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Decay in $W^{1,\infty}$ for the 1D semilinear damped wave equation on a bounded domain

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Abstract

In this Ph.D. thesis, we study a semilinear wave equation with nonlinear and time-dependent damping term.

After rewriting the equation as a first order system, we define a class of approximate solutions employing typical tools of hyperbolic systems of conservation laws, such as the Riemann problem. We prove that the initial-boundary value problem is well-posed for initial data in L^∞ space. By recasting the problem as a discrete-time nonhomogeneous system, which is related to a probabilistic interpretation of the solution, we provide a strategy to study its long-time behavior uniformly with respect to the mesh size parameter $\Delta x = 1/N \to 0$. The proof makes use of the Birkhoff decomposition of doubly stochastic matrices and of accurate estimates on the iteration system as $N \to \infty$.

Under appropriate assumptions on the nonlinearity, we prove the exponential convergence in L^∞ of the solution to the first order system towards a stationary solution, as $t \to +\infty$, as well as uniform error estimates for the approximate solutions.

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Chapter 1

Introduction

1.1 Introduction and motivation of the work

This Ph.D. thesis aims at studying the asymptotic behavior of solutions of a semilinear damped wave equation on a bounded 1D domain. Such equations arise in many contexts in mathematics, and they are important as they can describe real-world applications. For instance, the damped wave equations can be used to model a propagation of waves in dissipative media. Moreover, the damped wave equation first occurred in the mathematical description of the telegraph, and it is generally known as the equation of telegraphy. The telegrapher's equation, see [42], is a linear differential equation which represents a classical vibrating membrane with the resistance proportional to the velocity.

In this dissertation, we consider the following semilinear damped wave equation

$$\begin{cases} \partial_{tt}u - \partial_{xx}u + 2k(x)\alpha(t)g(\partial_t u) = 0\,, & (x,t) \in [0,1] \times \mathbb{R}^+\,, \\ u(0,x) = u_0(x)\,, \quad \partial_t u(0,x) = u_1(x)\,, & x \in [0,1]\,, \\ u(t,0) = u(t,1) = 0\,, & t \in \mathbb{R}^+\,. \end{cases} \tag{1.1}$$

Until recently, equation (1.1) was studied with either space- or time-dependent damping coefficients, and the methods used for these two types of problems are incompatible. In the time-independent case, $\alpha(t) = const.$, problem (1.1) has been considered in several papers, see for instance, [22, 32, 34, 39, 49, 21, 35, 19], the review paper [50], and the recent monograph [36]. It is well known that the initial-boundary value problem (1.1) is well-posed for initial data $(u_0, u_1) \in H_0^1([0,1]) \times L^2([0,1])$, for $k(x) \in L^\infty([0,1])$ with $k(x) \geq 0$. Moreover, suitable decay estimates for the energy are obtained, which are either exponential or polynomial. More precisely, for the decay to hold, it is sufficient that the non-negative damping potential $k(x)$ to be effective (strictly positive) in a subinterval. Whether the decay is exponential or polynomial depends in an essential manner on the behavior of the function g near $u_t = 0$, so that the decay is exponential when g behaves linearly near zero and polynomial when g degenerates in a power-like

fashion. For example, in [44], the author considered the case $g(s) = |s|^r s$, with $r > -1$, and under the assumption that the function $k(x)$ is supported only near the boundary, it is proved that the energy of the solution decays with rate $(1+t)^{-\mu}$ for some $\mu > 0$ provided a regular initial data $(u_0, u_1) \in (H^2 \cap H_0^1) \times H_0^1$. In [48], the author has generalized the results of [44] to include a large class of function g. We also refer to the recent paper [1], where sharp energy decay rates are obtained for a large class of nonlinearly first-order damped systems.

In [35, Theorem 3.1], L^p decay estimates with $2 \leq p \leq \infty$ are obtained under the assumptions of $k(x) = 1$, $\alpha(t) = 1$, g' that vanishes at 0, and by assuming sufficiently regular data, $(u_0, u_1) \in W^{2,\infty}([0,1]) \times W^{1,\infty}([0,1])$. This regularity restriction appears to be due to the lack of a Lyapunov functional, equivalent to the norm of $(u(\cdot,t), u_t(\cdot,t))$ in $W^{1,\infty}([0,1]) \times L^{\infty}([0,1])$, see also [20].

There are fewer results in the case of the wave equation with time-dependent dissipation. It is shown, in particular, that the effect of the time dependence on the decay rate is very delicate. Nakao in [45] generalizes the result of [44] to prove a precise decay estimate of the solutions of the initial-boundary value problem (1.1) for a regular data $(u_0, u_1) \in (H_2 \cap H_0^1) \times H_0^1$, with a localized nonlinear dissipation which depends on the time and on the space variable. In [11], the author generalizes Nakao work [45] to prove decay estimates for (1.1) without any assumption on the dynamics (that is, there is no assumption in the domain where the damping term is effective).

In this Ph.D. thesis, first, we prove the existence and stability of weak solutions to (1.1) with time-dependent damping term. Then, we assume that the damping is space-dependent and that $g' > 0$ to prove the decay of solutions in $W^{1,\infty}$. Our main contribution is to develop a novel approach that originates from the point of view of the hyperbolic systems of balance laws. In particular, after rewriting the equation as a first order system, we construct approximate solutions that allow us to get an accurate description of the solution, whose evolution is recast as a discrete time system. Then we find a strategy for the analysis of this system that makes use of a discrete representation formula (rather than Lyapunov functionals). This eventually leads to the decay in L^{∞} of the solution in terms of (u_x, u_t).

More precisely, we introduce the variables $J = \partial_t u$ and $\rho = -\partial_x u$, then the damped wave equation (1.1) is equivalent to the following first order 2×2 system in one space dimension

$$\begin{cases} \partial_t \rho + \partial_x J = 0, \\ \partial_t J + \partial_x \rho = -2k(x)\alpha(t)g(J), \end{cases} \tag{1.2}$$

where $x \in I = [0,1]$, $t \geq 0$ and

$$(\rho, J)(\cdot, 0) = (\rho_0, J_0)(\cdot)\,, \qquad J(0,t) = J(1,t) = 0\,, \tag{1.3}$$

for $(\rho_0, J_0) \in L^\infty(I)$.

About the terms k, α and g in (1.2), let

$$k \in L^1(I)\,, \quad k \geq 0 \; a.e.\,, \qquad g \in C^1(\mathbb{R})\,, \quad g(0) = 0\,, \quad g'(J) \geq 0\,,$$

and

$$\alpha \in BV_{loc} \cap L^\infty([0,\infty);[0,1])\,, \qquad \alpha(t) \geq 0\,.$$

We remark that the assumption on k includes the possibility of localized damping, for instance, $k(x) = \bar{k} > 0$ on some (α, β) with $[\alpha, \beta] \subset (0,1)$, and $k(x) = 0$ otherwise. In this thesis, part of the analysis is carried on for the case of localized damping, while for the proof of L^∞ estimate for the solutions of (1.2), we require that $k(x)$ is uniformly positive, see assumption (1.11).

The coefficient $\alpha(t)$ in (1.2), with values in $[0,1]$, plays the role of a time localization of the damping term. A specific time dependent case is the *intermittent damping* [43, 37], in which for some $0 < T_1 < T_2$ one has

$$\alpha(t) = \begin{cases} 1 & t \in [0, T_1), \\ 0 & t \in [T_1, T_2) \end{cases}, \qquad \alpha(t + T_2) = \alpha(t) \quad \forall\, t > 0\,. \tag{1.4}$$

In the time-independent case, $\alpha(t) = const.$, the large time behavior of solutions to (1.2)–(1.3) is governed by the stationary solution

$$J(x) = 0, \qquad \rho(x) = const. = \int_I \rho_0\,.$$

After possibly changing the variable ρ with $\rho - \int_I \rho_0$, it is not restrictive to assume that $\int_I \rho_0(x)\,dx = 0$.

This thesis gives an overall review of works contained in [3, 4, 5], in addition, we provide an ongoing study of the localized damping case. More precisely, in [3], which is a joint work with Debora Amadori and Edda Dal Santo, we consider initial data $(\rho_0, J_0) \in BV(I)$, and we assume that the damping is space dependent with $g' > 0$. Hence, under appropriate assumptions on the nonlinearity, we prove the exponential convergence in L^∞ of the solution to the first order system (1.2) towards the stationary solution, as $t \to +\infty$, as well as uniform error estimates for the approximate solutions. In [4], which is in collaboration with Debora Amadori, we prove that the initial-boundary value problem (1.2)-(1.3) with time-dependent source is well-posed for $L^\infty(I)$ initial data, as well as, we work on extending the

results of [3] by showing that the time-asymptotic stability of solutions $\rho = 0 = J$ holds for L^∞ initial data. Finally, we provide some numerical simulations and perspectives.

For the system (1.2) a class of approximations of Well-Balanced type to the Cauchy problem was studied in [28, 27] and in the papers [6, 7, 8]. In these last papers, suitable L^1 error estimates are derived by means of stability analysis for hyperbolic systems of conservation laws, obtained through a suitable adaptation of the Bressan-Liu-Yang functional [16, 15].

In this thesis, we adopt the same approach to define approximate solutions, for the initial-boundary value problem (1.2)–(1.3). We remark that these approximate solutions can be regarded as wave-front tracking solutions [15], with a special choice of the approximate initial data, having discontinuities uniformly distributed on a grid.

However, the analysis of the approximate solutions for the boundary value problem turns out to be very different from the one for the Cauchy problem. Indeed, thanks to the fact that the characteristic speeds are ± 1 and by approximating the initial data with a uniform mesh, the structure of the approximate solution (see Figure 3.4) is very special: each approximate solution is piecewise constant, with discontinuities that occur along polygonal lines that intersect each other at times which are multiple of $\Delta x = \Delta t = N^{-1}$, $N \in 2\mathbb{N}$.

As a consequence, based on the probabilistic interpretation of the solution (see [42, 23] and [17] for a semilinear hyperbolic system with relaxation), the problem can be recast as a discrete time evolution of a finite dimensional linear system of size N as follows,

$$\sigma(t^n+) = B(t^n)\sigma(t^{n-1}+) + (\bar{\alpha}_n - \bar{\alpha}_{n-1})\, \boldsymbol{G}_n\,, \tag{1.5}$$

where $\sigma(t)$, $t \neq t^n, t^{n+1/2}$ denotes a vector of *wave sizes* appearing in the approximate solution to (1.2), (1.3) at time t, while $B(t^n)$ is a doubly stochastic matrix (that is, a non-negative matrix for which the sum of all the elements by row is 1, as well as by column), the matrix $B(t^n) \in M_{2N}(\mathbb{R})$ depends on time: more precisely, its coefficients depend on $g'(J(x,t))$ and on $\alpha(t)$. The vector $\boldsymbol{G}_n$ appears in the time-dependent case that is $\alpha(t) \neq const.$, see (4.10) for its definition.

We remark that part of the analysis and the well posedness result in Theorem 1.2.1 are for the time- and space-dependent case, while for the proof of the decay of solutions in L^∞, we require that $\alpha(t) \equiv 1$. When $\alpha(t) \equiv 1$, we have the following representation:

$$\sigma(t^n+) = B(t^n)\sigma(t^{n-1}+) = B(t^n)B(t^{n-1})\cdots B(0+)\sigma(0+)\,. \tag{1.6}$$

For a review of the properties of non-negative and stochastic matrices, see references [10, 40, 47]. We refer the reader to Section 4.1 for more details on the derivation of (1.5) and on the structure of $B(t^n)$. The behavior of the vector $\sigma(t)$ as $n \to \infty$ depends on the spectral properties of the matrix B: whenever g is nonlinear (that is, B is not constant in time even if $\alpha(t) \equiv 1$), the behavior of (1.5) (or (1.6)) is not trivial and may require advanced matrix analysis' tools, such as the concept of Joint Spectral Radius ([41, 30]).

Also, a possible approach to the study of exponential stability of $\sigma(t) \to 0$ in (1.6) goes through the existence of a suitable Lyapunov functional. For N fixed it is certainly possible to construct it, for instance by constructing a suitable norm on $\mathbb{R}^{2N}$ which is contractive along the discrete trajectories of the system; this is possibly done by means of Schur triangularization theorem [40, Theorem 2.3.1, p. 101] and using the fact that the spectral radius of a square matrix A is the greatest lower bound of all the matrix norms of A [40, Lemma 5.6.10, p. 347]. See also the recent preprint [14].

However, following this strategy, it does not appear clear how to get the needed information on the size of the eigenvalues, uniformly on N. We overcame this difficulty by working on iterates of B in (1.6) having a constant balance between n and N, which is the relevant limit.

In the next section of this introductory chapter, we summarize the main results of the thesis. Section 1.3 is devoted to showing the organization of the thesis.

1.2 Main results

In this section, we introduce the main results of this thesis. The first one (Theorem 1.2.1) concerns the existence and stability of weak solutions to (1.2) with time-dependent source, while the second one (Theorem 1.2.2) concerns the asymptotic-time decay in L^∞ of the solutions with BV initial data under more specific assumptions.

From now on, we use the standard notation $\mathbb{R}_+ = [0, +\infty)$.

Definition 1.2.1. *Let $(\rho_0, J_0) \in L^\infty(I)$. A weak solution of the problem (1.2)–(1.3) is a function*

$$(\rho, J) : I \times \mathbb{R}_+ \to \mathbb{R}^2$$

that satisfies the following properties:

(a) the map $t \mapsto (\rho, J)(\cdot, t)$ is continuous from $\mathbb{R}_+$ to $L^\infty(I; \mathbb{R}^2)$, and it satisfies $(\rho, J)(\cdot, 0) = (\rho_0, J_0)$;

(b) the equation $(1.2)_1$ is satisfied in the distributional sense in $[0,1] \times (0, \infty)$, while the equation $(1.2)_2$ in the distributional sense in $(0,1) \times (0, \infty)$.

The boundary condition in (1.3) is taken into account by means of the first part of (b), that is, by requiring that for all test functions $\phi \in C^1([0,1] \times (0,+\infty))$ one has

$$\int_0^1 \int_0^\infty \{\rho \partial_t \phi + J \partial_x \phi\} \, dxdt = 0 \,.$$

Now we state the following well-posedness result.

Theorem 1.2.1. *Assume that*

$$k \in L^1(I) \,, \quad k \geq 0 \text{ a.e.} \,, \qquad g \in C^1(\mathbb{R}) \,, \quad g(0) = 0 \,, \quad g'(J) \geq 0 \tag{1.7}$$

and that

$$\alpha \in BV_{loc} \cap L^\infty([0,\infty);[0,1]) \,. \tag{1.8}$$

Let $(\rho_0, J_0) \in L^\infty(I)$ with $\int_I \rho_0 = 0$. Then there exists a unique function

$$(\rho, J) : I \times \mathbb{R}_+ \to \mathbb{R}^2$$

which is a weak solution of (1.2)–(1.3) in the sense of Definition 1.2.1. Let $J = f^+ - f^-$ and $\rho = f^+ + f^-$ (see system (3.5)), then for any pair of initial data (f_0^-, f_0^+) and $(\widetilde{f}_0^-, \widetilde{f}_0^+)$, the corresponding solutions $f^\pm$, $\widetilde{f}^\pm$ on $(0,1) \times (0,T)$ satisfy

$$\|(f^-, f^+)(\cdot,t) - (\widetilde{f}^-, \widetilde{f}^+)(\cdot,t)\|_{L^1(I)} \leq \|(f_0^-, f_0^+) - (\widetilde{f}_0^-, \widetilde{f}_0^+)\|_{L^1(I)} \,. \tag{1.9}$$

Moreover one has

$$\int_I \rho(x,t) \, dx = 0 \qquad \forall \, t > 0 \tag{1.10}$$

and there exists $D \subset \mathbb{R}^2$, $D_J \subset \mathbb{R}$ that are invariant domains for (ρ, J) and for J, respectively.

About the definition of D and D_J, see (3.36) and (3.37).

Next, we consider the case of time-independent damping. Assume that $\alpha(t) \equiv 1$,

$$k \in L^\infty(I) \,, \qquad \operatorname*{ess\,inf}_I k > 0 \,, \tag{1.11}$$

and

$$g \in C^1(\mathbb{R}) \,, \qquad g(0) = 0 \,, \qquad g'(J) > 0 \quad \forall \, J \,. \tag{1.12}$$

Let $0 < k_1 \leq k_2$ such that $k_1 \leq k(x) \leq k_2$ a.e. on I. Then define

$$d_1 \doteq k_1 \min_{J \in D_J} g'(J) > 0 \,, \qquad d_2 \doteq k_2 \max_{J \in D_J} g'(J) \,, \tag{1.13}$$

where $D_J \subset \mathbb{R}$ is a closed bounded interval, depending on the initial data, which is invariant for J (see Theorem 1.2.1). Clearly one has $0 < d_1 \leq d_2$, and we assume that

$$e^{d_2} - d_2 < e^{d_1}, \tag{1.14}$$

which consists in a moderate smallness requirement on the damping term that is this condition is valid if $k(x)$ and g' do not vary much. Now we state the following theorem.

Theorem 1.2.2. *Let $\alpha(t) \equiv 1$ and assume (1.11), (1.12), (1.14). Let $(\rho_0, J_0) \in BV(I)$, with $\int_I \rho_0 = 0$ and let $(\rho, J)(x,t)$ be the corresponding weak solution of the problem (1.2), (1.3).*

Then, there exist constant values $C_j > 0$, $j = 1, \cdots, 3$ that depend only on the coefficients of the equation and on the initial data, such that

$$\begin{aligned} \|J(\cdot,t)\|_\infty &\leq C_1 e^{-C_3 t}, \\ \|\rho(\cdot,t)\|_\infty &\leq C_2 e^{-C_3 t}. \end{aligned} \tag{1.15}$$

where C_3 is given by

$$C_3 = |\log C(d_1,d_2)|, \qquad C(d_1,d_2) = e^{-d_1}(e^{d_2} - d_2) < 1.$$

Remark 1.2.1. *We present some essential comments on Theorem 1.2.2.*

(a) *In terms of the damped wave equation (1.1), Theorem 1.2.2 provides a result on the decay in $W^{1,\infty}$ of the solution towards zero. Hence, our result is related to [35, Theorem 3.1], where a decay estimate for the solution of the semilinear wave equation with $(u_x(0,\cdot), u_t(0,\cdot)) \in W^{1,\infty}(0,\ell) \times W^{1,\infty}(0,\ell)$ is obtained.*

Some differences occur in the assumptions on the damping term:

– we assume that $g' > 0$, while in [35], the possibly more interesting case of a degeneracy of g' at $J = u_t = 0$ is considered;

– we consider a space-dependent damping term, $k(x)g(J)$;

– finally, we assume some restriction on the nonlinearity, namely (1.14).

(b) *Beyond the exponential stability estimates (1.15), we also provide asymptotic estimates for the approximate solutions $(\rho^{\Delta x}, J^{\Delta x})(x,t)$ of (1.2)-(1.3) defined in Subsection 3.2.1. For the precise estimates of the approximate solutions with initial data in $BV(I)$ space, see Section 5.2, estimates (5.20) and (5.21).*

(c) *We are working in extending the exponential stability estimates (1.15) for the solutions of (1.2)-(1.3) with L^∞ initial data. The proof is not trivial, since the constants C_1 and C_2 in (1.15) depend on the total variation of the initial data. A partial result in this direction is given by Theorem 5.3.4 for the linear case that is $k(x) \equiv \bar{k} > 0$ and $g'(J) \equiv \bar{C} > 0$ are constant (the telegrapher's equation). See Remark 5.3.2.*

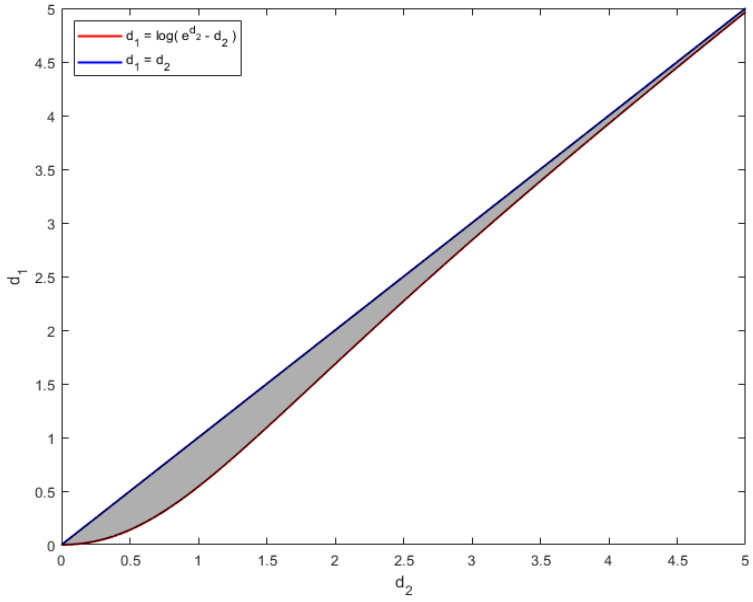

FIGURE 1.1: This graph shows that there exists a non-empty interval of values for $d_1 \leq d_2$ for which condition (1.14) holds.

Remark 1.2.2. *Hereby we list several other comments on Theorem 1.2.2.*

(i) *From (1.13), it is clear that $d_1 \leq d_2$. Also, one can see in Figure 1.1 that for every $d_1 > 0$ there exists a non-empty interval of values for $d_2 \geq d_1$ for which (1.14) holds. Moreover, if d_1 increases, then d_1 and d_2 have to be almost of the same value. While, if d_1 is close to zero, then the values of d_1 and d_2 have to be close to each other.*

(ii) *If $k(x) \equiv \bar{k} > 0$ and $g'(J) \equiv \bar{C} > 0$ are constant (as in the telegrapher's equation, [42]) then, $d_1 = d_2 = d$ and then (1.14) is satisfied for every $d = \bar{k}\bar{C} > 0$. Moreover, one has the exponential decay rate is estimated by*

$$C_3 = \left|\log(1 - de^{-d})\right| \sim d \qquad \text{as } d \to 0\,.$$

If $k(x) \equiv \bar{k} > 0$, then the condition (1.14) can be interpreted as a smallness condition on $\max_{J\in D_J} g'(J) - \min_{J\in D_J} g'(J)$.

On the other hand, by assuming that g is linear ($g'(J) \equiv \bar{C} > 0$), then, (1.14) implies that the upper and lower bound of $k(x)$, k_2 and k_1, are sufficiently close to each other.

(iii) *For (1.14) to hold, it is necessary that $d_1 > 0$ and hence that $g' > 0$ as in (1.12). Differently, if g' vanishes at $J = 0$, an exponential decay is no longer expected; see [35].*

(iv) *Condition* (1.14) *is a sufficient condition on the damping term* $k(x)g(J)$ *for the* L^∞ *estimate* (1.15) *to hold. This condition is required in order to get a suitable contractive estimate, see Proposition* 4.3.4.

In order to relax the assumption (1.14) *within our approach and to allow for a more general assumption on the damping term, it would be necessary to employ a more accurate Birkhoff decomposition of the matrices* $B(t^n)$*, rather than the one in* (4.70) *which involves just two matrices, and then to extend the result of Theorem* 4.2.3.

This technical extension is not immediate and it may be object of a future investigation.

(v) *(About localized damping) In the main theorem, we require that* k *satisfies the assumption* (1.11)*; in particular* $k(x)$ *has to be uniformly positive on* $(0,1)$*. On the other hand, the construction scheme in Section* 3.2 *and Theorem* 1.2.1 *work under the more general assumption* (1.7) *on* $k(x)$*, that include the case of a localized damping. In Section* 5.4*, we consider the localized damping case and we provide an estimate of the modulus of the eigenvalues.*

(vi) *(The time dependent damping) Through out the thesis, we require the general assumption* (1.8) *on* $\alpha(t)$*. While for the* L^∞ *decay to accrue, we require that* $\alpha(t) \equiv 1$*. In particular, the construction scheme in Section* 3.2 *and Theorem* 1.2.1 *work for the time dependent damping case, that is* $\alpha(t)$ *satisfies* (1.8)*, including the On-Off case* (1.4)*. We remark that the* L^∞ *decay of the case On-Off damping is investigated only numerically in Test* 3 *and we will consider it as a future work.*

1.3 Structure of the thesis

The thesis is organized as the following:

- In Chapter 2, we review earlier results for the damped wave equation on a bounded domain regarding the existence, uniqueness, and stability results. In particular, in Section 2.1, we recall the existence and uniqueness results for the semilinear damped wave equation with an exterior time-dependent force density that are proved by Haraux in [36]. Then, in Section 2.2, we present the proof of the stability for solution of semilinear damped wave equation (1.1) with $\alpha(t) = 1$ that is done by C. Dafermos in [22], which is based on LaSalle invariance principle 2.2.4.

- In Chapter 3, in Section 3.1, we recall some preliminary results on Riemann problems for hyperbolic system, and we prove interaction estimates that take into account of the time change of the damping term. In particular, we study the Riemann problem for the time-independent case in Subsection 3.1.1, while in Subsection 3.1.2, we study the interaction estimates for the

time-dependent case. Then, in Section 3.2, we construct a family of approximate solutions by adapting the approach in the case of the Cauchy problem (see for instance [6, p.607]), and we provide a-priori estimates on such approximations (Subsections 3.2.2– 3.2.4). In Section 3.3, we prove the convergence of the approximate solutions in the BV setting and we use the stability in L^1, together with a density argument, to show the existence and stability for L^∞ initial data (ρ_0, J_0), thus completing the proof of Theorem 1.2.1 .

- In Chapter 4, we describe our strategy to study the long-time behavior of the approximate solution. In Section 4.1, we introduce a vector representation of the approximate solution, then we study the evolution in time of this vector by means of finite-dimensional evolution system. In Section 4.2, we use Birkhoff decomposition Theorem A.1.1 for doubly stochastic matrices, and we prove an exponential-type formula in Theorem 4.2.3. Thanks to a careful expression of the first order in $1/N$, and to a refined decomposition of the vectors (see Subsection 4.3.1), we prove a contraction property for the ℓ^1 norm of the iterated matrix in (1.6) for the two cases: linear and non-linear damping, in Subsection 4.3.3 and Subsection 4.3.4, respectively.

- In Chapter 5, we combine all the results of Chapters 3 and 4 for proving the main result in Theorem 1.2.2. We start with a representation formula for the approximate solutions $(\rho^{\Delta x}, J^{\Delta x})$, see Section 5.1. Then, in Section 5.2, we relate the L^∞ norm of the approximate solutions with the ℓ_1 norm of the iterated matrix, see Proposition 5.2.1. Using the contraction property of the ℓ_1 norm that is proved in Subsection 4.3.4, we prove the L^∞ decay estimates for the approximate solutions $(\rho^{\Delta x}, J^{\Delta x})$. Finally, the L^∞ decay estimate (1.15) in Theorem 1.2.2 for the weak solution of the problem (1.2) is proved by passing to the limit (see Section 3.3) by means of Helly's Theorem 3.3.1, see the proof at the end of Section 5.2. We remark that in this case, the constants C_1 and C_2 in (1.15) depend on the total variation of the initial data which is finite for BV initial data.

  In Section 5.3, we extend partially the result of Theorem 1.2.2 to the case of initial data in L^∞ space. More precisely, we consider the telegrapher equation that is the case when $k(x)$ and g' are constants. Thanks to a careful study of the representation formula of the approximate solution in Section 5.1, Theorem 5.3.4 proves a contraction property of the invariant domain D defined at (3.36) for the approximate solutions. See Remark 5.3.2.

  Finally, Section 5.4 contains an ongoing work of the localized damping case. More precisely, we study the spectral properties of the evolution problem in (1.6) when $k(x) = \bar{k} > 0$ on some (α, β) with $[\alpha, \beta] \subset (0, 1)$, and $k(x) = 0$

otherwise. Using Rayleigh quotient Theorem A.1.4, we prove an estimate for the modulus of the eigenvalues of the matrix.

- Chapter 6 is devoted to performing some numerical simulations based on the numerical implementation of the algorithm presented in Chapter 3. We test some cases such as the On-Off time-dependent case, and the case of localized damping.

Chapter 2

Literature review of semilinear damped wave equation

This part of the thesis is devoted to reviewing earlier results for the damped wave equation on a bounded domain regarding the existence, uniqueness, and stability results. The existence and uniqueness issues for the semilinear damped wave equation with an exterior time-dependent force density are proved by Haraux in [36]. While the stability result for semilinear damped wave equation (1.1) with $\alpha(t) = 1$, is proved by C. Dafermos in [22].

2.1 Existence and uniqueness

Let Ω be a bounded and open subset of $\mathbb{R}^n, n \geq 1$. The initial value problem associated with the semilinear wave equation is given by

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u + g(\partial_t u) = h(t,x)\,, & (t,x) \in \mathbb{R}^+ \times \Omega\,, \\ u(0,x) = u_0, \quad \partial_t u(0,x) = u_1\,, & x \in \Omega\,, \\ u(t,x) = 0\,, & (t,x) \in \mathbb{R}^+ \times \partial\Omega\,. \end{cases} \tag{2.1}$$

Assume that g, h satisfy the following:

$$g : \mathbb{R} \to \mathbb{R}\,, \quad g \in C(\mathbb{R})\,, \quad g' \geq 0\,, \tag{2.2}$$

$$h \in C([0,T]; L^1(\Omega)) \cap L^1([0,T]; H_0^1(\Omega))\,. \tag{2.3}$$

Functional setting: we introduce proper spaces for the unknown u so that (2.1) is satisfied in a weak sense. If u is a smooth solution, with $u \in C(\overline{\Omega})$, then we can multiply by a test function $\varphi \in C_c^1(\Omega)$ and integrate on Ω to get

$$\int_\Omega \partial_t^2 u \cdot \varphi dx - \int_\Omega (\sum \partial_{x_j}^2 u)\varphi dx + \int_\Omega g(\partial u)\varphi dx = \int_\Omega h\varphi dx$$
$$\Rightarrow \int_\Omega \partial_t^2 u \cdot \varphi dx + \int_\Omega \nabla u \cdot \nabla \varphi dx + \int_\Omega g(\partial u)\varphi dx = \int_\Omega h\varphi dx\,, \tag{2.4}$$

where we used the fact that $\varphi = 0$ on $\partial\Omega$ in which we have

$$\int_\Omega (\sum \partial_{x_j}^2 u)\varphi dx = \underbrace{\int_\Omega \sum \partial_{x_j}((\partial_{x_j} u)\varphi)dx}_{=\int_{\partial\Omega}\varphi\nabla u\cdot u dx} - \int_\Omega \sum \partial_{x_j} u \partial_{x_j}\varphi dx$$
$$= -\int_\Omega \nabla u \nabla \varphi dx\,.$$

Therefore, natural setting for weak solution u: $u \in H_0^1$ where

$$H^1(\Omega) = \Big\{ v : \Omega \to \mathbb{R} : v \in L^2(\Omega), \exists q_1 \cdots q_n \in L^2(\Omega) :$$
$$\int_\Omega v\partial_{x_j}\varphi dx = -\int_\Omega q_i \varphi dx \quad \forall \varphi \in C_c^\infty(\Omega) \Big\}\,,$$

H^1 is a Hilbert space with norm

$$\|v\|_{H^1} = \|v\|_{L^2} + \sum_{j=1}^n \|\partial_{x_j} v\|_{L^2}\,.$$

On H_0^1, we can use the same norm. But for bounded sets we have

- Poincaré inequality: Ω bounded $\Rightarrow \exists C = C(\Omega) : \forall v \in H_0^1(\Omega)$

  $$\|v\|_{L^2} \le C\|\nabla v\|_{L^2}\,,$$

  which implies that on $H_0^1(\Omega)$ the norm $\left(\int_\Omega |\nabla v|^2 dx\right)^{\frac{1}{2}}$ is equivalent to $\|v\|_{H^1}$.
- $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$ such that

  $$H^{-1}(\Omega) = \{F : H_0^1(\Omega) \to \mathbb{R}, F \text{ linear and continuous}\}\,,$$

with norm

$$\|F\|_{H^{-1}} = \sup_{\|v\|_{H_0^1}\leq 1} < F, v >, \qquad v \to < F, v > \in \mathbb{R} \quad \forall v \in H_0^1 .$$

Recall the Riesz representation theorem, see [18], that is $\forall F \in H^{-1}$, $\exists v_0 \in H_0^1$ such that

$$< F, v >_{H^{-1},H_0^1} = < v_0, v >_{H_0^1} = \int \nabla v_0 \cdot \nabla v .$$

Hence, $H_0^1(\Omega) \subseteq L^2(\Omega) = (L^2(\Omega))' \subseteq H^{-1}(\Omega)$.

Now, we are in the position to define the weak solution for equation (2.1).

Definition 2.1.1. *(Weak solution) Let $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, g is Lipschitz continuous and $h \in C([0,T]; L^2(\Omega))$. We say that a function $u : [0,\infty) \times \Omega \to \mathbb{R}$ such that $u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega)) \cap C^2([0,\infty); H^{-1}(\Omega))$ is a weak solution to (2.1) if*

(a) $u(0,x) = u_0(x)$, $\partial_t u(0,x) = u_1(x)$ *a.e in* Ω.

(b) The equation

$$\partial_{tt} u - \Delta u + g(\partial_t u) = h$$

is satisfied in $H^{-1}(\Omega)$, that is $\forall t \geq 0$ and $\forall \varphi \in H_0^1(\Omega)$ we have

$$< u'', \varphi >_{H^{-1},H_0^1} + \int_\Omega \nabla u \nabla \varphi dx + \int_\Omega g(\partial_t u)\varphi dx = \int_\Omega h\varphi dx .$$

Motivation for definition of weak solution: We claim from (2.4) that

$$\varphi \in H_0^1(\Omega) :\to \int_\Omega h\varphi dx - \int_\Omega g(\partial_t u)\varphi dx - \int_\Omega \nabla u \nabla \varphi dx = \phi(\varphi),$$

is a linear functional, and continuous in $H_0^1(\Omega), \forall t$.

Proof of the claim: the following hold true

(i) $\left| \int_\Omega h(t,\cdot)\varphi dx \right| \leq \|h(t,\cdot)\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)}$;

(ii) $\int_\Omega \nabla u \nabla \varphi dx \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)}$;

(iii) We have that $g(\partial_t u) \in L^2(\Omega)$. Indeed, thanks to (2.2), there exists $M > 0$ such that

$$|g(\partial_t u)| \leq |g(0)| + M|\partial_t u|\,,$$

noting that $|g(0)| \in L^2$ since Ω bounded, and $|\partial_t u| \in L^2$, we get $g(\partial_t u) \in L^2(\Omega)$.

As a consequence,

- $\phi(\varphi)| \leq C\|\varphi\|_{H_0^1(\Omega)}$, so ϕ is linear.
- $|\phi(\varphi_1) - \phi(\varphi_2)| = |\phi(\varphi_1 - \varphi_2)| \leq C\|\varphi_1 - \varphi_2\|_{H_0^1(\Omega)}$, so ϕ is continuous on $H_0^1(\Omega)$.

Hence, it is natural to consider $u''(t) = \partial_{tt} u(t,\cdot)$ in $H^{-1}(\Omega)$.

Now, we recall the definition of contraction mapping and Banach's fixed point theorem that will be usful in the proof of the main result, see e.g., [24]:

Definition 2.1.2. *Let (X,d) be a metric space. Then a map $C : X \to X$ is called a contraction mapping on X if there exists $k \in [0,1)$ such that*

$$d(C(x), C(y)) \leq k\, d(x,y) \qquad \forall x, y \in X\,.$$

Theorem 2.1.1. *(Banach's fixed point theorem) Let (X,d) be a complete metric space. Let $C : X \to X$ be a contraction mapping on X. Then there exists a unique $\bar{x} \in X$ such that $C\bar{x} = \bar{x}$.*

Next, we state the existence and uniqueness theorem.

Theorem 2.1.2. *Assume $h \in C([0,\infty); L^2(\Omega))$, $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, and $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. Then there exists a unique solution $u : [0,\infty) \times \Omega \to \mathbb{R}$ that is a weak solution in the sense of Definition 2.1.1. Moreover*

$$E(t) = \frac{1}{2}\int_\Omega \left[|\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2\right] dx\,, \tag{2.5}$$

satisfies $E \in C^1([0,\infty))$ and

$$\frac{dE}{dt} = \int_\Omega [h(t,x) - g(\partial_t u)]\, \partial_t u dx\,. \tag{2.6}$$

Before we start the proof of this theorem, we state a preliminary result for the inhomogeneous wave equation that is the case where $g = 0$.

Proposition 2.1.3. *Let $g = 0$. Under the same assumptions of Theorem 2.1.2. There exists a unique weak solution u in the sense of Definition 2.1.1 that is a solution of the equation*

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = h(t,x). \tag{2.7}$$

Moreover,

$$\frac{dE}{dt} = \int_\Omega h(t,x)\partial_t u dx\,.$$

Proof. The proof can be done through Galerkin method, we omit the proof and we refer the reader to [25] or [36] for more details. □

Now we start with the proof of Theorem 2.1.2.

Proof. The proof is a standard application of the classical method combining a contraction mapping argument, see Theorem 2.1.1, and on a priori estimates on the maximal interval of existence. More precisely, we need to prove the following:

(I) Local in time existence by contraction mapping theorem.

(II) The existence for all time with the help of a priori estimates on the solution.

