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DESTABILISING NONNORMAL STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this article we address the stability of linear stochastic differential equations. In particular, we focus our attention on non-normality in stochastic differential equations. Following Higham and Mao we study a test problem for non-normal stochastic differential equations, that is stable without noise, and prove a property conjectured by Higham and Mao, that is that an exponentially small (in the dimension) noise term is able to destabilise in a mean-square sense the solution of the SDE.

1. Introduction.

For a normal matrix A, the spectrum governs in robust way the asymptotic behaviour of a homogeneous linear system of ODEs with constant coefficient of the form

$$\frac{d}{dt}x(t) = Ax(t), \quad x(t_0) = x_0. \tag{1}$$

In general, when A is non-normal, the eigenvalues do not determine the large transient behaviour of the system and the asymptotic stability of the system is susceptible to small perturbations. In [12], Higham and Mao proposed to extend such kind of issue to the SDEs setting, i.e., to examine the possibility of destabilising in the mean-square sense, a stable, but non-normal system like (1) by a small amount of noise. In particular, they constructed a bidimensional example, which we briefly recall in Section 3.1, to illustrate such idea in detail. The main ingredients in the analysis are ε-pseudospectra of nonnormal linear operators, see [5, 6, 7, 9, 20] and stability issues for SDEs, [1, 3, 4, 11, 14, 15, 17, 18]. Furthermore, several interesting applications in fluid dynamics, population biology, and stochastic control, based on non-normal drift-diffusion interactions in stochastic models, see [2] and references therein, are related to the presented topic. Providing an answer to the general

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matter of constructing a mean-square destabilising stochastic term for a stable non-normal is difficult, but paves the way to a class of fascinating related problems. In this work, we restrict our attention to an asymptotic aspect of such general framework. As a canonical way, the authors of [12] introduced non-normality fixing $b > 0$ in the bidiagonal matrix

$$a \begin{pmatrix} -1 & b & 0 & \dots & 0 \\ 0 & -1 & b & 0 & \dots \\ 0 & \dots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & -1 & b \\ 0 & \dots & \dots & 0 & -1 \end{pmatrix} \in \mathbb{R}^{n,n} \tag{2}$$

and consider $n \to \infty$. In this case the question of finding an appropriate form of noise that vanishes as $n \to \infty$ and destabilises the system appears to be open and was conjectured in [12].

We provide a positive answer to this conjecture.

In the sequel, $\mathbb{E}$ represents expectation and $\|\cdot\|_2$ denotes the 2-norm.

If we consider a linear systems of SDEs of the form

$$dx(t) = Ax(t)dt + Gx(t)dw(t), \qquad x(0) = x_0, \tag{3}$$

where $w(t)$ denotes a scalar Brownian motion, $A \in \mathbb{R}^{n,n}$ and $G \in \mathbb{R}^{n,n}$ are real matrices , We denote the solution of (3) as $x(t)$.

We shall make use of the following standard definition.

Definition 1.1. The zero solution of (3) is said to be mean-square stable, if for each $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that $\mathbb{E}\left[\|x(t)\|_2\right] < \varepsilon$ for all $t \geq 0$, whenever $\mathbb{E}\left[\|x_0\|_2\right] < \delta_\varepsilon$. If, in addition,

$$\lim_{t\to\infty} \mathbb{E}\left[\|x(t)\|_2\right] = 0$$

the zero solution of (3) is said to be asymptotically mean-square stable.

The following well-known result plays a crucial role in the analysis of the stability of (3) (see e,g [4]), whose proof relies on the fact that the expectation of the matrix-valued process $x(t)x(t)^\top$, whose vectorization is denoted by $y(t) \in \mathbb{R}^{n^2}$, is given by the unique solution of the differential equation

$$\frac{d}{dt}\mathbb{E}\left[y(t)\right] = S\mathbb{E}\left[y(t)\right]$$

where

$$S = \mathrm{Id} \otimes A + A \otimes \mathrm{Id} + G \otimes G. \tag{4}$$

Theorem 1.2. *The zero solution of (3) is asymptotically mean-square stable if and only if S is a Hurwitz matrix, that is, all eigenvalues of S lie in the left complex half-plane $\mathbb{C}^- = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$.*

We shall analyse mean-square stability/instability of linear systems of non-normal SDEs by means of Theorem 1.2.

