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High friction limit for Euler–Korteweg and Navier–Stokes–Korteweg models via relative entropy approach

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Abstract

The aim of this paper is to investigate the singular relaxation limits for the Euler–Korteweg and the Navier–Stokes–Korteweg system in the high friction regime. We shall prove that the viscosity term is present only in higher orders in the proposed scaling and therefore it does not affect the limiting dynamics, and the two models share the same equilibrium equation. The analysis of the limit is carried out using the relative entropy techniques in the framework of weak, finite energy solutions of the relaxation models converging toward smooth solutions of the equilibrium. The results proved here take advantage of the enlarged formulation of the models in terms of the *drift velocity* introduced in [6], generalizing in this way the ones proved in [15] for the Euler–Korteweg model.

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1. Introduction

The objective of this work is to study the high friction limit for the Euler–Korteweg and the Navier–Stokes–Korteweg systems, that is:

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$$\begin{cases} \partial_t \rho + \operatorname{div} m = 0 \\ \partial_t m + \operatorname{div} \left(\frac{m \otimes m}{\rho} \right) + \nabla p(\rho) = 2\nu \operatorname{div}(\mu_L(\rho) Du) + \nu \nabla(\lambda_L(\rho) \operatorname{div} u) \\ \quad + \rho \nabla \left(k(\rho) \Delta \rho + \frac{1}{2} k'(\rho) |\nabla \rho|^2 \right) - \xi \rho u, \end{cases} \tag{1.1}$$

where $t > 0$, $x \in \mathbb{T}^n$, the n -dimensional torus, ρ is the density, $m = \rho u$ is the momentum, and the constants $\xi > 0$ and $\nu \geq 0$ stand for the large friction and the viscosity coefficient ($\nu = 0$ for the case of Euler–Korteweg system). As usual, in the viscosity terms of (1.1)

$$Du = \frac{\nabla u + {}^t \nabla u}{2}$$

is the symmetric part of the gradient ∇u and the Lamé coefficients $\mu_L(\rho)$ and $\lambda_L(\rho)$ verifies

$$\mu_L(\rho) \geq 0; \quad \frac{2}{n} \mu_L(\rho) + \lambda_L(\rho) \geq 0. \tag{1.2}$$

Moreover, $p(\rho)$ stands for the pressure, connected to the internal energy $e(\rho)$ by the relations

$$e'(\rho) = \frac{p(\rho)}{\rho^2}; \quad h(\rho) = \rho e(\rho); \quad h''(\rho) = \frac{p'(\rho)}{\rho}; \quad p(\rho) = \rho h'(\rho) - h(\rho). \tag{1.3}$$

As a consequence, we readily obtain

$$\rho \nabla(h'(\rho)) = \rho h''(\rho) \nabla \rho = \nabla p(\rho).$$

In what follows, and we shall confine ourselves to the case of monotone pressure, and, for simplicity we shall consider the classical γ -law $p(\rho) = \rho^\gamma$ for $\gamma > 1$, for which the function h is given by

$$h(\rho) = \frac{1}{\gamma - 1} \rho^\gamma.$$

The literature concerning this kind of systems, which include in particular Quantum Hydrodynamic models, is very wide and a complete description of it is beyond the main interest of our present research, which is focused in the study of the relaxation limit for weak, finite energy solutions of (1.1). In particular, we are not interested here in investigating the existence of such solutions, but solely in understanding their behavior in the high friction regime. However, for some rigorous mathematical studies of such systems, regarding in particular the existence of weak solutions, the dedicated reader may refer to [1–4] and the reference therein.

The high friction regime, after an appropriate time scaling, in both cases is given by the following equation:

$$\rho_t = \operatorname{div}_x \left(\rho \nabla_x \left(h'(\rho) + k(\rho) \Delta \rho + \frac{1}{2} k'(\rho) |\nabla \rho|^2 \right) \right), \tag{1.4}$$

as one can easily check by performing the classical Hilbert expansion. Moreover, the rigorous study of this singular limit in terms of relative entropy techniques limit when $\nu > 0$ does not present significant differences, and therefore we shall first discuss the case of Euler–Korteweg system in full details, and leave the discussion of the Navier–Stokes–Korteweg for the last section, where we shall emphasize only how to control the new terms due to the presence of the viscosity in (1.1). Moreover, it is worth to observe here that, besides the natural condition (1.2) needed to guarantee the dissipative nature of the viscosity terms, we shall assume here only appropriate uniform integrability conditions on that functions (which can be deduced from a bound of their L^1 norm in terms of the energy), without a precise connection with the capillarity coefficient $k(\rho)$, as it is usually needed in the analysis of these models.

The kind of singular limits under investigation here enters in the realm of diffusive relaxations, for which hyperbolic systems of balance laws (as (1.1) for $\nu = 0$) converge in a diffusive scaling toward parabolic equilibrium systems. This kind of asymptotic analysis has been addressed in various frameworks and with several techniques; in particular we refer to [8] and the reference therein for the results concerning weak solutions and compactness arguments. More recently, this kind of limits has been also successfully addressed by means of relative entropy techniques, starting from the well-known case of the Euler system with friction (obtained by choosing $k(\rho) = 0$ in (1.1) in addition to $\nu = 0$) converging to the porous media equation [14]. It is worth recalling that, as already pointed out before, this asymptotic behavior has been analyzed also before under many different viewpoints, and in particular for this remarkable example we refer to [16, 11, 12]. However, the study of such limits with the present technique, even if it is confined to the case of smooth solutions at equilibrium, has the advantage of obtaining a stability estimate and hence a rate of convergence as the relaxation parameter goes to zero.

More recently, many other diffusive limits have been addressed following the same ideas; among others, see [5, 9, 13, 17, 7], and in particular here we recall the general framework introduced in [10, 15], where the relative entropy calculation and the analysis of the diffusive limits have been presented in the general framework of abstract Euler flows generated by the first variation of an energy functional $\mathcal{E}(\rho)$:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \rho \partial_t u + \rho u \cdot \nabla u = -\rho \nabla \frac{\delta \mathcal{E}}{\delta \rho} - \xi \rho u. \end{cases}$$

The system (1.1) under consideration here belongs to this class of abstract flows for the following particular choice for $\mathcal{E}(\rho)$:

$$\mathcal{E}(\rho) = \int \left(h(\rho) + \frac{1}{2} k(\rho) |\nabla \rho|^2 \right) dx. \tag{1.5}$$

Referring in particular to the analysis of the large limit, among other possible instances, we recall here that in the paper [15] the Authors showed the emergence of the (Cahn–Hilliard type) equation (1.4) as high friction limit of the Euler–Korteweg system solely in the case of constant capillarity $k(\rho) = C_k$. This result is based on the following general relative entropy relation for the aforementioned abstract Euler equations [10, 15]

$$\frac{d}{dt} \left(\mathcal{E}(\rho|\bar{\rho}) + \int \frac{1}{2} \rho |u - \bar{u}|^2 \right) + \xi \int \rho |u - \bar{u}|^2 dx =$$

$$\int \nabla \bar{u} : S(\rho|\bar{\rho})dx - \int \rho \nabla \bar{u} : (u - \bar{u}) \otimes (u - \bar{u})dx,$$

written here for (ρ, u) and $(\bar{\rho}, \bar{u})$ smooth solutions of this system. The stress tensor S appearing in the relation above can be defined in many examples of physical interest starting from the energy functional as follows:

$$-\rho \nabla \frac{\delta \mathcal{E}}{\delta \rho} = \text{div } S.$$

In the particular case under consideration here, this relation becomes

$$-\nabla p(\rho) + \rho \nabla \left(k(\rho)\Delta\rho + \frac{1}{2}k'(\rho)|\nabla\rho|^2 \right) = \text{div } S.$$

The relation recalled above, and thus the corresponding control of the diffusive limit can be improved if we confine our attention to the specific form of the Euler-Korteweg systems (1.1), as it has been recently proved in [6]. Indeed, the relative entropy techniques turns out to be more effective if one introduces the *drift velocity*

$$v = \frac{\nabla\mu(\rho)}{\rho},$$

where $\mu(\rho)$ satisfies $\mu'(\rho) = \sqrt{\rho k(\rho)}$. In this way, it is possible to obtain an augmented formulation of (1.1), which, for the Euler-Korteweg system (that is, with $v = 0$), reads as follows:

$$\begin{cases} \partial_t \rho + \text{div}(\rho u) = 0 \\ \partial_t(\rho u) + \text{div}(\rho u \otimes u) + \nabla p(\rho) = \text{div}(\mu(\rho)\nabla v) + \frac{1}{2}\nabla(\lambda(\rho)\text{div } v) - \xi \rho u \\ \partial_t(\rho v) + \text{div}(\rho v \otimes u) + \text{div}(\mu(\rho)^t \nabla u) + \frac{1}{2}\nabla(\lambda(\rho)\text{div } u) = 0, \end{cases} \tag{1.6}$$

where $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$ and, thanks to the Bohm identity (see [6]), we also have

$$\text{div}(\mu(\rho)\nabla v) + \frac{1}{2}\nabla(\lambda(\rho)\text{div } v) = \text{div } S_1$$

thus defining the new stress tensor S_1 in (1.6) due solely to the capillarity effects. As we shall prove in the sequel, this approach will lead us to control the high friction limits for non constant capillarities, obtaining the same advantages already pointed out in [6] also in the context of diffusive relaxation, thus generalizing the results of [15] for this particular system. More precisely, the strategy is to define a new *momentum* $J = \rho v$ to then estimate the following relative entropy:

$$\begin{aligned} \eta(\rho, m, J|\bar{\rho}, \bar{m}, \bar{J}) &= \eta(\rho, m, J) - \eta(\bar{\rho}, \bar{m}, \bar{J}) - \bar{\eta}_\rho(\rho - \bar{\rho}) - \bar{\eta}_m \cdot (m - \bar{m}) \\ &\quad - \bar{\eta}_J \cdot (J - \bar{J}) \\ &= \frac{1}{2}\rho|u - \bar{u}|^2 + \frac{1}{2}\rho|v - \bar{v}|^2 + h(\rho|\bar{\rho}). \end{aligned}$$

In the present analysis, which involves a relaxation limit between two different diffusive theories, the equilibrium (smooth) solution $(\bar{\rho}, \bar{m}, \bar{J})$ will solve the corresponding diffusive limiting equation, which shall then be recasted as an appropriate correction of the relaxing system (1.6), as already done in previous works [14,15].

The outline of this work is as follows. In Section 2, after the appropriate time scaling, we perform the Hilbert expansion of (1.6) in order to recognize the limit equation. Then we rewrite the latter as a correction of the relaxation system (1.6) to take full advantage of the relative entropy tools. Section 3 is devoted to obtaining the relative entropy inequality, which will be used as an yardstick to measure the distance between the two solutions in the relaxation limit of the subsequent section. Finally, in Section 5 we describe our all results can be adapted in a straightforward way to the case of the Navier–Stokes–Korteweg model (1.1) for $\nu > 0$.

2. Hilbert expansion and formal diffusive limit for the Euler–Korteweg model

In this section we shall present the correct scaling for which (1.1), and hence (1.6), exhibits the desired diffuse limit. More precisely, for $\xi = 1/\epsilon$, we rescale the time so that $\partial_t \rightarrow \epsilon \partial_t$ and (1.1) becomes:

$$\begin{cases} \partial_t \rho + \frac{1}{\epsilon} \operatorname{div} m = 0 \\ \partial_t m + \frac{1}{\epsilon} \operatorname{div} \left(\frac{m \otimes m}{\rho} \right) + \frac{1}{\epsilon} \nabla p(\rho) = \frac{1}{\epsilon} \operatorname{div} S_1 - \frac{1}{\epsilon^2} \rho u. \end{cases} \tag{2.1}$$

Accordingly, (1.6) reads

$$\begin{cases} \partial_t \rho + \frac{1}{\epsilon} \operatorname{div}(m) = 0 \\ \partial_t(m) + \frac{1}{\epsilon} \operatorname{div} \left(\frac{m \otimes m}{\rho} \right) + \frac{1}{\epsilon} \nabla p(\rho) = \frac{1}{\epsilon} \operatorname{div} S_1 - \frac{1}{\epsilon^2} \rho u \\ \partial_t(J) + \frac{1}{\epsilon} \operatorname{div} \left(\frac{J \otimes m}{\rho} \right) + \frac{1}{\epsilon} \operatorname{div} S_2 = 0, \end{cases} \tag{2.2}$$

where $J = \rho v$ and (see [6] for further details)

$$\operatorname{div} S_2 = \operatorname{div}(\mu(\rho)^t \nabla u) + \frac{1}{2} \nabla(\lambda(\rho) \operatorname{div} u).$$

In order to perform the Hilbert expansion, we need to introduce the asymptotic expansions of ρ and m in (2.2), and the one for J will follow, being $J = \rho v = \nabla \mu(\rho)$. To this end,

$$\begin{aligned} \rho &= \rho_0 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \dots \\ m &= m_0 + \epsilon m_1 + \epsilon^2 m_2 + \dots \end{aligned}$$

and collect the terms of the same order. From the mass conservation we get:

$$\begin{aligned} O(\epsilon^{-1}) : \quad & \operatorname{div} m_0 = 0; \\ O(1) : \quad & \partial_t \rho_0 + \operatorname{div} m_1 = 0; \\ O(\epsilon) : \quad & \dots \end{aligned}$$

from the momentum equation we get:

$$\begin{aligned} O(\epsilon^{-2}) : \quad & m_0 = 0; \\ O(\epsilon^{-1}) : \quad & -m_1 = \nabla p(\rho_0) - \operatorname{div} S_1(\rho_0); \\ O(1) : \quad & \dots \end{aligned}$$

Hence, from these first relations, we recover the equilibrium relation $m_0 = 0$, the Darcy’s law $m_1 = -\nabla_x p(\rho_0) + \operatorname{div}_x S_1(\rho_0)$, and the following gradient flow dynamic for ρ_0 :

$$\partial_t \rho_0 + \operatorname{div} (-\nabla p(\rho_0) + \operatorname{div} S_1(\rho_0)) = 0, \tag{2.3}$$

that is, the formal limit as $\epsilon \rightarrow 0$ of (2.2).

