



Relaxation Limit from the Quantum Navier–Stokes Equations to the Quantum Drift–Diffusion Equation

Paolo Antonelli¹ · Giada Cianfarani Carnevale² · Corrado Lattanzio²  · Stefano Spirito²

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Abstract

The relaxation time limit from the quantum Navier–Stokes–Poisson system to the quantum drift–diffusion equation is performed in the framework of finite energy weak solutions. No assumptions on the limiting solution are made. The proof exploits the suitably scaled a priori bounds inferred by the energy and BD entropy estimates. Moreover, it is shown how from those estimates the Fisher entropy and free energy estimates associated to the diffusive evolution are recovered in the limit. As a byproduct, our main result also provides an alternative proof for the existence of finite energy weak solutions to the quantum drift–diffusion equation.

Keywords Quantum-Navier-Stokes · Quantum-Drift diffusion · Relaxation limit · Weak solutions · BD entropy

1 Introduction

This paper studies the relaxation time limit for the quantum Navier–Stokes–Poisson (QNSP) system with linear damping, toward the quantum drift–diffusion equation.

✉ Corrado Lattanzio
corrado@univaq.it

Paolo Antonelli
paolo.antonelli@gssi.it

Giada Cianfarani Carnevale
giada.cianfaranicarnevale@graduate.univaq.it

Stefano Spirito
stefano.spirito@univaq.it

¹ GSSI-Gran Sasso Science Institute, L'Aquila, Italy

² Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università degli Studi dell'Aquila, L'Aquila, Italy

More precisely, in the three-dimensional torus \mathbb{T}^3 , we consider a compressible, viscous fluid, whose dynamics is prescribed by

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho u) &= 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\rho Du) + \nabla \rho^\gamma + \rho \nabla V &= 2\rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) - \xi \rho u \\ -\Delta V &= \rho - g. \end{aligned} \quad (1.1)$$

Here, the unknowns ρ , u , Du and V denote the particle density, the velocity field, the symmetric part of the gradient ∇u and the electrostatic potential, respectively. The function g is given and represents the doping profile.

The system arises in the macroscopic description of electron transport in nanoscale semiconductor devices (Jüngel 2009), where quantum mechanical effects must be taken into account. In this context the dissipative term $-\xi \rho u$ describes collisions between electrons and the semiconductor crystal lattice (see, for instance, Baccarani and Wordeman 1985), and $\tau = 1/\xi$ is the relaxation time. The advantage of using macroscopic models for quantum fluids, with respect to kinetic models, is their reduced complexity, especially from a computational point of view (Markowich et al. 1990). Moreover, hydrodynamic models correctly describe high field phenomena or submicronic devices. However, in certain regimes, as in particular for low carrier densities and small electric fields, these models can be further reduced to some simpler ones. In the context of semiconductor devices for instance, quantum transport of electrons can be effectively described by the quantum drift–diffusion (QDD) equation (Roosbroeck 1950), given by

$$\begin{aligned} \partial_t \rho + \operatorname{div} \left(2\rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) - \nabla \rho^\gamma - \rho \nabla V \right) &= 0 \\ -\Delta V &= \rho - g. \end{aligned} \quad (1.2)$$

The (QDD) equation can be formally recovered from system (1.1) as a relaxation limit. Precisely, by rescaling the time as follows

$$t' = \epsilon t, \quad (\rho^\epsilon, u^\epsilon)(t', x) = (\rho, u) \left(\frac{t'}{\epsilon}, x \right), \quad (1.3)$$

where $\epsilon := 1/\xi$, the scaled system reads

$$\begin{aligned} \partial_t \rho_\epsilon + \frac{1}{\epsilon} \operatorname{div}(\rho_\epsilon u_\epsilon) &= 0 \\ \partial_t(\rho_\epsilon u_\epsilon) + \frac{1}{\epsilon} \operatorname{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon) - \frac{1}{\epsilon} \operatorname{div}(\rho_\epsilon Du_\epsilon) + \frac{1}{\epsilon} \nabla \rho_\epsilon^\gamma + \frac{1}{\epsilon} \rho_\epsilon \nabla V_\epsilon &= 0 \\ &= \frac{1}{\epsilon} 2\rho_\epsilon \nabla \left(\frac{\Delta \sqrt{\rho_\epsilon}}{\sqrt{\rho_\epsilon}} \right) - \frac{1}{\epsilon^2} \rho_\epsilon u_\epsilon - \Delta V_\epsilon = \rho_\epsilon - g. \end{aligned} \quad (1.4)$$

Thus, in the limit $\epsilon \rightarrow 0$, we formally obtain that

$$\lim_{\epsilon \rightarrow 0} \rho_\epsilon \frac{u_\epsilon}{\epsilon} = 2\rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) - \nabla \rho^\gamma - \rho \nabla V \tag{1.5}$$

and therefore the (QDD) equation.

Remark 1.1 It is worth observing that, as already noted in Cianfarani and Lattanzio (2020), under the scaling (1.3), the diffusion term does not appear in the limit, in view of the value of the momentum at the limit given in (1.5). Indeed, if we perform the following Hilbert expansion:

$$\begin{aligned} \rho^\epsilon &= \rho^0 + \epsilon \rho^1 + \dots \\ u^\epsilon &= u^0 + \epsilon u^1 + \dots \end{aligned} \tag{1.6}$$

and substitute (1.6) in (1.4), we deduce:

$$\begin{aligned} O\left(\frac{1}{\epsilon^2}\right) : u^0 &= 0; \\ O\left(\frac{1}{\epsilon}\right) : \rho^0 u^1 &= 2\rho^0 \nabla \left(\frac{\Delta \sqrt{\rho^0}}{\sqrt{\rho^0}} \right) - \nabla \rho^{0\gamma} - \rho^0 \nabla V^0 \\ &= \operatorname{div}(\rho^0 \nabla^2 \log \rho^0) - \nabla \rho^{0\gamma} - \rho^0 \nabla V^0; \\ O(1) : \rho^0 u^2 + \rho^1 u^1 &= -\nabla(\gamma(\rho^1)^{\gamma-1} \rho^0) \\ &\quad + \operatorname{div}(\rho^1 \nabla^2 \log \rho^0 + \rho^0 \nabla^3 \log \rho^1) + \operatorname{div}(\rho^0 D(u^1)). \end{aligned}$$

Thus, the dissipation term ρDu does not appear in the zero relaxation limit, but it is effective only at higher orders.

The main purpose of our paper is to rigorously prove the above limit, that is to prove that scaled finite energy weak solutions to (1.1) converge to finite energy weak solutions to (1.2). To this aim, in the following we shall refer to (1.4) with initial datum (ρ^0, u^0) and doping profile g possibly depending in a suitable way on the relaxation parameter ϵ as well.

Theorem 1.2 *Let $(\rho_\epsilon, u_\epsilon, V_\epsilon)$ be a weak solution of (1.1) in the sense of Definition 2.1 with data $(\rho_\epsilon^0, u_\epsilon^0, g_\epsilon)$ satisfying*

- $\{\rho_\epsilon^0\}_\epsilon$ is bounded in $L^1 \cap L^\gamma(\mathbb{T}^3)$ such that $\rho_\epsilon^0 \rightarrow \rho^0$ in $L^q(\mathbb{T}^3)$, $q < 3$ and $\gamma > 1$
- $\{\nabla \sqrt{\rho_\epsilon^0}\}_\epsilon$ is bounded in $L^2(\mathbb{T}^3)$,
- $\{\sqrt{\rho_\epsilon^0} u_\epsilon^0\}_\epsilon$ is bounded in $L^2(\mathbb{T}^3)$,
- $\{g_\epsilon\}_\epsilon$ is bounded in $L^2(\mathbb{T}^3)$ such that $g_\epsilon \rightarrow g$ in $L^2(\mathbb{T}^3)$.

Then, up to subsequences, there exists $\rho \geq 0$ and V such that

$$\begin{aligned} \sqrt{\rho_\epsilon} &\rightarrow \sqrt{\rho} \text{ strongly in } L^2((0, T); H^1(\mathbb{T}^3)) \\ \nabla V_\epsilon &\rightarrow \nabla V \text{ strongly in } C([0, T]; L^2(\mathbb{T}^3)), \end{aligned}$$

and (ρ, V) is a finite energy weak solution of ρ of (1.2) with initial datum $\rho(0) = \rho^0$, in the sense of Definition 2.8. Namely there exist $\Lambda, \mathcal{S} \in L^2((0, T) \times \mathbb{T}^3)$ such that

$$\sqrt{\rho}\Lambda = 2 \operatorname{div}(\sqrt{\rho}\nabla^2\sqrt{\rho} - \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} - \rho^\gamma \mathbb{I}) - \rho\nabla V \text{ in } \mathcal{D}'((0, T) \times \mathbb{T}^3) \tag{1.7}$$

$$\sqrt{\rho}\mathcal{S} = 2\sqrt{\rho}\nabla^2\sqrt{\rho} - 2\nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} \text{ a.e. in } (0, T) \times \mathbb{T}^3 \tag{1.8}$$

and $C > 0$ such that for a.e. $t \in (0, T)$

$$\int_{\mathbb{T}^3} \left(|\nabla\sqrt{\rho}|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{1}{2}|\nabla V|^2 \right) (t)dx + \int_0^t \int_{\mathbb{T}^3} |\Lambda|^2 dsdx \leq C \tag{1.9}$$

$$\begin{aligned} \int_{\mathbb{T}^3} (\rho(\log \rho - 1) + 1)(t)dx + \int_0^t \int_{\mathbb{T}^3} |\mathcal{S}|^2 dsdx + \frac{4}{\gamma} \int_0^t \int_{\mathbb{T}^3} |\nabla\rho^{\frac{\gamma}{2}}|^2 dsdx \\ + \int_0^t \int_{\mathbb{T}^3} \rho(\rho - g) dsdx \leq \int_{\mathbb{T}^3} (\rho^0(\log \rho^0 - 1) + 1)dx. \end{aligned} \tag{1.10}$$

Moreover, if in addition, the initial data also satisfy

$$\begin{aligned} \sqrt{\rho_\epsilon^0}u_\epsilon^0 &\rightarrow 0 \text{ strongly in } L^2(\mathbb{T}^3) \\ \nabla\sqrt{\rho_\epsilon^0} &\rightarrow \nabla\sqrt{\rho^0} \text{ strongly in } L^2(\mathbb{T}^3) \\ \rho_\epsilon^0 &\rightarrow \rho^0 \text{ strongly in } L^\gamma(\mathbb{T}^3), \end{aligned}$$

then ρ is an energy dissipating weak solution, meaning that in addition to be a finite energy weak solution for a.e. $t \in (0, T)$ it holds

$$\begin{aligned} \int_{\mathbb{T}^3} \left(|\nabla\sqrt{\rho}|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{1}{2}|\nabla V|^2 \right) (t)dx + \int_0^t \int_{\mathbb{T}^3} |\Lambda|^2 dsdx \\ \leq \int_{\mathbb{T}^3} |\nabla\sqrt{\rho^0}|^2 dx + \int_{\mathbb{T}^3} \frac{(\rho^0)^\gamma}{\gamma-1} dx + \int_{\mathbb{T}^3} \frac{1}{2}|\nabla V(0)|^2 dx. \end{aligned}$$

Let us notice that the estimates (1.9) and (1.10) yield the boundedness of the Fisher entropy and the free energy, respectively. On the other hand, the quantities Λ and \mathcal{S} characterized in (1.7) and (1.8) provide the associated entropy dissipations, in a weaker sense than the estimates derived in Gianazza et al. (2009) and Jüngel and Matthes (2008). Indeed, formally

$$\Lambda = 2\sqrt{\rho}\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \frac{\nabla\rho^\gamma}{\sqrt{\rho}}, \quad \mathcal{S} = \sqrt{\rho}\nabla^2\log\rho, \quad (1.11)$$

but, due to the low regularity setting and the possible presence of vacuum regions it seems not possible to obtain the relations (1.11) in the limit, so the only available information we have is given by formulas (1.7) and (1.8). We refer to Remark 2.4 and Proposition 2.11 below for more details on the tensor Λ and \mathcal{S} .

An exhaustive list of all references concerning diffusive relaxation limits and asymptotic behavior for systems of conservation laws with friction, and in particular for hydrodynamic models for semiconductors, is beyond the interest of our presentation. For the theory of diffusive relaxation, we refer here to Donatelli and Marcati (2004), concerning in particular the case of multidimensional general semilinear systems, and the reference therein. Moreover, concerning in particular the case of high friction limits with relative entropy techniques in the context of Korteweg theories (Giesselmann et al. 2017), we refer to Lattanzio and Tzavaras (2017); Cianfarani and Lattanzio (2020); see also Lattanzio and Tzavaras (2013) for the case of Euler equations with friction. Finally, for the particular case of Euler–Poisson models for semiconductors, we recall that the rigorous analysis of the diffusive relaxation limits in the context of weak, entropic solutions started with the seminal paper (Marcati and Natalini 1995), where the one-dimensional case is treated using compensated compactness; see also Lattanzio and Marcati (1999); Lattanzio (2000) for the multi- d case.

Besides the modeling point of view, there are some other mathematical aspects which motivate our result. First of all, the study of this singular limit is related to the asymptotic behavior of solutions to (1.1) for large times. Let $(\rho^*, u^*) = (r, 0)$ be the stationary solution to (1.1), where, for g constant, $r = \int \rho = g$ is the mean value of the particle density. Then it can be shown that solutions to (1.1) exponentially converge toward (ρ^*, u^*) as $t \rightarrow \infty$, see Gualdani et al. (2003) for the one-dimensional problem (with suitable boundary conditions, see also Liang and Zheng (2008) for some extensions) and Bresch et al. (2020) for the proof of this result in the framework of finite energy weak solutions in the three-dimensional torus.