The article is organized as follows. In Section 2, we analyse (2) in the simple case $n = 2$, for which we determine by an analytic investigation the *perturbation* of A which provides the highest destabilisation. In Section 3, we revisit a result by Higham and Mao which relates non-normality and de-stabilisation. In Section 4, we set the problem to be investigated for an arbitrary size and give the main theoretical result of this paper. In Section 5, we investigate numerical discretization of the considered SDEs. Even if we are not able to provide rigorous results, we show some numerical results which suggest certain properties of the considered methods.

2. A guiding simple example.

We start recalling two definitions, which will be useful throughout the paper.

Definition 2.1. The spectral abscissa of a matrix $A \in \mathbb{C}^{n\times n}$, is defined as

$$\alpha(A) = \max\{\mathrm{Re}(z) : z \in \Lambda(A)\}$$

where, $\Lambda(A)$ denotes the spectrum of A.

Definition 2.2. The Henrici's distance to normality of a matrix $A \in \mathbb{C}^{n\times n}$ (see [10]) is defined as $\nu(A) = ||A||_F - \sum_{i=1}^n |\lambda_i(A)|^2$ where, $\lambda_i(A)$ is an eigenvalue of A, for $i = 1, ..., n$.

Consider the nonnormal matrix (for $a > 0$ and $b > 0$) given by the 2×2 Jordan block

$$A = a \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}.$$

The spectral abscissa $\alpha(A) = -a$ independently of b, so that the matrix is Hurwitz; the higher the value of b the higher the non-normality of A, since $\nu(A) = b$.

2.1. The smallest destabilising additive perturbation.

We look for Δ of smallest possible norm such that $\alpha(A + \Delta) = 0$. So we look at

$$\Delta^\star \longrightarrow \min_{\Delta:\alpha(A+\Delta)=0} \|\Delta\| \tag{5}$$

where $\|\cdot\|$ is the spectral norm. It is well-known [8, 20] that a matrix $\Delta^\star$ solving (5) has rank-1 and it is real in this case. So, we set

$$\Delta = ayx^\top$$

with $y, x \in \mathbb{R}^2$ defined according to the following normalisation

$$x = \begin{pmatrix} 1 \\ x_2 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ 1 \end{pmatrix}$$

For the Jordan block the ε-pseudospectral sets have circular form so that - for increasing ε - they first touch the imaginary axis at the origin.

Imposing the condition $\det(A + \Delta) = 0$ we obtain

$$\|\Delta\| = a\frac{1}{b + x_2 + y_1}$$

It is well-known (see e.g [8]) that for an extremal perturbation the following conditions hold:

$$\begin{aligned} (A + \Delta)x &= 0 \\ y^\top(A + \Delta) &= 0 \end{aligned}$$

This gives

$$\begin{aligned} y_1 &= \frac{1 - bx_2}{x_2} = \frac{b^2 - cb + 2}{c - b}, \qquad \text{with } c = \sqrt{b^2 + 4} \\ x_2 &= \frac{1}{2}(c - b) \end{aligned}$$

so that

$$\Delta^\star = a \begin{pmatrix} \dfrac{b^2 - bc + 2}{b^2 - bc + 4} & -\dfrac{(b-c)\left(b^2 - bc + 2\right)}{2\left(b^2 - bc + 4\right)} \\ -\dfrac{b-c}{b^2 - cb + 4} & \dfrac{b^2 - bc + 2}{b^2 - bc + 4} \end{pmatrix}$$