In order to compare weak solutions of (2.2) and strong solutions of its parabolic equilibrium (2.3) and take full advantage of the relative entropy estimate for hyperbolic systems, as already done in [14,15], we rewrite the latter as Euler–Korteweg system with friction plus an error term as follows. Let us denote by $\bar{\rho}$ the (smooth) solution of (2.3). Then $(\bar{\rho}, \bar{m} = \bar{\rho}\bar{u})$ solves

$$\begin{cases} \partial_t \bar{\rho} + \frac{1}{\epsilon} \operatorname{div} \bar{m} = 0 \\ \partial_t \bar{m} + \frac{1}{\epsilon} \operatorname{div} \left(\frac{\bar{m} \otimes \bar{m}}{\bar{\rho}} \right) + \frac{1}{\epsilon} \nabla p(\bar{\rho}) = \frac{1}{\epsilon} \operatorname{div} \bar{S}_1 - \frac{1}{\epsilon^2} \bar{m} + e(\bar{\rho}, \bar{m}), \end{cases} \tag{2.4}$$

where

$$\bar{m} = \epsilon (-\nabla p(\bar{\rho}) + \operatorname{div} S_1(\bar{\rho})).$$

Clearly, in (2.4), the error term $e(\bar{\rho}, \bar{m}) = \bar{e}$ is given by:

$$\begin{aligned} \bar{e} &= \frac{1}{\epsilon} \operatorname{div}_x \left(\frac{\bar{m} \otimes \bar{m}}{\bar{\rho}} \right) + \bar{m}_t \\ &= \epsilon \operatorname{div}_x ((-\nabla p(\bar{\rho}) + \operatorname{div} S_1(\bar{\rho})) \otimes (-\nabla p(\bar{\rho}) + \operatorname{div} S_1(\bar{\rho}))) \\ &\quad + \epsilon (-\nabla p(\bar{\rho}) + \operatorname{div} S_1(\bar{\rho}))_t \\ &= O(\epsilon). \end{aligned} \tag{2.5}$$

Introducing the notation $\bar{J} = \bar{\rho}\bar{v} = \nabla\mu(\bar{\rho})$, the equilibrium can be rewritten also as follows:

$$\begin{cases} \partial_t \bar{\rho} + \frac{1}{\epsilon} \operatorname{div} \bar{m} = 0 \\ \partial_t \bar{m} + \frac{1}{\epsilon} \operatorname{div} \left(\frac{\bar{m} \otimes \bar{m}}{\bar{\rho}} \right) + \frac{1}{\epsilon} \nabla p(\bar{\rho}) = \frac{1}{\epsilon} \operatorname{div} \bar{S}_1 - \frac{1}{\epsilon^2} \bar{m} + e(\bar{\rho}, \bar{m}) \\ \partial_t \bar{J} + \frac{1}{\epsilon} \operatorname{div} \left(\frac{\bar{J} \otimes \bar{m}}{\bar{\rho}} \right) + \frac{1}{\epsilon} \operatorname{div} \bar{S}_2 = 0. \end{cases} \tag{2.6}$$

As already done previously [14,15], in next section we as shall validate rigorously the large friction limit using relative entropy estimates, but this time using the enlarged reformulation in terms of the drift velocity, thus considering the singular limit from (2.2) to (2.6).

3. Relative entropy estimate for the Euler–Korteweg model

Let us start by we start by recalling the entropy–entropy flux pair (η, Q) associated to the original Euler-Korteweg system (1.1) with $\xi = 1/\epsilon$ and after the related time scaling. Using the notation of [10,15], we obtain the potential energy (see (1.5))

$$F(\rho, \nabla \rho) = h(\rho) + \frac{1}{2}k(\rho)|\nabla \rho|^2,$$

while the kinetic energy reads

$$E_K = \frac{1}{2}\rho|u|^2.$$

Moreover, the couple (η, Q) is defined in the following way:

$$\begin{aligned} \eta(\rho, m, \nabla \rho) &= \frac{1}{2}\rho|u|^2 + \frac{1}{2}k(\rho)|\nabla \rho|^2 + h(\rho); \\ Q(\rho, m, \nabla \rho) &= \frac{1}{2}\rho u|u|^2 + \rho u \left(h'(\rho) + \frac{1}{2}k'(\rho)|\nabla \rho|^2 - \operatorname{div}(k(\rho)\nabla \rho) \right) \\ &\quad + k(\rho)\nabla \rho \operatorname{div}(\rho u). \end{aligned}$$

Before the rigorous justification of the relative entropy calculation in the context of weak solutions we are interested in, let us first briefly present the (formal) computation leading to the desired expression in the case when both solutions (of the relaxation and the limiting equations) are regular. Let us emphasize once again that in the sequel we shall take advantage of the reformulation (2.2) in terms of the drift velocity, and the rewriting of the equilibrium equation in (2.6).

If we introduce $m = \rho u$, then (smooth) solutions of (1.1) in the diffusive regime satisfy

$$\begin{aligned} \partial_t \eta(\rho, m, \nabla \rho) + \frac{1}{\epsilon} \operatorname{div} \left(\frac{1}{2}m \frac{|m|^2}{\rho^2} + m \left(h'(\rho) + \frac{1}{2}k'(\rho)|\nabla \rho|^2 - \operatorname{div}(k(\rho)\nabla \rho) \right) + \right. \\ \left. k(\rho)\nabla \rho \operatorname{div} m \right) = -\frac{1}{\epsilon^2} \frac{|m|^2}{\rho} \leq 0, \end{aligned}$$

while (smooth) solutions of (2.4) satisfy the following energy dissipation identity:

$$\partial_t \eta(\bar{\rho}, \bar{m}, \nabla \bar{\rho}) + \frac{1}{\epsilon} \operatorname{div} Q(\bar{\rho}, \bar{m}, \nabla \bar{\rho}) = -\frac{1}{\epsilon^2} \frac{|\bar{m}|^2}{\bar{\rho}} + \frac{\bar{m}}{\bar{\rho}} \cdot \bar{e}. \tag{3.1}$$

It is worth to observe here that (3.1) is a rewriting of the classical energy relation valid for the solution $\bar{\rho}$ to the equilibrium gradient flow equation (2.3). At this point, the main difference here with respect to the arguments in [10,14,15] relies on the fact that we use the notation of [6]: we introduce a fictitious velocity $v = \sqrt{\frac{k(\rho)}{\rho}} \nabla \rho$ and correspondingly its transport equation along the velocity u (see (2.2)₃). This leads us to define a “new” entropy-entropy flux pair (η, Q) related to the “new” potential energy

$$F(\rho, J) = h(\rho) + \frac{1}{2} \frac{|J|^2}{\rho},$$

where $J = \rho v$. Hence, the entropy rewrites as follows:

$$\eta(\rho, m, J) = \frac{1}{2} \frac{|m|^2}{\rho} + h(\rho) + \frac{1}{2} \frac{|J|^2}{\rho},$$

while its flux Q is given by:

$$Q(\rho, m, J) = \frac{1}{2} m \frac{|m|^2}{\rho^2} + m h'(\rho) + \frac{1}{2} m \frac{|J|^2}{\rho^2}.$$

We get:

$$\partial_t \eta(\rho, m, J) + \frac{1}{\epsilon} \operatorname{div} Q(\rho, m, J) = \frac{1}{\epsilon} \frac{m}{\rho} \cdot \operatorname{div} S_1 - \frac{1}{\epsilon} \frac{J}{\rho} \cdot \operatorname{div} S_2 - \frac{1}{\epsilon^2} \frac{|m|^2}{\rho}, \tag{3.2}$$

while for the regular solution of the parabolic equation we get:

$$\partial_t \eta(\bar{\rho}, \bar{m}, \bar{J}) + \frac{1}{\epsilon} \operatorname{div} Q(\bar{\rho}, \bar{m}, \bar{J}) = \frac{1}{\epsilon} \frac{\bar{m}}{\bar{\rho}} \cdot \operatorname{div} \bar{S}_1 - \frac{1}{\epsilon} \frac{\bar{J}}{\bar{\rho}} \cdot \operatorname{div} \bar{S}_2 - \frac{1}{\epsilon^2} \frac{|\bar{m}|^2}{\bar{\rho}} + \bar{e} \cdot \frac{\bar{m}}{\bar{\rho}}. \tag{3.3}$$

Before formally prove the relative entropy relation in the context of weak solutions, here we sketch the derivation of (3.2) for the system (2.2) and state the final result. To this end, a direct computation shows

$$\partial_t \left(\frac{1}{2} \frac{|m|^2}{\rho} \right) + \frac{1}{\epsilon} \operatorname{div} \left(\frac{1}{2} m \frac{|m|^2}{\rho^2} \right) = -\frac{1}{\epsilon} u \cdot \nabla p(\rho) + \frac{1}{\epsilon} u \cdot \operatorname{div} S_1 - \frac{1}{\epsilon^2} \rho |u|^2,$$

and

$$\begin{aligned} \partial_t F(\rho, J) &= \partial_t \left(h(\rho) + \frac{1}{2} \frac{|J|^2}{\rho} \right) = -\frac{1}{\epsilon} \operatorname{div} \left(m \left(h'(\rho) + \frac{1}{2} |v|^2 \right) \right) \\ &\quad + \frac{1}{\epsilon} u \cdot \nabla p(\rho) - \frac{1}{\epsilon} v \cdot \operatorname{div} S_2, \end{aligned}$$

leading to (3.2). In this framework, the relative entropy is defined as:

$$\begin{aligned} \eta(\rho, m, J | \bar{\rho}, \bar{m}, \bar{J}) &= \eta(\rho, m, J) - \eta(\bar{\rho}, \bar{m}, \bar{J}) - \eta_\rho(\bar{\rho}, \bar{m}, \bar{J})(\rho - \bar{\rho}) \\ &\quad - \eta_m(\bar{\rho}, \bar{m}, \bar{J}) \cdot (m - \bar{m}) - \eta_J(\bar{\rho}, \bar{m}, \bar{J}) \cdot (J - \bar{J}). \end{aligned}$$

When both solutions are regular, it verifies the following relation:

$$\begin{aligned} \partial_t \eta(\rho, m, J | \bar{\rho}, \bar{m}, \bar{J}) &+ \frac{1}{\epsilon} \operatorname{div}_x Q(\rho, m, J | \bar{\rho}, \bar{m}, \bar{J}) = \\ &- \frac{1}{\epsilon} \rho \nabla \bar{u} : (u - \bar{u}) \otimes (u - \bar{u}) - \frac{1}{\epsilon^2} \rho |u - \bar{u}|^2 - \frac{\rho}{\bar{\rho}} \bar{e} \cdot (u - \bar{u}) - \frac{1}{\epsilon} p(\rho | \bar{\rho}) \operatorname{div} \bar{u} \\ &- \frac{1}{\epsilon} \rho \nabla \bar{u} : (v - \bar{v}) \otimes (v - \bar{v}) - \frac{1}{\epsilon} \rho (\mu''(\rho) \nabla \rho - \mu''(\bar{\rho}) \nabla \bar{\rho}) \cdot ((v - \bar{v}) \operatorname{div} \bar{u} \\ &- (u - \bar{u}) \operatorname{div} \bar{v}) \\ &- \frac{1}{\epsilon} \rho (\mu'(\rho) - \mu'(\bar{\rho})) ((v - \bar{v}) \cdot \nabla (\operatorname{div} \bar{u}) - (u - \bar{u}) \cdot \nabla (\operatorname{div} \bar{v})), \end{aligned}$$

where the relative flux is given by

$$\begin{aligned} Q(\rho, u, v | \bar{\rho}, \bar{u}, \bar{v}) &= \rho u \frac{1}{2} |u - \bar{u}|^2 + \rho u (h'(\rho) - h'(\bar{\rho})) + \frac{1}{2} \rho u |v - \bar{v}|^2 \\ &\quad - \mu(\rho) \nabla v (u - \bar{u}) - \frac{1}{2} \lambda(\rho) \operatorname{div} v (u - \bar{u}) - \\ &\quad \mu(\rho) \nabla u (\bar{v} - v) - \frac{1}{2} \lambda(\rho) \operatorname{div} u (\bar{v} - v) \\ &\quad - \mu(\bar{\rho}) \frac{\rho}{\bar{\rho}} \nabla \bar{v} (\bar{u} - u) + \mu(\bar{\rho}) \frac{\rho}{\bar{\rho}} \nabla \bar{u} (\bar{v} - v) \\ &\quad - \rho \left(\frac{\mu(\rho)}{\rho} - \frac{\mu(\bar{\rho})}{\bar{\rho}} \right) (\nabla \bar{u} (v - \bar{v}) - \nabla \bar{v} (u - \bar{u})) - \\ &\quad \frac{1}{2} \left(\lambda(\rho) - \frac{\rho}{\bar{\rho}} \lambda(\bar{\rho}) \right) ((v - \bar{v}) \operatorname{div} \bar{u} - (u - \bar{u}) \operatorname{div} \bar{v}) \end{aligned}$$

and the relative entropy can be also rewritten as

$$\eta(\rho, m, J | \bar{\rho}, \bar{m}, \bar{J}) = \frac{1}{2} \rho |u - \bar{u}|^2 + \frac{1}{2} \rho |v - \bar{v}|^2 + h(\rho | \bar{\rho}).$$

Now, to generalize this relation for weak solutions, let us first state the precise definition of the latter, based on the one introduced in [15]. We recall that we shall consider here γ -law pressures

$p(\rho) = \rho^\gamma$, while the capillarity coefficient $k(\rho)$ is given by $k(\rho) = \frac{(s+3)^2}{4}\rho^s$, for which we obtain $\mu(\rho) = \rho^{\frac{s+3}{2}}$, with the conditions $\gamma > 1, s + 2 \leq \gamma$ and $s \geq -1$.