On the other hand, it is also interesting to determine the asymptotic dynamics which governs the exponential convergence to equilibrium. This is indeed achieved by performing the scaling in (1.3); hence, the (QDD) equation (1.2) also gives the asymptotic dynamics we are interested in.

On a related subject, let us also comment on the inviscid counterpart of system (1.1), namely the quantum hydrodynamic (QHD) system (Antonelli and Marcati 2009, 2016). Due to the dissipative term $-\xi\rho u$, also in this case it is possible to show both the exponential convergence toward the stationary solution (Huang et al. 2006, 2010) and the relaxation limit (Jüngel et al. 2006), again toward the (QDD) equation. However the only available results here deal with small, regular perturbations around stationary solutions. This can be seen as due to the lack of regularizing effect of the viscosity, by means of the BD entropy estimates.

Notice that in Theorem 1.2 the only assumption needed is the initial energy associated to the system (1.1) to be uniformly bounded at the initial time. In particular, no assumptions on the limiting solution to (1.2) are given. Consequently, as a byprod-

uct our main theorem also provides an alternative proof for the existence of finite energy weak solutions to (1.2), see Jüngel and Matthes (2008) and Gianazza et al. (2009). Furthermore, in the proof of our main result it is possible to see how the energy and BD entropy estimates, respectively, associated to (1.1) yield, in the limit $\varepsilon \rightarrow 0$, the Fisher and free energy, respectively, associated to (1.2). Those facts were already noticed, in a similar context, in the recent proceeding (Bresch et al.), see also Bresch et al. (2019). More precisely, the authors in Bresch et al. consider the one-dimensional shallow water equations with a nonlinear damping term. By using a similar scaling as in (1.3), the authors study the convergence toward a lubrication type model. In particular, in Bresch et al. the authors emphasize how the BD entropy for the hydrodynamical system converges toward the so-called Bernis–Friedman Bernis and Friedman (1990) entropy, associated to the limiting diffusive equation.

Let us remark that also in the context of semiconductor device modeling it would make sense to consider a nonlinear damping term, as in Bresch et al. Indeed this would correspond to the case when the relaxation time τ is no longer a constant, but a function of the particle density. This is consistent with the derivation of the hydrodynamic system from kinetic theories, as in general the relaxation coefficient may depend on the particle density.

Finally, our result can also be seen as related to the derivation of (1.1) and (1.2) from kinetic equations. These macroscopic models for quantum transport are usually derived from collisional Wigner-type equations, with a suitable choice of the collision operator, see Degond and Ringhofer (2003); Degond et al. (2005) and Jüngel (2012) for a more comprehensive discussion about those issues. In particular, the QNS system with a linear damping was derived in Jüngel and Milišić (2011) by applying the moment method to a Wigner-type equation whose collisional operator is chosen to be the sum of a BGK and a Caldeira–Leggett-type operator, see also Jüngel et al. (2011) where an alternative derivation is given by avoiding the Chapman–Enskog expansion. Actually in Jüngel and Milišić (2011) and Jüngel et al. (2011) the authors derive the full QNS system, where also the dynamics of the energy density is given, in our paper we only consider the isentropic dynamics given by (1.1). We also mention Brull and Méhats (2010) where the QNS system without damping is derived. On the other hand, the QDD equation can also be derived from the same Wigner-type equation by using a diffusive scaling. In this sense our result, obtained by using the scaling (1.3), can be seen as linking the two different scalings used to derive (1.1) and (1.2) directly from kinetic models.

Organization of the Paper

The paper is organized as follows. In Sect. 2 we give the definition of weak solution for the quantum Navier–Stokes system and the quantum drift–diffusion equation, we give a formal proof of the ε -independent estimates and we state the main theorem and in Sect. 3 we prove the main result of the paper.

Notations

We denote with $L^p(\mathbb{T}^n)$ the standard Lebesgue spaces. The Sobolev space of L^p functions with k distributional derivatives in L^p is denoted $W^{k,p}$; in the case $p = 2$ we write $H^k(\mathbb{T}^n)$. The spaces $W^{-k,p}$ and H^{-k} denote the dual spaces of $W^{k,p'}$ and H^k where p' is the usual Hölder conjugate of p . Given a Banach space B , the classical Bochner space of real valued function with values in B is denoted by $L^p(0, T; B)$ and sometimes also the abbreviation $L^p_t(B_x)$ will be used. Given a function $f \in L^p(\mathbb{T}^3)$, we denote the average of f alternatively \bar{f} or \tilde{f} , and throughout the paper we can assume without loss of generality that $|\mathbb{T}^3| = 1$. We denote with $Du = (\nabla u + \nabla u^t)/2$ the symmetric part of the Jacobian matrix ∇u and with $Au = (\nabla u - \nabla u^t)/2$ the antisymmetric part. Moreover, for a generic tensor \mathcal{A} , we denote with \mathcal{A}^s and \mathcal{A}^a , respectively, its symmetric and its antisymmetric part.

Finally, the subscript ϵ used to denote sequences of functions has to be always understood running over a countable set.

2 Definition of Weak Solutions and Main Result

In this section we give the definition of weak solutions for the system (2.1) and (1.2). The existence of weak solutions of (2.1) is out of the scope of this paper. On the other hand, in Appendix A we provide an overview of the main ideas of the approximation procedure, which is based on a compactness argument as in Lacroix-Violet and Vasseur (2017); Antonelli and Spirito (2017), see also Antonelli and Spirito (2018, 2019a, b).

2.1 Weak Solutions of the Quantum Navier–Stokes–Poisson Equations

Let us consider the following system in $(0, T) \times \mathbb{T}^3$ for a given $g_\epsilon : \mathbb{T}^3 \rightarrow \mathbb{R}$,

$$\begin{aligned} \partial_t \rho_\epsilon + \frac{1}{\epsilon} \operatorname{div}(\rho_\epsilon u_\epsilon) &= 0 \\ \partial_t(\rho_\epsilon u_\epsilon) + \frac{1}{\epsilon} \operatorname{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon) - \frac{1}{\epsilon} \operatorname{div}(\rho_\epsilon Du_\epsilon) + \frac{1}{\epsilon} \nabla \rho_\epsilon^\gamma + \frac{1}{\epsilon} \rho_\epsilon \nabla V_\epsilon &= \rho_\epsilon - g_\epsilon \quad (2.1) \\ &= \frac{1}{\epsilon} 2\rho_\epsilon \nabla \left(\frac{\Delta \sqrt{\rho_\epsilon}}{\sqrt{\rho_\epsilon}} \right) - \frac{1}{\epsilon^2} \rho_\epsilon u_\epsilon - \Delta V_\epsilon = \rho_\epsilon - g_\epsilon \end{aligned}$$

with initial data

$$\begin{aligned} \rho_\epsilon(0, x) &= \rho_\epsilon^0(x), \\ \rho_\epsilon(0, x)u_\epsilon(0, x) &= \rho_\epsilon^0(x)u_\epsilon^0(x), \end{aligned} \quad (2.2)$$

on $\{t = 0\} \times \mathbb{T}^3$ and zero average condition for V_ϵ , namely

$$\int_{\mathbb{T}^3} V_\epsilon(x, t) dx = 0. \quad (2.3)$$

We emphasize that in order to solve (2.1)₃, (2.3) we need the following assumption on the doping profile $g_\epsilon: \int_{\mathbb{T}^3} g_\epsilon(x) dx = M_\epsilon$ and $M_\epsilon > 0$ where $M_\epsilon := \int_{\mathbb{T}^3} \rho_\epsilon^0(x)$; see also Remark 2.6 below.

The definition of weak solution is the following.

Definition 2.1 Given ρ_ϵ^0 positive and such that $\sqrt{\rho_\epsilon^0} \in H^1(\mathbb{T}^3)$ and $\rho_\epsilon^0 \in L^\gamma(\mathbb{T}^3)$, $g_\epsilon \in L^2(\mathbb{T}^3)$ and u_ϵ^0 such that $\sqrt{\rho_\epsilon^0} u_\epsilon^0 \in L^2(\mathbb{T}^3)$, then $(\rho_\epsilon, u_\epsilon, V_\epsilon)$ with $\rho_\epsilon \geq 0$ and V_ϵ with zero average is a weak solution of the Cauchy problem (2.1)–(2.2) if the following conditions are satisfied.

(1) Integrability conditions:

$$\begin{aligned} \sqrt{\rho_\epsilon} &\in L^\infty(0, T; H^1(\mathbb{T}^3)) \cap L^2(0, T; H^2(\mathbb{T}^3)), & \sqrt{\rho_\epsilon} u_\epsilon &\in L^\infty(0, T; L^2(\mathbb{T}^3)), \\ \rho_\epsilon^\gamma &\in L^\infty(0, T; L^1(\mathbb{T}^3)), & \mathcal{T}_\epsilon &\in L^2(0, T; L^2(\mathbb{T}^3)), \\ \rho_\epsilon &\in C(0, T; L^2(\mathbb{T}^3)), & V_\epsilon &\in C(0, T; H^2(\mathbb{T}^3)). \end{aligned}$$

(2) Continuity equation:

For any $\phi \in C^\infty([0, T) \times \mathbb{T}^3; \mathbb{R})$ such that $\phi(T) = 0$

$$\int \rho_\epsilon^0 \phi(0) dx + \iint \rho_\epsilon \phi_t + \frac{1}{\epsilon} \sqrt{\rho_\epsilon} \sqrt{\rho_\epsilon} u_\epsilon \nabla \phi \, ds dx = 0. \tag{2.4}$$

(3) Momentum equation:

For any $\psi \in C^\infty([0, T) \times \mathbb{T}^3; \mathbb{R}^3)$ such that $\psi(T) = 0$

$$\begin{aligned} &\int \rho_\epsilon^0 u_\epsilon^0 \psi(0) dx + \iint \sqrt{\rho_\epsilon} (\sqrt{\rho_\epsilon} u_\epsilon) \psi_t \, ds dx + \frac{1}{\epsilon} \iint \sqrt{\rho_\epsilon} u_\epsilon \sqrt{\rho_\epsilon} u_\epsilon \cdot \nabla \psi \, ds dx \\ &- \frac{1}{\epsilon} \iint \sqrt{\rho_\epsilon} \mathcal{T}_\epsilon^s : \nabla \psi \, ds dx + \frac{1}{\epsilon} \iint \rho_\epsilon^\gamma \operatorname{div} \psi \, ds dx - \frac{1}{\epsilon} \iint \rho_\epsilon \nabla V_\epsilon \psi \, ds dx \\ &- \frac{1}{\epsilon} \iint 2\sqrt{\rho_\epsilon} \nabla^2 \sqrt{\rho_\epsilon} : \nabla \psi \, ds dx + \frac{1}{\epsilon} \iint 2\nabla \sqrt{\rho_\epsilon} \otimes \nabla \sqrt{\rho_\epsilon} : \nabla \psi \, ds dx \\ &+ \frac{1}{\epsilon^2} \iint \rho_\epsilon u_\epsilon \psi \, ds dx = 0. \end{aligned} \tag{2.5}$$

(4) Poisson equation:

For a.e. $(t, x) \in (0, T) \times \mathbb{T}^3$ it holds that

$$-\Delta V_\epsilon = \rho_\epsilon - g_\epsilon. \tag{2.6}$$

(5) Energy dissipation:

For any $\varphi \in C^\infty([0, T) \times \mathbb{T}^3; \mathbb{R})$

$$\iint \sqrt{\rho_\epsilon} \mathcal{T}_{\epsilon, i, j} \varphi \, ds dx = - \iint \rho_\epsilon u_{\epsilon, i} \nabla_j \varphi \, ds dx - \iint 2\sqrt{\rho_\epsilon} u_{\epsilon, i} \otimes \nabla_j \sqrt{\rho_\epsilon} \varphi \, ds dx. \tag{2.7}$$

(6) Energy Inequality:

For a.e. $t \in (0, T)$

$$\begin{aligned} & \int_{\mathbb{T}^3} \left(\frac{1}{2} \rho_\epsilon |u_\epsilon|^2 + \frac{\rho_\epsilon^\gamma}{\gamma - 1} + 2|\nabla \sqrt{\rho_\epsilon}|^2 + \frac{1}{2} |\nabla V_\epsilon|^2 \right) (t) dx \\ & + \frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^3} |\mathcal{T}_\epsilon^s|^2 ds dx + \frac{1}{\epsilon^2} \int_0^t \int_{\mathbb{T}^3} \rho_\epsilon |u_\epsilon|^2 ds dx \\ & \leq \int_{\mathbb{T}^3} \left(\frac{1}{2} \rho_\epsilon^0 |u_\epsilon^0|^2 + \frac{(\rho_\epsilon^0)^\gamma}{\gamma - 1} + 2|\nabla \sqrt{\rho_\epsilon^0}|^2 + \frac{1}{2} |\nabla V_\epsilon(0)|^2 \right) dx. \end{aligned} \tag{2.8}$$

