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Existence of quasiequilibria in metric vector spaces

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ABSTRACT

This paper provides sufficient conditions for the existence of solutions for quasi-equilibrium problems and generalized game problems in the setting of infinite-dimensional metrizable spaces. To this purpose, we prove a modified version of a selection theorem due to Michael [15] by exploiting the fact that any compact set in a metric space is both complete and separable. Thereafter, by a fixed point technique which is based on the notion of inside point of a convex set, we provide some existence results without requiring the upper semicontinuity and the closed-valuedness of the feasibility maps.

1. Introduction

A great number of mathematical equilibrium models which are apparently different have a common structure that leads to a unified format: the Ky Fan inequality problem [7]. In these equilibrium problems the constraint set is fixed and hence the model can not be used in many cases where the constraints depend on the current analyzed point: quasivariational inequalities, generalized Nash equilibrium problems, equilibria in economics, network equilibrium problems and so on. This more general setting, commonly called quasiequilibrium problem, originates in [3] and it was extended to infinite dimensional spaces in [13] for the first time.

Formally, the quasiequilibrium problem consists in finding $x \in C$ such that

$$x \in K(x) \quad \text{and} \quad f(x,y) \geq 0, \quad \forall y \in K(x) \tag{1}$$

where C is a nonempty set, $f : C \times C \to \mathbb{R}$ is the equilibrium function, and the set-valued map $K : C \rightrightarrows C$ describes how the feasible region changes together with the considered point. Clearly, quasiequilibrium problems are modeled upon quasivariational inequalities.

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A generalized Nash equilibrium problem with a finite number of players can be reformulated through a quasiequilibrium problem as well. We recall that a generalized Nash equilibrium problem is a noncooperative game in which, in contrast to the standard Nash equilibrium problem, the strategy set of each player depends on the strategies of all the other players. Let N be a family of players. Each player i controls the variables $x_i \in C_i$, where C_i is a nonempty set. We denote by $x = (x_i) \in \prod_{i\in N} C_i = C$ the vector formed by all these decision variables and by x^{-i} we denote the strategy vector of all the players different from player i. The set of all such vectors will be denoted by C_{-i}. We sometimes write (x_i, x^{-i}) instead of x in order to emphasize the i-th player's variables within x. Each player i has an objective loss function $\theta_i : C \to \mathbb{R}$ that depends on all players' strategies. Furthermore, each player's strategy must belong to a set identified by the set-valued map $K_i : C_{-i} \rightrightarrows C_i$ in the sense that the strategy space of the player i is $K_i(x^{-i})$ which depends on the rival players' strategies x^{-i}. The generalized Nash equilibrium problem consists in finding $x \in C$ such that for each $i \in N$ one has

$$x_i \in K_i(x^{-i}) \quad \text{and} \quad \theta_i(x_i, x^{-i}) \leq \theta_i(y_i, x^{-i}), \quad \forall y_i \in K_i(x^{-i}). \tag{2}$$

When N is finite, finding a generalized Nash equilibrium amounts to solving a quasiequilibrium problem with the Nikaido-Isoda aggregate function

$$f(x, y) = \sum_{i\in N} \big(\theta_i(x^{-i}, y_i) - \theta(x)\big)$$

and $K(x) = \prod_{i\in N} K_i(x^{-i})$.

Usually most of the existence results for quasiequilibrium problems require both the upper and the lower semicontinuity of the set-valued map K together with its closed convex-valuedness (see for instance [4,6,16,17]). In [9] existence results in a finite dimensional space were proved, in which upper semicontinuity and closed-valuedness of the map K were replaced with closedness of the set of the fixed points of K. Cubiotti and Yao [10] extend this result in normed strategy spaces but at the expense of requiring the Hausdorff lower semicontinuity instead of the weaker notion of lower semicontinuity and assuming the nonemptiness of the interior of $K_i(x^{-i})$ with respect to the affine space generated by C_i. More recently, the authors of the present paper have succeeded in replacing the Hausdorff lower semicontinuity by the usual lower semicontinuity [8]. This provides a partial positive answer to a question raised in [10] since the result is proved assuming the completeness and the separability of the strategy spaces. The separability assumption has been dropped in [2], but adding the closed-valuedness of K.

In this paper we establish new existence results for problems (1) and (2) defined on convex compact subsets of metrizable locally convex vector spaces. In the same spirit of [2,8–10] we do not require the upper semicontinuity of the feasibility set-valued maps. Moreover we do not make use of the Hausdorff lower semicontinuity as in [10], the separability as in [8], or the completeness as in [2]. The key step is a selection theorem for lower semicontinuous set-valued maps whose values are convex but not necessarily closed sets. This result is comparable to [15, Theorem 3.1‴]: our theorem involves a metrizable locally convex vector space instead of a separable Banach space, but at the cost of requiring the compactness of the range space.

2. Preliminaries

We start this section by recalling some topological and algebraic notions that will appear throughout the remainder of this paper (see [19] for more details). Given two sets $A \subseteq C$ in a topological space, we denote by $\operatorname{int}_C A$ and $\operatorname{cl}_C A$ the interior and the closure of A in the relative topology of C while $\partial_C A$ indicates the boundary of A in C, i.e.

$$\partial_C A = \operatorname{cl}_C A \setminus \operatorname{int}_C A = \operatorname{cl}_C A \cap \operatorname{cl}_C(C \setminus A).$$

In order to simplify notation, in a vector space a closed line segment between x and y is denoted by square brackets

$$[x, y] = \{tx + (1 - t)y : 0 \leq t \leq 1\}$$

and the corresponding open line segment is denoted by parentheses

$$(x, y) = \{tx + (1 - t)y : 0 < t < 1\}.$$

Similarly for intervals closed on one side and open on the other $[x, y)$ and $(x, y]$.

