



Analytic Functions in Local Shift-Invariant Spaces and Analytic Limits of Level Dependent Subdivision

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Abstract

In this paper we characterize all subspaces of analytic functions in finitely generated shift-invariant spaces with compactly supported generators and provide explicit descriptions of their elements. We illustrate the differences between our characterizations and Strang-Fix or zero conditions on several examples. Consequently, we depict the analytic functions generated by scalar or vector subdivision with masks of bounded and unbounded support. In particular, we prove that exponential polynomials are indeed the only analytic limits of level dependent scalar subdivision schemes with finitely supported masks.

Keywords Analytic subspaces · Finitely generated shift-invariant spaces · Refinable functions · Level dependent (non-stationary) subdivision

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1 Introduction

Shift-invariant spaces

$$S_{\Phi} = \left\{ \sum_{j=1}^n \sum_{k \in \mathbb{Z}} c(j, k) \phi_j(\cdot - k) : c(j, \cdot) \in \ell(\mathbb{Z}) \right\}$$

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Sections 1 and 5 are written by M. Charina, Sections 2, 3, and 4 are written by V. Yu. Protasov. All results in this paper are products of authors collaborative work.

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generated by a finite family $\Phi = \{\phi_1, \dots, \phi_n\}$ of compactly supported distributions ϕ_1, \dots, ϕ_n , $n \in \mathbb{N}$, have been well studied in the literature in various contexts. On the one hand, such spaces arise naturally in the signal processing, where they model signals represented by integer shifts of some basic signals [1,2,54,56]. On the other hand, approximation properties of shift-invariant spaces have been put to good use in the study of representation systems, affine synthesis and related issues, see e.g. [27,55] and references therein.

In this paper, in the univariate setting, we characterize the structure of all analytic subspaces of S_Φ with or without the assumption on the refinability of ϕ_1, \dots, ϕ_n , $n \in \mathbb{N}$. We call a function *analytic* [19], if it is infinitely differentiable on \mathbb{R} and at each point in \mathbb{R} can be expressed as a power series with positive radius of convergence. The well known results from the literature describe fully the structure of polynomial or exponential polynomial subspaces of S_Φ [7,8,11,17,24,34]; intrinsically related approximation order of S_Φ [3–6,26,36,37,39,47,51,53]; favorable basis or frames for S_Φ [12,21,22,44,48,49]; regularity, stability and other properties of Φ [6,8,9,15,28,30,35,52] or the connection between subdivision, cascade algorithms and shift-invariant spaces [7,8,13,23,24,31,32,41,58]. However we are not aware of any results that describe the full variety of analytic functions in S_Φ .

The understanding of the fine structure of such subspaces of S_Φ is important since in applications various functions to be approximated are assumed to be analytic. For example, solutions of certain differential equations, special signals in signal processing etc. Furthermore some important classes of shift-invariant spaces, e.g. spaces of band-limited functions, consist of analytic functions. In this paper, we address the question of the existence of subspaces of S_Φ comprised of analytic functions other than polynomials or exponential polynomials. We show that the full variety of analytic functions in shift-invariant spaces is very rich, even for a shift-invariant space generated by one function $\phi = \phi_1$. Nevertheless, we classify this variety completely. In the following we present our main results:

- *Structure of all analytic subspaces of S_Φ*

Theorem 1 *Let $H \subset S_\Phi$ be the space of all analytic functions in S_Φ . Then, there exist $s \in \mathbb{N}$, $\lambda_j \in \mathbb{C}$, $d_j \in \mathbb{N}_0$ and 1-periodic analytic functions $\omega_{j,k} : \mathbb{R} \rightarrow \mathbb{C}$, $k = 0, \dots, d_j$, $j = 1, \dots, s$, such that*

$$H \subseteq \left\{ f \in S_\Phi : f(t) = \sum_{j=1}^s e^{\lambda_j t} \sum_{k=0}^{d_j} \omega_{j,k}(t) t^k \right\}. \quad (1)$$

Theorem 1 follows from Theorem 2.7 in Sect. 2.1.2. Thus, all analytic functions in a finitely generated space S_Φ are exponential polynomials of the form (1) with coefficients being arbitrary analytic 1-periodic functions $\omega_{j,k}$. We would like to emphasize that H is only a subspace of the space of all exponential polynomials defined by the sum in (1). The complete characterization of the structure of H is given in Theorem 2.8 in terms of special integer matrices. The full variety of spaces H has a non-trivial combinatorial structure illustrated by Example 2.10.

Theorem 2.8 also provides a characterization of all analytic functions generated by vector subdivision. Polynomial subspaces generated by vector subdivision schemes and approximation properties of the related multi-wavelets were thoroughly studied in the stationary case in [6,15,33,38,40,42,43] and references therein.

In Sect. 3, we consider the shift-invariant space S_ϕ generated by a single compactly supported not necessarily refinable distribution ϕ and show that the analytic subspace H of S_ϕ has a simpler structure.

- Structure of all analytic subspaces of S_ϕ generated by a single distribution ϕ .

Similarly to the Strang-Fix conditions, the structure of $H \subset S_\phi$ is characterized in terms of the zeros of the Fourier transform (analytically extended to \mathbb{C})

$$\widehat{\phi} = \int_{\mathbb{R}} \phi(t) e^{-2\pi i t \cdot} dt, \text{ of } \phi \text{ and its derivatives } \widehat{\phi}^{(k)}.$$

Theorem 2 *Let $\lambda \in \mathbb{C}$, $d \in \mathbb{N}_0$. The space S_ϕ contains a function $e^{\lambda t} \sum_{k=0}^d \omega_k(t) t^k$ for some 1-periodic analytic functions ω_k if and only if the sequences*

$$\left\{ \widehat{\phi}^{(k)} \left(-\frac{i\lambda}{2\pi} + \ell \right) : \ell \in \mathbb{Z} \right\}, \quad k = 0, \dots, d, \tag{2}$$

all decay exponentially as ℓ goes to infinity.

Theorem 2 follows from Theorem 3.1 in Subsect. 3. Theorem 3.1 in addition, classifies the analytic functions ω_k in terms of the sequences in (2). Theorem 3.1 also sheds light onto the generation properties of the level dependent subdivision scheme with masks of unbounded support e.g. by Rvachev in [50] or of the related constructions in [14,29].

In Sect. 4, we additionally assume that the Fourier transform of ϕ satisfies the generalized refinability property

$$\widehat{\phi}(y) = \prod_{j=1}^{\infty} a_j(2^{-j} y), \quad y \in \mathbb{R},$$

for some trigonometric polynomials a_j . In other words, ϕ is a limit of the level dependent subdivision scheme with masks a_j , $j \in \mathbb{N}$. In this case, Theorem 4.1 states that the sequences in (2) contain only finitely many non-zero elements. The well known Strang-Fix conditions is a special case of Theorem 4.1. An equivalent characterization of H in terms of the trigonometric polynomials a_j is given in Theorem 4.5. Examples 4.7–4.9 illustrate the differences between the well known zero conditions and the characterization in Theorem 4.5.

- Analytic subspaces of S_ϕ generated by level dependent scalar subdivision

Theorems 1 and 2 allow for the following characterization of analytic limits of level dependent subdivision.

Theorem 3 *Every analytic limit of a level dependent scalar subdivision scheme with finitely supported masks is an exponential polynomial.*

More precisely, Theorem 3 follows from Theorem 4.1 that says that in the refinable case the 1-periodic analytic functions ω_k are trigonometric polynomials. Moreover, if all trigonometric polynomials a_j are the same, then all ω_k are constant and as expected polynomials are the only analytic limits generated by stationary subdivision.

2 Analytic Functions in Finitely Generated Shift-Invariant Spaces

We start our analysis with finitely generated shift invariant spaces. There is a multitude of results in the literature about the properties of shift-invariant spaces generated by $\Phi = \{\phi_1, \dots, \phi_n\}$ with each ϕ_j of compact support, see [47] and references therein. We denote by S_Φ the corresponding shift-invariant space

$$S_\Phi = \left\{ \sum_{j=1}^n \sum_{k \in \mathbb{Z}} c(j, k) \phi_j(\cdot - k) : c(j, \cdot) \in \ell(\mathbb{Z}) \right\}$$

with $\ell(\mathbb{Z})$ being the space of sequences over \mathbb{C} . Such spaces for $n = 1$ arise in the context of stationary and level dependent subdivision, while for $n > 1$ in the context of vector subdivision schemes and multi-wavelets.

Our goal is to expose the classes of analytic functions that belong to S_Φ . Under an analytic function [19] we mean a function which is infinitely differentiable on \mathbb{R} and at each point in \mathbb{R} can be expressed as a power series with positive radius of convergence.

Our characterizations make use of the following exponential spaces.

Definition 2.1 Let $\Lambda \subset \mathbb{C}$ and \mathcal{P}_k the space of polynomials of degree less than or equal to k . The space $U \subset L_{1,loc}(\mathbb{R})$ is called *exponential* if

$$U = \text{span}\{p(\cdot)e^{\lambda \cdot} : p \in \mathcal{P}_{k(\lambda)}, \lambda \in \Lambda\} \quad \dim(U) = \sum_{\lambda \in \Lambda} (k(\lambda) + 1),$$

with the integer $k(\lambda)$ being the multiplicity of $\lambda \in \Lambda$.

In Subsect. 2.1, we characterize the set of all analytic functions that belong to S_Φ without any additional assumptions on the generators in Φ . In Sect. 3, we study the special case of $n = 1$ under the assumption that the integer shifts of $\phi = \phi_1$ are linearly independent. In Sect. 4, we additionally assume the generalized refinability of $\phi = \phi_1$.

2.1 Case $n \geq 1$

In this subsection, we study the structure of the subspace H of analytic functions in the shift-invariant space S_Φ . The main result of this section, Theorem 2.8, states that the expected contribution of the exponential spaces U to the structure of H has to be unexpectedly augmented by contributions of certain 1-periodic analytic functions. The proof of Theorem 2.8 is based on auxiliary results from Subsect. 2.1.1 and on Theorem 2.7.

The next Example shows that the presence of 1-periodic analytic functions in the subspace H is very natural.

Example 2.2 Let ω be 1-periodic analytic function. For a B-spline ψ , define

$$\phi = \psi \cdot \omega.$$

Then, due to the partition of unity property of ψ and periodicity of ω , we have

$$\sum_{k \in \mathbb{Z}} \phi(\cdot - k) = \sum_{k \in \mathbb{Z}} \psi(\cdot - k) \omega(\cdot - k) = \omega \sum_{k \in \mathbb{Z}} \psi(\cdot - k) = \omega,$$

i.e. $\omega \in S_\Phi$.

2.1.1 Auxiliary Results

To study the structure of the subspace H , we make use of the following well known difference operator.