Definition 3.1. (ρ, m, J) with $\rho \in C([0, \infty))$; $(L^1(\mathbb{T}^n)) (m, J) \in C([0, \infty))$; $(L^1(\mathbb{T}^n))^{2n}$, $\rho \geq 0$, is a weak (periodic) solution of (2.2) if

$$\sqrt{\rho}u, \sqrt{\rho}v \in L^\infty((0, T); L^2(\mathbb{T}^n)^n), \rho \in C([0, \infty)); (L^\gamma(\mathbb{T}^n)),$$

and (ρ, m, J) satisfy for all $\psi \in C_c^1([0, \infty)); C^1(\mathbb{T}^n)$ and for all $\phi, \varphi \in C_c^1([0, \infty)); C^1(\mathbb{T}^n)^n$:

$$\begin{aligned} & - \iint_{(0,+\infty) \times \mathbb{T}^n} \left(\rho\psi_t + \frac{1}{\epsilon}m \cdot \nabla_x \psi \right) dxdt = \int_{\mathbb{T}^n} \rho(x, 0)\psi(x, 0); \\ & - \iint_{(0,+\infty) \times \mathbb{T}^n} \left[m \cdot (\phi)_t + \frac{1}{\epsilon} \left(\frac{m \otimes m}{\rho} : \nabla_x \phi \right) + \frac{1}{\epsilon} p(\rho) \operatorname{div} \phi \right. \\ & \left. + \frac{1}{\epsilon} \left(\mu(\rho)v \cdot \nabla \operatorname{div}(\phi) + \nabla \mu(\rho) \cdot (\nabla \phi)v + \frac{1}{2} \nabla \lambda(\rho) \cdot v \operatorname{div} \phi + \frac{1}{2} \lambda(\rho)v \cdot \nabla \operatorname{div} \phi \right) \right] dxdt \\ & = -\frac{1}{\epsilon^2} \iint_{(0,+\infty) \times \mathbb{T}^n} m \cdot \phi dxdt + \int_{\mathbb{T}^n} m(x, 0) \cdot \phi(x, 0) dx, \end{aligned}$$

where we have used the identity

$$\begin{aligned} S &= -p(\rho)\mathbb{I} + S_1 = -p(\rho)\mathbb{I} + \mu(\rho)\nabla v + \frac{1}{2}\lambda(\rho) \operatorname{div} v\mathbb{I}; \\ & - \iint_{(0,+\infty) \times \mathbb{T}^n} \left[J \cdot \varphi_t + \frac{1}{\epsilon} \left(\frac{J \otimes m}{\rho} : \nabla_x \varphi \right) - \frac{1}{\epsilon} \left(\mu(\rho)u \cdot (\nabla \operatorname{div} \varphi) + \nabla \mu(\rho) \cdot (\nabla \varphi)u \right) \right. \\ & \left. + \frac{1}{2} \nabla \lambda(\rho) \cdot u \operatorname{div} \varphi + \frac{1}{2} \lambda(\rho)u \cdot \nabla \operatorname{div} \varphi \right] dxdt = \int_{\mathbb{T}^n} J(x, 0) \cdot \varphi(x, 0) dx, \end{aligned}$$

where we have used the identity

$$S_2 = \mu(\rho)^t \nabla u + \frac{1}{2} \lambda(\rho) \operatorname{div} u \mathbb{I}.$$

If in addition $\eta(\rho, m, J) \in C([0, \infty)); L^1(\mathbb{T}^n)$ and (ρ, m, J) satisfy

$$\iint_{(0,+\infty) \times \mathbb{T}^n} (\eta(\rho, m, J)) \dot{\theta}(t) dxdt \leq \int_{\mathbb{T}^n} (\eta(\rho, m, J))|_{t=0} \theta(0) dx$$

$$-\frac{1}{\epsilon^2} \iint_{(0,+\infty) \times \mathbb{T}^n} \frac{|m|^2}{\rho} \theta(t) dx dt \tag{3.4}$$

for any non-negative $\theta \in W^{1,\infty}[0, \infty)$ compactly supported on $[0, \infty)$, then (ρ, m, J) is called a *dissipative weak solution*.

If $\eta(\rho, m, J) \in C([0, \infty); L^1(\mathbb{T}^n))$ and (ρ, m, J) satisfy (3.4) as an equality, then (ρ, m, J) is called a *conservative weak solution*.

We say that a dissipative (or conservative) weak (periodic) solution (ρ, m, J) of (2.2) with $\rho \geq 0$ has finite total mass and energy if

$$\sup_{t \in (0, T)} \int_{\mathbb{T}^n} \rho dx \leq M < +\infty,$$

and

$$\sup_{t \in (0, T)} \int_{\mathbb{T}^n} \eta(\rho, m, J) dx \leq E_o < +\infty.$$

Theorem 3.2. *Let (ρ, m, J) be a dissipative (or conservative) weak solution of (2.2) with finite total mass and energy according to Definition 3.1, and let $\bar{\rho}$ be a smooth solution of (2.3). Then*

$$\begin{aligned} & \int_{\mathbb{T}^n} \eta(\rho, m, J | \bar{\rho}, \bar{m}, \bar{J})(t) dx \leq \int_{\mathbb{T}^n} \eta(\rho, m, J | \bar{\rho}, \bar{m}, \bar{J})(0) dx \\ & - \frac{1}{\epsilon^2} \iint_{(0,t) \times \mathbb{T}^n} \rho |u - \bar{u}|^2 dx d\tau - \frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \rho \nabla \bar{u} : (u - \bar{u}) \otimes (u - \bar{u}) dx d\tau \\ & - \frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} p(\rho | \bar{\rho}) \operatorname{div} \bar{u} dx d\tau - \frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \rho \nabla \bar{u} : (v - \bar{v}) \otimes (v - \bar{v}) dx d\tau \\ & - \iint_{(0,t) \times \mathbb{T}^n} e(\bar{\rho}, \bar{m}) \cdot \frac{\rho}{\bar{\rho}} (u - \bar{u}) dx d\tau \\ & - \frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \rho [(\mu''(\rho) \nabla \rho - \mu''(\bar{\rho}) \nabla \bar{\rho}) \cdot ((v - \bar{v}) \operatorname{div} \bar{u} - (u - \bar{u}) \operatorname{div} \bar{v})] dx d\tau \\ & - \frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \rho (\mu'(\rho) - \mu'(\bar{\rho})) [(v - \bar{v}) \cdot \nabla \operatorname{div} \bar{u} - (u - \bar{u}) \cdot \nabla \operatorname{div} \bar{v}] dx d\tau, \tag{3.5} \end{aligned}$$

where

$$\bar{m} = \bar{\rho} \bar{u} = \epsilon (-\nabla p(\bar{\rho}) + \operatorname{div} S_1(\bar{\rho})); \quad \bar{J} = \bar{\rho} \bar{v} = \nabla \mu(\bar{\rho}). \tag{3.6}$$

Proof. Let (ρ, m, J) be a weak dissipative (or conservative) weak solution of (2.2) according to Definition 3.1 and let $\bar{\rho}$ be a strong solution of (2.3), so that, using (3.6), $(\bar{\rho}, \bar{m}, \bar{J})$ satisfies (2.6). We consider the following function $\theta(\tau)$ in the energy (in)equality (3.4) of Definition 3.1:

$$\theta(\tau) = \begin{cases} 1, & \text{for } 0 \leq \tau < t, \\ \frac{t-\tau}{\mu} + 1, & \text{for } t \leq \tau < t + \tau, \\ 0, & \text{for } \tau \geq t + \mu. \end{cases}$$

Then, as $\mu \rightarrow 0$, we readily obtain:

$$\int_{\mathbb{T}^n} (\eta(\rho, m, J))|_{\tau=0}^t \leq -\frac{1}{\epsilon^2} \iint_{(0,t) \times \mathbb{T}^n} \frac{|m|^2}{\rho} dx d\tau.$$

Moreover, by a direct integration in $(0, t) \times \mathbb{T}^n$ of (3.3) we get:

$$\int_{\mathbb{T}^n} \eta(\bar{\rho}, \bar{m}, \bar{J})|_{\tau=0}^t = -\frac{1}{\epsilon^2} \iint_{(0,t) \times \mathbb{T}^n} \frac{|\bar{m}|^2}{\bar{\rho}} dx d\tau + \iint_{(0,t) \times \mathbb{T}^n} \frac{\bar{m}}{\bar{\rho}} \cdot \bar{e} \tag{3.7}$$

because

$$\begin{aligned} 0 &= -\frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} (\nabla \bar{u} : \bar{S}_1 - \nabla \bar{v} : \bar{S}_2) dx d\tau \\ &= \frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} (\bar{u} \cdot \text{div } \bar{S}_1 - \bar{v} \cdot \text{div } \bar{S}_2) dx d\tau \\ &= \frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} (\nabla \mu(\bar{\rho}) \cdot (\nabla \bar{v} \bar{u} - \nabla \bar{u} \bar{v}) + \mu(\bar{\rho})(\bar{u} \cdot \nabla \text{div } \bar{v} - \bar{v} \cdot \nabla \text{div } \bar{u})) dx d\tau \\ &\quad + \frac{1}{2\epsilon} \iint_{(0,t) \times \mathbb{T}^n} (\nabla \lambda(\bar{\rho}) \cdot (\bar{u} \text{div } \bar{v} - \bar{v} \text{div } \bar{u}) + \lambda(\bar{\rho})(\bar{u} \cdot \nabla \text{div } \bar{v} - \bar{v} \cdot \nabla \text{div } \bar{u})) dx d\tau, \tag{3.8} \end{aligned}$$

being $\bar{\rho}$ periodic and using the definitions of \bar{S}_1 and \bar{S}_2 :

$$\begin{aligned} \bar{S}_1 &= \mu(\bar{\rho}) \nabla \bar{v} + \frac{1}{2} \lambda(\bar{\rho}) \text{div } \bar{v} \mathbb{1}; \\ \bar{S}_2 &= \mu(\bar{\rho}) \nabla \bar{u} + \frac{1}{2} \lambda(\bar{\rho}) \text{div } \bar{u} \mathbb{1}. \end{aligned}$$

Indeed we have:

$$\frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \nabla \bar{u} : \bar{S}_1 dx d\tau = \frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \left(\mu(\bar{\rho}) \nabla \bar{u} : \nabla \bar{v} + \frac{1}{2} \lambda(\bar{\rho}) \text{div } \bar{v} \text{div } \bar{u} \right) dx d\tau,$$

and

$$\frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \nabla \bar{v} : \bar{S}_2 dx d\tau = \frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \left(\mu(\bar{\rho}) \nabla \bar{v} : {}^t \nabla \bar{u} + \frac{1}{2} \lambda(\bar{\rho}) \operatorname{div} \bar{u} \operatorname{div} \bar{v} \right) dx d\tau.$$

Therefore, since

$$\nabla \bar{v} = \nabla \left(\frac{\mu'(\bar{\rho})}{\bar{\rho}} \nabla \bar{\rho} \right) = \nabla^2 M(\bar{\rho})$$

is symmetric, it holds:

$$\nabla \bar{u} : \nabla \bar{v} - \nabla \bar{v} : {}^t \nabla \bar{u} = \nabla \bar{u} : \nabla \bar{v} - \nabla \bar{u} : {}^t \nabla \bar{v} = \nabla \bar{u} : \nabla \bar{v} - \nabla \bar{u} : \nabla \bar{v} = 0,$$

and the integral

$$\begin{aligned} & \frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} (\nabla \bar{u} : \bar{S}_1 - \nabla \bar{v} : \bar{S}_2) dx d\tau \\ &= \frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \left(\mu(\bar{\rho}) (\nabla \bar{u} : \nabla \bar{v} - \nabla \bar{v} : {}^t \nabla \bar{u}) + \frac{1}{2} \lambda(\bar{\rho}) (\operatorname{div} \bar{v} \operatorname{div} \bar{u} - \operatorname{div} \bar{v} \operatorname{div} \bar{u}) \right) dx d\tau \end{aligned}$$

vanishes.

