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Results in Mathematics

Sturm–Liouville Differential Inclusions with Set-Valued Reaction Term Depending on a Parameter

Tiziana Cardinali and Elisa Continelli

Abstract. In this paper we study the controllability for a Cauchy problem governed by a nonlinear differential inclusion driven by a Sturm–Liouville type operator. In particular, the considered second order differential inclusion involves a set-valued reaction term depending on a parameter. The key tool in the proof of the controllability result we provide is a multivalued version of the theorem recently proved by Haddad–Yarou, here established for an initial conditions problem monitored by a nonlinear second order differential inclusion presenting the sum of two multimaps on the right-hand side. We thereby deduce the existence of a local admissible pair for the considered control problem, that is the existence of a couple of functions consisting of a control, which is a measurable function, and the correspondent trajectory, which is an absolutely continuous function with absolutely continuous derivative. Secondly, under appropriate assumptions on the involved multimaps, we obtain an increased regularity for the solutions produced by our existence result. This regularity is the same of that recently tested by Bonanno, Iannizzotto and Marras for a different type of problem, which however involves the Sturm–Liouville operator.

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Keywords. Sturm–Liouville differential inclusions, nonlinear second order differential inclusions, Clarke subdifferential, controllability.

1. Introduction

In this paper, we investigate about the controllability of the following Cauchy problem governed by a nonlinear differential inclusion driven by a type Sturm–Liouville operator involving a set-valued reaction term depending on a parameter

$$(SLP)\begin{cases} -(p(t)x'(t))' + q(t)x(t) \in \lambda(H(x(t),x'(t)) - p(t)V(x(t),x'(t))) \\ \quad + u(x(t),x'(t)) \\ u(x(t),x'(t)) \in U(x(t),x'(t)) \\ x(0) = x_0 \\ x'(0) = y_0 \end{cases},$$

where Ω_1, Ω_2 are bounded subsets of $\mathbb{R}$, $\Omega = \Omega_1 \times \Omega_2$, $(x_0, y_0) \in \Omega$, p, q, H, V are suitable maps and U is the multimap which allows the control of the present problem. To be specific, we prove the existence of a local admissible pair for problem (SLP), that is the existence of a pair $\{x, u\}$ satisfying the initial conditions in (SLP) and such that $x : [0, T] \to \mathbb{R}$ is an absolutely continuous function with absolutely continuos derivative x' and $u : \Omega \to \mathbb{R}$ is measurable function with respect to the Borel σ-algebra on Ω. Initially, in a series of papers, Sturm and Liouville write any linear second order differential equation in the self adjoint form

$$-(p(t)x')' + q(t)x = 0$$

and study a problem monitored by the above equation having the boundary conditions

$$a_1x(0) - a_2x'(0) = 0$$

$$b_1x(T) - b_2x'(T) = 0,$$

later named Sturm–Liouville problem in the literature. Subsequently, in the theory of differential equations (inclusions) involving the Sturm–Liouville operator

$$L_x = -(p(t)x')' + q(t)x,$$

one denotes the considered differential equation/inclusion as a type Sturm–Liouville differential equation/inclusion. Although the subject of Sturm–Liouville is over 180 years old, a surprising number of reasults referring to Sturm–Louville differential equations have recent origin ([1,2,6–8]). In these works, the authors achieve the existence of solutions for different typologies of problems governed by differential equations or inclusions involving Sturm–Liouville operators. The investigations in this direction is stimulated by problems related to mathematical physics, physics, engineering and others. For instance, the development of quantum mechanics in the 1920s and 1930s (the one-dimensional time-indipendent Schrödinger equation is a special case of a

