



A Modified Michael's Selection Theorem with Application to Generalized Nash Equilibrium Problem

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Abstract

This paper provides a sufficient condition for the existence of solutions for generalized Nash equilibrium problems in the infinite-dimensional setting and with a countable (possibly infinite) number of players. The result has been achieved as a consequence of a modified version of Michael's selection theorem that works even when the range space is not metrizable and the set-valued map has not closed values.

Keywords Continuous selection · Fixed point · Generalized Nash equilibrium problem

Mathematics Subject Classification 46N10 · 49J53 · 54H25 · 91B02

1 Introduction

The generalized Nash equilibrium problem (GNEP for short) is a widely used model in many fields as telecommunications, electricity, transportation to name a few. In a GNEP, a number of players compete with each other and, in contrast to the standard Nash equilibrium problem, the strategy set of each player depends on the strategies of all the other players.

The continuity of the feasibility set-valued maps, together with the convexity, and the closedness of their values are typically needed when dealing with existence results for GNEPs. The convexity and the compactness of the strategy sets, the continuity of the loss functions and the convexity of each loss function with respect to the i -th strategy are also typically assumed (see for instance [3]).

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In [18], an equilibrium existence result was proved, in which the lower semicontinuity of the feasibility set-valued maps was not assumed by imposing a very general condition on the loss functions. In [7], the upper semicontinuity and closed-valuedness assumptions on the feasibility maps were dispensed by exploiting the closedness of the fixed points set of the product feasibility map but in the finite-dimensional setting. Subsequently Cubiotti and Yao [8] extended this result when the strategy spaces are normed but they assumed the Hausdorff lower semicontinuity of the feasibility set-valued maps that is stronger than the lower semicontinuity. They also assume that the feasibility set of each player has nonempty interior with respect to the affine space generated by his/her strategy set. The Hausdorff lower semicontinuity assumption has been weakened in [5] at the expense of more restrictive assumptions on the strategy spaces, i.e., their separability and completeness. To overcome the separability of the Banach spaces, in [2] the authors used the closed-valuedness of the feasibility maps. Recently [6] the completeness and separability of the spaces have been removed but not their metrizability. Clearly, this fact precludes the use of weakly compact strategy sets. Aim of this paper is to avoid the assumption of metrizability of the topological vector spaces and the closed-valuedness of the feasibility set-valued maps by establishing a new selection result for a set-valued map acting from a topological space to a topological vector space.

The famous Michael's Theorem 3.2'' in [13] affirms the following conditions are sufficient for the existence of a continuous selection: The domain is a paracompact space, the values are in a Banach space and the set-valued map is lower semicontinuous with closed convex values. One of the essential feature of this result is the closed valuedness of the map indeed, even if its domain is the closed unit interval, the result becomes false when the map has not closed values as depicted in [13, Example 6.3]. Subsequently, this result was generalized in [14] to the case of nonmetrizable range space. In particular, Michael showed that every lower semicontinuous set-valued map from a paracompact space to a compact convex subset of a locally convex topological vector space admits a continuous selection whenever such a compact convex set is metrizable also and the values of the map are closed and convex. Carrying on a research line started in preceding papers [5, 6], we are able to avoid the assumption of closed valuedness of the set-valued map requiring that its values belong to a suitable family of convex sets only.

2 Some Concepts and Facts of Topology and Set-Valued Analysis

In this section, we summarize some results from set-valued analysis and topology. The aim is to establish terminology and notation, to set out results needed at various stages in the paper.

In this paper, a topological vector space is always considered Hausdorff. We denote by $\text{cl } A$ the closure and by $\text{co } A$ the convex hull of a subset A of a topological vector space. We recall an interesting notion of relative interior for convex sets due to Michael [13].

Definition 2.1 Let C be a convex subset of a topological vector space. The convex set $S \subseteq C$ is a face of C if $x_1, x_2 \in C$, $t \in (0, 1)$ and $tx_1 + (1 - t)x_2 \in S$ imply $x_1, x_2 \in S$. A point $x \in C$ is an inside point if it is not in any proper closed face of $\text{cl } C$. Denote by $I(C)$ the set of the inside points of C .

A comparison with other notions of relative interior is given in [5] and [6]. In particular, it is worthy of note that if the usual relative interior of C , i.e., the interior of C within the closed affine hull of C , is nonempty then it coincides with $I(C)$. As a direct consequence, the two notions coincide when the space is finite-dimensional.

The next part deals with the family of convex sets $\mathcal{D}(X)$ defined as follows:

$$\mathcal{D}(X) = \{C \subseteq X : C \text{ is convex and } I(\text{cl } C) \subseteq C\},$$

In [13], where X was a Banach space, it was proved that $\mathcal{D}(X)$ contains all the convex sets which are either closed, or with nonempty interior. These results still hold if one replaces the Banach space with any topological vector space. Actually, taking advantage that the relative interior of a convex set, when nonempty, coincides with $I(C)$, this class contains all the convex sets with nonempty relative interior. In particular, when X is finite-dimensional, $\mathcal{D}(X)$ coincides with the family of all convex sets. Other properties of the class $\mathcal{D}(X)$ have been established in [5, 6].