A normal space is a topological space X such that every two disjoint closed sets of X have disjoint open neighborhoods. X is perfectly normal if it is normal and every open set is a F_σ set, i.e. countable union of closed sets. An open cover of a space X is locally finite if every point of the space has a neighborhood that intersects only finitely many sets in the cover. A topological space X is said to be paracompact if every open cover has a locally finite open refinement. X is metrizable if the topology is compatible with some metric. All metric spaces are perfectly normal. We also have to say that every metric space is paracompact and that compact metric spaces are complete and separable.

A topological vector space is, in this paper, a vector space X with a topology such that every point of X is a closed set and the vector space operations are continuous. Every topological vector space is a Hausdorff space. Associate to each $a \in X$ and to each scalar $\lambda \neq 0$ the translation operator $x \mapsto x + a$ and the multiplication operator $x \mapsto \lambda x$ are homeomorphisms of X onto X. In this context a local base will always mean a local base at 0 and a topological vector space is locally convex if there is a local base whose members are convex. To ensure no misunderstanding with the notion of boundedness we recall that a subset C of a topological vector space X is bounded if for every neighborhood U of 0 there exists $t > 0$ such that $C \subseteq sU$, for all $|s| \geq t$. We stress the fact that when X is metrizable this notion does not match the classical one in metric spaces. Anyway a compact subset of a topological vector space is bounded.

For the basic definitions and facts about set-valued maps, we refer to [1] and [18]. Let $\Phi : X \rightrightarrows Y$ be a set-valued map with X and Y two topological spaces. The domain of Φ is

$$\operatorname{dom} \Phi = \{x \in X : \Phi(x) \neq \emptyset\},$$

the graph of Φ is

$$\operatorname{gph} \Phi = \{(x, y) \in \operatorname{dom} \Phi \times Y : y \in \Phi(x)\},$$

and the lower section of Φ at $y \in Y$ is

$$\Phi^{-1}(y) = \{x \in X : y \in \Phi(x)\}.$$

The map Φ is said to be closed-valued if $\Phi(x)$ is a closed set for any $x \in X$. The terms compact-valued and convex-valued are similarly defined. The map Φ is lower semicontinuous at x if for each open set Ω such that $\Phi(x) \cap \Omega \neq \emptyset$ there exists a neighborhood U_x of x such that $\Phi(x') \cap \Omega \neq \emptyset$ for every $x' \in U_x$; instead it is upper semicontinuous at x if for each open set Ω such that $\Phi(x) \subseteq \Omega$ there exists a neighborhood U_x of x such that $\Phi(x') \subseteq \Omega$ for every $x' \in U_x$. A set-valued map with open graph has open lower sections and, in turn, if it has open lower sections then it is lower semicontinuous. Moreover the domain of a lower semicontinuous set-valued map is open.

The next propositions are classical results concerning the preservation of lower semicontinuity under various set theoretic operations on set-valued maps.

Lemma 2.1. *Let $\Phi : X \rightrightarrows Y$ be lower semicontinuous and $\Omega \subseteq Y$ be open then the map $\Phi_\Omega : X \rightrightarrows Y$ defined by $\Phi_\Omega(x) = \Phi(x) \cap \Omega$ is lower semicontinuous.*

Lemma 2.2. *Let $\Phi_1, \Phi_2 : X \rightrightarrows Y$ be such that $\operatorname{cl}\Phi_1 = \operatorname{cl}\Phi_2$, then Φ_1 is lower semicontinuous if and only if Φ_2 is lower semicontinuous.*

Lemma 2.3. *Let $\Phi_1 : X \rightrightarrows Y$ be lower semicontinuous. If $C \subseteq X$ is a closed set and $\Phi_2 : C \rightrightarrows Y$ is lower semicontinuous with $\Phi_2(x) \subseteq \Phi_1(x)$, for every $x \in C$, then the map $\Phi : X \rightrightarrows Y$ defined by*

$$\Phi(x) = \begin{cases} \Phi_1(x) & \text{if } x \notin C \\ \Phi_2(x) & \text{if } x \in C \end{cases}$$

is lower semicontinuous.

Let $\{X_i\}_{i\in N}$ be an arbitrary family of topological spaces and $\Phi_i : X \rightrightarrows X_i$ be set-valued maps defined on $X = \prod_{i\in N} X_i$. We denote by $\Phi : X \rightrightarrows X$ the set-valued map defined by $\Phi(x) = \prod_{i\in N} \Phi_i(x)$, for each $x \in X$. The upper semicontinuity of each Φ_i is inherited by Φ under a suitable assumption.

Lemma 2.4. *If each Φ_i is upper semicontinuous and compact-valued then Φ is upper semicontinuous.*

Proof. The map Φ is the composition of the continuous function $\Delta : X \to X^N$, defined by $(\Delta(x))_i = x$, for each i, and the product map $\prod_{i\in N} \Phi_i : X^N \rightrightarrows X$ which is upper semicontinuous by [1, Theorem 17.28]. The upper semicontinuity of Φ can be derived by [1, Theorem 17.23]. □

Lastly, a fixed point of a set-valued map $\Phi : X \rightrightarrows X$ is a point $x \in X$ satisfying $x \in \Phi(x)$ and the set of the fixed points of Φ is denoted by $\operatorname{fix}\Phi$.

3. A selection result

A selection of a set-valued map $\Phi : X \rightrightarrows Y$ is a function $\varphi : X \to Y$ that satisfies $\varphi(x) \in \Phi(x)$ for each $x \in X$. The Axiom of Choice guarantees that set-valued maps with nonempty domain always admit selections, but they may have no additional useful properties. Michael proved a series of results on the existence of continuous selections that assume the condition of lower semicontinuity of set-valued maps.