Sturm–Liouville equation) and the proof of the general spectral theorem due to von Neumann and Stone provide reasons for deepening the theory of Sturm–Lioville differential equations/inclusions. The aim of this paper is twofold. On one hand, we first obtain the existence of local absolutely continuous solutions with absolutely continuous derivative for a Cauchy problem governed by the following nonlinear second order differential inclusion presenting the sum of two multimaps on the right-hand side

$$x''(t) \in F(x(t), x'(t)) + G(t, x(t), x'(t)).$$

By doing so, we give a multivalued version of the Haddad–Yarou result proved in [10] (see Theorem 2.1). In the case in which the considered Hilbert space is $\mathbb{R}^n$, our Theorem 2.1 strictly contains the existence theorem of [10]. Moreover, we provide a result of the Lupulescu type (see [13], Corollary 2.2 and Remark 3.1). On the other hand, we reach the target of finding a local admissible pair for problem (SLP) by employing our Theorem 2.1 related to problem (P) and by relying on the classical Michael Selection Theorem. We also obtain a result, again concerning a Cauchy problem monitored by a Sturm–Liouville differential inclusion, which let us find solutions having a better regularity than the one recently established in [6–8], but having the same regularity recently tackled in [1] and [2] by Bonanno, Iannizzotto and Marras for another kind of problem, however guided by a type Sturm–Liouville operator. Finally, Appendix contains some background material with the intent of making the paper self-standing.

2. Existence of solutions for second order differential inclusions

In this section we present a result of existence of solutions for the following Cauchy problem governed by a nonlinear second order differential inclusion presenting the sum of two multimaps on the right-hand side

$$(P)\begin{cases} x''(t) \in F(x(t), x'(t)) + G(t, x(t), x'(t)) \\ x(0) = x_0 \\ x'(0) = y_0 \end{cases},$$

where Ω_1 and Ω_2 are open subsets of $\mathbb{R}^n$, $(x_0, y_0) \in \Omega_1 \times \Omega_2$, $F : \Omega_1 \times \Omega_2 \to \mathcal{P}_k(\mathbb{R}^n)$ and $G : [0, b] \times \Omega_1 \times \Omega_2 \to \mathcal{P}(\mathbb{R}^n)$, $b > 0$, are two maps satisfying the following assumptions (see Appendix for the involved definitions: Definitions 4.2, 4.4, 4.6, 4.8, 4.9):

(F_1) $F : \Omega_1 \times \Omega_2 \to \mathcal{P}_k(\mathbb{R}^n)$ is H-upper semicontinuous;

(F_2) there exist a linear application $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ with the property that $\langle x, \gamma(x)\rangle > 0$, for every $x \in \mathbb{R}^n \setminus \{0\}$, and a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ β-uniformly regular over Ω_2, with $\beta \geq 0$, such that

$$F(x, y) \subseteq \gamma(\partial^C f(y)), \quad \forall (x, y) \in \Omega_1 \times \Omega_2,$$

where $\partial^C f(y)$ is the Clarke subdifferential of f at y;

(G) $G : [0,b] \times \Omega_1 \times \Omega_2 \to \mathcal{P}(\mathbb{R}^n)$ is a Michael map which satisfies the lower Scorza–Dragoni property and such that for any bounded set $B \subseteq \Omega_1 \times \Omega_2$ there exists a compact set K_B in $\mathbb{R}^n$ such that

$$G(t,x,y) \subseteq K_B, \quad \forall (t,x,y) \in [0,b] \times B. \tag{1}$$

By *local (A.C.)-solution* of (P) we mean an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$, $T \in]0,b]$, with absolutely continuous derivative x' such that $x(0) = x_0$, $x'(0) = y_0$ and $x''(t) \in F(x(t), x'(t)) + G(t, x(t), x'(t))$, $a.e.\, t \in [0,T]$.

Remark 2.1. Let us note that, denoted with $\Gamma(\mathbb{R}^n)$ the set of all linear applications $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $\langle x, \gamma(x) \rangle > 0$, for every $x \in \mathbb{R}^n \backslash \{0\}$, in the case in which the considered separable Hilbert space H is $\mathbb{R}^n$, $\Gamma(\mathbb{R}^n)$ coincides with the set $\Gamma(H)$ defined in [10] and consisting of all linear applications $\gamma : H \to H$ such that

- (γ) for every $x \in H \backslash \{0\}$, $\langle x, \gamma(x) \rangle > 0$;
- $(\gamma)^*$ the restriction of γ to the closed unitary ball of H centered at 0, $\bar{B}(0,1)$, is continuous from $(\bar{B}(0,1), \tau_w)$ into H (where τ_w is the weak topology on H).