In the paper, we employ some facts of set-valued analysis which are reviewed in the next for the reader's convenience (see [1, 16] for details). Let X and Y be two topological spaces and $\Phi : X \rightrightarrows Y$ a set-valued map. We denote by $\text{dom } \Phi = \{x \in X : \Phi(x) \neq \emptyset\}$ the domain of Φ and by $\text{gph } \Phi = \{(x, y) \in \text{dom } \Phi \times Y : y \in \Phi(x)\}$ its graph. We say that Φ is convex-valued if $\Phi(x)$ is a convex set for each $x \in X$. The terms closed-valued and compact-valued are similarly defined. We define the convex hull map $\text{co } \Phi$ of Φ by $(\text{co } \Phi)(x) = \text{co } \Phi(x)$ and $\text{cl } \Phi$ is the set-valued map such that its images are the closure of the images of Φ .

The map Φ is lower semicontinuous at x if for each open set Ω such that $\Phi(x) \cap \Omega \neq \emptyset$ there exists a neighborhood U_x of x such that $\Phi(x') \cap \Omega \neq \emptyset$ for every $x' \in U_x$; it is upper semicontinuous at x if for each open set Ω such that $\Phi(x) \subseteq \Omega$ there exists a neighborhood U_x of x such that $\Phi(x') \subseteq \Omega$ for every $x' \in U_x$. The domain of a lower semicontinuous set-valued map is open.

Well-known results concerning the preservation of semicontinuity under various set theoretic operations are the following.

Lemma 2.1 Let $\Phi_1, \Phi_2 : X \rightrightarrows Y$ be such that $\text{cl } \Phi_1 = \text{cl } \Phi_2$, then Φ_1 is lower semicontinuous if and only if Φ_2 is lower semicontinuous.

Lemma 2.2 Let $\Phi : X \rightrightarrows Y$ be lower semicontinuous and $\Omega \subseteq Y$ be open, then the map $\Phi_\Omega : X \rightrightarrows Y$ defined by $\Phi_\Omega(x) = \Phi(x) \cap \Omega$ is lower semicontinuous.

Lemma 2.3 Let $\Phi_1 : X \rightrightarrows Y$ be lower semicontinuous. If $C \subseteq X$ is a closed set and $\Phi_2 : C \rightrightarrows Y$ is lower semicontinuous with $\Phi_2(x) \subseteq \Phi_1(x)$, for every $x \in C$, then the map $\Phi : X \rightrightarrows Y$ defined by

$$\Phi(x) = \begin{cases} \Phi_1(x) & \text{if } x \notin C \\ \Phi_2(x) & \text{if } x \in C \end{cases}$$

is lower semicontinuous.

A function $\varphi : X \rightarrow Y$ is a selection of the set-valued map $\Phi : X \rightrightarrows Y$ if $\varphi(x) \in \Phi(x)$ for each $x \in X$. The concept of locally selectionable maps has been introduced because these set valued maps do possess continuous selections when they are defined on paracompact topological spaces.

Definition 2.2 A set-valued map $\Phi : X \rightrightarrows Y$ is said to be locally (continuously) selectionable if for each $x \in \text{dom } \Phi$, there exist an open neighborhood U_x of x and a continuous map $\varphi : U_x \cap \text{dom } \Phi \rightarrow Y$ such that $\varphi(x') \in \Phi(x')$ for all $x' \in U_x \cap \text{dom } \Phi$.

The following result [15, Theorem 5] is a slight generalization of [4, Proposition 1.10.2].

Theorem 2.1 *Let X be a paracompact space, Y a topological vector space, and $\Phi : X \rightrightarrows Y$ a locally selectionable set-valued map with nonempty convex values. Then Φ has a continuous selection.*

The following result was established in [14], and it will play a fundamental role for proving our selection result.

Theorem 2.2 *Let X be a paracompact space, Y a locally convex topological vector space and $C \subseteq Y$ be a metrizable set such that $\text{cl co } K$ is compact for every compact $K \subseteq C$. If $\Phi : X \rightrightarrows C$ is a lower semicontinuous set-valued map with nonempty values such that, for some metric on C , every $\Phi(x)$ is complete then $\text{cl co } \Phi$ has a continuous selection.*

This theorem guarantees that every lower semicontinuous set-valued map Φ from a paracompact space X to a metrizable compact convex subset C of a locally convex topological vector space admits a continuous selection whenever the values of Φ are closed and convex. In the next section, we show that the closedness of $\Phi(x)$ can be weakened to requiring that $\Phi(x)$ belongs to $\mathcal{D}(Y)$, for every x .