Proof (I): We start by defining the set X: Let $\delta > 0$ and $I_\delta = [0,\delta]$. Let $P > 0$ such that

$$\frac{P}{\sqrt{2}} > \max\{\|u_0\|_{H_0^1}, \|u_1\|_{L^2}\}\,,$$

and define the set X as

$$X = \big\{u \in C(I_\delta, H_0^1) \cap C^1(I_\delta, L^2) : u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x)\,,\\ \max_{t\in I_\delta}\|u(t)\|_{H_0^1} \le P\,, \max_{t\in I_\delta}\|u'(t)\|_{L^2} \le P\big\}\,,$$

with norm

$$\|u\|_X = \max\Big\{\max_{t\in I_\delta}\|u(t)\|_{H_0^1}\,,\ \max_{t\in I_\delta}\|u'(t)\|_{L^2}\Big\}\,. \tag{2.8}$$

So that X is a metric space with metric $d(u_1,u_2) = \|u_1 - u_2\|_X$. Now, let $v \in X$ and let $C(v) = z(t,x)$ be the unique solution by Proposition 2.1.3 of the following

equation:

$$\frac{\partial^2 z}{\partial t^2} - \Delta z = \underbrace{h(t,x) - g(\frac{\partial v}{\partial t})}_{\bar{h}(t,x)},$$
$$z(0) = u_0\,, \qquad \frac{\partial z}{\partial t}(0) = u_1\,.$$

Claim: $\forall\ \frac{P}{\sqrt{2}} > \max\{\|u_0\|_{H_0^1}, \|u_1\|_{L^2}\}$, $\exists \delta = \delta(P)$ such that:

(1) $C(X) \subseteq X$.

(2) $C : X \to X$ is a strict contraction.

Proof of (1): It holds if and only if for δ small enough the following is true

$$\begin{cases} \max\limits_{t\in I_\delta} \|z(t)\|_{H_0^1} \leq P\,, \\ \max\limits_{t\in I_\delta} \|\partial_t z\|_{L^2} \leq P\,, \end{cases} \tag{2.9}$$

so we need to prove the above inequalities. Fix $\delta > 0$, $v \in X$, then the energy for $z(t,x)$ is:

$$E(t) = \frac{1}{2}\left(\|z(t,\cdot)\|_{H_0^1}^2 + \|\partial_t z(t,\cdot)\|_{L^2}^2\right)\,,$$

which implies that

$$\|z(t,\cdot)\|_{H_0^1}^2 \leq 2E(t)\,, \qquad \|\partial_t z(t,\cdot)\|_{L^2}^2 \leq 2E(t)\,. \tag{2.10}$$

Therefore, at time $t = 0$

$$\begin{aligned} E(0) &= \frac{1}{2}\left(\|z_0\|_{H_0^1}^2 + \|\partial_t z(0)\|_{L^2}^2\right) \\ &= \frac{1}{2}\left(\|u_0\|_{H_0^1}^2 + \|u_1\|_{L^2}^2\right) \\ &\leq \max\{\|u_0\|_{H_0^1}^2, \|u_1\|_{L^2}^2\} \\ &< \left(\frac{P}{\sqrt{2}}\right)^2 = \frac{P^2}{2}\,. \end{aligned}$$

Now, from Proposition 2.1.3, the energy estimate for z satisfy:

$$\begin{aligned}\frac{dE}{dt}(t) &= \int_\Omega \bar{h}(t,x)\frac{\partial z}{\partial t}(t,x)dx \leq \|\bar{h}(t,x)\|_{L^2(\Omega)}\|\frac{\partial z}{\partial t}(t,x)\|_{L^2(\Omega)}\\ &\leq \frac{1}{2}\left(\|\bar{h}(t,x)\|^2_{L^2(\Omega)} + \|\frac{\partial z}{\partial t}(t,x)\|^2_{L^2(\Omega)}\right)\\ &\leq \frac{1}{2}\max_{t\in I_\delta}\|\bar{h}(t,x)\|^2_{L^2(\Omega)} + E(t)\,,\end{aligned}$$

where

$$\begin{aligned}\max_{t\in I_\delta}\|\bar{h}(t,x)\|^2_{L^2(\Omega)} &\leq \max_{t\in I_\delta}\|h(t,x)\|^2_{L^2(\Omega)} + \max_{t\in I_\delta}\|g(\partial_t v)\|^2_{L^2(\Omega)}\\ &\leq \max_{t\in I_\delta}\|h(t,x)\|^2_{L^2(\Omega)} + M\underbrace{\max_{t\in I_\delta}\|\partial_t v(t,\cdot)\|^2_{L^2(\Omega)}}_{\leq P^2} + \|g(0)\|^2_{L^2(\Omega)}\,.\end{aligned} \tag{2.11}$$

Hence,

$$\begin{aligned}&\frac{dE}{dt}(t) \leq \frac{1}{2}\max_{t\in I_\delta}\|\bar{h}(t,x)\|^2_{L^2(\Omega)} + E(t)\,,\\ \Rightarrow\ &E(t) \leq E(0)e^t + \frac{1}{2}\max_{t\in I_\delta}\|\bar{h}(t,x)\|^2_{L^2(\Omega)}(e^t-1)\,.\end{aligned}$$

Since $E(0) < \frac{P^2}{2}$, then $\exists\delta > 0$ such that

$$E(t) \leq \frac{P^2}{2}\,, \qquad 0 \leq t \leq \delta\,.$$

Note that δ does not depend on the choice of v, it depends only on P and on the given function h, indeed, δ depends on the term $\max_{t\in I_\delta}\|\bar{h}(t,x)\|^2_{L^2(\Omega)}$ which depends only on the parameters of the problem, see (2.11).

So by (2.10), the inequalities in (2.9) are proved. In conclusion, $\exists\delta > 0:\ X \ni v \to C(v) = z \in X$ which proves claim (1).

Proof of (2): We want to prove that there exists δ' with $0 < \delta' < \delta$ such that $C : X \to X$ is strict contraction in the sense of Definition 2.1.2, i.e., we want to prove that there exists $k \in (0,1)$ such that for all $v_1, v_2 \in X$ we have

$$\|C(v_1) - C(v_2)\|_X \leq k\|v_1 - v_2\|_X\,.$$

Consider $v_1, v_2 \in X$ such that $z_1 = C(v_1)$, $z_2 = C(v_2)$ that are solutions of

$$\begin{cases} \dfrac{\partial^2 z_1}{\partial t^2} - \Delta z_1 = h(t,x) - g(\dfrac{\partial v_1}{\partial t}), \\ \dfrac{\partial^2 z_2}{\partial t^2} - \Delta z_2 = h(t,x) - g(\dfrac{\partial v_2}{\partial t}), \\ z_1(0) = z_2(0) = u_0, \quad \dfrac{\partial z_1}{\partial t}(0) = \dfrac{\partial z_2}{\partial t}(0) = u_1, \end{cases}$$

with

$$E_{(z_1-z_2)} = \frac{1}{2}\left(\|z_1(t,\cdot) - z_2(t,\cdot)\|^2_{H_0^1} + \|\partial_t \left(z_1(t,\cdot) - z_2(t,\cdot)\right)\|^2_{L^2} \right).$$

The energy estimate for $z_1 - z_2$ is:

$$\begin{aligned} \frac{dE_{(z_1-z_2)}}{dt}(t) &= \int_\Omega \left(g(\frac{\partial v_2}{\partial t}) - g(\frac{\partial v_1}{\partial t}) \right) \frac{\partial}{\partial t}(z_1 - z_2) dx \\ &\leq M \| \frac{\partial v_2}{\partial t} - \frac{\partial v_1}{\partial t} \|_{L^2(\Omega)} \| \frac{\partial}{\partial t}(z_1 - z_2) \|_{L^2(\Omega)} \\ &\leq \frac{M}{2} \left(\| \frac{\partial}{\partial t}(v_2 - v_1) \|^2_{L^2(\Omega)} + \| \frac{\partial}{\partial t}(z_1 - z_2) \|^2_{L^2(\Omega)} \right) \\ &\leq \frac{M}{2} \max_{t \in I_\delta} \| \frac{\partial}{\partial t}(v_2 - v_1) \|^2_{L^2(\Omega)} + M E_{(z_1-z_2)}. \end{aligned}$$

Note that $E_{z_1-z_2}(0) = 0$, so we have the following:

$$\begin{aligned} E_{(z_1-z_2)}(t) &\leq \frac{1}{2}(e^{Mt} - 1) \max_{t \in I_\delta} \| \frac{\partial}{\partial t}(v_2 - v_1) \|^2_{L^2(\Omega)} \\ &\leq \frac{1}{2}(e^{Mt} - 1) \|v_1 - v_2\|^2_X. \end{aligned}$$

Choose $\delta' \leq \delta$ and $k < 1$ such that $(e^{M\delta'} - 1) \leq k$, then

$$E_{(z_1-z_2)}(t) \leq \frac{1}{2} k \|v_1 - v_2\|^2_X,$$

and by (2.10), we conclude that

$$\begin{aligned} \|(z_1 - z_2)(t,\cdot)\|^2_{H_0^1} &\leq 2E_{(z_1-z_2)}(t) \leq k \|v_1 - v_2\|^2_X, \\ \| \frac{\partial}{\partial t}(z_1 - z_2)(t,\cdot) \|^2_{L^2} &\leq 2E_{(z_1-z_2)}(t) \leq k \|v_1 - v_2\|^2_X. \end{aligned}$$

Recalling (2.8), we have

$$\begin{aligned}\|C(v_1) - C(v_2)\|_X &= \|z_1 - z_2\|_X \\ &= \max\{\max_{t\in I_{\delta'}} \|z_1 - z_2\|_{H_0^1}, \max_{t\in I_{\delta'}} \|\frac{\partial}{\partial t}(z_1 - z_2)\|_{L^2}\} \\ &\leq k\|v_1 - v_2\|_X^2 .\end{aligned}$$

Hence, $C : X \to X$ is strict contraction. By the fixed point Theorem 2.1.1, there exists a unique fixed point $u \in X$ such that $u \in C(I_{\delta'}; H_0^1(\Omega)) \cap C^1(I_{\delta'}; L^2(\Omega)) \cap C^2(I_{\delta'}; H^{-1}(\Omega))$ and

$$\frac{dE}{dt}(t) \leq C_1 + C_2 E(t) ,$$

with C_1, C_2 independent in time.

Proof (II): To prove the existence globally in time, let $\widetilde{I} \subseteq [0, \infty)$ be the maximum interval of existence, we need to prove that $\widetilde{I} = [0, \infty)$. But the energy does not blow up in finite time, so the proof is concluded. □

2.2 Stability of the zero solution

There is an extensive literature on the stabilization of the damped wave equations. In [22], C. Dafermos proved a stabilization result for the wave equation on a bounded open domain, i.e., the energy of every solution of equation (1.1) tends to 0 as $t \to \infty$, under the assumptions that $k(x) \neq 1$, $\alpha(t) = 1$, and g is continuously differentiable and strictly increasing over $\mathbb{R}$. The proof is based on LaSalle invariance principle. This result has been generalized by A. Haraux in [33], where g is just a maximal monotone graph. However, this method does not show the rate of the decay. In [49], Zuazua proved the exponential stability of the energy for the semilinear wave equation with damping $k(x)\partial_t u$ that is localized, and with force function. Nakao in [44], extended the result of Zuazua by considering the case of a nonlinear damping term with a polynomial growth near the origin. For more works on the rate of decay for the energy, see for instance [49, 21, 11].

In this section, we only present an overview of the stability result that is proved by Dafermos in [22] for the semilinear damped wave equation in a bounded, open, smooth, and connected set $\Omega \subseteq \mathbb{R}^n (n \geq 1)$. The phenomenon "Stabilization" means that the self-oscillations induced by the wave equation are damped out asymptotically, and so we are left, when time tends to infinity, with an equilibrium

if the system is autonomous [33]. Dafermos employed the LaSalle invariance principle 2.2.4 to prove that the energy of every solution tends to zero as time goes to infinity. The advantage of this approach is that it is simple that it only requires quite weak assumptions on the dissipative mechanism. However, this method does not produce explicit estimates of the decay rates.

Consider the semilinear damped wave equation:

$$\begin{cases} \partial_{tt}u - \partial_{xx}u + k(x)g(\partial_t u) = 0\,, & (x,t) \in \Omega \times \mathbb{R}^+\,, \\ u(0,x) = u_0(x)\,, \quad \partial_t u(0,x) = u_1(x)\,, & x \in \Omega\,, \\ u = 0\,, & (x,t) \in \partial\Omega \times \mathbb{R}^+\,, \end{cases} \tag{2.12}$$

where Ω is an open , bounded, smooth, and connected set in $\mathbb{R}^n$. We assume that $k(x)$ is smooth and satisfies $k(x) \geq 0, x \in \Omega$, $k(x_0) > 0$ for some $x_0 \in \Omega$. In addition, we assume $g(x) \in C^1(\mathbb{R})$ and it is strictly increasing.

The energy of this problem is

$$E(u,u_t) = \frac{1}{2}\int_\Omega (|u_t|^2 + |\nabla u|^2)dx\,, \tag{2.13}$$

and it satisfies the following:

$$\frac{d}{dt}E(u,u_t) = -\int_\Omega k(x)g(u_t)u_t dx \leq 0\,, \tag{2.14}$$

and therefore, the energy is a non-increasing function. The goal is to show how LaSalle argument can verify the decay of the energy to zero as time goes to infinity.

Before we state the main theorem, we should recall some well-known results from elementary topological dynamics that we will use in the proof. Throughout this section, (X,d) denotes a complete metric space.

Definition 2.2.1. *A dynamical system (semigroup) on X is a family $\{S_t\}_{t>0}$ of mappings on X such that:*

(i) $S_t \in C(X,X), \forall t > 0$;

(ii) $S_0 = I$;

(iii) $S_{t+s} = S_t \circ S_s, \forall s,t \geq 0$;

(iv) the function $t \to S_t x$ is in $C([0,\infty),X)$ for all $x \in X$.

Definition 2.2.2. *For all $x \in X$, the continuous curve $t \to S_t x$ is called the trajectory from x.*

Definition 2.2.3. *Let $x \in X$. The set*

$$\omega(x) = \{y \in X \,:\, \exists t_n \to \infty,\ S_{t_n}x \to y \text{ as } n \to \infty\}\,,$$

is called the ω-limit set of x.

Proposition 2.2.1. *We have $\omega(x) = \bigcap\limits_{s>0} \overline{\bigcup\limits_{t\geq s} \{S_t x\}}$.*

Proposition 2.2.2. *For all $x \in X$ and all $t > 0$, we have*

$$\omega(S_t x) = \omega(x)\,, \qquad S_t(\omega(x)) \subset \omega(x)\,, \tag{2.15}$$

which means that $\omega(x)$ is positive invariant under S_t. In addition, if $\bigcup\limits_{t\geq 0} \{S_t x\}$ is relatively compact in X, then

$$S_t(\omega(x)) = \omega(x) \neq \emptyset. \tag{2.16}$$

Theorem 2.2.3. *For fixed $x \in X$, suppose that $\bigcup\limits_{t\geq 0} \{S_t x\}$ is relatively compact in X, then:*

(i) $S_t(\omega(x)) = \omega(x) \neq \emptyset$ *for all* $t \geq 0$;

(ii) $\omega(x)$ *is a compact connected subset of* X;

(iii) $d(S_t x, \omega(x)) \to 0$ *as* $t \to \infty$.

Definition 2.2.4. *(Lyapunov function) A function $\phi : X \to \mathbb{R}$ is called a Lyapunov function for $\{S_t\}_{t\geq 0}$ if ϕ is continuous and $\phi(S_t x) \leq \phi(x)$ for all $x \in X$ and all $t \geq 0$.*

Note that if ϕ is a Lyapunov function for $\{S_t\}_{t\geq 0}$ then, for all $x \in X$, the function $t \to \phi(S_t x)$ is non-increasing.

Theorem 2.2.4. *(LaSalle Invariance Principle) Let ϕ be a Lyapunov function for $\{S_t\}_{t\geq 0}$, and let $x \in X$ be such that $\bigcup\limits_{t\geq 0} \{S_t x\}$ is relatively compact in X. Then:*

(i) $\ell = \lim\limits_{t\to\infty} \phi(S_t x)$ *exists;*

(ii) $\phi(y) = \ell$, *for all* $y \in \omega(x)$.

Now, we are in the position to state the stability result.

Theorem 2.2.5. *Let $u(t,x)$ be the solution of (2.12) with $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Then*

$$(u(t,\cdot), u_t(t,\cdot)) \xrightarrow{H_0^1(\Omega)\times L^2(\Omega)} 0\,, \qquad \text{as } t \to \infty\,. \tag{2.17}$$

Proof. We rewrite the first equation of (2.12) as a first order system

$$\begin{cases} u_t = v\,, \\ v_t = u_{xx} - k(x)g(v)\,, \end{cases} \tag{2.18}$$

which generates a continuous semigroup S_t on $H_0^1(\Omega) \times L^2(\Omega)$. We start the proof with first assuming that the initial data $(u_0, u_1) \in (H^2(\Omega) \bigcap H_0^1(\Omega)) \times H_0^1(\Omega)$, which is the domain of the generator of S_t:

• **Step 1** We prove that

$$S_t((u_0, u_1)) \xrightarrow{H_0^1 \times L^2} \omega((u_0, u_1))\,, \qquad \text{as } t \to \infty\,, \tag{2.19}$$

where ω is the limit set defined in Definition 2.2.3 for which $X = H_0^1 \times L^2$. To do so, we need to prove that $\bigcup_{t\geq 0} \{S_t(u_0, u_1)\}$ is relatively compact in $H_0^1(\Omega) \times L^2(\Omega)$. Differentiating system (2.18) with respect to t, we get

$$v_{tt} = \Delta v - k(x)g'(v)v_t\,.$$

The energy integral of this system

$$\frac{d}{dt}E(v, v_t) = -\int_\Omega k(x)g'(v)v_t^2 dx \leq 0\,, \tag{2.20}$$

shows that $\{v(t,\cdot)_{t\geq 0}\}$ is bounded in $H_0^1(\Omega)$, and is therefore relatively compact in $L^2(\Omega)$, and that $\{v_t(t,\cdot)_{t\geq 0}\}$ is bounded in $L^2(\Omega)$. It follows from (2.12) that $\{\Delta u(t,\cdot)_{t\geq 0}\}$ is bounded in $L^2(\Omega)$ and, consequently, $\{u_t(t,\cdot)_{t\geq 0}\}$ is relatively compact in $H_0^1(\Omega)$. Thus, by Theorem 2.2.3, the proof of (2.19) is complete.

• **Step 2** By the energy integral (2.14), $E(u, v)$ is a Liapunov functional for S_t which is continuous on $H_0^1(\Omega) \times L^2(\Omega)$, so that by Theorem 2.2.4, $E(u, v)$ is constant on $\omega((u_0, u_1))$.

• **Step 3** Now, by Proposition 2.2.2 $\omega((u_0, u_1))$ is positive invariant. Using this property in conjunction with (2.14) and the properties of $k(x)$ and $g(v)$, we arrive at the following conclusion: If $(\bar{u}(t,\cdot), \bar{v}(t,\cdot)) = S_t(\bar{u}_0, \bar{u}_1)$ with $(\bar{u}_0, \bar{u}_1) \in \omega((u_0, u_1))$, then

$$\bar{v}(t, x) = 0, \qquad \forall t \geq 0 \quad x \in \text{supp}(k(x))\,. \tag{2.21}$$

In particular, $\bar{u}(t,\cdot)$ becomes a solution of the undamped wave equation so that we have the representation

$$(\bar{u}(t,x),\bar{v}(t,x)) = Re\sum_n e^{i\lambda_n t} w_n(x)(1,i\lambda_n)\,, \tag{2.22}$$

where

$$\begin{cases} \Delta w_n + \lambda_n^2 w_n = 0 & \text{on } \Omega\,, \\ w_n = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.23}$$

Substituting $\bar{v}(t,x)$ from (2.22) into (2.21) and using the elementary properties of almost periodic functions, we deduce that $w_n(x) = 0$ for $x \in \text{supp}(k(x))$. Since solutions of (2.23) are analytic in Ω and Ω connected, it follows that $w_n(x) = 0$ for $x \in \Omega$, so that $(\bar{u}(t,x),\bar{v}(t,x)) = 0$ for $t \geq 0$ and $x \in \Omega$.

Thus the solutions of (2.18) through $(H^2(\Omega)\bigcap H_0^1(\Omega)) \times H_0^1(\Omega)$ tend to zero as $t \to \infty$. Using simple completion argument, we extend the result to the case of initial data in $H_0^1(\Omega) \times L^2(\Omega)$. □

Chapter 3

Well-Balanced approximate solution

In this chapter, we recall some preliminary results on Riemann problems for hyperbolic system, and we prove interaction estimates that take into account of the time change of the damping term. Then, we construct a family of approximate solutions by adapting the approach in the case of the Cauchy problem (see for instance [6, p.607]), and we provide a-priori estimates on such approximations. Finally, we prove the convergence of the approximate solutions in the BV setting and use the stability in L^1, together with a density argument, to show the existence and stability for L^∞ initial data (ρ_0, J_0), thus completing the proof of Theorem 1.2.1 .

The Well-Balanced scheme, that is a scheme preserving some discrete version of stationary equation, has been introduced in [29] for one-dimensional scalar equations in order to improve the consistency of numerical schemes in the inhomogeneous case. Since then, the scheme has been widely used, see for instance [27] and references therein.

The core of this scheme is that instead of solving approximately generalized Riemann problem for the 1D scalar equation $\partial_t u + \partial_x f(u) = k(x)g(u)$ to derive a Godunov scheme, one can solve a modified one, by replacing $k(x)$ by an antiderivative $a(x)$ and rewriting the right-hand side of the equation as $g(u)\partial_x a$. At the discrete level, we approximate the variable $a(x)$ by piecewise constant functions, which implies that the effect of the source term is concentrated into a countable collection of Dirac masses in order to integrate it inside a Riemann solver.

3.1 Preliminaries

In terms of the diagonal variables $f^\pm$, defined by

$$\rho = f^+ + f^- , \qquad J = f^+ - f^- \tag{3.1}$$

the system (1.2) rewrites as a discrete-velocity kinetic model

$$\begin{cases} \partial_t f^- - \partial_x f^- = k(x)\alpha(t)\, g(f^+ - f^-), \\ \partial_t f^+ + \partial_x f^+ = -k(x)\alpha(t)\, g(f^+ - f^-)\,. \end{cases} \tag{3.2}$$

3.1.1 The time-independent case: the Riemann problem

In the following, we assume that $\alpha(t) \equiv 1$. We recall some preliminary results from [6] dealing with Riemann problems for system (3.4). Our approach is based on an alternative formulation of system (1.2) that is obtained by adding an equation for the antiderivative of k:

$$a = a(x) \doteq \int_0^x k(y)\, dy\,, \tag{3.3}$$

which by (1.7) satisfies $a \in AC(\mathbb{R})$ (Absolutely continuous), $a_x = k \geq 0$, $\mathrm{TV}\, a = a(1) - a(0) = \|k\|_{L^1} > 0$.

This leads to consider the following non-conservative homogeneous 3×3 system

$$\begin{cases} \partial_t \rho + \partial_x J & = 0\,, \\ \partial_t J + \partial_x \rho + 2g(J)\partial_x a & = 0\,, \\ \partial_t a & = 0\,, \end{cases} \tag{3.4}$$

which in diagonal variables (3.1) is written as

$$\begin{cases} \partial_t f^- - \partial_x f^- - g(f^+ - f^-)\partial_x a & = 0\,, \\ \partial_t f^+ + \partial_x f^+ + g(f^+ - f^-)\partial_x a & = 0\,, \\ \partial_t a & = 0\,. \end{cases} \tag{3.5}$$

Notice that the non-conservative product $g(J)\partial_x a$, which in principle is ambiguous across the discontinuities of $a(x)$, is well-defined since J is constant along stationary solutions.

Systems (3.4), (3.5) are introduced in order to be able to set up the WB algorithm: this procedure consists in localizing a source term of bounded extent into a countable collection of Dirac masses in order to integrate it inside a Riemann solver by means of an elementary wave, which is obviously linearly degenerate. The characteristic speed of system (3.5) are $\mp 1, 0$ with corresponding right eigenvectors $(0,1,0)^t$, $(1,0,0)^t$ and $(-g,-g,1)^t$.

We call 0-*wave curves* those characteristic curves corresponding to the speed 0; they are related to the stationary equations for $f^\pm$, that is

$$\partial_x f^\pm = -g(f^+ - f^-)\partial_x a\,. \tag{3.6}$$

We denote either by $(\rho_\ell, J_\ell, a_\ell)$, (ρ_r, J_r, a_r) or by $(f_\ell^-, f_\ell^+, a_\ell)$, (f_r^-, f_r^+, a_r) the left and right states corresponding to Riemann data for (3.4), (3.5) respectively.

Proposition 3.1.1. *[6] Assume that $k(x) \geq 0$, that $g(J)J \geq 0$ and consider the initial states*

$$U_\ell = (\rho_\ell, J_\ell, a_\ell)\,, \qquad U_r = (\rho_r, J_r, a_r)$$

with corresponding states $(f_\ell^-, f_\ell^+, a_\ell)$, (f_r^-, f_r^+, a_r) in the $(f^\pm, a)$ variables. Assume $a_\ell \leq a_r$ and set

$$\delta \doteq a_r - a_\ell \geq 0\,. \tag{3.7}$$

Then the following holds.

(i) The solution to the Riemann problem for system (3.4) and initial data U_ℓ, U_r is uniquely determined by

$$U(x,t) = \begin{cases} U_\ell & x/t < -1 \\ U_* = (\rho_{*,\ell}, J_*, a_\ell) & -1 < x/t < 0 \\ U_{**} = (\rho_{*,r}, J_*, a_r) & 0 < x/t < 1 \\ U_r & x/t > 1 \end{cases} \tag{3.8}$$

with

$$J_* + g(J_*)\delta = f_\ell^+ - f_r^-\,, \qquad \rho_{*,r} - \rho_{*,\ell} = -2g(J_*)\delta\,, \tag{3.9}$$

see Figure 3.1.

(ii) If $m < M$ are given real numbers, the square $[m, M]^2$ is invariant for the solution to the Riemann problem in the (f^-, f^+)-plane. That is, the solution $U(x,t)$ given in (3.8) satisfies

$$f^\pm(x,t) \in [m, M] \tag{3.10}$$

for any (f_ℓ^-, f_ℓ^+), $(f_r^-, f_r^+) \in [m, M]^2$ and for any $\delta \geq 0$.

(iii) For every pair U_ℓ, U_r with (f_ℓ^-, f_ℓ^+), $(f_r^-, f_r^+) \in [m, M]^2$, let $\sigma_{-1} = (J_* - J_\ell)$ and $\sigma_1 = (J_r - J_*)$. Hence,

$$\big||\sigma_1| - |f_r^+ - f_\ell^+|\big| \leq C_0\delta\,, \qquad \big||\sigma_{-1}| - |f_r^- - f_\ell^-|\big| \leq C_0\delta\,, \tag{3.11}$$

where

$$C_0 = \max\{g(M-m), -g(m-M)\}\,. \tag{3.12}$$

Proof. **Proof of** (i) The Riemann problem for system (3.4) is solved in terms of the three characteristic families, resulting in three waves: the two ± 1-waves, with

corresponding speed ± 1, where only $f^{\pm}$ can change its value; and the 0-wave, corresponding to the stationary field of (3.4), evolving along the stationary equations (3.6).

The intermediate states in the Riemann fan are U_*, U_{**} while the waves appearing in the solution are as follows: U_ℓ and U_* are connected by a -1-wave, U_* and U_{**} are connected by a 0-wave, and U_{**} and U_r are connected by a 1-wave.

Here the " $*$ " denotes the corresponding value related to the 0-wave: more precisely, $(\rho_{*,\ell}, J_*)$ and $(\rho_{*,r}, J_*)$ denote the left and right state along the 0-wave, respectively, in term of the variables (ρ, J). Notice that J is constant across the 0-wave. The 3 unknown values $(\rho_{*,\ell}, \rho_{*,r}, J_*)$ satisfy the system of conditions

$$\begin{cases} \rho_{*,\ell} + J_* = \rho_\ell + J_\ell = 2f_\ell^+ \\ \rho_{*,r} - J_* = \rho_r - J_r = 2f_r^- \\ \rho_{*,r} - \rho_{*,l} = -2g(J_*)\delta \end{cases}$$

Indeed, recalling (3.5), the first two equations state that $f^{\pm}$ is conserved along the ± 1-wave, while the third is relation that characterizes the 0-wave.

From the above system, an equation for J_* can be deduced:

$$J_* + g(J_*)\delta = f_\ell^+ - f_r^- \,. \tag{3.13}$$

Since the map

$$x \mapsto x + g(x)\delta \doteq \varphi_\delta(x)$$

is increasing for $\delta \geq 0$, then (3.13) implicitly defines a unique value $J_* = J_*(\delta, f_\ell^+ - f_r^-)$. Hence (i) and (3.9) are proved.

Proof of (ii) By (3.13), the values f_*^+, f_*^- are defined by the identity

$$\begin{aligned} f_*^+ - f_r^- &= f_\ell^+ - f_*^- = J_*\,, \\ \Rightarrow f_\ell^+ - f_*^+ &= f_*^- - f_r^- = g(J_*)\delta\,, \end{aligned} \tag{3.14}$$

then one find that

$$(f_\ell^+ - f_*^+)(f_*^+ - f_r^-) = (f_\ell^+ - f_*^-)(f_*^- - f_r^-) = J_* \cdot g(J_*)\delta. \tag{3.15}$$

Noticing that $u.g(u) \geq 0$ for all u, we conclude that, if $\delta \geq 0$, the new values $f_*^{\pm}$ do not leave the interval with extrema f_ℓ^+, f_r^- :

$$m \leq \min\{f_\ell^+, f_r^-\} \leq f_*^+, f_*^- \leq \max\{f_\ell^+, f_r^-\} \leq M,$$

therefore (3.10) is proved.

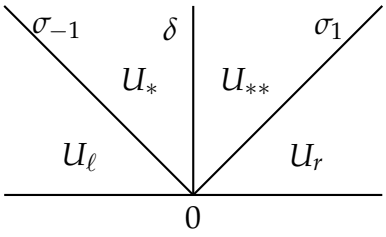

FIGURE 3.1: Structure of the solution to the Riemann problem.

PROOF OF (iii) Concerning (3.11), we use (3.9) to find that

$$\begin{aligned}
|f_r^+ - f_\ell^+| &\le |f_r^+ - f_*^+| + |f_*^+ - f_\ell^+| \\
&= |\sigma_1| + |f_*^+ - f_\ell^+| \\
&= |\sigma_1| + |g(J_*)|\delta \\
&\le |\sigma_1| + C_0\delta \\
&= |f_r^+ - f_*^+| + C_0\delta \\
&\le |f_r^+ - f_\ell^+| + |f_*^+ - f_\ell^+| + C_0\delta \\
&\le |f_r^+ - f_\ell^+| + 2C_0\delta\,,
\end{aligned}$$

with C_0 as in (3.12), so that

$$|f_r^+ - f_\ell^+| - C_0\delta \le |\sigma_1| \le |f_r^+ - f_\ell^+| + C_0\delta.$$

An analogous estimate holds for σ_{-1}. Hence we end up with (3.11).

□

We stress that, in (3.11)–(3.12), the quantity C_0 is independent of $\delta \ge 0$.

Here and in the following, we denote by $\Delta\phi(x)$ the difference $\phi(x+) - \phi(x-)$, where ϕ is a real-valued function defined on a subset of $\mathbb{R}$, and the limits $\phi(x\pm) = \lim_{y\to x\pm}\phi(y)$ exist. We define the amplitude of ±1–waves as follows:

$$\sigma_{\pm1} = \Delta J = \pm\Delta f^\pm = \pm\Delta\rho\,. \tag{3.16}$$

In particular, with the notation of Figure 3.1, we have

$$\begin{aligned}
J_r - J_\ell &= \sigma_1 + \sigma_{-1} \\
\rho_r - \rho_\ell &= \sigma_1 - \sigma_{-1} - 2g(J_*)\delta\,.
\end{aligned} \tag{3.17}$$

3.1.2 The time-dependent case: interaction estimates

As time evolves, the wave-fronts that stem from $t = 0$ propagate and interact between each other; also the coefficient $\alpha(t)$ changes in time. In order to get a-priori estimates on their total variation and L^∞–norm, we study the interactions of waves in the solutions to (3.5).

In [3, Proposition 3], the multiple interaction of two ± 1 waves with a single 0–wave of size $\delta > 0$ is studied. The following proposition extends such a statement to the case in which the 0–wave changes size at the time of the interaction.

Proposition 3.1.2. *(Multiple interactions, time-dependent case) Assume that at a time $\bar{t} > 0$ an interaction involving a $(+1)$–wave, a 0–wave and a (-1)–wave occurs, see Figure 3.2. Let δ be as in (3.7) and $\alpha^\pm \geq 0$ be given, so that $\alpha(t) = \alpha^+$ for $t > \bar{t}$ and $\alpha(t) = \alpha^-$ for $t < \bar{t}$. Assume that*

$$(\sup g')\delta\alpha^\pm < 1\,. \tag{3.18}$$

Let $\sigma^-_{\pm 1}$ be the sizes (see (3.16)) of the incoming waves and $\sigma^+_{\pm 1}$ be the sizes of the outgoing ones. Let $J^\pm_*$ be the intermediate values of J (which are constant across the 0–wave), before and after the interaction as in Figure 3.2, and choose a value $s \in (\min J^\pm_*, \max J^\pm_*)$ such that

$$g'(s) = \frac{g(J^+_*) - g(J^-_*)}{J^+_* - J^-_*}\,. \tag{3.19}$$

Then, for $\gamma^\pm \doteq g'(s)\delta\alpha^\pm$, it holds

$$\begin{pmatrix}\sigma^+_{-1}\\ \sigma^+_1\end{pmatrix} = \frac{1}{1+\gamma^-}\begin{pmatrix}1 & \gamma^-\\ \gamma^- & 1\end{pmatrix}\begin{pmatrix}\sigma^-_{-1}\\ \sigma^-_1\end{pmatrix} + \frac{g(J^+_*)(\alpha^+ - \alpha^-)\delta}{1+\gamma^-}\begin{pmatrix}-1\\ +1\end{pmatrix}\,, \tag{3.20}$$

and similarly

$$\begin{pmatrix}\sigma^+_{-1}\\ \sigma^+_1\end{pmatrix} = \frac{1}{1+\gamma^+}\begin{pmatrix}1 & \gamma^+\\ \gamma^+ & 1\end{pmatrix}\begin{pmatrix}\sigma^-_{-1}\\ \sigma^-_1\end{pmatrix} + \frac{g(J^-_*)(\alpha^+ - \alpha^-)\delta}{1+\gamma^+}\begin{pmatrix}-1\\ +1\end{pmatrix}\,. \tag{3.21}$$

Moreover,

$$\sigma^+_1 + \sigma^+_{-1} = \sigma^-_1 + \sigma^-_{-1} \tag{3.22}$$

$$|\sigma^+_{-1}| + |\sigma^+_1| \leq |\sigma^-_{-1}| + |\sigma^-_1| + 2C_0\delta|\alpha^+ - \alpha^-| \tag{3.23}$$

with $C_0 = \max\{g(M-m), -g(m-M)\}$ as in (3.12), together with

$$m = \min\left\{f^\pm_\ell, f^\pm_r\right\}, \qquad M = \max\left\{f^\pm_\ell, f^\pm_r\right\}\,.$$

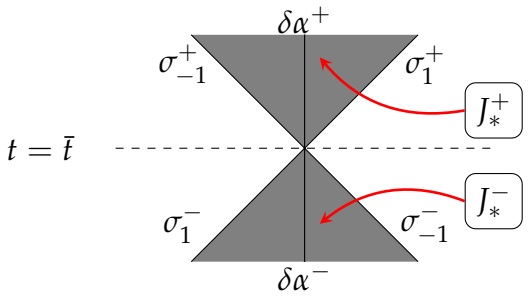

FIGURE 3.2: Multiple interaction, time-dependent case.

Remark 3.1.1. *(a) If $\alpha(t)$ is as in (1.4), the* ON–OFF *time corresponds to $\alpha^- = 1$, $\alpha^+ = 0$ while the* OFF–ON *time corresponds to $\alpha^- = 0$, $\alpha^+ = 1$.*

(b) With the notation of Proposition 3.1.2, one has

$$f^{\pm}_{*,\ell},\ f^{\pm}_{*,r} \in [m, M]\,, \quad |s| \le M - m \tag{3.24}$$

where $f^{\pm}_{*,\ell}, f^{\pm}_{*,r}$ are the intermediate states after the interaction time.

Indeed, as a consequence of Proposition 3.1.1-(ii), the values $f^{+}_{*,\ell}$, $f^{+}_{*,r}$ belong to $[m, M]$. Using the same argument of the proof of Proposition 3.1.1 in [6], one can conclude that the same property holds also for the intermediate state **before** *the interaction, that is, $f^{-}_{*,\ell}$, $f^{-}_{*,r} \in [m, M]$. As a consequence, both the intermediate values $J^{\pm}_*$ satisfy*

$$|J^{\pm}_*| \le M - m$$

and hence, by the intermediate value theorem used in (3.19), we obtain that $|s| \le M - m$.