Now we want to evaluate the linear part of the relative entropy for the difference $(\rho - \bar{\rho}, m - \bar{m}, J - \bar{J})$ choosing suitable test functions in the weak formulation (according to Definition 3.1) of the equation satisfied by these differences, namely:

$$\begin{aligned} & - \iint_{[0,\infty) \times \mathbb{T}^n} \left(\psi_t (\rho - \bar{\rho}) + \frac{1}{\epsilon} \psi_{x_i} (m_i - \bar{m}_i) \right) dx d\tau = \int_{\mathbb{T}^n} (\rho - \bar{\rho}) \psi|_{t=0} dx, \tag{3.9} \\ & - \iint_{[0,\infty) \times \mathbb{T}^n} \phi_t \cdot (m - \bar{m}) + \frac{1}{\epsilon} \left(\frac{m_i m_j}{\rho} - \frac{\bar{m}_i \bar{m}_j}{\bar{\rho}} \right) \partial_{x_j} \phi_i + \frac{1}{\epsilon} [p(\rho) - p(\bar{\rho})] \partial_{x_i} \phi_i dx d\tau \\ & - \frac{1}{\epsilon} \iint_{[0,\infty) \times \mathbb{T}^n} (\mu(\rho) v_i - \mu(\bar{\rho}) \bar{v}_i) \partial_{x_i} \partial_{x_j} \phi_j + (\partial_{x_i} \mu(\rho) v_j - \partial_{x_i} \mu(\bar{\rho}) \bar{v}_j) \partial_{x_j} \phi_i dx d\tau \\ & - \frac{1}{\epsilon} \iint_{[0,\infty) \times \mathbb{T}^n} \frac{1}{2} (\partial_{x_i} (\lambda(\rho)) v_i - \partial_{x_i} (\lambda(\bar{\rho})) \bar{v}_i) \partial_{x_j} \phi_j + \frac{1}{2} (\lambda(\rho) v_i - \lambda(\bar{\rho}) \bar{v}_i) \partial_{x_i} \partial_{x_j} \phi_j dx d\tau \\ & = -\frac{1}{\epsilon^2} \iint_{[0,\infty) \times \mathbb{T}^n} (m - \bar{m}) \cdot \phi dx d\tau - \iint_{[0,\infty) \times T^n} \bar{e} \cdot \phi dx d\tau + \int_{\mathbb{T}^n} (m - \bar{m}) \cdot \phi|_{t=0} dx \end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
 & - \iint_{[0, \infty) \times \mathbb{T}^n} (\varphi_t \cdot (J - \bar{J})) + \frac{1}{\epsilon} \left(\frac{J_i m_j}{\rho} - \frac{\bar{J}_i \bar{m}_j}{\bar{\rho}} \right) \partial_{x_j} \varphi_i dx d\tau \\
 & + \frac{1}{\epsilon} \iint_{[0, \infty) \times \mathbb{T}^n} (\mu(\rho) u_i - \mu(\bar{\rho}) \bar{u}_i) \partial_{x_i} \partial_{x_j} \varphi_j + (\partial_{x_i} \mu(\rho) u_j - \partial_{x_i} \mu(\bar{\rho}) \bar{u}_j) \partial_{x_j} \varphi_i dx d\tau \\
 & + \frac{1}{\epsilon} \iint_{[0, \infty) \times \mathbb{T}^n} \frac{1}{2} (\partial_{x_i} \lambda(\rho) u_i - \partial_{x_i} \lambda(\bar{\rho}) \bar{u}_i) \partial_{x_j} \varphi_j + \frac{1}{2} (\lambda(\rho) u_i - \lambda(\bar{\rho}) \bar{u}_i) \partial_{x_i} \partial_{x_j} \varphi_j dx d\tau \\
 & = \int_{\mathbb{T}^n} (J - \bar{J}) \cdot \varphi|_{t=0} dx,
 \end{aligned} \tag{3.11}$$

where ψ, ϕ, φ are Lipschitz test functions, ϕ, φ vector-valued, compactly supported in $[0, +\infty)$ in time and periodic in space. In the above relation we choose in particular

$$\begin{aligned}
 \psi &= \theta(\tau) \left(h'(\bar{\rho}) - \frac{1}{2} \frac{|\bar{m}|^2}{\bar{\rho}^2} - \frac{|\bar{J}|^2}{\bar{\rho}^2} \right) \quad \text{and} \\
 \Phi &= (\phi, \varphi) = \theta(\tau) \left(\frac{\bar{m}}{\bar{\rho}}, \frac{\bar{J}}{\bar{\rho}} \right), \text{ where } \theta(\tau) \text{ is defined above.}
 \end{aligned}$$

Then, letting $\mu \rightarrow 0$ in (3.9) we obtain

$$\begin{aligned}
 & \int_{\mathbb{T}^n} \left(h'(\bar{\rho}) - \frac{1}{2} \frac{|\bar{m}|^2}{\bar{\rho}^2} - \frac{|\bar{J}|^2}{\bar{\rho}^2} \right) (\rho - \bar{\rho})|_{\tau=0} dx \\
 & - \iint_{[0, t] \times \mathbb{T}^n} \left[\partial_\tau \left(h'(\bar{\rho}) - \frac{1}{2} \frac{|\bar{m}|^2}{\bar{\rho}^2} - \frac{|\bar{J}|^2}{\bar{\rho}^2} \right) (\rho - \bar{\rho}) dx d\tau \right] \\
 & - \frac{1}{\epsilon} \iint_{[0, t] \times \mathbb{T}^n} \nabla_x \left(h'(\bar{\rho}) - \frac{|\bar{m}|^2}{\bar{\rho}^2} - \frac{|\bar{J}|^2}{\bar{\rho}^2} \right) \cdot (m - \bar{m}) dx d\tau = 0.
 \end{aligned}$$

From (3.10):

$$\begin{aligned}
 & \int_{\mathbb{T}^n} \frac{\bar{m}}{\bar{\rho}} \cdot (m - \bar{m})|_{\tau=0} dx - \iint_{[0, t] \times \mathbb{T}^n} \partial_\tau \left(\frac{\bar{m}}{\bar{\rho}} \right) \cdot (m - \bar{m}) dx d\tau \\
 & - \frac{1}{\epsilon} \iint_{[0, t] \times \mathbb{T}^n} \left[\left(\frac{m_i m_j}{\rho} - \frac{\bar{m}_i \bar{m}_j}{\bar{\rho}} \right) \partial_{x_j} \left(\frac{\bar{m}_i}{\bar{\rho}} \right) + (p(\rho) - p(\bar{\rho})) \operatorname{div} \left(\frac{\bar{m}}{\bar{\rho}} \right) \right] dx d\tau \\
 & - \frac{1}{\epsilon} \iint_{[0, t] \times \mathbb{T}^n} \left[\mu(\rho) (v - \bar{v}) \cdot \nabla \operatorname{div} \left(\frac{\bar{m}}{\bar{\rho}} \right) + \nabla \mu(\rho) \cdot \nabla \left(\frac{\bar{m}}{\bar{\rho}} \right) (v - \bar{v}) \right] dx d\tau
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \left[(\mu(\rho) - \mu(\bar{\rho}))\bar{v} \cdot \nabla \operatorname{div} \left(\frac{\bar{m}}{\bar{\rho}} \right) + \nabla(\mu(\rho) - \mu(\bar{\rho})) \nabla \left(\frac{\bar{m}}{\bar{\rho}} \right) \bar{v} \right] dx d\tau \\
 & -\frac{1}{2\epsilon} \iint_{[0,\infty) \times \mathbb{T}^n} \left(\nabla \lambda(\rho) \cdot (v - \bar{v}) \operatorname{div} \left(\frac{\bar{m}}{\bar{\rho}} \right) + \lambda(\rho)(v - \bar{v}) \cdot \nabla \operatorname{div} \left(\frac{\bar{m}}{\bar{\rho}} \right) \right) dx d\tau \\
 & -\frac{1}{2\epsilon} \iint_{[0,\infty) \times \mathbb{T}^n} \left[\nabla(\lambda(\rho) - \lambda(\bar{\rho})) \cdot \bar{v} \operatorname{div} \left(\frac{\bar{m}}{\bar{\rho}} \right) + (\lambda(\rho) - \lambda(\bar{\rho}))\bar{v} \cdot \nabla \operatorname{div} \left(\frac{\bar{m}}{\bar{\rho}} \right) \right] dx d\tau \\
 & = -\frac{1}{\epsilon^2} \iint_{[0,t] \times \mathbb{T}^n} \frac{\bar{m}}{\bar{\rho}} \cdot (m - \bar{m}) dx d\tau - \iint_{[0,t] \times \mathbb{T}^n} \frac{\bar{m}}{\bar{\rho}} \cdot \bar{e} dx d\tau.
 \end{aligned}$$

Analogously, from (3.11):

$$\begin{aligned}
 & \int_{\mathbb{T}^n} \frac{\bar{J}}{\bar{\rho}} \cdot (J - \bar{J})|_{\tau=0} dx - \iint_{[0,t] \times \mathbb{T}^n} \partial_\tau \left(\frac{\bar{J}}{\bar{\rho}} \right) \cdot (J - \bar{J}) dx d\tau \\
 & -\frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \left(\frac{m_j J_i}{\rho} - \frac{\bar{m}_j \bar{J}_i}{\bar{\rho}} \right) \partial_{x_j} \left(\frac{\bar{J}_i}{\bar{\rho}} \right) dx d\tau \\
 & +\frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \left[\mu(\rho) \left(\frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right) \cdot \nabla(\operatorname{div} \bar{v}) + \nabla \mu(\rho) \cdot \nabla \bar{v} \left(\frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right) \right] dx d\tau \\
 & +\frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \left[(\mu(\rho) - \mu(\bar{\rho})) \frac{\bar{m}}{\bar{\rho}} \cdot \nabla \operatorname{div} \bar{v} + \nabla(\mu(\rho) - \mu(\bar{\rho})) \nabla \bar{v} \frac{\bar{m}}{\bar{\rho}} \right] dx d\tau \\
 & +\frac{1}{2\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \left[\nabla \lambda(\rho) \cdot \left(\frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right) \operatorname{div} \bar{v} + \lambda(\rho) \left(\frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right) \cdot \nabla \operatorname{div} \bar{v} \right] dx d\tau \\
 & +\frac{1}{2\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \nabla(\lambda(\rho) - \lambda(\bar{\rho})) \cdot \frac{\bar{m}}{\bar{\rho}} \operatorname{div} \bar{v} + (\lambda(\rho) - \lambda(\bar{\rho})) \frac{\bar{m}}{\bar{\rho}} \cdot \nabla \operatorname{div} \bar{v} dx d\tau = 0.
 \end{aligned}$$

Combining the above relations we get:

$$\begin{aligned}
 & \int_{\mathbb{T}^n} [\eta(\rho, m, J|\bar{\rho}, \bar{m}, \bar{J})]|_{\tau=0} dx \leq -\frac{1}{\epsilon^2} \iint_{[0,t] \times \mathbb{T}^n} [\rho|u|^2 - \bar{\rho}|\bar{u}|^2 - \bar{u}(\rho u - \bar{\rho} \bar{u})] dx d\tau \\
 & -\iint_{[0,t] \times \mathbb{T}^n} \left[\partial_\tau \left(h'(\bar{\rho}) - \frac{1}{2}|\bar{u}|^2 - \frac{1}{2}|\bar{v}|^2 \right) (\rho - \bar{\rho}) + \partial_\tau(\bar{u})(\rho u - \bar{\rho} \bar{u}) + \partial_\tau(\bar{v})(\rho v - \bar{\rho} \bar{v}) \right] dx d\tau \\
 & -\frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \nabla \left(h'(\bar{\rho}) - \frac{1}{2}|\bar{u}|^2 - \frac{1}{2}|\bar{v}|^2 \right) (\rho u - \bar{\rho} \bar{u}) dx d\tau
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{\epsilon} \iint_{[0, \infty) \times \mathbb{T}^n} [p(\rho) - p(\bar{\rho})] \operatorname{div} \bar{u} dx d\tau \\
 & -\frac{1}{\epsilon} \iint_{[0, t] \times \mathbb{T}^n} (\rho u_i u_j - \bar{\rho} \bar{u}_i \bar{u}_j) \partial_{x_j} (\bar{u}_i) + (\rho v_i u_j - \bar{\rho} \bar{v}_i \bar{u}_j) \partial_{x_j} (\bar{v}_i) dx d\tau \\
 & -\frac{1}{\epsilon} \iint_{[0, t] \times \mathbb{T}^n} \mu(\rho) [(v - \bar{v}) \nabla \operatorname{div} \bar{u} - (u - \bar{u}) \nabla \operatorname{div} \bar{v}] \\
 & + \nabla \mu(\rho) [\nabla \bar{u} (v - \bar{v}) - \nabla \bar{v} (u - \bar{u})] dx d\tau \\
 & -\frac{1}{2\epsilon} \iint_{[0, t] \times \mathbb{T}^n} \nabla \lambda(\rho) [(v - \bar{v}) \operatorname{div} \bar{u} - (u - \bar{u}) \operatorname{div} \bar{v}] \\
 & + \lambda(\rho) [(v - \bar{v}) \nabla \operatorname{div} \bar{u} - (u - \bar{u}) \nabla \operatorname{div} \bar{v}] dx d\tau \\
 & + \frac{1}{\epsilon} \iint_{[0, t] \times \mathbb{T}^n} ((\mu(\rho) - \mu(\bar{\rho})) (\bar{u} \cdot \nabla \operatorname{div} \bar{v} - \bar{v} \cdot \nabla \operatorname{div} \bar{u}) \\
 & + \nabla(\mu(\rho) - \mu(\bar{\rho})) \cdot (\nabla \bar{v} \bar{u} - \nabla \bar{u} \bar{v})) dx d\tau \\
 & + \frac{1}{2\epsilon} \iint_{[0, t] \times \mathbb{T}^n} ((\lambda(\rho) - \lambda(\bar{\rho})) (\bar{u} \cdot \nabla \operatorname{div} \bar{v} - \bar{v} \cdot \nabla \operatorname{div} \bar{u}) \\
 & + \nabla(\lambda(\rho) - \lambda(\bar{\rho})) \cdot (\bar{u} \operatorname{div} \bar{v} - \bar{v} \operatorname{div} \bar{u})) dx d\tau. \tag{3.12}
 \end{aligned}$$