3 A Continuous Selection Result

This section is completely devoted to prove that selection result we will use for establishing the existence of a Nash equilibrium.

Theorem 3.1 *Let X be a metrizable space, Y be a locally convex topological vector space and $C \subseteq Y$ be a metrizable compact convex set. If $\Phi : X \rightrightarrows C$ is a lower semicontinuous set-valued map with nonempty values in the class $\mathcal{D}(Y)$, then Φ admits a continuous selection.*

Since every compact metric space is complete, if Φ had closed values the existence of a continuous selection would be guaranteed by Theorem 2.2. Anyway the family $\mathcal{D}(Y)$ contains more than just closed convex sets. Also Theorem 3.2 in [6] cannot be directly applied since the metrizability of the whole topological vector space Y is required. Here, we show that a continuous selection exists even when Y is not metrizable. The idea is to consider the set-valued map $\text{cl } \Phi : X \rightrightarrows C$ which is lower semicontinuous (Lemma 2.1) with nonempty closed convex values inside the compact set C . Theorem 2.2 ensures the existence of a selection of $\text{cl } \Phi$ but this is not yet sufficient to guarantee

the existence of a selection of Φ . By exploiting the metrizable and compactness of C , we will find a sequence $\{\varphi_n\}$ of continuous selections for $\text{cl } \Phi$ such that $\{\varphi_n(x)\}$ is dense in $\text{cl } \Phi(x)$ for every $x \in X$.

Proof The first step is to show that there is a metric d compatible with the relative topology of C such that all balls are convex. Let $s : Y \times Y \rightarrow Y$ be the subtraction map $s(u, v) = u - v$, for each $u, v \in Y$, and put

$$M = s(C \times C) = \{u - v : u, v \in C\}.$$

Since C is metrizable and compact, so is $C \times C$. Moreover the image of a compact metric space in a Hausdorff space by a continuous function is metrizable [19, Corollary 23.2], then M is metrizable and so is also first countable.

Noticing that 0 lies in M , let $\{\Omega_n\}$ be a countable open neighborhood base at 0 in the relative topology of M . Write each Ω_n as $W_n \cap M$, where W_n is an open subset of Y . Since in a locally convex topological vector space the collection of all open convex balanced neighborhoods of zero is a neighborhood base at zero [1, Lemma 5.72], it is possible to choose an open convex balanced V_n , with $0 \in V_n \subseteq W_n$. Moreover, from the continuity of the addition map at $(0, 0)$, this can be done in order to have $V_{n+1} + V_{n+1} \subseteq V_n$. It is then evident that $U_n = V_n \cap M \subseteq W_n \cap M = \Omega_n$, so $\{U_n\}$ is an open neighborhood base at 0 in the relative topology of M .

Since the collection $\{V_n\}$ is a filter base, there is a unique topology τ' on the vector space Y that makes it a topological vector space having $\{V_n\}$ as a neighborhood base at zero [1, Theorem 5.6]. This space is locally convex (but not necessarily Hausdorff), and hence, τ' can be generated by the collection of the Minkowsky functionals p_n associated to V_n which are continuous seminorms. Since $\{V_n\}$ is countable, this topology can be defined by a pseudometric that is invariant under translations, and for which the open balls are convex. Indeed, for $u, v \in Y$, let

$$d(u, v) = \sup_n \min \left\{ p_n(u - v), n^{-1} \right\}.$$

We see immediately that d is a pseudometric invariant under translations and defining τ' . Furthermore, the quasicconvexity of p_n implies the quasicconvexity of $\min\{p_n, n^{-1}\}$ and hence the quasicconvexity of the pointwise supremum $d(\cdot, 0)$. It then follows that the balls are convex.

We next claim that (C, τ') is Hausdorff. To see this, let u and v be distinct points in C and denote $w = u - v \in M$. Since Y is Hausdorff there exists an open set $W \subseteq Y$, such that $0 \in W$ and $w \notin W$, and let n be such that $U_n \subseteq W \cap M$. Notice that $w \notin V_n$ since otherwise $w \in V_n \cap M = U_n \subseteq W \cap M$. It follows that $1 \leq p_n(w) = p_n(v - u)$, so u and v may be separated in the topology τ' generated by p_n , proving our claim.

Clearly τ' is coarser than the default topology τ of Y , so the identity map $id : (C, \tau) \rightarrow (C, \tau')$ is continuous. Given that (C, τ) is compact and that we now know that (C, τ') is Hausdorff, a well-known result in topology implies that the above map is a homeomorphism and hence, the topologies τ and τ' , once restricted to C , coincide. Since a pseudometric, once restricted to an Hausdorff subspace is a metric, then d is the desired metric on C .