2.2. **First order expansion.** For large b we have (with $\beta = \frac{1}{b}$)

$$x = \begin{pmatrix} 1 \\ \beta + \mathcal{O}\left(\beta^{-2}\right) \end{pmatrix}, \qquad y = \begin{pmatrix} \beta + \mathcal{O}\left(\beta^{-2}\right) \\ 1 \end{pmatrix}, \qquad \|\Delta^\star\| = \frac{1}{b} + \mathcal{O}\left(\beta^{-2}\right)$$

and finally

$$\Delta^\star = a \begin{pmatrix} \mathcal{O}\left(\beta^{-2}\right) & \mathcal{O}\left(\beta^{-3}\right) \\ \beta + \mathcal{O}\left(\beta^{-2}\right) & \mathcal{O}\left(\beta^{-2}\right) \end{pmatrix}$$

which is well approximated by

$$\Delta = \Delta_0 = a \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \tag{6}$$

In the next section we shall discuss (for an arbitrary dimension) the effect of Δ_0 as a multiplicative stochastic term.

3. Non-normality and destabilisation.

In this section we provide a class of non-normal test problems and investigate the stability of an associate SDE.

3.1. The 2×2 SDE studied by Higham and Mao.

In [12] Higham and Mao wrote: *An ODE system that is stable yet highly nonnormal may exhibit large transient growth. In the same way that such a system can become unstable when a nonlinear term is added, it should also be possible to de-stabilise by adding a small amount of noise.*

Including a multiplicative noise term they studied an Ito SDE system of the form

$$dx(t) = Ax(t)dt + Gx(t)dw(t)$$

where $w(t)$ denotes a scalar Brownian motion,

$$A = \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$$

and

$$G = \varepsilon E, \qquad E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varepsilon = b^{-1/4}$$

with

$$x(0) = x_0, \qquad \mathbb{E}\left[\|x_0\|_2\right] < +\infty$$

Their choice of the matrix G is motivated by the fact that skew-symmetric matrices have been observed to destabilise other types of SDEs.

Remarkably they proved the following result.

Theorem 3.1. *Given sufficiently large b, there exist positive constants $D(b)$ and $\delta(b)$ such that*

$$\mathbb{E}\left[\|x_0\|_2\right] \geq D(b)\mathbb{E}\left[\|x_0\|_2\right] \mathrm{e}^{\delta(b)t} \qquad \forall\ t > 0$$

4. **Higher dimension. General case.** We make explicit reference to the test problem for non-normality introduceded by Higham and Mao, in a general dimension n, considering the matrix A_{HM}

$$A_{\mathrm{HM}} = \begin{pmatrix} -1 & b & 0 & \dots & 0 \\ 0 & -1 & b & 0 & \dots \\ 0 & \dots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & -1 & b \\ 0 & \dots & \dots & 0 & -1 \end{pmatrix} \tag{7}$$

We focus our attention to problem (7) for two reasons:

(i) It well represents a strongly non-normal case, proposed in the literature, since the distance to normality is given by $\nu(A_{HM}) = b\sqrt{n-1}$, so that the non-normality is concentrated in the parameter b.

(ii) It allows us an analytic analysis which permits to prove that in the associated SDE mean-stability is lost for stochastic terms exponentially small (in the dimension), as was conjectured by Higham and Mao. It can be inferred (see Table 1) that the same is true for a wide class of strongly non-normal matrices.

We introduce the matrix

$$E = \mathrm{e}_n\, \mathrm{e}_1^\top = \begin{pmatrix} 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \dots & \vdots \\ 0 & 0 & \dots & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & \dots & 0 \end{pmatrix} \tag{8}$$

which has the same form of (6), that we have constructed in Section 2.

4.1. **An illustrative numerical investigation.** We start by providing a few numerical tests on the matrix $A = A_{\mathrm{HM}} + \varepsilon E$, which suggest the subsequent main theoretical result.