One of the most widely used tools for establishing the existence of generalized equilibria is the famous Michael selection Theorem that we state in the version given in [18].

Theorem 3.1. *Let X be a paracompact space, (Y, d) is a metric locally convex vector space and $\Phi : X \rightrightarrows Y$ a lower semicontinuous set-valued map with nonempty complete convex values. Then Φ admits a continuous selection.*

The local convexity is mentioned explicitly because the balls belonging to a given metric need not be convex. One has, however, that if Y is metrizable and locally convex, then there exists a metric for Y with convex balls [19, Theorem 1.24]. Note that the balls defined from a norm are convex.

The completeness of $\Phi(x)$ plays a fundamental role and it cannot be dropped without adding as described in [15, Example 6.2]. Anyway, by strengthening the requirements on X and Y Michael showed that it is possible to dispense with the completeness and still guarantee the existence of a continuous selection [15, Theorem 3.1′′′]. More precisely it is assumed that X is perfectly normal, Y is a separable Banach space and the values of Φ belong to a suitable class of convex sets, which will be defined in the sequel. In this section we obtain a modified version of this result by exploiting the fact that any compact set in a metric space is both complete and separable.

Before proceeding with the statement and the proof of our result, we review a particular notion of relative interior for convex sets which was introduced by Michael [15]. Let C be a convex subset of a topological vector space X. The convex set $S \subseteq C$ is a face of C if $x_1, x_2 \in C$, $t \in (0,1)$ and $tx_1 + (1-t)x_2 \in S$ imply $x_1, x_2 \in S$. Let $\mathcal{F}_C$ be the (possibly empty) collection of all proper closed faces of $\operatorname{cl} C$.

Definition 3.1. A point $x \in C$ is an inside point if it is not in any proper closed face of $\operatorname{cl} C$. Denote by

$$I(C) = C \setminus \bigcup_{S \in \mathcal{F}_C} S$$

the set of the inside points of C.

A comparison with other notions of relative interior is given in [8] when X is Banach space. Anyway, the nonemptiness of the relative interior $\operatorname{ri} C$ of a convex set C, i.e. the interior of C relative to the closed affine hull of C, still implies that $\operatorname{ri} C = I(C)$ in the more general setting of a topological vector space.

Lemma 3.1. *Let C be a convex set in a topological vector space such that $\operatorname{ri}(C) \neq \emptyset$; then $\operatorname{ri} C = I(C)$.*

Proof. The inclusion $\operatorname{ri}(C) \subseteq I(C)$ can be proved as in [8, Theorem 2.2]. Assume by contradiction there exists $x \in I(C)$ such that $x \notin \operatorname{ri} C$ and, without loss of generality, suppose that $x = 0$. Since the closed affine hull of C is a vector subspace V, then the convex set $\operatorname{ri} C$ and $x = 0$ can be properly separated in V, i.e. there exists a continuous linear functional $x^* \in V^*$ such that $\langle x^*, z \rangle < 0$ for each $z \in \operatorname{ri} C$. Since x^* is continuous and $\operatorname{cl} \operatorname{ri} C = \operatorname{cl} C$ then $\langle x^*, z \rangle \leq 0$ for each $z \in \operatorname{cl} C$. Clearly the set $\{z \in V : \langle x^*, z \rangle = 0\} \cap \operatorname{cl} C$ is a proper closed convex face of $\operatorname{cl} C$ which contains $x = 0$ and this contradicts $x \in I(C)$. □

Lemma 3.2. *Let C be a convex set in a topological vector space. If $x \in C$ and $y \in I(C)$ then $[y, x) \subseteq I(C)$. In particular the set $I(C)$ is convex, and dense in C if it is nonempty.*

Proof. Since $y \in C$ then $[y, x) \subseteq C$. Assume by contradiction there exists some proper closed face S of $\operatorname{cl} C$ such that $(y, x) \cap S \neq \emptyset$. Hence $y \in S$ which contradicts $y \in I(C)$. □

Lemma 3.3. *Let C_i be a convex set in a topological vector space X_i with $i \in N = \{1, \ldots, n\}$. Then $I(\prod_{i \in N} C_i) = \prod_{i \in N} I(C_i)$.*

Proof. Firstly we show that S is a face of the product of the C_i if and only if S is the product of faces of C_i. The "if" part is clear, then let S be a face of $\prod_{i \in N} C_i$ and

$$S_i = \left\{ x_i \in X_i : \exists x^{-i} \in \prod_{j \neq i} X_j \text{ with } (x_i, x^{-i}) \in S \right\}$$

the projection of S on X_i. Trivially S_i is a face of C_i and $S \subseteq \prod_{i \in N} S_i$. Fix $x \in \prod_{i \in N} S_i$. For each i there exists $x^{-i} = (x_j^{-i}) \in \prod_{j \neq i} C_j$ such that $y_i = (x_i, x^{-i}) \in S$. Then

$$\frac{1}{n} x + \frac{n-1}{n} z = \sum_{i \in N} \frac{1}{n} y_i \in S,$$

where $z = (z_j)$ with

$$z_j = \sum_{i \neq j} \frac{1}{n-1} x_j^{-i} \in C_j.$$

It follows that $x \in S$ and $S = \prod_{i \in N} S_i$. Since a product of sets is closed if and only if each set is closed, S is a closed face of $\operatorname{cl} \prod_{i \in N} C_i = \prod_{i \in N} \operatorname{cl} C_i$ if and only if S is the product of closed faces of $\operatorname{cl} C_i$. Hence the conclusion easily follows. □

A convex series of elements of a subset C of a topological vector space is a series of the form $\sum_{i \geq 1} \lambda_i x_i$, where $x_i \in C$ and $\lambda_i \geq 0$ for each i, and $\sum_{i \geq 1} \lambda_i = 1$. The set C is said to be CS-compact [12] if every convex series of its elements converges to a point of the set C.