First of all, let us observe that the last two lines of the relation above are indeed zero, as one can easily prove repeating the arguments leading to (3.8), with the differences $\mu(\rho) - \mu(\bar{\rho})$ and $\lambda(\rho) - \lambda(\bar{\rho})$ replacing $\mu(\bar{\rho})$ and $\lambda(\bar{\rho})$ inside the definition of the tensors S_1 and S_2 . Moreover, using the relation $h''(\bar{\rho}) = p'(\bar{\rho})/\bar{\rho}$ and the continuity equation for $\bar{\rho}$, we get

$$\begin{aligned}
 & - \iint_{[0, t] \times \mathbb{T}^n} \left(\partial_\tau h'(\bar{\rho})(\rho - \bar{\rho}) + \frac{1}{\epsilon} \nabla h'(\bar{\rho})(\rho u - \bar{\rho} \bar{u}) \right) dx d\tau \\
 & = \frac{1}{\epsilon} \iint_{[0, t] \times \mathbb{T}^n} \left(p'(\bar{\rho})(\rho - \bar{\rho}) \operatorname{div} \bar{u} + \frac{\rho}{\bar{\rho}} \nabla p(\bar{\rho})(\bar{u} - u) \right) dx d\tau.
 \end{aligned}$$

We multiply the transport equations of \bar{u} and \bar{v} , namely

$$\begin{aligned}
 \partial_\tau \bar{u} + \frac{1}{\epsilon} (\bar{u} \cdot \nabla \bar{u}) + \frac{1}{\epsilon} \frac{\nabla p(\bar{\rho})}{\bar{\rho}} - \frac{\operatorname{div} \bar{S}_1}{\bar{\rho}} - \frac{\bar{e}}{\bar{\rho}} &= -\frac{1}{\epsilon^2} \bar{u}, \\
 \partial_\tau \bar{v} + \frac{1}{\epsilon} (\bar{u} \cdot \nabla \bar{v}) + \frac{1}{\epsilon} \frac{\operatorname{div} \bar{S}_2}{\bar{\rho}} &= 0,
 \end{aligned}$$

by $\rho(\bar{u} - u)$ and $\rho(\bar{v} - v)$ respectively to conclude

$$\begin{aligned} & \partial_\tau \left(\frac{1}{2} |\bar{u}|^2 \right) (\rho - \bar{\rho}) + \frac{1}{\epsilon} \nabla \left(\frac{1}{2} |\bar{u}|^2 \right) \cdot (\rho u - \bar{\rho} \bar{u}) - \partial_\tau \bar{u} \cdot (\rho u - \bar{\rho} \bar{u}) - \frac{1}{\epsilon} \partial_{x_j} \bar{u}_i (\rho u_i u_j - \bar{\rho} \bar{u}_i \bar{u}_j) \\ &= -\frac{1}{\epsilon} \rho \nabla \bar{u} : [(u - \bar{u}) \otimes (u - \bar{u})] - \frac{1}{\epsilon} \frac{\nabla p(\bar{\rho})}{\bar{\rho}} \rho \cdot (\bar{u} - u) + \frac{1}{\epsilon} \frac{\rho}{\bar{\rho}} \operatorname{div} \bar{S}_1 \cdot (\bar{u} - u) + \bar{e} \frac{\rho}{\bar{\rho}} \cdot (\bar{u} - u) \\ & \quad - \frac{1}{\epsilon^2} \rho \bar{u} \cdot (\bar{u} - u) \end{aligned}$$

and

$$\begin{aligned} & \partial_\tau \left(\frac{1}{2} |\bar{v}|^2 \right) (\rho - \bar{\rho}) + \frac{1}{\epsilon} \nabla \left(\frac{1}{2} |\bar{v}|^2 \right) \cdot (\rho v - \bar{\rho} \bar{v}) - \partial_\tau \bar{v} \cdot (\rho v - \bar{\rho} \bar{v}) - \frac{1}{\epsilon} \partial_{x_j} \bar{v}_i (\rho v_i u_j - \bar{\rho} \bar{v}_i \bar{u}_j) \\ &= -\frac{1}{\epsilon} \rho \nabla \bar{v} : [(v - \bar{v}) \otimes (u - \bar{u})] - \frac{1}{\epsilon} \frac{\rho}{\bar{\rho}} \operatorname{div} \bar{S}_2 \cdot (\bar{v} - v). \end{aligned}$$

In view of the calculation above, (3.12) rewrites as follows:

$$\begin{aligned} & \int_{\mathbb{T}^n} [\eta(\rho, m, J|\bar{\rho}, \bar{m}, \bar{J})] \Big|_{\tau=0}^t dx \leq \\ & - \frac{1}{\epsilon^2} \iint_{[0,t] \times \mathbb{T}^n} \rho |u - \bar{u}|^2 dx d\tau - \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho \nabla \bar{u} : [(u - \bar{u}) \otimes (u - \bar{u})] dx d\tau \\ & - \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho \nabla \bar{v} : [(v - \bar{v}) \otimes (u - \bar{u})] dx d\tau + \iint_{[0,t] \times \mathbb{T}^n} \bar{e} \cdot \frac{\rho}{\bar{\rho}} (\bar{u} - u) dx d\tau \\ & - \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} p(\rho|\bar{\rho}) \operatorname{div} \bar{u} dx d\tau - \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} [\mu(\rho)((v - \bar{v}) \cdot \nabla \operatorname{div} \bar{u} - (u - \bar{u}) \cdot \nabla \operatorname{div} \bar{v}) \\ & \quad + \nabla \mu(\rho) \cdot (\nabla \bar{u}(v - \bar{v}) - \nabla \bar{v}(u - \bar{u}))] dx d\tau \\ & - \frac{1}{2\epsilon} \iint_{[0,t] \times \mathbb{T}^n} [\nabla \lambda(\rho) \cdot ((v - \bar{v}) \operatorname{div} \bar{u} - (u - \bar{u}) \operatorname{div} \bar{v}) + \lambda(\rho)((v - \bar{v}) \cdot \nabla \operatorname{div} \bar{u} \\ & \quad - (u - \bar{u}) \cdot \nabla \operatorname{div} \bar{v})] dx d\tau \\ & + \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \frac{\rho}{\bar{\rho}} \left[(\mu(\bar{\rho}) \operatorname{div} \nabla \bar{v} + {}^t \nabla \mu(\bar{\rho}) {}^t \nabla \bar{v}) \cdot (\bar{u} - u) \right. \\ & \quad \left. + \frac{1}{2} (\nabla \lambda(\bar{\rho}) \operatorname{div} \bar{v} + \lambda(\bar{\rho}) \nabla \operatorname{div} \bar{v}) \cdot (\bar{u} - u) \right] dx d\tau \\ & - \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \frac{\rho}{\bar{\rho}} \left[(\mu(\bar{\rho}) \operatorname{div} {}^t \nabla \bar{u} + {}^t \nabla \mu(\bar{\rho}) \nabla \bar{u}) \cdot (\bar{v} - v) \right. \\ & \quad \left. + \frac{1}{2} (\nabla \lambda(\bar{\rho}) \operatorname{div} \bar{u} + \lambda(\bar{\rho}) \nabla \operatorname{div} \bar{u}) \cdot (\bar{v} - v) \right] dx d\tau. \end{aligned} \tag{3.13}$$

We recall that $\operatorname{div}^t \nabla \bar{u} = \nabla \operatorname{div} \bar{u}$ and therefore $\operatorname{div} \nabla \bar{v} = \nabla \operatorname{div} \bar{v}$ being $\nabla \bar{v}$ symmetric. Hence, we can collect terms as follows:

$$\begin{aligned}
 I_1 := & -\frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \left(\mu(\rho) - \frac{\rho}{\bar{\rho}} \mu(\bar{\rho}) \right) (\nabla \operatorname{div} \bar{u} \cdot (v - \bar{v}) - \nabla \operatorname{div} \bar{v} \cdot (u - \bar{u})) dx d\tau \\
 & - \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \left(\nabla \mu(\rho) - \frac{\rho}{\bar{\rho}} \nabla \mu(\bar{\rho}) \right) \cdot (\nabla \bar{u}(v - \bar{v}) - \nabla \bar{v}(u - \bar{u})) dx d\tau.
 \end{aligned}$$

In addition, recalling also the definition of $v = \frac{\nabla \mu(\rho)}{\rho}$, we conclude:

$$\begin{aligned}
 I_1 = & -\frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho \left(\frac{\mu(\rho)}{\rho} - \frac{\mu(\bar{\rho})}{\bar{\rho}} \right) (\nabla \operatorname{div} \bar{u} \cdot (v - \bar{v}) - \nabla \operatorname{div} \bar{v} \cdot (u - \bar{u})) dx d\tau \\
 & - \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho (v - \bar{v}) \cdot (\nabla \bar{u}(v - \bar{v}) - \nabla \bar{v}(u - \bar{u})) dx d\tau.
 \end{aligned}$$

Moreover, we define

$$\begin{aligned}
 I_2 := & -\frac{1}{2\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \left(\lambda(\rho) - \frac{\rho}{\bar{\rho}} \lambda(\bar{\rho}) \right) ((v - \bar{v}) \cdot \nabla \operatorname{div} \bar{u} - (u - \bar{u}) \cdot \nabla \operatorname{div} \bar{v}) dx d\tau \\
 & - \frac{1}{2\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \left(\nabla \lambda(\rho) - \frac{\rho}{\bar{\rho}} \nabla \lambda(\bar{\rho}) \right) \cdot ((v - \bar{v}) \operatorname{div} \bar{u} - (u - \bar{u}) \operatorname{div} \bar{v}) dx d\tau.
 \end{aligned}$$

Since $\lambda(\rho) = 2(\rho \mu'(\rho) - \mu(\rho))$, one has

$$\begin{aligned}
 I_2 = & -\frac{1}{2\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho \left(\frac{\lambda(\rho)}{\rho} - \frac{\lambda(\bar{\rho})}{\bar{\rho}} \right) ((v - \bar{v}) \cdot \nabla \operatorname{div} \bar{u} - (u - \bar{u}) \cdot \nabla \operatorname{div} \bar{v}) dx d\tau \\
 & - \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho (\mu''(\rho) \nabla \rho - \mu''(\bar{\rho}) \nabla \bar{\rho}) \cdot ((v - \bar{v}) \operatorname{div} \bar{u} - (u - \bar{u}) \operatorname{div} \bar{v}) dx d\tau.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 I_1 + I_2 = & -\frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho ((\mu''(\rho) \nabla \rho - \mu''(\bar{\rho}) \nabla \bar{\rho}) \cdot ((v - \bar{v}) \operatorname{div} \bar{u} + (\bar{u} - u) \operatorname{div} \bar{v})) dx d\tau \\
 & - \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho (\mu'(\rho) - \mu'(\bar{\rho})) ((v - \bar{v}) \cdot \nabla \operatorname{div} \bar{u} + (\bar{u} - u) \cdot \nabla \operatorname{div} \bar{v}) dx d\tau \\
 & - \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} [\rho (v - \bar{v}) \cdot \nabla \bar{u}(v - \bar{v}) - \rho (v - \bar{v}) \nabla \bar{v}(u - \bar{u})] dx d\tau \tag{3.14}
 \end{aligned}$$

Finally, using (3.14) in (3.13) we obtain (3.5) and the proof is complete. \square

4. Stability result and convergence of the diffusive limit

With the relative entropy estimate (3.5) of Theorem 3.2 at hand, we are now able to control our diffusive relaxation limit in terms of the quantity

$$\Psi(t) := \int_{\mathbb{T}^n} \left(h(\rho|\bar{\rho}) + \frac{1}{2}\rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 + \frac{1}{2}\rho \left| \frac{J}{\rho} - \frac{\bar{J}}{\bar{\rho}} \right|^2 \right) dx. \tag{4.1}$$

The proof of our convergence result will follow the blueprint of [14,15], in particular generalizing the results of the latter to our more general case in terms of the capillarity coefficient, thanks to the enlarged reformulation of the system due to [6]. To this end, let us first remark that, since we are dealing here to γ -law gases, $\gamma > 1$, we have

$$h(\rho) = \frac{1}{\gamma - 1} \rho^\gamma.$$

Therefore

$$p(\rho|\bar{\rho}) = (\gamma - 1)h(\rho|\bar{\rho}), \tag{4.2}$$

and the error term in (3.5) involving the pressure will be then controlled in terms of the relative entropy, namely in terms of the “distance” Ψ defined in (4.1). It is worth observing that the same kind of control can be obtained for general monotone pressure laws, with h given as in (1.3) and satisfying appropriate conditions, and for positive densities; see [14,10,15] for details, as well as for discussions about the metric induced by (4.1). Moreover, to control the last two terms of (3.5), we take advantage of the results contained in [6], and in particular the following one, that we report here below for the sake of completeness.

Lemma 4.1. [6, Lemma 35] *Let assume $\mu(\rho) = \rho^{\frac{s+3}{2}}$ with $\gamma \geq s + 2$ and $s \geq -1$. We have*

$$\rho|\mu'(\rho) - \mu'(\bar{\rho})|^2 \leq C(\bar{\rho})h(\rho|\bar{\rho}),$$

with $C(\bar{\rho})$ uniformly bounded for $\bar{\rho}$ belonging to compact sets in $\mathbb{R}^+ \times \mathbb{T}^n$.