(7) BD entropy: Let $w_\epsilon = u_\epsilon + \nabla \log \rho_\epsilon$, then there exists a tensor $\mathcal{S}_\epsilon \in L^2(0, T; L^2(\mathbb{T}^3))$ such that

$$\sqrt{\rho_\epsilon} \mathcal{S}_\epsilon = 2\sqrt{\rho_\epsilon} \nabla^2 \sqrt{\rho_\epsilon} - 2\nabla \sqrt{\rho_\epsilon} \otimes \nabla \sqrt{\rho_\epsilon} \text{ a.e. in } (0, T) \times \mathbb{T}^3 \tag{2.9}$$

we have for a.e. $t \in (0, T)$

$$\begin{aligned} & \int_{\mathbb{T}^3} \left(\frac{1}{2} \epsilon \rho_\epsilon |w_\epsilon|^2 + \epsilon \frac{\rho_\epsilon^\gamma}{\gamma - 1} + \epsilon 2|\nabla \sqrt{\rho_\epsilon}|^2 + \frac{1}{2} \epsilon |\nabla V_\epsilon|^2 + (\rho_\epsilon (\log \rho_\epsilon - 1) + 1) \right) (t) dx \\ & + \int_0^t \int_{\mathbb{T}^3} |\mathcal{T}_\epsilon^a|^2 ds dx + \int_0^t \int_{\mathbb{T}^3} |\mathcal{S}_\epsilon|^2 ds dx + \frac{4}{\gamma} \int_0^t \int_{\mathbb{T}^3} |\nabla \rho_\epsilon^{\frac{\gamma}{2}}|^2 ds dx \\ & + \frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^3} \rho_\epsilon |u_\epsilon|^2 ds dx + \int_0^t \int_{\mathbb{T}^3} \rho_\epsilon (\rho_\epsilon - g_\epsilon) ds dx \\ & \leq \int_{\mathbb{T}^3} \left(\frac{1}{2} \epsilon \rho_\epsilon^0 |w_\epsilon^0|^2 + \epsilon \frac{(\rho_\epsilon^0)^\gamma}{\gamma - 1} + \epsilon 2|\nabla \sqrt{\rho_\epsilon^0}|^2 + \frac{1}{2} \epsilon |\nabla V_\epsilon(0)|^2 + (\rho_\epsilon^0 (\log \rho_\epsilon^0 - 1) + 1) \right) dx. \end{aligned} \tag{2.10}$$

(8) There exists an absolute constant C such that

$$\begin{aligned} & \int_0^t \int_{\mathbb{T}^3} |\nabla^2 \sqrt{\rho_\epsilon}|^2 + |\nabla \rho_\epsilon^{\frac{1}{4}}|^4 ds dx \leq C \int_{\mathbb{T}^3} \left(\frac{1}{2} \epsilon \rho_\epsilon^0 |w_\epsilon^0|^2 + \epsilon \frac{(\rho_\epsilon^0)^\gamma}{\gamma - 1} + \epsilon 2|\nabla \sqrt{\rho_\epsilon^0}|^2 \right) dx \\ & + C \int_{\mathbb{T}^3} \left(\frac{1}{2} \epsilon |\nabla V_\epsilon(0)|^2 + (\rho_\epsilon^0 (\log \rho_\epsilon^0 - 1) + 1) \right) dx \end{aligned} \tag{2.11}$$

The following remarks aim at explaining some peculiar points of the definition of weak solutions. In particular, we explain the presence of the tensors \mathcal{T}_ϵ and \mathcal{S}_ϵ in the following remarks, and the reason why we must further assume the bound (2.11).

Remark 2.2 (Weak formulation of the quantum term) We emphasize that in the weak formulation introduced above, the third-order term in the momentum equation can be written in different ways:

$$2\rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = \operatorname{div}(\rho \nabla^2 \log \rho) = \operatorname{div}(2\sqrt{\rho} \nabla^2 \sqrt{\rho} - 2\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}), \tag{2.12}$$

and we are using the last expression to give a distributional meaning to the third-order tensor term.

Remark 2.3 (*The velocity field and the vacuum set*) We stress that in the definition of weak solutions the vacuum region can be of positive measure. As a consequence, the velocity it is not uniquely defined in the vacuum set, namely if we change its value on $\{\rho = 0\}$ we would still have the same weak solution of (2.1)–(2.2). Moreover, even if we choose $u = 0$ on $\{\rho = 0\}$ we cannot deduce a priori bound in any Lebesgue space since, from the energy estimate (2.8) and the BD entropy (2.10), the uniform bounds on the velocity are always weighted by a power of ρ_ϵ .

Remark 2.4 (*About the tensors \mathcal{T}_ϵ and \mathcal{S}_ϵ*)

The presence of the tensor \mathcal{T}_ϵ is due to the possible presence of vacuum regions. Indeed, if the density is bounded away from zero, (2.7) implies that $\mathcal{T}_\epsilon = \sqrt{\rho_\epsilon} \nabla u_\epsilon$. On the other hand, even in the case when the vacuum has zero Lebesgue measure, ∇u_ϵ also cannot be defined in a distributional sense. Therefore, the tensor \mathcal{T}_ϵ arises as a weak L^2 -limit of the sequence $\{\sqrt{\rho_\epsilon^n} \nabla u_\epsilon^n\}_n$ of approximations, see Lacroix-Violet and Vasseur (2017); Antonelli and Spirito (2017). Analogously, \mathcal{S}_ϵ is again motivated by the presence of vacuum regions. Indeed, also in this case, if the density is bounded away from zero, by using the identity (2.12), we have that $\mathcal{S}_\epsilon = \sqrt{\rho_\epsilon} \nabla^2 \log \rho_\epsilon$. As for \mathcal{T}_ϵ , the tensor \mathcal{S}_ϵ arises as an L^2 weak limit of the sequence $\{\sqrt{\rho_\epsilon^n} \nabla^2 \log \rho_\epsilon^n\}_n$, where again $\{\rho_\epsilon^n, u_\epsilon^n\}_n$ is a suitable sequence of approximations. Notice that the fact that $\{\sqrt{\rho_\epsilon^n} \nabla^2 \log \rho_\epsilon^n\}_n$ is bounded in $L^2((0, T) \times \mathbb{T}^3)$ because of the BD entropy, see Proposition 2.7. We refer to Appendix A, where a description of the approximation scheme and a more precise explanation of the tensors \mathcal{T}_ϵ and \mathcal{S}_ϵ are provided.

Remark 2.5 (*About the inequality (2.11)*) Regarding (2.11) we recall that in Jüngel and Matthes (2008) it is proved there exists $C > 0$ depending only on the dimension such that for any smooth function $\rho > 0$ it holds

$$\iint |\nabla \rho^{\frac{1}{4}}|^4 ds dx + \iint |\nabla^2 \sqrt{\rho}|^2 ds dx \leq C \iint \rho |\nabla^2 \log \rho|^2 ds dx. \quad (2.13)$$

The inequality (2.11) follows by applying (2.13) to the approximation ρ_ϵ^n , which is smooth and such that the quantity $\sqrt{\rho_\epsilon^n} \nabla^2 \log \rho_\epsilon^n$ is well-defined, and then using the bound on $\sqrt{\rho_\epsilon^n} \nabla^2 \log \rho_\epsilon^n$ inferred from the approximated version of the BD entropy and using the weak lower-semicontinuity of the norms. For completeness we give more details on the inequality (2.11) in the Appendix A and we give the full proof of (2.13) in Appendix B.

Remark 2.6 (*About the integrability conditions and the Poisson equation*) We notice that by using the integrability hypothesis and (2.4) the average of the density is conserved, namely $\int_{\mathbb{T}^3} \rho_\epsilon(t, x) dx = M_\epsilon$ for any $t \in (0, T)$. Therefore, $\rho_\epsilon - g$ has zero average for any $t \in (0, T)$ and then the compatibility for the Poisson equation (2.1)₃ is always satisfied.

Next, from the minimum regularity required for ρ_ϵ to be a weak, finite energy solution, we readily obtain $\rho_\epsilon \in C([0, T]; L^2(\mathbb{T}^3))$; this (redundant) condition is thus listed

explicitly in Definition 2.1. Hence, $\rho_\epsilon|_{t=0} = \rho_\epsilon^0$ in $L^2(\mathbb{T}^3)$ and we have that $V_\epsilon(0)$ is well-defined and coincides with the solution of the corresponding elliptic equation at $t = 0$, that is:

$$-\Delta V_\epsilon(0) = \rho_\epsilon^0 - g_\epsilon. \tag{2.14}$$

Finally, since ρ_ϵ^0 and g_ϵ are both bounded in $L^2(\mathbb{T}^3)$ we have that $V_\epsilon|_{t=0} = V_\epsilon(0)$ is bounded in $H^2(\mathbb{T}^3)$.

Note that, since weak solutions have a limited amount of regularity, the energy inequality and the BD entropy must be included in the definition. In particular, being the proof of the existence based on a compactness argument, (2.8) and (2.10) are proved for approximate solutions and then are obtained through a limiting argument; we refer again to the Appendix A. On the other hand, as usual in PDE they are motivated by formal estimate. This is exactly the content of the next proposition.

Proposition 2.7 *Let $\epsilon > 0$ and $(\rho_\epsilon, u_\epsilon, V_\epsilon)$ be a sequence of smooth solutions of (2.1) with initial data (2.2). Define $w_\epsilon = u_\epsilon + \nabla \log \rho_\epsilon$. Then, for any $t \in [0, T)$ the pair $(\rho_\epsilon, u_\epsilon, V_\epsilon)$ satisfies the following estimates:*

(1) (Energy Estimate)

$$\begin{aligned} & \int_{\mathbb{T}^3} \left(\frac{1}{2} \rho_\epsilon |u_\epsilon|^2 + \frac{\rho_\epsilon^\gamma}{\gamma - 1} + 2|\nabla \sqrt{\rho_\epsilon}|^2 + \frac{1}{2} |\nabla V_\epsilon|^2 \right) (t) dx \\ & + \frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^3} \rho_\epsilon |Du_\epsilon|^2 ds dx + \frac{1}{\epsilon^2} \int_0^t \int_{\mathbb{T}^3} \rho_\epsilon |u_\epsilon|^2 ds dx \tag{2.15} \\ & = \int_{\mathbb{T}^3} \left(\frac{1}{2} \rho_\epsilon^0 |u_\epsilon^0|^2 + \frac{(\rho_\epsilon^0)^\gamma}{\gamma - 1} + 2|\nabla \sqrt{\rho_\epsilon^0}|^2 + \frac{1}{2} |\nabla V_\epsilon(0)|^2 \right) dx. \end{aligned}$$

(2) (BD entropy)

$$\begin{aligned} & \int_{\mathbb{T}^3} \left(\frac{1}{2} \epsilon \rho_\epsilon |w_\epsilon|^2 + \epsilon \frac{\rho_\epsilon^\gamma}{\gamma - 1} + \epsilon 2|\nabla \sqrt{\rho_\epsilon}|^2 + \epsilon \frac{1}{2} |\nabla V_\epsilon|^2 \right. \\ & \left. + (\rho_\epsilon (\log \rho_\epsilon - 1) + 1) \right) (t) dx \\ & + \int_0^t \int_{\mathbb{T}^3} \rho_\epsilon |Au_\epsilon|^2 ds dx \\ & + \int_0^t \int_{\mathbb{T}^3} \rho_\epsilon |\nabla^2 \log \rho_\epsilon|^2 ds dx + \frac{4}{\gamma} \int_0^t \int_{\mathbb{T}^3} |\nabla \rho_\epsilon^{\frac{\gamma}{2}}|^2 ds dx \tag{2.16} \\ & + \int_0^t \int_{\mathbb{T}^3} \rho_\epsilon (\rho_\epsilon - g_\epsilon) ds dx + \frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^3} \rho_\epsilon |u_\epsilon|^2 ds dx \\ & = \int_{\mathbb{T}^3} \left(\frac{1}{2} \epsilon \rho_\epsilon^0 |w_\epsilon^0|^2 + \epsilon \frac{(\rho_\epsilon^0)^\gamma}{\gamma - 1} + \epsilon 2|\nabla \sqrt{\rho_\epsilon^0}|^2 + \frac{1}{2} \epsilon |\nabla V_\epsilon(0)|^2 \right. \\ & \left. + (\rho_\epsilon^0 (\log \rho_\epsilon^0 - 1) + 1) \right) dx. \end{aligned}$$

Proof Let us start by recalling the following alternative ways to write the dispersive term:

$$2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \operatorname{div}(\rho\nabla^2\log\rho) = \operatorname{div}(2\sqrt{\rho}\nabla^2\sqrt{\rho} - 2\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}).$$

Then, for smooth solutions $(\rho_\epsilon, u_\epsilon)$ of (2.1), as usual we multiply the continuity equation (2.1)₁ by $\gamma\rho_\epsilon^{\gamma-1}/(\gamma-1)$ and the momentum equation (2.1)₂ by u_ϵ and integrate in space to get:

$$\frac{d}{dt}\int_{\mathbb{T}^3}\frac{\rho_\epsilon^\gamma}{\gamma-1}dx - \frac{1}{\epsilon}\int_{\mathbb{T}^3}\nabla\rho_\epsilon^\gamma u_\epsilon dx = 0. \tag{2.17}$$

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{T}^3}\left(\frac{1}{2}\rho|u_\epsilon|^2\right)dx + \frac{1}{\epsilon}\int_{\mathbb{T}^3}\rho|Du_\epsilon|^2dx + \frac{1}{\epsilon}\int_{\mathbb{T}^3}\nabla\rho^\gamma\cdot u_\epsilon dx \\ + \frac{1}{\epsilon}\int_{\mathbb{T}^3}\rho_\epsilon\nabla^2\log\rho_\epsilon:\nabla u_\epsilon dx + \frac{1}{\epsilon^2}\int_{\mathbb{T}^3}\rho|u_\epsilon|^2dx + \frac{1}{\epsilon}\int_{\mathbb{T}^3}\rho_\epsilon\nabla V_\epsilon u_\epsilon dx = 0, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{T}^3}\frac{1}{2}|\nabla V_\epsilon|^2 dx = -\int_{\mathbb{T}^3}(\Delta V_\epsilon)_t V_\epsilon dx = \int_{\mathbb{T}^3}\partial_t\rho_\epsilon V_\epsilon dx \\ = -\frac{1}{\epsilon}\int_{\mathbb{T}^3}\operatorname{div}(\rho_\epsilon u_\epsilon)V_\epsilon dx = \frac{1}{\epsilon}\int_{\mathbb{T}^3}\rho_\epsilon\nabla V_\epsilon u_\epsilon dx, \end{aligned} \tag{2.19}$$