Lemma 3.4. *Let C be a nonempty compact convex and metrizable set in a topological vector space, then $I(C) \neq \emptyset$.*

Proof. Since every compact metric space is separable, there exists a sequence $\{x_i\}$ dense in C. Moreover the compactness of C implies that C is bounded and complete, hence C is CS-compact [12, Proposition 2]. The convex series $\sum_{i \geq 1} 2^{-i} x_i$ converges to a point $x \in C$ and we show that x belongs to $I(C)$. Assume by contradiction that there exists a proper closed face S of C such that $x \in S$. For every fixed i the convex series

$$\sum_{j \geq 1, j \neq i} \frac{2^{-j}}{1 - 2^{-i}} x_j$$

converges to $\hat{x}_i \in C$ and $x = 2^{-i} x_i + (1 - 2^{-i}) \hat{x}_i$. Two cases are possible: either $x = x_i$ or x is an interior point of the segment $[x_i, \hat{x}_i]$. In both cases $x_i \in S$. Since $\{x_i\}$ is dense in C and S is closed, this implies $S = C$, which is impossible. □

The last part of this introduction deals with the family of convex sets $\mathcal{D}(X)$ defined as follows:

$$\mathcal{D}(X) = \{C \subseteq X : C \text{ is convex and } I(\operatorname{cl} C) \subseteq C\}.$$

In [15], where X was a Banach space, it was proved that $\mathcal{D}(X)$ contains all the convex sets which are either closed, or with nonempty interior, or finite dimensional. In particular, when X is finite dimensional the class $\mathcal{D}(X)$ coincides with the family of all convex sets. More recently in [8] it has been proved that if $C \in \mathcal{D}(X)$ and $\Omega \subset X$ is open convex, then $C \cap \Omega \in \mathcal{D}(X)$. These results still hold if one replaces the Banach space with any topological vector space. Moreover the class $\mathcal{D}(X)$ has other properties of which are worthy of note.

Lemma 3.5. *Let C be a convex set in a topological vector space X. Then*

$$C \in \mathcal{D}(X) \text{ if and only if } I(C) = I(\operatorname{cl} C). \tag{3}$$

Moreover

a. *if* $\operatorname{ri}(C) \neq \emptyset$ *then* $C \in \mathcal{D}(X)$*;*
b. *if* $C \in \mathcal{D}(X)$ *then* $I(C) \in \mathcal{D}(X)$*;*
c. *if* $C \in \mathcal{D}(X)$ *then* $I(I(C)) = I(C)$*;*
d. *if* $C_1, C_2 \in \mathcal{D}(X)$ *with* $I(C_1) \cap I(C_2) \neq \emptyset$*, then* $\operatorname{cl}(C_1 \cap C_2) = \operatorname{cl} C_1 \cap \operatorname{cl} C_2$*;*
e. *if* $C_i \in \mathcal{D}(X)$ *with* $i \in N = \{1, \ldots, n\}$ *then* $\prod_{i \in N} C_i \in \mathcal{D}(X^n)$*.*

Proof. If $C \in \mathcal{D}(X)$ and $x \in I(\operatorname{cl} C)$ then $x \in C$ and $x \notin S$ for all $S \in \mathcal{F}_C$; hence $x \in I(C)$. The other implications in (3) are trivially true.

a. From Lemma 3.1 we have $I(C) = \operatorname{ri} C = \operatorname{ri} \operatorname{cl} C = I(\operatorname{cl} C)$ and (3) implies $C \in \mathcal{D}(X)$.
b. If $I(C) = \emptyset$ there is nothing to prove. Hence we may assume that $I(C) \neq \emptyset$, and $I(\operatorname{cl} I(C)) = I(\operatorname{cl} C) = I(C)$ where the first equality follows from Lemma 3.2 and the second from (3).
c. If $I(C) = \emptyset$ there is nothing to prove. Hence we may assume that $I(C) \neq \emptyset$, then $I(C) \in \mathcal{D}(X)$ from b. and $I(I(C)) = I(\operatorname{cl} I(C)) = I(\operatorname{cl} C) = I(C)$ where the first and the third equality follow from (3) and the second one follows from Lemma 3.2.
d. Let $x \in \operatorname{cl} C_1 \cap \operatorname{cl} C_2$ and $y \in I(C_1) \cap I(C_2) = I(\operatorname{cl} C_1) \cap I(\operatorname{cl} C_2)$ where the equality follows from (3). From Lemma 3.2 the segment $[y, x) \subseteq I(\operatorname{cl} C_1) \cap\ I(\operatorname{cl} C_2) \subseteq C_1 \cap C_2$ which implies that $x \in \operatorname{cl}(C_1 \cap C_2)$.
e. It follows from Lemma 3.3. □

Now we formulate the main result of this section.

Theorem 3.2. *Let X be a metric space, Y be a metrizable locally convex vector space and $\Phi : X \rightrightarrows Y$ be a compact lower semicontinuous set-valued map with nonempty values in the class $\mathcal{D}(Y)$. Then Φ admits a continuous selection.*

Theorem 3.2 involves a metrizable locally convex vector space instead of a separable Banach space [15, Theorem 3.1‴] but at the cost of requiring the compactness of the range space. Anyhow, Theorem 3.2 will be applied in the next section to set-valued maps of a compact set into itself, and hence the condition of compactness of the range space is automatically fulfilled. The proof of this selection result is based on the following.