Example 1. We set $a = 1, b = 2, \varepsilon = \gamma^{-n}, \gamma = 1.5$ and consider a perturbation εE, with $E = \varepsilon \mathrm{e}_n\, \mathrm{e}_1^\top$ and let us observe its effects on the spectral abscissa of $A_{\mathrm{HM}} + \varepsilon E$, for various values of ε.

n	ε	$\alpha(A_{\mathrm{HM}} + \varepsilon E)$
5	1.5^{-5}	$1.607340 \cdot 10^{-1}$
10	1.5^{-10}	$2.440439 \cdot 10^{-1}$
20	1.5^{-20}	$2.879151 \cdot 10^{-1}$
40	1.5^{-40}	$3.104275 \cdot 10^{-1}$

TABLE 1. Computed values for the spectral abscissa of the perturbed matrix $A_{\mathrm{HM}} + \varepsilon E$.

Example 1 indicates that matrices of the form (8) with exponentially (in the dimension) small norm are able to destabilise or quasi-destabilise the considered non-normal matrix A_{HM}.

4.2. **Deterministic destabilisation.** Let us consider first the Higham-Mao ODE

$$\dot{x}(t) = (A + \varepsilon E)\, x(t), \qquad x(0) = x_0 \tag{9}$$

with $A = A_{\mathrm{HM}}$ and E defined according to (7) and (8).

A straightforward calculation yields

$$\det\left(A_{\mathrm{HM}} + \varepsilon E - \lambda \mathrm{Id}\right) = -\left(\lambda + 1\right)^n + \varepsilon b^{n-1}$$

which determines the rightmost eigenvalue as

$$\lambda_R = -1 + \varepsilon b^{n-1}$$

which implies that if $\varepsilon = \gamma^{-n+1}$ with $1 < \gamma < b$, $\lambda_R > 0$, that is the zero solution of (9) is unstable.

This means that a suitably educated exponentially small perturbation (where the exponential dependence refers to the dimension) is able to destabilise the nominally stable system $\dot{x} = Ax$.

4.3. **Mean-square destabilisation of the Higham-Mao SDE.** Let us consider the SDE

$$dx(t) = Ax(t)dt + \varepsilon Ex(t)dw(t), \qquad x(0) = x_0. \tag{10}$$

with A and E defined according to (7) and (8).

Our main result is that exponentially small (in n) perturbations εE destabilise (in a mean square sense) the system.

Theorem 4.1. *Consider the SDE (10), with A and E defined according to (7) and (8). Let $\varepsilon = \gamma^{-n}$ with $1 < \gamma < b$; then there exists $\bar{n}$ such that for dimension $n \geq \bar{n}$, the solution $x(t)$ of the SDE is not mean-square asymptotically stable.*

Proof. According to the result in [4]

$$S = \mathcal{A} + \mathcal{E}$$

with $\mathcal{A} = A \otimes \mathrm{Id}_n + \mathrm{Id}_n \otimes A$ and $\mathcal{E} = \varepsilon^2 E \otimes E$

$$E = \mathrm{e}_n \mathrm{e}_1^\top,$$

with e_k the k-th unit vector.

We let $N = n^2$, and look at the determinant of $S - \lambda \mathrm{Id}_N$ (Id_N denoting the identity in dimension N).

We let

$$\mathcal{A}(z) = \mathcal{A} - \lambda \mathrm{Id}_N, \qquad z = -\lambda - 2$$

i.e. the matrix coinciding to $\mathcal{A}$ with z on the diagonal replacing -1.