Indeed, in the case in which the considered Hilbert space is $\mathbb{R}^n$, a linear application $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ immediately satisfies $(\gamma)^*$ (see [4], Proposition 11.2). In addition, let us underline the fact that the set $\Gamma(\mathbb{R}^n)$ also coincides with the set of all automorphisms of $\mathbb{R}^n$ associated to positive definite matrices.

Firstly, we prove the following result

Theorem 2.1. *If $F : \Omega_1 \times \Omega_2 \to \mathcal{P}_k(\mathbb{R}^n)$, $G : [0,b] \times \Omega_1 \times \Omega_2 \to \mathcal{P}(\mathbb{R}^n)$ satisfy assumptions (F_1), (F_2) and (G), then for every $(x_0, y_0) \in \Omega_1 \times \Omega_2$ there exist $T > 0$ and a local (A.C.)-solution $x : [0,T] \to \mathbb{R}^n$ of problem (P).*

Proof. First of all we observe that the multifunction $G : [0,b] \times \Omega_1 \times \Omega_2 \to \mathcal{P}(\mathbb{R}^n)$ satisfies the assumptions of Theorem 4.2 in Appendix. Therefore, we can say that G has a Carathéodory selection. It means that there exists a function $g : [0,b] \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ such that: for every $(x,y) \in \Omega_1 \times \Omega_2,\ t \to g(t,x,y)$ is measurable; for every $t \in [0,b],\ (x,y) \to g(t,x,y)$ is continuous on $\Omega_1 \times \Omega_2$ and

$$g(t,x,y) \in G(t,x,y),\ a.e.\, t \in [0,b], \forall (x,y) \in \Omega_1 \times \Omega_2. \tag{2}$$

Then, if we denote with N the null measure subset of $[0,b]$ given by

$$N = \{t \in [0,b] : g(t,x,y) \notin G(t,x,y), (x,y) \in \Omega_1 \times \Omega_2\},$$

we can write (2) in this form

$$g(t,x,y) \in G(t,x,y),\ \forall t \in [0,b] \backslash N, \forall (x,y) \in \Omega_1 \times \Omega_2.$$

Now, fixed a point $\bar{z} \in \mathbb{R}^n$, we can construct the function $\hat{g} : \mathbb{R}_0^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\hat{g}(t,x,y) = \begin{cases} g(t,x,y), & t \in [0,b] \backslash N, (x,y) \in \Omega_1 \times \Omega_2 \\ \bar{z}, & t \in (\mathbb{R}_0^+ \backslash [0,b]) \cup N, (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \end{cases} .$$

Since g is a Carathéodory function, it immediately follows that $\hat{g}$ too is a Carathéodory function. Furthermore, fixed a bounded set $B \subseteq \Omega_1 \times \Omega_2$, from assumption (G) we have that there exists a compact set K_B such that

$$G(t,x,y) \subseteq K_B, \quad \forall (t,x,y) \in [0,b] \times B.$$

Then, if we set $K_B^* = K_B \cap \{\bar{z}\}$, we can say that K_B^* is a compact set in $\mathbb{R}^n$ such that

$$\hat{g}(t,x,y) \subseteq K_B^*, \quad \forall (t,x,y) \in \mathbb{R}_0^+ \times B.$$

Now, we can note that the function γ of assumption (F_2) belongs to the set $\Gamma(\mathbb{R}^n)$ (see Remark 2.1). Therefore, combining the fact that F assumes compact values and has properties (F_1), (F_2) with what have been just proved for the function $\hat{g}$, we can claim that F and $\hat{g}$ satisfy all hypothesis of Theorem 3.1 in [10]. Hence, there exist $T \in]0,b]$ and a local (A.C.)-solution $x : [0,T] \to \mathbb{R}^n$ for the Cauchy problem

$$(\hat{P}) \begin{cases} x'' \in F(x(t), x'(t)) + \hat{g}(t, x(t), x'(t)) \\ x(0) = x_0 \\ x'(0) = y_0 \end{cases} .$$

Consequently, by using the fact that $\hat{g}(t,\cdot,\cdot) = g(t,\cdot,\cdot)$, for a.e. $t \in [0,b]$, and (2), we can claim that $x : [0,T] \to \mathbb{R}^n$ is a local (A.C.)-solution for problem (P). □

Remark 2.2. Since the property (M) is difficult to check, we want to remember a class of functions that surely satisfies it. It is enough to consider multimaps $G : [0,b] \times \Omega_1 \times \Omega_2 \to \mathcal{P}(\mathbb{R}^n)$ that assume closed and convex values. For such multimaps the property (M) is obviously satisfied. Indeed, if G is a multimap assuming values in $\mathcal{P}_{fc}(\mathbb{R}^n)$, then for every close set $Z \subseteq [0,b] \times \Omega_1 \times \Omega_2$ such that G restricted to Z is lower semicontinuous, from Michael Theorem we can say that G has a continuous selection on Z.

