - \beta_m}{\beta_m^2} \quad (20)$$

Moreover, the sensitivity coefficients $\tilde{X}_{i,X42}(\tilde{x}, \tilde{t} - \tilde{t}_h, P)$ may be derived from equations (18a)-(18f) by simply replacing \tilde{t} with $\tilde{t} - \tilde{t}_h$.

For early times (less than the deviation time \tilde{t}_d) the sensitivity coefficients may be evaluated analytically by using equations (7a) and (7b).

5. Results

Figures 3-5 show the dimensionless sensitivity coefficients \tilde{X}_k , \tilde{X}_C and \tilde{X}_{C_t} as a function of time for different P values. They are computed for a heating duration of $\tilde{t}_h = 1$ and for two temperature sensor locations: 1) at the interface between the heater and the sample ($\tilde{x} = 0$), and 2) at the sample backside ($\tilde{x} = 1$). In addition, in these figures a comparison with the sensitivities for $P=0$ (when the heater is completely neglected - X22B50T0 case [6, Chap. 8]) is also provided.

As shown by figure 3 the sensitivity to thermal conductivity of the sample \tilde{X}_k increases in absolute

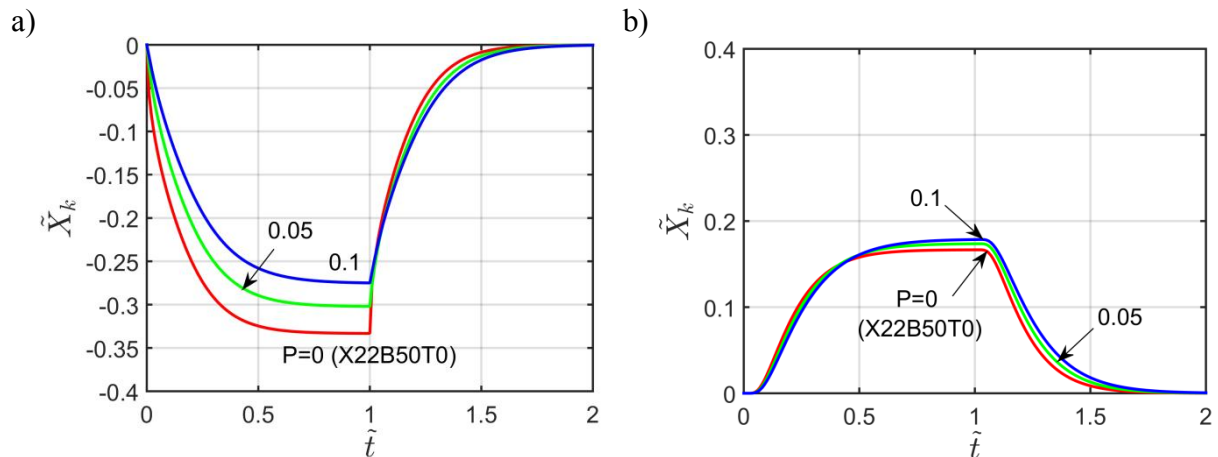


Figure 3. Dimensionless scaled sensitivity coefficient to the thermal conductivity as a function of the dimensionless time, for different P values, at a) $\tilde{x} = 0$ and at b) $\tilde{x} = 1$.