Therefore we have to compute the determinant of

$$\mathcal{A}(z) + \varepsilon^2 \mathrm{e}_N \mathrm{e}_1^\top \in \mathbb{R}^{N,N}.$$

Using the Shermann Morrison formula

$$\det\left(\mathcal{A}(z) + \varepsilon^2 \mathrm{e}_N \mathrm{e}_1^\top\right) = \det(\mathcal{A}(z))\left(1 + \varepsilon^2 e_1^\top \mathcal{A}(z)^{-1} e_N\right)$$

being $e_1^\top \mathcal{A}(z)^{-1} e_N$ the first component of the solution of the linear system

$$\mathcal{A}(z)x = e_N.$$

Obviously

$$\det\left(\mathcal{A}(z)\right) = z^N = z^{n^2}.$$

Tedious algebraic computations arising from backward substitution for solving the upper triangular linear system leads to the formula

$$e_1^\top \mathcal{A}(z)^{-1} e_N = nC_n \frac{b^{2(n-1)}}{z^{2n-1}}$$

with

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!}$$

the so-called Catalan numbers. For large n we have

$$C_n = \frac{2^{2n}}{\sqrt{\pi}n^{3/2}} + \dots \quad \Longrightarrow \quad nC_n \approx \frac{2^{2n}}{\sqrt{\pi}n^{1/2}}$$

The computation of the characteristic polynomial gives us:

$$p(z) = z^{n^2} + \varepsilon^2 nC_n b^{2(n-1)} z^{(n-1)^2}$$

which has roots $z = 0$ (of multiplicity $(n-1)^2$) and

$$z^{2n-1} = -\varepsilon^2 nC_n b^{2(n-1)} < 0$$

that implies, for the leftmost (real) root

$$-z_R = b^{\frac{2n-2}{2n-1}} \varepsilon^{2/(2n-1)} (nC_n)^{1/(2n-1)}$$

and for large n

$$-z_R \approx 2b\varepsilon^{1/n}$$

If $\varepsilon = \gamma^{-n}$, with $\frac{b}{\gamma} > 1$, then

$$\lambda_R = -2 + 2\frac{b}{\gamma} > 0$$

and the system is destabilised. □

5. **Numerical behaviour of one-step methods.**

In this section, we experiment a few θ-Maruyama methods, a class of simple one-step methods which is largely used in the numerical discretization of SDEs. When we apply a one-step method for stochastic equations to our system (3), we get a difference equation, which can be rewritten as

$$X_{n+1} = M_n X_n, \tag{11}$$

where M_n is a sequence of independent random matrices. In general, the entries of the matrix M_n depend on the entries of the drift and diffusion matrices, on the parameters θ, on the stepsize h and on the Brownian increment ΔW_n. Let us recall the result of [4].

Lemma 5.1. *The zero solution of the linear system of difference equations (11) is meam square symptotically stable if and only if $\rho(S) < 1$, where $S = \mathbb{E}(M_n \otimes M_n)$.*

A classical θ-Maruyama method, applied to system (3) with fixed stepsize h reads

$$X_{n+1} = X_n + h[\theta A X_{n+1} + (1-\theta) A X_n] + G X_n \Delta W_n.$$

In this case $M_n = C + D\Delta W_n$, where

$$C = (\mathrm{Id} - h\theta A)^{-1}(\mathrm{Id} + h(1-\theta)A), \qquad D = (\mathrm{Id} - h\theta A)\sqrt{h}G.$$

Theorem 5.2 (see [4]). *The mean-square stability matrix T for the θ-Maruyama method applied to (3) is given by*

$$T = C \otimes C + D \otimes D \tag{12}$$

In our case we have that A is of the form (7), therefore we have

$$\mathrm{Id} - h\theta A = \mathrm{diag}(1 + h\theta) + \mathrm{diag}(-h\theta b, +1),$$

$$\mathrm{Id} + h(1-\theta)A = \mathrm{diag}(1 + h(1-\theta)) + \mathrm{diag}(bh(1-\theta), +1).$$