Lemma 3.6. *Let X be a metric space, Y be a metrizable locally convex vector space and $\Phi : X \rightrightarrows Y$ be a compact lower semicontinuous set-valued map with nonempty closed convex values. Then there exists a countable collection $\mathcal{F}$ of continuous selections for Φ such that, for every $x \in C$, $\{\varphi(x)\}_{\varphi\in\mathcal{F}}$ is dense in $\Phi(x)$.*

Proof. Let d be a metric for Y with convex balls and $\{y_j\}$ be a countable, dense subset of $\operatorname{cl} \Phi(X)$. For each j and k, the nonempty set

$$U_{j,k} = \left\{x \in X : \Phi(x) \cap B(y_j, k^{-1}) \neq \emptyset\right\};$$

is open by the lower semicontinuity of Φ. In metrizable spaces, every open set is an F_σ set, then there exists a countable family of closed sets $\{C_{i,j,k}\}$ such that

$$U_{j,k} = \bigcup_{i\geq 1} C_{i,j,k}$$

Let $\Phi_{i,j,k} : X \rightrightarrows Y$ be defined by

$$\Phi_{i,j,k}(x) = \begin{cases} \Phi(x) & \text{if } x \notin C_{i,j,k} \\ \operatorname{cl}\left(\Phi(x) \cap B(y_j, k^{-1})\right) & \text{if } x \in C_{i,j,k} \end{cases}$$

The set-valued map $\operatorname{cl}\left(\Phi \cap B(y_j, k^{-1})\right)$ is lower semicontinuous by Lemma 2.1 and Lemma 2.2, then, since $C_{i,j,k}$ is closed, Lemma 2.3 implies that $\Phi_{i,j,k}$ is lower semicontinuous with $\operatorname{dom} \Phi_{i,j,k} = X$. Since every metric space is paracompact and the values of $\Phi_{i,j,k}$ are compact and hence complete, Theorem 3.1 guarantees the existence of a continuous selection $\varphi_{i,j,k}$ for each $\Phi_{i,j,k}$. The family $\{\varphi_{i,j,k}\}$ is a countable collection of continuous selections for Φ and it only remains to check that $\{\varphi_{i,j,k}(x)\}$ is dense in $\Phi(x)$ for every $x \in X$. Let $x \in X$, $y \in \Phi(x)$, and k positive integer. From the density of $\{y_j\}$ there exists j such that $d(y_j, y) \leq \frac{1}{2k}$. Then $x \in U_{j,2k}$ and, in particular, $x \in C_{i,j,2k}$ for some i. Hence $d(\varphi_{i,j,2k}(x), y_j) \leq \frac{1}{2k}$ which implies

$$d(\varphi_{i,j,2k}(x), y) \leq d(\varphi_{i,j,2k}(x), y_j) + d(y_j, y) < \frac{1}{k}$$

and the proof is complete. □

Proof of Theorem 3.2. Let $\overline{\Phi} : X \rightrightarrows Y$ be defined by $\overline{\Phi}(x) = \operatorname{cl} \Phi(x)$; we will find a continuous $\varphi : X \to Y$ such that $\varphi(x) \in I(\overline{\Phi}(x))$ for every $x \in X$. The map $\overline{\Phi}$ is lower semicontinuous (Lemma 2.2) and it has nonempty closed convex values inside the compact set $\operatorname{cl} \Phi(X)$. Lemma 3.6 guarantees the existence of a sequence $\{\varphi_i\}$ of selections for $\overline{\Phi}$ such that $\{\varphi_i(x)\}$ is dense in $\overline{\Phi}(x)$ for every $x \in X$. Fixed $x \in X$, arguing as in the proof of Lemma 3.4, the convex series $\sum_{i \geq 1} 2^{-i} \varphi_i(x)$ converges to a point $\varphi(x) \in I(\overline{\Phi}(x))$.

Moreover the range space $\operatorname{cl} \Phi(X)$ is bounded in the topological vector space Y, and therefore, for each neighborhood U of the origin there exists a scalar $\delta > 0$ such that $\lambda \operatorname{cl} \Phi(X) \subseteq U$ whenever $|\lambda| \leq \delta$. Take $p > -\log_2 \delta$; then, for every $x \in X$, taking into account that $\overline{\Phi}(x) \subseteq \operatorname{cl} \Phi(X)$ is convex,

$$\sum_{i \geq 1} 2^{-i} \varphi_i(x) - \sum_{1 \leq i \leq p} 2^{-i} \varphi_i(x) = \sum_{i \geq p+1} 2^{-i} \varphi_i(x) \in 2^{-p} \overline{\Phi}(x) \subseteq U.$$

Therefore the series $\sum_{i \geq 1} 2^{-i} \varphi_i(x)$ is uniformly convergent and the limit φ is a continuous function [14, Theorem 7.9]. □

4. Existence results for equilibria and quasiequilibria

Theorem 3.1‴ in [15] has been employed by the authors [8] to get the existence of solutions of the quasiequilibrium problem (1) in a separable Banach space. Later, making use of the Michael selection Theorem 3.1, Alleche and Rădulescu [2] avoided the assumption of separability of the space, though at the cost of requiring the values of the feasibility set-valued map K to be closed. Now, we formulate the following result which exploits the selection Theorem 3.2 and it allows to avoid the completeness and separability of the space, and the closed valuedness of K.

Taking a cue from [2] we recall that a set-valued map $\Phi : X \rightrightarrows Y$ between topological spaces is locally selectionable if for all $(x, y) \in \operatorname{gph} \Phi$ there exist a neighborhood U_x of x and a continuous function $\varphi : U_x \to Y$ such that $(x, y) \in \operatorname{gph} \varphi$ and $\varphi(x') \in \Phi(x')$ for every $x' \in U_x$. Any locally selectionable set-valued map is lower semicontinuous. Moreover these set-valued maps admit continuous selections whenever they are convex valued and defined on paracompact topological spaces [5].