Definition 2.3 The exponential difference operator ∇_λ on S_Φ is defined by

$$\nabla_\lambda : S_\Phi \rightarrow S_\Phi, \quad \nabla_\lambda(f) = e^{-\lambda} f(\cdot + 1) - f, \quad \lambda \in \mathbb{C}.$$

In Lemma 2.4, we recall the action of the powers of the difference operator ∇_λ on the products of the form $\omega(t)p(t)e^{\lambda t}$ with a 1-periodic function ω and an algebraic polynomial p . Such products are shown in the sequel to be the building blocks of the elements of H . Clearly, powers of ∇_λ annihilate $\omega(t)p(t)e^{\lambda t}$. On the contrary, ∇_λ does not affect the structure of such products, if we replace the term $e^{\lambda t}$ by $e^{\mu t}$ with $\mu \neq \lambda$. We present the proof of this straightforward fact to illustrate that, even in the presence of the 1-periodic function ω , the action of the difference operator ∇_λ is inherited from its action on the exponential spaces U .

Lemma 2.4 Let $n \in \mathbb{N}$ and $\lambda, \mu \in \mathbb{C}, \lambda \neq \mu$. Then for every 1-periodic function $\omega : \mathbb{R} \rightarrow \mathbb{C}$ and every polynomial $p \in \mathcal{P}_{n-1}$, we have

- (i) $\nabla_\lambda^n(\omega p e^{\lambda t}) = 0,$
- (ii) $\nabla_\lambda(\omega p e^{\mu t}) = \omega \tilde{p} e^{\mu t}$ for some $\tilde{p} \in \mathcal{P}_{n-1}$ with $\deg(\tilde{p}) = \deg(p).$

Proof Part (i): By Definition 2.3, we get

$$\nabla_\lambda(\omega(t)p(t)e^{\lambda t}) = \omega(t)p(t+1)e^{\lambda t} - \omega(t)p(t)e^{\lambda t} = \omega(t)e^{\lambda t} (p(t+1) - p(t)), \quad t \in \mathbb{R},$$

where the polynomial $p(\cdot + 1) - p \in \mathcal{P}_{n-2}$, since the leading coefficients of $p(\cdot + 1)$ and p are equal. Applying ∇_λ iteratively, we obtain the claim.

Part (ii): Similarly to (i), the claim follows, due to

$$\begin{aligned} \nabla_\lambda(\omega(t)p(t)e^{\mu t}) &= \omega(t)p(t+1)e^{\mu(t+1)-\lambda} - \omega(t)p(t)e^{\mu t} \\ &= \omega(t)e^{\mu t} (e^{\mu-\lambda} p(t+1) - p(t)), \end{aligned}$$

where the polynomial $\tilde{p} = e^{\mu-\lambda} p(\cdot + 1) - p \in \mathcal{P}_{n-1}$, since $\lambda \neq \mu$. □

Our next result, Lemma 2.6, states that any 1-periodic function, appearing in the representation of $f \in H$, is analytic. It also exposes the finer structure of H . We first illustrate the idea of the proof of Lemma 2.6 on the following example.

Example 2.5 Assume that, for 1-periodic functions $\omega_{1,0}, \omega_{2,0}, \omega_{2,1} : \mathbb{R} \rightarrow \mathbb{C}$ and $\lambda, \mu \in \mathbb{C}, \lambda \neq \mu$, the function

$$f(t) = \omega_{1,0}(t) e^{\lambda t} + (\omega_{2,0}(t) + \omega_{2,1}(t) t) e^{\mu t}, \quad t \in \mathbb{R},$$

belongs to H . Then, applying ∇_λ to eliminate the term with $e^{\lambda t}$, we obtain that

$$\begin{aligned} \nabla_\lambda(f)(t) &= \omega_{1,0}(t) e^{\lambda t} + (\omega_{2,0}(t) + \omega_{2,1}(t) (t + 1)) e^{\mu(t+1)-\lambda} - \omega_{1,0}(t) e^{\lambda t} \\ &\quad - (\omega_{2,0}(t) + \omega_{2,1}(t) t) e^{\mu t} \\ &= \left(\omega_{2,0}(t) (e^{\mu-\lambda} - 1) + \omega_{2,1}(t) e^{\mu-\lambda} + \omega_{2,1}(t) (e^{\mu-\lambda} - 1) t \right) e^{\mu t} \end{aligned}$$

belongs to H , due to the shift-invariance of S_Φ . Similarly, we apply ∇_μ to eliminate the "constant" term in $\nabla_\lambda(f)$ and get that

$$\nabla_\mu(\nabla_\lambda(f))(t) = e^{-\mu} \nabla_\lambda(f)(t + 1) - \nabla_\lambda(f)(t) = \omega_{2,1}(t) (e^{\mu-\lambda} - 1) e^{\mu t}$$

is in H . This implies also that the 1-periodic function $\omega_{2,1}$ is analytic, due to the analyticity of $\nabla_\mu(\nabla_\lambda(f))$ and $e^{\mu t}$. Moreover, the analyticity of $\nabla_\lambda(f)$, $\omega_{2,1}(t) e^{\mu t}$ and $\omega_{2,1}(t) t e^{\mu t}$, implies that the 1-periodic function $\omega_{2,0}$ is analytic. And, finally, considering

$$f(t) - (\omega_{2,0}(t) + \omega_{2,1}(t) t) e^{\mu t} = \omega_{1,0}(t) e^{\lambda t}$$

yields that $\omega_{1,0}$ is analytic.

Now we are ready to formulate and prove Lemma 2.6.

Lemma 2.6 *Let $H \subset S_\Phi$ be a subspace of all analytic functions. If there exist $s \in \mathbb{N}$, $\lambda_j \in \mathbb{C}$ (pairwise distinct modulo $2\pi i$) and $d_j \in \mathbb{N}_0, j = 1, \dots, s$, such that*

$$f(t) = \sum_{j=1}^s e^{\lambda_j t} \sum_{k=0}^{d_j} \omega_{j,k}(t) t^k, \quad \omega_{j,k} : \mathbb{R} \rightarrow \mathbb{C} \text{ are } 1\text{-periodic}, \quad (3)$$

belongs to H , then, for $j = 1, \dots, s$,

- (i) $\omega_{j,k}, k = 0, \dots, d_j$, are analytic and
- (ii) there exist polynomials $p_{j,k}$ with $\deg(p_{j,k}) = k, k = 0, \dots, d_j$, such that

$$e^{\lambda_j t} \sum_{k=0}^{d_j} p_{j,k}(t) \omega_{j,k}(t) \text{ belong to } H.$$

Proof Let $\ell \in \{1, \dots, s\}$. Due to Lemma 2.4, we can eliminate the summands in f with exponential factors $e^{\lambda_j t}$, $j \in \{1, \dots, s\} \setminus \{\ell\}$, by applying the corresponding difference operators ∇_{λ_j} to f each $d_j + 1$ times, respectively. By the shift-invariance of S_Φ and Definition 2.3, the resulting function

$$\tilde{f}(t) := e^{\lambda_\ell t} \sum_{k=0}^{d_\ell} \left(\sum_{m=k}^{d_\ell} c_{k,m} \omega_{\ell,m}(t) \right) t^k = e^{\lambda_\ell t} \sum_{k=0}^{d_\ell} p_{\ell,k}(t) \omega_{\ell,k}(t) \text{ belongs to } H$$

and all its coefficients $c_{k,m}$ are non-zero, since $c_{k,m}$ are products of the factors e^{λ_j} , $j = 1, \dots, d$, or $e^{\lambda_j - \lambda_\ell} - 1$, $j \neq \ell$, and of the binomial coefficients in the expansions of $(t + 1)^k$, $k = 0, \dots, d_\ell$. Therefore, $\deg(p_{\ell,k}) = k$, $k = 0, \dots, d_\ell$. Note also that the leading term of \tilde{f} for $k = d_\ell$ is of the form $c_{d_\ell,d_\ell} \omega_{\ell,d_\ell}(t) t^{d_\ell} e^{\lambda_\ell t}$. Thus, by Lemma 2.4, applying $\nabla_{\lambda_\ell}^{d_\ell}$ to \tilde{f} leaves us with $d_\ell! c_{d_\ell,d_\ell} \omega_{\ell,d_\ell}(t) e^{\lambda_\ell t} \in H$. The analyticity of $\omega_{\ell,d_\ell}(t) e^{\lambda_\ell t}$ and $e^{\lambda_\ell t}$ implies that the function ω_{ℓ,d_ℓ} is analytic. Next we apply to \tilde{f} the operator $\nabla_{\lambda_\ell}^{d_\ell-1}$ to obtain

$$\left((d_\ell - 1)! c_{d_\ell-1,d_\ell-1} \omega_{\ell,d_\ell-1}(t) + \tilde{c}_{d_\ell-1,d_\ell} \omega_{\ell,d_\ell}(t) + \tilde{c}_{d_\ell,d_\ell} \omega_{\ell,d_\ell}(t) t \right) e^{\lambda_\ell t} \in H.$$

Its analyticity and the analyticity of ω_{ℓ,d_ℓ} and $e^{\lambda_\ell t}$ imply that $\omega_{\ell,d_\ell-1}$ is analytic. Continuing iteratively yields the claim. \square

2.1.2 Structure of H

We show first that every $f \in H$ is indeed of the form (3) with 1-periodic analytic functions $\omega_{j,k}$. We also give an upper bound for the dimension of H in terms of

$$\lceil \text{supp}(\phi_j) \rceil = \lceil |\text{supp}(\phi_j)| \rceil, \quad j = 1, \dots, n.$$

Theorem 2.7 *Let $H \subset S_\Phi$ be the space of all analytic functions in S_Φ . Then, there exist $s \in \mathbb{N}$, $\lambda_j \in \mathbb{C}$ (pairwise distinct modulo $2\pi i$), $d_j \in \mathbb{N}_0$ and 1-periodic analytic functions $\omega_{j,k} : \mathbb{R} \rightarrow \mathbb{C}$, $k = 0, \dots, d_j$, $j = 1, \dots, s$, such that*

$$H \subseteq \left\{ f \in S_\Phi : f(t) = \sum_{j=1}^s e^{\lambda_j t} \sum_{k=0}^{d_j} \omega_{j,k}(t) t^k \right\}. \tag{4}$$

Moreover,

$$\dim(H) \leq \sum_{j=1}^n \lceil \text{supp}(\phi_j) \rceil.$$

Proof Due to the compact supports of $\phi_j, j = 1, \dots, n$, there are only finitely many functions $\phi_j(\cdot - \ell), j = 1, \dots, n, \ell \in \mathbb{Z}$, whose supports intersect with the interval $[0, 1]$. This finite number we denote by

$$N = \dim (S_\Phi|_{[0,1]}) \leq \sum_{j=1}^n \lceil \text{supp}(\phi_j) \rceil.$$