Proof of Proposition 3.1.2. Let J^-_*, J^+_* be the intermediate values of J before and after the interaction, respectively. By (3.9) these values satisfy

$$J^+_* + g(J^+_*)\delta\alpha^+ = f^+_\ell - f^-_r\,, \qquad J^-_* - g(J^-_*)\delta\alpha^- = f^+_r - f^-_\ell\,.$$

Since the quantity $J_r - J_\ell$ remains constant across the interaction, we get

$$J_r - J_\ell = (J_r - J^+_*) + (J^+_* - J_\ell) = (J_r - J^-_*) + (J^-_* - J_\ell)\,.$$

Then, by the definition (3.16) of the sizes ($\sigma_{\pm 1} = \Delta J$) we deduce the identity (3.22). Using again (3.9) and (3.16), the same procedure applied to $\rho_r - \rho_\ell$ and the fact

that $\sigma_{\pm 1} = \pm\Delta\rho$ lead to the following identity:

$$\sigma_1^+ - \sigma_{-1}^+ - 2g(J_*^+)\delta\alpha^+ = \sigma_1^- - \sigma_{-1}^- - 2g(J_*^-)\delta\alpha^-\,,$$

that can be rewritten as

$$\begin{aligned}\sigma_1^+ - \sigma_{-1}^+ &= \sigma_1^- - \sigma_{-1}^- + 2\left[g(J_*^+) - g(J_*^-)\right]\delta\alpha^- + 2g(J_*^+)\delta(\alpha^+ - \alpha^-)\\ &= \sigma_1^- - \sigma_{-1}^- + 2g'(s)\left[J_*^+ - J_*^-\right]\delta\alpha^- + 2g(J_*^+)\delta(\alpha^+ - \alpha^-)\end{aligned} \tag{3.25}$$

for s as in (3.19). Notice that

$$J_*^+ - J_*^- = (J_*^+ - J_r) + (J_r - J_*^-) = -\sigma_1^+ + \sigma_{-1}^-$$

and, replacing J_r with J_ℓ, one has

$$J_*^+ - J_*^- = \sigma_{-1}^+ - \sigma_1^-\,.$$

Since both equations are true, then one can combine them and write

$$J_*^+ - J_*^- = \frac{1}{2}\left(\sigma_{-1}^+ - \sigma_1^+ + \sigma_{-1}^- - \sigma_1^-\right)\,.$$

By substitution into (3.25), we get

$$\sigma_1^+ - \sigma_{-1}^+ = \sigma_1^- - \sigma_{-1}^- + g'(s)\left(\sigma_{-1}^+ - \sigma_1^+ + \sigma_{-1}^- - \sigma_1^-\right)\delta\alpha^- + 2g(J_*^+)\delta(\alpha^+ - \alpha^-)\,,$$

which, for $\gamma^- \doteq g'(s)\delta\alpha^-$ leads to

$$\left(1+\gamma^-\right)\left(\sigma_1^+ - \sigma_{-1}^+\right) = \left(1-\gamma^-\right)\left(\sigma_1^- - \sigma_{-1}^-\right) + 2g(J_*^+)\delta(\alpha^+ - \alpha^-)\,.$$

In conclusion, recalling (3.22), we have the following 2×2 linear system

$$\begin{aligned}&\sigma_1^+ + \sigma_{-1}^+ = \sigma_1^- + \sigma_{-1}^-\\ &\sigma_1^+ - \sigma_{-1}^+ = \frac{1-\gamma^-}{1+\gamma^-}\left(\sigma_1^- - \sigma_{-1}^-\right) + \frac{2g(J_*^+)\delta(\alpha^+ - \alpha^-)}{1+\gamma^-}\end{aligned}$$

whose solution is given by (3.20). The proof of (3.21) is completely similar. Finally, by taking the absolute values in (3.20), we get (3.23).

This concludes the proof of Proposition 3.1.2. □

3.2 Approximate solutions

In this section, we construct a family of approximate solutions (Subsection 3.2.1) and we provide a-priori estimates on such approximations (Subsections 3.2.2–3.2.4).

3.2.1 Construction of the approximate solutions

In this subsection, we construct a family of approximate solutions for the initial–boundary value problem associated to system (3.4) and initial, boundary conditions (1.3) with

$$\int_I \rho_0(x)\,dx = 0\,. \tag{3.26}$$

Step 1: approximation of initial data, $k(x)$ and of $\alpha(t)$. Let $N \in 2\mathbb{N}$ and set

$$\Delta x = \Delta t = \frac{1}{N}\,, \qquad x_j = j\Delta x\ (j = 0,\dots,N)\,, \qquad t^n = n\Delta t\ (n \geq 0)\,.$$

The size of the 0-wave at a point $0 < x_j < 1$ is given by

$$\delta_j = \int_{x_{j-1}}^{x_j} k(x)dx\,, \qquad j = 1,\dots,N-1\,. \tag{3.27}$$

Assume $\Delta x = 1/N$ small enough so that for $C_1 = \sup g'(J)$,

$$C_1\|\alpha\|_\infty \cdot \delta_j < 1\,. \tag{3.28}$$

We approximate the initial data $f_0^\pm$ and $a(x)$ as

$$(f_0^\pm)^{\Delta x}(x) = f_0^\pm(x_j+)\,, \qquad a^{\Delta x}(x) = a(x_j) = \int_0^{x_j} k\,, \qquad x \in (x_j, x_{j+1})\,. \tag{3.29}$$

Recalling that $\int \rho_0\,dx = 0$ and that $\rho = f^+ + f^-$, we easily deduce the following inequality:

$$\left|\int_I \left[(f_0^+)^{\Delta x} + (f_0^-)^{\Delta x}\right] dx\right| \leq \Delta x \mathrm{TV}\,\rho_0\,. \tag{3.30}$$

Finally, we approximate $\alpha(t)$ in a natural way as follows:

$$\alpha_n(t) = \bar\alpha_n := \alpha(t^n+) \qquad \text{for } t \in [t_n, t_{n+1})\,, \quad n \geq 0. \tag{3.31}$$

Step 2: solution at $t > 0$, small t. At $t = 0$ each Riemann problem that arises at $0 < x_j < 1$ is solved using Proposition 3.1.1 adapted to the time-dependence of

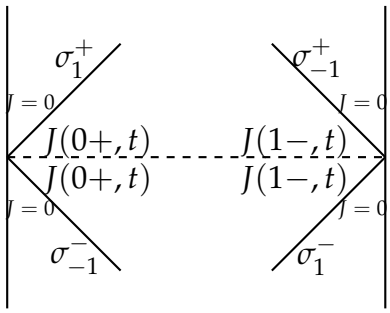

FIGURE 3.3: Interactions with the boundaries $x = 0, 1$ at time $t > 0$.

the source term in (1.2), that is the missing values in (3.2) satisfy the following:

$$J_* + g(J_*)\delta\bar{\alpha}_n = f_\ell^+ - f_r^- , \qquad \rho_{*,r} - \rho_{*,\ell} = -2g(J_*)\delta\bar{\alpha}_n . \tag{3.32}$$

Moreover, at $x = 0$ and $x = 1$ we have to deal with two *boundary Riemann problems*. For instance, at $x = 0$, $t = 0$ one has to solve the problem with $(f_0^-, f_0^+)(0+)$ as initial data and $J = 0$ as boundary datum. The solution consists of a single $(+1)$-wave and the intermediate state (f_*^-, f_*^+) between $x = 0$ and the $(+1)$-wave is uniquely determined by

$$f_*^- = f_0^- , \qquad f_*^+ - f_*^- = 0 \quad \Rightarrow \quad f_*^+ = f_0^- . \tag{3.33}$$

The size of the outgoing wave is given by

$$\sigma_1 = \Delta J = (f_0^+ - f_0^-) = J_0(0+) . \tag{3.34}$$

Step 3: solution at $t > 0$, general t. At $t = t^n = n\Delta t$ with $n \geq 1$, multiple interactions of waves occur at $0 < x_j < 1$ and the newly generated Riemann problems are again solved as in Proposition 3.1.1 adapted to the time-dependence of the source term in (1.2). At $x = 0$, let σ_{-1}^- be the size of a (-1)–wave that hits the boundary. Clearly, on the left of this wave the boundary condition $J = 0$ is satisfied. Being J_r the value of J on the right of the incoming wave, its size σ_{-1}^- satisfies

$$\sigma_{-1}^- = \Delta J = J_r .$$

The boundary Riemann problem is solved as before and a new $(+1)$–wave is issued at the point $x = 0$, $t = t^n$. Since the boundary condition is still satisfied after the interaction, the size of the new wave will be equal to

$$\sigma_1^+ = \Delta J = J_r = \sigma_{-1}^- . \tag{3.35}$$

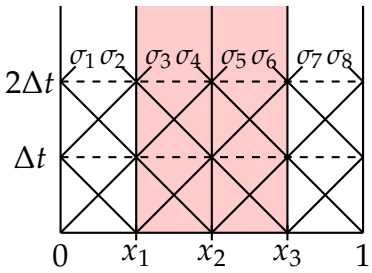

FIGURE 3.4: Well-balanced scheme in the case $N = 4$.

Hence, the total variation does not change under reflection of waves at the boundaries. See Figure 3.3 for a picture of this interaction. The construction that is presented above leads to the definition of an approximate solution $(f^{\pm})^{\Delta x}(x,t)$ and hence of $\rho^{\Delta x}$, $J^{\Delta x}$. In the rest of this chapter, as far as there is no ambiguity in the notation, we will drop the Δx and will refer to $(f^{\pm})(x,t)$ as an approximate solution with fixed parameter $\Delta x > 0$.

3.2.2 Invariant domains

Recalling Proposition 3.1.1-(ii), the set

$$D = [m, M] \times [m, M]\,, \qquad M = \sup_I f_0^{\pm}\,, \quad m = \inf_I f_0^{\pm} \tag{3.36}$$

is an invariant domain for the solution to the Riemann problem in the (f^-, f^+)-variables. Let

$$J_{\max} = M - m\,, \qquad D_J = [-J_{\max}, J_{\max}]\,. \tag{3.37}$$

Here D_J denotes the closed interval which is the projection of D on the J-axis.

It is easy to verify that D is invariant also under the solution to the Riemann problem at the boundary. Indeed, assume that there is a -1-wave impinging on the boundary $x = 0$ at a certain time $\bar{t}$ with a $+1$ reflected wave. Let $(\bar{f}^-, \bar{f}^+) \in D$ be the state on the right of the impinging/reflected wave. Hence

- the state between $x = 0$ and the impinging wave, for $t < \bar{t}$, is $(\bar{f}^+, \bar{f}^+)$,
- the state between $x = 0$ and the reflected wave, for $t > \bar{t}$, is $(\bar{f}^-, \bar{f}^-)$,

and both these states belong to D.

Proposition 3.2.1. *Under the assumptions of Theorem 1.2.1, for every $t \geq 0$ the following holds:*

$$m \leq f^{\pm}(x,t) \leq M\,, \tag{3.38}$$

and hence, by means of (3.1),

$$2m \leq \rho(x,t) \leq 2M\,, \qquad |J(x,t)| \leq M - m\,, \tag{3.39}$$

with m, M given in (3.36).

As a consequence of the properties above, the solution satisfies $J(x,t) \in D_J$ outside discontinuities. We remark that, given $m < M$, the bound above holds

- for every choice of source term coefficients $k(x)$, $g(J)$, $\alpha(t)$ as in (1.7), (1.8);
- for every (approximate) solution such that the initial data satisfies (3.36).

We also remark that, in case of no source term (for instance if $k(x) \equiv 0$), by the analysis of the Riemann problems one finds that the invariant domain is smaller than the square D, being the rectangle $[m^-, M^-] \times [m^+, M^+]$:

$$m^\pm \leq f^\pm(x,t) \leq M^\pm\,,$$

where

$$m^\pm \doteq \inf_I f_0^\pm\,, \qquad M^\pm \doteq \sup_I f_0^\pm\,.$$

3.2.3 Conservation of mass

In this subsection, we prove that the total mass of $\rho^{\Delta x}$ is conserved in time.

Proposition 3.2.2. *Under the assumptions of Theorem 1.2.1, one has*

$$\frac{d}{dt}\int_I \rho^{\Delta x}(x,t)\,dx = 0\,, \tag{3.40}$$

and

$$\left|\int_I \rho^{\Delta x}(x,t)\,dx\right| \leq \Delta x \mathrm{TV}\,\rho_0\,. \tag{3.41}$$

Proof. Let

$$y_1(t) < y_2(t) < \ldots < y_{2N}(t) \qquad \forall\, t > 0\,,\ t \neq t^n,\ t \neq t^{n+1/2} \tag{3.42}$$

be the location of the ± 1 waves at time t, that is, the location of all the possible discontinuities (see Figure 3.5). Note that the $y_j(t)$ does not necessarily correspond to a discontinuity. Observe that, by the Rankine-Hugoniot condition of the first equation in (1.2), which is satisfied in the approximate solution, we have

$$\Delta J(y_j(t)) = \Delta\rho(y_j(t))\dot{y}_j\,. \tag{3.43}$$

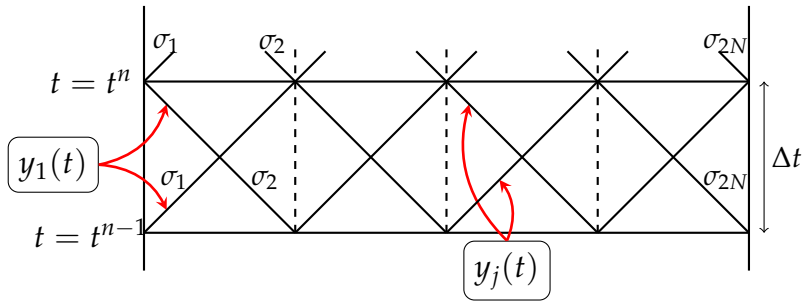

FIGURE 3.5: Illustration of the polygonals $y_j(t)$ and of the wave strengths $\sigma_j(t)$

Now, observe that the function

$$t \mapsto \int_I \rho^{\Delta x}(x,t)\,dx\,;$$

is continuous and piecewise linear on $\mathbb{R}_+$, and that its derivative is given by

$$\begin{aligned}\frac{d}{dt}\int_I \rho^{\Delta x}(x,t)\,dx &= -\sum_{j=1}^{2N}\Delta\rho(y_j)\dot{y}_j \\ &= -\sum_{j=1}^{2N}\Delta J(y_j(t)) = -J(1-,t)+J(0+,t) = 0 \end{aligned} \tag{3.44}$$

for every $t \neq t^n, t^{n+1/2}$, where we used (3.43) and the boundary conditions $J(1-,t) = J(0+,t) = 0$, which are satisfied exactly for every $t \neq t^n$. Hence (3.40) is proved.

Finally, the inequality (3.41) follows from (3.40), (3.30) and recalling that $\rho = f^+ + f^-$. The proof is complete. □

3.2.4 Uniform bounds on the Total Variation

We define

$$L_\pm(t) = \sum_{(\pm 1)-waves} |\Delta f^\pm|\,, \tag{3.45}$$

$$L_0(t) = \frac{1}{2}\left(\sum_{0-waves} |\Delta f^+| + |\Delta f^-|\right) \tag{3.46}$$

that by (3.16) are related to ρ and J as

$$L_\pm(t) = \mathrm{TV}\, J(\cdot,t)\,, \qquad\qquad L_\pm(t) + L_0(t) = \mathrm{TV}\,\rho(\cdot,t)\,.$$

As in the case of the Cauchy problem [6] and as in [3], the functional $L_\pm(t)$ may change only at the times t^n, due to the interactions with the $(\pm 1) - waves$ with the $0 - waves$. Let us evaluate the total possible increase of $L_\pm$. At each time t^n, by using the inequality (3.23), we get

$$L_\pm(t^n+) \le L_\pm(t^n-) + 2C_0\,|\bar\alpha_n - \bar\alpha_{n-1}| \sum_{j=1}^{N-1} \delta_j \le L_\pm(t^n-) + 2C_0\,|\bar\alpha_n - \bar\alpha_{n-1}|\,\|k\|_{L^1}\,.$$

Summing up the previous inequality, one gets

$$L_\pm(t^n+) \le L_\pm(0+) + 2C_0 \mathrm{TV}\,\{\alpha;[0,t_n]\}\|k\|_{L^1}\,. \tag{3.47}$$

Hence for every $T > 0$ the function $[0,T] \ni t \mapsto L_\pm(t)$ is uniformly bounded in t and Δx. Moreover one has

$$\begin{aligned} L_\pm(0+) \le{}& \mathrm{TV}\, f^+(\cdot,0) + \mathrm{TV}\, f^-(\cdot,0) + |J_0(0+)| + |J_0(1-)| + 2C_0\alpha(0+)\|k\|_{L^1}\,, \\ L_0(t) \le{}& \|\alpha\|_\infty \sum_j |g(J_*(x_j))|\Delta a(x_j) \le C_0\|\alpha\|_\infty\|k\|_{L^1}\,. \end{aligned} \tag{3.48}$$

In conclusion,

$$\begin{aligned} \mathrm{TV}\, f^+(\cdot,t) + \mathrm{TV}\, f^-(\cdot,t) ={}& L_\pm(t) + 2L_0(t) \\ \le{}& \mathrm{TV}\, f^+(\cdot,0) + \mathrm{TV}\, f^-(\cdot,0) + |J_0(0+)| + |J_0(1-)| \\ & + 4C_0\left(\|\alpha\|_\infty + \mathrm{TV}\,\{\alpha;[0,T]\}\right)\,\|k\|_{L^1} \\ \doteq{}& M\,, \end{aligned} \tag{3.49}$$

and hence the total variation of $t \mapsto (\rho^{\Delta x}, J^{\Delta x})(\cdot,t)$ is uniformly bounded on all finite time intervals $[0,T]$, with $T > 0$, uniformly in Δx.

3.3 Strong convergence as $\Delta x \to 0$ and well-posedness

This section is devoted to proving Theorem 1.2.1. The core of the proof is an adaptation of Helly's compactness theorem that we recall in the following ([15, Theorem 2.4, p. 15]).

Theorem 3.3.1. *(Helly's Theorem). Consider a sequence of functions* $\{u_N\}_{N\in\mathbb{N}} : [0,\infty) \times \mathbb{R} \to \mathbb{R}^n$ *with the following properties:*

$$\begin{aligned} &\mathrm{TV}\,(u_N(t,\cdot)) \le C\,, \qquad |u_N(t,x)| \le M\,, \qquad \forall (t,x) \in [0,\infty)\times\mathbb{R}\,,\\ &\int_{-\infty}^{\infty} |u_N(t,x) - u_N(s,x)|dx \le L|t-s|\,, \qquad \forall t,s \ge 0\,, \end{aligned}$$

for some constants C, M, L*. Then there exists a subsequence* u_μ *which converges to some function* u *in* $L^1_{loc}([0,\infty)\times\mathbb{R};\mathbb{R}^n)$*. This limit function satisfies*

$$\int_{-\infty}^{\infty} |u(t,x) - u(s,x)|dx \le L|t-s|\,, \qquad \forall t,s \ge 0\,.$$

The point value of the limit function u *can be uniquely determined by requiring that*

$$u(t,x) = u(t,x+) \doteq \lim_{y\to x+} u(t,y) \qquad \forall (t,x).$$

In this case, one has

$$\mathrm{TV}\,(u(t,\cdot)) \le C\,, \qquad |u(t,x)| \le M\,, \qquad \forall (t,x) \in [0,\infty)\times\mathbb{R}\,.$$

Now, we prove Theorem 1.2.1, we start by proving it for $(\rho_0, J_0) \in BV(I)$.

In this case, it is possible to pass to the limit thanks to Helly's compactness Theorem 3.3.1 adapted to a bounded interval). To prove this statement, we observe that the approximate solutions are uniformly bounded (with respect to t and $N = (\Delta x)^{-1}$) in the L^∞–norm and their total variation is uniformly bounded as well. Also, the following property holds: for M defined in (3.49),

$$\int_0^1 |(f^\pm)^{\Delta x}(x,t) - (f^\pm)^{\Delta x}(x,s)|\,dx \le M|t-s| \qquad \text{for all } \Delta x \quad \text{and } t,s \ge 0\,. \tag{3.50}$$

Indeed, let t and s be in the time intervals where no interactions exist, that is

$$t^n \le s < t \le t^{n+\frac{1}{2}} \qquad \text{or} \qquad t^{n+\frac{1}{2}} \le s < t \le t^{n+1}\,, \tag{3.51}$$

then

$$\begin{aligned}\int_0^1 |(f^\pm)^{\Delta x}(x,t) - (f^\pm)^{\Delta x}(x,s)|\,dx &= \sum_j |\Delta(f^\pm)^{\Delta x}(x_j,\cdot)||\Delta x_j| \\ &= \sum_j |\Delta(f^\pm)^{\Delta x}(x_j,\cdot)||\dot{x}_j|\;|t-s| \\ &= \sum_j |\sigma_j(t)|\;|t-s| \\ &= \mathrm{TV}\,(f^\pm)^{\Delta x}(\cdot,t)\;|t-s| \le M\,|t-s|\,,\end{aligned}$$

where (3.49) is used in the last inequality. Note that for t, s in larger intervals than (3.51), the map $t \to (f^\pm)^{\Delta x}(\cdot,t) \in L^1(0,1)$ is continuous.

Hence, by Helly's theorem, there exists a subsequence $(\Delta x)_j \to 0$ such that $f^{\pm(\Delta x)_j} \to f^\pm$ in $L^1_{loc}(0,1)\times[0,\infty)$ for some functions $f^\pm : (0,1)\times[0,\infty) \to \mathbb{R}$, that are weak solutions to (3.2).

More precisely, the time-Lipschitz inequality (3.50) is satisfied in the limit as $\Delta x \to 0$, and hence functions $f^\pm(x,t) \in L^\infty((0,1)\times[0,\infty))$ are Lipschitz continuous as functions of t in $L^1(0,1)$:

$$\int_0^1 |f^\pm(x,t) - f^\pm(x,s)|\,dx \le M|t-s| \qquad \text{for all } t,s \ge 0\,.$$

Up to a choice of a representative of $f^\pm$ (the one which is continuous from the right, in space) one has $f^\pm(\cdot,t) \in BV(I)$, where the function $t \to \mathrm{TV}\, f^\pm(\cdot,t)$ is non increasing. Also, the L^∞ bounds which are valid for $(f^\pm)^{\Delta x}$ are also valid for $f^\pm$.

In terms of $\rho^{\Delta x}$, $J^{\Delta x}$, the identity

$$\int_0^1\int_0^\infty \left\{\rho^{\Delta x}\partial_t\phi + J^{\Delta x}\partial_x\phi\right\}\,dxdt = 0 \tag{3.52}$$

holds for every $\phi \in C^1([0,1]\times(0,T))$ (that is, up to the boundaries of I) since $J^{\Delta x}(0+,t) = 0 = J^{\Delta x}(1-,t)$ for every $t \neq t^n$. Hence the identity (3.52) is satisfied by the strong limit (ρ, J). Moreover, by passing to the limit as $(\Delta x)_j \to 0$ in (3.41) one obtains that (1.10) holds, that is

$$\int_I \rho(x,t)\,dx = 0 \qquad \forall\, t > 0\,.$$

To obtain the stability in L^1 with respect to the initial data, one can observe that the coupling in system (3.2) is *quasimonotone*, in the sense that the equations

$$\partial_t f^{\pm} \pm \partial_x f^{\pm} = \mp G\,, \qquad G(x,t,f^{\pm}) = k(x)\alpha(t)\, g(f^{+} - f^{-})$$

satisfy, thanks to the assumptions (1.8) and (1.7),

$$\mp \frac{\partial G}{\partial f^{\pm}} \leq 0\,.$$

By adapting the arguments in [31], which rely on Kružkov techniques, one can prove the L^1 stability estimate (1.9), that is, for any pair of initial data (f_0^-, f_0^+) and $(\widetilde{f}_0^-, \widetilde{f}_0^+)$, the corresponding solutions $f^{\pm}$, $\widetilde{f}^{\pm}$ on $(0,1)\times(0,T)$ satisfy

$$\|(f^-, f^+)(\cdot,t) - (\widetilde{f}^-, \widetilde{f}^+)(\cdot,t)\|_{L^1(I)} \leq \|(f_0^-, f_0^+) - (\widetilde{f}_0^-, \widetilde{f}_0^+)\|_{L^1(I)}\,.$$

Therefore, the weak solution to (1.2)–(1.3) is unique on $(0,1)\times(0,T)$ and can be prolonged for all times, $t \in \mathbb{R}^+$.

Finally, let $(\rho_0, J_0) \in L^\infty(I)$. Then there exists a sequence $\{(\rho_0, J_0)_n\}_{n\in\mathbb{N}} \subset BV(I)$ such that $(\rho_0, J_0)_n \to (\rho_0, J_0) \in L^1(I)$. By the L^1 stability estimate (1.9), the limit in L^1 of $f_n^{\pm}(\cdot,t)$ is well defined and hence for $(\rho, J)(\cdot,t)$. Since the identity

$$\int_0^1 \int_0^\infty \{\rho_n \partial_t \phi + J_n \partial_x \phi\}\, dxdt = 0 \tag{3.53}$$

holds for every $\phi \in C^1([0,1]\times(0,\infty))$ and for every n, then (3.53) is valid also for the strong limit (ρ, J), as well as (1.10). This completes the proof of Theorem 1.2.1 .

Chapter 4

A finite-dimensional representation of the approximate solutions

In this chapter, we study the evolution in time of the approximate solution by means of finite-dimensional evolution system, of size $2N = 2\Delta x^{-1}$ for the vector $\sigma(t)$ given in (4.1). Then, we prove an exponential-type formula for a given doubly stochastic matrix. The proof relies on a detailed study of the expansion of the power whose coefficients are described by hypergeometric functions, and their sum is computed through modified Bessel functions. Thanks to a careful expression of the first order in $1/N$ and to a convenient decomposition of the vectors, a cancellation property is identified, see Proposition 4.3.4.

4.1 The iteration matrix

In this section, we will study the evolution in time of the approximate solution by means of finite-dimensional evolution system, of size $2N = 2\Delta x^{-1}$. Let's introduce a vector representation of the approximate solution that will be the basis of our subsequent analysis. Define

$$\mathcal{T} = \{t \geq 0 : \ t = t^n = n\Delta t \text{ or } t = t^{n+\frac{1}{2}} = \left(n + \frac{1}{2}\right)\Delta t\,, \quad n = 0, 1, \ldots\}$$

the set of possible interaction times. At every time $t \notin \mathcal{T}$, we can introduce the vector of the sizes

$$\sigma(t) = (\sigma_1, \ldots, \sigma_{2N})(t) \in \mathbb{R}^{2N}\,, \qquad N \in 2\mathbb{N} \tag{4.1}$$

where, recalling (3.16) and the notation in Proposition 3.2.2, especially (3.42) and (3.43), one has

$$\sigma_j \doteq \Delta J(y_j) = \Delta\rho(y_j)\dot{y}_j\,. \tag{4.2}$$

Let's examine its evolution in the following steps.

(1) At time $t = 0+$, $\sigma(0+)$ is given by the size of the waves that arise at $x_j = j\Delta x$, with $j = 0, \dots, N$. In particular, a (+1) wave arises at $x = 0$, two (± 1) waves arise at each x_j with $j = 1, \dots, N-1$ and finally a (-1) wave arises at $x = 1$.

(2) At every time $t^{n+\frac{1}{2}}$, $n \geq 0$, the vector $\sigma(t)$ evolves by exchanging positions of each pair $\sigma_{2j-1}, \sigma_{2j}$:

$$(\sigma_{2j-1}, \sigma_{2j}) \mapsto (\sigma_{2j}, \sigma_{2j-1}) \qquad j = 1, \dots, N \tag{4.3}$$

that results into

$$\sigma(t+) = B_1 \sigma(t-), \quad B_1 \doteq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{4.4}$$

(3) At each time $t^n = n\Delta t$, $n \geq 1$, the interactions with the Dirac masses at each x_j of the source term occur, and we have to take into account the relations introduced in Proposition 3.1.2.

For each $j = 1, \dots, N-1$ we will rely on the identity (3.21). Define the *transition coefficients* γ_j^n as follows:

$$\gamma_j^n = g'(s_j^n)\delta_j \bar{\alpha}_n, \qquad j = 1, \dots, N-1, \quad n \geq 1, \tag{4.5}$$

where δ_j is given in (3.27), $\bar{\alpha}_n$ in (3.31) and s_j^n satisfies a relation as in (3.19); more precisely

$$g'(s_j^n) = \frac{g\left(J(x_j, t^n+)\right) - g\left(J(x_j, t^n-)\right)}{J(x_j, t^n+) - J(x_j, t^n-)},$$

such that $J(x_j, t^n)$ is computed by the implicit equations (3.32). Moreover introduce the terms

$$p_{j,n} = g\left(J(x_j, t^n-)\right) \frac{\delta_j}{1 + \gamma_j^n}, \qquad j = 1, \dots, N-1, \quad n \geq 1. \tag{4.6}$$

Then, the local interaction is described as follows:

$$\boxed{\begin{pmatrix}\sigma_{2j}\\ \sigma_{2j+1}\end{pmatrix} \mapsto \frac{1}{1+\gamma_j^n}\begin{pmatrix}\gamma_j^n\sigma_{2j}+\sigma_{2j+1}\\ \sigma_{2j}+\gamma_j^n\sigma_{2j+1}\end{pmatrix} + (\bar{\alpha}_n-\bar{\alpha}_{n-1})\,p_{j,n}\begin{pmatrix}-1\\ +1\end{pmatrix}.} \tag{4.7}$$

To recast it in a global matrix form, we define

$$\boldsymbol{\gamma}^n = (\gamma_1^n,\dots,\gamma_{N-1}^n) \in \mathbb{R}^{N-1} \tag{4.8}$$

and set

$$B_2(\boldsymbol{\gamma}^n) = \begin{bmatrix} 1 & & & \\ & \boxed{\hat{A}_1^n} & & 0 \\ & & \ddots & \\ & 0 & & \boxed{\hat{A}_{N-1}^n} \\ & & & 1 \end{bmatrix}, \qquad \hat{A}_j^n = \frac{1}{1+\gamma_j^n}\begin{bmatrix}\gamma_j^n & 1\\ 1 & \gamma_j^n\end{bmatrix}. \tag{4.9}$$

The matrix $B_2(\boldsymbol{\gamma})$ is tridiagonal with diagonal components as follows,

$$\left(1, \frac{\gamma_1^n}{1+\gamma_1^n}, \frac{\gamma_1^n}{1+\gamma_1^n}, \frac{\gamma_2^n}{1+\gamma_2^n}, \dots, \frac{\gamma_{N-2}^n}{1+\gamma_{N-2}^n}, \frac{\gamma_{N-1}^n}{1+\gamma_{N-1}^n}, \frac{\gamma_{N-1}^n}{1+\gamma_{N-1}^n}, 1\right) \in \mathbb{R}^{2N}$$

and subdiagonals

$$\left(0, \frac{1}{1+\gamma_1^n}, 0, \frac{1}{1+\gamma_2^n}, 0, \dots, \frac{1}{1+\gamma_{N-1}^n}, 0\right) \in \mathbb{R}^{2N-1}.$$

Hence $\boldsymbol{\sigma}(t)$ evolves according to

$$\boldsymbol{\sigma}(t^n+) = B_2(\boldsymbol{\gamma}^n)\boldsymbol{\sigma}(t^n-) + (\bar{\alpha}_n-\bar{\alpha}_{n-1})\,\boldsymbol{G}_n$$

with

$$\boldsymbol{G}_n = (0, -p_{1,n}, +p_{1,n}, \dots, -p_{N-1,n}, +p_{N-1,n}, 0)^t. \tag{4.10}$$

Remark 4.1.1. *We give a couple of remarks about the use of the local interaction estimates* (3.20), (3.21)

(i) *If, in place of* (3.21), *the relation* (3.20) *is used, the quantities* (4.5) *and* (4.6) *are defined by*

$$\gamma_j^n = g'(s_j^n)\delta_j\bar{\alpha}_{n-1}\,, \qquad p_{j,n} = g\left(J(x_j, t^n+)\right)\frac{\delta_j}{1+\gamma_j^n}\,.$$

(ii) *In the definition of* (4.7), *notice that we consider the* space order *instead of the* family order, *that was used in* (3.20). *That is,*

$$(\sigma_{2j}, \sigma_{2j+1}) = \begin{cases} (\sigma_1^-, \sigma_{-1}^-) & \textit{before the interaction} \\ (\sigma_{-1}^+, \sigma_1^+) & \textit{after the interaction}\,. \end{cases}$$

We summarize the previous identities to get the following statement.

Proposition 4.1.1. *At time $t = t^n$, let B_1, $B_2(\gamma)$, $\boldsymbol{G}_n$ be defined by* (4.4), (4.9), (4.10) *respectively. Define*

$$B(\gamma) := B_2(\gamma)B_1\,. \tag{4.11}$$

Then the following relation holds,

$$\boxed{\boldsymbol{\sigma}(t^n+) = B(\boldsymbol{\gamma}^n)\boldsymbol{\sigma}(t^{n-1}+) + (\bar{\alpha}_n - \bar{\alpha}_{n-1})\,\boldsymbol{G}_n}\,, \qquad n \geq 1\,. \tag{4.12}$$

The matrix B in (4.11) *is doubly stochastic for every $\boldsymbol{\gamma}$; we will call it* transition matrix. *Notice that it is non-negative provided that all the γ_j^n (see* (4.5)*) are non-negative, which relies on the assumption that $g' \geq 0$. Let's summarize some properties:*

(i) The determinant of B is

$$\det(B) = -\left(1 - \frac{2\gamma_1}{1+\gamma_1}\right)\cdots\left(1 - \frac{2\gamma_{N-1}}{1+\gamma_{N-1}}\right)\,. \tag{4.13}$$

(ii) Its eigenvalues λ_i satisfy $|\lambda_i| \leq 1$ for all $i = 1, \ldots, 2N$;

(iii) The values $\lambda = \pm 1$ are eigenvalues with corresponding (left and right) eigenvectors

$$\begin{aligned} \lambda_- &= -1\,, \qquad v_- = (1,-1,-1,1,\ldots,1,-1,-1,1)\,, \\ \lambda_+ &= 1\,, \qquad e = (1,1,\ldots,1,1)\,. \end{aligned} \tag{4.14}$$

(iv) If $\gamma_j \cdot \gamma_{j+1} > 0$ for some j, then the eigenvalues with maximum modulus are exactly two ($\lambda = \pm 1$) and they are simple.

Moreover, $B(\mathbf{0})$ is a normal matrix, since it is a permutation and hence

$$B(\mathbf{0})^t B(\mathbf{0}) = B(\mathbf{0})B(\mathbf{0})^t = I_{2N}\,.$$

This property does not hold if $\boldsymbol{\gamma} \neq \mathbf{0}$.

Proof. Proof of (4.12) is obvious by the evolution of $\sigma(t)$ that is presented before, and from the definitions of B_1, $B_2(\boldsymbol{\gamma})$, $\boldsymbol{G}_n$ defined by (4.4), (4.9), (4.10) respectively.

Proof of (i) By the Binet Theorem, see [40, p. 28], we have

$$\det(B) = \det(B_2)\det(B_1)$$

where

$$\det(B_1) = 1\,, \qquad \det(B_2) = (\frac{2\gamma_1}{1+\gamma_1} - 1)\cdots(\frac{2\gamma_{N-1}}{1+\gamma_{N-1}} - 1)\,.$$

Since $(N-1)$ is odd, we obtain (4.13).

Proof of *(ii)* and *(iii)* By Gershgorin Theorem A.1.2, see [40, p. 387], all the eigenvalues of the matrix B are located in the circle of center 0 and radius 1 in the complex plane. Indeed, all the terms on the diagonal are 0 and

$$\sum_{j=1\,,i\neq j}^{2N} |B_{ij}| = 1\,, \qquad \forall\, i\,.$$

Hence *(ii)* follows. About *(iii)* it is immediate to check that

$$Bv_- = -v_-\,, \qquad v_-^t B = -v_-^t\,,$$

while $Be = e$ and $e^t B = e^t$ follow by the double stochastic character of B.

Proof of *(iv)* It remains to prove that $\lambda_\pm$ are the *only* eigenvalues of B with modulus 1, while all the other have modulus < 1.

We claim that B satisfies the hypotheses of Romanovsky Theorem, see [46] and [40, p. 541]. The latter result states that a non-negative irreducible matrix $A \in M_n(\mathbb{R})$ has exactly $p \in \mathbb{N}$ eigenvalues with maximum modulus if, for any node of the corresponding directed graph, p is the greatest common divisor of the lengths of all the directed paths that both start and end at a same node.

See Figure 4.1 for a picture of the graph related to the matrix $B = [B_{ij}]$ with $i, j = 1, \ldots 2N$, where each node correspond to a row i and each directed arc (i, j)

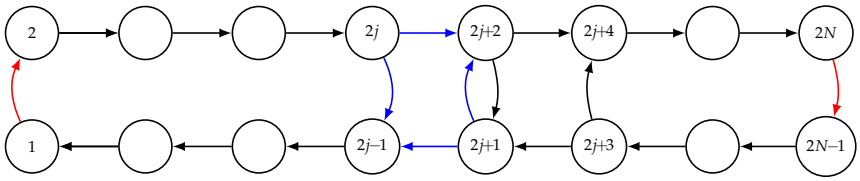

FIGURE 4.1: The graph corresponding to B when γ_j, $\gamma_{j+1} > 0$. The red arcs correspond to the first and final row of the matrix, while the blue arcs connecting the nodes $2j-1, 2j, 2j+1, 2j+2$ correspond to the submatrix $\hat{B}_j$.

corresponds to a non-zero element B_{ij}. Remark that the graph of B can be deduced by noticing that the first row is represented by the arc $(1,2)$, the last row by the arc $(2N, 2N-1)$ and that each 2×4 submatrix occupying the block of rows $2j, 2j+1$ and columns $2j-1, \dots, 2j+2$,

$$\hat{B}_j = \frac{1}{1+\gamma_j} \begin{bmatrix} \gamma_j & 0 & 0 & 1 \\ 1 & 0 & 0 & \gamma_j \end{bmatrix} \qquad j = 1, \dots, N-1,$$

corresponds to a squared subgraph made of the arcs $(2j, 2j-1)$, $(2j, 2j+2)$, $(2j+1, 2j-1)$, $(2j+1, 2j+2)$. Notice that, if $\gamma_j = 0$, then only the upper arc $(2j, 2j+2)$ and the lower one $(2j+1, 2j-1)$ survive in the squared subgraph related to $\hat{B}_j$. The whole graph is then obtained by juxtaposing the arcs $(1,2)$, $(2N, 2N-1)$ to the subgraphs representing $\hat{B}_j$, for $j = 1, \dots, N-1$.