Remark 2.3. Let us note that, in the case in which the considered Hilbert space is $\mathbb{R}^n$, our Theorem 2.1 strictly contains the result established in [10] by Haddad–Yarou. Indeed, given any single valued function $g : \mathbb{R}_0^+ \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ having the property (H_3) required in Theorem 3.1 by the authors of [10], we can consider the multimap $G : [0,b] \times \Omega_1 \times \Omega_2 \to \mathcal{P}_k(\mathbb{R}^n)$ defined as follows

$$G(t,x,y) = \{g(t,x,y)\}, \quad \forall (t,x,y) \in [0,b] \times \Omega_1 \times \Omega_2.$$

Obviously G is a Michael map and, furthermore, G has the lower Scorza–Dragoni property (see [12], Proposition 2.7.16). Thus, G satisfies assumption (G) of Theorem 2.1.

Next, as a consequence of Theorem 2.1, we establish the following

Corollary 2.2. *Let* $F : \Omega_1 \times \Omega_2 \to \mathcal{P}_k(\mathbb{R}^n)$ *be a multimap that satisfies properties* (F_1) *and*

$(\hat{F}_2)$ *there exists a proper, convex, locally Lipschitz and lower semicontinuous function* $f : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$F(x, y) \subseteq \partial f(y), \quad \forall (x, y) \in \Omega_1 \times \Omega_2, \tag{3}$$

where $\partial f(y)$ *is the subdifferential of* f *at* y.

Let $G : \mathbb{R}_0^+ \times \Omega_1 \times \Omega_2 \to \mathcal{P}(\mathbb{R}^n)$ *be a multimap satisfying assumption* (G). *Then, for every* $(x_0, y_0) \in \Omega_1 \times \Omega_2$, *there exist* $T > 0$ *and a local (A.C.)-solution* $x : [0, T] \to \mathbb{R}^n$ *of problem* (P).

Proof. First of all, we prove that the multimap F satisfies assumptions of Theorem 2.1. Of course, property (F_1) holds. So, all that remains is to check the validity of hypothesis (F_2). Since the function f of hypothesis $(\hat{F}_2)$ is proper, convex and lower semicontinuous, we can say that f is β-uniformly regular over Ω_2 with $\beta \equiv 0$ (see [10], Example 2.2). Therefore, being in addition the function f lower semicontinuous and locally Lipschitz, by using Proposition 4.1 of Appendix, we can write

$$\partial^C f(y) = \partial^P f(y), \quad \forall y \in \Omega_2, \tag{4}$$

where $\partial^P f(y)$ is the proximal subdifferential of f at y (see Appendix, Definition 4.3). Furthermore, being f a convex function too, one has that

$$\partial^P f(y) = \partial f(y), \quad \forall y \in \Omega_2. \tag{5}$$

Indeed, from Definitions 4.1 and 4.3, it is obviously true that

$$\partial f(x) \subseteq \partial^P f(x).$$

Conversely, let $\xi \in \partial^P f(x)$. Then we can write that there exist $\sigma, \eta > 0$ for which

$$\langle \xi, y - x \rangle \leq f(y) - f(x) + \sigma \|y - x\|^2, \quad \forall y \in B_{\mathbb{R}^n}(x, \eta). \tag{6}$$

Let $y \in \mathbb{R}^n$. Then there exists $\rho \in]0, 1[$ such that $ty + (1 - t)x \in B_{\mathbb{R}^n}(x, \eta)$, for every $t \in]0, \rho]$. By using (6), we have that, for every $t \in]0, \rho]$,

$$\langle \xi, ty + (1 - t)x - x \rangle \leq f(ty + (1 - t)x) - f(x) + \sigma \|ty + (1 - t)x - x\|^2,$$

that is

$$t\langle \xi, y - x \rangle \leq f(ty + (1 - t)x) - f(x) + \sigma t^2 \|y - x\|^2, \quad \forall t \in]0, \rho]. \tag{7}$$