Moreover, using the equation satisfied by the so-called *effective velocity* $\nabla\log\rho_\epsilon$, (see Bresch et al. 2019 for its formal derivation)

$$(\rho_\epsilon\nabla\log\rho_\epsilon)_t + \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon\nabla\log\rho_\epsilon\otimes u_\epsilon) + \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon^t\nabla u_\epsilon) = 0,$$

after multiplication by $\nabla\log\rho_\epsilon$ itself and space integration we obtain

$$\frac{d}{dt}\int_{\mathbb{T}^3}2|\nabla\sqrt{\rho_\epsilon}|^2 dx - \frac{1}{\epsilon}\int_{\mathbb{T}^3}\rho_\epsilon\nabla^2\log\rho_\epsilon:\nabla u_\epsilon dx = 0. \tag{2.20}$$

Hence, summing (2.18, (2.17)) and (2.20) and integrating time we end up to (2.15). For the BD entropy relation, we take once again advantage of the effective velocity, and in particular introducing the quantity $w_\epsilon = u_\epsilon + \nabla\log\rho_\epsilon$. Then, using the relations

$$\begin{aligned} \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon\nabla\log\rho_\epsilon) &= \frac{1}{\epsilon}\Delta\rho_\epsilon, \\ (\rho_\epsilon\nabla\log\rho_\epsilon)_t &= -\frac{1}{\epsilon}\nabla\operatorname{div}(\rho_\epsilon u_\epsilon), \\ \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon u_\epsilon\otimes\nabla\log\rho_\epsilon + \rho_\epsilon\nabla\log\rho_\epsilon\otimes u_\epsilon) &= \frac{1}{\epsilon}\Delta(\rho_\epsilon u_\epsilon) - \frac{2}{\epsilon}\operatorname{div}(\rho_\epsilon Du_\epsilon) + \frac{1}{\epsilon}\nabla\operatorname{div}(\rho_\epsilon u_\epsilon), \\ \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon\nabla\log\rho_\epsilon\otimes\nabla\log\rho_\epsilon) &= \frac{1}{\epsilon}\Delta(\rho_\epsilon\nabla\log\rho_\epsilon) - \frac{1}{\epsilon}\operatorname{div}(\rho_\epsilon\nabla^2\log\rho_\epsilon). \end{aligned}$$

we obtain the following alternative version of (2.1):

$$\begin{aligned} \partial_t \rho_\epsilon + \frac{1}{\epsilon} \operatorname{div}(\rho_\epsilon w_\epsilon) &= \frac{1}{\epsilon} \Delta \rho_\epsilon \\ \partial_t(\rho_\epsilon w_\epsilon) + \frac{1}{\epsilon} \operatorname{div}(\rho_\epsilon w_\epsilon \otimes w_\epsilon) + \frac{1}{\epsilon} \nabla \rho_\epsilon^\gamma &+ \frac{1}{\epsilon} \operatorname{div}(\rho_\epsilon D w_\epsilon) - \frac{1}{\epsilon} \Delta(\rho_\epsilon w_\epsilon) \\ - \frac{1}{\epsilon} \operatorname{div}(\rho_\epsilon \nabla^2 \log \rho_\epsilon) + \frac{1}{\epsilon} \rho_\epsilon \nabla V_\epsilon + \frac{1}{\epsilon^2} \rho_\epsilon u_\epsilon &= 0. \end{aligned} \tag{2.21}$$

Now we pass to compute the energy associated to the system (2.21). To this end, as already done for (2.1), we use the continuity equation (2.21)₁ and multiply the equation (2.21)₂ by w_ϵ to conclude

$$\begin{aligned} \partial_t \left(\frac{1}{2} \rho |w_\epsilon|^2 \right) + \frac{1}{\epsilon} \operatorname{div} \left(\rho_\epsilon w_\epsilon \frac{1}{2} |w_\epsilon|^2 \right) + \frac{1}{\epsilon} \left(\frac{1}{2} \Delta \rho_\epsilon |w_\epsilon|^2 - \Delta(\rho_\epsilon w_\epsilon) \cdot w_\epsilon \right) \\ + \frac{1}{\epsilon} \nabla \rho_\epsilon^\gamma \cdot w_\epsilon + \frac{1}{\epsilon} \operatorname{div}(\rho_\epsilon D w_\epsilon) \cdot w_\epsilon - \frac{1}{\epsilon} \operatorname{div}(\rho_\epsilon \nabla^2 \log \rho_\epsilon) \cdot w_\epsilon \\ + \frac{1}{\epsilon^2} \rho_\epsilon u_\epsilon w_\epsilon + \frac{1}{\epsilon} \rho_\epsilon \nabla V_\epsilon w_\epsilon = 0. \end{aligned}$$

Since

$$\frac{1}{\epsilon} \left(\frac{1}{2} \Delta \rho_\epsilon |w_\epsilon|^2 - \Delta(\rho_\epsilon w_\epsilon) \cdot w_\epsilon \right) = -\frac{1}{\epsilon} \left(\frac{1}{2} \operatorname{div} \left(\nabla \rho_\epsilon |w_\epsilon|^2 \right) + \operatorname{div}(\rho_\epsilon \nabla w_\epsilon) \cdot w_\epsilon \right),$$

when we integrate the equality above over \mathbb{T}^3 we get:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{T}^3} \frac{1}{2} \rho_\epsilon |w_\epsilon|^2 dx + \frac{1}{\epsilon} \int_{\mathbb{T}^3} \rho_\epsilon |\nabla w_\epsilon|^2 dx + \frac{1}{\epsilon} \int_{\mathbb{T}^3} \nabla \rho_\epsilon^\gamma \cdot w_\epsilon dx \\ + \frac{1}{\epsilon} \int_{\mathbb{T}^3} \rho_\epsilon \nabla V_\epsilon w_\epsilon dx - \frac{1}{\epsilon} \int_{\mathbb{T}^3} \rho_\epsilon |D(w_\epsilon)|^2 dx + \frac{1}{\epsilon^2} \int_{\mathbb{T}^3} \rho_\epsilon u_\epsilon w_\epsilon dx \\ + \frac{1}{\epsilon} \int_{\mathbb{T}^3} \rho_\epsilon \nabla^2 \log \rho_\epsilon : \nabla w_\epsilon dx = 0. \end{aligned} \tag{2.22}$$

We observe that $|\nabla w_\epsilon|^2 - |D(w_\epsilon)|^2 = |A(w_\epsilon)|^2 = |A(u_\epsilon)|^2$, and, from the definition of w_ϵ , we infer

$$\begin{aligned} \nabla \rho_\epsilon^\gamma \cdot w_\epsilon &= \nabla \rho_\epsilon^\gamma \cdot u_\epsilon + \frac{4}{\gamma} |\nabla \rho_\epsilon^{\gamma/2}|^2; \\ \rho_\epsilon u_\epsilon w_\epsilon &= \rho_\epsilon u_\epsilon^2 + \rho_\epsilon u_\epsilon \nabla \log \rho_\epsilon; \\ \rho_\epsilon \nabla^2 \log \rho_\epsilon : \nabla w_\epsilon &= \rho_\epsilon \nabla^2 \log \rho_\epsilon : \nabla u_\epsilon + \rho_\epsilon |\nabla^2 \log \rho_\epsilon|^2 \\ \rho_\epsilon \nabla V_\epsilon w_\epsilon &= \rho_\epsilon \nabla V_\epsilon u_\epsilon + \rho_\epsilon \nabla V_\epsilon \nabla \log \rho_\epsilon \end{aligned}$$

Moreover, we recall (2.17), (2.20) and (2.19) to compute:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{T}^3} (\rho_\epsilon (\log \rho_\epsilon - 1) + 1) dx &= \frac{1}{\epsilon} \int_{\mathbb{T}^3} \rho_\epsilon u_\epsilon \nabla \log \rho_\epsilon \, dx, \\ \frac{1}{\epsilon} \int_{\mathbb{T}^3} \rho_\epsilon \nabla V_\epsilon w_\epsilon \, dx &= \frac{d}{dt} \int_{\mathbb{T}^3} \frac{1}{2} |\nabla V_\epsilon|^2 \, dx + \frac{1}{\epsilon} \int_{\mathbb{T}^3} \rho_\epsilon \nabla V_\epsilon \nabla \log \rho_\epsilon \, dx, \end{aligned}$$

the last term can be rewritten:

$$\begin{aligned} \frac{1}{\epsilon} \int_{\mathbb{T}^3} \rho_\epsilon \nabla V_\epsilon \nabla \log \rho_\epsilon \, dx &= \frac{1}{\epsilon} \int_{\mathbb{T}^3} \nabla V_\epsilon \nabla \rho_\epsilon \, dx = -\frac{1}{\epsilon} \int_{\mathbb{T}^3} \Delta V_\epsilon \rho_\epsilon \, dx \\ &= \frac{1}{\epsilon} \int_{\mathbb{T}^3} \rho_\epsilon (\rho_\epsilon - g_\epsilon) \, dx \end{aligned}$$

Therefore, we multiply (2.22) by ϵ and, using the previous identities, we obtain the final estimate

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{T}^3} \left[\epsilon \left(\frac{1}{2} \rho |w_\epsilon|^2 + \frac{\rho_\epsilon^\gamma}{\gamma - 1} + 2 |\nabla \sqrt{\rho_\epsilon}|^2 + \frac{1}{2} |\nabla V_\epsilon|^2 \right) + \rho_\epsilon (\log \rho_\epsilon - 1) + 1 \right] dx \\ + \int_{\mathbb{T}^3} \rho |A(u_\epsilon)|^2 \, dx + \frac{4}{\gamma} \int_{\mathbb{T}^3} |\nabla \rho_\epsilon^{\frac{\gamma}{2}}|^2 \, dx + \int_{\mathbb{T}^3} \rho_\epsilon |\nabla^2 \log \rho_\epsilon|^2 \, dx \\ + \frac{1}{\epsilon} \int_{\mathbb{T}^3} \rho_\epsilon u_\epsilon^2 \, dx + \int_{\mathbb{T}^3} \rho_\epsilon (\rho_\epsilon - g_\epsilon) \, dx = 0 \end{aligned}$$

which gives (2.16) upon time integration. □

2.2 Weak Solution of the Quantum Drift–Diffusion Equation

Next, we consider the quantum drift–diffusion–Poisson equation in $(0, T) \times \mathbb{T}^3$ with $g : \mathbb{T}^3 \rightarrow \mathbb{R}$

$$\begin{aligned} \partial_t \rho + \operatorname{div} \left(2\rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) - \nabla \rho^\gamma - \rho \nabla V \right) &= 0 \\ - \Delta V &= \rho - g, \end{aligned} \tag{2.23}$$

with initial datum

$$\rho|_{t=0} = \rho^0 \geq 0, \tag{2.24}$$

and V such that

$$\int_{\mathbb{T}^3} V \, dx = 0. \tag{2.25}$$

As before, to solve (2.23)₂, (2.25) we need the following assumption for g : defining $M := \int_{\mathbb{T}^3} \rho^0$, we assume that $\int_{\mathbb{T}^3} g = M$.

The following definition specifies the framework of weak solutions to (2.23) obtained in the zero relaxation limit.