First, notice that B is irreducible, which is equivalent to say that the graph is totally connected, namely that each node can be reached from any other node via a path made of arcs present in the graph: this holds true since one can always follow the circuit $(1, 2, 4, \dots, 2j, 2j+2, \dots, 2N, 2N-1, \dots, 2j+1, 2j-1, \dots, 3, 1)$ from any node in the graph. Secondly, the length of any path in the graph connecting a node to itself can be divided at most by 2, which means that in this case $p = 2$. Indeed, there is no way to obtain a path of odd length because there are no diagonal arcs. Moreover, by assumption there exists an index j such that γ_j, γ_{j+1} are not zero as in Figure 4.1.

Then, it is easy to see that there are at least two paths connecting the node 1 to itself of lengths $2j$ and $2j+2$ and the great common divisor must be 2.

Now, by the Romanovsky Theorem we can conclude that $\lambda_\pm$ are the only two eigenvalues with modulus 1 and the proof of (iv) is complete. □

Remark 4.1.2. *Notice that in general B_2 depends on t^n, since the coefficients γ_j depend on $g'(J)$ and on $\bar{\alpha}_n$. However, in the time-independent case, $\alpha(t) = const.$, the structure*

of the matrix B (the coefficients which are $\neq 0$) does not change with n, in the sense that, for a fixed j, either $\gamma_j^n \neq 0$ for every n or $\gamma_j^n = 0$ for every n.

It is well known that doubly stochastic matrices can be written as a convex combination of permutations by Birkhoff Theorem A.1.1, which are at most $4N^2 - 4N + 2 = (2N-1)^2 + 1$, see ([40, Theorem 8.7.2]). In the next proposition, for γ constant we give an explicit Birkhoff decomposition of the matrix $B(\gamma)$.

Proposition 4.1.2. *Let $\gamma = \gamma\,(1,\dots,1) \in \mathbb{R}^{N-1}$, for some constant $\gamma \in [0,1)$. Then the matrix B can be decomposed as*

$$B(\gamma) = \frac{1}{1+\gamma}\left(B(\mathbf{0}) + \gamma B_1\right). \tag{4.15}$$

Proof. Since γ is constant, then the matrix $B_2(\gamma)$ in (4.9) can be written as

$$B_2(\gamma) = \frac{1}{1+\gamma}\left(B_2(\mathbf{0}) + \gamma I_{2N}\right). \tag{4.16}$$

Recalling that $B(\gamma) = B_2(\gamma)B_1$ and substituting (4.16), we obtain (4.15). □

Remark 4.1.3. *The properties established in Subsections 3.2.2–3.2.4 can be rewritten in terms of the vectorial representation of the solution (4.1).*

(i) (Boundary conditions) *From equation (3.44) it follows that*

$$\sigma(t)\cdot e = 0 \tag{4.17}$$

for every t at which $\sigma(t)$ is defined, that is, t being not interaction time.
Indeed,

$$\sigma(t)\cdot e = \sum_{j=1}^{2N}\sigma_j(t) = \sum_{j=1}^{2N}\Delta J(y_j(t)) = J(1-,t) - J(0+,t) = 0\,.$$

(ii) (Total variation) *The quantity $L_\pm(t)$ coincides with $\|\sigma(t)\|_{\ell_1}$. In particular, from (3.47)–(3.48) we obtain*

$$\|\sigma(0+)\|_{\ell_1} \leq \mathrm{TV}\, f^+(\cdot,0) + \mathrm{TV}\, f^-(\cdot,0) + |J_0(0+)| + |J_0(1-)| + 2C_0\alpha(0+)\|k\|_{L^1}\,, \tag{4.18}$$

$$\|\sigma(t)\|_{\ell_1} \leq \|\sigma(0+)\|_{\ell_1} + 2C_0\mathrm{TV}\left\{\alpha;[0,t_n]\right\}\|k\|_{L^1}, \qquad t^n < t < t^{n+1}\,.$$

(iii) *The following property holds,*

$$|\sigma(t)\cdot v_-| \leq |\sigma(0+)\cdot v_-| \leq \mathrm{TV}\left\{\bar{J}_0;[0,1]\right\} \qquad \forall\, t \notin \mathcal{T} \tag{4.19}$$

where v_- is the eigenvector corresponding to $\lambda = -1$, see (4.14), and

$$\bar{J}_0(x) = \begin{cases} J_0(x) & x \in (0,1) \\ 0 & x \in 0 \text{ or } 1\,. \end{cases} \tag{4.20}$$

To prove the first inequality in (4.19), we first consider $t \in (t^n, t^{n+1/2})$ and use the iteration formula (4.12) to obtain

$$\boldsymbol{\sigma}(t) \cdot v_- = \boldsymbol{\sigma}(t^n) \cdot v_- = B(\boldsymbol{\gamma}^n)\boldsymbol{\sigma}(t^{n-1}+) \cdot v_- + \boldsymbol{G}_n \cdot v_-\,.$$

By recalling the definition of (4.10), we immediately deduce that

$$\boldsymbol{G}_n \cdot v_- = 0 \qquad \forall\, n\,,$$

and therefore that

$$\begin{aligned} \boldsymbol{\sigma}(t) \cdot v_- &= \boldsymbol{\sigma}(t^{n-1}+) \cdot B(\boldsymbol{\gamma}^n)^t v_- \\ &= -\boldsymbol{\sigma}(t^{n-1}+) \cdot v_- \\ &= -B(\boldsymbol{\gamma}^{n-1})\boldsymbol{\sigma}(t^{n-2}+) \cdot v_- \\ &= (-1)^n \boldsymbol{\sigma}(0+) \cdot v_-\,, \end{aligned}$$

from which (4.19) follows for $t \in (t^n, t^{n+1/2})$. Secondly, for $t \in (t^{n+1/2}, t^{n+1})$, by using (4.4) we have that

$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}(t^{n+1/2}+) = B_1\boldsymbol{\sigma}(t^{n+1/2}-) = B_1\boldsymbol{\sigma}(t^n+)\,, \qquad t \in (t^{n+1/2}, t^{n+1})$$

and hence

$$\boldsymbol{\sigma}(t) \cdot v_- = \boldsymbol{\sigma}(t^n+) \cdot B_1 v_- = -\boldsymbol{\sigma}(t^n+) \cdot v_-$$

from which it follows again (4.19).

To prove the second inequality in (4.19), by recalling the definition of v_- in (4.14), we observe that

$$\boldsymbol{\sigma}(0+) \cdot v_- = \sigma_1^0 + \sum_{j=1}^{N-1} (-1)^j \left(\sigma_{2j}^0 + \sigma_{2j+1}^0\right) + \sigma_{2N}^0\,.$$

Recalling that σ_{2j}^0, σ_{2j+1}^0 are the two outgoing waves at $x_j = j\Delta x$ and time $t = 0$, then by (3.17) it holds

$$\sigma_{2j}^0 + \sigma_{2j+1}^0 = J(x_j+, 0) - J(x_j-, 0)\,.$$

Moreover, since the approximate solution satisfies the boundary conditions $J = 0$, for small t we have

$$\sigma_1^0 = J(x_1-,0) - J(0+,t) = J(x_1-,0) = J(0+,0)\,, \qquad \sigma_{2N}^0 = -J(1-,0)\,.$$

Therefore,

$$\begin{aligned}\boldsymbol{\sigma}(0+)\cdot v_- = {} & J(x_1-,0)+ \\ & + \sum_{j=1}^{N-1}(-1)^j(J(x_j+,0) - J(x_j-,0)) - J(x_{N-1}+,0)\,, \end{aligned} \tag{4.21}$$

and then, by recalling (3.29), we find that

$$|\boldsymbol{\sigma}(0+)\cdot v_-| \le |J_0(0+)| + \mathrm{TV}\, J_0 + |J_0(1-)|\,,$$

that gives the second inequality in (4.19).

(iv) *The* undamped *equation: $k(x) \equiv 0$.*

In this case, each vector $\boldsymbol{G}_n$ vanishes and $\boldsymbol{\gamma}^n = \mathbf{0}$. Therefore from (4.12) and (4.3) we obtain

$$\boldsymbol{\sigma}(t) = \begin{cases} B(\mathbf{0})^n\boldsymbol{\sigma}(0+) & t^n < t < t^{n+\frac{1}{2}} \\ B_1 B(\mathbf{0})^n\boldsymbol{\sigma}(0+) & t^{n+\frac{1}{2}} < t < t^{n+1}\,. \end{cases}$$

Since every wave-front issued at $t = 0$, reflect on the two boundaries and gets back to the initial position after a time $T = 2 = 2N\Delta t$, it is clear that

$$B(\mathbf{0})^{2N} = I_{2N} \tag{4.22}$$

that is, $B(\mathbf{0})^{2N}$ coincides with the identity matrix in M_{2N}. As a consequence, the powers of $B(\mathbf{0})$ are periodic with period $2N$:

$$B(\mathbf{0})^{n+2N} = B(\mathbf{0})^n\,, \qquad n \in \mathbb{Z}\,.$$

With a similar argument one can prove that

$$(B(\mathbf{0})^N)_{ij} = \begin{cases} 1 & \text{if } i+j = 2N+1 \\ 0 & \text{otherwise,} \end{cases} \tag{4.23}$$

that is, $B(\mathbf{0})^N$ is the matrix with component 1 on the antidiagonal positions $(i, 2N+1-i)$ and 0 otherwise. It is clear that $(B(\mathbf{0})^N)^2 = B(\mathbf{0})^{2N} = I_{2N}$.

Remark 4.1.4. *Let us consider the Cauchy problem for system (1.2) with $\alpha(t) = 1$ and periodic initial data of the form*

$$\begin{cases} \rho_0(x) = \rho_0(x+1), \\ J_0(x) = J_0(x+1), \end{cases} \qquad x \in \mathbb{R}. \tag{4.24}$$

This problem can be studied following the same approach as for the initial-boundary value problem considered in the thesis. Indeed, constructing approximate solutions by means of the well-balanced scheme outlined in Section 3.2, we describe the evolution of $\boldsymbol{\sigma}(t)$ (vector of the waves present at time t in the space interval $[0,1]$) as follows.

At time $t = (n - \frac{1}{2})\Delta t$, $n \geq 1$, each pair of components σ_{2j-1} and σ_{2j} are switched, $j = 1, \dots, N$. Hence, $\boldsymbol{\sigma}(t+) = B_1\boldsymbol{\sigma}(t-) = B_1\boldsymbol{\sigma}(t^{n-1}+)$, where B_1 is defined in (4.4).

At time $t = n\Delta t$, recalling (3.20), we have

$$\boldsymbol{\sigma}(t+) = \widetilde{B}_2\boldsymbol{\sigma}(t-)$$

where

$$\widetilde{B}_2(\boldsymbol{\gamma}^n) = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & \boxed{\hat{A}_1^n} & & \boldsymbol{0} & 0 \\ \vdots & & \ddots & & \\ 0 & \boldsymbol{0} & & \boxed{\hat{A}_{N-1}^n} & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}, \qquad \hat{A}_j^n = \frac{1}{1+\gamma_j^n} \begin{bmatrix} \gamma_j^n & 1 \\ 1 & \gamma_j^n \end{bmatrix}.$$

Remark that $\widetilde{B}_2$ differs from B_2 defined in (4.9) only for the first and final rows. Indeed, by the periodicity of the initial data we have

$$\sigma_1(t^n+) = \sigma_{2N}(t^n-) \quad \text{and} \quad \sigma_{2N}(t^n+) = \sigma_1(t^n-).$$

Finally, we compute $\widetilde{B} \doteq \widetilde{B}_2 B_1$, which differs from B defined in (4.11) just for the first and last rows. Hence, the evolution of the waves is completely determined by

$$\boldsymbol{\sigma}(t^n+) = \widetilde{B}\boldsymbol{\sigma}(t^{n-1}+) \qquad n \geq 0,$$

where $\widetilde{B} \in M_{2N}(\mathbb{R})$ is a non-negative, doubly stochastic matrix. Moreover, by an analogous of Proposition 4.1.1–(iv) we can prove that there are only two eigenvalues with maximum modulus equal to 1.

4.2 Expansion formula for the matrix

In this section, we will assume that $k(x) \equiv \bar{k} > 0$, $g'(J) \equiv C_1$, $\alpha(t) \equiv 1$ and then γ is a vector where all the components are equal:

$$\gamma = \gamma(1, \dots, 1)\,, \qquad \gamma = d\Delta x = \frac{d}{N}\,, \qquad d = \bar{k}C_1\,, \tag{4.25}$$

which corresponds to the standard telegrapher's equation:

$$\begin{cases} \partial_t \rho + \partial_x J = 0, \\ \partial_t J + \partial_x \rho = -2dJ\,. \end{cases} \tag{4.26}$$

By Proposition 4.1.2 and Birkhoff Theorem A.1.1, if γ is as in (4.25), then the matrix $B(\gamma)$ can be decomposed as the sum of only **two** matrices,

$$B(\gamma) = \frac{1}{1+\gamma}\left(B(\mathbf{0}) + \gamma B_1\right)\,.$$

Thanks to this decomposition, we can analyze the powers of $B(\gamma)$. For a generic $n \in \mathbb{N}$ one has

$$B(\gamma)^n = (1+\gamma)^{-n}\left[B(\mathbf{0}) + \gamma B_1\right]^n \tag{4.27}$$

The factor $(1+\gamma)^{-n}$ provides an exponentially decreasing term with respect to time. Indeed, let $T > 0$ and recalling that $\Delta t = N^{-1}$, we have

$$\left(1 + \frac{d}{N}\right)^{-[TN]} \to \mathrm{e}^{-dT} \qquad N \to \infty\,. \tag{4.28}$$

Let us focus on the second factor in (4.27), that is

$$\left[B(\mathbf{0}) + \gamma B_1\right]^n = \sum_{k=0}^{n} \gamma^k S_k(B(\mathbf{0}), B_1), \tag{4.29}$$

where each term $S_k(B(\mathbf{0}), B_1)$ is the sum of all products of n matrices which are either B_1 or $B(\mathbf{0})$, and in which B_1 appears exactly k times, that is

$$\begin{cases} S_k(B(\mathbf{0}), B_1) = \displaystyle\sum_{(\ell_1,\dots,\ell_{k+1})} B(\mathbf{0})^{\ell_1} \cdot B_1 \cdot B(\mathbf{0})^{\ell_2} \cdot B_1 \cdots B(\mathbf{0})^{\ell_k} \cdot B_1 \cdot B(\mathbf{0})^{\ell_{k+1}} \\ \qquad 0 \le \ell_j \le n-k\,, \qquad \displaystyle\sum_{j=1}^{k+1} \ell_j = n-k\,. \end{cases} \tag{4.30}$$

In what follows, we extensively use the fact that $B_1^2 = I_{2N} = B(0)^{2N}$ and the commutation property described in the next proposition.

Proposition 4.2.1. *The following identity holds for any $\ell \in \mathbb{N}$:*

$$B(\mathbf{0})^{\pm\ell} B_1 = B_1 B(\mathbf{0})^{\mp\ell}. \tag{4.31}$$

Proof. Recalling (4.4) and (4.11) with $\boldsymbol{\gamma} = \mathbf{0}$, we have that $B(\mathbf{0})^{-1} = (B_2(\mathbf{0})B_1)^{-1} = B_1 B_2(\mathbf{0})$. Then for every $\ell \ge 0$ we have

$$\begin{aligned} B(\mathbf{0})^{-\ell} B_1 &= \underbrace{(B_1 B_2(\mathbf{0})) \cdots (B_1 B_2(\mathbf{0}))}_{\ell \text{ times}} \cdot B_1 \\ &= B_1 \cdot \underbrace{(B_2(\mathbf{0}) B_1) \cdots (B_2(\mathbf{0}) B_1)}_{\ell \text{ times}} \\ &= B_1 \cdot B(\mathbf{0})^{\ell}\,. \end{aligned}$$

As for the identity for $+\ell$, notice that

$$\begin{aligned} B(\mathbf{0})^{\ell} B_1 &= B(\mathbf{0})^{2N-(2N-\ell)} B_1 = B(\mathbf{0})^{2N} B(\mathbf{0})^{-(2N-\ell)} B_1 \\ &= B(\mathbf{0})^{-(2N-\ell)} B_1, \end{aligned}$$

where we used that $B(\mathbf{0})^{2N} = I_{2N}$. Hence, by the first identity we get

$$B(\mathbf{0})^{\ell} B_1 = B_1 \cdot B(\mathbf{0})^{2N-\ell} = B_1 \cdot B(\mathbf{0})^{-\ell}\,.$$

□

By means of (4.31) and using that $B_1^2 = I_{2N}$, the generic term in the sum S_k in (4.30) can be conveniently rewritten. Indeed, one has $S_0 = B(\mathbf{0})^n$. For $k = 1, \dots, n$, we have to distinguish the case of even/odd k.

- For k even, we have

$$B(\mathbf{0})^{\ell_1} \cdot B_1 \cdot B(\mathbf{0})^{\ell_2} \cdot B_1 \cdots B(\mathbf{0})^{\ell_k} \cdot B_1 \cdot B(\mathbf{0})^{\ell_{k+1}} = B(\mathbf{0})^{\alpha-\beta}\,, \tag{4.32}$$

where

$$\alpha = \sum_{j=1,\, j \text{ odd}}^{k+1} \ell_j\,, \qquad \beta = \sum_{j=2,\, j \text{ even}}^{k+1} \ell_j = n - k - \alpha\,. \tag{4.33}$$

Now let us count how many vectors $(\ell_1, \dots, \ell_{k+1})$ lead, thanks to (4.32), to the same matrix

$$B(\mathbf{0})^{\alpha-\beta} = B(\mathbf{0})^{2\alpha+k-n}\,.$$

In the first sum of (4.33) the indices are $k/2+1$, while in the second sum they are $k/2$. Hence, for a given α, the number of the distinct vectors $(\ell_1, \dots, \ell_{k+1})$ for which (4.33) holds is [1]

$$\binom{\alpha + \frac{k}{2}}{\frac{k}{2}} \binom{n - \alpha - 1 - \frac{k}{2}}{\frac{k}{2} - 1}\,, \qquad \alpha = 0, \dots, n-k\,.$$

If we perform a change of variable $j = \alpha + k/2$, we get

$$\binom{j}{\frac{k}{2}} \binom{n-j-1}{\frac{k}{2}-1}\,, \qquad j = \frac{k}{2}, \dots, n - \frac{k}{2}\,,$$

and

$$\boxed{S_k(B(\mathbf{0}), B_1) = \sum_{j=\frac{k}{2}}^{n-\frac{k}{2}} \binom{j}{\frac{k}{2}} \binom{n-j-1}{\frac{k}{2}-1} B(\mathbf{0})^{2j-n}\,, \qquad k = 2, 4, \dots, n\,.} \tag{4.34}$$

- For k odd, we have

$$\begin{aligned} B(\mathbf{0})^{\ell_1} \cdot B_1 \cdot B(\mathbf{0})^{\ell_2} \cdot B_1 \cdots B(\mathbf{0})^{\ell_k} \cdot B_1 \cdot B(\mathbf{0})^{\ell_{k+1}} &= B(\mathbf{0})^{\alpha-\beta} B_1 \\ &= B(\mathbf{0})^{2\alpha+k} B(\mathbf{0})^{-n} B_1 \\ &= B(\mathbf{0})^{2\alpha+k-1} B_2(\mathbf{0}) B(\mathbf{0})^{n}\,, \end{aligned}$$

[1] Given $M \geq 0$ and $a_j \geq 0$ integers such that $\sum_{j=1}^{n} a_j = M$, the number of distinct $(a_1, \dots, a_n)$ is

$$\binom{M+n-1}{n-1} = \binom{M+n-1}{M}\,.$$

where α, $\beta = n - k - \alpha$ are given in (4.33).

Here, the number of vectors $(\ell_1, \dots, \ell_{k+1})$ for which (4.33) holds are counted as follows. The indices ℓ_j are in total $(k+1)/2$ for both sums, hence for a given α the number of terms is

$$\binom{\alpha + \frac{k-1}{2}}{\frac{k-1}{2}} \binom{n - \alpha - \frac{k-1}{2} - 1}{\frac{k-1}{2}}, \qquad \alpha = 0, \dots, n-k.$$

If we perform a change of variable $j = \alpha + \frac{k-1}{2}$, we get

$$\binom{j}{\frac{k-1}{2}} \binom{n-j-1}{\frac{k-1}{2}}, \qquad j = \frac{k-1}{2}, \dots, n - \frac{k+1}{2}.$$

Hence,

$$\boxed{S_k(B(\mathbf{0}), B_1) = \sum_{j=\frac{k-1}{2}}^{n-\frac{k+1}{2}} \binom{j}{\frac{k-1}{2}} \binom{n-j-1}{\frac{k-1}{2}} B(\mathbf{0})^{2j-n} B_2(\mathbf{0}) \qquad k = 1, 3, \dots, n-1.} \tag{4.35}$$

For later use, in (4.35), it is convenient to rewrite the term $B(\mathbf{0})^{2j-n} B_2(\mathbf{0})$ as follows. Recalling that $B(\mathbf{0})$ is given by $B(\mathbf{0}) = B_2(\mathbf{0}) B_1$, we obtain

$$B_2(\mathbf{0}) = B_2(\mathbf{0}) B_1^2 = B(\mathbf{0}) B_1$$

and hence, by means of (4.31),

$$B(\mathbf{0})^{2j-n} B_2(\mathbf{0}) = B(\mathbf{0})^{2j-n+1} B_1 = B_1 B(\mathbf{0})^{n-2j-1}. \tag{4.36}$$

For later use, we define the following sets of indices

$$\mathcal{I}' \doteq \{1, 4, 5, 8, \dots, 2N-3, 2N\}, \qquad \mathcal{I}'' \doteq \{2, 3, 6, 7 \dots, 2N-2, 2N-1\}. \tag{4.37}$$

The next proposition gives an explicit formula for the sum of the powers of $B(0)$.

Proposition 4.2.2. *Let $\widehat{P}$ be the matrix defined by*

$$\widehat{P} \doteq \frac{1}{2N} \left(e e^t + v_- v_-^t \right), \tag{4.38}$$

which is the matrix composed by $N^2/4$ squared blocks as

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Then, the following identity holds:

$$\frac{1}{N}\sum_{j=0}^{N-1} B(\mathbf{0})^{2j} = \frac{1}{N}\sum_{j=1}^{N} B(\mathbf{0})^{2j} = \widehat{P}\,. \tag{4.39}$$

Proof. The first equality in (4.39) follows from the following identity:

$$\left(I_{2N} - B(0)^2\right)\left(\sum_{j=0}^{N-1} B(0)^{2j}\right) = 0\,.$$

Indeed,

$$\begin{aligned}\left(I_{2N} - B(0)^2\right)\left(\sum_{j=0}^{N-1} B(\mathbf{0})^{2j}\right) &= \left(\sum_{j=0}^{N-1} B(\mathbf{0})^{2j}\right) - \left(\sum_{j=1}^{N} B(\mathbf{0})^{2j}\right) \\ &= I_{2N} - B(\mathbf{0})^{2N} = 0.\end{aligned}$$

To prove the second identity in (4.39), observe that the matrix $B(\mathbf{0})^2$ contains the following two separated "cycles" of length N,

$$\begin{aligned}&1 \to 5 \to 9 \to \ldots \to 2N-3 \to 2N \to 2N-4 \to \ldots \to 4 \to 1\\ &2 \to 3 \to 7 \to \ldots \to 2N-1 \to 2N-2 \to 2N-6 \to \ldots \to 6 \to 2\,.\end{aligned}$$

In the first, second case, the indexes are precisely the ones in $\mathcal{I}'$, $\mathcal{I}''$ respectively, see (4.37).

By summing all the permutations $B(\mathbf{0})^2, \ldots, B(\mathbf{0})^{2N} = I_{2N}$, one obtains that every i^{th} row, with $i \in \mathcal{I}'$, has value =1 exactly at every index $\in \mathcal{I}'$ and value $= 0$ otherwise. The same holds for every i^{th} row with $i \in \mathcal{I}''$. Hence (4.39) holds. □

The next theorem provides an estimate on the components of $B(\gamma)^{hN}$, $h \in \mathbb{N}$ in terms of d, N.

Theorem 4.2.3. *Let $N \in 2\mathbb{N}$, $h \in \mathbb{N}$ and $d \geq 0$. Then the following identity holds*

$$\left[B(\mathbf{0}) + \frac{d}{N}B_1\right]^{hN} = B(\mathbf{0})^{hN} + dh\widehat{P} + R_N(h,d) \tag{4.40}$$

where the remainder $R_N(h,d)$ is given by

$$R_N(h,d) = \sum_{j=0}^{hN-1} \zeta_{j,hN} B_1 B(\mathbf{0})^{hN-2j-1} + \sum_{j=1}^{hN-1} \eta_{j,hN} B(\mathbf{0})^{2j-hN}\,. \tag{4.41}$$

The coefficients $\zeta_{j,hN}$ and $\eta_{j,hN}$ satisfy the following estimate:

$$0 \leq \sum_{j=0}^{hN} \zeta_{j,hN} + \sum_{j=1}^{hN} \eta_{j,hN} \leq \mathrm{e}^{hd} - hd - 1 + \frac{K}{N} \tag{4.42}$$

where $K = K(h,d) \geq 0$ is independent on N, and $K(h,d) \to 0$ as $d \to 0$, for every $h \in \mathbb{N}$.

Proof. By (4.34), (4.35), and (4.36), after exchange the sum in k and j, we can write (4.29) for any n as the following

$$\begin{aligned}[B(\mathbf{0}) + \gamma B_1]^n = B(\mathbf{0})^n + \gamma \sum_{j=0}^{n-1} B_1 B(\mathbf{0})^{n-2j-1} \\ + \sum_{j=0}^{n-1} \zeta_{j,n} B_1 B(\mathbf{0})^{n-2j-1} + \sum_{j=1}^{n-1} \eta_{j,n} B(\mathbf{0})^{2j-n}\,,\end{aligned} \tag{4.43}$$

where $\gamma = \frac{d}{N}$ and

$$\zeta_{j,n} = \sum_{\ell=1}^{\min\{j,n-j-1\}} \gamma^{2\ell+1} \binom{j}{\ell} \binom{n-j-1}{\ell}\,, \tag{4.44}$$

$$\eta_{j,n} = \sum_{i=1}^{\min\{j,n-j\}} \gamma^{2i} \binom{j}{i} \binom{n-j-1}{i-1}\,. \tag{4.45}$$

In the expansion above, the term with the $\zeta_{j,n}$ accounts for the odd powers, ≥ 3, of γ while the term with the $\eta_{j,n}$ accounts for the even powers ≥ 2 of γ.

From now on, we assume that $n = hN$. We recall the identity (4.39), and some immediate identities,

$$\widehat{P}B_2(\mathbf{0}) = \widehat{P}\,, \qquad B(\mathbf{0})^2\widehat{P} = \widehat{P}B(\mathbf{0})^2 = \widehat{P}\,.$$

Therefore

$$\sum_{j=0}^{hN-1} B_1 B(\mathbf{0})^{hN-2j-1} = hB_1 \sum_{j=0}^{N-1} B(\mathbf{0})^{hN-2j-1} = hN\widehat{P}\,,$$

and the identity (4.43) rewrites as

$$\begin{aligned}[B(\mathbf{0}) + \gamma B_1]^{hN} &= B(\mathbf{0})^{hN} + hd\widehat{P} + R_N(h,d) \\ R_N(h,d) &= \sum_{j=0}^{hN-1} \zeta_{j,hN} B_1 B(\mathbf{0})^{hN-2j-1} + \sum_{j=1}^{hN-1} \eta_{j,hN} B(\mathbf{0})^{2j-hN}\,.\end{aligned}$$

To complete the proof, we need to estimate the sums of $\zeta_{j,hN}$, $\eta_{j,hN}$. We claim that

$$0 \le \sum_{j=0}^{hN} \zeta_{j,hN} \le \sinh(hd) - hd + \frac{1}{N} f_0(h,d) \tag{4.46}$$

$$0 \le \sum_{j=1}^{hN} \eta_{j,hN} \le \cosh(hd) - 1 + \frac{1}{N} f_1(h,d) \tag{4.47}$$

where

$$\begin{aligned}f_0(h,d) &\doteq \sum_{\ell=1}^{\infty} \left(\frac{h}{2}\right)^{2\ell} \frac{d^{2\ell+1}}{(\ell!)^2} = d\,[I_0(hd) - 1] \\ f_1(h,d) &\doteq \sum_{i=1}^{\infty} \left(\frac{h}{2}\right)^{2i-1} \frac{(d)^{2i}}{i!(i-1)!} = dI_1(hd)\,,\end{aligned}$$

and

$$I_\alpha(2x) = \sum_{m=0}^{\infty} \frac{x^{2m+\alpha}}{m!(m+\alpha)!}\,, \qquad \alpha = 0, 1$$

is a modified Bessel function of the first type. It is clear that, once the claim above is proved, then it follows that (4.42) holds with

$$K(h,d) = f_0(h,d) + f_1(h,d)\,. \tag{4.48}$$

We start with $\zeta_{j,hN}$ defined in (4.44). Using the inequality

$$\binom{n}{k} \leq \frac{n^k}{k!}, \qquad 0 \leq k \leq n$$

and the definition $\gamma = d/N$, we find that

$$\zeta_{j,hN} \leq \frac{1}{N} \sum_{\ell=1}^{\infty} \frac{(d)^{2\ell+1}}{(\ell!)^2} \frac{j^\ell}{N^\ell} \frac{(hN-j-1)^\ell}{N^\ell}. \tag{4.49}$$

Then we introduce the change of variable

$$x_j = \frac{j}{hN}, \qquad j = 0, \ldots, hN-1. \tag{4.50}$$

Thanks to the inequality (4.49) we get

$$\begin{aligned} 0 \leq \zeta_{j,hN} &\leq \frac{1}{hN} \sum_{\ell=1}^{\infty} \frac{(hd)^{2\ell+1}}{(\ell!)^2} x_j^\ell \left(1 - x_j - \frac{1}{hN}\right)^\ell \\ &\leq \frac{1}{hN} \sum_{\ell=1}^{\infty} \frac{(hd)^{2\ell+1}}{(\ell!)^2} x_j^\ell (1-x_j)^\ell. \end{aligned}$$

As a consequence, we deduce an estimate for the sum of the $\zeta_{j,hN}$:

$$\begin{aligned} 0 \leq \sum_{j=0}^{hN-1} \zeta_{j,hN} &\leq \frac{1}{hN} \sum_{j=0}^{hN-1} \sum_{\ell=1}^{\infty} \frac{(hd)^{2\ell+1}}{(\ell!)^2} x_j^\ell (1-x_j)^\ell \\ &= \sum_{\ell=1}^{\infty} \frac{(hd)^{2\ell+1}}{(\ell!)^2} \left\{ \frac{1}{hN} \sum_{j=0}^{hN-1} x_j^\ell (1-x_j)^\ell \right\} \end{aligned}$$

Using the definition (4.50), we observe that

$$\frac{1}{hN} \sum_{j=0}^{hN-1} x_j^\ell (1-x_j)^\ell \;\to\; \int_0^1 x_j^\ell (1-x_j)^\ell \, dx \quad \text{as } N \to \infty, \qquad \ell \geq 1;$$

more precisely the following estimate holds,

$$\sum_{j=0}^{hN-1} x_j^\ell (1-x_j)^\ell \frac{1}{hN} = \left(\sum_{j=0}^{(hN/2)-1} + \sum_{j=(hN/2)+1}^{hN-1} \right) x_j^\ell (1-x_j)^\ell \frac{1}{hN} + \frac{1}{hN}\left(\frac{1}{2}\right)^{2\ell}$$
$$\leq \int_0^1 x_j^\ell (1-x_j)^\ell \, dx + \frac{1}{hN}\left(\frac{1}{2}\right)^{2\ell}. \tag{4.51}$$

It is easy to check the following identities

$$\int_0^1 x_j^\ell (1-x_j)^\ell \, dx = \frac{(\ell!)^2}{(1+2\ell)!}, \qquad \ell \geq 1. \tag{4.52}$$

By plugging the previous estimates into the sum of the $\zeta_{j,n}$ we get

$$0 \leq \sum_{j=0}^{hN-1} \zeta_{j,n} \leq \sum_{\ell=1}^{\infty} \frac{(hd)^{2\ell+1}}{(\ell!)^2} \frac{(\ell!)^2}{(1+2\ell)!} + \frac{1}{N} \underbrace{\sum_{\ell=1}^{\infty} \left(\frac{h}{2}\right)^{2\ell} \frac{(d)^{2\ell+1}}{(\ell!)^2}}_{=f_0(h,d)}$$
$$= \sum_{\ell=1}^{\infty} \frac{(hd)^{2\ell+1}}{(1+2\ell)!} + \frac{1}{N} f_0(h,d)$$
$$= \sinh(hd) - hd + \frac{1}{N} f_0(h,d).$$

Therefore (4.46) follows.

Similarly to the estimate (4.49) for $\zeta_{j,hN}$ and using the change of variables (4.50), for $\eta_{j,hN}$ defined in (4.45) we find that

$$\eta_{j,hN} \leq \frac{1}{hN} \sum_{i=1}^{\infty} \frac{(hd)^{2i}}{i!(i-1)!} x_j^i \left(1 - x_j - \frac{1}{hN}\right)^{i-1}$$
$$\leq \frac{1}{hN} \sum_{i=1}^{\infty} \frac{(hd)^{2i}}{i!(i-1)!} x_j^i \left(1 - x_j\right)^{i-1}.$$

The sum of the $\eta_{j,hN}$ can be estimated as follows,

$$\sum_{j=1}^{hN-1} \eta_{j,hN} \leq \sum_{i=1}^{\infty} \frac{(hd)^{2i}}{i!(i-1)!} \left\{ \frac{1}{hN} \sum_{j=1}^{hN-1} x_j^i \left(1 - x_i\right)^{i-1} \right\}.$$

while by (4.51) with $\ell = i-1$ and by (4.52) we find that

$$\frac{1}{hN}\sum_{j=1}^{hN-1} x_j^i\,(1-x_j)^{i-1} \le \int_0^1 x_j^i\,(1-x_j)^{i-1}\,dx \;+\; \frac{1}{hN}\left(\frac{1}{2}\right)^{2i-1}$$
$$= \frac{(i-1)!(i)!}{(2i)!} \;+\; \frac{1}{hN}\left(\frac{1}{2}\right)^{2i-1}.$$

Therefore

$$\sum_{j=1}^{hN-1}\eta_{j,N} \le \sum_{i=1}^{\infty}\frac{(hd)^{2i}}{i!(i-1)!}\frac{(i-1)!(i)!}{(2i)!} \;+\; \frac{1}{N}\underbrace{\sum_{i=1}^{\infty}\left(\frac{h}{2}\right)^{2i-1}\frac{(d)^{2i}}{i!(i-1)!}}_{=f_1(h,d)}$$
$$= \sum_{i=1}^{\infty}\frac{(hd)^{2i}}{(2i)!} \;+\; \frac{1}{N}f_1(h,d)$$
$$= \cosh(hd) - 1 \;+\; \frac{1}{N}f_1(h,d)\,,$$

that leads to (4.47). This completes the proof of Theorem 4.2.3. □

Remark 4.2.1. *For $a \in \mathbb{R}$ and $n \ge 0$, n integer, we introduce the notation (shifted factorial, see [9, p. 2]):*

$$(a)_n = \begin{cases} 1 & n = 0 \\ a(a+1)\cdots(a+n-1) & n \ge 1\,. \end{cases} \tag{4.53}$$

With this notation we can write $(1)_n = n!$. Observe that, if a is a negative integer, then $(a)_n$ vanishes for every $n \ge |a|+1$.