Therefore, for every $f \in S_\Phi$, the $N + 1$ functions $f(\cdot + \ell) \in S_\Phi, \ell = 0, \dots, N$, are linearly dependent over $[0, 1]$, namely

$$\sum_{\ell=0}^N a_\ell f(t + \ell) = 0, \quad t \in [0, 1],$$

where some of the coefficients $a_\ell \in \mathbb{C}$ are non-zero. If, furthermore, $f \in H$, then, due to the analyticity of this linear combination, we have

$$\sum_{\ell=0}^N a_\ell f(t + \ell) = 0, \quad t \in \mathbb{R}. \tag{5}$$

The identity (5) implies that, for every $\tau \in [0, 1]$, the sequence of numbers $\{f(\tau + \ell) : \ell \in \mathbb{Z}\}$ satisfies the linear difference equation with constant coefficients a_0, \dots, a_N . Let $\alpha_j \in \mathbb{C} \setminus \{0\}, j = 1, \dots, s, s \leq N$, be the non-zero roots of the characteristic polynomial corresponding to (5) and $\mu_j \in \mathbb{N}$ be the multiplicity of α_j . Then, the solution of (5), for $\tau \in [0, 1]$, has the form

$$f(\tau + \ell) = \sum_{j=1}^s e^{\lambda_j \ell} \sum_{k=0}^{d_j} b_{j,k}(\tau) \ell^k, \quad d_j := \mu_j - 1,$$

$$\lambda_j := \ln(\rho_j) + i \theta_j, \quad b_{j,k} : [0, 1] \rightarrow \mathbb{R}, \quad \ell \in \mathbb{Z}.$$

Extend $b_{j,k}$ to a 1-periodic function over \mathbb{R} . Then substitution $t = \tau + \ell$ and the binomial identity lead to

$$f(t) = \sum_{j=1}^s e^{\lambda_j(t-\tau)} \sum_{k=0}^{d_j} b_{j,k}(t) (t - \tau)^k$$

$$= \sum_{j=1}^s e^{\lambda_j t} \sum_{k=0}^{d_j} \left(\sum_{m=0}^{d_j-k} c_{m,k} b_{j,k+m}(t) \tau^m e^{-\lambda_j \tau} \right) t^k, \quad c_m \in \mathbb{R}, \quad t \in \mathbb{R}.$$

The functions $\omega_{j,k} : [0, 1] \rightarrow \mathbb{C}$, $\omega_{j,k}(\tau) = \sum_{m=0}^{d_j-k} c_{m,k} b_{j,k+m}(\tau) \tau^m e^{-\lambda_j \tau}$, can be extended to \mathbb{R} to be 1-periodic. By Lemma 2.6 (i), due to $f \in H$, all the functions $\omega_{j,k}$ are analytic. \square

Finally, in Theorem 2.8, we observe that the intrinsic building blocks of H depend on the invariant spaces of the shift operators A_{d_j} with $d_j, j = 1, \dots, s$, in (4). The operator A_{d_j} acts on the space

$$M_{d_j} = \bigoplus_{k=0}^{d_j} \mathcal{P}_k,$$

i.e. on the direct sum of the spaces \mathcal{P}_k of algebraic polynomials of degrees at most k , by

$$A_{d_j}(p_0, \dots, p_{d_j}) = (p_0(\cdot + 1), \dots, p_{d_j}(\cdot + 1))$$

M_{d_j} is a linear space of dimension $\sum_{k=0}^{d_j} (k + 1)$ and A_{d_j} is a linear operator with the block-diagonal matrix representation

$$A_{d_j} = \begin{pmatrix} B_0 & 0 & \dots & 0 \\ 0 & B_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & B_{d_j} \end{pmatrix} \quad \text{with} \quad B_k = \begin{pmatrix} 1 & 0 & \dots & 0 \\ k & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 1 \end{pmatrix} \quad (6)$$

of size $k + 1, k = 0, \dots, d_j$ (B_k maps the vector of coefficients of p_k to the coefficients of $p_k(\cdot + 1)$). More precisely, B_k is a lower triangular matrix with ones on the main diagonal and the ℓ -th column of B_k contains (starting with the main diagonal element) the binomial coefficients of the expansions of $(t + 1)^{k+1-\ell}, \ell = 1, \dots, k + 1$, respectively.

Theorem 2.8 *Let $H \subseteq S_\Phi$ be the space of all analytic functions in a finitely generated shift-invariant space S_Φ . Then there exist $s \in \mathbb{N}, \lambda_j \in \mathbb{C}$ (pairwise distinct modulo $2\pi i$), $d_j \in \mathbb{N}_0$, 1-periodic analytic functions $\omega_{j,k} : \mathbb{R} \rightarrow \mathbb{C}$, and subspaces $N_j \subset M_{d_j}, j = 1, \dots, s$, each N_j is an invariant subspace of the block-diagonal matrix A_{d_j} in (6) such that H is a linear span of spaces L_1, \dots, L_s with*

$$L_j = \left\{ e^{\lambda_j t} \sum_{k=0}^{d_j} p_{j,k} \omega_{j,k}(t) \mid p_{j,k}(t) \in \mathcal{P}_k, (p_{j,0}, \dots, p_{j,d_j}) \in N_j \right\}, \quad (7)$$

Moreover, every subspace of H has the same form (7) with the same λ_j and $\omega_{j,k}$ but with some subspaces $N'_j \subset N_j, j = 1, \dots, s$.

Proof For f in H , by Lemma 2.6 (ii), we get that

$$g(t) = g_j(t) = e^{\lambda_j t} \sum_{k=0}^{d_j} p_{j,k}(t) \omega_{j,k}(t) \in H, \quad j = 1, \dots, s,$$

with $\deg(p_{j,k}) = k, k = 0, \dots, d_j$. Note that the operators $e^{\lambda_j - \mu} A_{d_j} - I, \mu \in \{\lambda_\ell : \ell = 1, \dots, s\} \setminus \{\lambda_j\}$, are non degenerate (invertible) on $M_{d_j}, j = 1, \dots, s$ and describe the transformation $f \mapsto g_j$ in Lemma 2.6 (ii). Moreover, we observe that the shift operator $g(\cdot) \mapsto g(\cdot + 1)$ leaves the functions $\omega_{j,k}$ unchanged and maps the vector $(p_{j,0}, \dots, p_{j,d_j}) \in M_{d_j}$ to the vector $e^{\lambda_j} A_{d_j} (p_{j,0}, \dots, p_{j,d_j}) \in M_{d_j}$. Since H is shift-invariant, it contains the linear span of all integer shifts of g . Hence,

it contains all functions $e^{\lambda_j t} \sum_{k=0}^{d_j} \tilde{p}_{j,k} \omega_{j,k}$ with $(\tilde{p}_{j,0}, \dots, \tilde{p}_{j,d_j})$ from the minimal invariant subspace of the operator A_{d_j} that contains the vector $(p_{j,0}, \dots, p_{j,d_j})$. The invertibility of A_{d_j} and $e^{\lambda_j - \mu} A_{d_j} - I, \lambda_j \neq \mu$, on $M_{d_j}, j = 1, \dots, s$, completes the proof. \square

Remark 2.9 Theorem 2.8 classifies all possible spaces of analytic functions in finitely generated shift-invariant spaces S_Φ . One takes a finite set of complex numbers $\lambda_1, \dots, \lambda_s$ and non-negative integers $d_j, j = 1, \dots, s$. Then, for each $j = 1, \dots, s$, one chooses an arbitrary invariant subspace N_j of the corresponding block-diagonal matrices A_{d_j} . This defines the functional space (7). The direct sum of those s spaces is the space of all analytic functions in a shift-invariant space S_Φ . It is interesting to note that the matrices A_{d_j} defined in (6) may have a very rich variety of invariant spaces. It would be interesting to obtain the description of such invariant spaces at least for A_{d_j} with small number of diagonal-blocks. In the example below we consider the simplest case of two diagonal blocks of sizes 1 and 2 and show that already in this case there are four possible spaces N_1 .

The following example shows that the structure of the invariant subspaces of the matrices A_{d_j} in (6) is highly nontrivial.

Example 2.10 By Theorem 2.8, every subspace H of analytic functions generated by the integer shifts of a finite set of compactly supported functions is a direct sum of spaces L_j of the form (7). Consider the simplest case $s = 1$, i.e., $H = L_1, d_1 = 3$ and the 3×3 matrix M_3 has two blocks, of sizes one and two. For the sake of simplicity in what follows we denote $\lambda_1 = \lambda, M_3 = M$ and $N_1 = N$. Let $\lambda \in \mathbb{C}, (a, b, c) \in \mathbb{R}^3$ and $\omega_{1,0}, \omega_{2,2} : \mathbb{R} \rightarrow \mathbb{C}$ be 1-periodic analytic functions. We classify all invariant subspaces $N \subseteq M = \mathcal{P}_0 \oplus \mathcal{P}_1$ of the linear operator $A : M \rightarrow M$ which, by (6), has the matrix representation

$$A = \begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix} \quad \text{with } B_0 = 1 \quad \text{and } B_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

By Theorem 2.8 there is a one-to-one correspondence between these invariant subspaces N of A and the subspaces $L_1 = L$ in (7).

(1). The first subspace of the matrix A is $N^{(1)} = \left\{ (0, b, c) \mid (b, c) \in \mathbb{R}^2 \right\}$ and the corresponding subspace of the analytic functions is given by

$$L = L^{(1)} = \left\{ e^{\lambda t} \omega_{j,1}(t) (bt + c) \mid (b, c) \in \mathbb{R}^2 \right\}.$$

(2). The second invariant subspace of the matrix A is $N^{(2)} = \left\{ (a, 0, c) \mid (a, c) \in \mathbb{R}^2 \right\}$ with

$$L = L^{(2)} = \left\{ e^{\lambda t} (a \omega_{j,0}(t) + c \omega_{j,1}(t)) \mid (a, c) \in \mathbb{R}^2 \right\}.$$

(3). Moreover, for every vector $(a_0, c_0) \in \mathbb{R}^2 \setminus \{0\}$, the matrix A has the following one-dimensional invariant subspace $N^{(a_0, c_0)} = \left\{ \tau (a_0, 0, c_0) \mid \tau \in \mathbb{R} \right\}$ with

$$L = L^{(a_0, c_0)} = \left\{ e^{\lambda t} (\tau a_0 \omega_{j,0}(t) + \tau c_0 \omega_{j,1}(t)) \mid \tau \in \mathbb{R} \right\}.$$

Surprisingly, these are not all invariant subspaces of A . There is one more family of invariant subspaces.

(4). For every $(u, v) \in \mathbb{R}^2 \setminus \{0\}$, the matrix A has a two-dimensional invariant subspace

$$N^{(u,v)} = \left\{ (a, b, c) \in \mathbb{R}^3 \mid ua + vb = 0 \right\}$$

with

$$L = L^{(u,v)} = \left\{ e^{\lambda t} (a \omega_{j,0}(t) + \omega_{j,1}(t)(bt + c)) \mid (a, b, c) \in \mathbb{R}^3, ua + vb = 0 \right\}.$$

Thus, there are four possible choices for the corresponding space L_1 .

Note that even in this simple example the classification of invariant subspaces of the matrix A is nontrivial.

Remark 2.11 A natural question arises to what degree can our results be extended to infinitely generated shift-invariant spaces? We do not know the answer to this question at the moment and can only remark that this is a challenging problem, since in the proofs we rely significantly on the finiteness of the generator set. In the following sections we go in the opposite direction and consider singly generated shift-invariant spaces, due to their importance in wavelet theory, subdivisions, etc.