The next step consists in finding a countable family of continuous selections of $\text{cl } \Phi$. The method of proof follows the same line of reasoning as in [6, Lemma 3.6]. Let $\{y_j\}$ be a countable, dense subset of C . For each j and k positive integers, the nonempty set

$$U_{j,k} = \left\{ x \in X : \text{cl } \Phi(x) \cap B(y_j, k^{-1}) \neq \emptyset \right\}$$

is open since $\text{cl } \Phi$ is lower semicontinuous. Here $B(y_j, k^{-1})$ is the open convex ball centered at y_j with radius k^{-1} defined by the pseudometric d . Moreover, the metrizable-ability of X guarantees that there exists a countable family of closed sets $\{C_{i,j,k}\}$ such that

$$U_{j,k} = \bigcup_{i \geq 1} C_{i,j,k}.$$

Let $\Phi_{i,j,k} : X \rightrightarrows C$ be defined by

$$\Phi_{i,j,k}(x) = \begin{cases} \Phi(x) & \text{if } x \notin C_{i,j,k} \\ \text{cl } \Phi(x) \cap B(y_j, k^{-1}) & \text{if } x \in C_{i,j,k} \end{cases}.$$

The set-valued map $\text{cl } \Phi \cap B(y_j, k^{-1})$ is lower semicontinuous by Lemmas 2.1 and 2.2. Moreover, since $C_{i,j,k}$ is closed, Lemmas 2.1 and 2.3 imply that $\text{cl } \Phi_{i,j,k}$ is lower semicontinuous with $\text{dom } \text{cl } \Phi_{i,j,k} = X$.

Every metric space is paracompact and the values of $\text{cl } \Phi_{i,j,k}$ are complete. The existence of a continuous selection $\varphi_{i,j,k}$ for each $\text{cl } \Phi_{i,j,k}$ follows from Theorem 2.2 and therefore the family $\{\varphi_{i,j,k}\}$ is a countable collection of continuous selections for $\text{cl } \Phi$.

We claim that $\{\varphi_{i,j,k}(x)\}$ is dense in $\text{cl } \Phi(x)$ for every $x \in X$. Fix $x \in X$, $y \in \text{cl } \Phi(x)$, and k positive integer. From the density of $\{y_j\}$ there exists j such that $d(y_j, y) \leq (2k)^{-1}$. Then $x \in U_{j,2k}$ and, in particular, $x \in C_{i,j,2k}$ for some i . Hence, $d(\varphi_{i,j,2k}(x), y_j) \leq (2k)^{-1}$ which implies

$$d(\varphi_{i,j,2k}(x), y) \leq d(\varphi_{i,j,2k}(x), y_j) + d(y_j, y) < k^{-1}.$$

For the sake of simplicity, after a reindexing, let's denote the family of continuous selection by $\{\varphi_n\}$.

Now we show that there exists a continuous function $\varphi : X \rightarrow C$ such that $\varphi(x) \in I(\text{cl } \Phi(x))$ for every $x \in X$. We recall that a convex series of elements of a subset C of a topological vector space is a series of the form $\sum_{n \geq 1} \lambda_n y_n$, where $y_n \in C$ and $\lambda_n \geq 0$ for each n , and $\sum_{n \geq 1} \lambda_n = 1$. The compactness of C implies that C is complete and (von Neumann) bounded, i.e., every neighborhood of the zero vector can be inflated to include the set. Therefore [10, Proposition 2] ensures that every convex series of elements of C converges to a point of C . Hence, fixed $x \in X$, the convex series $\sum_{n \geq 1} 2^{-n} \varphi_n(x)$ converges to a point $\varphi(x) \in C$. We show that $\varphi(x)$ belongs to

$I(\text{cl } \Phi(x))$. Assume by contradiction that there exists a proper closed face S of $\text{cl } \Phi(x)$ such that $\varphi(x) \in S$. For every fixed m the convex series

$$\sum_{n \geq 1, n \neq m} \frac{2^{-n}}{1 - 2^{-m}} \varphi_n(x)$$

converges to $\varphi_{-m}(x) \in C$ and $\varphi(x) = 2^{-m}\varphi_m(x) + (1 - 2^{-m})\varphi_{-m}(x)$. Two cases are possible: either $\varphi(x) = \varphi_m(x)$ or $\varphi(x)$ is an interior point of the segment $[\varphi_m(x), \varphi_{-m}(x)]$. In both cases $\varphi_m(x) \in S$. Since $\{\varphi_n(x)\}$ is dense in $\text{cl } \Phi(x)$ and S is closed, this implies $S = \text{cl } \Phi(x)$, which is impossible.