Then the product of the binomial coefficients in (4.44) can be rewritten as follows,

$$\binom{j}{\ell}\binom{hN-j-1}{\ell} = \frac{1}{(\ell!)^2}(-j)_\ell\cdot(-hN+j+1)_\ell\,, \qquad \ell \ge 0\,,$$

and it is clear that the above quantity vanishes for $\ell > \min\{j, hN-j-1\}$. Therefore, starting from $\ell = 0$, the coefficients $\zeta_{j,N}$ is rewritten as

$$\zeta_{j,N} = \sum_{\ell=0}^{\infty}\frac{\gamma^{2\ell+1}}{\ell!}\frac{(-j)_\ell(-hN+j+1)_\ell}{(1)_\ell}\,. \tag{4.54}$$

Then, it can be rewritten in terms of the hypergeometric function, see [9],

$$ {}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}\,, \qquad a,b,c \in \mathbb{R}\,. $$

In conclusion we have

$$ \zeta_{j,N} = \gamma\, {}_2F_1(-j, -hN + j + 1, 1; \gamma^2)\,, \qquad \gamma = \frac{d}{N} $$

and hence, from (4.35), we obtain:

$$ \sum_{\substack{k=1 \\ k\,odd}}^{hN-1} \gamma^k S_k = \gamma \sum_{j=0}^{hN-1} {}_2F_1(-j, -hN + j + 1, 1; \gamma^2) B(0)^{2j-hN} B_2(0)\,. $$

4.3 ℓ_1 contraction

Let us define the operator $\mathcal{B}_n$ to be

$$ \mathcal{B}_n \doteq \left[B^{(n)} B^{(n-1)} \cdots B^{(2)} B^{(1)} \right], \quad B^{(n)} = B(\gamma^n)\,, \qquad n \in \mathbb{N} \tag{4.55} $$

In this section, we prove that the $\|\cdot\|_{\ell_1}$ is contractive for $\mathcal{B}_n$ with $n = N$, where

$$ \|A\|_{\ell_1} = \max_j \sum_{i=1}^{n} |a_{ij}|\,, \qquad A = (a_{ij}) \in M_n $$

is the maximum column sum matrix norm, which is induced by the ℓ_1-norm on $\mathbb{R}^n$. More precisely, we, first, write a convenient decomposition of the vectors on the subspace

$$ E_- \doteq < e, v_- >^{\perp}, \tag{4.56} $$

the notation $< e, v_- >^{\perp}$ is the perpendicular of the span of e and v_-, so that E_- is the space generated by all the eigenvectors of those eigenvalues λ such that $|\lambda| < 1$, along which a suitable cancellation occurs later on. Then, we prove a contractive estimate for the linear damping case, that is the case k, g' and α constants. Finally, we prove the same estimate for the non-linear damping.

4.3.1 A first decomposition of the strength vector

We decompose the initial vector $\sigma(0+)$ as follows:

$$\sigma(0+) = \frac{(\sigma(0+)\cdot e)}{2N}\, e + \frac{(\sigma(0+)\cdot v_-)}{2N}\, v_- + \widetilde{\sigma}(0+),$$

where e, v_- are the eigenvectors defined at (4.14) and $\widetilde{\sigma}(0+) \in E_-$.

Recalling (4.17), that is $\sigma(0+)\cdot e = 0$, hence the decomposition of $\sigma(0+)$ reduces to

$$\sigma(0+) = \frac{(\sigma(0+)\cdot v_-)}{2N}\, v_- + \widetilde{\sigma}(0+). \tag{4.57}$$

Let $\alpha(t) = const.$, consider the matrix $\mathcal{B}_n$ defined at (4.55), obtained by iterating the step (4.12). By means of (4.57) and using again (4.14) for v_-, we get that

$$\sigma(t^n+) = \mathcal{B}_n\sigma(0+) = (-1)^n \frac{(\sigma(0+)\cdot v_-)}{2N}\, v_- + \mathcal{B}_n\widetilde{\sigma}(0+)\,. \tag{4.58}$$

4.3.2 A refined decomposition of the strength vector

In this subsection, we focus on the analysis of $\|\mathcal{B}_n\widetilde{\sigma}(0+)\|_{\ell_1}$. In particular, we analyze the sequence $\{\mathcal{B}_n\widetilde{\sigma}\}_{n\in\mathbb{N}}$ whenever $\widetilde{\sigma}$ belongs to the subspace $E_- =< e, v_- >^{\perp}$.

Let $N \in 2\mathbb{N}$ and consider $\widetilde{\sigma} \in E_-$. By definition (4.14) of e, v_- then $\widetilde{\sigma}$ satisfies

$$\begin{cases} \widetilde{\sigma}_1 + \widetilde{\sigma}_2 + \cdots + \widetilde{\sigma}_{2N} = 0, \\ \widetilde{\sigma}_1 - \widetilde{\sigma}_2 - \widetilde{\sigma}_3 + \widetilde{\sigma}_4 + \widetilde{\sigma}_5 - \cdots + \widetilde{\sigma}_{2N} = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} \widetilde{\sigma}_1 + \widetilde{\sigma}_4 + \cdots + \widetilde{\sigma}_{2N-3} + \widetilde{\sigma}_{2N} = 0, \\ \widetilde{\sigma}_2 + \widetilde{\sigma}_3 + \cdots + \widetilde{\sigma}_{2N-2} + \widetilde{\sigma}_{2N-1} = 0. \end{cases}$$

Here we recall the sets $\mathcal{I}''$ and $\mathcal{I}'$ defined at (4.37),

$$\mathcal{I}' \doteq \{1,4,5,8,\ldots,2N-3,2N\}\,, \qquad \mathcal{I}'' \doteq \{2,3,6,7\ldots,2N-2,2N-1\}\,.$$

We introduce the following subspaces in $\mathbb{R}^{2N}$, each of dimension $N-1$:

$$H_1 \doteq \{(x_1,\ldots,x_{2N}) :\ x_1 + x_4 + \cdots + x_{2N-3} + x_{2N} = 0,\ x_j = 0,\ j \in \mathcal{I}''\},$$
$$H_2 \doteq \{(x_1,\ldots,x_{2N}) :\ x_2 + x_3 + \cdots + x_{2N-2} + x_{2N-1} = 0,\ x_j = 0,\ j \in \mathcal{I}'\}.$$

Hence we can write

$$\widetilde{\sigma} = \widetilde{\sigma}' + \widetilde{\sigma}'' \,, \qquad \widetilde{\sigma}' \in H_1 \,, \quad \widetilde{\sigma}'' \in H_2 \,. \tag{4.59}$$

Notice that, since H_1 and H_2 are complementary, we have

$$\|\widetilde{\sigma}\|_{\ell_1} = \|\widetilde{\sigma}' + \widetilde{\sigma}''\|_{\ell_1} = \|\widetilde{\sigma}'\|_{\ell_1} + \|\widetilde{\sigma}''\|_{\ell_1} \,. \tag{4.60}$$

Let us define the vectors $\mathbf{v}_{ij} \in E_-$ for either $i, j \in \mathcal{I}'$ or $i, j \in \mathcal{I}''$ as follows,

$$(\mathbf{v}_{ij})_i = 1 \qquad (\mathbf{v}_{ij})_j = -1 \qquad (\mathbf{v}_{ij})_k = 0 \quad \forall\, k \neq i, j \,, \quad i \neq j \,, \tag{4.61}$$

such vectors $\mathbf{v}_{ij}$ do not form basis. Remark that $\widetilde{\sigma}'$ and $\widetilde{\sigma}''$ can be written as a linear combination of suitable $\mathbf{v}_{ij}$'s, i.e. we can identify $\beta'_{ij}, \beta''_{ij} \in \mathbb{R}$ such that

$$\widetilde{\sigma}' = \sum_{i,j \in \mathcal{I}'} \beta'_{ij} \mathbf{v}_{ij} \,, \qquad \widetilde{\sigma}'' = \sum_{i,j \in \mathcal{I}''} \beta''_{ij} \mathbf{v}_{ij} \,. \tag{4.62}$$

By the triangular inequality, one has that

$$\|\widetilde{\sigma}'\|_{\ell_1} \leq \sum_{ij} |\beta'_{ij}| \, \|\mathbf{v}_{ij}\|_{\ell_1} = 2 \sum_{ij} |\beta'_{ij}| \,, \qquad \|\widetilde{\sigma}''\|_{\ell_1} \leq 2 \sum_{ij} |\beta''_{ij}| \,.$$

In the next Lemma we prove that, for a suitable choice of the decomposition, the sum above can be made an equality.

Lemma 4.3.1. *(i) There exists a choice of the vectors $\mathbf{v}_{ij}$ such that (4.62) holds together with*

$$\|\widetilde{\sigma}'\|_{\ell_1} = 2 \sum_{ij} |\beta'_{ij}|, \tag{4.63}$$

$$\|\widetilde{\sigma}''\|_{\ell_1} = 2 \sum_{ij} |\beta''_{ij}|. \tag{4.64}$$

(ii) The following estimate holds,

$$\|\mathcal{B}_n \widetilde{\sigma}\|_{\ell_1} \leq \sup_{i,j} \left\| \mathcal{B}_n \frac{\mathbf{v}_{ij}}{\|\mathbf{v}_{ij}\|_{\ell_1}} \right\|_{\ell_1} \cdot \|\widetilde{\sigma}\|_{\ell_1}, \qquad \forall\, \widetilde{\sigma} \in E_-. \tag{4.65}$$

Proof. We start with (i), it suffices to prove (4.63), since (4.64) is analogous.

First, we have to find a suitable linear decomposition of $\widetilde{\sigma}'(0+)$ in a basis of vectors of the form $\mathbf{v}_{ij}$, with $i, j \in \mathcal{I}'$. By construction we have

$$\widetilde{\sigma}' = \left(\widetilde{\sigma}'_1, 0, 0, \widetilde{\sigma}'_4, \widetilde{\sigma}'_5, 0 \ldots, 0, \widetilde{\sigma}'_{2N-3}, 0, 0, \widetilde{\sigma}'_{2N}\right),$$

i.e. the components corresponding to indices in $\mathcal{I}''$ are zero. Therefore, we can simplify the notation and in place of $\widetilde{\sigma}'$ consider

$$\mathbf{x} = (x_1, x_2, \ldots, x_N) = \left(\widetilde{\sigma}'_1, \widetilde{\sigma}'_4, \ldots, \widetilde{\sigma}'_{2N}\right) \in \mathbb{R}^N,$$

the vector obtained erasing from $\widetilde{\sigma}'$ the zero components and satisfying $x_1 + x_2 + \cdots + x_N = 0$. Below we describe an algorithm to decompose $\mathbf{x}$ along a basis of $\mathbf{v}_{ij}$'s, for $i, j \in \mathcal{I}'$.

Step 1. Let $\mathbf{x} \neq 0$. Hence there exists a pair of indices $k_1, h_1 \in \{1, \ldots, N\}$ such that

$$x_{k_1} \cdot x_{h_1} < 0\,, \qquad 0 < |x_{k_1}| = \min_{k=1,\ldots,N;\, x_k \neq 0} |x_k|\,.$$

In particular one has that $|x_{h_1}| \geq |x_{k_1}|$.

Step 2. Define the vector

$$\mathbf{x}^{(1)} \doteq \mathbf{x} - x_{k_1}\mathbf{v}_{k_1 h_1} \in \mathbb{R}^N,$$

and notice that it satisfies

$$\left(\mathbf{x}^{(1)}\right)_k = \begin{cases} 0 & k = k_1 \\ x_{h_1} + x_{k_1} & k = h_1 \\ x_k & k \neq k_1,\, h_1\,. \end{cases}$$

In particular,

$$\left|\left(\mathbf{x}^{(1)}\right)_{h_1}\right| = |x_{h_1}| - |x_{k_1}| \geq 0$$

and hence

$$\left\|\mathbf{x}^{(1)}\right\|_{\ell_1} = \|\mathbf{x}\|_{\ell_1} - 2|x_{k_1}| < \|\mathbf{x}\|_{\ell_1}.$$

Step 3. We apply the same procedure to $\mathbf{x}^{(1)}$, namely we choose suitable indexes $k_2, h_2 \in \{1, \ldots, N\}$ such that

$$\left(\mathbf{x}^{(1)}\right)_{k_2} \cdot \left(\mathbf{x}^{(1)}\right)_{h_2} < 0\,, \qquad 0 < \left|\left(\mathbf{x}^{(1)}\right)_{k_2}\right| = \min_{k=1,\ldots,N\,,\left(\mathbf{x}^{(1)}\right)_k \neq 0} \left|\left(\mathbf{x}^{(1)}\right)_k\right|\,.$$

Notice that, since $\left(\mathbf{x}^{(1)}\right)_{k_1} = 0$, one has that k_2, h_2 are different from k_1. Moreover one has $\left|\left(\mathbf{x}^{(1)}\right)_{h_2}\right| \geq \left|\left(\mathbf{x}^{(1)}\right)_{k_2}\right|$.

As in Step **2**, we define

$$\begin{aligned}\mathbf{x}^{(2)} &\doteq \mathbf{x}^{(1)} - \left(\mathbf{x}^{(1)}\right)_{k_2}\mathbf{v}_{k_2h_2} \\ &= \mathbf{x} - x_{k_1}\mathbf{v}_{k_1h_1} - \left(\mathbf{x}^{(1)}\right)_{k_2}\mathbf{v}_{k_2h_2},\end{aligned}$$

that is

$$\left(\mathbf{x}^{(2)}\right)_k = \begin{cases} 0 & k = k_2 \\ \left(\mathbf{x}^{(1)}\right)_{h_2} + \left(\mathbf{x}^{(1)}\right)_{k_2} & k = h_2 \\ \left(\mathbf{x}^{(1)}\right)_k & k \neq k_2,\, h_2\,. \end{cases}$$

Notice that

$$\left(\mathbf{x}^{(2)}\right)_k = 0 \qquad \text{for } k = k_1,\, k_2$$

and that

$$\left|\left(\mathbf{x}^{(2)}\right)_{h_2}\right| = \left|\left(\mathbf{x}^{(1)}\right)_{h_2}\right| - \left|\left(\mathbf{x}^{(1)}\right)_{k_2}\right| \geq 0\,.$$

Observe that $|x_{k_1}| + \left|\left(\mathbf{x}^{(1)}\right)_{k_2}\right| \leq |x_{k_1}| + |x_{k_2}|$ and

$$\begin{aligned}\left\|\mathbf{x}^{(2)}\right\|_{\ell_1} &= \left\|\mathbf{x}^{(1)}\right\|_{\ell_1} - 2\left|\left(\mathbf{x}^{(1)}\right)_{k_2}\right| \\ &= \|\mathbf{x}\|_{\ell_1} - 2\left(|x_{k_1}| + \left|\left(\mathbf{x}^{(1)}\right)_{k_2}\right|\right)\,.\end{aligned}$$

Step 4. Proceeding by induction, after at most $N-1$ iterations of the method we get

$$\begin{aligned}\mathbf{x}^{(N-1)} &\doteq \mathbf{x} - x_{k_1}\mathbf{v}_{k_1h_1} - \left(\mathbf{x}^{(1)}\right)_{k_2}\mathbf{v}_{k_2h_2} - \cdots - \left(\mathbf{x}^{(N-2)}\right)_{k_{N-1}}\mathbf{v}_{k_{N-1}h_{N-1}} \\ &= (0,\ldots,0) \in \mathbb{R}^N.\end{aligned}$$

Thus,

$$0 = \left\|\mathbf{x}^{(N-1)}\right\|_{\ell_1} = \|\mathbf{x}\|_{\ell_1} - 2\left(\sum_{i=1}^{N-1} |x_{k_i}|\right). \tag{4.66}$$

and hence

$$\|\mathbf{x}\|_{\ell_1} = \frac{1}{2}\sum_{i=1}^{N-1} |x_{k_i}|\,.$$

Since we can write that $\sum_{i=1}^{N-1} |x_{k_i}| = \sum_{ij} |\beta'_{ij}|$, then the proof of (4.63) is complete.

Proof of (ii) By using (4.59), we have

$$\mathcal{B}_n\widetilde{\sigma} = \mathcal{B}_n\widetilde{\sigma}' + \mathcal{B}_n\widetilde{\sigma}''\,.$$

By means of (4.62) and (i) we find that

$$\begin{aligned}\left\|\mathcal{B}_n\widetilde{\sigma}\right\|_{\ell_1} &\le \left\|\mathcal{B}_n\widetilde{\sigma}'\right\|_{\ell_1} + \left\|\mathcal{B}_n\widetilde{\sigma}''\right\|_{\ell_1}\\ &\le \sum_{\mathcal{J}'}|\beta'_{ij}|\left\|\mathcal{B}_n\mathbf{v}_{ij}\right\|_{\ell_1} + \sum_{\mathcal{J}''}|\beta''_{ij}|\left\|\mathcal{B}_n\mathbf{v}_{ij}\right\|_{\ell_1}\\ &\le \left(\sum_{\mathcal{J}'}|\beta'_{ij}| + \sum_{\mathcal{J}''}|\beta''_{ij}|\right)\sup_{i,j}\left\|\mathcal{B}_n\mathbf{v}_{ij}\right\|_{\ell_1}\\ &\le \frac{1}{2}\left(\left\|\widetilde{\sigma}'\right\|_{\ell_1} + \left\|\widetilde{\sigma}''\right\|_{\ell_1}\right)\sup_{i,j}\left\|\mathcal{B}_n\mathbf{v}_{ij}\right\|_{\ell_1}.\end{aligned}$$

As $\|\mathbf{v}_{ij}\|_{\ell_1} = 2$ and by using (4.60), the proof of (4.65) is complete. □

4.3.3 ℓ_1 contraction: Linear damping

Next, we want to prove a contractive estimate for $\|B(\gamma)^N\mathbf{v}_{ij}\|_{\ell_1}$. Then, thanks to (4.65), the contractive estimate holds also for every $\widetilde{\sigma} \in E_-$. We recall that here $\gamma = \gamma(1,\dots,1) \in \mathbb{R}^{N-1}$ with $\gamma = d/N$ for some $d > 0$.

Theorem 4.3.2. *For $N \in 2\mathbb{N}$. For every $d > 0$ and $\widetilde{\sigma} \in E_-$ there is a constant $C_N(d) > 0$ such that*

$$\left\|B(\gamma)^N\widetilde{\sigma}\right\|_{\ell_1} \le C_N(d)\left\|\widetilde{\sigma}\right\|_{\ell_1}, \tag{4.67}$$

where $\widetilde{\sigma} \in E_-$ and

$$C_N(d) \to (1 - de^{-d}) < 1\,, \qquad N \to \infty\,. \tag{4.68}$$

Proof. Thanks to Lemma 4.3.1, especially estimate (4.65), that is

$$\left\|B_N\widetilde{\sigma}\right\|_{\ell_1} \le \sup_{i,j}\left\|B_N\frac{\mathbf{v}_{ij}}{\|\mathbf{v}_{ij}\|_{\ell_1}}\right\|_{\ell_1}\cdot\left\|\widetilde{\sigma}\right\|_{\ell_1}, \qquad \forall\widetilde{\sigma} \in E_-,$$

it is then sufficient to study the behavior of

$$\sup_{i,j}\left\|B_N\frac{\mathbf{v}_{ij}}{\|\mathbf{v}_{ij}\|_{\ell_1}}\right\|_{\ell_1}$$

as $n \to \infty$ for every $\mathbf{v}_{ij}$, as defined in (4.61), with either $i, j \in \mathcal{I}'$ or $i, j \in \mathcal{I}''$.

Notice that

$$B(\gamma)^N \mathbf{v}_{ij} = B(\gamma)^N e_i - B(\gamma)^N e_j = B(\gamma)^N[i] - B(\gamma)^N[j],$$

where e_i, e_j are vectors of the canonical basis of $\mathbb{R}^{2N}$ and $B(\gamma)^N[i], B(\gamma)^N[j]$ denote the i-th and j-th column of the matrix $B(\gamma)^N$. Hence, $\|B(\gamma)^N \mathbf{v}_{ij}\|_{\ell_1}$ corresponds to the distance between two columns of $B(\gamma)^N$ indicized by either $i, j \in \mathcal{I}'$ or $\in \mathcal{I}''$.

Assume that $i, j \in \mathcal{I}'$, the other case being completely similar. We use the expression (4.27) for $B(\gamma)^N$ and Theorem 4.2.3 to get

$$\left\| B(\gamma)^N[i] - B(\gamma)^N[j] \right\|_{\ell_1} = \left(1 + \frac{d}{N}\right)^{-N} \sum_{\ell=1}^{2N} |b_{\ell i} - b_{\ell j}|,$$

where $b_{\ell i}$ denotes the generic element of the matrix $[B(0) + \gamma B_1]^N$ and where $b_{\ell i}$, $b_{\ell j} = 0$ if $\ell \notin \mathcal{I}'$.

Then a key observation is that, the contribution in formula (4.40) from the term $\widehat{P}$ is zero because

$$\widehat{P}[i] - \widehat{P}[j] = 0 \in \mathbb{R}^{2N}, \qquad i, j \in \mathcal{I}'.$$

The same property holds if $i, j \in \mathcal{I}''$. The matrix $B(0)^N$ defined at (4.23) has $b_{2N+1-i,i} = 1$ and 0 otherwise. Therefore

$$\begin{aligned}
\sum_{\ell=1}^{2N} |b_{\ell i} - b_{\ell j}| &\le |b_{2N+1-i,i} - b_{ij}| + |b_{ji} - b_{2N+1-j,j}| + \sum_{\ell \neq i,j}^{2N} |b_{\ell i} - b_{\ell j}| \\
&\le 2 \left(1 + \sum_{j=0}^{N-1} \zeta_{j,N} + \sum_{j=1}^{N-1} \eta_{j,N}\right) \\
&\le \|\mathbf{v}_{ij}\|_{\ell_1} \left[e^d - d + \frac{K}{N}\right],
\end{aligned}$$

where $K = K(d) > 0$ is a constant independent of N, and the last inequality holds by the estimate (4.42). By denoting

$$C_N(d) \doteq \left(1 + \frac{d}{N}\right)^{-N} \left[e^d - d + \frac{K}{N}\right], \tag{4.69}$$

we easily get that $C_N(d) \to (1 - d e^{-d})$ as $N \to \infty$. This completes the proof of Theorem 4.3.2. □

The formula (4.67) indicates that, as $N \to \infty$, the matrix norm induced by the

ℓ_1–norm is asymptotically contractive for the power $B(\gamma)^N$ on the subspace E_-. Of course, for d and N fixed, the sequence of matrices $B(\gamma)^n$ will converge to zero as $n \to \infty$ on the subspace E_- (that is, every vector $B(\gamma)^n\sigma$ with $\sigma \in E_-$), see Gelfand Theorem A.1.5. Hence, every matrix norm will become contractive after a sufficiently large number n of iterations.

4.3.4 ℓ_1 contraction: Nonlinear damping

In this subsection, we prove Theorem 4.3.2 for the case of non-linear damping. We will assume that the damping term is time-independent, with $\alpha(t) \equiv 1$, and that (1.11), (1.12) hold. Let us define d_1, d_2 as in (1.13), that is

$$d_1 = k_1 \min_{J\in D_J} g'(J) > 0\,, \qquad d_2 = k_2 \max_{J\in D_J} g'(J)\,.$$

Lemma 4.3.3. *Under the assumptions of Theorem 1.2.2, the following entrywise inequality holds:*

$$B(\gamma^n) \le \left(1+\frac{d_1}{N}\right)^{-1}\left[B(\mathbf{0}) + \frac{d_2}{N}B_1\right] \qquad \forall\, n \ge 1\,. \tag{4.70}$$

Proof. Recalling the definitions (3.27), (4.5) for δ_j and γ_j^n respectively, the following bounds hold for every j and n:

$$\frac{k_1}{N} \le \delta_j \le \frac{k_2}{N}\,, \qquad \frac{d_1}{N} \le \gamma_j^n \le \frac{d_2}{N}\,.$$

Now, recalling (4.9), the 2×2 matrices $\hat{A}_j^n$ satisfy the following entrywise bounds,

$$\hat{A}_j^n = \frac{1}{1+\gamma_j^n}\left\{\begin{bmatrix}0&1\\1&0\end{bmatrix} + \gamma_j^n\begin{bmatrix}1&0\\0&1\end{bmatrix}\right\} \le \left(1+\frac{d_1}{N}\right)^{-1}\left\{\begin{bmatrix}0&1\\1&0\end{bmatrix} + \frac{d_2}{N}\begin{bmatrix}1&0\\0&1\end{bmatrix}\right\}$$

that are independent of n, j. By using again (4.9) and the definition (4.11) of $B(\gamma^n)$, we get (4.70). □

Recall the definition of the operator $\mathcal{B}_n$ defined in (4.55), that is

$$\mathcal{B}_n \doteq \left[B^{(n)}B^{(n-1)}\cdots B^{(2)}B^{(1)}\right]\,, \quad B^{(n)} = B(\gamma^n)\,, \qquad n\in\mathbb{N}$$

Since the bound in (4.70) is independent of time, we have

$$\mathcal{B}_n \le \left(1+\frac{d_1}{N}\right)^{-n}\left[B(\mathbf{0}) + \frac{d_2}{N}B_1\right]^n\,. \tag{4.71}$$

Proposition 4.3.4. *There exists a constant $C_N(d_1, d_2)$ such that as $N \to \infty$*

$$C_N(d_1, d_2) \to \mathrm{e}^{-d_1}(\mathrm{e}^{d_2} - d_2) \doteq C(d_1, d_2) \tag{4.72}$$

and

$$\left\|\mathcal{B}_N \widetilde{\sigma}\right\|_{\ell_1} \leq C_N(d_1, d_2) \left\|\widetilde{\sigma}\right\|_{\ell_1}.$$

In particular, if d_1 and d_2 satisfy (1.14), then $C_N(d_1, d_2) < 1$ for N large enough.

Proof. The term $\left[B(0) + \frac{d_2}{N} B_1\right]^N$ in (4.71) can be estimated as in the proof of Theorem 4.2.3. Then, as in the proof of Theorem 4.3.2, the conclusion follows easily with

$$C_N(d_1, d_2) \doteq \left(1 + \frac{d_1}{N}\right)^{-N} \left[\mathrm{e}^{d_2} - d_2 + \frac{K(d_1, d_2)}{N}\right].$$

□

Chapter 5

Long time behavior of the approximate solutions

This chapter is devoted to proving Theorem 1.2.2. In Section 5.1, we provide a pointwise representation of the approximate solutions. In Section 5.2, Theorem 1.2.2 is proved for BV initial data using the ℓ^1 contraction property and all the results of the previous chapters. In Section 5.3, we study the long time behavior of the approximate solutions with L^∞ initial data. More precisely, we consider the linear damping case and we prove partial results by showing a contraction property of the invariant domain $[m, M]^2$ for the approximate solutions, see Theorem 5.3.4. Section 5.4 is devoted to studying the localized damping case, where we provide an estimate for the eigenvalues of the matrix $B(\gamma)$ using Rayleigh quotient Theorem.

5.1 A representation formula for ρ and J

In this section, we provide a pointwise representation of $\rho(x,t)$, $J(x,t)$ by means of the vectorial quantity $\sigma(t)$. It is based on the key properties (4.2) and $(3.9)_2$, that we recall here for convenience: for y_j given in (3.42),

$$\begin{cases} \sigma_j = \Delta J(y_j) = \Delta\rho(y_j)\dot{y}_j & x = y_j(t)\,, \\ \Delta\rho(x_j) = -2g(J(x_j))\delta_j & x = x_j = j\Delta x \end{cases} \qquad j = 1, \dots, 2N \tag{5.1}$$

Therefore we can reconstruct the functions $x \to \rho(x,t)$ and $x \to J(x,t)$ as stated in the following Proposition. We define

$$\boldsymbol{v}_0 = \mathbf{0}_{2N}\,, \qquad \boldsymbol{v}_\ell = (\underbrace{1, \cdots, 1}_{\ell}, 0, \cdots, 0) \in \mathbb{R}^{2N}, \quad \ell = 1, \cdots, 2N \tag{5.2}$$

and

$$H = \left\{ \boldsymbol{v}_\ell \in \mathbb{R}^{2N}, \quad \ell = 0, \cdots, 2N \right\} . \tag{5.3}$$

Lemma 5.1.1. *(Representation formula for ρ, J, $f^\pm$)*
For every (x,t) with $x \neq y_j(t)$ and $t \in (t^n, t^{n+1})$, the following holds.

1. *There exists $\boldsymbol{v} = \boldsymbol{v}(x) \in H$ such that*

$$J(x,t) = \boldsymbol{\sigma}(t) \cdot \boldsymbol{v}(x) . \tag{5.4}$$

In particular

$$\boldsymbol{v}(x_j) = \boldsymbol{v}_{2j} , \qquad j = 0, \dots, N . \tag{5.5}$$

2. *If moreover $x \neq x_j$, then*

$$\rho(x,t) = \widehat{\boldsymbol{\sigma}}(t) \cdot \boldsymbol{v}(x) + \rho(0+,t) - 2\bar{\alpha}_n \sum_{j:\ x_j < x} g(J(x_j,t))\delta_j , \tag{5.6}$$

with

$$\widehat{\boldsymbol{\sigma}}(t) = \pm \Pi \boldsymbol{\sigma}(t) = \begin{cases} \Pi\boldsymbol{\sigma} & t \in \left(t^n, t^{n+1/2}\right) \\ -\Pi\boldsymbol{\sigma}(t) & t \in \left(t^{n+1/2}, t^{n+1}\right) \end{cases} \tag{5.7}$$

and

$$\Pi = \mathrm{diag}(1, -1, 1, -1, \dots, 1, -1) \in M_{2N} . \tag{5.8}$$

3. *Finally, for $j = 0, \dots, N-1$ one has that*

$$f^\pm(x_j+,t) = \boldsymbol{\sigma}(t) \cdot \boldsymbol{v}_{2j}^\pm + \frac{1}{2}\rho(0+,t) - \bar{\alpha}_n \sum_{0 \le \ell \le j} g(J(x_\ell,t))\delta_\ell \tag{5.9}$$

where

$$\begin{aligned} \boldsymbol{v}_{2j}^+ &= \frac{1}{2}\left(\Pi + I_{2N}\right)\boldsymbol{v}_{2j} = (\underbrace{1,0,\dots,1,0}_{2j},0,0,\dots,0,0) \\ \boldsymbol{v}_{2j}^- &= \frac{1}{2}\left(\Pi - I_{2N}\right)\boldsymbol{v}_{2j} = -(\underbrace{0,1,\dots,0,1}_{2j},0,0,\dots,0,0) . \end{aligned} \tag{5.10}$$

Proof. (1) About (5.4), it is enough to observe that

$$J(x,t) = \underbrace{J(0+,t)}_{=0} + \sum_{y_\ell(t) < x} \Delta J(y_\ell) = \sum_{y_\ell < x} \sigma_\ell(t) .$$

Hence

$$J(x,t) = \sigma(t) \cdot v_{\bar{\ell}}$$

with $\bar{\ell} \in \{0, 1, \dots, 2N-1\}$ such that

$$y_{\bar{\ell}} < x < y_{\bar{\ell}+1}\,. \tag{5.11}$$

In particular, if $x_j = j\Delta x$, then

$$J(x_j,t) = \underbrace{J(0+,t)}_{=0} + \sum_{y_\ell(t)<x_j} \Delta J(y_\ell) = \sum_{\ell=1}^{2j} \sigma_\ell(t) = \sigma(t) \cdot v_{2j}\,.$$

Hence (5.5) is proved.

(2) To prove (5.6), let's write $\rho(x,t)$ for $x \neq x_j$ and $x \neq y_\ell$ as follows:

$$\rho(x,t) = \rho(0+,t) + \underbrace{\sum_{y_\ell<x} \Delta\rho(y_\ell,t)}_{(a)} + \underbrace{\sum_{x_j<x} \Delta\rho(x_j,t)}_{(b)}\,.$$

Indeed, differently from J, the component ρ varies also along the 0-waves. About (a), by recalling the first relation in (5.1), we get

$$\sum_{y_\ell<x} \Delta\rho(y_\ell,t) = \sum_{y_\ell<x} \sigma_\ell\, \dot{y}_\ell\,.$$

Now, notice that (see Figure 3.5)

$$\dot{y}_j(t) = \begin{cases} 1 & j \text{ odd} \\ -1 & j \text{ even} \end{cases} \qquad t \in \left(t^n, t^n + \frac{\Delta t}{2}\right)$$

as well as

$$\dot{y}_j(t) = \begin{cases} -1 & j \text{ odd} \\ 1 & j \text{ even} \end{cases} \qquad t \in \left(t^n + \frac{\Delta t}{2}, t^{n+1}\right)\,.$$

Therefore (a) is of the form

$$\sum_{y_\ell<x} \Delta\rho(y_\ell,t) = \widehat{\sigma}(t) \cdot v_{\bar{\ell}}\,.$$

Concerning (b), since $\Delta\rho(x_j) = -2g(J(x_j))\delta_j$ we immediately get

$$\sum_{x_j<x} \Delta\rho(x_j,t) = -2\bar{\alpha}_n \sum_{x_j<x} g(J(x_j,t))\delta_j\,.$$

Therefore the proof of (5.6) is complete.

(3) Finally, about (5.9), we use the relation $f^{\pm} = \frac{\rho \pm J}{2}$ to get

$$f^{\pm}(x_j+,t) = \frac{\widehat{\sigma}(t) \pm \sigma(t)}{2} \cdot v(x_j) + \frac{1}{2}\rho(0+,t) - \bar{\alpha}_n \sum_{0\le\ell\le j} g(J(x_\ell,t))\delta_\ell\,.$$

We rewrite the first term as follows,

$$\begin{aligned}\frac{\widehat{\sigma}(t) \pm \sigma(t)}{2} \cdot v(x_j) &= \frac{1}{2}\left(\Pi \pm I_{2N}\right)\sigma(t)\cdot v(x_j)\\ &= \sigma(t)\cdot \underbrace{\frac{1}{2}\left(\Pi \pm I_{2N}\right) v_{2j}}_{=v_{2j}^{\pm}}\end{aligned}$$

where we used (5.5) and the fact that the matrices $\Pi \pm I_{2N}$,

$$\begin{aligned}\frac{1}{2}\left(\Pi + I_{2N}\right) &= \mathrm{diag}(1,0,1,0,\dots,1,0)\,,\\ \frac{1}{2}\left(\Pi - I_{2N}\right) &= -\mathrm{diag}(0,1,0,1,\dots,0,1)\end{aligned}$$

are symmetric. The proof of (5.9) is complete. □

Remark 5.1.1. *Here is a list of remarks about the representation formulas in Lemma 5.1.1.*

(a) The value of $\rho(0+,t)$ in (5.6) is determined by the conservation of mass identity: $\int_I \rho^{\Delta x}(x,t)\,dx = \int_I \rho^{\Delta x}(x,0)\,dx$.

(b) By the definitions (5.10), (4.4) of v_{2j}^{+} and B_1, respectively, it is immediate to find that

$$B_1 v_{2j}^{\pm} = -v_{2j}^{\mp}\,. \tag{5.12}$$

(c) *The last term in* (5.9)*, which is related to the variation of* $f^{\pm}$ *across the point sources* x_j*, can be also conveniently expressed as a scalar product with* $\boldsymbol{v}_{2j}^{\pm}$*. Indeed, if we define*

$$\begin{aligned}\widehat{p}_j(t) &= g(J(x_j,t))\delta_j\\ \widehat{\boldsymbol{G}}(t) &= (0,-\widehat{p}_1,\widehat{p}_1,\dots,-\widehat{p}_{N-1},\widehat{p}_{N-1},0)^t\end{aligned}$$

then it is immediate to verify the following identity holds:

$$\sum_{0\le\ell\le j} g(J(x_\ell,t))\delta_\ell = \widehat{\boldsymbol{G}}(t)\cdot\boldsymbol{v}_{2j}^{-} = \widehat{\boldsymbol{G}}(t)\cdot\boldsymbol{v}_{2j+2}^{+}\,. \tag{5.13}$$

Notice the similarity between $\widehat{\boldsymbol{G}}$*, for time* $t = t^n-$*, and the* vector source term $\boldsymbol{G}_n$ *defined at* (4.10)*. In general, the map* $t\mapsto\widehat{\boldsymbol{G}}(t)$ *is nonlinear with respect to* $\boldsymbol{\sigma}(t)$ *because of the nonlinearity of* $J\mapsto g(J)$*.*

5.2 Long time behavior of the solutions: BV initial data

In this section, we combine the results of the previous chapters to prove the main Theorem 1.2.2 with initial data $(\rho_0, J_0)\in BV(I)$.

- First, we prove Proposition 5.2.1 that relates the L^∞-norm of the approximate solutions $J(\cdot,t^n)$, $\rho(\cdot,t^n)$ as $n\to\infty$ to the evolution of the ℓ_1–norm of the operator $\mathcal{B}_n$ (4.55), that is

$$\mathcal{B}_n \doteq \left[B^{(n)}B^{(n-1)}\cdots B^{(2)}B^{(1)}\right]\,,\quad B^{(n)} = B(\boldsymbol{\gamma}^n)\,,\qquad n\in\mathbb{N}$$

  on the eigenspace $E_-\doteq <e, v_->^{\perp}$.

- Then, we employ Theorem 4.2.3, Lemma 4.3.1, Proposition 4.3.4, and Proposition 5.2.1 to prove the decay in L^∞ of the approximate solutions.

- Finally, by the proof of the strong convergence of the approximate solution towards the exact solution of the problem in Section 3.3, we conclude the L^∞ estimate in (1.15) for the exact solution of the problem (1.2)-(1.3) with initial data $(\rho_0, J_0)\in BV(I)$.

Recall the decomposition of $\boldsymbol{\sigma}(t^n+)$ (4.58), that is

$$\boldsymbol{\sigma}(t^n+) = \mathcal{B}_n\boldsymbol{\sigma}(0+) = (-1)^n\frac{(\boldsymbol{\sigma}(0+)\cdot v_-)}{2N}\,v_- + \mathcal{B}_n\widetilde{\boldsymbol{\sigma}}(0+)\,. \tag{5.14}$$

In the following proposition, we employ (5.14) to obtain L^∞-bounds on $J = J^{\Delta x}$, $\rho = \rho^{\Delta x}$.