We are now ready to state our main convergence theorem.

Theorem 4.2. *Let $T > 0$ be fixed and let (ρ, m, J) be as in Definition 3.1 and $\bar{\rho}$ be a smooth solution of (2.3) with $\bar{\rho} \geq \delta > 0$, and define \bar{m} and \bar{J} by (3.6). Assume the pressure $p(\rho)$ is given by the γ -law ρ^γ , $\gamma > 1$, and assume $\mu(\rho) = \rho^{\frac{s+3}{2}}$ with $\gamma \geq s + 2$ and $s \geq -1$. Then, for any $t \in [0, T]$, the stability estimate*

$$\Psi(t) \leq C(\Psi(0) + \epsilon^4), \tag{4.3}$$

holds true, where C is a positive constant depending on T, M , the L^1 bound for ρ , assumed to be uniform in $\epsilon, \bar{\rho}$, and its derivatives. Moreover, if $\Psi(0) \rightarrow 0$ as $\epsilon \rightarrow 0$, then as $\epsilon \rightarrow 0$

$$\sup_{t \in [0, T]} \Psi(t) \rightarrow 0.$$

Proof. In view of the definition of Ψ in (4.1), from the relative entropy estimate given by Theorem 3.2 we get:

$$\Psi(t) + \frac{1}{\epsilon^2} \iint_{[0, t] \times \mathbb{T}^n} \rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 dx d\tau \leq \Psi(0) + \iint_{[0, t] \times \mathbb{T}^n} (|Q| + |E|) dx d\tau, \tag{4.4}$$

where the terms Q and E are given by

$$E := \bar{v} \cdot \frac{\rho}{\bar{\rho}} \left(\frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right), \quad Q = Q_1 + Q_2,$$

with

$$\begin{aligned} Q_1 &:= -\frac{1}{\epsilon} \iint_{[0, t] \times \mathbb{T}^n} \rho \nabla \bar{u} : [(u - \bar{u}) \otimes (u - \bar{u})] dx d\tau \\ &\quad - \frac{1}{\epsilon} \iint_{[0, t] \times \mathbb{T}^n} \rho \nabla \bar{v} : [(v - \bar{v}) \otimes (v - \bar{v})] dx d\tau - \frac{1}{\epsilon} \iint_{[0, t] \times \mathbb{T}^n} p(\rho | \bar{\rho}) \operatorname{div} \bar{u} dx d\tau \\ Q_2 &:= -\frac{1}{\epsilon} \iint_{[0, t] \times \mathbb{T}^n} \rho [(\mu''(\rho) \nabla \rho - \mu''(\bar{\rho}) \nabla \bar{\rho}) \cdot ((v - \bar{v}) \operatorname{div} \bar{u} - (u - \bar{u}) \operatorname{div} \bar{v})] dx d\tau \\ &\quad - \frac{1}{\epsilon} \iint_{[0, t] \times \mathbb{T}^n} \rho (\mu'(\rho) - \mu'(\bar{\rho})) ((v - \bar{v}) \cdot \nabla \operatorname{div} \bar{u} - (u - \bar{u}) \cdot \nabla \operatorname{div} \bar{v}) dx d\tau. \end{aligned}$$

We use the Young inequality and the previous results to estimate E and Q_1 (as in [14,15]) and Q_2 (following [6]) in terms of the relative entropy itself. We start from the error term E :

$$\begin{aligned} \iint_{[0, t] \times \mathbb{T}^n} |E| dx d\tau &\leq \frac{\epsilon^2}{2} \iint_{[0, t] \times \mathbb{T}^n} \left| \frac{\bar{v}}{\bar{\rho}} \right|^2 \rho dx d\tau + \frac{1}{2\epsilon^2} \iint_{[0, t] \times \mathbb{T}^n} \rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 dx d\tau \\ &\leq CT\epsilon^4 + \frac{1}{4\epsilon^2} \iint_{[0, t] \times \mathbb{T}^n} \rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 dx d\tau, \end{aligned}$$

using the bounds for $\bar{\rho}$, the L^1 bound for ρ , and in view of the fact that, as shown in (2.5), the error term \bar{v} is $O(\epsilon)$. For the term Q_1 we use again the fact that $\nabla \bar{u} = O(\epsilon)$ to conclude

$$\begin{aligned} \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho \nabla \bar{u} : [(u - \bar{u}) \otimes (u - \bar{u})] dx d\tau &\leq C_1 \iint_{[0,t] \times \mathbb{T}^n} \rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 dx d\tau, \\ \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho \nabla \bar{v} : [(v - \bar{v}) \otimes (v - \bar{v})] dx d\tau &\leq C_2 \iint_{[0,t] \times \mathbb{T}^n} \rho \left| \frac{J}{\bar{\rho}} - \frac{\bar{J}}{\bar{\rho}} \right|^2 dx d\tau, \\ \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho(\rho|\bar{\rho}) \operatorname{div} \bar{u} dx d\tau &\leq C_3 \iint_{[0,t] \times \mathbb{T}^n} h(\rho|\bar{\rho}) dx d\tau, \end{aligned}$$

the latter thanks to (4.2) as well. For the new term Q_2 coming from the formulation of the relative entropy estimate of [6], the strategy is the same: we shall take advantage of the estimates from that paper, by carefully taking into account of the singular coefficient in terms of the relaxation parameter ϵ . For the first term we define

$$\begin{aligned} \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho(\mu''(\rho)\nabla\rho - \mu''(\bar{\rho})\nabla\bar{\rho}) \cdot ((v - \bar{v}) \operatorname{div} \bar{u} + (\bar{u} - u) \operatorname{div} \bar{v}) dx d\tau \\ = Q_{21} + Q_{22}, \end{aligned}$$

where

$$\begin{aligned} Q_{21} &:= \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \sqrt{\bar{\rho}}(\mu''(\rho)\nabla\rho - \mu''(\bar{\rho})\nabla\bar{\rho}) \cdot \sqrt{\bar{\rho}}(v - \bar{v}) \operatorname{div} \bar{u} dx d\tau, \\ Q_{22} &:= \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \sqrt{\bar{\rho}}(\mu''(\rho)\nabla\rho - \mu''(\bar{\rho})\nabla\bar{\rho}) \cdot \sqrt{\bar{\rho}}(\bar{u} - u) \operatorname{div} \bar{v} dx d\tau. \end{aligned}$$

Again, $\operatorname{div} \bar{u} = O(\epsilon)$ and, since $\mu''(\rho)\nabla\rho - \mu''(\bar{\rho})\nabla\bar{\rho} = \frac{s+1}{2}(v - \bar{v})$, we readily obtain

$$\begin{aligned} Q_{21} &\leq C_4 \iint_{[0,t] \times \mathbb{T}^n} \rho \left| \frac{J}{\rho} - \frac{\bar{J}}{\bar{\rho}} \right|^2 dx d\tau, \\ Q_{22} &\leq C_5 \iint_{[0,t] \times \mathbb{T}^n} \rho \left| \frac{J}{\rho} - \frac{\bar{J}}{\bar{\rho}} \right|^2 dx d\tau + \frac{1}{8\epsilon^2} \iint_{[0,t] \times \mathbb{T}^n} \rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 dx d\tau, \end{aligned}$$

using Young’s inequality for the second estimate. Analogously, we split the second term in Q_2 in two:

$$\begin{aligned} \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho(\mu'(\rho) - \mu'(\bar{\rho}))((v - \bar{v}) \cdot \nabla \operatorname{div} \bar{u} + (\bar{u} - u) \cdot \nabla \operatorname{div} \bar{v}) dx d\tau \\ = \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \sqrt{\bar{\rho}}(\mu'(\rho) - \mu'(\bar{\rho}))\sqrt{\bar{\rho}}(v - \bar{v}) \cdot \nabla \operatorname{div} \bar{u} dx d\tau \end{aligned}$$

$$+ \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \sqrt{\rho}(\mu'(\rho) - \mu'(\bar{\rho}))\sqrt{\bar{\rho}}(\bar{u} - u) \cdot \nabla \operatorname{div} \bar{v} dx d\tau.$$

Hence, we use Young’s inequality and Lemma 4.1 to bound the first term in view of $\nabla \operatorname{div} \bar{u} = O(\epsilon)$, while for the second one we take advantage of the control given by the friction term:

$$\begin{aligned} & \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho(\mu'(\rho) - \mu'(\bar{\rho}))((v - \bar{v}) \cdot \nabla \operatorname{div} \bar{u} + (\bar{u} - u) \cdot \nabla \operatorname{div} \bar{v}) dx d\tau \\ & \leq C_6 \iint_{[0,t] \times \mathbb{T}^n} \left(h(\rho|\bar{\rho}) + \rho \left| \frac{J}{\rho} - \frac{\bar{J}}{\bar{\rho}} \right|^2 \right) dx d\tau + \frac{1}{8\epsilon^2} \iint_{[0,t] \times \mathbb{T}^n} \rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 dx d\tau. \end{aligned}$$

Finally the relative entropy inequality becomes:

$$\Psi(t) + \frac{1}{2\epsilon^2} \iint_{[0,t] \times \mathbb{T}^n} \rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 dx d\tau \leq \Psi(0) + \tilde{C}\epsilon^4 + C \int_0^t \Psi(\tau) d\tau,$$

and the Gronwall’s Lemma gives the desired result. \square

5. The high friction limit of Navier–Stokes–Korteweg system

In the same spirit of the previous discussions, in this section we want to study the high-friction limit in the case of the Navier–Stokes–Korteweg system, which, in the (enlarged) formulation and after the scaling described above, rewrites as follows:

$$\begin{cases} \partial_t \rho + \frac{1}{\epsilon} \operatorname{div} m = 0 \\ \partial_t m + \frac{1}{\epsilon} \operatorname{div} \left(\frac{m \otimes m}{\rho} \right) + \frac{1}{\epsilon} \nabla p(\rho) - \frac{2\nu}{\epsilon} \operatorname{div}(\mu_L(\rho) Du) - \frac{\nu}{\epsilon} \nabla(\lambda_L(\rho) \operatorname{div} u) \\ \quad = \frac{1}{\epsilon} \operatorname{div} S_1 - \frac{1}{\epsilon^2} m \\ \partial_t J + \frac{1}{\epsilon} \operatorname{div} \left(\frac{J \otimes m}{\rho} \right) + \frac{1}{\epsilon} \operatorname{div} S_2 = 0. \end{cases} \tag{5.1}$$

In system (5.1), $m = \rho u$, $J = \rho v$, the viscosity coefficient ν is positive, and, as denoted above,

$$Du = \frac{\nabla u + {}^t \nabla u}{2}$$

is the symmetric part of the gradient ∇u and we recall that the Lamé coefficient verifies

$$\mu_L(\rho) \geq 0; \quad \frac{2}{n} \mu_L(\rho) + \lambda_L(\rho) \geq 0. \tag{5.2}$$

Moreover, the effective velocity $v = \nabla\mu(\rho)/\rho$ and the stresses S_1 and S_2 are the same of the Euler–Korteweg system, namely

$$\operatorname{div} S_1 = \operatorname{div}(\mu(\rho)\nabla v) + \frac{1}{2\epsilon}\nabla(\lambda(\rho)\operatorname{div} v)$$

and

$$\operatorname{div} S_2 = (\mu(\rho))^t \nabla u + \frac{1}{2\epsilon}\nabla(\lambda(\rho)\operatorname{div} u),$$

as well as the definition of the functions $\mu(\rho)$ and $\lambda(\rho)$. As already pointed out above, we stress once again that the coefficients $\mu_L(\rho)$ and $\lambda_L(\rho)$ need not to coincide with $\mu(\rho)$ and $\lambda(\rho)$, and we shall only assume their L^1 norm is bounded uniformly in ϵ , which can be viewed as a control of them in terms of the pressure term ρ^γ and using the energy bound E_ρ .

The Hilbert expansion applied to system (5.1) will give us the same formal limit of the previous case, that is the viscosity term will affect the expansion only for higher terms, and therefore, the limit solution $\bar{\rho}$ as $\epsilon \rightarrow 0$ satisfies the following equation:

$$\partial_t \bar{\rho} + \operatorname{div}(-\nabla p(\bar{\rho}) + \operatorname{div} S_1(\bar{\rho})) = 0, \tag{5.3}$$

while the (nonzero) leading term for the momentum is given by

$$\bar{m} = \epsilon(-\nabla p(\rho) + \operatorname{div} \bar{S}_1). \tag{5.4}$$

Indeed, we introduce the asymptotic expansion of the state variables:

$$\begin{aligned} \rho &= \rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \dots \\ m &= m_0 + \epsilon m_1 + \epsilon^2 m_2 + \dots \end{aligned}$$

in the system (5.1) and collect the terms of the same order; the expansion for J will clearly come from the one of ρ . Then, from the mass conservation we get:

$$\begin{aligned} O(\epsilon^{-1}) : & \quad \operatorname{div} m_0 = 0; \\ O(1) : & \quad \partial_t \rho_0 + \operatorname{div} m_1 = 0; \\ O(\epsilon) : & \quad \partial_t \rho_1 + \operatorname{div} m_2 = 0; \\ O(\epsilon^2) : & \quad \dots \end{aligned}$$

while, from the momentum equation we get:

$$\begin{aligned} O(\epsilon^{-2}) : & \quad m_0 = 0; \\ O(\epsilon^{-1}) : & \quad -m_1 = \nabla p(\rho_0) - \operatorname{div} S_1(\rho_0); \\ O(1) : & \quad -m_2 = \nabla(p'(\rho_0)\rho_1) - \operatorname{div}(\mu'(\rho_0)\rho_1\nabla v_0 + \mu(\rho)\nabla v_1) \\ & \quad + \nabla(\lambda'(\rho_0)\rho_1\operatorname{div} v_0 + \lambda(\rho_0)\operatorname{div} v_1) \end{aligned}$$

$$- 2\nu \operatorname{div} \left(\mu_L(\rho_0) D \left(\frac{m_1}{\rho_0} \right) \right) - \nu \nabla \left(\lambda_L(\rho) \operatorname{div} \frac{m_1}{\rho_0} \right);$$

$O(\epsilon) : \quad \dots$

Hence, from these first relations, we recover the equilibrium relation $m_0 = 0$, the Darcy’s law $m_1 = -\nabla_x p(\rho_0) + \operatorname{div}_x S_1(\rho_0)$, and the gradient flow dynamic (5.3) for the leading term ρ_0 .