Definition 2.8 Given $g \in L^2(\mathbb{T}^3)$, we say that (ρ, V) with $\rho \geq 0$ is a finite energy weak solution of (2.23) if

(1) Integrability condition:

$$\begin{aligned} \sqrt{\rho} &\in L^\infty(0, T; H^1(\mathbb{T}^3)) \cap L^2(0, T; H^2(\mathbb{T}^3)); & \rho &\in L^\infty(0, T; L^\gamma(\mathbb{T}^3)) \\ V &\in C((0, T); H^2(\mathbb{T}^3)) \end{aligned}$$

(2) Continuity equation:

$$\begin{aligned} \int_{\mathbb{T}^3} \rho^0 \phi(0) \, dx + \iint (\rho \phi_t - \rho^\gamma \operatorname{div} \nabla \phi - 2 \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} : \nabla^2 \phi \\ + 2 \sqrt{\rho} \nabla^2 \sqrt{\rho} : \nabla^2 \phi + \rho \nabla V \cdot \phi) \, ds dx = 0, \end{aligned}$$

for any $\phi \in C_c^\infty([0, T]; C^\infty(\mathbb{T}^3))$ such that $\phi(T) = 0$,

(3) Poisson equation:

For a.e. $(t, x) \in (0, T) \times \mathbb{T}^3$ it holds that

$$-\Delta V = \rho - g \tag{2.26}$$

(4) Entropy inequalities:

there exist $\Lambda, \mathcal{S} \in L^2((0, T) \times \mathbb{T}^3)$ such that

$$\sqrt{\rho} \Lambda = \operatorname{div}(2 \sqrt{\rho} \nabla^2 \sqrt{\rho} - 2 \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} - \rho^\gamma \mathbb{I}) - \rho \nabla V \text{ in } \mathcal{D}'((0, T) \times \mathbb{T}^3) \tag{2.27}$$

$$\sqrt{\rho} \mathcal{S} = 2 \sqrt{\rho} \nabla^2 \sqrt{\rho} - 2 \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} \text{ a.e. in } (0, T) \times \mathbb{T}^3 \tag{2.28}$$

and constant $C > 0$ such that for a.e. $t \in (0, T)$

$$\int_{\mathbb{T}^3} \left(2 |\nabla \sqrt{\rho}|^2 + \frac{\rho^\gamma}{\gamma - 1} + \frac{1}{2} |\nabla V|^2 \right) (t) \, dx + \int_0^t \int_{\mathbb{T}^3} |\Lambda|^2 \, ds dx \leq C, \tag{2.29}$$

$$\int_{\mathbb{T}^3} (\rho (\log \rho - 1) + 1) (t) \, dx + \int_0^t \int_{\mathbb{T}^3} |\mathcal{S}|^2 \, ds dx + \frac{4}{\gamma} \int_0^t \int_{\mathbb{T}^3} |\nabla \rho^{\frac{\gamma}{2}}|^2 \, ds dx \tag{2.30}$$

$$+ \int_0^t \int_{\mathbb{T}^3} \rho (\rho - g) \, ds dx \leq \int_{\mathbb{T}^3} (\rho^0 (\log \rho^0 - 1) + 1) \, dx.$$

In addition, (ρ, V) is called *energy dissipating weak solution* if in particular it holds that for a.e. $t \in (0, T)$:

$$\begin{aligned} \int_{\mathbb{T}^3} \left(2 |\nabla \sqrt{\rho}|^2 + \frac{\rho^\gamma}{\gamma - 1} + \frac{1}{2} |\nabla V|^2 \right) (t) \, dx + \int_0^t \int_{\mathbb{T}^3} |\Lambda|^2 \, ds dx \\ \leq \int_{\mathbb{T}^3} 2 |\nabla \sqrt{\rho^0}|^2 \, dx + \int_{\mathbb{T}^3} \frac{\rho^{0\gamma}}{\gamma - 1} \, dx + \int_{\mathbb{T}^3} \frac{1}{2} |\nabla V(0, x)|^2. \end{aligned} \tag{2.31}$$

Remark 2.9 Note that Definition 2.8, although motivated by the relaxation limit, it is very natural and based on the formal *a priori estimate* given in Proposition 2.11. In particular, the inequalities (2.29) and (2.31) are also crucial in the arguments of Jüngel and Matthes (2008) and Gianazza et al. (2009).

Remark 2.10 The terminology finite energy and energy dissipating weak solutions is motivated by the fact that both (2.29) and (2.31) arise as a weak limit of the energy inequality (2.8). As in Proposition 2.7 we derive the formal estimates for smooth solutions.

Proposition 2.11 *Let (ρ, V) be a smooth solution of (2.23) with data satisfying (2.24),(2.25). Then, for any $t \in [0, T)$ the pair (ρ, V) satisfies the following estimates:*

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{T}^3} \left(2|\nabla\sqrt{\rho}|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{1}{2}|\nabla V|^2 \right) dx \\ & + \int_{\mathbb{T}^3} \rho \left| 2\nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) - \gamma\rho^{\gamma-2}\nabla\rho - \nabla V \right|^2 dx = 0, \end{aligned} \tag{2.32}$$

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{T}^3} \rho(\log\rho - 1) + 1 + dx \int_{\mathbb{T}^3} \rho|\nabla^2 \log\rho|^2 dx \\ & + \frac{4}{\gamma} \int_{\mathbb{T}^3} |\nabla\rho^{\frac{\gamma}{2}}|^2 dx + \int_{\mathbb{T}^3} \rho(\rho - g) dx = 0. \end{aligned} \tag{2.33}$$

Proof The energy (2.33) is achieved by multiplying (2.23) by $\log\rho$ and integrating by parts:

$$\begin{aligned} & \int_{\mathbb{T}^3} \rho_t \log\rho dx = \frac{d}{dt} \int_{\mathbb{T}^3} ((\rho(\log\rho - 1)) + 1) dx, \\ & \int_{\mathbb{T}^3} \operatorname{div} \left(2\rho\nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) - \nabla\rho^\gamma \right) \log\rho dx = \int_{\mathbb{T}^3} \operatorname{div} (\operatorname{div}(\rho\nabla^2 \log\rho) - \nabla\rho^\gamma) \log\rho dx \\ & = \int_{\mathbb{T}^3} \rho|\nabla^2 \log\rho|^2 dx + \frac{4}{\gamma} \int_{\mathbb{T}^3} |\nabla\rho^{\frac{\gamma}{2}}|^2 dx, \\ & - \int_{\mathbb{T}^3} \operatorname{div}(\rho\nabla V) \log\rho dx = \int_{\mathbb{T}^3} \rho\nabla V \nabla \log\rho dx = - \int_{\mathbb{T}^3} \Delta V \rho dx = \int_{\mathbb{T}^3} \rho(\rho - g) dx. \end{aligned}$$

Moreover, if we multiply (2.23) by $-2\Delta\sqrt{\rho}/\sqrt{\rho} + \gamma\rho^{\gamma-1}/(\gamma-1)$, after integrating by parts we get

$$\begin{aligned} & \int_{\mathbb{T}^3} \rho_t \left(\frac{-2\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) dx = - \int_{\mathbb{T}^3} \operatorname{div} \left(2\frac{\nabla\sqrt{\rho}}{\sqrt{\rho}} \rho_t \right) dx + \int_{\mathbb{T}^3} \nabla\sqrt{\rho} \nabla \left(2\frac{\rho_t}{\sqrt{\rho}} \right) dx \\ & = \frac{d}{dt} \int_{\mathbb{T}^3} 2|\nabla\sqrt{\rho}|^2 dx, \int_{\mathbb{T}^3} \rho_t \left(\frac{\gamma}{\gamma-1} \rho^{\gamma-1} \right) dx = \frac{d}{dt} \int_{\mathbb{T}^3} \frac{\rho^\gamma}{\gamma-1} dx, \\ & \int_{\mathbb{T}^3} \operatorname{div} \left(2\rho\nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) \right) \left(-2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} + \frac{\gamma}{\gamma-1} \rho^{\gamma-1} \right) dx \end{aligned}$$

$$\begin{aligned}
 &= \int_{\mathbb{T}^3} \left(4\rho \left| \nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) \right|^2 - 2\gamma\rho \nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) \rho^{\gamma-2} \nabla \rho \right) dx, \\
 &\int_{\mathbb{T}^3} \operatorname{div}(-\nabla\rho^\gamma) \left(-2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} + \frac{\gamma}{\gamma-1}\rho^{\gamma-1} \right) dx \\
 &= \int_{\mathbb{T}^3} -\nabla\rho^\gamma \nabla \left(2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) + \rho|\gamma\rho^{\gamma-2}\nabla\rho|^2 dx \\
 &= \int_{\mathbb{T}^3} -2\gamma\rho\rho^{\gamma-2}\nabla\rho \nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) + \rho|\gamma\rho^{\gamma-2}\nabla\rho|^2 dx, \\
 &\int_{\mathbb{T}^3} -\operatorname{div}(\rho\nabla V) \frac{\gamma}{\gamma-1}\rho^{\gamma-1} dx = \int_{\mathbb{T}^3} \rho\nabla V \gamma\rho^{\gamma-2}\nabla\rho dx \\
 &= \int_{\mathbb{T}^3} \nabla V \nabla\rho^\gamma dx = - \int_{\mathbb{T}^3} \Delta V \rho^\gamma dx \\
 &\int_{\mathbb{T}^3} \operatorname{div}(\rho\nabla V) 2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} dx = - \int_{\mathbb{T}^3} \rho\nabla V 2\nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) dx \\
 &= - \int_{\mathbb{T}^3} \nabla V \operatorname{div}(\rho\nabla^2 \log \rho) dx
 \end{aligned}$$

Finally, we multiply (2.23) by V ; integrating by parts, we get:

$$\begin{aligned}
 \int_{\mathbb{T}^3} \rho_t V dx &= - \int_{\mathbb{T}^3} \Delta V_t V dx = \int_{\mathbb{T}^3} \nabla V_t \nabla V dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\nabla V|^2 dx, \\
 \int_{\mathbb{T}^3} \operatorname{div} \left(2\rho \nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) \right) V dx &= - \int_{\mathbb{T}^3} 2\rho \nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right) \nabla V dx \\
 &= - \int_{\mathbb{T}^3} \nabla V \operatorname{div}(\rho\nabla^2 \log \rho) dx, \\
 \int_{\mathbb{T}^3} -\operatorname{div}(\nabla\rho^\gamma) V dx &= \int_{\mathbb{T}^3} \nabla\rho^\gamma \nabla V dx = - \int_{\mathbb{T}^3} \rho^\gamma \Delta V dx
 \end{aligned}$$

and (2.32) follows by summing up all terms. □

Remark 2.12 It is worth to observe that if we perform the Hilbert expansion of (2.1) the limit solution $(\bar{\rho}, \bar{u})$ formally satisfies at the first nontrivial order $O(1/\epsilon)$ (the order $O(1/\epsilon^2)$ tells us the momentum expansion starts from the power one in ϵ) the following identities:

$$\begin{aligned}
 \bar{\rho}\bar{u} &= \epsilon\bar{\rho} \left(2\nabla \left(\frac{\Delta\sqrt{\bar{\rho}}}{\sqrt{\bar{\rho}}} \right) - \gamma\bar{\rho}^{\gamma-2}\nabla\bar{\rho} - \nabla\bar{V} \right), \\
 \partial_t \bar{\rho} + \operatorname{div} \left(2\bar{\rho}\nabla \left(\frac{\Delta\sqrt{\bar{\rho}}}{\sqrt{\bar{\rho}}} \right) - \nabla\bar{\rho}^\gamma - \bar{\rho}\nabla\bar{V} \right) &= 0,
 \end{aligned}$$

which is exactly the quantum drift–diffusion equation (2.23). We underline also that, with the above definition for \bar{u} , namely

$$(\bar{\rho}, \bar{u}) = \left(\bar{\rho}, \epsilon \left(2\nabla \left(\frac{\Delta\sqrt{\bar{\rho}}}{\sqrt{\bar{\rho}}} \right) - \gamma\bar{\rho}^{\gamma-2}\nabla\bar{\rho} - \nabla\bar{V} \right) \right),$$

in the formulation for the mechanical energy (2.15) for (2.1), at the limit the latter reduces to (2.32) after a time integration, that is the corresponding conservation of the mechanical energy for the quantum drift–diffusion equation. The same happens for the BD entropy, namely (2.10) for $\epsilon \rightarrow 0$ reduces to (the time integrated version of) (2.33). This fact is coherent with our analysis and it will be validated by Theorem 3.1 below.

Remark 2.13 We remark that if ρ is a weak solution in the sense of the previous definition the initial datum ρ^0 is attained for example in the strong topology of $L^2(\mathbb{T}^3)$.

3 Main Result

In this section we prove Theorem 1.2, which we rewrite for reader’s convenience.

Theorem 3.1 *Let $(\rho_\epsilon, u_\epsilon, V_\epsilon)$ be a weak solution of (2.1) in the sense of Definition 2.1 with data (2.2) satisfying*

$$\begin{aligned} \{\rho_\epsilon^0\}_\epsilon &\text{ is bounded in } L^1 \cap L^\gamma(\mathbb{T}^3) \text{ such that } \rho_\epsilon^0 \rightarrow \rho^0 \text{ in } L^q(\mathbb{T}^3), \quad q < 3 \\ \{\nabla\sqrt{\rho_\epsilon^0}\}_\epsilon &\text{ is bounded in } L^2(\mathbb{T}^3), \\ \{\sqrt{\rho_\epsilon^0}u_\epsilon^0\}_\epsilon &\text{ is bounded in } L^2(\mathbb{T}^3), \\ \{g_\epsilon\}_\epsilon &\in L^2(\mathbb{T}^3) \text{ and } g_\epsilon \rightharpoonup g \text{ weakly in } L^2(\mathbb{T}^3). \end{aligned} \tag{3.1}$$

Then, up to subsequences, there exist $\rho \geq 0$ and V such that

$$\begin{aligned} \sqrt{\rho_\epsilon} &\rightarrow \sqrt{\rho} \text{ strongly in } L^2((0, T); H^1(\mathbb{T}^3)) \\ \nabla V_\epsilon &\rightarrow \nabla V \text{ strongly in } C([0, T]; L^2(\mathbb{T}^3)) \end{aligned}$$

and (ρ, V) is a finite energy weak solution of (2.23–(2.24)) in the sense of Definition 2.8. If in addition to (3.1), it also holds that

$$\begin{aligned} \sqrt{\rho_\epsilon^0}u_\epsilon^0 &\rightarrow 0 \text{ strongly in } L^2(\mathbb{T}^3) \\ \nabla\sqrt{\rho_\epsilon^0} &\rightarrow \nabla\sqrt{\rho^0} \text{ strongly in } L^2(\mathbb{T}^3) \\ \rho_\epsilon^0 &\rightarrow \rho^0 \text{ strongly in } L^\gamma(\mathbb{T}^3) \end{aligned} \tag{3.2}$$

then (ρ, V) is an energy dissipating weak solution.

The proof of Theorem 3.1 requires several preliminaries, collected in the following section.

3.1 Preliminary Results

We start by proving the uniform bounds obtained by the energy estimate and BD entropy.