Proposition 5.2.1. *For every $t \in (t^n, t^{n+1})$ one has*

$$\|J(\cdot,t)\|_\infty \le \frac{1}{2N}\mathrm{TV}\,\bar{J}_0 + \|\mathcal{B}_n\widetilde{\sigma}(0+)\|_{\ell^1} \tag{5.15}$$

$$\|\rho(\cdot,t)\|_\infty \le \frac{2}{N}(1+C_1\|k\|_{L^1})\,\mathrm{TV}\,\bar{J}_0 + \\ + 2(1+2C_1\|k\|_{L^1})\|\mathcal{B}_n\widetilde{\sigma}(0+)\|_{\ell^1} + \frac{1}{N}\mathrm{TV}\,\rho_0\,. \tag{5.16}$$

Proof. **Proof of (5.15).** Recall the representation formula (5.4) for $J(x,t^n+)$, that is:

$$J(x,t^n+) = \sigma(t^n+)\cdot v$$

where $v \in H$. By (5.14) we obtain

$$\sigma(t^n+)\cdot v = (-1)^n\frac{1}{2N}(\sigma(0+)\cdot v_-)(v_-\cdot v) + \mathcal{B}_n\widetilde{\sigma}(0+)\cdot v\,. \tag{5.17}$$

Recalling the definition of (4.14), observe that $v_-\cdot v \in \{\pm 1, 0\}$ and hence

$$\begin{aligned}|J(x,t^n+)| &= |\sigma(t^n+)\cdot v| \\ &\le \frac{1}{2N}|\sigma(0+)\cdot v_-| + |\mathcal{B}_n\widetilde{\sigma}(0+)\cdot v| \\ &\le \frac{1}{2N}\mathrm{TV}\,\bar{J}_0 + \|\mathcal{B}_n\widetilde{\sigma}(0+)\|_{\ell_1}\end{aligned}$$

where (4.19) is used for $|\sigma(0+)\cdot v_-|$ and an $\ell_1-\ell_\infty$ estimate is used for $\mathcal{B}_n\widetilde{\sigma}(0+)\cdot v$.

To complete the proof of (5.15), it remains to bound the values of J at times $t \in (t^n+\Delta t/2, t^{n+1})$, since it may change due to the linear interaction of the waves. Recalling (4.4), we have

$$\sigma(t^{n+1}-) = B_1\sigma(t^n+) = (-1)^n\frac{1}{2N}(\sigma(0+)\cdot v_-)B_1v_- + B_1\mathcal{B}_n\widetilde{\sigma}(0+)$$

with $B_1v_- = -v_-$. By proceeding as before, we obtain

$$\begin{aligned}|J(x,t^{n+1}-)| = |\sigma(t^{n+1}-)\cdot v| &\le \frac{1}{2N}\mathrm{TV}\,\bar{J}_0 + \|B_1\mathcal{B}_n\widetilde{\sigma}(0+)\|_{\ell_1} \\ &\le \frac{1}{2N}\mathrm{TV}\,\bar{J}_0 + \|\mathcal{B}_n\widetilde{\sigma}(0+)\|_{\ell_1}\,,\end{aligned}$$

where it is used that multiplication by B_1 leaves unaltered the ℓ_1 norm (being a permutation matrix). Therefore, (5.15) is completely proved.

Proof of (5.16). For $x \neq x_j$ and $x \neq y_\ell$, by (5.6), for $t \in (t^n, t^n + \Delta t/2)$ we have

$$\rho(x,t) = \widehat{\sigma}(t) \cdot \boldsymbol{v} + \rho(0+,t) - 2 \sum_{x_j<x} g(J(x_j,t))\delta_j \,,$$

where $\widehat{\sigma}(t)$ is defined at (5.7).

Recalling (3.30), we have

$$\left| \int_0^1 \rho(x,t^n+)\,dx \right| = \left| \int_0^1 \rho(x,0)\,dx \right| \leq \Delta x \mathrm{TV}\,\rho_0 \,,$$

then

$$\begin{aligned} |\rho(0+,t^n+)| &\leq \left| \int_0^1 [\rho(0+,t^n+) - \rho(x,t^n+)]\,dx \right| \;+\; \Delta x \mathrm{TV}\,\rho_0 \\ &\leq \sup_x \left| \sum_{y_\ell<x} \Delta\rho(y_\ell,t^n+) \right| + \sup_x \left| \sum_{x_j<x} \Delta\rho(x_j,t^n+) \right| \;+\; \Delta x \mathrm{TV}\,\rho_0 \\ &= |\widehat{\sigma}(t^n+)\cdot\boldsymbol{v}| + 2 \left| \sum_{x_j<x} g(J(x_j,t))\delta_j \right| \;+\; \Delta x \mathrm{TV}\,\rho_0 \,, \end{aligned}$$

and hence

$$|\rho(x,t^n+)| \leq 2\,|\widehat{\sigma}(t^n+)\cdot\boldsymbol{v}| + 4 \left| \sum_{x_j<x} g(J(x_j,t))\delta_j \right| \;+\; \Delta x \mathrm{TV}\,\rho_0 \,. \tag{5.18}$$

- To estimate $\widehat{\sigma}(t^n+)\cdot\boldsymbol{v}$, we use (5.7) and then we proceed similarly to (5.17):

$$\begin{aligned} \widehat{\sigma}(t^n+)\cdot\boldsymbol{v} &= \Pi\sigma(t^n+)\cdot\boldsymbol{v} \\ &= (-1)^n \frac{1}{2N} (\sigma(0+)\cdot v_-)(\Pi v_-\cdot\boldsymbol{v}) + \Pi\mathcal{B}_n\widetilde{\sigma}(0+)\cdot\boldsymbol{v} \,, \end{aligned}$$

where $\Pi v_- = (1,1,-1,-1,\cdots,1,1,-1,-1)$, hence $|\Pi v_-\cdot\boldsymbol{v}| \leq 2$. Then, by using (4.19) and the fact that $\|\Pi\|_{\ell_1} = 1$, we get:

$$\begin{aligned} |\widehat{\sigma}(t^n+)\cdot\boldsymbol{v}| &\leq \frac{1}{N}|\sigma(0+)\cdot v_-| + \|\mathcal{B}_n\widetilde{\sigma}(0+)\|_{\ell_1} \\ &\leq \frac{1}{N}\mathrm{TV}\,\bar{J}_0 + \|\mathcal{B}_n\widetilde{\sigma}(0+)\|_{\ell_1} \,. \end{aligned}$$

- To estimate $\sum_{x_j<x} g(J(x_j,t))\delta_j$, we have

$$\left|\sum_{x_j<x} g(J(x_j,t^n+))\delta_j\right| \le C_1 \max_j |J(x_j,t^n+)| \cdot \left(\sum_{j=1}^{N-1}\delta_j\right)$$
$$\le C_1\|k\|_{L^1}\left(\frac{1}{2N}\mathrm{TV}\,\bar{J}_0 + \|\mathcal{B}_n\widetilde{\sigma}(0+)\|_{\ell_1}\right).$$

In conclusion, for every $x\in(0,1)$ we find that

$$|\rho(x,t^n+)| \le 2\Delta x\,(1+C_1\|k\|_{L^1})\,\mathrm{TV}\,\bar{J}_0 + 2\,(1+2C_1\|k\|_{L^1})\,\|\mathcal{B}_n\widetilde{\sigma}(0+)\|_{\ell_1} \;+\; \Delta x\mathrm{TV}\,\rho_0$$

which is (5.16) for $t\in(t^n,t^n+\Delta t/2)$. The estimate for $t\in(t^n+\Delta t/2,t^{n+1})$ is done similarly as the one for J. □

Remark 5.2.1. *(*On the total variation of J*). We remark that the total variation of $J^{\Delta x}$, being*

$$\mathrm{TV}\,J^{\Delta x}(\cdot,t) = \|\sigma(t)\|_{\ell_1},$$

does not necessarily vanish at $t\to\infty$. Indeed, from (4.58) it follows that

$$\|\sigma(t^n+)\|_{\ell_1} \ge \frac{1}{2N}|\sigma(0+)\cdot v_-|\,\|v_-\|_{\ell_1} - \|\mathcal{B}_n\widetilde{\sigma}(0+)\|_{\ell_1}$$
$$= |\sigma(0+)\cdot v_-| - \|\mathcal{B}_n\widetilde{\sigma}(0+)\|_{\ell_1}$$

where it is used that $\|v_-\|_{\ell_1}=2N$ (see the definition of v_- at (4.14)). By means of (4.21), and using the notation

$$J_\ell = J(x_{\ell-1}+,0) = J(x_\ell-,0) = J_0(x_{\ell-1}+) \qquad \ell=1,\dots,N$$

we have

$$|\sigma(0+)\cdot v_-| = \left|J_1 - J_N + \sum_{\ell=1}^{N-1}(-1)^\ell(J_{\ell+1}-J_\ell)\right|$$
$$= 2\left|J_1 - J_N + \sum_{\ell=2}^{N-1}(-1)^{\ell-1}J_\ell\right| = 2\left|\sum_{\ell=1}^{N/2}(J_{2\ell-1}-J_{2\ell})\right|.$$

If the initial datum $J_0(x)$ is strictly monotone, then

$$|\sigma(0+)\cdot v_-| = 2\,|J_N - J_1| \;\to\; 2\,|J_0(1-) - J_0(0+)| = 2\mathrm{TV}\,J_0 > 0\,, \quad N\to\infty\,.$$

About the second term in the sum, when γ is constant in time we have $\mathcal{B}_n = B(\gamma)^n$ and

$$\|B^n \widetilde{\sigma}(0+)\|_{\ell_1} \to 0 \qquad \text{as } n \to +\infty$$

since $\widetilde{\sigma}(0+)$ belongs to the subspace $E_- =< e, v_- >^{\perp}$ corresponding to the eigenvalues with modulus < 1. Therefore $\mathrm{TV}\, J(\cdot, t)$ does not tend to zero as $t \to +\infty$ for J_0 strictly monotone, and the limit is uniformly positive as $\Delta x = 1/N \to 0$. However, in (5.20), it will turn out that the L^∞-norm of J is of order Δx for large t.

Proof of Theorem 1.2.2. To prove (1.15) in Theorem 1.2.2, we employ Theorem 4.2.3, Lemma 4.3.1, Proposition 4.3.4, and Proposition 5.2.1 together with Helly's theorem. About the estimate for J, we proceed as follows:

- We start from (5.15), that is

$$\|J^{\Delta x}(\cdot, t)\|_\infty \leq \frac{1}{2N} \mathrm{TV}\, \bar{J}_0 + \|\mathcal{B}_n \widetilde{\sigma}(0+)\|_{\ell^1}\,.$$

- Let $n \in \mathbb{N}$, $0 \leq h \in \mathbb{N}$ and $Nh \leq n < N(h+1)$, so that

$$h \leq \frac{n}{N} = n\Delta t = t^n < (h+1)\,, \qquad h \geq 0\,. \tag{5.19}$$

  Since E_- is an invariant subspace for all $B^{(n)}$, we have

$$\widetilde{\sigma}(t^n) = \mathcal{B}_n \widetilde{\sigma}(0+) \in E_- \quad \forall\, n\,.$$

  Hence by Proposition 4.3.4 and using that $\left\|B^{(n)} v\right\|_{\ell_1} \leq \left\|v\right\|_{\ell_1}$ for all $v \in \mathbb{R}^{2N}$, the following holds

$$\begin{aligned}
\left\|\widetilde{\sigma}(t^n)\right\|_{\ell_1} = \left\|\mathcal{B}_n \widetilde{\sigma}(0+)\right\|_{\ell_1} &\leq \left\|\mathcal{B}_{Nh} \widetilde{\sigma}(0+)\right\|_{\ell_1} \\
&= \left\|\mathcal{B}_N \left(\mathcal{B}_{N(h-1)} \widetilde{\sigma}(0+)\right)\right\|_{\ell_1} \\
&\leq C_N \left\|\mathcal{B}_{N(h-1)} \widetilde{\sigma}(0+)\right\|_{\ell_1} \\
&\leq C_N^h \left\|\widetilde{\sigma}(0+)\right\|_{\ell_1}\,.
\end{aligned}$$

  Let $\delta > 0$ satisfy $[C - \delta, C + \delta] \subset (0, 1)$, and choose N large enough so that

$C_N(d_1, d_2) \in [C - \delta, C + \delta]$. One can easily get

$$\begin{aligned} |C_N(d_1, d_2) - C(d_1, d_2)| &\leq \frac{1}{N}\left(1 + \frac{d_1}{N}\right)^{-N} [f_0(d_2) + f_1(d_2)] \\ &\quad + \left(\mathrm{e}^{d_2} - d_2\right) \cdot \mathrm{e}^{-d_1} \left(\left(1 + \frac{d_1}{N}\right) - 1\right) \\ &\leq \frac{1}{N} \hat{C}(d_1, d_2)\,, \end{aligned}$$

for a suitable constant $\hat{C}(d_1, d_2) > 0$. Therefore, one has

$$\left|C_N^h - C^h\right| \leq |C_N - C| \cdot h|\xi|^{h-1}, \qquad \forall\, h \geq 1\,,$$

for some $\xi \in [C - \delta, C + \delta] \subset (0, 1)$. Since the quantity $h|\xi|^{h-1}$ is uniformly bounded for $h \geq 1$ and $\xi \in [C - \delta, C + \delta]$, then we deduce that for some $\hat{C}_0 > 0$ one has

$$\left\|\mathcal{B}_n \widetilde{\sigma}(0+)\right\|_{\ell_1} \leq \left(C^h + \frac{\hat{C}_0}{N}\right) \left\|\widetilde{\sigma}(0+)\right\|_{\ell_1}\,,$$

where n, N, h satisfy (5.19).

- From (4.57) we have that

$$\widetilde{\sigma}(0+) = \sigma(0+) - \frac{(\sigma(0+) \cdot v_-)}{2N}\, v_-,$$

and then

$$\left\|\widetilde{\sigma}(0+)\right\|_{\ell_1} \leq \left\|\sigma(0+)\right\|_{\ell_1} + \frac{\left\|\sigma(0+)\right\|_{\ell_1}}{2N} 2N = 2\left\|\sigma(0+)\right\|_{\ell_1}.$$

Moreover, using (3.11) and (3.34), we have

$$\left\|\sigma(0+)\right\|_{\ell_1} \leq \mathrm{TV}\,\rho_0 + \mathrm{TV}\,\bar{J}_0 + 2C_0\|k\|_{L^1}$$

where $\bar{J}_0$ is defined at (4.20). Therefore it holds, for $h \leq t^n \leq (h+1)$:

$$\left\|\mathcal{B}_n \widetilde{\sigma}(0+)\right\|_{\ell_1} \leq 2\left(C^h + \frac{\hat{C}_0}{N}\right) \left(\mathrm{TV}\,\rho_0 + \mathrm{TV}\,\bar{J}_0 + 2C_0\|k\|_{L^1}\right)\,.$$

Using the relation (5.19) for h, n and N, we have

$$C^h \leq C^{t^n-1} = \frac{1}{C}\mathrm{e}^{-|\log C|t^n}.$$

In conclusion, we prove the following result:

$$\begin{aligned}\|J^{\Delta x}(\cdot,t^n)\|_\infty \leq \frac{1}{2N}\left\{\mathrm{TV}\,\bar{J}_0 + 4\hat{C}_0\left(\mathrm{TV}\,\rho_0 + \mathrm{TV}\,\bar{J}_0 + 2C_0\|k\|_{L^1}\right)\right\} \\ + \frac{2}{C}\mathrm{e}^{-|\log C|t^n}\left(\mathrm{TV}\,\rho_0 + \mathrm{TV}\,\bar{J}_0 + 2C_0\|k\|_{L^1}\right),\end{aligned} \tag{5.20}$$

that leads to decay in L^∞ for the approximate solution $J^{\Delta x}(\cdot,t^n)$.

Starting from (5.16), the estimate for the approximate solution $\rho^{\Delta x}(\cdot,t^n)$, is obtained in a similar way. Hence, we have

$$\begin{aligned}\|\rho^{\Delta x}(\cdot,t^n)\|_\infty &\leq \frac{2}{N}(1+C_1\|k\|_{L^1})\,\mathrm{TV}\,\bar{J}_0 + \frac{1}{N}\mathrm{TV}\,\rho_0 \\ &\quad + 2(1+2C_1\|k\|_{L^1})\|\mathcal{B}_n\widetilde{\sigma}(0+)\|_{\ell^1} \\ &\leq \frac{2}{N}(1+C_1\|k\|_{L^1})\,\mathrm{TV}\,\bar{J}_0 + \frac{1}{N}\mathrm{TV}\,\rho_0 \\ &\quad + 4(1+2C_1\|k\|_{L^1})\left(\frac{1}{C}\mathrm{e}^{-|\log C|t^n} + \frac{\hat{C}_0}{N}\right)\left(\mathrm{TV}\,\rho_0 + \mathrm{TV}\,\bar{J}_0 + 2C_0\|k\|_{L^1}\right) \\ &\leq \frac{2}{N}\left((1+C_1\|k\|_{L^1})\,\mathrm{TV}\,\bar{J}_0 + \mathrm{TV}\,\rho_0\right) \\ &\quad + \frac{8}{N}\hat{C}_0(1+2C_1\|k\|_{L^1})\left(\mathrm{TV}\,\rho_0 + \mathrm{TV}\,\bar{J}_0 + 2C_0\|k\|_{L^1}\right) \\ &\quad + \frac{4}{C}\mathrm{e}^{-|\log C|t^n}\left(\mathrm{TV}\,\rho_0 + \mathrm{TV}\,\bar{J}_0 + 2C_0\|k\|_{L^1}\right)(1+2C_1\|k\|_{L^1}),\end{aligned} \tag{5.21}$$

Then, passing to the limit by means of Helly's theorem, see Section 3.3, the estimates in (1.15) hold for suitable constants C_j, $j=1,2$ which are independent of Δx and t. The constant C_3 in (1.15) is given by

$$C_3 \doteq |\log C(d_1,d_2)| \qquad C(d_1,d_2) = \mathrm{e}^{-d_1}(\mathrm{e}^{d_2}-d_2).$$

This completes the proof of Theorem 1.2.2 for BV initial data. □

5.3 Long time behavior of solutions: L^∞ initial data

Here, we consider the problem of the long time behavior of the approximate solutions with L^∞ initial data. We provide 'incomplete' arguments suggesting that Theorem 1.2.2 is valid for L^∞ initial data. In particular, we consider the telegrapher's equation (4.26) in which we prove a contraction property of the invariant domain $[m, M]^2$ for the approximate solutions, see Theorem 5.3.4.

In this section we will assume that

$$k(x) \equiv \bar{k} > 0\,, \qquad g'(J) \equiv C_1\,, \qquad \alpha(t) \equiv 1$$

and then γ (4.8), has all the components that are equal to (4.25),

$$\gamma = \gamma(1, \dots, 1)\,, \quad d = \bar{k}C_1 > 0\,, \quad \gamma = d\Delta x = \frac{d}{N}\,,$$

which corresponds to the case of the standard telegrapher's equation (4.26). In this case, the iteration formula (4.12) leads to

$$\sigma(t^n+) = B(\gamma)^n \sigma(0+)\,. \tag{5.22}$$

In the following, the analysis will be based on the equation (5.22) for $n = hN$. By recalling (4.27) and the expansion formula (4.40), we get

$$\boxed{\begin{aligned}\sigma(t^{hN}+) &= B(\gamma)^{hN}\sigma(0+)\\ &= \left(1+\frac{d}{N}\right)^{-hN}\left(B(\mathbf{0})^{hN} + dh\widehat{P} + R_N(h,d)\right)\sigma(0+)\,.\end{aligned}} \tag{5.23}$$

Recalling (5.7), one obtains a similar expression for

$$\widehat{\sigma}(t^{hN}+) = \Pi B(\gamma)^{hN}\sigma(0+)\,. \tag{5.24}$$

In the formula (5.23), an expansion in powers of d is obtained, since $R_N(h,d)$ can be expressed in terms of powers d^ℓ with $\ell \geq 2$. A key point is the identification of the first order term $\widehat{P}$, that will lead us to a cancellation property stated in the following proposition.

Proposition 5.3.1. *The following identity holds,*

$$\widehat{P}\sigma(0+) = \frac{1}{2N}(\sigma(0+)\cdot v_-)v_-\,. \tag{5.25}$$

Proof. By recalling the definition of $\widehat{P}$ in (4.38), one has that

$$\widehat{P}\boldsymbol{w} = \frac{1}{2N}\big((\boldsymbol{w}\cdot e)\, e + (\boldsymbol{w}\cdot v_-)\, v_-\big) \qquad \forall\, \boldsymbol{w}\in \mathbb{R}^{2N}\,. \tag{5.26}$$

By setting $\boldsymbol{w} = \sigma(0+)$, from (4.17) we immediately get (5.25). □

In Theorem 4.3.2, it is proved that the matrix norm induced by $\|\cdot\|_{\ell_1}$ is contractive for $B(\gamma)^n$ on the subspace $E_- \doteq < e, v_- >^{\perp}$, (4.56). Here we provide a similar version of Theorem 4.3.2, whose proof is simplified by the use of expansion formula (4.40).

Proposition 5.3.2. *Let $N \in 2\mathbb{N}$, $h \in \mathbb{N}$ and $d \geq 0$. There exists a constant $C_N(h,d)$ (see (5.29) below) such that*

$$C_N(h,d) \to (1 - hd\mathrm{e}^{-hd}) \doteq C(h,d) < 1\,, \qquad N\to\infty \tag{5.27}$$

and that, for all $\boldsymbol{w}\in\mathbb{R}^{2N}$,

$$\big\|B(\gamma)^{hN}\boldsymbol{w}\big\|_{\ell_1} \leq C_N(h,d)\big\|\boldsymbol{w}\big\|_{\ell_1} + dh\left(1+\frac{d}{N}\right)^{-hN}\left(|\boldsymbol{w}\cdot e| + |\boldsymbol{w}\cdot v_-|\right)\,. \tag{5.28}$$

In particular, for N large enough such that $C_N(h,d) < 1$, the ℓ_1–norm is contractive on the subspace E_- defined at (4.56).

Proof. Let $\boldsymbol{w}\in\mathbb{R}^{2N}$. By means of the formula (4.27) and the expansion formula (4.40), we obtain

$$\begin{aligned} B(\gamma)^{hN}\boldsymbol{w} &= \left(1+\frac{d}{N}\right)^{-hN}\left[B(\mathbf{0}) + \frac{d}{N}B_1\right]^{hN}\boldsymbol{w} \\ &= \left(1+\frac{d}{N}\right)^{-hN}\left[B(\mathbf{0})^{hN}\boldsymbol{w} + \frac{dh}{2N}\big((\boldsymbol{w}\cdot e)\, e + (\boldsymbol{w}\cdot v_-)\, v_-\big) + R_N(h,d)\boldsymbol{w}\right] \end{aligned}$$

where we used (5.26).

Let $||\cdot||$ be a vector norm that is invariant under components permutation of the vectors. Since $B(\mathbf{0})^N$ is permutation matrix and $R_N(h,d)$ is a linear combination of permutation matrices, we use (4.42) to get that

$$\begin{aligned} ||B(\gamma)^{hN}\boldsymbol{w}|| \leq{}& \left(1+\frac{d}{N}\right)^{-hN}||\boldsymbol{w}||\left(1+\mathrm{e}^{hd}-hd-1+\frac{K}{N}\right) \\ &+ \left(1+\frac{d}{N}\right)^{-hN}\frac{dh}{2N}\left(|\boldsymbol{w}\cdot e|\cdot||e|| + |\boldsymbol{w}\cdot v_-|\cdot||v_-||\right)\,. \end{aligned}$$

The above estimate holds, in particular, for $||\cdot|| = \|\cdot\|_{\ell_1}$. Since $\|e\|_{\ell_1} = \|v_-\|_{\ell_1} = 2N$, and by setting

$$C_N(h,d) \doteq \left(1+\frac{d}{N}\right)^{-hN}\left[e^{hd} - hd \ + \ \frac{1}{N}K(h,d)\right] \tag{5.29}$$

then the estimate (5.28) follows. The proof of Proposition 5.3.2 is complete. □

Remark 5.3.1. *About the estimate* (5.28)*, a proper choice for h can be done in order to optimize the contraction constant $C_N(h,d)$.*

Indeed, since the function $x \mapsto (1 - xde^{-xd})$ has a global minimum at $x = 1/d$, it is clear that there is an integer value $\bar{h} \geq 1$ such that

$$C(\bar{h},d) = \min\{1 - hde^{-hd}\,;\ h = 1,2,\dots\}\,.$$

Also, for $d \geq 1$ one has that $\bar{h} = 1$.

Next, under the assumptions (4.25), we prove a contractivity property of the invariant domain $[m,M]^2$ for the approximate solutions.

Proposition 5.3.3. *Given $\bar{\boldsymbol{w}} \in \mathbb{R}^{2N}$ such that $\bar{\boldsymbol{w}} \cdot \boldsymbol{v}_{2N} = 0$, and given $d \geq 0$, let*

$$\boldsymbol{w}(d) = \bar{\boldsymbol{w}} + \frac{d}{N}\left(1+\frac{d}{N}\right)^{-1}\Phi(\bar{\boldsymbol{w}})$$

where,

$$\Phi(\boldsymbol{w}) = (\boldsymbol{w}\cdot\boldsymbol{v}_{2N}, -\boldsymbol{w}\cdot\boldsymbol{v}_2, \boldsymbol{w}\cdot\boldsymbol{v}_2, \dots, -\boldsymbol{w}\cdot\boldsymbol{v}_{2N-2}, \boldsymbol{w}\cdot\boldsymbol{v}_{2N-2}, -\boldsymbol{w}\cdot\boldsymbol{v}_{2N})\ , \tag{5.30}$$

for $\boldsymbol{w} \in \mathbb{R}^{2N}$ and for $\boldsymbol{v}_{2\ell}$, $\ell = 0,\dots,N$ defined as in (5.2)*. Then one has*

$$\bar{\boldsymbol{w}} = \boldsymbol{w}(d) - \frac{d}{N}\Phi(\boldsymbol{w}(d)) \tag{5.31}$$

and

$$B(\mathbf{0})^N\bar{\boldsymbol{w}} = B(\mathbf{0})^N\boldsymbol{w}(d) - \frac{d}{N}\Phi(B(\mathbf{0})^N\boldsymbol{w}(d))\,. \tag{5.32}$$

Moreover, let $m \leq 0 \leq M$ be such that

$$m \leq \bar{\boldsymbol{w}}\cdot\boldsymbol{v}_{2\ell}^{\pm} \leq M \qquad \ell = 0,\dots,N\,. \tag{5.33}$$

Then one has, for every $d \geq 0$ and j,k:

$$B(d)\bar{\boldsymbol{w}}\cdot(\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm}) \leq M - m\,. \tag{5.34}$$

Proof. Thanks to the definition of $v_{2\ell}$,

$$v_0 = \mathbf{0}\,, \qquad v_{2\ell} = (\underbrace{1,\cdots,1}_{2\ell},0,\cdots,0) \qquad \ell = 1,\dots,N\,,$$

we easily find that

$$\Phi(w)\cdot v_{2\ell} = \sum_{j=1}^{2\ell}\Phi(w)_j = -w\cdot v_{2\ell}\,, \quad \ell = 1,\dots,N\,.$$

Then we claim that the map Φ satisfies the following property:

$$\Phi(\Phi(w)) = -\Phi(w).$$

Indeed

$$\begin{aligned}\Phi(\Phi(w)) &= \left(0,\underbrace{-\Phi(w)\cdot v_2}_{=w\cdot v_2},\Phi(w)\cdot v_2,\dots,\underbrace{-\Phi(w)\cdot v_{2N-2}}_{=w\cdot v_{2N-2}},\Phi(w)\cdot v_{2N-2},0\right)\\ &= -\Phi(w)\,.\end{aligned}$$

Since Φ is linear, one has

$$\begin{aligned}\Phi(w(d)) &= \Phi(\bar w) + \frac{d}{N}\left(1+\frac{d}{N}\right)^{-1}\underbrace{\Phi(\Phi(\bar w))}_{-\Phi(\bar w)}\\ &= \Phi(\bar w)\left[1-\frac{d}{N}\left(1+\frac{d}{N}\right)^{-1}\right] = \left(1+\frac{d}{N}\right)^{-1}\Phi(\bar w)\,.\end{aligned}$$

This proves (5.31). To prove (5.32), it is sufficient to prove that

$$\Phi(B(\mathbf{0})^N w(d)) = B(\mathbf{0})^N\Phi(w(d))\,. \tag{5.35}$$

Indeed, if (5.35), from (5.31) we find immediately that

$$B(\mathbf{0})^N\bar w = B(\mathbf{0})^N w(d) - \frac{d}{N}B(\mathbf{0})^N\Phi(w(d)) = B(\mathbf{0})^N w(d) - \frac{d}{N}\Phi(B(\mathbf{0})^N w(d))\,,$$

hence (5.32) holds.

To prove (5.35), let $\boldsymbol{w}$ any vector in $\mathbb{R}^{2N}$ such that $\boldsymbol{w} \cdot \boldsymbol{v}_{2N} = 0$. We recall (4.23) to find that

$$\begin{aligned} B(\mathbf{0})^N \boldsymbol{w} \cdot \boldsymbol{v}_{2\ell} &= \boldsymbol{w} \cdot B(\mathbf{0})^N \boldsymbol{v}_{2\ell} \\ &= \boldsymbol{w} \cdot (\boldsymbol{v}_{2N} - \boldsymbol{v}_{2N-2\ell}) = \boldsymbol{w} \cdot \boldsymbol{v}_{2N} - \boldsymbol{w} \cdot \boldsymbol{v}_{2N-2\ell} \\ &= -\boldsymbol{w} \cdot \boldsymbol{v}_{2N-2\ell} \end{aligned}$$

and hence

$$\Phi(B(\mathbf{0})^N \boldsymbol{w}) = (0, \boldsymbol{w} \cdot \boldsymbol{v}_{2N-2}, -\boldsymbol{w} \cdot \boldsymbol{v}_{2N-2}, \dots, \boldsymbol{w} \cdot \boldsymbol{v}_2, -\boldsymbol{w} \cdot \boldsymbol{v}_2, 0) = B(\mathbf{0})^N \Phi(\boldsymbol{w}) .$$

Since $\boldsymbol{w}(d) \cdot \boldsymbol{v}_{2N} = 0$ for every $d \geq 0$, the previous identity applies and (5.35) holds.

To prove (5.34), recall (4.15), then we have

$$B(d)\bar{\boldsymbol{w}} \cdot (\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm}) = \frac{1}{1+d} \left(\underbrace{B(\mathbf{0})\bar{\boldsymbol{w}} \cdot (\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm})}_{(I)} + d \underbrace{B_1 \bar{\boldsymbol{w}} \cdot (\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm})}_{(II)} \right)$$

Estimate of (I),

$$(I) = \bar{\boldsymbol{w}} \cdot B(\mathbf{0})^t (\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm}) ,$$

and one can check that the following holds true

$$\begin{aligned} B(\mathbf{0})^t (\boldsymbol{v}_{2j}^{+} - \boldsymbol{v}_{2k}^{+}) &= \boldsymbol{v}_{2j-2}^{+} - \boldsymbol{v}_{2k-2}^{+} \\ B(\mathbf{0})^t (\boldsymbol{v}_{2j}^{-} - \boldsymbol{v}_{2k}^{-}) &= \boldsymbol{v}_{2j+2}^{+} - \boldsymbol{v}_{2k+2}^{+} . \end{aligned}$$

Therefore, by (5.33), we get

$$(I) = \begin{cases} \bar{\boldsymbol{w}} \cdot (\boldsymbol{v}_{2j-2}^{-} - \boldsymbol{v}_{2k-2}^{-}) \leq M - m \\ \bar{\boldsymbol{w}} \cdot (\boldsymbol{v}_{2j+2}^{+} - \boldsymbol{v}_{2k+2}^{+}) \leq M - m \end{cases}$$

Estimate of (II), one has the following

$$\begin{aligned} (II) &= \bar{\boldsymbol{w}} \cdot B_1 (\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm}) \\ &= -\bar{\boldsymbol{w}} \cdot (\boldsymbol{v}_{2j}^{\mp} - \boldsymbol{v}_{2k}^{\mp}) \\ &\leq M - m \end{aligned}$$

The last inequality holds by (5.33). Hence,

$$B(d)\bar{\boldsymbol{w}} \cdot (\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm}) \leq \frac{1}{1+d}\left((M-m) + d(M-m)\right) = M - m\,.$$

The proof of (5.34) is complete. □

Theorem 5.3.4. *Let $f^{\pm}$ be the approximate solution corresponding to the linear problem (4.26). Let $N \in 2\mathbb{N}$, $h \in \mathbb{N}$ and let $m \leq 0 \leq M$ be the constant values defined at (3.36). Then there exists a constant $\widehat{C} > 0$ such that*

$$\sup f^{\pm}(\cdot, t^{hN}) - \inf f^{\pm}(\cdot, t^{hN}) \leq C_N(h,d)(M-m) + \frac{\widehat{C}}{N} \tag{5.36}$$

for $C_N(h,d)$ defined at (5.29).

Proof. The proof employs the representation formula (5.9) for $f^{\pm}$ and the expansion formula (5.23).

- We start from the representation formula (5.9) for $t = t^{hN}+$. Thanks to the assumptions (4.25), it reads as:

$$f^{\pm}(x_j+, t) = \boldsymbol{\sigma}(t) \cdot \boldsymbol{v}_{2j}^{\pm} + \frac{1}{2}\rho(0+,t) - \frac{d}{N}\sum_{0\leq \ell \leq j} J(x_\ell, t)\,, \qquad j = 0, \dots, N-1 \tag{5.37}$$

where $x_j = j\Delta x = \frac{j}{N}$ and $\boldsymbol{v}_{2j}^{\pm}$ are defined at (5.10).

We remark that $f^{\pm}$ is possibly discontinuous only at $x = x_j$ and along (± 1)–waves, and hence their image is given by the values at $x = 0+$, $x = 1-$ and $x = x_j\pm$ with $j = 1, \dots, N-1$. At $x = x_j-$ one has that

$$f^{\pm}(x_j-, t) = \boldsymbol{\sigma}(t) \cdot \boldsymbol{v}_{2j}^{\pm} + \frac{1}{2}\rho(0+,t) - \frac{d}{N}\sum_{0\leq \ell < j} J(x_\ell, t)\,, \qquad j = 1, \dots, N$$

and hence

$$|f^{\pm}(x_j+, t) - f^{\pm}(x_j-, t)| \leq \sup |J(\cdot, t)|\frac{d}{N} \leq (M-m)\frac{d}{N}\,,$$

that vanishes as $N \to \infty$.

Therefore, in the following we will consider only the values of $f^{\pm}$ at $x = x_j+$.