In the same spirit of Section 2, we rewrite the scalar equation (5.3) in the same form of the “hyperbolic part” of system (5.1) by adding an appropriate error term. To this end, let us consider $\bar{\rho}$ a smooth solution of (5.3) and assume \bar{m} is given by (5.4) and, as said before, $\bar{J} = \nabla \mu(\bar{\rho})$. Then $(\bar{\rho}, \bar{m}, \bar{J})$ satisfies

$$\begin{cases} \partial_t \bar{\rho} + \frac{1}{\epsilon} \operatorname{div} \bar{m} = 0 \\ \partial_t \bar{m} + \frac{1}{\epsilon} \operatorname{div} \left(\frac{\bar{m} \otimes \bar{m}}{\bar{\rho}} \right) + \frac{1}{\epsilon} \nabla p(\bar{\rho}) = \frac{1}{\epsilon} \operatorname{div} \bar{S}_1 - \frac{1}{\epsilon^2} \bar{m} + \bar{e} \\ \partial_t \bar{J} + \frac{1}{\epsilon} \operatorname{div} \left(\frac{\bar{J} \otimes \bar{m}}{\bar{\rho}} \right) + \frac{1}{\epsilon} \operatorname{div} \bar{S}_2 = 0, \end{cases} \tag{5.5}$$

where

$$\bar{e} = \frac{1}{\epsilon} \operatorname{div} \left(\frac{\bar{m} \otimes \bar{m}}{\bar{\rho}} \right) + \bar{m}_t = O(\epsilon).$$

The idea of introducing the system (5.5) is that of reconstructing the same first–order part of the relaxing system, to take advantage of the properties which link the entropy and the convective terms, and then obtain in a more direct way the relative energy estimate, as already done in Section 3, and therefore there is no need to introduce viscosity terms (and thus extra errors) in this reformulation of the equilibrium dynamics. As consequence, the structure of the two systems is the same of the one considered above, and hence we shall emphasize here below only the differences with respect to the previous calculations in obtaining the desired relative entropy inequality.

Let us start by recalling the constitutive relations for the functions involved in (5.1), that is the γ –law pressure $p(\rho) = \rho^\gamma$, $\mu(\rho) = \rho^{\frac{s+3}{2}}$ with the conditions $\gamma > 1$, $s + 2 \leq \gamma$, $s \geq -1$ and $\lambda(\rho) = 2(\rho \mu'(\rho) - \mu(\rho))$. The mechanical energy associated to (5.1) is given by

$$\eta(\rho, m, J) = \frac{1}{2} \frac{|m|^2}{\rho} + \frac{1}{2} \frac{|J|^2}{\rho} + h(\rho),$$

and, proceeding as in the previous sections, we (formally) obtain

$$\frac{d}{dt} \int_{\mathbb{T}^n} \eta(\rho, m, J) dx + \frac{2\nu}{\epsilon} \int_{\mathbb{T}^n} \mu_L(\rho) |D(u)|^2 dx + \frac{\nu}{\epsilon} \int_{\mathbb{T}^n} \lambda_L(\rho) |\operatorname{div} u|^2 dx = -\frac{1}{\epsilon^2} \int_{\mathbb{T}^n} \frac{|m|^2}{\rho} dx.$$

In particular, as it is well known, condition (5.2) implies the mechanical energy dissipates along solutions of (5.1). On the other hand, the entropy $\bar{\eta}(\bar{\rho}, \bar{m}, \bar{J})$ associated to (5.5) satisfies:

$$\frac{d}{dt} \int_{\mathbb{T}^n} \bar{\eta}(\bar{\rho}, \bar{m}, \bar{J}) dx = -\frac{1}{\epsilon^2} \int_{\mathbb{T}^n} \frac{|\bar{m}|^2}{\rho} dx + \int_{\mathbb{T}^n} \bar{e} \frac{\bar{m}}{\bar{\rho}} dx. \tag{5.6}$$

We state here below the definition of weak solutions we shall consider in the study of our relaxation limit

Definition 5.1. (ρ, m, J) with $\rho \in C([0, \infty); (L^1(\mathbb{T}^n)))$ $(m, J) \in C([0, \infty); (L^1(\mathbb{T}^n))^{2n})$, $\rho \geq 0$, is a weak (periodic) solution of (5.1) if

$$\begin{aligned} \sqrt{\rho}u, \sqrt{\rho}v &\in L^\infty((0, T); L^2(\mathbb{T}^n)^n), \quad \rho \in C([0, \infty); (L^y(\mathbb{T}^n))), \\ \mu_L(\rho)D(u) &\in L^1((0, T); L^1(\mathbb{T}^n)^{2n}), \quad \lambda_L(\rho) \operatorname{div} u \in L^1((0, T); L^1(\mathbb{T}^n)) \end{aligned}$$

and (ρ, m, J) satisfy for all $\psi \in C_c^1([0, \infty); C^1(\mathbb{T}^n))$ and for all $\phi, \varphi \in C_c^1([0, \infty); C^1(\mathbb{T}^n)^n)$:

$$\begin{aligned} & - \iint_{(0,+\infty) \times \mathbb{T}^n} \left(\rho \psi_t + \frac{1}{\epsilon} m \cdot \nabla_x \psi \right) dx dt = \int_{\mathbb{T}^n} \rho(x, 0) \psi(x, 0); \\ & - \iint_{(0,+\infty) \times \mathbb{T}^n} \left[m \cdot (\phi)_t + \frac{1}{\epsilon} \left(\frac{m \otimes m}{\rho} : \nabla_x \phi \right) + \frac{1}{\epsilon} p(\rho) \operatorname{div} \phi - \frac{2\nu}{\epsilon} \mu_L(\rho) D(u) : \nabla \phi \right. \\ & - \frac{\nu}{\epsilon} \lambda_L(\rho) \operatorname{div} u \operatorname{div} \phi + \frac{1}{\epsilon} (\mu(\rho)v \cdot \nabla \operatorname{div}(\phi) + \nabla \mu(\rho) \cdot (\nabla \phi v)) + \\ & \left. \frac{1}{\epsilon} \left(\frac{1}{2} \nabla \lambda(\rho) \cdot v \operatorname{div} \phi + \frac{1}{2} \lambda(\rho) v \cdot \nabla \operatorname{div} \phi \right) \right] dx dt = \\ & - \frac{1}{\epsilon^2} \iint_{(0,+\infty) \times \mathbb{T}^n} m \cdot \phi dx dt + \int_{\mathbb{T}^n} m(x, 0) \cdot \phi(x, 0) dx, \end{aligned}$$

where we have used the identity

$$\begin{aligned} S &= -p(\rho)\mathbb{I} + S_1 = -p(\rho)\mathbb{I} + \mu(\rho)\nabla v + \frac{1}{2}\lambda(\rho) \operatorname{div} v \mathbb{I}, \\ & - \iint_{(0,+\infty) \times \mathbb{T}^n} \left[J \cdot \varphi_t + \frac{1}{\epsilon} \left(\frac{J \otimes m}{\rho} : \nabla_x \varphi \right) - \frac{1}{\epsilon} \left(\mu(\rho)u \cdot (\nabla \operatorname{div} \varphi) + \nabla \mu(\rho) \cdot (\nabla \varphi u) \right. \right. \\ & \left. \left. + \frac{1}{2} \nabla \lambda(\rho) \cdot u \operatorname{div} \varphi + \frac{1}{2} \lambda(\rho)u \cdot \nabla \operatorname{div} \varphi \right) \right] dx dt = \int_{\mathbb{T}^n} J(x, 0) \cdot \varphi(x, 0) dx, \end{aligned}$$

where we have used the identity

$$S_2 = \mu(\rho)^t \nabla u + \frac{1}{2} \lambda(\rho) \operatorname{div} u \mathbb{I}.$$

If in addition $\eta(\rho, m, J) \in C([0, \infty); L^1(\mathbb{T}^n))$ and (ρ, m, J) satisfy

$$\begin{aligned} & \iint_{(0,+\infty) \times \mathbb{T}^n} (\eta(\rho, m, J)) \dot{\theta}(t) dx dt \leq \int_{\mathbb{T}^n} (\eta(\rho, m, J))|_{t=0} \theta(0) dx \\ & - \frac{1}{\epsilon^2} \iint_{(0,+\infty) \times \mathbb{T}^n} \frac{|m|^2}{\rho} \theta(t) dx dt - \frac{1}{\epsilon} \iint_{(0,\infty) \times \mathbb{T}^n} \mu_L(\rho) |D(u)|^2 \theta(t) dx dt \\ & - \frac{1}{\epsilon} \iint_{(0,\infty) \times \mathbb{T}^n} \lambda_L(\rho) |\operatorname{div} u|^2 \theta(t) dx dt \end{aligned} \tag{5.7}$$

for any non-negative $\theta \in W^{1,\infty}[0, \infty)$ compactly supported on $[0, \infty)$, then (ρ, m, J) is called a *dissipative* weak solution.

If $\eta(\rho, m, J) \in C([0, \infty); L^1(\mathbb{T}^n))$ and (ρ, m, J) satisfy (5.7) as an equality, then (ρ, m, J) is called a *conservative* weak solution.

We say that a dissipative (or conservative) weak (periodic) solution (ρ, m, J) of (5.1) with $\rho \geq 0$ has finite total mass and energy if

$$\sup_{t \in (0,T)} \int_{\mathbb{T}^n} \rho dx \leq M < +\infty,$$

and

$$\sup_{t \in (0,T)} \int_{\mathbb{T}^n} \eta(\rho, m, J) dx \leq E_o < +\infty.$$

The relative entropy calculation is contained in the next theorem.

Theorem 5.2. *Let (ρ, m, J) be a dissipative (or conservative) weak solution of (5.1) with finite total mass and energy according to Definition 5.1, and $\bar{\rho}$ smooth solution of (5.3). Then*

$$\begin{aligned} & \int_{\mathbb{T}^n} \eta(\rho, m, J | \bar{\rho}, \bar{m}, \bar{J})(t) dx \leq \int_{\mathbb{T}^n} \eta(\rho, m, J | \bar{\rho}, \bar{m}, \bar{J})(0) dx \\ & - \frac{2\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \mu_L(\rho) |D(u - \bar{u})|^2 dx d\tau - \frac{\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \lambda_L(\rho) |\operatorname{div}(u - \bar{u})|^2 dx d\tau \\ & - \frac{1}{\epsilon^2} \iint_{(0,t) \times \mathbb{T}^n} \rho |u - \bar{u}|^2 dx d\tau - \frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \rho \nabla \bar{u} : (u - \bar{u}) \otimes (u - \bar{u}) dx dt \\ & - \iint_{(0,t) \times \mathbb{T}^n} e(\bar{\rho}, \bar{m}) \cdot \frac{\rho}{\bar{\rho}} (u - \bar{u}) dx d\tau - \frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} p(\rho | \bar{\rho}) \operatorname{div} \bar{u} dx d\tau \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \rho \nabla \bar{u} : (v - \bar{v}) \otimes (v - \bar{v}) dx d\tau \\
 & -\frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \rho [(\mu''(\rho) \nabla \rho - \mu''(\bar{\rho}) \nabla \bar{\rho}) \cdot ((v - \bar{v}) \operatorname{div} \bar{u} - (u - \bar{u}) \operatorname{div} \bar{v})] dx d\tau \\
 & -\frac{1}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \rho (\mu'(\rho) - \mu'(\bar{\rho})) [(v - \bar{v}) \cdot \nabla \operatorname{div} \bar{u} - (u - \bar{u}) \cdot \nabla \operatorname{div} \bar{v}] dx d\tau. \\
 & -\frac{2\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \mu_L(\rho) D(\bar{u}) : D(u - \bar{u}) dx d\tau - \frac{\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \lambda_L(\rho) \operatorname{div} \bar{u} (\operatorname{div} u - \operatorname{div} \bar{u}) dx d\tau \tag{5.8}
 \end{aligned}$$

where

$$\bar{m} = \bar{\rho} \bar{u} = \epsilon (-\nabla p(\bar{\rho}) + \operatorname{div} S_1(\bar{\rho})); \quad \bar{J} = \bar{\rho} \bar{v} = \nabla \mu(\bar{\rho})$$

are defined as in (3.6)