Then, by using the convexity of f and by dividing for $t \in]0, \rho]$, (7) becomes

$$\langle \xi, y - x \rangle \leq f(y) - f(x) + \sigma t \|y - x\|^2, \quad \forall t \in]0, \rho]. \tag{8}$$

Letting $t \downarrow 0$ in (8), we have that

$$\langle \xi, y - x \rangle \leq f(y) - f(x),$$

that is $\xi \in \partial f(x)$. Then also

$$\partial^p f(y) \subseteq \partial f(y).$$

Hence, by using (4) and (5), we can say that

$$\partial f(y) = \partial^C f(y), \quad \forall y \in \Omega_2. \tag{9}$$

Now, setting $\gamma = 1_{\mathbb{R}^n} \in \Gamma(\mathbb{R}^n)$, where $1_{\mathbb{R}^n}$ the identical function on $\mathbb{R}^n$, (3) becomes (see (9))

$$F(x,y) \subseteq \partial^C f(y), \quad \forall (x,y) \in \Omega_1 \times \Omega_2.$$

So assumption (F_2) of Theorem 2.1 holds. Then, by applying Theorem 2.1, we can deduce the existence of a local (A.C.)-solution for problem (P). □

Remark 2.4. Let us note that Corollary 2.2 is a multivalued version of the Lupulescu Theorem proved in [13]. Indeed, using again Theorem 4.2 of Appendix, we have the existence of a Carathéodory selection g of the multimap G, that is g is a function of the type of [13].

3. Controllability for Sturm–Liouville differential inclusions

Now we present a controllability result concerning the following Cauchy problem governed by a Sturm–Liouville differential inclusion

$$(SLP)\begin{cases} -(p(t)x'(t))' + q(t)x(t) \in \lambda(H(x(t),x'(t)) - p(t)V(x(t),x'(t))) \\ \qquad\qquad\qquad\qquad + u(x(t),x'(t)) \\ u(x(t),x'(t)) \in U(x(t),x'(t)) \\ x(0) = x_0 \\ x'(0) = y_0 \end{cases},$$

where Ω_1 and Ω_2 are two open and bounded subsets of $\mathbb{R}$, $\Omega = \Omega_1 \times \Omega_2$, $(x_0, y_0) \in \Omega$, $\lambda, b \in \mathbb{R}^+$ and $p : [0,b] \to \mathbb{R}$, $q : [0,b] \to \mathbb{R}$, $H : \Omega \to \mathcal{P}_{fc}(\mathbb{R})$, $V : \Omega \to \mathcal{P}_k(\mathbb{R})$, $U : \Omega \to \mathcal{P}_{fc}(\mathbb{R})$ satisfy the following properties:

- (p) $p \in C^1([0,b])$ and $p(t) > 0$, for every $t \in [0,b]$;
- (q) $q \in C([0,b])$;
- (V_1) $V : \Omega \to \mathcal{P}_k(\mathbb{R})$ is H-upper semicontinuous;
- (V_2) there exists $\gamma \in \Gamma(\mathbb{R})$ and a locally Lipschitz, β-uniformly regular, with $\beta \geq 0$, function $f : \mathbb{R} \to \mathbb{R}$ over Ω_2 such that

  $$V(x,y) \subseteq \gamma(\partial^C f(y)), \quad \forall (x,y) \in \Omega_1 \times \Omega_2,$$

  where $\partial^C f(y)$ is the Clarke subdifferential of f at y;
- (HU) H and U are bounded and lower semicontinuous.

We recall that *local admissible pair* for problem (SLP) is a couple $\{x,u\}$ satisfying (SLP) such that $x : [0,T] \to \mathbb{R}$ is an absolutely continuous function with absolutely continuous derivative x' and $u : \Omega \to \mathbb{R}$ is a $B(\Omega)$-measurable function.

Theorem 3.1. *Under the assumptions* (p), (q), (V_1), (V_2) *and* (HU) *there exists a local admissible pair* $\{x, u\}$ *for problem* (SLP).