5.1. **Explicit Euler-Maruyama method.** Setting $\theta = 0$, we recast Euler-Maruyama method and the factors C and D of the corresponding stability matrix (12) take the form

$$C = \mathrm{diag}(1-h) + \mathrm{diag}(bh, +1), \qquad D = \sqrt{h}B.$$

Similarly to what we have done previously, we assume $B = \varepsilon E$ with E given by (8). The stability matrix T, for $n = 3$ and $h \neq 0$, is given by

$$T = (h-1)^2 \begin{pmatrix} 1 & -d & 0 & -d & d^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & -d & 0 & -d & d^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -d & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -d & 0 & -d & d^2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -d & 0 & -d & d^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -d \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -d \\ \frac{\varepsilon^2 h}{(h-1)^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (13)$$

where $d = \frac{b\,h}{h-1}$. We notice that the matrix (13) has $(h-1)^2$ as eigenvalue. In this case, we cannot get an easy formula for the characteristic polynomial of T, so that it is very technical to determine the whole spectrum of the matrix, so we will make some experimental considerations.

Remark 1. In the very special case, in which $h = 1$, the characteristic polynomial has the particularly simple following form

$$p(\lambda) = \lambda^{(n(n-1))}(\lambda^n - \varepsilon^2 b^{(2n-2)})$$

Therefore, if $\varepsilon = \gamma^{-n}$, as $n \longrightarrow \infty$, the discrete solution remains stable if $b/\gamma < 1$, i.e. the same stability condition found for the continuous problem (Theorem 4.1).

Let us analyse the stability matrix of the explicit Euler Maruyama method. Because of the particular structures of C and D, the part $C \otimes C$ of the stability matrix S is given by

$$\begin{pmatrix} (1-h)C & bhC & \mathbf{0} & \cdots \\ \mathbf{0} & (1-h)C & bhC & \\ & \cdots & & \\ \mathbf{0} & \mathbf{0} & (1-h)C & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and $D \otimes D = h\varepsilon^2 E \otimes E$.

From this explicit representations, we can observe that the Gershgorin row and column circles are given by

$$\mathcal{C}_1 = \{|z - (1-h)^2| \leq 2bh|1-h| + b^2h^2\}$$

$$\mathcal{C}_2 = \{|z - (1-h)^2| \leq bh|1-h|\}$$

and

$$\mathcal{C}_3 = \{|z - (1-h)^2| \leq \varepsilon^2 h\}$$

Therefore, the union of the circles is the circle

$$\mathcal{C} = \cup_{i=1}^{3} C_i = \{z : |z - (1-h)^2| \leq kh\}$$

where

$$k = \max\{2b|1-h| + b^2 h,\ \varepsilon^2\}$$

Notice that, this considerations are valid for any dimension n. It appears clear that for growing h it is not possible to have stability. On the other hand, a way to guarantee stability may be obtained by choosing the stepsize h - when possible - such that the circle $\mathcal{C}$, is included in the unit circle, centred in the origin; i.e. $h \in (0,2) \cap \{2bh|1-h| + b^2h^2 < \min\{(1-h)^2, 1-(1-h)^2\}\}$.

5.2. **Implicit Euler-Maruyama.** The implicit Euler-Maruyama method is obtained setting $\theta = 1$. In this case, the matrices C and D to construct the stability matrix of method are

$$C = \operatorname{diag}\left(\frac{1}{(1+h)^2}\right) + \sum_{j=2}^{n} \operatorname{diag}\left(\frac{b^{j-1}h^{j-1}}{(1+h)^j}, +j\right)$$

and

$$D(i,j) = \begin{cases} \varepsilon \dfrac{b^{n-i} h^{\frac{2(n-i)+1}{2}}}{(1+h)^{n+1-i}}, & \text{if } j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

In this case, the form of the stability matrix S is much more difficult to describe and the Gerschgorin circles are not invariants, with respect to the dimension, as in the explicit Euler-Maruyama method.