3 Analytic Functions in Singly Generated Shift-Invariant Spaces

In this section, we characterize the structure of the subspace H of analytic functions in a singly generated shift-invariant space

$$S_\phi = \left\{ \sum_{k \in \mathbb{Z}} c(k) \phi(\cdot - k) : c \in \ell(\mathbb{Z}) \right\}.$$

This characterization, stated in Theorem 3.1, relates the structure of H to the exponential decay of the sequences derived from the Fourier transform (analytically extended to \mathbb{C})

$$\widehat{\phi}(y) = \int_{\mathbb{R}} \phi(x) e^{-2\pi i x \cdot y} dx, \quad y \in \mathbb{R}.$$

of a compactly supported distribution ϕ . In this section, w.l.g. we assume that the integer shifts of ϕ are linearly independent. This assumption is very natural due to the existence of the “canonical” generator for S_ϕ , see e.g. [47, Theorem 15].

Due to Theorem 2.8, we consider analytic functions $g(t) = e^{\lambda t} \sum_{k=0}^d p_k(t) \omega_k(t)$, $t \in \mathbb{R}$, in H with polynomials p_k satisfying $\deg p_k = k$, $k = 0, \dots, d$, $d \in \mathbb{N}_0$. If $p_j \equiv 0$, $j \in \{0, \dots, d\}$, we set the corresponding 1-periodic analytic function ω_j to be identically zero on \mathbb{R} . We assume that $\omega_d(t) \neq 0$.

Theorem 3.1 *Let ϕ be the generator of the shift-invariant space S_ϕ .*

(i) *If for $\lambda \in \mathbb{C}$ and $d \in \mathbb{N}_0$, the space S_ϕ contains $g(t) = e^{\lambda t} \sum_{k=0}^d p_k(t) \omega_k(t)$, where the functions ω_k are 1-periodic analytic, $\omega_d \neq 0$ and the polynomials p_k are of degree k , then the sequences*

$$\left\{ \widehat{\phi}^{(k)} \left(-\frac{i\lambda}{2\pi} + \ell \right) : \ell \in \mathbb{Z} \right\}, \quad k = 0, \dots, d \tag{8}$$

decay exponentially as $|\ell|$ goes to infinity.

(ii) *Conversely, if the sequences in (8) decay exponentially for some $\lambda \in \mathbb{C}$ and $d \in \mathbb{N}_0$, then the space S_ϕ contains the $(d + 1)$ -dimensional subspace of analytic functions*

$$H_\lambda = \left\{ \sum_{\ell \in \mathbb{Z}} e^{\lambda \ell} p(\ell) \phi(\cdot - \ell) : p \in \mathcal{P}_d \right\} \tag{9}$$

spanned by

$$\sum_{\ell \in \mathbb{Z}} e^{\lambda \ell} \ell^k \phi(t - \ell) = e^{\lambda t} \sum_{j=0}^k \binom{k}{j} t^{k-j} (-1)^j \omega_j(t), \quad t \in \mathbb{R}, \quad k = 0, \dots, d, \tag{10}$$

where ω_k are 1-periodic analytic functions given by

$$\omega_k(t) = - \left(\frac{-1}{2\pi i} \right)^k \sum_{\ell \in \mathbb{Z}} \widehat{\phi}^{(k)} \left(-\frac{i \lambda}{2\pi} + \ell \right) e^{2\pi i \ell t} \quad k = 0, \dots, d, \quad t \in \mathbb{R}. \tag{11}$$

Proof Let $\psi(t) = e^{-\lambda t} \phi(t), t \in \mathbb{R}$.

We first prove (ii). Assume that the sequences in (8) decay exponentially. Then, by Payley-Wiener theorem, the 1-periodic functions in (11) are analytic and, by the Poisson summation formula ¹, we have

$$\begin{aligned} \omega_k(t) &= - \left(\frac{-1}{2\pi i} \right)^k \sum_{\ell \in \mathbb{Z}} \widehat{\phi}^{(k)} \left(-\frac{i \lambda}{2\pi} + \ell \right) e^{2\pi i \ell t} \\ &= - \left(\frac{-1}{2\pi i} \right)^k \sum_{\ell \in \mathbb{Z}} \widehat{\psi}^{(k)}(\ell) e^{2\pi i \ell t} = \sum_{\ell \in \mathbb{Z}} (t - \ell)^k \psi(t - \ell) \end{aligned}$$

for $k = 0, \dots, d$ and $t \in \mathbb{R}$. Hence, for $k = 0, \dots, d$, the functions

$$\begin{aligned} \sum_{\ell \in \mathbb{Z}} \ell^k \psi(t - \ell) &= \sum_{\ell \in \mathbb{Z}} \left(t - (t - \ell) \right)^k \psi(t - \ell) \\ &= \sum_{\ell \in \mathbb{Z}} \sum_{j=0}^k \binom{k}{j} t^{k-j} (-1)^j (t - \ell)^j \psi(t - \ell) \\ &= \sum_{j=0}^k \binom{k}{j} t^{k-j} (-1)^j \sum_{\ell \in \mathbb{Z}} (t - \ell)^j \psi(t - \ell) \\ &= \sum_{j=0}^k \binom{k}{j} t^{k-j} (-1)^j \omega_j(t), \quad t \in \mathbb{R}, \end{aligned}$$

are analytic and belong to $H_\lambda \subseteq S_\phi$.

¹ To keep the standard notation for S_ϕ , we use here the Poisson summation formula with the “-” sign in the summation. We hope this will not confuse the reader.

The proof of (i) is by induction on d . In the case $d = 0$, the polynomial p_0 is constant, w.l.g $p_0(t) \equiv 1$. Then $g(t) = e^{\lambda t} \omega(t) \in S_\phi$ if and only if $\omega \in S_\psi$, i.e.

$$\omega(t) = \sum_{\ell \in \mathbb{Z}} a_\ell \psi(t - \ell), \quad a_\ell, t \in \mathbb{R}. \tag{12}$$

The periodicity of ω implies that

$$\sum_{\ell \in \mathbb{Z}} a_\ell \psi(t + 1 - \ell) = \sum_{\ell \in \mathbb{Z}} a_\ell \psi(t - \ell), \quad t \in \mathbb{R},$$

which is equivalent to the identity

$$\sum_{\ell \in \mathbb{Z}} (a_{\ell+1} - a_\ell) \psi(t - \ell) = 0, \quad t \in \mathbb{R}.$$

Due to the linear independence of the integer shifts of ψ , we obtain $a_{\ell+1} - a_\ell = 0$ for all $\ell \in \mathbb{Z}$. Or, equivalently, w.l.g. $a_\ell = 1$ for all $\ell \in \mathbb{Z}$. Therefore, by the Poisson summation formula, we obtain

$$\omega(t) = \sum_{\ell \in \mathbb{Z}} \psi(t - \ell) = - \sum_{\ell \in \mathbb{Z}} \widehat{\psi}(\ell) e^{2\pi i \ell t} = - \sum_{\ell \in \mathbb{Z}} \widehat{\phi} \left(-\frac{i\lambda}{2\pi} + \ell \right) e^{2\pi i \ell t}, \quad t \in \mathbb{R}.$$

Due to the analyticity of ω , the above identity holds if and only if there exists a constant $C > 0$ and $q \in (0, 1)$ such that

$$\left| \widehat{\phi} \left(-\frac{i\lambda}{2\pi} + \ell \right) \right| \leq C q^{|\ell|} \quad \text{for all } \ell \in \mathbb{Z}.$$

We assume that the hypothesis is true for $d - 1$ and prove (i) for $d \in \mathbb{N}_0$. Note that the function $g = \sum_{k=0}^d p_k \omega_k, \omega_d \neq 0$ belongs to S_ψ if and only if $g = \sum_{\ell \in \mathbb{Z}} a_\ell \psi(\cdot - \ell)$. Thus, by the periodicity of $\omega_k, k = 0, \dots, d$, we get

$$g(t + 1) - g(t) = \sum_{\ell \in \mathbb{Z}} (a_{\ell+1} - a_\ell) \psi(t - \ell) = \sum_{k=0}^d (p_k(t - 1) - p_k(t)) \omega_k(t), \quad t \in \mathbb{R}.$$

Define $q_k := p_k(\cdot - 1) - p_k, k = 1, \dots, d$. Due to $p_0(t) \equiv 1$ and $\deg(q_k) = k - 1$ for $k = 1, \dots, d$, the function

$$\tilde{g}(t) := \sum_{\ell \in \mathbb{Z}} (a_{\ell+1} - a_\ell) \psi(t - \ell) = \sum_{k=0}^{d-1} q_{k+1}(t) \omega_{k+1}(t), \quad t \in \mathbb{R},$$

satisfies the inductive assumption. Therefore, the sequences in (8) for $k = 0, \dots, d - 1$, decay exponentially. Secondly, by (ii), the structure of the d -dimensional H_λ and the analyticity of \tilde{g} , imply that

$$\sum_{\ell \in \mathbb{Z}} (a_{\ell+1} - a_\ell) \psi(\cdot - \ell) = \sum_{\ell \in \mathbb{Z}} p(\ell) \psi(\cdot - \ell), \quad \text{for some } p \in \mathcal{P}_{d-1}.$$

The linear independence of the integer shifts of ψ implies that $a_{\ell+1} - a_\ell = p(\ell)$ for all $\ell \in \mathbb{Z}$. Theory of difference equations ensures that every solution of this difference equation is given by $a_\ell = \tilde{p}(\ell)$, $\ell \in \mathbb{Z}$, for some polynomial \tilde{p} of degree d . We write $\tilde{p}(\ell) = \alpha \ell^d + q(\ell)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\deg(q) \leq d - 1$, and have

$$g(t) = \sum_{\ell \in \mathbb{Z}} (\alpha \ell^d + q(\ell)) \psi(t - \ell) = \sum_{k=0}^d p_k(t) \omega_k(t), \quad t \in \mathbb{R},$$

and, hence, the function

$$\alpha \sum_{\ell \in \mathbb{Z}} \ell^d \psi(t - \ell) = - \sum_{\ell \in \mathbb{Z}} q(\ell) \psi(t - \ell) + \sum_{k=0}^d p_k(t) \omega_k(t), \quad t \in \mathbb{R}, \tag{13}$$

is analytic due to the analyticity of g and, by the inductive assumption, analyticity of $\sum_{\ell \in \mathbb{Z}} q(\ell) \psi(\cdot - \ell)$. Consequently and due to the inductive assumption, the function

$$\begin{aligned} \sum_{\ell \in \mathbb{Z}} (t - \ell)^d \psi(t - \ell) &= (-1)^d \sum_{\ell \in \mathbb{Z}} \ell^d \psi(t - \ell) \\ &\quad + \sum_{k=1}^d (-1)^{d-k} t^k \sum_{\ell \in \mathbb{Z}} \ell^{d-k} \psi(t - \ell), \quad t \in \mathbb{R}, \end{aligned}$$

is analytic as well. Therefore, by the Poisson summation formula

$$\begin{aligned} \sum_{\ell \in \mathbb{Z}} (t - \ell)^d \psi(t - \ell) &= - \left(\frac{-1}{2\pi i} \right)^d \sum_{\ell \in \mathbb{Z}} \widehat{\psi}^{(d)}(\ell) e^{2\pi i \ell t} \\ &= - \left(\frac{-1}{2\pi i} \right)^d \sum_{\ell \in \mathbb{Z}} \widehat{\phi}^{(d)} \left(-\frac{i\lambda}{2\pi} + \ell \right) e^{2\pi i \ell t}, \quad t \in \mathbb{R}, \end{aligned}$$

and the analyticity implies that the sequence $\{\widehat{\phi}^{(d)}(-\frac{i\lambda}{2\pi} + \ell) : \ell \in \mathbb{Z}\}$ decays exponentially. Thus, we have shown that (8) is satisfied for $k = 0, \dots, d$. □