We prove that φ is continuous by showing that the series converges uniformly. Since C is bounded in Y , for each neighborhood U of the origin there is $\delta > 0$ such that $\lambda C \subseteq U$ whenever $|\lambda| \leq \delta$. Take $m > -\log_2 \delta$. For each $x \in X$, since $\text{cl } \Phi(x) \subseteq C$ is convex, we have

$$\sum_{n \geq 1} 2^{-n} \varphi_n(x) - \sum_{1 \leq n \leq m} 2^{-n} \varphi_n(x) = \sum_{n \geq m+1} 2^{-n} \varphi_n(x) \in 2^{-m} \text{cl } \Phi(x) \subseteq U.$$

Therefore the series $\sum_{n \geq 1} 2^{-n} \varphi_n(x)$ is uniformly convergent and the limit φ is continuous [11, Theorem 7.9].

Summarizing, we have found a continuous function φ such that $\varphi(x) \in I(\text{cl } \Phi(x))$ for every $x \in X$. Since the values $\Phi(x)$ belong to $\mathcal{D}(Y)$, then φ is a continuous selection of Φ . □

A theoretical example is outlined for the applicability of the previous selection result.

Example 3.1 Let $\{y_n\}$ with $n \in \mathbb{N}$ be an orthogonal normalized system of a separable infinite-dimensional Hilbert space Y . Consider $X = [0, 1]$ and define $\varphi : X \rightarrow Y$ as

$$\varphi(x) = \begin{cases} n(n+1)[(\frac{1}{n} - x)y_{n+1} + (x - \frac{1}{n+1})y_n] & \text{if } x \in (\frac{1}{n+1}, \frac{1}{n}] \\ 0 & \text{if } x = 0. \end{cases}$$

The function φ is a continuous piecewise linear curve on $(0, 1]$ which maps segments $[\frac{1}{n+1}, \frac{1}{n}]$ onto segments $[y_{n+1}, y_n]$. If Y is endowed with its weak topology, then φ is continuous on X . Indeed, when x goes to zero, n must go to $+\infty$, the sequence $\{y_n\}$ weakly converges to 0 for the Bessel’s inequality, and therefore, all the points of $[y_{n+1}, y_n]$ weakly converge to 0.

Now define the set-valued map $\Phi : X \rightrightarrows Y$ as $\Phi(x) = 2\varphi(x) + xB$ where B is the open unit ball. Clearly $\Phi(x) \in \mathcal{D}(Y)$ for each x , since it is open when $x \neq 0$ and it is $\{0\}$ when $x = 0$. Moreover Φ is not lower semicontinuous at $x = 0$ with respect to the norm topology: indeed $\|y\| > 1$ for all $y \in \Phi(\frac{1}{n})$ and hence $\Phi(\frac{1}{n}) \cap \Omega = \emptyset$ for all $n \in \mathbb{N}$ where $\Omega = B$. Thus, all the classical selection results do not apply since they require the lower semicontinuity.

Anyway, if Y is endowed with its weak topology, the set-valued map Φ is lower semicontinuous as sum of the continuous function 2φ and the lower semicontinuous

map $x \mapsto xB$. We show that all the other assumptions of Theorem 3.1 are verified and the existence of a continuous selection is guaranteed. First, it is easy to see that $\|y\| \leq 3$ for all $x \in X$ and $y \in \Phi(x)$. The set $C = \{y \in Y : \|y\| \leq 3\}$ is weakly compact (since Y is reflexive) and metrizable (since Y is reflexive and separable). Moreover, since a convex set is weakly closed if and only if it is strongly closed, then the family of convex sets $\mathcal{D}(Y)$ does not change when the weak topology is considered. Hence $\Phi(x) \in \mathcal{D}(Y)$ even if $\Phi(x)$ is not weakly open. From the construction, it follows that 2φ is a continuous selection of Φ .

It is well known that the weak topology of an infinite dimensional space is not metrizable; hence, the classical Michael selection theorem does not apply. Analogously, the requirements of Theorem 3.1''' in [13] and Theorem 3.2 in [6] fail.

Furthermore other requirements can fail even when the metrizability of the space Y is not required. For instance, the previous Theorem 2.2 does not even apply since $\Phi(x)$ is not complete for each $x \neq 0$. The famous Theorem 3.1 in [20] requires $\Phi^{-1}(y) = \{x \in X : y \in \Phi(x)\}$ to be open for each $y \in Y$. It is an easy calculation to check that $\|\varphi(x)\| \geq \sqrt{2}/2$ for each $x \neq 0$ and then $\Phi^{-1}(0) = \{0\}$ which is not open.

4 An Existence Result for Generalized Nash Equilibrium Problems

Gale and Mas-Colell proved a result [9, Section 2] which allows them to show the existence of equilibrium for a game without ordered preferences. Let N be a family of players and for each $i \in N$, C_i be a nonempty set of actions for player i . Let $C = \prod_{i \in N} C_i$ and $F_i : C \rightrightarrows C_i$ be the preference set-valued map of the player i . An element $x \in C$ is an equilibrium of the game if $F_i(x) = \emptyset$, for each $i \in N$. Gale and Mas-Colell assumed that N is finite; the strategy sets are finite-dimensional compact convex sets and that F_i are convex-valued with open graph. Next result generalizes the one in [9] to the infinite-dimensional case assuming that the preference set-valued maps are lower semicontinuous only.