- Recalling the identity (5.13) for the variation of $f^{\pm}$ across the point sources x_j, we find that

$$\widehat{p}_{j,n} = \frac{d}{N} J(x_j, t) = \frac{d}{N} \sigma(t) \cdot v_{2j}\,, \qquad \widehat{G}_n = \frac{d}{N} \Phi(\sigma(t))$$

where $\Phi : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is the linear map defined at (5.30). The map Φ has the following property:

$$\sum_{\ell=1}^{j} w \cdot v_{2\ell} = \Phi(w) \cdot v_{2j}^{-} = \Phi(w) \cdot v_{2j+2}^{+}\,, \qquad j \geq 1 \tag{5.38}$$

where $v_{2\ell}$ is defined as in (5.2). Therefore, as in (5.13), we can write

$$\sum_{0 \leq \ell \leq j} J(x_\ell, t) = \Phi(\sigma(t)) \cdot v_{2j}^{-} = \Phi(\sigma(t)) \cdot v_{2j+2}^{+}\,. \tag{5.39}$$

- Let $j, k \in \{0, \dots, N-1\}$, $j > k$. We combine (5.37) and (5.39) to get

$$f^{-}(x_j+, t) - f^{-}(x_k+, t) = \left(\sigma(t) - \frac{d}{N}\Phi(\sigma(t))\right) \cdot \left(v_{2j}^{-} - v_{2k}^{-}\right)$$
$$f^{+}(x_j+, t) - f^{+}(x_k+, t) = \sigma(t) \cdot \left(v_{2j}^{+} - v_{2k}^{+}\right) - \frac{d}{N}\Phi(\sigma(t)) \cdot \left(v_{2j+2}^{+} - v_{2k+2}^{+}\right)\,.$$

We claim that

$$f^{\pm}(x_j+, t) - f^{\pm}(x_k+, t) \leq \underbrace{\left(\sigma(t) - \frac{d}{N}\Phi(\sigma(t))\right) \cdot \left(v_{2j}^{\pm} - v_{2k}^{\pm}\right)}_{=(*)} + \frac{2d}{N}(M - m)\,. \tag{5.40}$$

Indeed, to prove (5.40) it is enough to check that

$$\left|\Phi(\sigma(t)) \cdot \left(v_{2j+2}^{+} - v_{2j}^{+} - v_{2k+2}^{+} + v_{2k}^{+}\right)\right| \leq 2(M - m)$$

which is true since

$$|\Phi(\sigma(t)) \cdot \left(v_{2j+2}^{+} - v_{2j}^{+}\right)| = |\sigma(t) \cdot v_{2j}| = |J(x_j, t)| \leq M - m\,.$$

Let's start with $(*)$. By applying the identity (5.23), the right hand side above can be written as a sum of three terms, corresponding to $B(\mathbf{0})^{hN}$, $\widehat{P}$ and $R_N(h, d)$

respectively:

$$(*) = \left(1 + \frac{d}{N}\right)^{-hN} [A + B + C]$$

where

$$A = \left[B(\mathbf{0})^{hN}\boldsymbol{\sigma}(0+) - \frac{d}{N}\Phi\left(B(\mathbf{0})^{hN}\boldsymbol{\sigma}(0+)\right)\right] \cdot (\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm})$$

$$B = dh\left[\widehat{P}\boldsymbol{\sigma}(0+) - \frac{d}{N}\Phi\left(\widehat{P}\boldsymbol{\sigma}(0+)\right)\right] \cdot (\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm})$$

$$C = \left[R_N(h,d)\boldsymbol{\sigma}(0+) - \frac{d}{N}\Phi\left(R_N(h,d)\boldsymbol{\sigma}(0+)\right)\right] \cdot (\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm}).$$

- **Estimate for B.** We claim that

$$|B| \leq \frac{dh}{N}|\boldsymbol{\sigma}(0+) \cdot v_-|.$$

To prove this claim, it is sufficient to prove that

$$(i) \qquad \widehat{P}\boldsymbol{\sigma}(0+) \cdot (\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm}) \in \{\pm 1, 0\},$$
$$(ii) \qquad \Phi\left(\widehat{P}\boldsymbol{\sigma}(0+)\right) \cdot (\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm}) = 0.$$

To prove (i), we use (5.25) to write that

$$\widehat{P}\boldsymbol{\sigma}(0+) \cdot \boldsymbol{v}_{2\ell}^{\pm} = \frac{1}{2N}\left(\boldsymbol{\sigma}(0+) \cdot v_-\right)\left(v_- \cdot \boldsymbol{v}_{2\ell}^{\pm}\right),$$

where v_- is the eigenvector in (4.14):

$$v_- = (1, -1, -1, 1, \ldots, 1, -1, -1, 1).$$

From (5.10), it is immediate to check that

$$v_- \cdot \boldsymbol{v}_{2\ell}^{+} = (1, -1, -1, 1, \ldots, 1, -1, -1, 1) \cdot (\underbrace{1, 0, \cdots, 1, 0}_{2\ell}, 0, \cdots, 0) \in \{0, 1\},$$

and similarly

$$v_- \cdot \boldsymbol{v}_{2\ell}^{-} = -(1, -1, -1, 1, \ldots, 1, -1, -1, 1) \cdot (\underbrace{0, 1, \cdots, 0, 1}_{2\ell}, 0, \cdots, 0) \in \{0, 1\}.$$

More precisely,

$$v_- \cdot \boldsymbol{v}_{2\ell}^+ = v_- \cdot \boldsymbol{v}_{2\ell}^- = \begin{cases} 1 & \text{if } \ell \text{ odd} \\ 0 & \text{if } \ell \text{ even}\,. \end{cases}$$

Therefore, it is immediate to conclude that (i) holds.

To prove (ii), we use the identity (5.38) to find that

$$\begin{aligned} \Phi\left(\widehat{P}\boldsymbol{\sigma}(0+)\right) \cdot \boldsymbol{v}_{2j}^- &= \sum_{\ell=1}^{j} \widehat{P}\boldsymbol{\sigma}(0+) \cdot \boldsymbol{v}_{2\ell} \\ &= \frac{1}{2N}\left(\boldsymbol{\sigma}(0+) \cdot v_-\right) \sum_{\ell=1}^{j} \underbrace{v_- \cdot \boldsymbol{v}_{2\ell}}_{=0} \\ &= 0\,. \end{aligned}$$

Here above we used the fact that $v_- \cdot \boldsymbol{v}_{2\ell} = v_- \cdot \left(\boldsymbol{v}_{2\ell}^+ - \boldsymbol{v}_{2\ell}^-\right) = 0$. The proof for

$$\Phi\left(\widehat{P}\boldsymbol{\sigma}(0+)\right) \cdot \boldsymbol{v}_{2j}^+ = \sum_{\ell=1}^{j-1} \widehat{P}\boldsymbol{\sigma}(0+) \cdot \boldsymbol{v}_{2\ell}$$

is totally analogous. The claim is proved.

- **Towards an estimate for A and C.** Consider the initial-boundary value problem with the same initial data and boundary condition as the one corresponding to $\boldsymbol{\sigma}(t)$, but for $k(x) \equiv 0$. Hence the problem is linear and undamped.

The corresponding evolution vector, that we denote with $\overline{\boldsymbol{\sigma}}(t)$, is defined inductively by

$$\begin{aligned} \overline{\boldsymbol{\sigma}}(t^n+) &= B(\mathbf{0})^n \overline{\boldsymbol{\sigma}}(0+)\,, \\ \overline{\boldsymbol{\sigma}}(t^{n+\frac{1}{2}}+) &= B_1 \overline{\boldsymbol{\sigma}}(t^n+)\,, \end{aligned} \qquad n \geq 1\,. \tag{5.41}$$

About $\overline{\boldsymbol{\sigma}}(0+)$ we claim that

$$\overline{\boldsymbol{\sigma}}(0+) = \boldsymbol{\sigma}(0+) - \frac{d}{N}\Phi(\boldsymbol{\sigma}(0+)) \tag{5.42}$$

where

$$\begin{aligned} \Phi(\boldsymbol{\sigma}(0+)) &= (0, -\boldsymbol{\sigma}(0+) \cdot \boldsymbol{v}_2, \boldsymbol{\sigma}(0+) \cdot \boldsymbol{v}_2, \ldots, -\boldsymbol{\sigma}(0+) \cdot \boldsymbol{v}_{2N-2}, \boldsymbol{\sigma}(0+) \cdot \boldsymbol{v}_{2N-2}, 0) \\ &= (0, -J(x_1, 0+), +J(x_1, 0+), \ldots, -J(x_{N-1}, 0+), +J(x_{N-1}, 0+), 0)^t\,. \end{aligned}$$

To prove the claim,

- we observe that $\overline{\sigma}_1 = \sigma_1$ and $\overline{\sigma}_{2N} = \sigma_{2N}$.

- at every x_j, $j = 1, \dots, N-1$ we compare $(\overline{\sigma}_{2j}, \overline{\sigma}_{2j+1})$ with $(\sigma_{2j}, \sigma_{2j+1})$. In the notation of Proposition 3.1.2, let J_* the middle value for J in the solution to the Riemann problem with $d = \bar{k} > 0$ and $J_m = f_\ell^+ - f_r^-$ the middle value for J when $\bar{k} = 0$. Using (3.9), we have the following identity:

$$J_* + \frac{d}{N} J_* = J_m \,,$$

from which we deduce

$$\overline{\sigma}_{2j} = J_m - J_\ell = \underbrace{(J_* - J_\ell)}_{=\sigma_{2j}} + \frac{d}{N} J_* = \sigma_{2j} + \frac{d}{N} J(x_j, 0+) \,.$$

Similarly one has

$$\overline{\sigma}_{2j+1} = J_r - J_m = \underbrace{(J_r - J_*)}_{=\sigma_{2j+1}} - \frac{d}{N} J_* = \sigma_{2j+1} - \frac{d}{N} J(x_j, 0+) \,.$$

Therefore (5.42) holds. The claim is proved.

It is easy to check that (5.42) can be inverted as follows:

$$\boldsymbol{\sigma}(0+) = \overline{\boldsymbol{\sigma}}(0+) + \frac{d}{N} \left(1 + \frac{d}{N}\right)^{-1} \Phi(\overline{\boldsymbol{\sigma}}(0+)) \,,$$

see Proposition 5.3.3.

- **Estimate for A.** We apply (5.31) and (5.32), for h even and odd, respectively, to find that

$$A = B(\mathbf{0})^{hN} \overline{\boldsymbol{\sigma}}(0+) \cdot (\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm}) \leq M - m \,.$$

- **Estimate for C.** By using (4.41) we get

$$\begin{aligned} &R_N(h,d)\sigma(0+) - \frac{d}{N}\Phi\left(R_N(h,d)\sigma(0+)\right) \\ &= \sum_{j=0}^{hN-1} \zeta_{j,hN}\left\{B_1 B(\mathbf{0})^{hN-2j-1}\sigma(0+) - \frac{d}{N}\Phi\left(B_1 B(\mathbf{0})^{hN-2j-1}\sigma(0+)\right)\right\} \\ &\quad + \sum_{j=1}^{hN-1} \eta_{j,hN}\left\{B(\mathbf{0})^{2j-hN}\sigma(0+) - \frac{d}{N}\Phi\left(B(\mathbf{0})^{2j-hN}\sigma(0+)\right)\right\}. \end{aligned}$$

Bound on the terms

$$\begin{aligned} B(\mathbf{0})^n\sigma(0+) - \frac{d}{N}\Phi\left(B(\mathbf{0})^n\sigma(0+)\right) = &\underbrace{B(\mathbf{0})^n\left[\sigma(0+) - \frac{d}{N}\Phi\left(\sigma(0+)\right)\right]}_{(I)} \\ &- \frac{d}{N}\underbrace{\left[\Phi\left(B(\mathbf{0})^n\sigma(0+)\right) - B(\mathbf{0})^n\Phi\left(\sigma(0+)\right)\right]}_{(II)} \end{aligned}$$

By (5.34), the term (I) has the property

$$(I)\cdot(\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm}) \le M - m\,.$$

About (II), it is obvious that

$$(II)\cdot(\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm}) \le \sup_{j,k,n\in\{1,\cdots,N-1\}}\left|(II)\cdot(\boldsymbol{v}_{2j}^{\pm} - \boldsymbol{v}_{2k}^{\pm})\right| \doteq \widetilde{C}(\sigma(0+))\,. \tag{5.43}$$

The same hold for the terms containing B_1. Therefore, by (4.42),

$$C \le \left(\mathrm{e}^{hd} - hd - 1 + \frac{K}{N}\right)\left((M-m) + \frac{d}{N}\widetilde{C}(\sigma(0+))\right)$$

Hence, collecting the bounds on the terms A, B and C, and using (4.19), we get

$$\begin{aligned} (*) \le\; & C_N(h,d)(M-m) + \frac{dh}{N}\left(1+\frac{d}{N}\right)^{-hN}\mathrm{TV}\,\bar{J}_0 \\ &+ \frac{d}{N}\left(1+\frac{d}{N}\right)^{-hN}\left(\mathrm{e}^{hd} - hd - 1 + \frac{K}{N}\right)\widetilde{C}(\sigma(0+))\,. \end{aligned}$$

Hence,

$$0 \leq \sup f^{\pm}(\cdot, t^{hN}) - \inf f^{\pm}(\cdot, t^{hN}) \leq C_N(h,d)(M-m) + \frac{2d(M-m)}{N}$$
$$+ \left(1 + \frac{d}{N}\right)^{-hN} \frac{d}{N} \left(h\mathrm{TV}\, \bar{J}_0 + \left(\mathrm{e}^{hd} - hd - 1 + \frac{K}{N} \right) \widetilde{C}(\sigma(0+)) \right)$$

which is (5.36). The proof of Theorem 5.3.4 is complete. □

Remark 5.3.2. *We remark that the constant $\widehat{C}$ in formula (5.36) may depend on N. This comes from estimating the term (II) in (5.43), we should provide a finite bound that independent on N for this term. Then, to prove the L^∞ decay of the approximate solution of the problem (4.26), we should iterate (5.36) to have the contraction property of the invariant domain. Finally, we pass to the limit by means of Theorem 1.2.1 to get the L^∞ decay of the solution to (4.26) with L^∞ initial data.*

5.4 Localized damping case

In this section, we consider the case of localized damping, that is, $k(x) \geq \bar{k} > 0$ on some (α, β) with $[\alpha, \beta] \subset (0,1)$, and $k(x) = 0$ otherwise. We used the Rayleigh quotient Theorem A.1.4 to obtain an estimate on the modulus of the eigenvalues of the matrix $B(\gamma)$ different from $\lambda_\pm$ (4.14). This could be a way to study the L^∞ decay of problem (1.2)–(1.3) with localized damping source. Let $\alpha(t) = 1$, and for the function $k = k(x)$, we assume that there exist $\bar{k} > 0$ and $0 \leq \alpha < \beta \leq 1$ such that

$$k \in L^1(I)\,, \qquad \begin{cases} k(x) \geq \bar{k} & \text{on } (\alpha, \beta)\,, \\ k(x) = 0 & \text{on } I \setminus (\alpha, \beta)\,; \end{cases} \tag{5.44}$$

while $g = g(J)$ satisfies (1.12), that is $g \in C^1(\mathbb{R})$, $g(0) = 0$ and it is strictly increasing function. Due to assumption (5.44) on k, the 0-waves δ_j (3.27) are positive in a certain range of indices and zero otherwise, i.e. for N large enough we can assume that there exist two indices j_α, j_β with $j_\alpha = \lfloor N\alpha \rfloor$, $j_\beta = \lfloor N\beta \rfloor$ and $1 \leq j_\alpha < j_\beta \leq N-1$ ($\lfloor \cdot \rfloor$ denotes the floor function, that is $\lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\}$), such that

$$\delta_j > 0 \quad \text{if } j_\alpha \leq j \leq j_\beta\,, \qquad \delta_j = 0 \quad \text{otherwise}\,. \tag{5.45}$$

Recall the *transition coefficients* (4.5), i.e. $\gamma_j^n = g'(s_j^n)\delta_j$, then

$$\gamma_j > 0 \quad \text{if} \quad j_\alpha \leq j \leq j_\beta\,, \qquad \gamma_j = 0 \quad \text{otherwise.} \tag{5.46}$$

Notice that the matrix $B(\boldsymbol{\gamma})$ (4.11) reduces to a permutation matrix $B(0)$ for $\boldsymbol{\gamma} = \mathbf{0} \in \mathbb{R}^{N-1}$ and we can write $B(\boldsymbol{\gamma}) = B(\mathbf{0}) + E(\boldsymbol{\gamma})$, where

$$E(\boldsymbol{\gamma}) \doteq B(\boldsymbol{\gamma}) - B(\mathbf{0}) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \frac{\gamma_1}{1+\gamma_1} & 0 & 0 & -\frac{\gamma_1}{1+\gamma_1} & \cdots & 0 & 0 & 0 & 0 \\ -\frac{\gamma_1}{1+\gamma_1} & 0 & 0 & \frac{\gamma_1}{1+\gamma_1} & & \vdots & \vdots & & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & \frac{\gamma_{N-1}}{1+\gamma_{N-1}} & 0 & 0 & -\frac{\gamma_{N-1}}{1+\gamma_{N-1}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{\gamma_{N-1}}{1+\gamma_{N-1}} & 0 & 0 & \frac{\gamma_{N-1}}{1+\gamma_{N-1}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This means that $B(\boldsymbol{\gamma})$ is obtained as a perturbation of $B(\mathbf{0})$.

Remark 5.4.1. *The* amplitude *of this perturbation is computed by the operator norm of* $E(\boldsymbol{\gamma})$*, that is*

$$|||E(\boldsymbol{\gamma})||| = \max_{\|v\|=1} \|E(\boldsymbol{\gamma})v\|,$$

where $\|\cdot\|$ *denotes the Euclidean norm in* $\mathbb{R}^{2N}$*. More precisely, we have*

$$|||E(\boldsymbol{c})||| = 2 \max_{j=1,\dots,N-1} \frac{\gamma_j}{\gamma_j + 1}. \tag{5.47}$$

Indeed, let $v \in \mathbb{R}^{2N}$ *and* $j^* \in \{1, \dots, N-1\}$ *be such that*

$$\|v\| = 1, \qquad \gamma_{j^*} = \max_{j=1,\dots,N-1} \frac{\gamma_j}{\gamma_j + 1}.$$

Then,

$$\begin{aligned} \|E(\boldsymbol{c})v\| &= \sqrt{2\frac{\gamma_1^2}{(\gamma_1+1)^2}(v_1 - v_4)^2 + \cdots + 2\frac{\gamma_{N-1}^2}{(\gamma_{N-1}+1)^2}(v_{N-3} - v_{2N})^2} \\ &\le \sqrt{2}\gamma_j^* \sqrt{(v_1 - v_4)^2 + \cdots + (v_{N-3} - v_{2N})^2} \\ &= \sqrt{2}\gamma_j^* \sqrt{v_1^2 + v_4^2 + \cdots + v_{N-3}^2 + v_{2N}^2 - 2(v_1 v4 + \cdots + v_{2N-3}v_{2N})} \\ &\le \sqrt{2}\gamma_j^* \sqrt{2\sum_{j=1}^{2N} v_j^2} = 2\gamma_j^*, \end{aligned}$$

where we used $\sum_{j=1}^{2N} v_i^2 \geq -2\sum_{j\neq i} v_i v_j$. *In particular, if we choose* $v \in \mathbb{R}^{2N}$ *with*

$$v_{2j^*-1} = \frac{1}{\sqrt{2}}, \qquad v_{2j^*+2} = -\frac{1}{\sqrt{2}}, \qquad v_i = 0 \ \text{otherwise},$$

we get $\|v\| = 1$ *and*

$$\|E(\gamma)v\| = \sqrt{2}\gamma_{j^*}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 2\gamma_{j^*},$$

hence (5.47) *holds true.*

In the next proposition, we present an iterative way to compute the characteristic polynomial of $B(\gamma)$. In particular, we prove that the characteristic polynomial of $B(\mathbf{0})$ is $\lambda^{2N} - 1$, hence its eigenvalues are the $2N$-th roots of unity in $\mathbb{C}$.

Proposition 5.4.1. *Assume* (5.45) *and consider the* 2×2 *matrix*

$$M_j = M(\lambda, \gamma_j) \doteq \frac{1}{\gamma_j + 1}\begin{bmatrix} -(\gamma_j+1)\lambda^2 & \gamma_j \\ -\gamma_j\lambda^2 & \gamma_j - 1\,, \end{bmatrix} \qquad j = 1, \dots, N-1\,. \tag{5.48}$$

Then, the characteristic polynomial $p(\lambda, \gamma) = p_{2N}(\lambda, \gamma)$ *of* B *is given by*

$$p_{2N} = p_{2N-1} - \lambda^2 p_{2N-2}\,, \tag{5.49}$$

where for $j = 0, \dots, N-1$ *the polynomials* p_{2j}, p_{2j+1} *are defined inductively as follows:*

$$\begin{cases} p_0 = p_1 \equiv 1\,, \\ p_{2j} = \frac{\gamma_j}{\gamma_j+1}\, p_{2j-1} - \lambda^2\, p_{2j-2}\,, \\ p_{2j+1} \doteq \frac{\gamma_j - 1}{\gamma_j+1} p_{2j-1} - \frac{\gamma_j}{\gamma_j+1}\lambda^2 p_{2j-2}\,. \end{cases}$$

In particular,

$$\begin{pmatrix} p_{2N-2} \\ p_{2N-1} \end{pmatrix} = M_{N-1} \cdot M_{N-2} \cdots M_2 \cdot M_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.50}$$

Moreover, we have

$$p_{2N}(\lambda, \mathbf{0}) = \lambda^{2N} - 1 \tag{5.51}$$

and the eigenvalues of $B(\mathbf{0})$ *are the* $2N$*-th roots of unity in* $\mathbb{C}$.

Proof. Since $B_1^2 = I_{\mathbb{R}^{2N\times 2N}}$, we can write

$$B - \lambda I = (B_2 - \lambda B_1)\, B_1\,,$$

and, since $\det B_1 = 1$, the eigenvalues λ of B satisfy

$$\det(B - \lambda I) = \det(B_2 - \lambda B_1) = 0\,.$$

The matrix $(B_2 - \lambda B_1)$ is symmetric and tridiagonal, with

$$1, \frac{\gamma_1}{\gamma_1+1}, \frac{\gamma_1}{\gamma_1+1}, \dots \frac{\gamma_{N-1}}{\gamma_{N-1}+1}, \frac{\gamma_{N-1}}{\gamma_{N-1}+1}, 1$$

on the diagonal and

$$-\lambda, \frac{1}{\gamma_1+1}, \dots, -\lambda, \frac{1}{\gamma_{N-1}+1}, -\lambda$$

on the first subdiagonal and superdiagonal. More precisely, we have

$$B_2 - \lambda B_1 = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_1 & a_2 & b_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & b_2 & a_3 & b_3 & & \vdots & & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & a_{2N-2} & b_{2N-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & b_{2N-2} & a_{2N-1} & b_{2N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{2N-1} & a_{2N} \end{bmatrix}$$

where $(a_1, a_2, \cdots, a_{2N})$ is defined by

$$a_1 = a_{2N} = 1, \qquad a_{2j} = a_{2j+1} = \frac{\gamma_j}{\gamma_j+1} \quad \forall\, j = 1, \dots, N-1;$$

and $(b_1, \dots, b_{2N-1})$ is defined by

$$b_{2j-1} = -\lambda \quad \forall\, j = 1, \dots, N,$$
$$b_{2j} = \frac{1}{\gamma_j+1} \quad \forall\, j = 1, \dots, N-1.$$

Following [40, p.35], we have that $p_{2N}(\lambda) = \det(B_2 - \lambda B_1)$ can be computed inductively as follows. For $j = 1, \dots, N-1$, we define the polynomials

$$p_0 = p_1 \equiv 1\,, \tag{5.52}$$

$$p_{2j} = a_{2j}\, p_{2j-1} - (b_{2j-1})^2\, p_{2j-2} = \frac{\gamma_j}{\gamma_j + 1}\, p_{2j-1} - \lambda^2\, p_{2j-2}\,, \tag{5.53}$$

$$p_{2j+1} = a_{2j+1}\, p_{2j} - (b_{2j})^2\, p_{2j-1} = \frac{\gamma_j}{\gamma_j + 1}\, p_{2j} - \frac{1}{(\gamma_j+1)^2} p_{2j-1}\,, \tag{5.54}$$

$$p_{2N} = p_{2N-1} - \lambda^2 p_{2N-2}\,. \tag{5.55}$$

By substitution of (5.53) into (5.54) we find

$$p_{2j+1} = \frac{\gamma_j - 1}{\gamma_j + 1} p_{2j-1} - \frac{\gamma_j}{\gamma_j + 1}\lambda^2 p_{2j-2} \qquad j = 1, \dots, N-1$$

Hence,

$$\begin{pmatrix} p_{2j} \\ p_{2j+1} \end{pmatrix} = M_j \begin{pmatrix} p_{2j-2} \\ p_{2j-1} \end{pmatrix} = M_j \cdot M_{j-1} \cdots M_2 \cdot M_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.56}$$

where M_j is explicitly given in (5.48). Finally, we obtain (5.50) and then (5.49).

If $\gamma_j = 0$ for every $j = 1, \dots, N-1$, then

$$\begin{pmatrix} p_{2j} \\ p_{2j+1} \end{pmatrix} = \begin{bmatrix} -\lambda^2 & 0 \\ 0 & -1 \end{bmatrix}^j \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (-1)^j \begin{pmatrix} \lambda^{2j} \\ 1 \end{pmatrix}$$

and, as a consequence, (5.51) holds. □

Let us call

$$\mu_\ell \doteq e^{\frac{2i\pi\ell}{2N}} = e^{\frac{i\pi\ell}{N}}\,, \qquad \ell = 0, \dots, 2N-1\,,$$

the eigenvalues of $B(\mathbf{0})$ which correspond to the $2N$-th roots of unity. Recall that $B(\mathbf{0})$ is a permutation matrix, hence it is normal. As a consequence, following the same arguments as in [12, pag. 185], we can prove that the eigenvalues of $B(\gamma)$ belong to the union of closed disks

$$\bigcup_{\ell=0}^{2N-1} \overline{D}(\mu_\ell, |||E(\gamma)|||). \tag{5.57}$$

Moreover, it is possible to show that for source of moderate strength the eigenvalues of $B(\gamma)$ are all distinct.

Proposition 5.4.2. *Let $\gamma \in K$, compact set of $\mathbb{R}^{N-1}$. If we assume*

$$C_1\|k\|_\infty < \frac{1}{2}, \tag{5.58}$$

then $B(\gamma)$ has $2N$ distinct eigenvalues.

Proof. We claim that all the disks in (5.57) are disjoint under assumption (5.58). Indeed, recall that all the μ_ℓ are located at the vertices of a regular $2N$-sided polygon in the complex plane, centered at $(0,0)$ and circumscribed by the unit circle. Hence, the minimal distance between two vertices is given by the length of the sides of the polygon, namely

$$\min_{\ell \neq \ell'} |\mu_\ell - \mu_{\ell'}| = 2\sin\left(\frac{\pi}{N}\right).$$

We have that

$$|||E(\gamma)||| = 2 \max_{j=1,\dots,N-1} \frac{\gamma_j}{\gamma_j + 1} \leq 2\,C_1\|k\|_\infty \Delta x = \frac{2\,C_1\|k\|_\infty}{N}$$

and we need to require that

$$|||E(\gamma)||| < \sin\left(\frac{\pi}{N}\right),$$

in order that the disks are disjoint. This holds true if in particular

$$\frac{2\,C_1\|k\|_\infty}{N} < \sin\left(\frac{\pi}{N}\right), \tag{5.59}$$

which is eventually ensured by assumption (5.58). Indeed, define

$$f(N) := \frac{2\,C_1\|k\|_\infty}{N} - \sin\left(\frac{\pi}{N}\right), \qquad N \geq 1.$$

We have that $f(1) = 2\,C_1\|k\|_\infty - 1 < 0$ by (5.58), $f(N) \to 0$ as $N \to +\infty$ and

$$\begin{aligned} f'(N) &= \frac{1}{N^2}\left(-2C_1\|k\|_\infty + \pi\cos\left(\frac{\pi}{N}\right)\right) \\ &> \frac{1}{N^2}\left(-1 + \pi\cos\left(\frac{\pi}{N}\right)\right) > 0. \end{aligned}$$

The function f is strictly increasing and tends to zero from below, hence it is strictly negative for each $N \geq 1$ and (5.59) is verified. □

For $\ell = 0, \dots, 2N$, let us denote by λ_ℓ the distinct eigenvalues of $B(\gamma)$ (under assumption (5.58)) that satisfy

$$\lambda_\ell = \mu_\ell + \frac{\boldsymbol{y}_\ell^H E(\boldsymbol{c})\boldsymbol{x}_\ell}{\boldsymbol{y}_\ell^H \boldsymbol{x}_\ell} + \mathcal{O}\left(|||E(\gamma)|||^2\right), \tag{5.60}$$

where $\boldsymbol{x}_\ell, \boldsymbol{y}_\ell \in \mathbb{C}^{2N}$ are respectively the right and left eigenvector associated to μ_ℓ (see [40, Theorem 6.3.12]).

We observe that $\lambda_+ = 1 = \mu_0$ and $\lambda_- = -1 = \mu_N$ belong to the spectrum of both $B(\gamma)$ and $B(\mathbf{0})$, while the remaining eigenvalues of $B(\gamma)$ lie in the interior of the unit circle in $\mathbb{C}$ and can be understood as a pertubation of

$$\mu_1, \dots, \mu_{N-1}, \mu_{N+1}, \dots, \mu_{2N-1}\,.$$

Below we use (5.60) to *locate* the eigenvalues of $B(\gamma)$. First, a general formula for the quantity $\boldsymbol{y}_\ell^H E(\gamma)\boldsymbol{x}_\ell / \boldsymbol{y}_\ell^H \boldsymbol{x}_\ell$ is computed in the following lemma.

Lemma 5.4.3. *Let $\mu_\ell \in \mathbb{C}$ be a fixed eigenvalue of $B(\mathbf{0})$, $\ell = 0, \dots, 2N-1$. If $\boldsymbol{x}_\ell \in \mathbb{C}^{2N}$ and $\boldsymbol{y}_\ell \in \mathbb{C}^{2N}$ denote respectively its associated right and left eigenvectors, then*

$$\frac{\boldsymbol{y}_\ell^H E(\gamma)\boldsymbol{x}_\ell}{\boldsymbol{y}_\ell^H \boldsymbol{x}_\ell} = -\frac{\mu_\ell}{N} \sum_{j=1}^{N-1} \frac{\gamma_j}{\gamma_j + 1} \left[1 - \cos\left(\frac{2\pi\ell j}{N}\right)\right]. \tag{5.61}$$

Proof. We first compute $\boldsymbol{x}_\ell$ and $\boldsymbol{y}_\ell$. By definition we have that

$$\boldsymbol{x}_\ell = (x_1, x_2, \dots, x_{2N})^T, \quad \text{and} \quad \boldsymbol{y}_\ell^H = (y_1, y_2, \dots, y_{2N})$$

satisfy $B(\mathbf{0})\boldsymbol{x}_\ell = \mu_\ell \boldsymbol{x}_\ell$ and $\boldsymbol{y}_\ell^H B(\mathbf{0}) = \mu_\ell \boldsymbol{y}_\ell^H$, namely:

$$\begin{pmatrix} x_2 \\ x_4 \\ x_1 \\ \vdots \\ x_{2N} \\ x_{2N-3} \\ x_{2N-1} \end{pmatrix} = \mu_\ell \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{2N-2} \\ x_{2N-1} \\ x_{2N} \end{pmatrix},$$

$$\begin{pmatrix} y_3 & y_1 & y_5 & \dots & y_{2N-4} & y_{2N} & y_{2N-2} \end{pmatrix} = \mu_\ell \begin{pmatrix} y_1 & y_2 & \dots & y_{2N-1} & y_{2N} \end{pmatrix}.$$

By letting $x_1 = y_1 = 1 = \lambda^{2N}$, we get

$$\boldsymbol{x}_\ell = \begin{pmatrix} 1 \\ \mu_\ell \\ \mu_\ell^{2N-1} \\ \mu_\ell^2 \\ \mu_\ell^{2N-2} \\ \vdots \\ \mu_\ell^{N+1} \\ \mu_\ell^N \end{pmatrix}, \quad \boldsymbol{y}_\ell^H = \begin{pmatrix} 1 & \mu_\ell^{2N-1} & \mu_\ell & \mu_\ell^{2N-2} & \mu_\ell^2 & \cdots & \mu_\ell^{N-1} & \mu_\ell^N \end{pmatrix}.$$

Hence,

$$E(\boldsymbol{\gamma})\boldsymbol{x}_\ell = \begin{pmatrix} 0 \\ \frac{\gamma_1}{\gamma_1+1}(x_1 - x_4) \\ -\frac{\gamma_1}{\gamma_1+1}(x_1 - x_4) \\ \frac{\gamma_2}{\gamma_2+1}(x_3 - x_6) \\ -\frac{\gamma_2}{\gamma_2+1}(x_3 - x_6) \\ \vdots \\ \frac{\gamma_{N-1}}{\gamma_{N-1}+1}(x_{2N-3} - x_{2N}) \\ -\frac{\gamma_{N-1}}{\gamma_{N-1}+1}(x_{2N-3} - x_{2N}) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\gamma_1}{\gamma_1+1}\left(\mu_\ell^{2N} - \mu_\ell^2\right) \\ -\frac{\gamma_1}{\gamma_1+1}\left(\mu_\ell^{2N} - \mu_\ell^2\right) \\ \frac{\gamma_2}{\gamma_2+1}\left(\mu_\ell^{2N-1} - \mu_\ell^3\right) \\ -\frac{\gamma_2}{\gamma_2+1}\left(\mu_\ell^{2N-1} - \mu_\ell^3\right) \\ \vdots \\ \frac{\gamma_{N-1}}{\gamma_{N-1}+1}\left(\mu_\ell^{2N-(N-2)} - \mu_\ell^N\right) \\ -\frac{\gamma_{N-1}}{\gamma_{N-1}+1}\left(\mu_\ell^{2N-(N-2)} - \mu_\ell^N\right) \\ 0 \end{pmatrix}$$

and

$$
\begin{aligned}
&\boldsymbol{y}_\ell^H E(\boldsymbol{\gamma})\boldsymbol{x}_\ell \\
&\quad= \frac{\gamma_1}{\gamma_1+1}\left(\mu_\ell^{2N}-\mu_\ell^2\right)\left(\mu_\ell^{2N-1}-\mu_\ell\right)+\frac{\gamma_2}{\gamma_2+1}\left(\mu_\ell^{2N-1}-\mu_\ell^3\right)\left(\mu_\ell^{2N-2}-\mu_\ell^2\right) \\
&\quad+\cdots+\frac{\gamma_{N-1}}{\gamma_{N-1}+1}\left(\mu_\ell^{N+2}-\mu_\ell^N\right)\left(\mu_\ell^{N+1}-\mu_\ell^{N-1}\right) \\
&=\frac{\gamma_1}{\gamma_1+1}\left(\mu_\ell^{4N-1}-2\mu_\ell^{2N+1}+\mu_\ell^3\right)+\frac{\gamma_2}{\gamma_2+1}\left(\mu_\ell^{4N-3}-2\mu_\ell^{2N+1}+\lambda^5\right) \\
&\quad+\cdots+\frac{\gamma_{N-1}}{\gamma_{N-1}+1}\left(\mu_\ell^{2N+3}-2\mu_\ell^{2N+1}+\mu_\ell^{2N-1}\right) \\
&=\frac{\gamma_1}{\gamma_1+1}\left(\mu_\ell^{2N-1}-2\mu_\ell+\mu_\ell^3\right)+\frac{\gamma_2}{\gamma_2+1}\left(\mu_\ell^{2N-3}-2\mu_\ell+\mu_\ell^5\right) \\
&\quad+\cdots+\frac{\gamma_{N-1}}{\gamma_{N-1}+1}\left(\mu_\ell^3-2\mu_\ell+\mu_\ell^{2N-1}\right) \\
&=\mu_\ell\Big[\frac{\gamma_1}{\gamma_1+1}\left(\mu_\ell^{2N-2}-2+\mu_\ell^2\right)+\frac{\gamma_2}{\gamma_2+1}\left(\mu_\ell^{2N-4}-2+\mu_\ell^4\right) \\
&\quad+\cdots+\frac{\gamma_{N-1}}{\gamma_{N-1}+1}\left(\mu_\ell^2-2+\mu_\ell^{2N-2}\right)\Big].
\end{aligned}
$$

Since $\boldsymbol{y}_\ell^H\boldsymbol{x}_\ell = 2N$, we obtain

$$
\begin{aligned}
\frac{\boldsymbol{y}_\ell^H E(\boldsymbol{\gamma})\boldsymbol{x}_\ell}{\boldsymbol{y}_\ell^H\boldsymbol{x}_\ell} &= \frac{\mu_\ell}{2N}\Big[\frac{\gamma_1}{\gamma_1+1}\left(\mu_\ell^{2N-2}-2+\mu_\ell^2\right)+\frac{\gamma_2}{\gamma_2+1}\left(\mu_\ell^{2N-4}-2+\mu_\ell^4\right) \\
&\quad+\cdots+\frac{\gamma_{N-1}}{\gamma_{N-1}+1}\left(\mu_\ell^2-2+\mu_\ell^{2N-2}\right)\Big] \\
&= -\frac{\mu_\ell}{2N}\sum_{j=1}^{N-1}\frac{\gamma_j}{\gamma_j+1}\left(2-\mu_\ell^{-2j}-\mu_\ell^{2j}\right).
\end{aligned}
$$

Notice that each term $\frac{\gamma_j}{\gamma_j+1}(2-\mu_\ell^{-2j}-\mu_\ell^{2j})$ appearing in the sum above is a positive real term. Indeed, μ_ℓ^{-2j} and μ_ℓ^{2j} are complex conjugate and $\mu_\ell^{-2j}+\mu_\ell^{2j} = 2\,\Re\left(\mu_\ell^{2j}\right) < 2$ since $|\mu_\ell^{2j}| = 1$. More precisely,

$$
\mu_\ell^j = e^{\frac{2i\pi\ell\cdot j}{2N}} = e^{\frac{i\pi\ell\cdot j}{N}}, \qquad \overline{\mu_\ell^j} = e^{-\frac{i\pi\ell j}{N}} = \mu_\ell^{-j}, \qquad j = 1,\dots,N,
$$

hence

$$\begin{aligned}(\mu_\ell^j)^2 + (\overline{\mu_\ell^j})^2 =& \left(\cos\left(\frac{\pi\ell j}{N}\right) + i\sin\left(\frac{\pi\ell j}{N}\right)\right)^2 + \left(\cos\left(\frac{\pi\ell j}{N}\right) - i\sin\left(\frac{\pi\ell j}{N}\right)\right)^2 \\ =& 2\left(\cos^2\left(\frac{\pi\ell j}{N}\right) - \sin^2\left(\frac{\pi\ell j}{N}\right)\right) \\ =& 2\cos\left(\frac{2\pi\ell j}{N}\right).\end{aligned}$$

Finally, we obtain (5.61). □

Recall that, since $k(x) \geq \bar{k}\chi_{[\alpha,\beta]}(x)$, by the definition of γ_j at (4.5) and by (5.46) we have that

$$\frac{\gamma_j}{\gamma_j + 1} \geq \frac{\inf g'}{2}\delta_j \geq \frac{\bar{k}\inf g'}{2N} \qquad \text{for all} \quad j_\alpha + 1 \leq j \leq j_\beta\,.$$

Using (5.61), we get

$$\frac{1}{\mu_\ell}\frac{\boldsymbol{y}_\ell^H E(\gamma)\boldsymbol{x}_\ell}{\boldsymbol{y}_\ell^H \boldsymbol{x}_\ell} \leq -\frac{\bar{k}\inf g'}{2N}\frac{1}{N}\sum_{j=j_\alpha+1}^{j_\beta}\left[1 - \cos\left(\frac{2\pi\ell j}{N}\right)\right] = -\frac{\bar{k}\inf g'}{2N} I_{N,\ell}\,,$$

where we set

$$I_{N,\ell} \doteq \frac{1}{N}\sum_{j=j_\alpha+1}^{j_\beta}\left[1 - \cos\left(\frac{2\pi\ell j}{N}\right)\right], \qquad \ell = 1, \ldots, \frac{N}{2}\,. \tag{5.62}$$

Remark that it is sufficient to consider $I_{N,\ell}$ just for $\ell \leq N/2$ because of the symmetry of the roots of unity. In this framework we can prove the following proposition.