Corollary 3.2 *The set of analytic functions spanned by the shifts of a compactly supported function ϕ is a linear span of spaces H_λ in (9) over all $\lambda \in \mathbb{C}$ such that the sequences in (8) decay exponentially.*

4 Singly Generated Shift-Invariant Spaces with Generalized Refinability

Additional assumption on the generalized refinability of ϕ , i.e. the property

$$\widehat{\phi}(y) = \prod_{j=1}^{\infty} a_j(2^{-j}y), \tag{14}$$

for some trigonometric polynomials

$$a_j(y) = \sum_{m \in \mathbb{Z}} a_{j,m} e^{-2\pi i m y}, \quad a_{j,m} \in \mathbb{R} \quad y \in \mathbb{R},$$

replaces the requirement in Theorem 3.1 on the exponential decay of sequences in (8) by a requirement that only finitely many of the sequence elements are non-zero (i.e. the corresponding 1-periodic analytic functions in (11) are trigonometric polynomials).

The main result of this section finalizes our knowledge about the structure of H .

Theorem 4.1 *Let S_ϕ be defined by $\widehat{\phi} = \prod_{j=1}^{\infty} a_j(2^{-j}\cdot)$ with the trigonometric polynomials a_j satisfying*

$$\deg(a_j) \leq N, \quad a_j(0) = 1 \quad \text{and} \quad \|a_j\|_\infty \leq C < \infty, \quad j \in \mathbb{N}.$$

If, for some $\lambda \in \mathbb{C}$ and $d \in \mathbb{N}_0, d \leq N$, the analytic function $e^{\lambda t} \sum_{k=0}^d p_k \omega_k, \omega_d \neq 0$ belongs to S_ϕ , then the sequences

$$\left\{ \widehat{\phi}^{(k)} \left(-\frac{i\lambda}{2\pi} + \ell \right) : \ell \in \mathbb{Z} \right\}, \quad k = 0, \dots, d \tag{15}$$

contain (all together) at most N non-zero elements.

The inductive proof of Theorem 4.1 follows from Propositions 4.3 and 4.4 and Lemma 4.2. Proposition 4.3 provides the base of the inductive proof of Theorem 4.1 in the case $d = 0$. The inductive step in Proposition 4.4 is proven similarly to Proposition 4.3, yet there are crucial differences that we point out. Both Propositions 4.3 and 4.4 rely on the result of Lemma 4.2.

In the proof of Proposition 4.3 we use the idea of the method of counting of zeros elaborated in [45]. The essence of the method is the following: if the infinite product of trigonometric polynomials has too many zeros on a segment $[0, r]$, then one of the polynomials must have more than N zeros which leads to the contradiction. However, for proving Proposition 4.3, this idea should be significantly modified since here we have to count not zeros but in a sense “almost zeros” of polynomials. That is why we begin with Lemma 4.2, which states that the maximum norm of a trigonometric

polynomial of degree N is small, if its point evaluations at arbitrary (well-separated) $N + 1$ pairwise distinct points in $[0, 1)$ are small. This result generalizes the well known fact that an algebraic polynomial of degree N is identically zero, if it vanishes at $N + 1$ points.

Lemma 4.2 *Let $a(y) = \sum_{m=0}^N a_m e^{-i 2\pi m y}$, $y \in \mathbb{R}$, $N \in \mathbb{N}$ and $y_m \in [0, 1)$, $m = 0, \dots, N$ be pairwise distinct. Then*

$$\|a\|_\infty \left(\min_{\substack{m,k=0,\dots,N \\ m \neq k}} |y_m - y_k| \right)^N \leq 2^{-N} (N + 1) \max_{m=0,\dots,N} |a(y_m)| \quad (16)$$

Proof By the Lagrange interpolation formula, the trigonometric polynomial a satisfies

$$a(y) = \sum_{m=0}^N a(y_m) \prod_{\substack{k=0 \\ k \neq m}}^N \frac{z - z_k}{z_m - z_k}, \quad z = e^{-i2\pi y}, \quad z_m = e^{-i2\pi y_m}, \quad m = 0, \dots, N.$$

Thus, due to $\prod_{\substack{k=0 \\ k \neq m}}^N |z_m - z_k| \geq \left(\min_{\substack{m,k=0,\dots,N \\ m \neq k}} |z_m - z_k| \right)^N$, on the unit circle we have

$$\begin{aligned} |a(y)| \left(\min_{\substack{m,k=0,\dots,N \\ m \neq k}} |z_m - z_k| \right)^N &\leq \max_{m=0,\dots,N} |a(y_m)| \sum_{m=0}^N \prod_{\substack{k=0 \\ k \neq m}}^N |z - z_k| \\ &\leq 2^N (N + 1) \max_{m=0,\dots,N} |a(y_m)|, \end{aligned}$$

where the chord length $|z - z_k| \leq 2$ for $|z| = 1$. On the other hand, the length of an arbitrary chord of a unit circle is at least the length of the shortest arc defined by this chord multiplied by $\frac{2}{\pi}$ (this estimate is achieved for diameters). Therefore, $|z_m - z_k| \geq \frac{2}{\pi} \cdot 2\pi \cdot |y_m - y_k| = 4 |y_m - y_k|$. Thus,

$$|a(y)| \left(\min_{\substack{m,k=0,\dots,N \\ m \neq k}} |z_m - z_k| \right)^N \geq |a(y)| 4^N \left(\min_{\substack{m,k=0,\dots,N \\ m \neq k}} |y_m - y_k| \right)^N.$$

Consequently, for every $y \in [0, 1)$, we have

$$|a(y)| \left(\min_{\substack{m,k=0,\dots,N \\ m \neq k}} |y_m - y_k| \right)^N \leq 2^{-N} (N + 1) \max_{m=0,\dots,N} |a(y_m)|.$$

Taking maximum over $y \in [0, 1)$, we arrive at the desired estimate (16). □

Now we are ready to prove Theorem 4.1.

Proposition 4.3 *The statement of Theorem 4.1 holds for $d = 0$.*

Proof Let $d = 0$ and assume that there are at least $N + 1$ non-zero elements in the corresponding sequence in (8) with $\alpha = -\frac{i\lambda}{2\pi}$, $\lambda \in \mathbb{C}$.

1.Step: W.l.g. $\widehat{\phi}(\alpha) \neq 0$. Then by (14), for every $\varepsilon > 0$ there exists $r_\alpha \in \mathbb{N}$ such that for all $r \geq r_\alpha$

$$\left| \prod_{j=1}^r a_j(2^{-j}\alpha) - \widehat{\phi}(\alpha) \right| < \varepsilon, \tag{17}$$

thus, this product is bounded away from zero uniformly for all $r \geq r_\alpha$. The fact that $a_j(0) = 1$, $j \in \mathbb{N}$, implies that there exists $R \in \mathbb{N}$ such that for all $r \geq r_\alpha$

$$\left| \prod_{j=r+R+1}^\infty a_j(2^{-j}\alpha + 2^{-j+r}) - 1 \right| < \varepsilon, \tag{18}$$

i.e. the above product is also uniformly bounded away from zero. Next, we split the infinite product appearing in the definition of $\widehat{\phi}(\alpha + \ell)$ into three products accordingly to the properties in (17)–(18). For $\ell = 2^r$, $r \geq r_\alpha$, due to the 1-periodicity of the trigonometric polynomials a_j , we have

$$\begin{aligned} \widehat{\phi}(\alpha + 2^r) &= \prod_{j=1}^r a_j(2^{-j}\alpha) \prod_{j=r+1}^{r+R} a_j(2^{-j}\alpha + 2^{-j+r}) \\ &\quad \prod_{j=r+R+1}^\infty a_j(2^{-j}\alpha + 2^{-j+r}). \end{aligned}$$

Due to (17)–(18), the exponential decay of the sequence in (8) implies that

$$\left| \prod_{j=r+1}^{r+R} a_j(2^{-j}\alpha + 2^{-j+r}) \right| \leq Cq^{2^r}, \quad q \in (0, 1). \tag{19}$$

Hence, at least one of the factors in (19) (*has an almost zero*) is in the absolute value smaller than or equal to $Cq^{2^r/R}$. Repeating the argument with $\ell = 2^{r+n}$, $n \in \mathbb{N}$, we conclude that J trigonometric polynomials $a_{r+1}, \dots, a_{r+1+J}$, $r \geq r_\alpha$, $J \gg R$, have at least $J - R$ almost zeros. The possible almost zeros for each $a_k, k \in \{r + 1, \dots, r + 1 + J\}$ are at the distinct complex points in (19)

$$2^{-k}\alpha + 2^{-1}, 2^{-k}\alpha + 2^{-2}, \dots, 2^{-k}\alpha + 2^{-R}. \tag{20}$$

2.Step: By assumption, there exist other N distinct $\alpha_\ell = \alpha + \ell, \ell \in L \subset \mathbb{N}$, such that $\hat{\phi}(\alpha_\ell) \neq 0$. We repeat the argument in 1.Step with $\alpha = \alpha_\ell$ for these N distinct α_ℓ and conclude that J trigonometric polynomials $a_{r+1}, \dots, a_{r+1+J}, r \geq \max\{r_\alpha, r_{\alpha_\ell}\}, J \gg R$, have (together with the almost zeros from 1.Step) at least $(N + 1)(J - R)$ almost zeros. The possible almost zeros for each $a_k, k \in \{r + 1, \dots, r + 1 + J\}$ are at the distinct complex points

$$2^{-k}\alpha_\ell + 2^{-1}, 2^{-k}\alpha_\ell + 2^{-2}, \dots, 2^{-k}\alpha_\ell + 2^{-R}. \tag{21}$$

Thus, due to $J \gg R$, on average there are at least

$$(N + 1) \left(1 - \frac{R}{J}\right) > N$$

almost zeros for each $a_{r+1}, \dots, a_{r+1+J}$ and, by the pigeonhole principle, there exists $a_k, k \in \{r + 1, \dots, r + 1 + J\}$ with $N + 1$ almost zeros of the form in (20)-(21).