Theorem 4.1 *Let N be a countable set and for each $i \in N$, C_i be a metrizable compact convex subset of a locally convex topological vector space X_i and F_i be lower semicontinuous. Then there exists a point $x \in C$ such that for each i , either $x_i \in I(\text{cl co } F_i(x))$ or $F_i(x) = \emptyset$.*

The countability of the family of players is needed in order to preserve the metrizability of the product C . The proof is analogous to the one given in [6, Theorem 4.2] but using Theorem 3.1 instead of [6, Theorem 3.2] to show that each map $G_i(x) = I(\text{cl co } F_i(x))$ has a continuous selection on its domain.

Theorem 4.1 is a composition of a fixed point theorem and an optimization theorem. When the values of F_i belong to the class $\mathcal{D}(X_i)$ and $\text{dom } F_i = C$ for each $i \in N$, Theorem 4.1 provides the existence of a fixed point $x \in F(x)$. Anyhow, we emphasize that Theorem 4.1 and [6, Lemma 3.3] ensure the existence of a fixed point x belonging to the relative interior of $F(x)$ whenever N is finite, F_i are nonempty convex-valued and X_i are finite dimensional spaces.

Now we are in position to establish a sufficient condition for the existence of a solution of a GNEP. Precisely, we consider a set N of players, each player i controls

the variables $x_i \in C_i$, where C_i is a nonempty strategy set. We denote by $x = (x_i) \in \prod_{i \in N} C_i = C$ the vector of all these decision variables and by x^{-i} the strategy vector of all the players different from player i . Denote by C_{-i} the set of all such vectors and write (x_i, x^{-i}) instead of x to emphasize the i -th player’s variables. An objective loss function $\theta_i : C \rightarrow \mathbb{R}$ that depends on all players’ strategies is fixed for each player i . Finally, each player’s strategy set depends on the rival players’ strategies; hence, a feasibility set-valued map $K_i : C_{-i} \rightrightarrows C_i$ is given for each player i . The GNEP consists in finding $x \in C$ such that for each $i \in N$ one has

$$x_i \in K_i(x^{-i}) \quad \text{and} \quad \theta_i(x_i, x^{-i}) \leq \theta_i(y_i, x^{-i}), \quad \forall y_i \in K_i(x^{-i}).$$

Since in this model K_i is independent of player i ’s choice we can consider the set-valued map $\widehat{K}_i : C \rightrightarrows C_i$ defined by $\widehat{K}_i(x) = K_i(x^{-i})$ as a technical convenience. As before we denote by $\widehat{K} : C \rightrightarrows C$ the product map.

Theorem 4.2 *Let N be a countable set and for each $i \in N$, C_i be a metrizable compact convex subset of a locally convex topological vector space X_i and K_i lower semicontinuous with nonempty values in the class $\mathcal{D}(X_i)$. Suppose that $\text{fix } \widehat{K}$ is closed and, for each $i \in N$,*

- i) $x \in \text{fix } \widehat{K}$ and A_i finite subset of $K_i(x_{-i})$ with $x_i \in \text{co } A_i$ imply there exists $a_i \in A_i$ such that $\theta_i(a_i, x^{-i}) \geq \theta_i(x_i, x^{-i})$;
- ii) θ_i is continuous.

Then the GNEP has a solution.

Proof The set-valued maps \widehat{K}_i are lower semicontinuous with nonempty values in the class $\mathcal{D}(X_i)$. Now we show that $F_i : \text{fix } \widehat{K} \rightrightarrows C_i$ defined by

$$F_i(x) = \widehat{K}_i(x) \cap \{y_i \in C_i : \theta_i(y_i, x^{-i}) < \theta_i(x_i, x^{-i})\}$$

is locally selectionable. Fix $(\bar{x}, \bar{y}_i) \in \text{gph } F_i$ and define $\overline{K}_i : C \rightrightarrows C_i$ as

$$\overline{K}_i(x) = \begin{cases} \widehat{K}_i(x) & \text{if } x \neq \bar{x}, \\ \{\bar{y}_i\} & \text{if } x = \bar{x}. \end{cases}$$