Theorem 4.1. *Let C be a compact convex subset of a metrizable locally convex vector space X, K a lower semicontinuous set-valued map with nonempty values in the class $\mathcal{D}(X)$, and $\operatorname{fix} K$ closed. Suppose that $f(x, x) \geq 0$ for all $x \in \operatorname{fix} K$, and that*

i) $\{y \in C : f(x, y) < 0\}$ *is convex, for all* $x \in \operatorname{fix} K$;
ii) $\{(x, y) \in \operatorname{fix} K \times C : f(x, y) \geq 0\}$ *is closed.*

Then the quasiequilibrium problem (1) has a solution.

Proof. We start proving that K is locally selectionable. Fix $(x_0, y_0) \in \operatorname{gph} K$ and define $K_0 : C \rightrightarrows X$ as

$$K_0(x) = \begin{cases} K(x) & \text{if } x \neq x_0 \\ \{y_0\} & \text{if } x = x_0 \end{cases}$$

K_0 is compact and lower semicontinuous (Lemma 2.3), and $K_0(x) \in \mathcal{D}(X)$ for any $x \in C$. From Theorem 3.2 K_0 admits a continuous selection and hence K is locally selectionable. The set-valued map $F : \operatorname{fix} K \rightrightarrows C$ defined by

$$F(x) = \{y \in C : f(x, y) < 0\}$$

has convex values (assumption i)) and open graph (assumption ii)). By contradiction assume that $F(x) \cap K(x) \neq \emptyset$ for all $x \in \operatorname{fix} K$. From [5, Proposition 1.10.4] $F \cap K$ is locally selectionable. Since $F \cap K$ is convex-valued it admits a continuous selection $g : \operatorname{fix} K \to C$ [5, Proposition 1.10.2]. The set-valued map $\Phi : C \rightrightarrows C$ defined as

$$\Phi(x) = \begin{cases} K(x) & \text{if } x \notin \operatorname{fix} K \\ \{g(x)\} & \text{if } x \in \operatorname{fix} K \end{cases}$$

is compact and lower semicontinuous, once again by Lemma 2.3 using the fact that $\operatorname{fix} K$ is closed. Moreover the values of Φ are in the class $\mathcal{D}(X)$ and Theorem 3.2 guarantees that g can be extended to a continuous selection φ for Φ. From the Schauder fixed point Theorem it follows that φ has a fixed point, i.e. there exists $x \in C$ such that $x = \varphi(x) \in \Phi(x)$. Clearly $x \in \operatorname{fix} K$ and this implies $x = g(x) \in F(x)$ which contradicts the fact that $f(x, x) \geq 0$, and the proof is complete. □

Theorem 4.1 encompasses [8, Theorem 3.3] where X is assumed to be a separable Banach space, and it does not require, as does [2, Theorem 4.1], that K is closed-valued. We would also say that the convexity of the level set in assumption i) can be deduced from the quasiconvexity of $f(x, \cdot)$ for all $x \in \operatorname{fix} K$. Instead the upper semicontinuity of f implies the closedness of the level set in assumption ii).

Example 4.1. Consider the quasiequilibrium problem (1) with $C = [0, 3]$, $f(x, y) = y - x$ and $K : [0, 3] \rightrightarrows [0, 3]$ defined by

$$K(x) = \begin{cases} (x, 3] & \text{if } x \in (0, 1) \\ (1, 2) & \text{if } x \in (1, 2) \\ [0, x) & \text{if } x \in (2, 3) \\ [0, 3] & \text{if } x \in \{0, 3\} \\ [1, 2] & \text{if } x \in \{1, 2\} \end{cases}$$

The set $\operatorname{fix} K = \{0\} \cup [1, 2] \cup \{3\}$ is closed and all the assumptions of Theorem 4.1 are fulfilled. Then the quasiequilibrium has solution and it is easy to show that the solution set is given by $\{0, 1\}$. Anyhow K is not upper semicontinuous at $x = 1$ and $x = 2$ and moreover [2, Theorem 4.1] can't be applied since K is not closed-valued.

Let us now deal with existence of solutions of generalized games. First, we show a version of the fixed-point theorem of Gale and Mas-Colell [11] based on the lower semicontinuity of the preference maps. Let N be a countable family of agents and for each $i \in N$, C_i be a nonempty set of actions for agent i. Set $C = \prod_{i \in N} C_i$. Let $F_i : C \rightrightarrows C_i$ be the preference set-valued map of agent i. For such a game an equilibrium is an element $x \in C$ such that $F_i(x) = \emptyset$, for each $i \in N$.

Theorem 4.2. *For each $i \in N$, let C_i be a compact convex subset of a metrizable locally convex vector space X_i, and F_i be a lower semicontinuous and convex-valued map. Then there exists a point $x \in C$ such that for each i, either $x_i \in I(\operatorname{cl} F_i(x))$ or $F_i(x) = \emptyset$.*

Proof. Since C is a countable product of metric spaces, then it is metrizable. Let $G_i : C \rightrightarrows X_i$ be defined by $G_i(x) = I(\operatorname{cl} F_i(x))$, for each $x \in C$ and $i \in N$. Then each G_i is a compact set-valued map, indeed

$$G_i(x) = I(\operatorname{cl} F_i(x)) \subseteq \operatorname{cl} F_i(x) \subseteq C_i, \quad \forall x \in C,$$

and $\operatorname{dom} G_i = \operatorname{dom} F_i$ thanks to Lemma 3.4. Moreover G_i is convex-valued and $\operatorname{cl} G_i(x) = \operatorname{cl} F_i(x)$, for every $x \in C$ (Lemma 3.2). Hence G_i is lower semicontinuous (Lemma 2.2). Furthermore the values of G_i are in $\mathcal{D}(X_i)$ (item b. of Lemma 3.5) and Theorem 3.2 implies that each G_i has a continuous selection φ_i on the open set $\operatorname{dom} F_i$. Define $\Phi_i : C \rightrightarrows C_i$ via

$$\Phi_i(x) = \begin{cases} \{\varphi_i(x)\} & \text{if } x \in \operatorname{dom} F_i \\ C_i & \text{if } x \notin \operatorname{dom} F_i \end{cases}$$