3.Step: We use Lemma 4.2 to get a contradiction to the fact that $a_k(0) = 1$. First note that all the points (we set $\alpha_0 = \alpha$)

$$w_\ell = 2^{-k}\alpha_\ell + 2^{-s}, \quad s \in \{1, \dots, R\}, \quad \ell \in L \cup \{0\}, \quad |L| = N + 1.$$

have, due to $\alpha = -\frac{i\lambda}{2\pi}, \lambda \in \mathbb{C}$, the same imaginary part

$$\text{Im}(w_\ell) = -2^{-k}(2\pi)^{-1} \text{Re}(\lambda), \quad \ell \in L \cup \{0\}.$$

Moreover, these points w_ℓ are separated by the distance of at least 2^{-k} for $k > R$. Indeed, let $n, s \in \{1, \dots, R\}, n \neq s$, and $\ell, \tilde{\ell} \in L \cup \{0\}, \ell \neq \tilde{\ell}$. Then, for $k > R$, due to $|\alpha_\ell - \alpha_{\tilde{\ell}}| = |\ell - \tilde{\ell}|$ being an integer bigger than or equal to 1, we have

$$\begin{aligned} |2^{-k}(\alpha_\ell - \alpha_{\tilde{\ell}}) + 2^s - 2^n| &\geq |2^s - 2^n| - 2^{-k}|\alpha_\ell - \alpha_{\tilde{\ell}}| \\ &\geq 2^{-R} - 2^{-k} \geq 2^{-k+1} - 2^{-k} = 2^{-k}. \end{aligned}$$

Secondly, for

$$a_k(y) = \sum_{m=0}^N a_{k,m} e^{-i2\pi m y}, \quad y \in \mathbb{R},$$

define the polynomial

$$\tilde{a}_k(y) = \sum_{m=0}^N \tilde{a}_{k,m} e^{-i2\pi m y}, \quad \tilde{a}_{k,m} = a_{k,m} e^{-2^{-k} \text{Re}(\lambda) m}, \quad m = 0, \dots, N.$$

Note that $\lim_{k \rightarrow \infty} \|a_k - \tilde{a}_k\|_\infty = 0$ and that the minimal distance between the real points (which are real parts of w_ℓ 's)

$$y_\ell = 2^{-k} \left(\frac{\text{Im}(\lambda)}{2\pi} + \ell \right) + 2^{-s}, \quad \ell \in L \cup \{0\},$$

is given by 2^{-k} , due to all w_ℓ 's having the same imaginary part. Also note that the almost zeros w_ℓ of a_k are closely related to the almost zeros of \tilde{a}_k by $a_k(w_\ell) = \tilde{a}_k(y_\ell)$, $\ell \in L \cup \{0\}$. Therefore, by Lemma 4.2, we get

$$\|\tilde{a}_k\|_\infty 2^{-k} \leq 2^{-N} (N + 1) C q^{2^{-k}/R}.$$

On the other hand, $a_k(0) = 1$ and $\lim_{k \rightarrow \infty} \|a_k - \tilde{a}_k\|_\infty = 0$ lead to a contradiction. \square

Next we provide the inductive step that completes the proof of Theorem 4.1.

Proposition 4.4 *The statement of Theorem 4.1 holds for $d \in \mathbb{N}$.*

Proof The base of the induction follows from Proposition 4.3. We assume that, for $k = 0, \dots, d - 1$, the sequences in (8) with $\alpha = -\frac{i\lambda}{2\pi}$, $\lambda \in \mathbb{C}$, have in total finitely many non-zero elements. This implies the existence of r_0 such that for all $r > r_0$ we have

$$\hat{\phi}^k(\alpha + 2^r) = 0, \quad k = 0, \dots, d - 1.$$

The inductive step we prove by contradiction assuming that there are at least $N + 1$ non zero elements in the sequences in (8) for $k = 0, \dots, d$.

1.Step By the argument in Proposition 4.3 *1.Step*, $\hat{\phi}(\alpha) \neq 0$ and, for arbitrary $r \geq r_\alpha$, the trigonometric polynomials $a_{r+1}, \dots, a_{r+1+J}$, $r \geq r_\alpha$, $J \gg R$, have at least $J - R$ almost zeros of the form in (20).

2.Step For the same α , the additional information about the exponential decay of the other sequences in (8) for $k = 1, \dots, d$ supplies another $d(J - R)$ almost zeros. Indeed, due to $\hat{\phi}(\alpha) \neq 0$, there exist $\rho \in (0, 1)$ and a constant $C_0 > 0$ such that

$$|\hat{\phi}(\alpha + t)| \geq C_0 \geq C_0 t^d > 0, \quad t \in (0, \rho). \tag{22}$$

Furthermore, for $\varepsilon > 0$ there exists $r_{\alpha,t} \in \mathbb{N}$ such that for all $r \geq r_{\alpha,t}$

$$\left| \prod_{j=1}^r a_j(2^{-j}\alpha + 2^{-j}t) - \hat{\phi}(\alpha + t) \right| < \varepsilon, \tag{23}$$

thus, this product decays slower than t^d uniformly for all $r \geq r_{\alpha,t}$. Making use of the Taylor expansion of $\hat{\phi}$ at $\alpha + 2^r + t$, $r \geq \max\{r_0, r_{\alpha,t}\}$, we obtain

$$|\hat{\phi}(\alpha + 2^r + t)| \leq C_2 q^{2^r} t^d, \quad q \in (0, 1) \tag{24}$$

with the constant $C_2 > 0$ depending on the constant $C > 0$ that governs the exponential decay in (8) and on the error term in the Taylor expansion. For $\ell = 2^r$, $r \geq \max\{r_0, r_{\alpha,t}\}$, due to the 1-periodicity of the trigonometric polynomials a_j , we have

$$\widehat{\phi}(\alpha + 2^r + t) = \prod_{j=1}^r a_j(2^{-j}\alpha + 2^{-j}t) \prod_{j=r+1}^{r+R} a_j(2^{-j}\alpha + 2^{-j+r} + 2^{-j}t) \prod_{j=r+R+1}^{\infty} a_j(2^{-j}\alpha + 2^{-j+r} + 2^{-j}t).$$

Due to (22), (23) and similar argument to (18), the decay in (24) implies that

$$\left| \prod_{j=r+1}^{r+R} a_j(2^{-j}\alpha + 2^{-j+r} + 2^{-j}t) \right| \leq C_2 q^{2^r}, \quad q \in (0, 1). \tag{25}$$

Hence, at least one of the factors in (19) (*has an almost zero*) is in the absolute value smaller than $C_2 q^{2^r/R}$. Repeating the argument with $\ell = 2^{r+n}$, $n \in \mathbb{N}$, we conclude that J trigonometric polynomials $a_{r+1}, \dots, a_{r+1+J}$, $r \geq \max\{r_0, r_{\alpha}, r_{\alpha,t}\}$, $J \gg R$, have at least $2(J - R)$ almost zeros. The possible almost zeros for each a_k , $k \in \{r + 1, \dots, r + 1 + J\}$ are at the distinct complex points in (25)

$$2^{-k}\alpha + 2^{-1} + 2^{-k}t, \quad 2^{-k}\alpha + 2^{-2} + 2^{-k}t, \quad \dots \quad 2^{-k}\alpha + 2^{-R} + 2^{-k}t. \tag{26}$$

3.Step We choose the natural numbers $s_1 < s_2 < \dots < s_d$ and real numbers $t_1 = 2^{-s_1}$, $t_2 = 2^{-s_2}$, \dots , $t_d = 2^{-s_d}$ such that t_j satisfy (22) and generate (together with almost zeros in *1.Step*) in total $(d + 1)(J - R)$ distinct almost zeros in (26) for the J trigonometric polynomials $a_{r+1}, \dots, a_{r+1+J}$, $r \geq \max\{r_0, r_{\alpha}, r_{\alpha,t}\}$.

4.Step For the other non-zero values in the sequences in (8) at points $\alpha_{\ell} = \alpha + \ell$, $\ell \in L \subset \mathbb{N}$, $|L| \leq N$, we repeat the argument in *1.Step-2.Step* with the corresponding t_j in *3.Step*. Hence, we conclude that J trigonometric polynomials $a_{r+1}, \dots, a_{r+1+J}$, $r \geq \max_{\ell \in L \cup \{0\}} \{r_0, r_{\alpha_{\ell}}, r_{\alpha_{\ell},t}\}$, $J \gg R$, have at least $(N + 1)(J - R)$ almost zeros of the form in (20) and in (26). Repeating the argument in *2.Step* of Proposition 4.3, by the pigeonhole principle, there exists a_k , $k \in \{r + 1, \dots, r + 1 + J\}$ with $N + 1$ distinct almost zeros of the form in (20) and in (26).

The claim follows by the argument similar to the one in *3.Step* of Proposition 4.3 with the minimal distance of 2^{-k-s_d} between the almost zeros in (20) and in (26). \square

4.1 Theorem 4.1 in Terms of the Masks

Theorem 4.1 provides a necessary condition for a level dependent subdivision scheme

to generate a function $e^{\lambda t} \sum_{k=0}^d p_k \omega_k$, where $\lambda \in \mathbb{C}$, p_k are certain algebraic poly-

nomials of degree k , and ω_k are certain periodic analytic functions. This necessary condition, in view of Theorem 3.1, becomes a criterion for construction of subdivision schemes with prescribed generation properties. This relates our results to well known sum rules or zero conditions or their generalized versions. Indeed, if the sequences (15) contain in total N non-zero elements, then they decay exponentially, and hence the space S_ϕ contains a subspace of analytic functions of the form mentioned above, see Theorem 3.1 part (ii).

The result of Theorem 4.1 is formulated in terms of the refinable function ϕ : the derivatives up to order d of its Fourier transform $\widehat{\phi}$ must have at most N nonzero values in points of the arithmetic progression $\{-\frac{i\lambda}{2\pi} + \ell\}_{\ell \in \mathbb{Z}}$. Of course, it would be natural to formulate an equivalent criterion in terms of the masks a_j . Indeed it is possible, see Theorem 4.5, and is done in terms of zeros of the masks a_j . The statement of Theorem 4.5 is not simple and rather technical. Its result however leads to generalized sum rules or zero conditions, see Examples 4.7–4.9.

Denote $\alpha = -\frac{i\lambda}{2\pi}$, $\lambda \in \mathbb{C}$. We consider the following binary tree \mathcal{T}_α . In the root we place the number $s_{0,0} = \alpha$ and in what follows we identify the vertices of this tree with the corresponding numbers $s_{j,k}$. The children of $s_{0,0}$ are two vertices of the first level $s_{1,0} = \frac{\alpha}{2}$ and $s_{1,1} = \frac{\alpha}{2} + \frac{1}{2}$. In general, for every non-negative integer j , the j th level contains 2^j vertices $s_{j,k} = 2^{-j}(\alpha + k)$, $k = 0, \dots, 2^j - 1$. Each vertex $s_{j,k}$ has two children in the next level $s_{j+1,k} = \frac{1}{2}s_{j,k}$ and $s_{j+1,k+2^j} = \frac{1}{2}s_{j,k} + \frac{1}{2}$.