Notice that \overline{K}_i is lower semicontinuous (Lemma 2.3) and $\overline{K}_i(x) \in \mathcal{D}(X_i)$ for any $x \in C$. From Theorem 3.1 \overline{K}_i admits a continuous selection $\varphi_i : C \rightarrow C_i$. Since (\bar{x}, \bar{y}_i) belongs to set $\{(x, y_i) \in \text{fix } \widehat{K} \times C_i : \theta_i(y_i, x^{-i}) < \theta_i(x_i, x^{-i})\}$ which is open in $\text{fix } \widehat{K} \times C_i$ from assumption ii), there exist two open neighborhoods U of \bar{x} and V of \bar{y}_i such that

$$\theta_i(y_i, x^{-i}) < \theta_i(x_i, x^{-i}), \quad \forall (x, y_i) \in (U \times V) \cap (\text{fix } \widehat{K} \times C_i).$$

Therefore φ_i restricted to the neighborhood $U \cap \varphi_i^{-1}(V) \cap \text{fix } \widehat{K}$ is a local selection of F_i . So F_i is also locally selectionable. Let $\Phi_i : C \rightrightarrows C_i$ be defined by

$$\Phi_i(x) = \begin{cases} \widehat{K}_i(x) & \text{if } x \notin \text{fix } \widehat{K}, \\ \{\varphi_i(x)\} & \text{if } x \in \text{dom } F_i, \\ \emptyset & \text{otherwise.} \end{cases}$$

Once again, $\{(x, y_i) \in \text{fix } \widehat{K} \times C_i : \theta_i(y_i, x^{-i}) < \theta_i(x_i, x^{-i})\}$ is open in $\text{fix } \widehat{K} \times C_i$, then F_i which is the intersection of the lower semicontinuous map \widehat{K}_i with an open graph map, is lower semicontinuous and its domain is open. Hence, Φ_i is lower semicontinuous (Lemma 2.3). Theorem 4.1 guarantees the existence of an element $x \in C$ such that, for each i , either $x_i \in I(\text{cl } \Phi_i(x))$ or $\Phi_i(x) = \emptyset$. If $x \notin \text{fix } \widehat{K}$ then

$$x_i \in I(\text{cl } \Phi_i(x)) = I(\text{cl } \widehat{K}_i(x)) \subseteq \widehat{K}_i(x), \quad \forall i \in N$$

that leads to a contradiction. Hence $x \in \text{fix } \widehat{K}$. Now assume by contradiction that there exists an index $i \in N$ such that $x \in \text{dom } F_i$, i.e., $\Phi_i(x) \neq \emptyset$ or, equivalently,

$$x_i \in I(\text{cl } \Phi_i(x)) = \varphi_i(x) \in \text{co } F_i(x).$$

Hence exists a finite set $A_i \subseteq \widehat{K}_i(x) = K_i(x_{-i})$ such that $\theta_i(a, x^{-i}) < \theta_i(x_i, x^{-i})$ for all $a \in A_i$ and $x_i \in \text{co } A_i$. This fact contradicts i). Therefore $x \in \text{fix } \widehat{K}$ and solves GNEP. □

We conclude this section dealing with the assumptions of Theorem 4.2. The assumption i) is quite technical but it is milder than the usual ones. We recall that a function $\varphi : X \rightarrow \mathbb{R}$ is quasiconvex on a convex set $C \subseteq X$ if for each A finite subset of C and for each $x \in \text{co } A$ we have

$$\varphi(x) \leq \max_{a \in A} \varphi(a).$$

Assumption i) reminds of this notion: in particular the quasiconvexity on C_i of the $\theta_i(\cdot, x^{-i})$ for each $x^{-i} \in C_{-i}$ is clearly sufficient but not necessary for assumption i) as the following example shows.

Example 4.1 Consider the GNEP between two players acting in $[-2, 2] \subseteq \mathbb{R}$. The strategy set of the first agent is described by the set-valued map $K_1(x_2) = \left[-\frac{|x_2|}{2}, \frac{|x_2|}{2}\right]$, instead the strategy set of the second agent is given by

$$K_2(x_1) = \begin{cases} [-2 + |x_1|, 2 - |x_1|] & \text{if } 1 < |x_1| \leq 2, \\ [-1, 1] & \text{if } |x_1| \leq 1. \end{cases}$$

Clearly, the two set-valued maps are lower semicontinuous and the set

$$\text{fix } \widehat{K} = \{(x_1, x_2) \in [-2, 2] \times [-2, 2] : 2|x_1| \leq |x_2| \leq 1\}$$

is closed. Take the same loss function $\theta(x_1, x_2) = x_1^2 - x_2^4 + 2x_2^2$ for both players which is continuous but not quasiconvex with respect to the second variable. Anyway, the assumption i) of Theorem 4.2 is satisfied. Indeed let $(x_1, x_2) \in \text{fix } \widehat{K}$ and $x_2 \in \text{co } A$ with $A \subseteq K_2(x_1) = [-1, 1]$. Then, there exists $a_m, a_M \in A$ such that $x_2 \in [a_m, a_M]$ and $\theta(x_1, a_m) \geq \theta(x_1, x_2)$ if $x_2 \in [-1, 0]$ and $\theta(x_1, a_M) \geq \theta(x_1, x_2)$ if $x_2 \in [0, 1]$. The unique solution of the GNEP is $(0, 0)$.