Proof. First of all we note that, thanks to assumption (p), in order to prove our existence result it will be enough to find a local adsmissible pair for problem

$$(SLP)^* \begin{cases} x''(t) \in -\dfrac{p'(t)}{p(t)}x'(t) + \dfrac{q(t)}{p(t)}x(t) - \dfrac{\lambda}{p(t)}H(x(t), x'(t)) \\ \qquad\qquad - \lambda V(x(t), x'(t)) - \dfrac{1}{p(t)}u(x(t), x'(t)) \\ u(x(t), x'(t)) \in U(x(t), x'(t)) \\ x(0) = x_0 \\ x'(0) = y_0 \end{cases}.$$

Now, it is immediate to note that the multimap U satisfies assumptions of classical Michael selection Theorem (see [14], Theorem 3.2). So there exists a continuous selection $\bar{u} : \Omega \to \mathbb{R}$ of U. The function $\bar{u}$ is also $B(\Omega)$-measurable from Corollary 3.11.5 in [11].
Therefore, let us focus on the Cauchy problem

$$(SLC) \begin{cases} x''(t) \in -\dfrac{p'(t)}{p(t)}x'(t) + \dfrac{q(t)}{p(t)}x(t) - \dfrac{\lambda}{p(t)}H(x(t), x'(t)) \\ \qquad\qquad - \lambda V(x(t), x'(t)) - \dfrac{1}{p(t)}\bar{u}(x(t), x'(t)) \\ x(0) = x_0 \\ x'(0) = y_0 \end{cases}.$$

Next, our goal will be to establish the existence of a local (A.C.)-solution for (SLC). To this aim, let $G : [0, b] \times \Omega \to \mathcal{P}(\mathbb{R})$ be the so defined multimap

$$G(t, x, y) = \left\{-\frac{p'(t)}{p(t)}y + \frac{q(t)}{p(t)}x - \frac{1}{p(t)}\bar{u}(x, y)\right\} - \frac{\lambda}{p(t)}H(x, y), \forall (t, x, y) \in [0, b] \times \Omega.$$

We want to show that G satisfies hypothesis (G) of Theorem 2.1. Since the multimap H takes values in $\mathcal{P}_{fc}(\mathbb{R})$, we can immediately say that G assumes convex and closed values. Then, from Remark 2.2, one has that G is a Michael map. Moreover, the multifunction G is lower semicontinuous in $[0, b] \times \Omega$. Indeed, by using assumptions (p), (q) and the continuity of the function $\bar{u}$, the so defined function

$$g(t, x, y) = -\frac{p'(t)}{p(t)}y + \frac{q(t)}{p(t)}x - \frac{1}{p(t)}\bar{u}(x, y), \ \ \forall (t, x, y) \in [0, b] \times \Omega,$$

is obviously continuous in $[0, b] \times \Omega$. On the other hand, the multimap defined by setting

$$K(t, x, y) = -\frac{\lambda}{p(t)}H(x, y), \ \ \forall (t, x, y) \in [0, b] \times \Omega, \tag{10}$$

is lower semicontinuous in $[0,b]\times\Omega$. To see this, fix $(\bar t,\bar x,\bar y)\in[0,b]\times\Omega$. Let $((t_n,x_n,y_n))_n$ be a sequence in $[0,b]\times\Omega$ with the property that $(t_n,x_n,y_n)\to(\bar t,\bar x,\bar y)$ in $[0,b]\times\Omega$. We want to prove that

$$\begin{aligned} K(\bar t,\bar x,\bar y) &\subseteq \underline{\lim} K(t_n,x_n,y_n)\\ &= \{u\in\mathbb{R}:\, u=\lim_{n\to+\infty}u_n,\, u_n\in K(t_n,x_n,y_n)\ \forall n\in\mathbb{N}\}. \end{aligned} \tag{11}$$

To show that (11) holds, fix $\bar z\in K(\bar t,\bar x,\bar y)$. Then we can write (see (10))

$$\bar z=-\frac{\lambda}{p(\bar t)}\bar w,$$

where $\bar w\in H(\bar x,\bar y)$. Of course, since $(t_n,x_n)\to(\bar t,\bar x)$ in $[0,b]\times\Omega_1$, by using (H), we have that

$$H(\bar x,\bar y)\subseteq\underline{\lim}H(x_n,y_n)=\{w\in\mathbb{R}:\, w=\lim_{n\to+\infty}w_n,\, w_n\in H(x_n,y_n)\ \forall n\in\mathbb{N}\}. \tag{12}$$