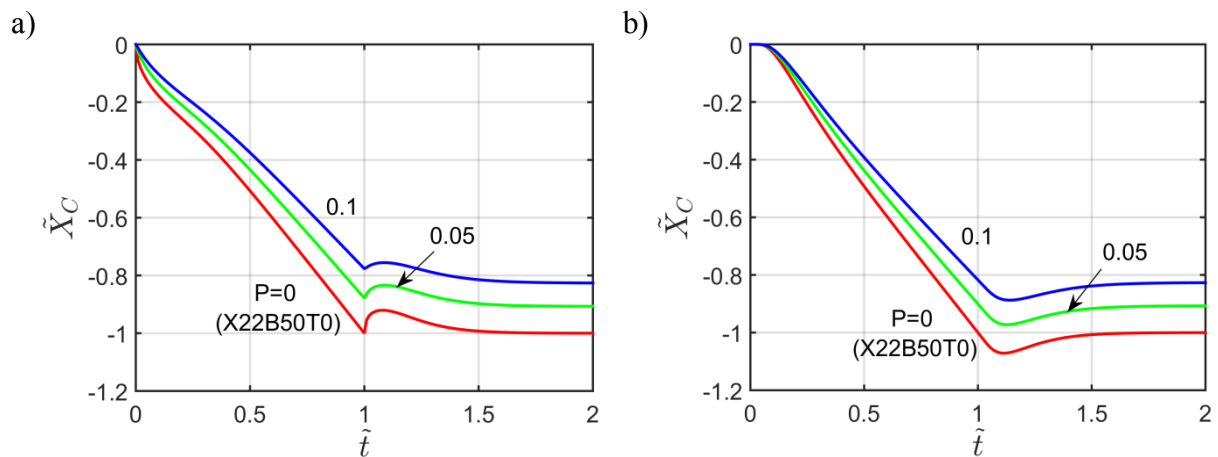


Figure 4. Dimensionless scaled sensitivity coefficient to the heat capacity of the sample as a function of the dimensionless time, for different P values, at a) $\tilde{x} = 0$ and at b) $\tilde{x} = 1$.

value with time during the heating period; then, when the heater is turn off ($\tilde{t}_h = 1$) it decreases approaching zero for large times. In particular, figure 4a shows that during the heating the sensitivity to k at $\tilde{x} = 0$ is lower in absolute value when P is higher, while the opposite occurs when the heating is off. At the sample backside the behaviour is quite different (see figure 3b). In fact, after a certain time (of about 0.4), \tilde{X}_k increases when P increases even during the heating period. Note, also, that the greatest sensitivity to k occurs at the interface heater-sample ($\tilde{x} = 0$).

The sensitivity to the volumetric heat capacity of the sample \tilde{X}_C is shown by figure 4. In particular, in figure 4a the sensitivity to C at $\tilde{x} = 0$ increases in absolute value with time during the heating period, while it exhibits a slight fall when the heater is turn off ($\tilde{t}_h = 1$); then it continues to increase until a constant value is approached for large times. On the contrary, the sensitivity at $\tilde{x} = 1$ continues in absolute value to rise even during the cooling period and, then, it decreases to reach the same constant value occurring at $\tilde{x} = 0$, as shown by figure 4b. In addition, the sensitivity to C at $\tilde{x} = 1$ is larger than that at $\tilde{x} = 0$ for some times. Unlike the sensitivity to k , the coefficient \tilde{X}_C is lower in absolute

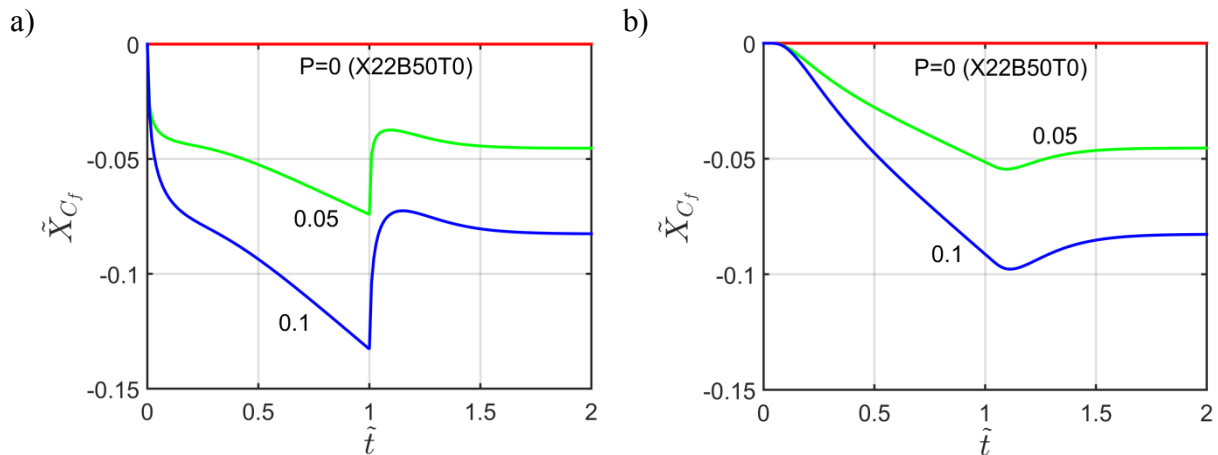


Figure 5. Dimensionless scaled sensitivity coefficient to the heat capacity of the heater as a function of the dimensionless time, for different P values, at a) $\tilde{x} = 0$ and at b) $\tilde{x} = 1$.