Lemma 3.2 *There exists a constant $C > 0$ depending on the data such that the following bounds, uniform with respect to ϵ , hold*

$$\begin{aligned} \sup_t \int_{\mathbb{T}^3} \rho_\epsilon dx &\leq C; \quad \sup_t \int_{\mathbb{T}^3} \rho_\epsilon u_\epsilon^2 dx \leq C; \quad \int_0^T \int_{\mathbb{T}^3} |\nabla \rho_\epsilon^{\frac{\gamma}{2}}|^2 ds dx \leq C; \\ \sup_t \int_{\mathbb{T}^3} |\nabla \sqrt{\rho_\epsilon}|^2 ds dx &\leq C; \\ \frac{1}{\epsilon} \int_0^T \int_{\mathbb{T}^3} |T_\epsilon^s|^2 ds dx &\leq C; \quad \sup_t \int_{\mathbb{T}^3} \rho_\epsilon^\gamma dx \leq C; \quad \frac{1}{\epsilon^2} \int_0^T \int_{\mathbb{T}^3} \rho_\epsilon u_\epsilon^2 ds dx \leq C; \\ \int_0^T \int_{\mathbb{T}^3} |\nabla^2 \sqrt{\rho_\epsilon}|^2 ds dx &\leq C; \quad \int_0^T \int_{\mathbb{T}^3} |S_\epsilon|^2 ds dx \leq C; \\ \sup_t \int_{\mathbb{T}^3} |\nabla^2 V_\epsilon|^2 dx &\leq C \end{aligned} \tag{3.3}$$

Proof We first notice that under the hypothesis (3.1) we have that

$$\{\rho_\epsilon^0 \log \rho_\epsilon^0\}_\epsilon \text{ is bounded in } L^1(\mathbb{T}^3). \tag{3.4}$$

This is obtained very easily under the hypothesis (3.1). Indeed, one can easily show that for $q > p \geq 1$ there exists $C > 0$ such that for $s \geq 0$:

$$|s \log s|^q \leq C(1 + s^p)$$

and therefore, by taking $q = \gamma$ and $1 \leq p < \gamma$ we have that

$$\{\rho_\epsilon^0 \log \rho_\epsilon^0\}_\epsilon \text{ is bounded in } L^q(\mathbb{T}^3). \tag{3.5}$$

which implies (3.4).

Next, as shown in Remark 2.6, $\{\nabla V_\epsilon|_{t=0}\}_\epsilon$ is bounded in $L^2(\mathbb{T}^3)$ and we have proved that the right-hand sides of (2.8) and (2.10) are bounded uniformly in ϵ . Then, to obtain (3.3) is enough to use the bound

$$\int_0^t \int_{\mathbb{T}^3} \rho_\epsilon (\rho_\epsilon - g_\epsilon) ds dx \geq \frac{1}{2} \int_0^t \int_{\mathbb{T}^3} |\rho_\epsilon|^2 ds dx - \frac{1}{2} \int_0^t \int_{\mathbb{T}^3} |g_\epsilon|^2 ds dx$$

in the energy inequalities (2.8) and (2.10). Finally, the bound on the potential follows from the Poisson equation. □

In the following lemma we prove the convergence needed to pass to the limit as $\epsilon \rightarrow 0$.

Lemma 3.3 *There exist $\rho \geq 0$ such that $\sqrt{\rho} \in L^\infty(0, T); H^1(\mathbb{T}^3) \cap L^2(0, T; H^2(\mathbb{T}^3))$ and the following hold:*

$$\rho_\epsilon \rightarrow \rho \text{ in } C([0, T]; L^q(\mathbb{T}^3)), \quad q < 3, \tag{3.6}$$

$$\sqrt{\rho_\epsilon} \rightharpoonup \sqrt{\rho} \text{ weakly in } L^2(0, T; H^2(\mathbb{T}^3)), \tag{3.7}$$

$$\sqrt{\rho_\epsilon} \overset{*}{\rightharpoonup} \sqrt{\rho} \text{ weakly* in } L^\infty(0, T; H^1(\mathbb{T}^3)), \tag{3.8}$$

$$\sqrt{\rho_\epsilon} \rightarrow \sqrt{\rho} \text{ strongly in } L^2(0, T; H^1(\mathbb{T}^3)). \tag{3.9}$$

$$\rho_\epsilon (\log \rho_\epsilon + 1) + 1 \rightarrow \rho (\log \rho + 1) + 1 \text{ strongly in } L^1(0, T; L^1(\mathbb{T}^3)), \tag{3.10}$$

$$\rho_\epsilon^\gamma \rightarrow \rho^\gamma \text{ in } L^1(0, T; L^1(\mathbb{T}^3)) \quad \gamma > 1. \tag{3.11}$$

Proof From (3.3) and Sobolev embedding $H^1(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$, we have that:

$$\{\sqrt{\rho_\epsilon}\}_\epsilon \text{ is bounded in } L^\infty(0, T; L^q(\mathbb{T}^3)) \quad q \in [1, 6] \tag{3.12}$$

Therefore, since $\nabla \rho_\epsilon = 2\nabla \sqrt{\rho_\epsilon} \sqrt{\rho_\epsilon}$ by using (3.12) and (3.3), we can infer that

$$\{\rho_\epsilon\}_\epsilon \text{ is bounded in } L^\infty(0, T; W^{1, \frac{3}{2}}(\mathbb{T}^3)).$$

Next, by using the weak formulation of the continuity equation and the bounds (3.3) we get that

$$\{\partial_t \rho_\epsilon\}_\epsilon \text{ is bounded in } L^2(0, T; W^{-1, \frac{3}{2}}(\mathbb{T}^3)). \tag{3.13}$$

Indeed, it is enough to note that we have

$$\int_0^T \|\partial_t \rho_\epsilon\|_{W^{-1, \frac{3}{2}}}^2 ds \leq \frac{1}{\epsilon^2} \int_0^T \|\rho_\epsilon u_\epsilon\|_{L^{\frac{3}{x}}}^2 ds \leq \int_0^T \|\sqrt{\rho_\epsilon}\|_{L_x^6}^2 \left\| \sqrt{\rho_\epsilon} \frac{u_\epsilon}{\epsilon} \right\|_{L_x^2}^2 ds.$$

Since for $q < 3$ we have $W^{1, 3/2}(\mathbb{T}^3) \subset L^q(\mathbb{T}^3)$ with compact embedding and $L^q(\mathbb{T}^3) \subset W^{-1, 3/2}(\mathbb{T}^3)$, we can apply Aubin–Lions lemma to deduce that there exists a subsequence not relabeled and $\rho \geq 0$ such that (3.6) holds. Moreover, the strong convergence (3.6) implies that $\rho_\epsilon \rightarrow \rho$ strongly in $L^q((0, T); L^q(\mathbb{T}^3))$ for any $q < 3$, and thus, passing to subsequence if necessary, we have that

$$\rho_\epsilon \rightarrow \rho \quad \text{a.e in } (0, T) \times \mathbb{T}^3. \tag{3.14}$$

Then, the convergence (3.7) and (3.11) follows from (3.14), the uniform bounds (3.3) and standard weak compactness considerations. Next, note that since (3.14) holds, for any $M > 0$ such that $\{\rho = M\}$ has zero measure, it holds that

$$\chi_{\{\rho_\epsilon \leq M\}} \rightarrow \chi_{\{\rho \leq M\}} \text{ a.e. in } (0, T) \times \mathbb{T}^3.$$

Therefore, since ρ is summable, we can always find a sequence $\{M_k\}_k$ of positive numbers such that $M_k \rightarrow \infty$ and

$$\chi_{\{\rho_\epsilon < M_k\}} \rightarrow \chi_{\{\rho < M_k\}} \text{ a.e. in } (0, T) \times \mathbb{T}^3. \tag{3.15}$$

Then, to prove (3.9) we start by proving that

$$\sqrt{\rho_\epsilon} \rightarrow \sqrt{\rho} \text{ strongly in } L^2(0, T; L^2(\mathbb{T}^3)). \tag{3.16}$$

First, notice that since the function $t \mapsto \sqrt{t}$ is continuous on $[0, \infty)$, (3.14) and (3.15) imply that

$$\sqrt{\rho_\epsilon} \chi_{\{\rho_\epsilon < M_k\}} \rightarrow \sqrt{\rho} \chi_{\{\rho < M_k\}} \text{ a.e in } (0, T) \times \mathbb{T}^3.$$

Then

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^3} |\sqrt{\rho_\epsilon} - \sqrt{\rho}|^2 ds dx &\leq \int_0^T \int_{\{\rho_\epsilon > M_k\}} |\sqrt{\rho_\epsilon}|^2 ds dx \\ &\quad + \int_0^T \int_{\mathbb{T}^3} |\sqrt{\rho_\epsilon} \chi_{\{\rho_\epsilon \leq M_k\}} - \sqrt{\rho} \chi_{\{\rho \leq M_k\}}|^2 ds dx \\ &\quad + \int_0^T \int_{\{\rho > M_k\}} |\sqrt{\rho}|^2 ds dx. \end{aligned}$$

Then, by using (3.3), (3.12), the fact that $\sqrt{\rho} \in L^\infty(0, T; L^6(\mathbb{T}^3))$, we have that

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^3} |\sqrt{\rho_\epsilon} - \sqrt{\rho}|^2 ds dx &\leq \frac{1}{M_k^2} \int_0^T \int |\sqrt{\rho_\epsilon}|^6 ds dx \\ &\quad + \int_0^T \int_{\mathbb{T}^3} |\sqrt{\rho_\epsilon} \chi_{\{\rho_\epsilon \leq M_k\}} - \sqrt{\rho} \chi_{\{\rho \leq M_k\}}|^2 ds dx \\ &\quad + \frac{1}{M_k^2} \int_0^T \int |\sqrt{\rho}|^6 ds dx \\ &\leq \int_0^T \int_{\mathbb{T}^3} |\sqrt{\rho_\epsilon} \chi_{\{\sqrt{\rho_\epsilon} \leq M_k\}} \\ &\quad - \sqrt{\rho} \chi_{\{\sqrt{\rho} \leq M_k\}}|^2 ds dx + \frac{2C}{M_k^2}. \end{aligned}$$

Then, we conclude by first sending $\epsilon \rightarrow 0$ in the second term, where we use Dominated Convergence Theorem, and then by sending $k \rightarrow \infty$.

The strong convergence (3.9) of $\sqrt{\rho_\epsilon}$ in $L^2(0, T; H^1(\mathbb{T}^3))$ is a consequence of the following simple interpolation inequality:

$$\|\sqrt{\rho_\epsilon}(t) - \sqrt{\rho}(t)\|_{H^1} \leq C \|\sqrt{\rho_\epsilon}(t) - \sqrt{\rho}(t)\|_{L^2}^{\frac{1}{2}} \|\sqrt{\rho_\epsilon}(t) - \sqrt{\rho}(t)\|_{H^2}^{\frac{1}{2}}. \tag{3.17}$$

Indeed, by integrating in time and using Hölder inequality, we have that

$$\begin{aligned} \int_0^T \|\sqrt{\rho_\epsilon}(t) - \sqrt{\rho}(t)\|_{H^1}^2 dt &\leq \left(\int_0^T \|\sqrt{\rho_\epsilon}(t) - \sqrt{\rho}(t)\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^T \|\sqrt{\rho_\epsilon}(t) - \sqrt{\rho}(t)\|_{H^2}^2 dt \right)^{\frac{1}{2}} \end{aligned}$$

and we conclude by using (3.3) and (3.7). Next, we prove (3.10). We first notice that by using (3.3) and by the very same argument used to deduce (3.5), we easily have that for some $p > 1$

$$\begin{aligned} \rho \log \rho &\in L^p((0, T) \times \mathbb{T}^3), \\ \{\rho_\epsilon \log \rho_\epsilon\}_\epsilon &\text{ is bounded in } L^p((0, T) \times \mathbb{T}^3). \end{aligned}$$

Moreover, since the function $s \rightarrow s \log s$ is continuous on $[0, \infty)$, by using (3.14) and (3.15), we have that

$$\rho_\epsilon \log \rho_\epsilon \chi_{\{\rho_\epsilon \leq M_k\}} \rightarrow \rho \log \rho \chi_{\{\rho \leq M_k\}} \text{ a.e. in } (0, T) \times \mathbb{T}^3.$$

Then

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^3} |\rho_\epsilon \log \rho_\epsilon - \rho \log \rho| ds dx &\leq \int_0^T \int_{\{\rho_\epsilon > M_k\}} |\rho_\epsilon \log \rho_\epsilon| ds dx \\ &\quad + \int_0^T \int_{\mathbb{T}^3} |\rho_\epsilon \log \rho_\epsilon \chi_{\{\rho_\epsilon \leq M_k\}} \\ &\quad - \rho \log \rho \chi_{\{\rho \leq M_k\}}| ds dx \\ &\quad + \int_0^T \int_{\{\rho > M_k\}} |\rho \log \rho| ds dx. \\ &\leq \int_0^T \int_{\mathbb{T}^3} |\rho_\epsilon \log \rho_\epsilon \chi_{\{\rho_\epsilon \leq M_k\}} \\ &\quad - \rho \log \rho \chi_{\{\rho \leq M_k\}}| ds dx \\ &\quad + \frac{2C}{(M_k \log M_k)^{p-1}} \end{aligned}$$

and we conclude as in before. Finally, we prove the convergence of the pressure term. We first note that from (3.3) we have that

$$\{\rho_\epsilon^{\frac{\gamma}{2}}\}_\epsilon \text{ is bounded in } L^\infty(0, T; L^2(\mathbb{T}^3)) \cap L^2(0, T; H^1(\mathbb{T}^3)).$$