Next example highlights the key contribution made by Theorem 4.2 also in comparison with some results from the literature.

Example 4.2 Consider the GNEP between two players acting in two different spaces, the former in $C_1 = [0, 1] \subseteq \mathbb{R}$, the latter in $C_2 = 4\overline{B}$ which is the closed ball with radius 4 in a separable infinite dimensional Hilbert space X_2 . Let $\{y_n\}$ with $n \in \mathbb{N}$ be an orthogonal normalized system of X_2 and $\varphi : [0, 1] \rightarrow X_2$ be defined as in Example 3.1. The feasibility set-valued maps are $K_1 : 4\overline{B} \rightrightarrows [0, 1]$ defined by

$$K_1(x_2) = \begin{cases} [0, 3/4] & \text{if } x_2 \in 4\overline{B} \setminus 3\overline{B}, \\ [0, 1/2] & \text{if } x_2 \in 3\overline{B} \setminus \{0\}, \\ [0, 1/4] & \text{if } x_2 = 0, \end{cases}$$

and $K_2 : [0, 1] \rightrightarrows 4\overline{B}$ defined by $K_2(x_1) = 2\varphi(x_1) + x_1\overline{B}$. As done in Example 3.1, K_2 is not lower semicontinuous at $x_1 = 0$ with respect to the norm topology, but it becomes lower semicontinuous if the norm topology of X_2 is replaced with the weak one. In fact, it is continuous since sum of two continuous set-valued maps. Moreover, K_1 is lower semicontinuous with respect to the weak topology (Lemma 2.3) and $4\overline{B}$ is a weakly compact metrizable set. The set

$$\text{fix } \widehat{K} = \{(x_1, x_2) \in [0, 1] \times 4\overline{B} : x_1 \in [0, 1/2] \text{ and } x_2 \in K_2(x_1)\}$$

is weakly closed thanks to the closed graph theorem. Fixed $x^* \in X_2$ with $\|x^*\| = 1$, take the same loss function $\theta(x_1, x_2) = -x_1 + \langle x^*, x_2 \rangle$ for both players. Endowing the space X_2 with the weak topology, all the assumptions of Theorem 4.2 are fulfilled and the problem has at least a solution. An easy calculation shows that the unique solution of the GNEP is $(1/2, -x^*)$.

Notice that K_1 is not upper semicontinuous and Theorem 2 in [18] does not apply. Also the more recent Theorem 1.1 in [8] cannot be applied since the continuity assumptions fail in the norm topology. Moreover all these results require the closed valuedness of the feasibility set-valued maps and K_1 does not satisfy this assumption. Such an assumption is not required by Theorem 4.4 in [6] but this result cannot be used since the metrizable of the whole vector space is needed.

Very recently, Scalzo in [17] has shown the existence of Nash equilibria in generalized games where the players' preferences are neither continuous nor ordered. The setting is more general but with a finite number of players. Moreover the result is based on the upper semicontinuity of the feasibility set-valued maps K_i .

We conclude citing an analogous existence result for GNEP [12, Theorem 4.3] which has weaker assumptions than ours. Anyway, the argumentation in the proof

is not correct because it is based on a false result (Lemma 4.5). In this lemma, the authors prove that the intersection of two set-valued maps, one of which is lower semicontinuous and the other has open values, is lower semicontinuous. This fact is clearly not true as the following trivial example shows.

Example 4.3 Let $\Phi_1, \Phi_2 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $\Phi_1(x) = [-1, 1]$ and

$$\Phi_2(x) = \begin{cases} (0, +\infty) & \text{if } x > 0, \\ \mathbb{R} & \text{if } x = 0, \\ (-\infty, 0) & \text{if } x < 0. \end{cases}$$

The former map is lower semicontinuous, and the latter one has open values but the intersection

$$(\Phi_1 \cap \Phi_2)(x) = \begin{cases} (0, 1] & \text{if } x > 0, \\ [-1, 1] & \text{if } x = 0, \\ [-1, 0) & \text{if } x < 0. \end{cases}$$

is not lower semicontinuous at $x = 0$.

5 Conclusions

In this paper, we have presented a modified version of a famous Michael's selection theorem that works even when the range space is not metrizable and the set-valued map has not closed values. As an application, we showed the existence of generalized Nash equilibria in locally convex topological vector spaces and with a countable (possibly infinite) number of players by dispensing the Hausdorff lower semicontinuity [8] of the feasibility set-valued maps which has been replaced by the standard lower semicontinuity. The closed-valuedness of the feasibility maps [2] as well as the metrizability of the spaces [6] have been avoided. In this setting, the weak topologies are allowed and future work on this topic may continue in the direction of developing more tractable coercivity conditions for the existence of solutions.

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