For each $j \in \mathbb{N}$, we denote by S_j the set of roots of the mask a_j on the j th level of the tree, i.e., the set of the numbers $\{s_{j,k}\}_{k=0}^{2^j-1}$, all roots are counted with multiplicities. Denote $S = \cup_{j \in \mathbb{N}} S_j$.

Now we are ready to formulate the characterization of analytic subspaces of S_ϕ .

Theorem 4.5 *Under assumption of Theorem 4.1, if an analytic function $e^{\lambda t} \sum_{k=0}^d p_k \omega_k$, belongs to S_ϕ , then all but at most $N \leq d$ infinite paths without backtracking along the tree \mathcal{T}_α , $\alpha = -\frac{i\lambda}{2\pi}$, have at least d vertices from the set S . Conversely, if this condition on \mathcal{T}_α is satisfied, then S_ϕ contains the $(d + 1)$ -dimensional subspace of analytic functions*

$$H_\lambda = \left\{ \sum_{\ell \in \mathbb{Z}} e^{\lambda \ell} p(\ell) \phi(\cdot - \ell) : p \in \mathcal{P}_d \right\}$$

spanned by functions from the family (10) where the 1-periodic functions ω_k in (11) are trigonometric polynomials with in total at most N nonzero coefficients.

Remark 4.6 Thus, if a level dependent subdivision scheme generates a function

$$e^{\lambda t} \sum_{k=0}^d p_k \omega_k, \quad \lambda = 2\pi i \alpha,$$

then each path from the roof of \mathcal{T}_α , except for at most N of them, contains at least d vertices from S . For $d = 0$, this means that if a non-stationary subdivision generates a

function $e^{\lambda t} \omega_0$, where ω_0 is 1-periodic analytic, then there are at most N paths whose vertices avoid the set S .

Proof of Theorem 4.5 To an arbitrary integer ℓ we associate the following path along the tree \mathcal{T}_α . Denote by k_j the remainder of the division of ℓ by 2^j , $j \in \mathbb{N}$. Thus, $0 \leq k_j < 2^j$ and $\ell - k_j$ is divisible by 2^j . Then the path $s_{0,0} \rightarrow s_{1,k_1} \rightarrow \dots \rightarrow s_{j,k_j} \rightarrow \dots$ is associated to the number ℓ . Conversely, for each finite path from the root there exist an integer ℓ whose associated path begins with this path. Since each function a_j is 1-periodic, it follows that for every ℓ and for its associated path, we have $a_j(2^{-j}(\alpha + \ell)) = a_j(2^{-j}(\alpha + k_j)) = a_j(s_{j,k_j})$. Therefore,

$$\widehat{\phi}(h + \ell) = \prod_{j=1}^{\infty} a_j(2^{-j}(\alpha + \ell)) = \prod_{j=1}^{\infty} a_j(s_{j,k_j}).$$

Thus, if the path $s_{0,0} \rightarrow s_{1,k_1} \rightarrow \dots \rightarrow s_{j,k_j} \rightarrow \dots$ avoids the set S , then $\widehat{\phi}(\alpha + \ell) \neq 0$. If this path intersects S , then the order of the zero of the function $\widehat{\phi}$ at the point $\alpha + \ell$ is equal to the number of vertices (counting multiplicities) of this path that belong to the set S . Thus, Theorem 4.1 implies that all but at most N of those paths have less than d vertices from S .

Conversely, if all but at most N infinite paths in \mathcal{T}_α have at least d vertices from the set S , then all the sequences (15) contain in total at most N non-zero elements. Therefore, they all decay exponentially. Theorem 3.1 part (ii) concludes the proof. \square

In spite of its rather involved formulation Theorem 4.5 is simple to apply, see the following Examples.

Example 4.7 The simplest situation is when each mask a_j has a root at the point $s_{j,2^{j-1}} = 2^{-j}\alpha + \frac{1}{2}$. Thus, S contains the set $\{s_{j,2^{j-1}}\}_{j \in \mathbb{N}}$. In this case there exists a unique path $s_{0,0} \rightarrow s_{1,0} \rightarrow \dots \rightarrow s_{j,0} \rightarrow \dots$ avoiding S . All other paths intersect S , see Fig. 1. It is easily seen that in this case $\widehat{\phi}(\alpha + \ell) = 0$ whenever $\ell \neq 0$. If, in addition, $a_j(2^{-j}\alpha) \neq 0$ for all $j \in \mathbb{N}$, then $\widehat{\phi}(\alpha) \neq 0$ and, by Theorem 3.1, $e^{\lambda t} \in S_\phi$ with $\lambda = 2\pi i\alpha$.

If each a_j has a root of multiplicity d at the point $s_{j,2^{j-1}} = 2^{-j}\alpha + \frac{1}{2}$ and $a_j(2^{-j}\alpha) \neq 0$, then $e^{\lambda t} p(t) \in S_\phi$ for every polynomial p of degree at most d .

Example 4.8 We assume that the mask a_1 has a zero at $s_{1,1} = \frac{\alpha}{2} + \frac{1}{2}$, a_2 has a zero at $s_{2,0} = \frac{\alpha}{4}$ and each a_j , $j \geq 3$, has a zero at $s_{j,2^{j-1}+2} = 2^{-j}\alpha + \frac{1}{2} + 2^{1-j}$. In this case, there exist a unique path $s_{0,0} \rightarrow s_{1,0} \rightarrow s_{2,2} \rightarrow s_{3,2} \rightarrow \dots \rightarrow s_{j,2} \rightarrow \dots$ avoiding S , see Fig. 2. We have $\widehat{\phi}(\alpha + \ell) = 0$, $\ell \neq 2$ and $\widehat{\phi}(\alpha + 2) = C \neq 0$. Hence, $C e^{\lambda t} \in S_\phi$ and therefore $e^{\lambda t} \in S_\phi$. If we add an extra zero $s_{j,2^{j-1}} = 2^{-j}\alpha + \frac{1}{2}$ to each mask a_j , $j \in \mathbb{N}$, then $e^{\lambda t} (at + b)e^{\lambda t} \in S_\phi$ for $a, b \in \mathbb{C}$.

In Examples 4.7 and 4.8, the sequence $\widehat{\phi}(\alpha + \ell)$, $\ell \in \mathbb{Z}$, contains only one non-zero element. Example 4.9 illustrates the case when the sequence $\widehat{\phi}(\alpha + \ell)$, $\ell \in \mathbb{Z}$ contains two non-zero elements. Similarly, one can construct sequences with arbitrary, but finite number of non-zero elements.

Fig. 1 Path from Example 4.7 avoiding S comprised of \circ

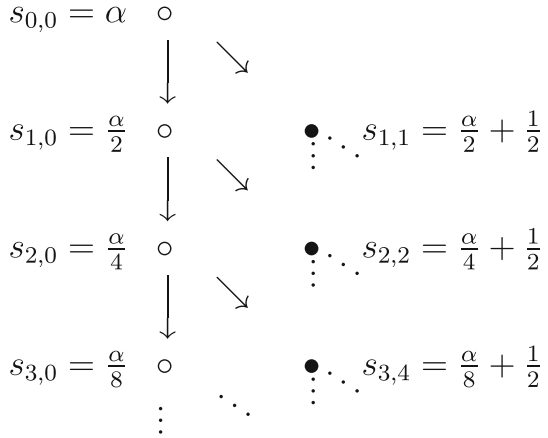
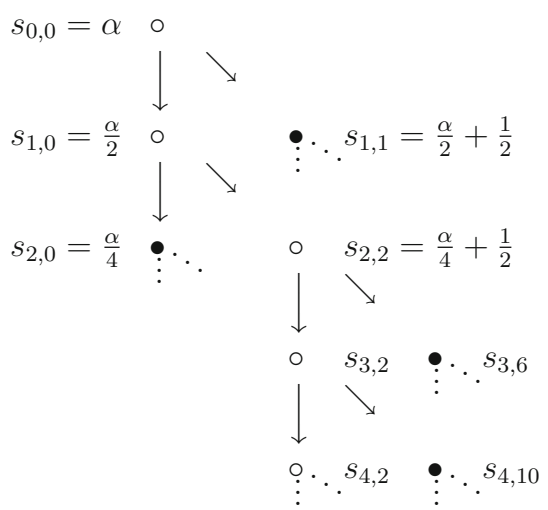


Fig. 2 Path from Example 4.8 avoiding S comprised of \circ



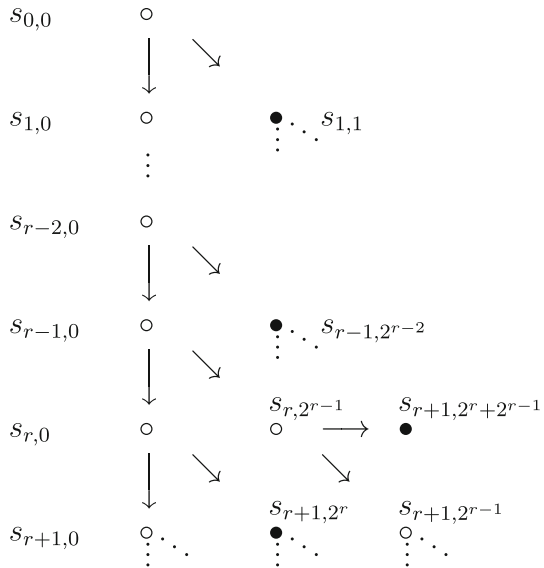
Example 4.9 Let $r \geq 2$ and assume that each mask $a_j, j \neq r$, has a zero at $s_{j,2^{j-1}} = 2^{-j}\alpha + \frac{1}{2}$. Furthermore, we assume that each mask $a_j, j \geq r + 1$, has a zero at $s_{j,2^{j-1}+2^{r-1}} = 2^{-j}\alpha + \frac{1}{2} + 2^{r-j-1}$. Thus, each of the masks a_1, \dots, a_{r-1} has only one zero, the mask a_r has no zeros and each of the masks $a_j, j \geq r + 1$ has two zeros. Hence, there are two paths

$$s_{0,0} \rightarrow s_{1,0} \rightarrow \dots \rightarrow s_{j,0} \rightarrow \dots$$

and

$$s_{0,0} \rightarrow s_{1,0} \rightarrow \dots \rightarrow s_{r-1,0} \rightarrow s_{r,2^{r-1}} \dots s_{r+1,2^{r-1}} \rightarrow s_{r+2,2^{r-1}} \rightarrow \dots$$

Fig. 3 Two paths from Example 4.9 avoiding S comprised of \circ



avoiding the set S , see Fig. 3. We have $\widehat{\phi}(\alpha + \ell) = 0, \ell \neq 0, \ell \neq 2^{r-1}$. Denote $A = \widehat{\phi}(\alpha)$ and $B = \widehat{\phi}(\alpha + 2^{r-1})$. Then $e^{\lambda t} (A + B e^{2^r \pi i t})$ belongs to S_ϕ

A consequence of Theorem 4.1 states that the analytic subspaces of the shift-invariant space S_ϕ , in the case all the trigonometric polynomials $a_j = a, j \in \mathbb{N}$, are the same, consist only of polynomials.