Proposition 5.4.4. *Let $\ell \geq 1$ and $I_{N,\ell}$ be defined as in (5.62). Then,*

$$\lim_{N\to+\infty} I_{N,\ell} \to I_\ell\,, \tag{5.63}$$

where

$$I_\ell \doteq (\beta - \alpha)\left[1 - \frac{\sin(2\pi\beta\ell) - \sin(2\pi\alpha\ell)}{2\pi(\beta - \alpha)\ell}\right].$$

In particular, let ℓ_0 be the largest integer $< 1/(\beta-\alpha)$, then there exist a constant $C_{\alpha,\beta} < 1$ such that

$$\inf_{\ell\in\mathbb{N}} I_\ell \geq (\beta-\alpha)\left[1 - C_{\alpha,\beta}\right], \tag{5.64}$$

where

$$C_{\alpha,\beta} = \max\left\{\frac{1}{\pi}, \max_{\ell=1,\dots,\ell_0} \frac{\sin(2\pi\beta\ell) - \sin(2\pi\alpha\ell)}{2\pi(\beta-\alpha)\ell}\right\} \tag{5.65}$$

Proof. Let $\ell \geq 1$ be fixed. From (5.62) we get

$$I_{N,\ell} = \frac{j_\beta - j_\alpha}{N}\left[1 - \frac{1}{j_\beta - j_\alpha}\sum_{j=j_\alpha+1}^{j_\beta}\cos(j\theta)\right],$$

where $\theta = \theta_{N,\ell} = 2\pi\ell/N$. By the definition of floor function, we have $j_\beta \leq \beta N < j_\beta + 1$ and $j_\alpha \leq \alpha N < j_\alpha + 1$, so that as $N \to \infty$ we get

$$0 \leq (\beta-\alpha) - \frac{j_\beta - j_\alpha}{N} < \frac{1}{N} \to 0\,. \tag{5.66}$$

By a useful trigonometric identity (Lagrange identity) we can write

$$\begin{aligned}\sum_{j=j_\alpha+1}^{j_\beta}\cos(j\theta) &= \sum_{j=1}^{j_\beta}\cos(j\theta) - \sum_{j=1}^{j_\alpha}\cos(j\theta)\\ &= \frac{1}{2\sin(\frac{\theta}{2})}\left[\sin\left(j_\beta\theta + \frac{\theta}{2}\right) - \sin\left(j_\alpha\theta + \frac{\theta}{2}\right)\right].\end{aligned}$$

Notice that, since we are taking the limit of $I_{N,\ell}$ as $N \to \infty$, we can consider $N > 2\ell$, so that $0 < \theta/2 < \pi/2$ and the above quantity is well-defined because $\sin(\theta/2)$ is not zero. Since

$$\begin{aligned}\sin\left(j_\beta\theta + \frac{\theta}{2}\right) - \sin\left(j_\alpha\theta + \frac{\theta}{2}\right) &= \left[\sin(j_\beta\theta) - \sin(j_\alpha\theta)\right]\cos(\theta/2)\\ &\quad + \left[\cos(j_\beta\theta) - \cos(j_\alpha\theta)\right]\sin(\theta/2)\,,\end{aligned}$$

we have

$$\frac{1}{j_\beta - j_\alpha} \sum_{j=j_\alpha+1}^{j_\beta} \cos(j\theta) = \frac{\sin(j_\beta\theta) - \sin(j_\alpha\theta)}{2(j_\beta - j_\alpha)\tan(\frac{\theta}{2})} + \frac{\cos(j_\beta\theta) - \cos(j_\alpha\theta)}{2(j_\beta - j_\alpha)}\,.$$

The last term vanishes as $N \to \infty$, since $j_\beta - j_\alpha = [\beta N] - [\alpha N] \to \infty$. On the other hand, we can write

$$\frac{\sin(j_\beta\theta) - \sin(j_\alpha\theta)}{2(j_\beta - j_\alpha)\tan(\frac{\theta}{2})} = \frac{\sin(j_\beta\theta) - \sin(j_\alpha\theta)}{(j_\beta - j_\alpha)\theta} \cdot \frac{\frac{\theta}{2}}{\tan(\frac{\theta}{2})}$$

and we get

$$\frac{\frac{\theta}{2}}{\tan(\frac{\theta}{2})} = \frac{\frac{\pi\ell}{N}}{\tan(\frac{\pi\ell}{N})} \to 1\,,$$

$$\frac{\sin(j_\beta\theta) - \sin(j_\alpha\theta)}{(j_\beta - j_\alpha)\theta} = \frac{\sin(j_\beta\frac{2\pi\ell}{N}) - \sin(j_\alpha\frac{2\pi\ell}{N})}{(j_\beta - j_\alpha)\frac{2\pi\ell}{N}} \to \frac{\sin(2\pi\beta\ell) - \sin(2\pi\alpha\ell)}{2\pi(\beta-\alpha)\ell}\,,$$

as $N \to \infty$. Hence,

$$\frac{\sin(j_\beta\theta) - \sin(j_\alpha\theta)}{2(j_\beta - j_\alpha)\tan(\frac{\theta}{2})} \to \frac{\sin(2\pi\beta\ell) - \sin(2\pi\alpha\ell)}{2\pi(\beta-\alpha)\ell} \qquad N \to \infty\,.$$

Hence, by (5.66) and the formula above we get (5.63).

We observe that this quotient is always < 1 and we distinguish two cases:

(i) if $\ell \geq 1/(\beta - \alpha)$, then

$$\sin(2\pi\beta\ell) - \sin(2\pi\alpha\ell) \leq 2 \leq 2(\beta-\alpha)\ell$$

and we get

$$\frac{\sin(2\pi\beta\ell) - \sin(2\pi\alpha\ell)}{2\pi(\beta-\alpha)\ell} \leq \frac{1}{\pi} < 1\,;$$

(ii) otherwise, for $\ell_0 \geq \ell$ being the largest integer $< 1/(\beta - \alpha)$, we have that

$$\frac{\sin(2\pi\beta\ell) - \sin(2\pi\alpha\ell)}{2\pi(\beta-\alpha)\ell} = \frac{1}{2\pi(\beta-\alpha)\ell} \int_{2\pi\alpha\ell}^{2\pi\beta\ell} \cos(\xi)\, d\xi < 1\,.$$

Hence,

$$C_{\alpha,\beta} \doteq \max\left\{\frac{1}{\pi}, \max_{\ell=1,\dots,\ell_0} \frac{\sin(2\pi\beta\ell) - \sin(2\pi\alpha\ell)}{2\pi(\beta-\alpha)\ell}\right\} < 1$$

and (5.64) holds true. □

We remark that $C_{\alpha,\beta}$ depends only on α, β and is always strictly less than 1. For instance, if $\alpha = 1/4$ and $\beta = 1/2$, then $\ell_0 < 4 = 1/(\beta-\alpha)$ and

$$\begin{aligned} C_{\alpha,\beta} &= \max\left\{\frac{1}{\pi}, \max_{\ell=1,2,3} \frac{\sin(2\pi\beta\ell) - \sin(2\pi\alpha\ell)}{2\pi(\beta-\alpha)\ell}\right\} \\ &= \max\left\{\frac{1}{\pi}, -\frac{2}{\pi}, 0, \frac{2}{3\pi}\right\} = \frac{1}{\pi} < 1. \end{aligned}$$

As a consequence of (5.60), (5.63) and (5.64), we obtain an estimate on the modulus of the eigenvalues of $B(\gamma)$ different from $\lambda_\pm$.

Corollary 5.4.5. *Let $\ell = 1, \dots, 2N-1$, $\ell \neq N$ (i.e. consider the eigenvalues $\lambda_\ell \neq \lambda_\pm$) and call*

$$\delta_N \doteq \frac{\bar{k}\inf g'}{2N}(\beta-\alpha)\left[1 - C_{\alpha,\beta}\right].$$

Then

$$\begin{aligned} |\lambda_\ell| &= |\mu_\ell|\left(1 - \frac{1}{N}\sum_{j=j_\alpha+1}^{j_\beta} \frac{\gamma_j}{\gamma_j+1}\left[1 - \cos\left(\frac{2\pi\ell j}{N}\right)\right]\right) + \mathcal{O}\left(|||E(\gamma)|||^2\right) \\ &\leq 1 - \delta_N + \mathcal{O}\left(|||E(\gamma)|||^2\right). \end{aligned} \tag{5.67}$$

Remark 5.4.2. *Let*

$$\mathcal{O}\left(|||E(\gamma)|||^2\right) \leq \mathcal{O}\left(\frac{1}{N^2}\right).$$

Then, we are interested in the constant

$$\left(1 - \delta_N + \mathcal{O}\left(|||E(\gamma)|||^2\right)\right)^N \to e^{-\delta}, \qquad N \to \infty$$

where

$$\delta = \frac{\bar{k}\inf g'}{2}(\beta-\alpha)\left[1 - C_{\alpha,\beta}\right].$$

In Chapter (6), we analyze the case of localized damping in Test 2, where we take

$$k(x) = \chi_{[0.4,0.6]}(x), \qquad g(J) = e^J - 1, \qquad \alpha(t) \equiv 1$$

and initial data

$$J_0(x) = 0\,, \qquad \rho_0(x) = x^2 - 1/3 \qquad x \in (0,1)\,.$$

In this case, $\inf g' = e^{-\frac{1}{6}}$, $\alpha = 0.4\,,\ \beta = 0.6$ *and*

$$\begin{aligned} C_{\alpha,\beta} &= \max\left\{\frac{1}{\pi}, \max_{\ell=1,\dots,4} \frac{\sin(2\pi\beta\ell) - \sin(2\pi\alpha\ell)}{2\pi(\beta-\alpha)\ell}\right\} \\ &= \frac{\sin(4\pi\beta) - \sin(4\pi\alpha)}{4\pi(\beta-\alpha)} = 0.75682\,. \end{aligned}$$

Hence,

$$\delta = \frac{e^{-\frac{1}{6}}}{2}(0.6 - 0.4)\left[1 - C_{\alpha,\beta}\right] = 0.02058.$$

While numerically, we got that the rate of decay is ≈ 0.05. *This difference is due to the oscillation we have in the numerical approximation, so we could not compute the precise value of the decay using the following equation:*

$$\frac{log(\|J(\cdot,t^{n+1})\|_{L^\infty}) - log(\|J(\cdot,t^n)\|_{L^\infty})}{\Delta t}\,,$$

as n varies. Instead, what we do is to plot the $log(\|J(\cdot,t^n)\|_{L^\infty})$, *then we use the fitting app, that provide a linear function plot* $C_1 + C_2 t$, *with* $C_2 = 0.05$ *which is the rate of decay.*

Chapter 6

Numerical simulations and perspectives

In this chapter we present some numerical proofs, based on the numerical implementation of the algorithm presented in Chapter 3. Define

$$f^{\pm}_{j,n} = (f^{\pm})^{\Delta x}(x, t^n)\,, \quad x \in (x_{j-1}, x_j)\,, \qquad j = 1, \dots, N\,, \quad n \geq 0$$

with the initial data:

$$f^{\pm}_{j,0} = f^{\pm}_0(x_{j-1}+)\,, \qquad j = 1, \dots, N\,. \tag{6.1}$$

The implicit equations

$$J^*_{j,n} + g(J^*_{j,n})\bar{\alpha}_n\delta_j = f^+_{j,n} - f^-_{j+1,n}\,, \qquad j = 1, \dots, N-1\,, \tag{6.2}$$

with δ_j as in (3.27) and $\bar{\alpha}_n$ as in (3.31), define uniquely the values

$$J^*_{1,n}, \dots, J^*_{N-1,n} \qquad n \geq 0\,.$$

To establish the values $f^{\pm}_{j,n+1}$ for every $n \geq 0$, based on the construction of the scheme presented in Subsection 3.2.1, one proceeds as follows:

- At the boundaries, we have to deal with two *boundary Riemann problems*, see **Step 2** in Subsection 3.2.1 and (3.33) for solving the problem at $x = 0$, $t = 0$. In general, the conditions are

  $$f^+_{1,n+1} = f^-_{1,n}\,, \qquad f^-_{N,n+1} = f^+_{N,n} \tag{6.3}$$

  which imply that the boundary conditions $J(0,t) = J(1,t) = 0$ are satisfied;

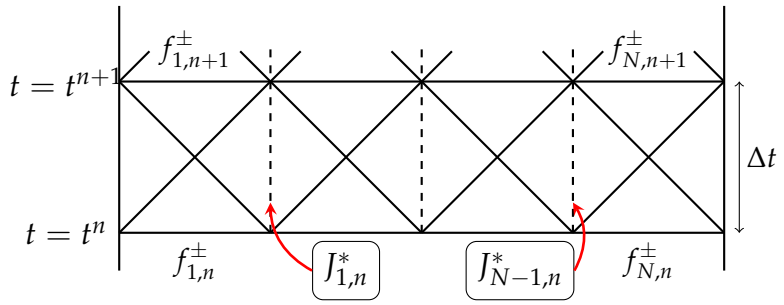

FIGURE 6.1: A time interval

- While the other values, for $j = 1, \dots, N-1$, are given by

$$f^+_{j+1,n+1} = f^+_{j,n} - g(J^*_{j,n})\bar{\alpha}_n\delta_j\,, \qquad f^-_{j,n+1} = f^-_{j+1,n} + g(J^*_{j,n})\bar{\alpha}_n\delta_j\,, \tag{6.4}$$

  which are a discrete version of (3.6). More precisely, given the approximate initial data (6.1), then the solution to the Riemann problem for (6.4) is uniquely established by (3.8). In the equation (6.4), $J^*_{j,n}$ should be computed by the implicit equations (6.2).

Test 1. Here we set

$$k(x) = 1\,, \qquad g(J) = J\,,$$

and

$$\rho_0(x) = -\pi\cos(\pi x)\,, \qquad J_0(x) = 0\,, \qquad x \in [0,1]\,. \tag{6.5}$$

Using separation of variables method (for the damped wave equation (1.1)), the exact solution of equation (1.2) is found to be

$$J_{ex}(x,t) = -\left(\sqrt{|1-\pi^2|} + \frac{1}{\sqrt{|1-\pi^2|}}\right) e^{-t}\sin(\pi x)\sin(\sqrt{|1-\pi^2|}t)\,,$$

$$\rho_{ex}(x,t) = -\pi e^{-t}\cos(\pi x)\left(\cos(\sqrt{|1-\pi^2|}t) + \frac{1}{\sqrt{|1-\pi^2|}}\sin(\sqrt{|1-\pi^2|}t)\right).$$

(I) We want to prove that the Well-Balanced approximation solutions converge to the exact solution as $\Delta x \to 0$, see Section 3.3. This can be shown numerically, take $t = 0.5$ and $N = 40, \cdots, 2000$. See Figure 6.2, where this graph show the values of $\|J^{\Delta x}(t,\cdot) - J_{ex}(t,\cdot)\|_{L^\infty}$ and $\|\rho^{\Delta x}(t,\cdot) - \rho_{ex}(t,\cdot)\|_{L^\infty}$ (red color).

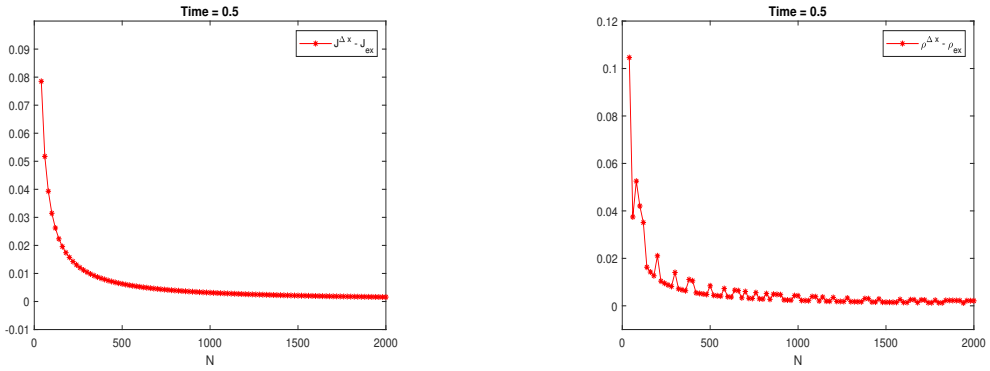

FIGURE 6.2: Test 1: Convergence of the approximate solutions as $\Delta x \to 0$.

(II) Here we aim to prove numerically what we proved in Remark 1.2.2–[ii]. In this test, the stationary solutions are $J(x) = \rho(x) = 0$, and $d = g'k(x) = k(x)$. The goal now is to see the rate of decay for the approximate solutions towards the zero stationary solutions in case $d \to 0$ that is $k(x)$ is small. In Figure 6.3, we take $k(x) = 0.1$, then the figure show the L^∞ decay of the approximate solutions to the zero stationary solutions (blue color). We use the exponential models in the Curve Fitting Toolbox in MATLAB on the left-hand side of Figure 6.3 (red color plots), and it appears that the decay is exponential with rate very close to 0.1 as proved in Remark 1.2.2–[ii]. See also Figure 6.4, where we take $k(x) = 0.05$.

Moreover, if the decay is exponential, then it should satisfies the following

$$\|J(\cdot,t)\|_{L^\infty} = C_1 e^{-C_2 t} \Longrightarrow \log(\|J(\cdot,t)\|_{L^\infty}) = \log(C_1) - C_2 t\,.$$

So, we plot the $\log(\|J(\cdot,t)\|_{L^\infty})$ and $\log(\|\rho(\cdot,t)\|_{L^\infty})$. These plots show that the decay is exponential with rate close to $C_2 = 0.1$ in case $k(x) = 0.1$, and $C_2 = 0.05$ in case $k(x) = 0.05$. See Figures 6.3-6.4.

Test 2. Here we choose

$$k(x) = \chi_{[0.4,0.6]}(x)\,, \qquad g(J) = e^J - 1\,, \qquad \alpha(t) \equiv 1$$

and initial data

$$J_0(x) = 0\,, \qquad \rho_0(x) = x^2 - 1/3 \qquad x \in (0,1)\,. \tag{6.6}$$

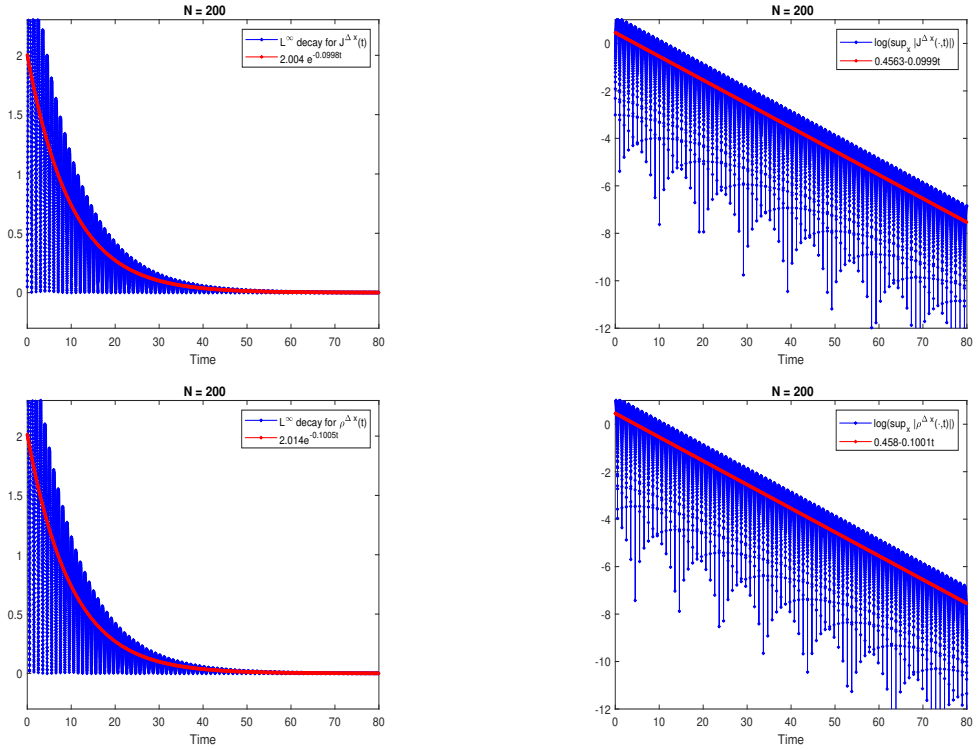

FIGURE 6.3: Test 1: Decay of approximate solutions with $k(x) = 0.1$.

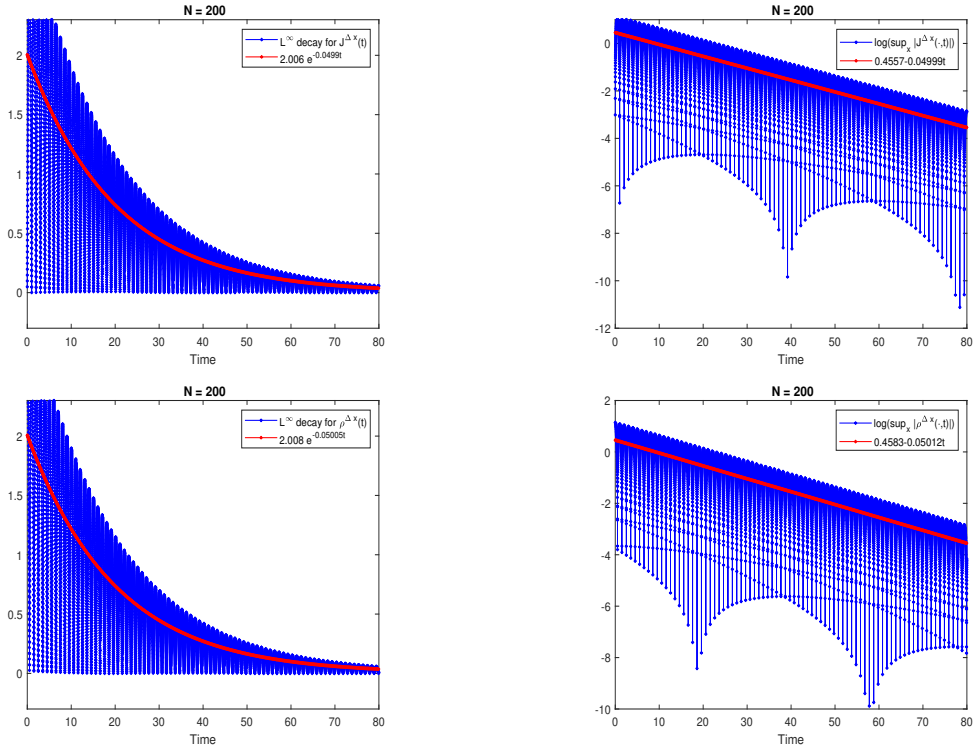

FIGURE 6.4: Test 1: Decay of approximate solutions with $k(x) = 0.05$.

N	**Error**	**Order of convergence** $= \left\lvert \frac{\log(\text{Error})}{\log N} \right\rvert$
600	0.002825	0.9175
800	0.00204	0.9267
1000	0.01548	0.9367

TABLE 6.1: Test 2: The order of convergence for $J^{\Delta x}$ at $T = 60$.

We use the Lambert W function (see [13] or [27, p. 256]) to find the value of $J^*_{j,n}$ in equation (6.2), hence

$$J^\star_{j,n} = \delta_j - W(\delta_j e^{\delta_j + f^+_{j-1,n} - f^-_{j,n}}) + f^+_{j-1,n} - f^-_{j,n}\,.$$

In Figure 6.5, we show the quantities $\|J^{\Delta x}(\cdot,t)\|_{L^\infty}$ and $\|\rho^{\Delta x}(\cdot,t)\|_{L^\infty}$ (blue color) for $t = t^n$ with $n = 1, \cdots, 14000$ and $N = 200$ with initial data (6.6). Using the linear model in the Curve Fitting Toolbox in MATLAB on the right-hand side of Figure 6.5 (red color plots), that fit the $\log(\|J^{\Delta x}(\cdot,t)\|_{L^\infty})$ and $\log(\|\rho^{\Delta x}(\cdot,t)\|_{L^\infty})$, the decay seems to be exponential with rate ≈ 0.05. See Remark 5.4.2 for connecting this result with the theoretical results of Section 5.4.

Moreover, one can compute the order of convergence for the approximate solution. Indeed, denote the order of convergence by α, for a fixed large time T, the following holds true

$$\|J^{\Delta x}(\cdot,T)\|_{L^\infty} = \frac{1}{N^\alpha} \quad \Longrightarrow \quad \alpha = -\frac{\log \|J^{\Delta x}(\cdot,T)\|_{L^\infty}}{\log N}\,.$$

In Table 6.1, we compute the error and the order of convergence of the approximate solution $J^{\Delta x}$ for $N = 600, 800, 1000$ at fixed time $T = 60$.

Test 3. (On-Off damping) Here we choose the same $k(x)$, $g(J)$ and the initial data as in Test 2, while we assume that $\alpha(t)$ is given by (1.4), with $T_1 = 1$, $T_2 = 2$.

As in Test 2, we use Lambert W function to find the value of $J^*_{j,n}$ in equation (6.2). In Figure 6.6, we plot the quantities $\|J^{\Delta x}(\cdot,t)\|_{L^\infty}$ and $\|\rho^{\Delta x}(\cdot,t)\|_{L^\infty}$ (blue color) for $t = t^n$ with $n = 1, \cdots, 10000$ and $N = 100$ with initial data (6.6). Again, plotting the $\log(\|J^{\Delta x}(\cdot,t)\|_{L^\infty})$ and $\log(\|\rho^{\Delta x}(\cdot,t)\|_{L^\infty})$ the rate of decay seems to be exponential with rate ≈ 0.03. In this test we have investigated the L^∞ decay of case On-Off damping, the theoretical proof will be in a next work. See Remark 1.2.2–(vi).

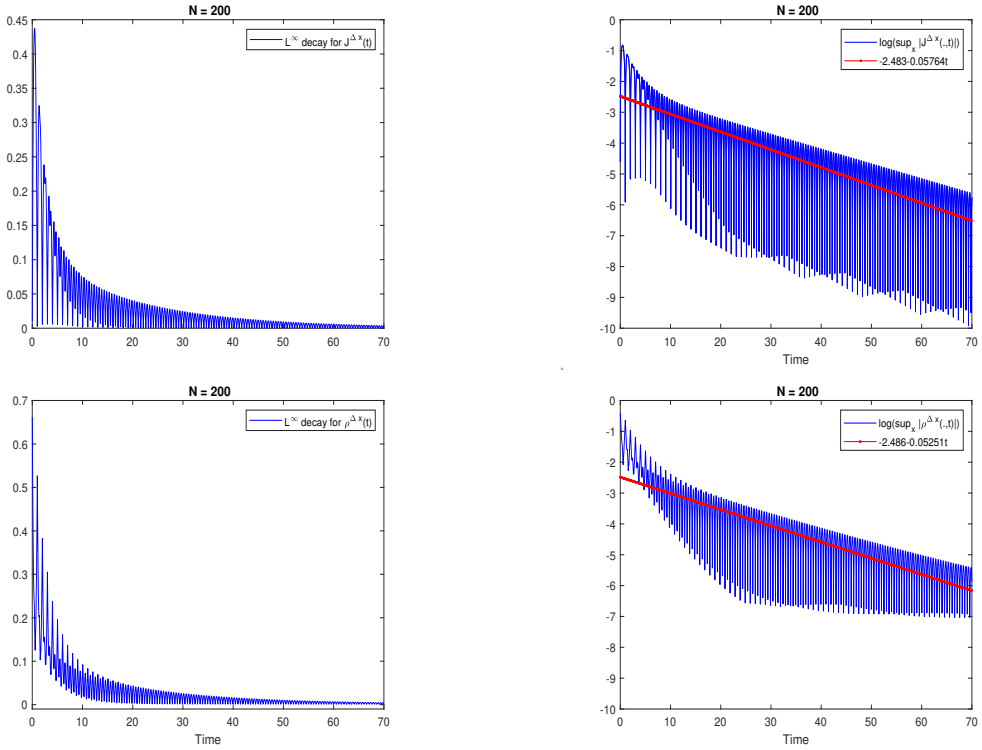

FIGURE 6.5: Test 2: Decay of approximate solutions.

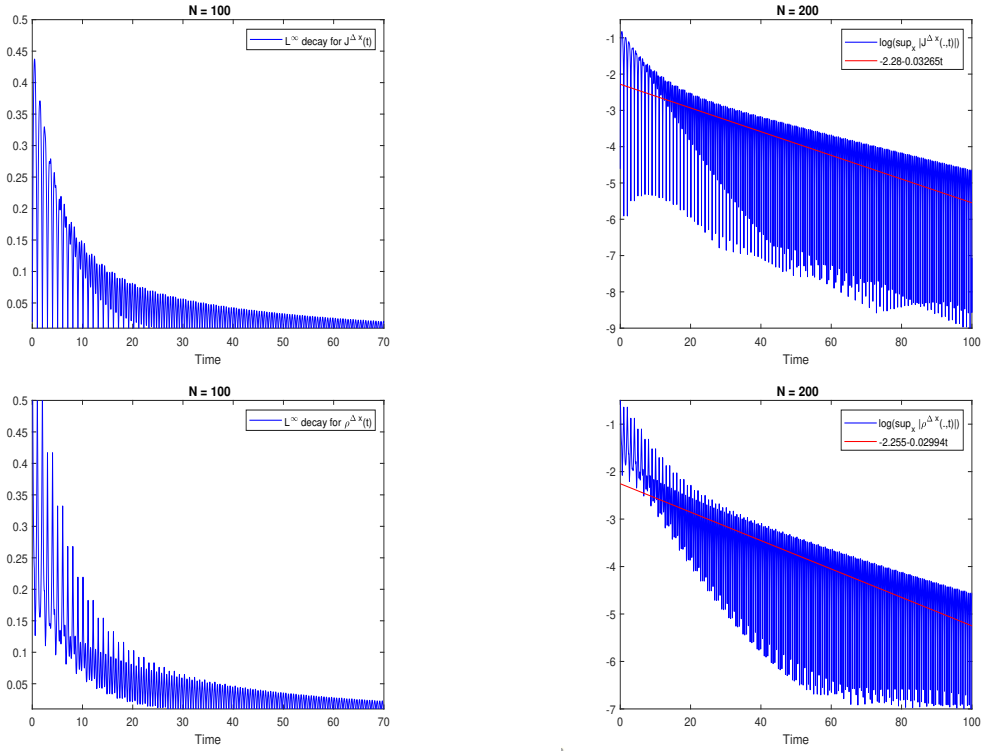

FIGURE 6.6: Test 3: Decay of approximate solutions in the case On-Off damping.

Appendix A

A.1 Results from matrix theory

In this Appendix, we recall some earlier results related to Matrix theory that we used to study the properties of the matrix $B(\gamma^n)$. For more details, we refer the reader to the books [40, 47, 41, 12, 10].

Definition A.1.1. *(Permutation matrices). A square matrix P is a permutation matrix if exactly one entry in each row and column is equal to 1 and all other entries are 0. Multiplication by such matrices effects a permutation of the rows or columns of the matrix multiplied.*

Left multiplication of a matrix $A \in M_{m,n}$ by an m-by-m permutation matrix P permutes the rows of A, while right multiplication of A by an n-by-n permutation matrix P permutes the columns of A. A permutation matrix applied to a vector v, gives a vector whose components are a permutation of v.

The determinant of a permutation matrix is ± 1, so permutation matrices are nonsingular. Although permutation matrices need not commute, the product of two permutation matrices is again a permutation matrix. Since the identity is a permutation matrix and $P^t = P^{-1}$ for every permutation matrix P, the set of n-by-n permutation matrices is a subgroup of $GL(n, C)$ with cardinality $n!$.

Definition A.1.2. *(Stochastic matrix). A nonnegative matrix $A \in M_n$ with the property that $Ae = e$, that is, all its row sums are $+1$, is said to be a (row) stochastic matrix; each row may be thought of as a discrete probability distribution on a sample space with n points. A column stochastic matrix is the transpose of a row stochastic matrix, that is, $e^t A = e^t$.*

Such matrices arise in the intercity population migration model. Stochastic matrices also arise in the study of Markov chains and in a variety of modeling problems in economics and operations research.

Definition A.1.3. *(Doubly stochastic matrix). A stochastic matrix $A \in M_n$ such that A^t is also stochastic is said to be doubly stochastic; all row and column sums are $+1$.*

A nonnegative matrix $A \in M_n$ is doubly stochastic if and only if both $Ae = e$ and $e^t A = e^t$. The permutation matrices are the fundamental and prototypical doubly stochastic matrices, for Birkhoff's theorem says that any doubly stochastic matrix is a convex combination of finitely many permutation matrices.

In the next two theorems, some properties of doubly stochastic matrix are studied. Birkhoff theorem concerns writing the doubly stochastic matrix as a finite sum of permutations matrices. While, Gershgorin theorem consider the properties of the eigenvalues of such matrix.

Theorem A.1.1. *(Birkhoff). A matrix $A \in M_n$ is doubly stochastic if and only if there are permutation matrices $P_1, \cdots, P_N \in M_n$ and positive scalars $t_1, \cdots, t_N \in \mathbb{R}$ such that $t_1 + \cdots + t_N = 1$ and*

$$A = t_1 P_1 + \cdots + t_N P_N \,. \tag{A.1}$$

Moreover, $N \leq n^2 - n + 1$.

Theorem A.1.2. *(Gershgorin). Let $A = [a_{ij}] \in M_n$, let*

$$R_i'(A) = \sum_{j \neq i} |a_{ij}| \,, \qquad i = 1, \cdots, n \,, \tag{A.2}$$

denote the deleted absolute row sums of A, and consider the n Gershgorin discs

$$\{z \in \mathbb{C} : |z - a_{ii}| \leq R_i'(A)\} \,, \qquad i = 1, \cdots, n \,.$$

The eigenvalues of A are in the union of Gershgorin discs

$$G(A) = \bigcup_{i=1}^{n} \{z \in \mathbb{C} : |z - a_{ii}| \leq R_i'(A)\} \,. \tag{A.3}$$

Furthermore, if the union of k of the n discs that comprise $G(A)$ forms a set $G_k(A)$ that is disjoint from the remaining $n - k$ discs, then $G_k(A)$ contains exactly k eigenvalues of A, counted according to their algebraic multiplicities.

Theorem A.1.3. *(The Cauchy-Binet formula). Let $A \in M_{m,k}$, $B \in M_{k,n}$, and $C = AB$. Furthermore, let $1 \leq r \leq \min\{m, k, n\}$, and let $\alpha \subseteq \{1, \cdots, m\}$ and $\beta \subseteq \{1, \cdots, n\}$ be index sets, each of cardinality r. An expression for the α, β minor of C is*

$$\det C[\alpha, \beta] = \det A[\alpha, \gamma] \det B[\gamma, \beta] \,,$$

in which the sum is taken over all index sets $\gamma \subseteq \{1, \cdots, k\}$ of cardinality r.

Theorem A.1.4. *(Rayleigh quotient). Let $A \in M_n$ be Hermitian, let the eigenvalues of A be ordered as $\lambda_{\min} = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n = \lambda_{\max}$. Let $i_1, \ldots, i_k$ be given integers with $1 \leq i_1 < \cdots < i_k \leq n$, let $x_{i_1}, \ldots, x_{i_k}$ be orthonormal and such that $Ax_{i_p} = \lambda_{i_p} x_{i_p}$ for each $p = 1, \ldots, k$, and let $S = span\{x_{i_1}, \ldots, x_{i_k}\}$. Then*

(i)

$$\begin{aligned}\lambda_{i_1} &= \min_{x:0\neq x\in S} \frac{x^*Ax}{x^*x} = \min_{x:x\in S \text{ and } \|x\|_2=1} x^*Ax \\ &\leq \max_{x:x\in S \text{ and } \|x\|_2=1} x^*Ax = \max_{x:0\neq x\in S} \frac{x^*Ax}{x^*x} = \lambda_{i_k}.\end{aligned}$$

(ii) $\lambda_{i_1} \leq x^*Ax \leq \lambda_{i_k}$ *for any unit vector $x \in S$ with equality in the right-hand (respectively, left-hand) inequality if and only if $Ax = \lambda_{i_k} x$ (respectively, $Ax = \lambda_{i_1} x$).*

(iii) $\lambda_{\min} \leq x^*Ax \leq \lambda_{\max}$ *for any unit vector $x \in \mathbb{C}^n$, with equality in the right-hand (respectively, left-hand) inequality if and only if $Ax = \lambda_{\max} x$ (respectively, $Ax = \lambda_{min} x$); moreover,*

$$\lambda_{\max} = \max_{x\neq 0} \frac{x^*Ax}{x^*x}, \quad \text{and} \quad \lambda_{\min} = \min_{x\neq 0} \frac{x^*Ax}{x^*x}$$

Theorem A.1.5. *(Gelfand formula). Let $|||\cdot|||$ be a matrix norm on M_n and let $A \in M_n$ with spectral radius $\rho(A)$. Then,*

$$\rho(A) = \lim_{k\to\infty} |||A^k|||^{1/k}.$$

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