Then, by Sobolev embedding

$$\{\rho_\epsilon^{\frac{\gamma}{2}}\}_\epsilon \text{ is bounded in } L^2(0, T; L^6(\mathbb{T}^3))$$

and therefore for a.e t :

$$\|\rho_\epsilon^{\frac{\gamma}{2}}(t)\|_{\frac{10}{3}} \leq \|\rho_\epsilon^{\frac{\gamma}{2}}(t)\|_2^{\frac{2}{5}} \|\rho_\epsilon^{\frac{\gamma}{2}}(t)\|_6^{\frac{3}{5}}.$$

Therefore, by integrating in time and using (3.3) we have that

$$\{\rho_\epsilon^{\frac{\gamma}{2}}\}_\epsilon \text{ is bounded in } L^{\frac{10}{3}}((0, T) \times \mathbb{T}^3),$$

which is equivalent to say that

$$\{\rho_\epsilon^\gamma\}_\epsilon \text{ is bounded in } L^{\frac{5}{3}}((0, T) \times \mathbb{T}^3).$$

Moreover, by (3.14) and since $t \mapsto t^\gamma$ is continuous on $[0, \infty)$ we have that also $\rho_\epsilon^\gamma \rightarrow \rho^\gamma$ a.e. in $(0, T) \times \mathbb{T}^3$ and by Fatou Lemma we have that

$$\rho^\gamma \in L^{\frac{5}{3}}((0, T) \times \mathbb{T}^3).$$

Moreover, by (3.14) and (3.15) we also have that

$$\rho_\epsilon^\gamma \chi_{\{\rho_\epsilon \leq M_k\}} \rightarrow \rho^\gamma \chi_{\{\rho \leq M_k\}} \text{ a.e. in } (0, T) \times \mathbb{T}^3.$$

Then,

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^3} |\rho_\epsilon^\gamma - \rho^\gamma| \, dsdx &\leq \int_0^T \int_{\{\rho_\epsilon > M_k\}} \rho_\epsilon^\gamma \, dsdx \\ &+ \int_0^T \int_{\mathbb{T}^3} |\rho_\epsilon^\gamma \chi_{\{\rho_\epsilon \leq M_k\}} - \rho^\gamma \chi_{\{\rho \leq M_k\}}| \, dsdx \\ &+ \int_0^T \int_{\{\rho > M_k\}} \rho^\gamma \, dsdx \\ &\leq \int_0^T \int_{\mathbb{T}^3} |\rho_\epsilon^\gamma \chi_{\{\rho_\epsilon \leq M_k\}} - \rho^\gamma \chi_{\{\rho \leq M_k\}}| \, dsdx \\ &+ \frac{2C}{M_k^{\frac{2}{3}\gamma}} \end{aligned}$$

and we conclude as before. □

3.2 Proof of Theorem 3.1

First, we note that from (3.3) and (3.1), by using (2.6) we have that

$$\{V_\epsilon\}_\epsilon \text{ is bounded in } L^\infty(0, T; H^2(\mathbb{T}^3)).$$

Moreover, since g_ϵ is not depending on time, by using (3.13), we have that

$$\{\partial_t V_\epsilon\}_\epsilon \text{ is bounded in } L^2(0, T; W^{-1, \frac{3}{2}}(\mathbb{T}^3)).$$

Since $H^2 \subset\subset H^1 \subset W^{-1, \frac{3}{2}}(\mathbb{T}^3)$, by using Aubin–Lions lemma we have that there exists V such that

$$\nabla V_\epsilon \rightarrow \nabla V \text{ strongly in } C([0, T]; L^2(\mathbb{T}^3)). \tag{3.18}$$

Moreover, the Poisson equation is satisfied pointwise. Regarding the momentum equation, let $\psi \in C^\infty([0, T]; C^\infty(\mathbb{T}^3))$ and consider the weak formulation of the momentum equations in Definition 2.1, multiplied by ϵ ,

$$\begin{aligned} & \epsilon \int \rho_\epsilon^0 u_\epsilon^0 \psi(0) dx + \epsilon^2 \iint \sqrt{\rho_\epsilon} \sqrt{\rho_\epsilon} \frac{u_\epsilon}{\epsilon} \psi_t ds dx + \epsilon^2 \iint \sqrt{\rho_\epsilon} \frac{u_\epsilon}{\epsilon} \otimes \sqrt{\rho_\epsilon} \frac{u_\epsilon}{\epsilon} : \nabla \psi ds dx \\ & - \sqrt{\epsilon} \iint \sqrt{\rho_\epsilon} \frac{T_\epsilon^s}{\sqrt{\epsilon}} : \nabla \psi ds dx + \iint \rho_\epsilon^\gamma \operatorname{div} \psi ds dx - \iint 2\sqrt{\rho_\epsilon} \nabla^2 \sqrt{\rho_\epsilon} : \nabla \psi ds dx \\ & - \iint \rho_\epsilon \nabla V_\epsilon \psi ds dx + \iint 2\nabla \sqrt{\rho_\epsilon} \otimes \nabla \sqrt{\rho_\epsilon} : \nabla \psi ds dx + \iint \rho_\epsilon \frac{u_\epsilon}{\epsilon} \psi ds dx = 0. \end{aligned} \tag{3.19}$$

We study the convergence in the limit of $\epsilon \rightarrow 0$ of all the terms separately. By using (3.1) and Hölder inequality, we conclude

$$\epsilon \left| \int \rho_\epsilon^0 u_\epsilon^0 \psi(0) dx \right| \leq \epsilon \|\psi\|_{L_{t,x}^\infty} \|\sqrt{\rho_\epsilon^0}\|_{L^2} \|\sqrt{\rho_\epsilon^0} u_\epsilon^0\|_{L^2} \leq \epsilon C \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

Analogously, from (3.3) and Hölder inequality, we get for $\epsilon \rightarrow 0$:

$$\begin{aligned} \epsilon^2 \left| \iint \sqrt{\rho_\epsilon} \sqrt{\rho_\epsilon} \frac{u_\epsilon}{\epsilon} \psi_t ds dx \right| & \leq \epsilon^2 \|\psi_t\|_{L_{t,x}^\infty} \|\sqrt{\rho_\epsilon}\|_{L_{t,x}^2} \left\| \sqrt{\rho_\epsilon} \frac{u_\epsilon}{\epsilon} \right\|_{L_{t,x}^2} \leq \epsilon^2 C \rightarrow 0, \\ \epsilon^2 \left| \iint \sqrt{\rho_\epsilon} \frac{u_\epsilon}{\epsilon} \otimes \sqrt{\rho_\epsilon} \frac{u_\epsilon}{\epsilon} : \nabla \psi ds dx \right| & \leq \epsilon^2 \|\nabla \psi\|_{L_{t,x}^\infty} \left\| \sqrt{\rho_\epsilon} \frac{u_\epsilon}{\epsilon} \right\|_{L_{t,x}^2}^2 \leq \epsilon^2 C \rightarrow 0, \\ \sqrt{\epsilon} \left| \iint \sqrt{\rho_\epsilon} \frac{T_\epsilon^s}{\sqrt{\epsilon}} : \nabla \psi ds dx \right| & \leq \sqrt{\epsilon} \|\nabla \psi\|_{L_{t,x}^\infty} \|\sqrt{\rho_\epsilon}\|_{L_{t,x}^2} \left\| \frac{T_\epsilon^s}{\sqrt{\epsilon}} \right\|_{L_{t,x}^2} \leq \sqrt{\epsilon} C \rightarrow 0. \end{aligned}$$

Next, by using (3.7) and (3.9) it follows that for $\epsilon \rightarrow 0$

$$\begin{aligned} \iint 2\sqrt{\rho_\epsilon} \nabla^2 \sqrt{\rho_\epsilon} : \nabla \psi \, dsdx &\rightarrow \iint 2\sqrt{\rho} \nabla^2 \sqrt{\rho} : \nabla \psi \, dsdx, \\ \iint 2\nabla \sqrt{\rho_\epsilon} \otimes \nabla \sqrt{\rho_\epsilon} : \nabla \psi \, dsdx &\rightarrow \iint 2\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} : \nabla \psi \, dsdx. \end{aligned}$$

Moreover, the convergence of ρ_ϵ^γ in (3.11) implies the continuity of the pressure term

$$\iint \rho_\epsilon^\gamma \operatorname{div} \psi \, dsdx \rightarrow \iint \rho^\gamma \operatorname{div} \psi \, dsdx$$

as $\epsilon \rightarrow 0$. Next, we consider the potential term and, by using (3.6) and (3.18), one gets:

$$\iint \rho_\epsilon \nabla V_\epsilon \psi \, dsdx \rightarrow \iint \rho \nabla V \psi \, dsdx \text{ as } \epsilon \rightarrow 0.$$

For the damping term, we first note that, that from (3.3), we can infer that there exists Λ such that

$$\sqrt{\rho_\epsilon} \frac{u_\epsilon}{\epsilon} \rightharpoonup \Lambda \text{ weakly in } L^2((0, T) \times \mathbb{T}^3), \tag{3.20}$$

by using also (3.9) we get that

$$\iint \sqrt{\rho_\epsilon} \sqrt{\rho_\epsilon} \frac{u_\epsilon}{\epsilon} \psi \, dsdx \rightarrow \iint \sqrt{\rho} \Lambda \psi \, dsdx \text{ as } \epsilon \rightarrow 0$$

Therefore, from (3.19) we conclude

$$\begin{aligned} \iint \sqrt{\rho} \Lambda \psi \, dsdx &= -2 \iint \nabla \sqrt{\rho} \otimes \sqrt{\rho} : \nabla \psi \, dsdx + 2 \iint \sqrt{\rho} \nabla^2 \sqrt{\rho} : \nabla \psi \, dsdx \\ &\quad - \iint \rho^\gamma \operatorname{div} \psi \, dsdx + \iint \rho \nabla V \psi \, dsdx, \end{aligned}$$

that is

$$\sqrt{\rho} \Lambda = \operatorname{div}(-2\nabla \sqrt{\rho} \otimes \sqrt{\rho} + 2\sqrt{\rho} \nabla^2 \sqrt{\rho} - \rho^\gamma \mathbb{I}) + \rho \nabla V \text{ in } \mathcal{D}'((0, T) \times \mathbb{T}^3). \tag{3.21}$$

Finally, for the continuity equation we similarly have for $\epsilon \rightarrow 0$:

$$\begin{aligned} \int \rho_\epsilon^0 \phi(0) \, dx + \iint \rho_\epsilon \phi_t + \sqrt{\rho_\epsilon} \sqrt{\rho_\epsilon} u_\epsilon \nabla \phi \, dsdx &\rightarrow \int \rho^0 \phi(0) \, dx \\ + \iint \rho \phi_t + \sqrt{\rho} \Lambda \nabla \phi \, dsdx, \end{aligned}$$

and therefore taking into account (3.21), we get that ρ satisfies

$$\iint \rho \phi_t + \sqrt{\rho} \Lambda \nabla \phi \, ds dx = 0, \tag{3.22}$$

or any $\phi \in C_c^\infty([0, T]; C^\infty(\mathbb{T}^3))$. Next, we prove (2.28). Again, from (3.3) we have that there exists \mathcal{S} such that

$$\mathcal{S}_\epsilon \rightharpoonup \mathcal{S} \text{ weakly in } L^2((0, T) \times \mathbb{T}^3), \tag{3.23}$$

Therefore, for any $\phi \in C^\infty(\mathbb{T}^3)$, we have that

$$\begin{aligned} \iint \sqrt{\rho_\epsilon} \mathcal{S}_\epsilon \phi \, ds dx &\rightarrow \iint \sqrt{\rho} \mathcal{S} \phi \, ds dx \\ \iint \sqrt{\rho_\epsilon} \nabla^2 \sqrt{\rho_\epsilon} \phi - \nabla \sqrt{\rho_\epsilon} \otimes \nabla \sqrt{\rho_\epsilon} \, ds dx &\rightarrow \iint \sqrt{\rho} \nabla^2 \sqrt{\rho} \phi - \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} \, ds dx \end{aligned}$$

where we have used (3.9), (3.7) and (3.23). Finally, by using (2.9) we get (2.28). Next, we prove the entropy inequalities. By lower semicontinuity we have that for a.e. $t \in (0, T)$

$$\begin{aligned} \int_{\mathbb{T}^3} \left(\frac{\rho(t, x)^\gamma}{\gamma - 1} + 2|\nabla \sqrt{\rho}(t, x)|^2 + \frac{1}{2}|\nabla V(t, x)|^2 \right) dx + \int_0^t \int_{\mathbb{T}^3} |\Lambda(t, x)|^2 \, ds dx \\ \leq \liminf_{\epsilon \rightarrow 0} \int_{\mathbb{T}^3} \left(\frac{1}{2} \rho_\epsilon^0 |u_\epsilon^0|^2 + \frac{(\rho_\epsilon^0)^\gamma}{\gamma - 1} + 2|\nabla \sqrt{\rho_\epsilon^0}|^2 + \frac{1}{2}|\nabla V_\epsilon^0|^2 \right) dx, \end{aligned} \tag{3.24}$$

and then (2.29) and (2.31) follows by using (3.1) and (3.2), respectively, and (3.18).