Corollary 4.10 *Let S_ϕ be defined by $\widehat{\phi} = \prod_{j=1}^\infty a(2^{-j}\cdot)$ with the trigonometric polynomial a satisfying $\deg(a) \leq N, a(0) = 1$ and $\|a\|_\infty \leq C < \infty$. If, for some $\lambda \in \mathbb{C}$ and $d \in \mathbb{N}_0, d \leq N$, the analytic function $e^{\lambda t} \sum_{k=0}^d p_k \omega_k, \omega_d \neq 0$, belongs to S_ϕ , then $\lambda = 0$ and $\omega_k(t) \equiv \text{constant}, k = 0, \dots, d$.*

Proof Let $\alpha = -\frac{\lambda i}{2\pi}$. By Theorem 4.1, there are only finitely many $\ell \in \mathbb{Z}$ such that $\widehat{\phi}(\alpha + \ell) \neq 0$.

Case $\lambda \neq 0$ is impossible. There exists at least one $\ell \in \mathbb{Z}$ (w.l.g $\ell = 0$) such that $\widehat{\phi}(\alpha + \ell) \neq 0$. Otherwise, if $\widehat{\phi}(\alpha + \ell) = 0, \ell \in \mathbb{Z}$, then (11) implies that $\omega_0(t) \equiv 0$ and, by (10), the integer shifts of ϕ are linearly dependent. Choose $R \in \mathbb{N}$ such that for all $r \geq R$ we have $\widehat{\phi}(\alpha + 2^r) = 0$. Ensuring these conditions we arrive at the contradiction to the fact that the trigonometric polynomial a has degree N . Indeed, by the 1-periodicity of a , we have

$$\widehat{\phi}(\alpha + 2^r) = \underbrace{\prod_{j=1}^r a(2^{-j}\alpha)}_{\neq 0} \prod_{j=r+1}^\infty a(2^{-j}\alpha + 2^{-j+r}) = 0, \tag{27}$$

which implies that at least one of the factors $a(2^{-j}\alpha + 2^{-j+r}) = 0$ for some $j \in \mathbb{N}$, $j \geq r + 1$. None of such factors, however, occur again for different r . Thus, to ensure that $\widehat{\phi}(\alpha + 2^r) = 0$, $r \in \mathbb{N}$, we are forced to choose a with infinitely many different zeros.

Case $\lambda = 0$: The claim that $\omega_k(t) \equiv \text{constant}$ for $k = 0, \dots, d$ is equivalent to $H_\lambda = \text{span}\{1, \dots, t^d\}$. By the assumption and by Theorem 3.1 (ii), the subspace H_λ is $d + 1$ dimensional. Induction on d . For $d = 0$, $\phi \in L_p(\mathbb{R})$ and the linear independence of its integer shifts imply that the zero condition $a(2^{-1}) = 0$ of order one is satisfied. This, together with the standard normalization condition $a(0) = 1$, imply that $\sum_{\ell \in \mathbb{Z}} \phi(t - \ell) = 1$. Next, we assume that $H_\lambda = \text{span}\{1, \dots, t^{d-1}\}$. Therefore, by [8],

$$D^k a(2^{-1}) = 0, \quad k = 0, \dots, d - 1,$$

and, by the 1-periodicity of a , we have, for $\ell = 2\ell' + 1$, $\ell' \in \mathbb{Z} \setminus \{0\}$,

$$D^k a(2^{-1}(2\ell' + 1)) = D^k a(2^{-1}) = 0, \quad k = 0, \dots, d - 1.$$

Hence, the expression for the d -th derivative of $\widehat{\phi}$ at such ℓ reduces to

$$\begin{aligned} \widehat{\phi}^{(d)}(\ell) &= 2^{-d} D^d a(2^{-1}) \prod_{j=2}^{\infty} a(2^{-j+1}\ell' + 2^{-j}) \\ &= 2^{-d} D^d a(2^{-1}) \prod_{j=1}^{\infty} a(2^{-j}\ell' + 2^{-j} \cdot 2^{-1}) \\ &= 2^{-d} D^d a(2^{-1}) \widehat{\phi}(\ell' + 2^{-1}). \end{aligned} \tag{28}$$

By Theorem 4.1, there exists infinitely many ℓ in (28) such that $\widehat{\phi}^{(d)}(\ell) = 0$. Assuming that $\widehat{\phi}(\ell' + 2^{-1}) = 0$ for all $\ell' \in \mathbb{Z}$ would contradict the linear independence of the integer shifts of ϕ , see [44, Chapter 3.4]. Indeed, it would require that a has zeros at complex roots of -1 , which implies polynomial structure of H_λ , but contradicts the linear independence. Therefore, $D^d a(2^{-1}) = 0$, which together with the inductive assumption implies that $H_\lambda = \text{span}\{1, \dots, t^d\}$. \square

Remark 4.11 Note that the case $\lambda \neq 0$ is possible in the setting of generalized refinability as the infinitely many zeros in (27) can be redistributed among the trigonometric polynomials $a_j(y)$, $j \in \mathbb{N}$.

5 Generation Properties of Level Dependent Subdivision

In this section, we discuss the analytic limits of level dependent subdivision schemes. In Subsect. 5.1, we link the results from Sect. 4 with generation properties of such subdivision schemes which are iterative algorithms

$$c_{j+1} = S_{a_j}c_j, \quad j \in \mathbb{N}, \tag{29}$$

mapping sequences $c_j = \{c_{j,k} : k \in \mathbb{Z}\}$ from $\ell(\mathbb{Z})$ into $\ell(\mathbb{Z})$. The linear subdivision operators S_{a_j} depend on the finite sequences (*masks*) $a_j = \{a_{j,k} : k \in \mathbb{Z}\}$ of real numbers and are defined by

$$(S_{a_j}c_j)_k = \sum_{m \in \mathbb{Z}} a_{j,k-2m}c_{j,m}, \quad k \in \mathbb{Z}, \quad j \in \mathbb{N}.$$

If the level dependent scheme associated with the sequence $\{a_j : j \in \mathbb{N}\}$ of masks converges, then all its limits belong to S_ϕ with

$$\phi = \phi_1 = \lim_{j \rightarrow \infty} S_{a_j} \dots S_{a_1} \delta, \quad \delta_k = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise.} \end{cases}$$

We would like to point out that there is a close connection between level dependent subdivision and cascade algorithms see e.g. [31,32] for details.

5.1 Analytic Limits

We say that a level dependent scheme associated with the mask sequence $\{a_j : j \in \mathbb{N}\}$ generates U in Definition 2.1, if the subdivision limit $\lim_{j \rightarrow \infty} S_{a_j} \dots S_{a_1} c_1$ belongs to U for every starting sequence c_1 sampled from a function in U .

The *generation* properties of subdivision schemes are well understood and are characterized in terms of so-called *zero conditions* or *generalized zero conditions*, see e.g. [8,25,34], on the mask symbols

$$a^{[j]}(z) = \sum_{k \in \mathbb{Z}} a_{j,k} z^k, \quad j \in \mathbb{N} \quad z \in \mathbb{C} \setminus \{0\}.$$

The zero conditions determine uniquely if the subdivision limit belongs to the exponential function space U in Definition 2.1 or not.

Note that the requirement in (15) boils down to the generalized zero conditions (or, equivalently, to the generalized Strang-Fix conditions [37,57]) on the trigonometric polynomials a_j (or subdivision symbols $a^{[j]}$). We first illustrate this fact on the following example.

Example 5.1 It is well known [25] that the generation of two exponential polynomials $e^{\lambda t}$ and $t e^{\lambda t}$, $\lambda \in \mathbb{C}$, by a level dependent subdivision scheme is equivalent to the requirement that the corresponding symbols satisfy the generalized zero conditions (for $\lambda = 0$, zero conditions at -1)

$$D^k a^{[j]}(-e^{-\lambda 2^{-j}}) = 0, \quad j \in \mathbb{N}, \quad k = 0, 1. \tag{30}$$

In this case, generalized refinability (14) together with a standard assumptions (for $\lambda = 0$, conditions at 1)

$$D^k a^{[j]}(e^{-\lambda 2^{-j}}) \neq 0, \quad j \in \mathbb{N}, \quad k = 0, 1, \tag{31}$$

imply that the only non-zero elements of the sequences in (8) for $k = 0, 1$ are

$$\widehat{\phi}\left(-\frac{i\lambda}{2\pi}\right) \neq 0 \quad \text{and} \quad \widehat{\phi}^{(1)}\left(-\frac{i\lambda}{2\pi}\right) \neq 0.$$

In other words, by Theorem 3.1, the 1-periodic analytic functions ω_0 and ω_1 are constant. Indeed, a straightforward computation yields

$$\widehat{\phi}\left(-\frac{i\lambda}{2\pi}\right) = \prod_{j=1}^{\infty} a_j\left(-2^{-j} \frac{i\lambda}{2\pi}\right) = \prod_{j=1}^{\infty} a^{[j]}(e^{-\lambda 2^{-j}}) \neq 0 \tag{32}$$

and

$$\widehat{\phi}^{(1)}\left(-\frac{i\lambda}{2\pi}\right) = \sum_{\ell \in \mathbb{N}} 2^{-\ell} D a_{\ell}^* (e^{-\lambda 2^{-j}}) \prod_{\substack{j=1 \\ j \neq \ell}}^{\infty} a^{[j]}(e^{-\lambda 2^{-j}}) \neq 0. \tag{33}$$

Furthermore, for $\beta \in \mathbb{Z} \setminus \{0\}$, let $j' \in \mathbb{N}$ be the number of 2's in the prime number decomposition of $|\beta|$. Define $j = j' + 1$. Then we have

$$D^k a_j\left(-\frac{i\lambda}{2\pi} + \beta\right) = D^k a^{[j]}(-e^{-\lambda 2^{-j}}), \quad k = 0, 1.$$

Therefore, by (30), we arrive at the generalized Strang-Fix conditions

$$D^{(k)} \widehat{\phi}\left(-\frac{i\lambda}{2\pi} + \beta\right) = 0 \quad \text{for } k = 0, 1 \text{ and } \beta \in \mathbb{Z} \setminus \{0\}. \tag{34}$$

Similarly, if (32)–(34) are satisfied, then, by Theorem 3.1, H_{λ} is spanned by $e^{\lambda t}$ and $t e^{\lambda t}$, which is equivalent to (30).

The main result of Subsect. 4, Theorem 4.1, essentially states that if a function belongs to the subspace $H \subseteq S_{\phi}$ of analytic functions, then it must satisfy the generalized zero conditions. Thus, completing the quest for exhibiting all possible analytic functions generated by level dependent subdivision. We restate Theorem 4.1.

Theorem 5.2 *Every analytic limit of a level dependent subdivision scheme is an exponential polynomial.*

In the special level independent (stationary) case, i.e. $a_j = a$ for all $j \in \mathbb{N}$. The Corollary 4.10 can be restated as follows.

Corollary 5.3 *Every analytic limit of a level independent subdivision scheme is a polynomial.*

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