Since $\operatorname{dom} F_i$ is open, Φ_i is upper semicontinuous with nonempty compact and convex values, and thus so is $\Phi : C \rightrightarrows C$ defined by $\Phi(x) = \prod_{i \in N} \Phi_i(x)$, for all $x \in C$ (Lemma 2.4). The Kakutani fixed point Theorem guarantees that there exists $x \in C$ such that $x \in \Phi(x)$. Hence, for each $i \in N$, either $x \in \operatorname{dom} F_i$ which implies $x_i = \varphi_i(x) \in G_i(x) = I(\operatorname{cl} F_i(x))$ or $x \notin \operatorname{dom} F_i$, i.e. $F_i(x) = \emptyset$. □

Theorem 4.2 is clearly a composition of a fixed point theorem and an optimization theorem. When F_i has values in the class $\mathcal{D}(X_i)$ and $\operatorname{dom} F_i = C$ for each $i \in N$, Theorem 4.2 provides the existence of a fixed point $x \in F(x)$. Anyhow we stress the fact that unlike for the upper semicontinuous case, where the existence of an interior fixed point is not guaranteed, if N is finite $x \in I(\operatorname{cl} F(x))$ (Lemma 3.3), regardless of the values of F_i are in $\mathcal{D}(X_i)$. While $x_i \notin I(\operatorname{cl} F_i(x))$ for each $i \in N$ and $x \in C$ implies that the game has a solution.

As observed by Arrow and Debreu [3] in a game the domain from which strategies are to be chosen is given to each player independently of the strategies chosen by other players. A generalized game (called also abstract economy) is a game in which the choice of an action by one agent affects the domain of actions of other agents. Formally the strategy space of the ith agent is described by a set-valued map $K_i : C \rightrightarrows C_i$ and an equilibrium of a generalized game is a point $x \in C$ such that $x_i \in K_i(x)$ and $F_i(x) \cap K_i(x) = \emptyset$, for each $i \in N$. Theorem 4.2 can be generalized to this setting.

Theorem 4.3. *For each $i \in N$, let C_i be a compact convex subset of a metrizable locally convex vector space X_i, K_i a lower semicontinuous set-valued map with nonempty values in the class $\mathcal{D}(X_i)$ and $\operatorname{fix} K$ closed. Suppose that for every $i \in N$,*

i) *F_i is convex-valued on $\operatorname{fix} K$,*
ii) *F_i is lower semicontinuous on $\operatorname{fix} K$,*
iii) *$F_i \cap K_i$ is lower semicontinuous on $\partial_C \operatorname{fix} K$.*

Then there exists a point $x \in \operatorname{fix} K$ such that for each i, either $x_i \in I(\operatorname{cl} F_i(x)) \cup I(\operatorname{cl}(F_i(x) \cap K_i(x)))$ or $F_i(x) \cap K_i(x) = \emptyset$.

Proof. For each $i \in N$, define $\Phi_i : C \rightrightarrows C_i$ via

$$\Phi_i(x) = \begin{cases} F_i(x) & \text{if } x \in \operatorname{int}_C \operatorname{fix} K \\ F_i(x) \cap K_i(x) & \text{if } x \in \partial_C \operatorname{fix} K \\ K_i(x) & \text{if } x \notin \operatorname{fix} K \end{cases}$$

where the closed set $\operatorname{fix} K$ is nonempty by Theorem 4.2, since the maps K_i are lower semicontinuous with nonempty values in the class $\mathcal{D}(X_i)$. Clearly each Φ_i is convex-valued and assumptions ii) and iii) together with Lemma 2.3 imply that Φ_i is lower semicontinuous on $\operatorname{fix} K$. Then, since $\operatorname{fix} K$ is closed, applying Lemma 2.3 a second time, it is found that Φ_i is lower semicontinuous on C. Therefore all the assumptions of Theorem 4.2 are satisfied and there exists $x \in C$ such that for each i, either $x_i \in I(\operatorname{cl} \Phi_i(x))$ or $\Phi_i(x) = \emptyset$. We show by contradiction that $x \in \operatorname{fix} K$. Take $i \in N$ such that $x_i \notin K_i(x)$. Then $\Phi_i(x) = K_i(x) \neq \emptyset$ which implies

$$x_i \in I(\operatorname{cl} \Phi_i(x)) = I(\operatorname{cl} K_i(x)),$$

and since $K_i(x) \in \mathcal{D}(X_i)$, we deduce the absurd conclusion that $x \in K_i(x)$. Hence $x \in \operatorname{fix} K$ and for each index i, $x \in \partial_C \operatorname{fix} K$ implies that either $x_i \in I(\operatorname{cl}(F_i(x) \cap K_i(x)))$ or $F_i(x) \cap K_i(x) = \emptyset$, while $x \in \operatorname{int}_C \operatorname{fix} K$ implies that either $x_i \in I(\operatorname{cl} F_i(x))$ or $F_i(x) = \emptyset$ and the proof is complete. □

The last part of this work deals with generalized Nash equilibrium problems as introduced within Section 1. The generalized Nash equilibrium problem (2) consists in finding $x \in C$ such that for each $i \in N$ one has

$$x_i \in K_i(x^{-i}) \quad \text{and} \quad f_i(y_i, x) \geq 0, \quad \forall y_i \in K_i(x^{-i}),$$

where $f : C_i \times C \to \mathbb{R}$ is defined by

$$f_i(y_i, x) = \theta_i(y_i, x^{-i}) - \theta_i(x_i, x^{-i}).$$

An abstract economy comes down to a generalized Nash equilibrium problem when the preferences set-valued maps are defined by

$$F_i(x) = \{y_i \in C_i : f_i(y_i, x) < 0\}, \quad \forall x \in C.$$

Since in this model K_i is independent of player i's choice we can consider the set-valued map $\widehat{K}_i : C \rightrightarrows C_i$ defined by $\widehat{K}_i(x) = K_i(x^{-i})$ as a technical convenience. As before we denote by $\widehat{K} : C \rightrightarrows C$ the product map.