Example 2. For the continuous problem, thanks to the results of Section 4.3 , we are able to compute a stochastic perturbation of the form εE, with $\varepsilon = \gamma^{-n}$ and $E = e_n e_1^T$, that keeps the system stable, but makes it close to destabilisation.

We consider a test problem with $n = 20$, $b = 2$ and $\gamma^* = 1.81$, for which the spectral abscissa of the stability matrix of the system S (4), is given by $\alpha^* = -0.0071$. We consider some values for γ close to this critical value and provide in Table 2 the corresponding values of the spectral abscissa of the stability matrix S. In order to understand the behaviour of numerical methods in these threshold values, we evaluate, for different values of the integration step, the spectral radius of their numerical stability matrices, constructed according to (12). In particular, in Table 3 we present the results for the explcit Euler-Maruyama method. and in Table 4, we show the results for the implicit Euler-Maruyama method..

We can comment the results expressed by these tables, sayng that for what concerns the explicit Euler-Maruyama method, its stability behaviour follows that of the analytical system, for values of the stepsize smaller than one. Regarding the implicit Euler-Maruyama method, we observe instead numerical damping effects.

6. **Conclusions.**

The importance of strong non-normality in the theory of linear systems of ODEs has been extensively studied. In this work, we have investigated a particular instance of the general problem presented by Higham and Mao in [12] of destabilising a non-normal linear homogeneous system by a noisy term, which yileds the study of a linear system of SDEs. For a prototype strongly non-normal test problem (the Jordan block (2)), we have constructed analytically a mean-square

γ	α
1.78	0.0274
1.79	0.0158
1.80	0.0043
1.81	-0.0071
1.82	-0.0183
1.83	-0.0294
1.84	-0.0404

TABLE 2. Values of the spectral abscissa α of the stability matrix S for the threshold perturbations γ.

$h \setminus \gamma$	1.78	1.79	1.8	1.81	1.82	1.83	1.84
1	1.1779	1.1648	1.1519	1.1392	1.1267	1.1144	1.1024
0.5	1.0228	1.0169	1.0111	1.0054	0.9997	0.9942	0.9887
0.1	1.0030	1.0018	1.0007	0.9996	0.9984	0.9973	0.9962
0.01	1.0003	1.0002	1	0.9999	0.9998	0.9997	0.9996

TABLE 3. Values of the spectral radius ρ of the stability matrix of the explicit Euler-Maruyama method, in correspondence of the perturbation εE, with $\varepsilon = \gamma^{-n}$ and $E = e_n e_1^T$, for different stepsize h.

$h \setminus \gamma$	1.78	1.79	1.8	1.81	1.82	1.83	1.84
2	0.9987	0.09777	0.9577	0.9384	0.9199	0.9021	0.8848
1.5	1.0055	0.9893	0.9736	0.9585	0.9438	0.9297	0.9159
1	1.0093	0.9981	0.9873	0.9767	0.9664	0.9564	0.9467
0.5	1.0084	1.0026	0.9970	0.9915	0.9861	0.9809	0.9757
0.1	1.0025	1.0013	1.0002	0.9991	0.9979	0.9968	0.9958
0.01	1.0003	1.0002	1	0.9999	0.9998	0.9997	0.9996

TABLE 4. Values of the spectral radius ρ of the stability matrix of the implicit Euler-Maruyama method, in correspondence of the perturbation εE, with $\varepsilon = \gamma^{-n}$ and $E = e_n e_1^T$, for different stepsize h.

destabilising perturbation which vanishes as the dimension of the system goes to infinity, proving a conjecture stated in [12].

Moreover we have shortly considered numerical discretization of the considered system of SDEs and provided a few numerical experiments with explicit and implicit Euler-Maruyama methods, which suggest - as expected - a certain role of the stepsize h in the stability behavior.

Naturally one would like to obtain an answer to the problem fr a more general non-normality setting. Working on further interesting structures of the drift matrix may be an interesting subject of future research.

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