Proof. To prove the relation (5.8) we underline here only the differences coming from the presence of viscosity term in the momentum equation m . To this end, we recall that from energy inequality (5.7), using the test function $\theta(\tau)$:

$$\theta(\tau) = \begin{cases} 1, & \text{for } 0 \leq \tau < t, \\ \frac{t-\tau}{\mu} + 1, & \text{for } t \leq \tau < t + \tau, \\ 0, & \text{for } \tau \geq t + \mu, \end{cases}$$

as $\mu \rightarrow 0$, one has:

$$\begin{aligned}
 \int_{\mathbb{T}^n} (\eta(\rho, m, J))|_{\tau=0}^t & \leq -\frac{1}{\epsilon^2} \iint_{(0,t) \times \mathbb{T}^n} \frac{|m|^2}{\rho} dx d\tau - \frac{2\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \mu_L(\rho) |D(u)|^2 dx d\tau \\
 & - \frac{\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \lambda_L(\rho) |\operatorname{div} u|^2 dx d\tau.
 \end{aligned}$$

On the other hand, integrating over $(0, t)$ the relation (5.6) we get:

$$\int_{\mathbb{T}^n} \bar{\eta}(\bar{\rho}, \bar{m}, \bar{J})|_{\tau=0}^t dx = -\frac{1}{\epsilon^2} \iint_{(0,t) \times \mathbb{T}^n} \frac{|\bar{m}|^2}{\rho} dx + \iint_{(0,t) \times \mathbb{T}^n} \bar{e} \frac{\bar{m}}{\bar{\rho}} dx.$$

To control the linear correction of the entropy we choose, as in Theorem 3.2, the following test functions in the weak formulation for the differences $(\rho - \bar{\rho}, m - \bar{m}, J - \bar{J})$:

$$\psi = \theta(\tau) \left(h'(\bar{\rho}) - \frac{1}{2} \frac{|\bar{m}|^2}{\bar{\rho}^2} - \frac{|\bar{J}|^2}{\bar{\rho}^2} \right) \quad \text{and}$$

$$\Phi = (\phi, \varphi) = \theta(\tau) \left(\frac{\bar{m}}{\bar{\rho}}, \frac{\bar{J}}{\bar{\rho}} \right),$$

where $\theta(\tau)$ is defined above. Since $D(u) : \nabla\phi = D(u) : D(\phi)$ and the equation for \bar{m} does not involve viscosity terms, the new terms due to the viscosity in the weak formulation of the equation for $m - \bar{m}$ re given solely by:

$$+\frac{\nu}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} (2\mu_L(\rho)D(u) : D(\bar{u})dx d\tau + \lambda_L(\rho) \operatorname{div} u \operatorname{div} \bar{u})dx d\tau.$$

Hence, the new terms we need to handle here with respect to Theorem 3.2 are the following integrals:

$$\begin{aligned} &\frac{2\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \mu_L(\rho)[D(u) : D(\bar{u})]dx d\tau + \frac{\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \lambda_L(\rho) \operatorname{div} u \operatorname{div} \bar{u}dx \tau \\ &- \frac{2\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \mu_L(\rho)|D(u)|^2 dx d\tau - \frac{\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \lambda_L(\rho)|\operatorname{div} u|^2 dx d\tau, \end{aligned}$$

which can be rearranged as follows:

$$\begin{aligned} &-\frac{2\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \mu_L(\rho)|D(u - \bar{u})|^2 dx d\tau - \frac{2\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \mu_L(\rho)D(\bar{u}) : D(u - \bar{u})dx d\tau \\ &- \frac{\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \lambda_L(\rho)|\operatorname{div}(u - \bar{u})|^2 dx d\tau - \frac{\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \lambda_L(\rho) \operatorname{div} \bar{u}(\operatorname{div} u - \operatorname{div} \bar{u})dx d\tau. \end{aligned}$$

Hence, repeating the same calculation of Theorem 3.2 for all remaining terms we readily obtain (5.8) and the proof is complete. \square

Now we use Theorem 5.2 to measure the distance between the two solutions in terms of the relative entropy as in Section 4. To this end, we recall the definition (4.1) of the “distance” $\Psi(t)$:

$$\Psi(t) = \int_{\mathbb{T}^n} \left(h(\rho|\bar{\rho}) + \frac{1}{2}\rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 + \frac{1}{2}\rho \left| \frac{J}{\rho} - \frac{\bar{J}}{\bar{\rho}} \right|^2 \right) dx.$$

Then the following theorem holds.

Theorem 5.3. *Let $T > 0$ be fixed, let (ρ, m, J) be as in Definition 5.1 and $\bar{\rho}$ be a smooth solution of (5.3) such that $\bar{\rho} \geq \delta > 0$, \bar{m} and \bar{J} defined as (3.6). Assume the pressure $p(\rho)$ is given by the γ -law ρ^γ with $\gamma > 1$. Assume $\mu(\rho) = \rho^{\frac{s+3}{2}}$ with $\gamma \geq s + 2$ and $s \geq -1$, and*

$$\|\mu_L(\rho)\|_{L^\infty((0,t);L^1(\mathbb{T}^n))}, \|\lambda_L(\rho)\|_{L^\infty((0,t);L^1(\mathbb{T}^n))} \leq \tilde{E}, \tag{5.9}$$

for a positive constant \tilde{E} independent from ϵ . Then, for $t \in [0, T]$, the stability estimate

$$\Psi(t) \leq C(\Psi(0) + \epsilon^4 + \nu\epsilon), \tag{5.10}$$

holds true, where C is a positive constant depending on T, M , the L^1 bound for ρ , and E_0 , the energy bound, both assumed to be uniform in $\epsilon, \bar{\rho}$ and its derivatives. Moreover, if $\Psi(0) \rightarrow 0$ as $\epsilon \rightarrow 0$, then as $\epsilon \rightarrow 0$

$$\sup_{t \in [0, T]} \Psi(t) \rightarrow 0. \tag{5.11}$$

Proof. From the definition of $\Psi(t)$ and from the relative entropy estimate given by Theorem 5.2 we obtain for $t \in [0, T]$:

$$\begin{aligned} \Psi(t) &+ \frac{1}{\epsilon^2} \iint_{[0,t] \times \mathbb{T}^n} \rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 dx d\tau + \frac{2\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \mu_L(\rho) |D(u) - D(\bar{u})|^2 dx d\tau \\ &+ \frac{\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \lambda_L(\rho) [\operatorname{div} u - \operatorname{div} \bar{u}]^2 dx d\tau \leq \Psi(0) + \iint_{(0,t) \times \mathbb{T}^n} (|E| + |Q| + |E_2|) dx d\tau. \end{aligned}$$

The terms Q and E are exactly the same of Section 4, that is

$$E = \bar{e} \cdot \frac{\rho}{\bar{\rho}} \left(\frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right)$$

and

$$\begin{aligned} Q &= -\frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho \nabla \bar{u} : [(u - \bar{u}) \otimes (u - \bar{u})] dx d\tau \\ &- \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} \rho \nabla \bar{u} : [(v - \bar{v}) \otimes (v - \bar{v})] dx d\tau - \frac{1}{\epsilon} \iint_{[0,t] \times \mathbb{T}^n} p(\rho|\bar{\rho}) \operatorname{div} \bar{u} dx d\tau \\ &- \frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^n} \rho [(\mu''(\rho) \nabla \rho - \mu''(\bar{\rho}) \nabla \bar{\rho}) ((v - \bar{v}) \operatorname{div} \bar{u} - (u - \bar{u}) \operatorname{div} \bar{v})] dx dt \\ &- \frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^n} \rho (\mu'(\rho) - \mu'(\bar{\rho})) [(v - \bar{v}) \nabla \operatorname{div} \bar{u} - (u - \bar{u}) \nabla \operatorname{div} \bar{v}] dx d\tau, \end{aligned}$$

while the new error term E_2 is defined as follows:

$$\begin{aligned}
 E_2 &:= -\frac{2\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \mu_L(\rho) D(\bar{u}) : D(u - \bar{u}) dx d\tau \\
 &\quad - \frac{\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \lambda_L(\rho) \operatorname{div} \bar{u} (\operatorname{div} u - \operatorname{div} \bar{u}) dx d\tau \\
 &=: E_{21} + E_{22}.
 \end{aligned}$$

Clearly, the terms Q and E can be bounded as in Theorem 4.2, namely

$$\iint_{[0,t] \times \mathbb{T}^n} |E| dx d\tau \leq CT\epsilon^4 + \frac{1}{4\epsilon^2} \iint_{[0,t] \times \mathbb{T}^n} \rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 dx d\tau,$$

where C depends on the bound for $\bar{\rho}$ and on the (uniform) L^1 bound for ρ . Here we also used the fact that $\bar{v} = O(\epsilon)$. Moreover, we recall the estimate for Q as well:

$$\iint_{[0,t] \times \mathbb{T}^n} |Q| dx d\tau \leq \frac{1}{4\epsilon^2} \iint_{[0,t] \times \mathbb{T}^n} \rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 dx d\tau + \tilde{C} \int_{[0,t]} \Psi(\tau) d\tau,$$

where \tilde{C} depends on the bounds $\operatorname{div} \bar{u}/\epsilon = O(1)$ and $\nabla \operatorname{div} \bar{u}/\epsilon = O(1)$.

To bound the new terms E_{21} and E_{22} we shall use the uniform bound (5.2) as follows.

$$\begin{aligned}
 E_{21} &= -\frac{2\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \mu_L(\rho) D(\bar{u}) : [D(u - \bar{u})] dx d\tau \leq \\
 &\frac{4\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \mu_L(\rho) |D(\bar{u})|^2 dx d\tau + \frac{\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \mu_L(\rho) |D(u - \bar{u})|^2 dx d\tau \leq \\
 &\frac{\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \mu_L(\rho) |D(u - \bar{u})|^2 dx d\tau + \nu C_2 T \epsilon,
 \end{aligned}$$

where we have used $D(\bar{u}) = O(\epsilon)$, and C_2 depends also on E_o in view of (5.9). The estimate for E_{22} is analogous: we use the fact that $\operatorname{div} \bar{u} = O(\epsilon)$ as follows

$$\begin{aligned}
 E_{22} &= -\frac{\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \lambda_L(\rho) \operatorname{div} \bar{u} (\operatorname{div} u - \operatorname{div} \bar{u}) dx d\tau \leq \\
 &\frac{\nu}{2\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \lambda_L(\rho) |\operatorname{div}(u - \bar{u})|^2 dx d\tau + \frac{2\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \lambda_L(\rho) |\operatorname{div} \bar{u}|^2 dx d\tau \leq \\
 &+ \nu C_4 T \epsilon + \frac{\nu}{2\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \lambda_L(\rho) |\operatorname{div}(u - \bar{u})|^2 dx d\tau,
 \end{aligned}$$

where C_4 depends also on E_o , again using (5.9).

Finally we get:

$$\begin{aligned} \Psi(t) + \frac{1}{2\epsilon^2} \iint_{[0,t] \times \mathbb{T}^n} \rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 dx d\tau + \frac{\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \mu_L(\rho) |D(u) - D(\bar{u})|^2 dx d\tau \\ + \frac{\nu}{2\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \lambda_L(\rho) [\operatorname{div} u - \operatorname{div} \bar{u}]^2 dx d\tau \leq \Psi(0) + \tilde{C}\epsilon^4 + \nu\bar{C}\epsilon + \int_0^t \Psi(\tau) d\tau, \end{aligned}$$

and, since from the relation (5.2) we obtain

$$\begin{aligned} 0 \leq \frac{\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \frac{1}{2} \left(\lambda_L(\rho) + \frac{2}{n} \mu_L(\rho) \right) |\operatorname{div}(u - \bar{u})|^2 dx d\tau \\ \leq \frac{\nu}{\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \mu_L(\rho) |D(u) - D(\bar{u})|^2 dx d\tau + \frac{\nu}{2\epsilon} \iint_{(0,t) \times \mathbb{T}^n} \lambda_L(\rho) [\operatorname{div} u - \operatorname{div} \bar{u}]^2 dx d\tau, \end{aligned}$$

the Gronwall’s Lemma gives the result. \square

Remark 5.4. Let us emphasize that the choice $\mu_L(\rho) = \mu(\rho)$ and $\lambda_L(\rho) = \lambda(\rho)$ is compatible with (5.2) and (5.9) in the range of exponents considered here. Indeed, we have $\mu(\rho) = \rho^{\frac{s+3}{2}}$ with $s \geq -1$, $s + 2 \leq \gamma$, and $\gamma > 1$, and $\lambda(\rho) = \frac{s+1}{2}\mu(\rho)$. Then $\mu(\rho)$ and $\lambda(\rho)$ are both nonnegative and

$$\|\mu_L(\rho)\|_{L^\infty((0,t); L^1(\mathbb{T}^n))}, \|\lambda_L(\rho)\|_{L^\infty((0,t); L^1(\mathbb{T}^n))} \leq \bar{C}\|\rho\|_{L^\infty((0,t); L^\gamma(\mathbb{T}^n))} \leq \bar{C}E_o^{\frac{1}{\gamma}}.$$

Moreover, it is worth to observe the difference between the stability estimate (4.3) obtained for the Euler–Korteweg model and (5.10) of Theorem 5.3. Besides the common control of the initial relative entropy $\Psi(0)$, the latter gives a control of the errors of the form $O(\epsilon^4) + O(\nu\epsilon)$, which is consistent with the one in (4.3) as $\nu \rightarrow 0+$. In other words, the stability estimate obtained in the Euler–Korteweg case is better, nevertheless it is recovered by the one obtained in the presence of the viscosity terms. The leeway which allows us to perform this estimate in the case of the high friction limit for the Navier–Stokes–Korteweg system is linked to the fact the viscosity terms appear at an intermediate order in the Hilbert expansion and they are “less singular” with respect to the ones coming from the friction term, and therefore they can be controlled in the relative entropy estimate.

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