Finally, regarding (2.30), we recall that we only assume (3.1). We first note that (3.1) implies that, up to a subsequence,

$$\rho_\epsilon^0 \rightarrow \rho^0 \text{ a.e. in } (0, T) \times \mathbb{T}^3$$

then by using (3.4) and the very same argument used in Lemma 3.3 to prove (3.10) we get that

$$\rho_\epsilon^0 (\log \rho_\epsilon^0 + 1) + 1 \rightarrow \rho^0 (\log \rho^0 + 1) + 1 \text{ strongly in } L^1(\mathbb{T}^3).$$

Moreover, we have that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_0^t \int_{\mathbb{T}^3} \rho_\epsilon (\rho_\epsilon - g_\epsilon) \, ds dx &= \lim_{\epsilon \rightarrow 0} \int_0^t \int_{\mathbb{T}^3} \rho_\epsilon^2 \, ds dx - \lim_{\epsilon \rightarrow 0} \int_0^t \int_{\mathbb{T}^3} \rho_\epsilon g_\epsilon \, ds dx \\ &= \int_0^t \int_{\mathbb{T}^3} \rho (\rho - g) \, ds dx, \end{aligned}$$

but this follows directly from (3.6) and the weak convergence of g_ϵ . Therefore, considering (2.10) and arguing exactly as done to deduce (3.24) we get (2.30).

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Appendix A.

In this part we explain the main steps in the construction of weak solutions in the sense of Definition 2.1. Two approximations are known for system (2.1) in the literature, namely the one in Antonelli and Spirito (2017) and the one in Lacroix-Violet and Vasseur (2017). It is still an open problem finding other approximation schemes, other than the ones in Antonelli and Spirito (2017) and Lacroix-Violet and Vasseur (2017), converging to weak solutions of (2.1). The main difficulty in providing an approximation scheme for the system (2.1) is due to the fact that the BD entropy, which is crucial in the theory, is obtained by a nonlinear transformation of the unknowns. The approximation scheme in Antonelli and Spirito (2017) requires for technical reasons some restrictions on the coefficients γ , ν and κ : precisely, the coefficients satisfy $\gamma \in (1, 3)$ and $\nu \sim \kappa$. In addition, the approximation scheme introduced in Lacroix-Violet and Vasseur (2017) is less rigid and does not require any restriction on the coefficients, and therefore, we sketch here the main ideas of the latter. The complete proof of the existence requires several approximation steps, and the last one is given formally by the following system:

$$\begin{aligned} \partial_t \rho + \frac{1}{\epsilon} \operatorname{div}(\rho u) &= 0 \\ \partial_t(\rho u) + \frac{1}{\epsilon} \operatorname{div}(\rho u \otimes u) - \frac{1}{\epsilon} \operatorname{div}(\rho Du) + \frac{1}{\epsilon} \nabla \rho^\gamma + \frac{1}{\epsilon} \rho \nabla V & \quad (\text{A.1}) \\ &= \frac{1}{\epsilon} \operatorname{div}(\rho \nabla^2 \log \rho) - \frac{1}{\epsilon^2} \rho u - r_0 u - r_0 \rho |u|^2 u - \Delta V = \rho - g, \end{aligned}$$

where, for $r_0 > 0$ fixed, we have introduced the artificial drag terms $-r_0 u - r_0 \rho |u|^2 u$. The solutions (ρ, u, V) of (A.1) formally satisfy approximated versions of the energy inequality (2.8) and the BD entropy (2.9), which ensure uniform bounds, with respect to r_0 , in the integrability spaces listed in Definition 2.1, provided natural bounds on the initial data are assumed. At this stage the main obstacle in the passage to the limit of $r_0 \rightarrow 0$ is due to possible concentration in the convective term. The obstacle is overcome by a truncation argument: one first performs the limit of $r_0 \rightarrow 0$ for a

modified momentum equation satisfied by $(\rho, \rho\beta_i^\delta(u_i))$, where $\beta_i^\delta(u_i)$ is the truncation of u_i , and after takes the limit for $\delta \rightarrow 0$. Note that solutions of the system (A.1) are only known to satisfy a limited amount of regularity, as in Definition 2.1; therefore, computing the equations satisfied by $(\rho, \rho\beta_i^\delta(u_i))$ requires some technical argument, which are inspired by the theory of renormalized solutions of transport equations with irregular coefficients of DiPerna and Lions (Di Perna and Lions 1989). In particular, it is exactly at this point that the additional terms $r_0\rho|u|^2u$ and r_0u play a major role. Moreover, we note that, although it can be proved the vacuum set has zero Lebesgue measure, the stress and dispersive tensor can not be defined. As a consequence, for any fixed $r_0 > 0$, and thus already at this stage of the approximation, the tensors \mathcal{T} and \mathcal{S} , which are replacing $\sqrt{\rho}\nabla u$ and $\sqrt{\rho}\nabla^2 \log \rho$, need to be considered. In order to prove existence for (A.1) for any $r_0 > 0$ fixed, one may consider the following approximating system:

$$\begin{aligned} \partial_t \rho + \frac{1}{\epsilon} \operatorname{div}(\rho u) &= \tau \Delta \rho, \\ \partial_t(\rho u) + \frac{1}{\epsilon} \operatorname{div}(\rho u \otimes u) - \frac{1}{\epsilon} \operatorname{div}(\rho Du) + \frac{1}{\epsilon} \nabla \rho^\gamma - \eta \nabla \rho^{-10} - \mu \Delta^2 u + \tau \nabla \rho \cdot \nabla u \\ &+ \frac{1}{\epsilon} \rho \nabla V = \frac{1}{\epsilon} \operatorname{div}(\rho \nabla^2 \log \rho) - \frac{1}{\epsilon^2} \rho u - r_0 u - r_0 \rho |u|^2 u + \delta \rho \nabla \Delta^9 \rho, \\ - \Delta V &= \rho - g, \end{aligned} \tag{A.2}$$

where τ, μ, δ , and η are positive constants. The existence of solutions for this system is proved by Faedo–Galerkin method, following with minor changes the arguments in Vasseur and Cheng (2016). Thanks to the additional regularizing terms in (A.2), it can be proved that their solutions satisfy an approximated version of the BD entropy. In addition, the presence of the terms $\eta \nabla \rho^{-10}$ and $\delta \rho \nabla \Delta^9 \rho$ guarantees that the density is bounded away from zero, namely there exists $C = C(\delta, \eta) > 0$ such that $\rho(x, t) \geq C(\delta, \eta) > 0$. At this point, it is possible to perform the limit of the parameters τ, μ, δ , and η going to zero. In particular, one first performs the limit as τ and μ going to zero, and then the limit of δ and η going to zero. In the latter limit there may be a possible loss of strict positivity of the density and, due to this, the tensors \mathcal{T} and \mathcal{S} appear, and, clearly, they depend on r_0 . More precisely, the tensor \mathcal{T} is obtained by standard compactness argument as a weak limit, namely:

$$\sqrt{\rho}_{\delta,\eta} \nabla u_{\delta,\eta} \rightharpoonup \mathcal{T} \text{ in } L^2((0, T) \times \mathbb{T}^3), \text{ as } \delta, \eta \rightarrow 0.$$

On the other hand, since it holds that

$$\rho_{\delta,\eta} \nabla u_{\delta,\eta} = \nabla(\rho_{\delta,\eta} u_{\delta,\eta}) - 2\sqrt{\rho}_{\delta,\eta} u_{\delta,\eta} \otimes \nabla \sqrt{\rho}_{\delta,\eta},$$

one can show that the latter converges for $\delta, \eta \rightarrow 0$ to

$$\sqrt{\rho} \mathcal{T} = \nabla(\rho u) - 2\sqrt{\rho} u \otimes \nabla \sqrt{\rho} \text{ in } \mathcal{D}'((0, T) \times \mathbb{T}^3).$$

Thus, the tensor \mathcal{T} is uniquely identified by (ρ, u) . The same compactness arguments apply for \mathcal{S} , which arises as a weak limit of $\sqrt{\rho_{\delta,\eta}} \nabla^2 \log \rho_{\delta,\eta}$, namely:

$$\sqrt{\rho_{\delta,\eta}} \nabla^2 \log \rho_{\delta,\eta} \rightharpoonup \mathcal{S} \text{ in } L^2((0, T) \times \mathbb{T}^3), \text{ as } \delta, \eta \rightarrow 0.$$

The equation (2.9) is therefore a consequence of the passage to the limit into the following identity

$$\rho_{\delta,\eta} \nabla^2 \log \rho_{\delta,\eta} = 2\sqrt{\rho_{\delta,\eta}} \nabla^2 \sqrt{\rho_{\delta,\eta}} - 2\nabla \sqrt{\rho_{\delta,\eta}} \otimes \nabla \sqrt{\rho_{\delta,\eta}}.$$

Indeed, as already observed for the tensor \mathcal{T} , one can show that, sending $\delta, \eta \rightarrow 0$, it holds

$$\sqrt{\rho} \mathcal{S} = 2\sqrt{\rho} \nabla^2 \sqrt{\rho} - 2\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} \text{ in } \mathcal{D}'((0, T) \times \mathbb{T}^3).$$

Hence, the final tensors \mathcal{T} and \mathcal{S} appearing in Definition 2.1 are the limit as $r_0 \rightarrow 0$ of the ones constructed above.

Finally, we explain why (2.11) has to be included in the definition of weak solutions. Note that, for δ, η and r_0 fixed, from (2.13) we obtain

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^3} |\nabla^2 \sqrt{\rho_{\delta,\eta,r_0}}|^2 + |\nabla \rho_{\delta,\eta,r_0}^{\frac{1}{4}}|^4 dx dt &\leq C \int_0^T \int_{\mathbb{T}^3} \rho_{\delta,\eta,r_0} |\nabla^2 \log \rho_{\delta,\eta,r_0}|^2 dx dt \\ &\leq C(\rho_{\delta,\eta,r_0}^0, u_{\delta,\eta,r_0}^0, V_{\delta,\eta,r_0}^0, g_{\delta,\eta,r_0}^0), \end{aligned}$$

where the constant on the right-hand side stands for the approximated version of the BD entropy at time $t = 0$. Then, in the limit of η and δ vanishing, we can use lower semicontinuity on the left-hand side and natural convergence hypotheses on the initial data to conclude

$$\int_0^T \int_{\mathbb{T}^3} |\nabla^2 \sqrt{\rho_{r_0}}|^2 + |\nabla \rho_{r_0}^{\frac{1}{4}}|^4 dx dt \leq C(\rho_{r_0}^0, u_{r_0}^0, V_{r_0}^0, g_{r_0}^0),$$

and, in the limit $r_0 \rightarrow 0$, one eventually gets exactly (2.11).

Appendix B.

For sake of completeness, in this appendix we present the proof of (2.13).

Lemma B.1 *There exists $C > 0$ depending only on the dimension such that for any smooth function $\rho > 0$ it holds*

$$\int |\nabla \rho^{\frac{1}{4}}|^4 + \int |\nabla^2 \sqrt{\rho}|^2 \leq C \int \rho |\nabla^2 \log \rho|^2.$$

Proof We first notice that

$$\int \rho \left| \nabla \left(\frac{\nabla \sqrt{\rho}}{\sqrt{\rho}} \right) \right|^2 = \int \rho \left| -\frac{1}{2} \nabla \log \rho \otimes \nabla \log \rho + \frac{1}{2\rho} \nabla^2 \rho \right|^2 = \frac{1}{4} \int \rho |\nabla^2 \log \rho|^2. \tag{B.1}$$

On the other hand we also have

$$\int \rho \left| \nabla \left(\frac{\nabla \sqrt{\rho}}{\sqrt{\rho}} \right) \right|^2 = \int \frac{1}{\rho} |\nabla \sqrt{\rho}|^4 + |\nabla^2 \sqrt{\rho}|^2 - 2 \frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} : \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}. \tag{B.2}$$

We have:

$$\begin{aligned} & \int \frac{1}{\sqrt{\rho}} \partial_{x_i} \partial_{x_j} \sqrt{\rho} \partial_{x_i} \sqrt{\rho} \partial_{x_j} \sqrt{\rho} = \int \partial_{x_i} \left(\partial_{x_j} \sqrt{\rho} \frac{\partial_{x_i} \sqrt{\rho}}{\sqrt{\rho}} \partial_{x_j} \sqrt{\rho} \right) \\ & - \int \partial_{x_j} \sqrt{\rho} \partial_{x_i} \left(\frac{\partial_{x_i} \sqrt{\rho}}{\sqrt{\rho}} \right) \partial_{x_j} \sqrt{\rho} - \int \frac{1}{\sqrt{\rho}} \partial_{x_i} \partial_{x_j} \sqrt{\rho} \partial_{x_i} \sqrt{\rho} \partial_{x_j} \sqrt{\rho}. \end{aligned}$$

The first term is zero, and thus, we get

$$2 \int \frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} : \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} = - \int |\nabla \sqrt{\rho}|^2 \operatorname{div} \left(\frac{\nabla \sqrt{\rho}}{\sqrt{\rho}} \right).$$

Then, we use Young inequality

$$\begin{aligned} 2 \left| \int \frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} : \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} \right| & \leq \frac{1}{2} \int \frac{|\nabla \sqrt{\rho}|^4}{\rho} + \frac{1}{2} \int \rho \left| \operatorname{div} \left(\frac{\nabla \sqrt{\rho}}{\sqrt{\rho}} \right) \right|^2 \\ & \leq \frac{1}{2} \int \frac{|\nabla \sqrt{\rho}|^4}{\rho} + \frac{1}{2} \int \rho \left| \nabla \left(\frac{\nabla \sqrt{\rho}}{\sqrt{\rho}} \right) \right|^2, \end{aligned}$$

and finally we get, using (B.1) and (B.2):

$$\int \frac{|\nabla \sqrt{\rho}|^4}{\rho} + |\nabla^2 \sqrt{\rho}|^2 \leq C \int \rho |\nabla^2 \log \rho|^2$$

that gives (2.13), being

$$\int |\nabla \rho^{\frac{1}{4}}|^4 = 16 \int \frac{|\nabla \sqrt{\rho}|^4}{\rho}.$$

□

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