The standard existence results for (2) usually require convexity and compactness of the strategy sets C_i and both the upper and the lower semicontinuity of the set-valued maps K_i, together with the convexity and the closedness of their values. In the context of finite-dimensional strategy spaces, the upper semicontinuity and closed-valuedness assumptions on K_i were avoided by making use of the closedness of $\operatorname{fix} K$ [9]. Cubiotti and Yao [10] extended the result to infinite-dimensional normed spaces assuming the Hausdorff lower semicontinuity of the K_i and the nonemptiness of the interior of their values. More recently [8] the Hausdorff lower semicontinuity of the maps K_i has been replaced by the usual lower semicontinuity, under the assumption of completeness and separability of the normed spaces X_i. Exploiting Theorem 4.3, we are able to avoid using completeness and separability of the spaces that are unnecessarily restrictive. Moreover this direct approach does not rely on the Nikaido-Isoda function [17] and allows us to prove the result for a countable family of players with quasiconvex loss functions.

Theorem 4.4. *For each $i \in N$, let C_i be a compact convex subset of a metrizable locally convex vector space X_i, K_i a lower semicontinuous set-valued map with nonempty values in the class $\mathcal{D}(X_i)$, and* fix $\widehat{K}$ *closed. Suppose that for each $i \in N$, the loss function θ_i is defined on the whole space X and*

i) $\theta_i(\cdot, x^{-i})$ *is quasiconvex on X_i for every $x^{-i} \in C_{-i}$;*
ii) θ_i *is continuous.*

Then the generalized Nash equilibrium problem (2) has a solution.

Proof. We verify all the assumptions of Theorem 4.3. The set-valued maps $\widehat{K}_i$ are lower semicontinuous with nonempty values in the class $\mathcal{D}(X_i)$. The convexity of the values of F_i descends from assumption i). Assumption ii) guarantees that the graph of F_i is open which implies that F_i and $F_i \cap \widehat{K}_i$ are lower semicontinuous maps. Moreover $I(\operatorname{cl} F_i(x)) \subseteq F_i(x)$, for all $x \in C$, since the values

$$F_i(x) = \{y_i \in X_i : f_i(y_i, x) < 0\} \cap C_i$$

are the intersection of an open convex set and C_i, and hence in the class $\mathcal{D}(X_i)$ ([8, Lemma 2.2]). Similarly the values of $F_i \cap \widehat{K}_i$ are in the class $\mathcal{D}(X_i)$ since

$$F_i(x) \cap \widehat{K}_i(x) = \{y_i \in X_i : f_i(y_i, x) < 0\} \cap K_i(x^{-i}).$$

Therefore

$$I(\operatorname{cl} F_i(x)) \cup I(\operatorname{cl}(F_i(x) \cap \widehat{K}_i(x))) \subseteq F_i(x)$$

and the fact that $f_i(x_i, x) = 0$ ensures that $x_i \notin I(\operatorname{cl} F_i(x)) \cup I(\operatorname{cl}(F_i(x) \cap \widehat{K}_i(x)))$, for all $x \in C$. Theorem 4.3 gives the existence of at least one solution to the Nash equilibrium problem. □

The following example shows a case in which Theorem 4.4 can be applied even if the strategy set-valued map of a player is not upper semicontinuous.

Example 4.2. Consider the generalized Nash equilibrium problem (2) between two players acting in two different spaces, the former in a (not necessarily complete or separable) normed space $(X, \|\cdot\|)$, the latter in $\mathbb{R}$. More precisely let $\overline{B}$ be the closed unit ball. The strategy set of the first agent is described by the set-valued map $K_1 : [0, 2] \rightrightarrows 2\overline{B}$ such that $K_1(x_2) = \overline{B}$ for all $x_2 \in [0, 2]$, instead the strategy set of the second agent is given by $K_2 : 2\overline{B} \rightrightarrows [0, 2]$ defined as

$$K_2(x_1) = \begin{cases} [\|x_1\|, 1] & \text{if } x_1 \in \overline{B} \\ [0, 2) & \text{if } x_1 \in 2\overline{B} \setminus \overline{B} \end{cases}$$

Thanks to Lemma 2.3 the set-valued map K_2 is lower semicontinuous and the set

$$\operatorname{fix} \widehat{K} = \{(x_1, x_2) \in \overline{B} \times [0, 2] : \|x_1\| \le x_2 \le 1\}$$

is closed. Taking the loss functions $\theta_1(x_1, x_2) = \theta_2(x_1, x_2) = \|x_1\| + x_2$, all the assumptions of Theorem 4.4 are fulfilled and the problem has a solution. Notice that K_2 is not upper semicontinuous on the unit sphere and hence the existence cannot be ensured by Arrow-Debreu-Nash Theorem (see for instance [4]). Moreover, even if K_2 is Hausdorff lower semicontinuous, the recent result in [10] does not apply due to the lack of closedness of the values of K_2 and the fact that the interior of $K_2(x_1)$ with respect to the affine space generated by $[0, 2]$ is empty when x_1 belongs to the unit sphere.

The unique solution of the generalized Nash equilibrium problem is $\{(0, 0)\}$.

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