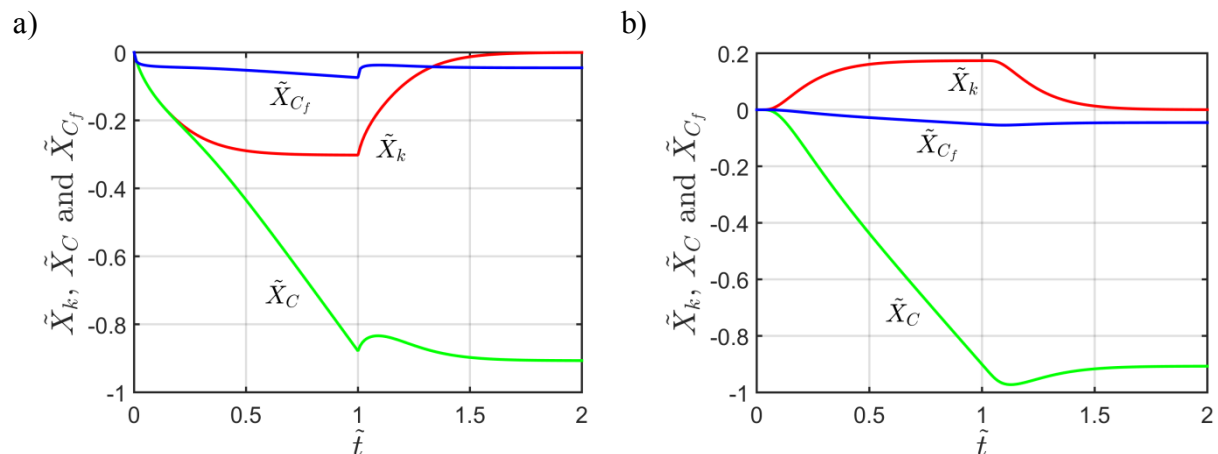


Figure 6. Comparison among dimensionless scaled sensitivity coefficients for $P = 0.05$ and $\tilde{t}_h = 1$ at a) $\tilde{x} = 0$ and at b) $\tilde{x} = 1$.

value when P is larger for all the experiment duration.

Figure 5 shows the sensitivity to the volumetric heat capacity of the heater \tilde{X}_{C_f} . Similarly to \tilde{X}_C , the sensitivity to C_f at $\tilde{x} = 0$ drops steeply as soon as the heater is turned off (see figure 5a), whereas the sensitivity at $\tilde{x} = 1$ continues to rise as shown by figure 5b. However, at both locations the same constant value is approached for large times. During the heating period the largest sensitivity occurs at $\tilde{x} = 0$, while for $\tilde{t} > 1$ it is larger at the sample backside.

Moreover, a comparison among the sensitivity coefficients computed at both $\tilde{x} = 0$ and $\tilde{x} = 1$ for $P = 0.05$ is given by Fig. 6. It suggests that, at early times, the sensitivities \tilde{X}_k and \tilde{X}_C are nearly equal, that is, correlated while, for larger times, they behave quite differently and, hence, they are uncorrelated. In addition, when the steady state is reached the sensitivity to the thermal conductivity k approaches zero, while the sensitivity to the heat capacity of the sample C approaches a not null steady value. It is also evident that the sensitivity function \tilde{X}_{C_f} is very small at both $\tilde{x} = 0$ and $\tilde{x} = 1$. In

other words the temperature of the sample is more sensitive to its heat capacity than to the heat capacity of the heater.

A comparison with the sensitivity obtained when a constant heat flux is applied for a unlimited period of time [7] is now discussed. As shown by figures 3-6 the sensitivity coefficients changes in shape when the heating is off. These changes in shape do not occur when a constant heat flux is applied for a unlimited period. In fact, in such a case the sensitivity to the thermal conductivity would approach a constant value, while the sensitivities to the volumetric heat capacity of the heater and the sample would continue to increase monotonically with time. Also, this is valid at both locations ($\tilde{x} = 0$ and $\tilde{x} = 1$). However, sensitivity coefficients having changes in shape or in sign are useful to reduce the degree of correlation among parameters and also to maximize the determinant of the $X^T X$ matrix (where X is the sensitivity matrix) involved in the parameter estimation procedure. For these reasons a finite heating period plays a key role on sensitivity coefficients.

6. Conclusions

The temperature solution for the X42B50T00 problem has been derived by using the superposition principle. Then, the sensitivity coefficients with respect to the thermal properties of the sample (thermal conductivity k and volumetric heat capacity C) and the volumetric heat capacity of the heater have been investigated in order to establish the best location for the temperature sensor.

The analysis shows that the sensitivity coefficients with respect to k and to C are uncorrelated and, therefore, these thermal properties can be estimated simultaneously. However, the maximum sensitivity to the thermal conductivity occurs the interface between the heater and the sample, while the sensitivity to the volumetric heat capacity of the specimen exhibits its maximum at the sample backside. For this reason, in order to estimate k and C simultaneously it should be more effective using two sensors (one located at $\tilde{x} = 0$ and the other at $\tilde{x} = 1$) rather than using only one sensor.

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