Then, since $\bar w\in H(\bar x,\bar y)$, by using (12), there exists a sequence $(w_n)_n$, $w_n\in H(x_n,y_n)$ for every $n\in\mathbb{N}$, such that $w_n\to\bar w$. In addition, since $(t_n)_n$ converges to $\bar t$, the continuity of p given by (p) implies that $p(t_n)\to p(\bar t)$. At this point, the sequence $(u_n)_n$ where

$$u_n=-\frac{\lambda}{p(t_n)}w_n,\ \ \forall n\in\mathbb{N},$$

is such that $u_n\in K(t_n,x_n,y_n)$, for every $n\in\mathbb{N}$ and $u_n\to-\frac{\lambda}{p(\bar t)}\bar w=\bar z$ in $\mathbb{R}$. Hence, we can conclude that $\bar z\in\underline{\lim}K(t_n,x_n,y_n)$ and, as a consequence, that (11) holds. This means that K is lower semicontinuous at the point $(\bar t,\bar x,\bar y)\in[0,b]\times\Omega$ (see Proposition 1.2.6 in [12]). Thus, from the arbitrariness of $(\bar t,\bar x,\bar y)$, we can deduce the lower semicontinuity of the multimap K in $[0,b]\times\Omega$. By applying Proposition 1.2.59 in [12], we can say that G is lower semicontinuous in $[0,b]\times\Omega$. Therefore, G satisfies the lower Scorza–Dragoni property (see, here Definition 4.8). Finally, since the set Ω is bounded, it is sufficient to prove the existence of a compact subset K of $\mathbb{R}$ such that

$$G(t,x,y)\subseteq K,\quad\forall(t,x,y)\in[0,b]\times\Omega. \tag{13}$$

By taking into account of (p), (q), (HU) and by recalling that Ω_1, Ω_2 are bounded, it is easy to chech that there exists $M>0$ such that

$$\|G(t,x,y)\|\le M,\quad\forall(t,x,y)\in[0,b]\times\Omega.$$

Hence (13) holds with $K=\bar B_{\mathbb{R}}(0,M)$. We have completely shown that the multimap G satisfies hypothesis (G) of Theorem 2.1. Now, turn our attention to the multimap

$$F(x,y)=\lambda V(x,y),\quad\forall(x,y)\in\Omega.$$

First of all, the multimap F is H-upper semicontinuous in Ω. Indeed, fixed $(\bar x,\bar y)\in\Omega$ and $\epsilon>0$, since V is H-upper semicontinuous, in correspondence of

$\frac{\epsilon}{\lambda} > 0$, there exists a neighbourhood I of $(\bar{x}, \bar{y})$ such that

$$V(x, y) \subseteq B\left(V(\bar{x}, \bar{y}), \frac{\epsilon}{\lambda}\right), \quad \forall (x, y) \in I. \tag{14}$$

Hence, multiplying by λ in (14), we have that

$$F(x, y) = \lambda V(x, y) \subseteq \lambda B\left(V(\bar{x}, \bar{y}), \frac{\epsilon}{\lambda}\right) = B(F(\bar{x}, \bar{y}), \epsilon), \quad \forall (x, y) \in I,$$

which means the H-upper semicontinuity of F at $(\bar{x}, \bar{y})$. From the arbitrariness of $(\bar{x}, \bar{y})$, we can claim that assumption (F_1) of Theorem 2.1 holds. Now, we know from hypothesis (V_2) that

$$V(x, y) \subseteq \gamma(\partial^C f(y)), \quad \forall (x, y) \in \Omega_1 \times \Omega_2.$$

So, if we define $\bar{\gamma} : \mathbb{R} \to \mathbb{R}$ given by $\bar{\gamma} = \lambda\gamma$, we have that

$$F(x, y) \subseteq \bar{\gamma}(\partial^C f(y)), \quad \forall (x, y) \in \Omega_1 \times \Omega_2,$$

where $\bar{\gamma}$ is a linear application belonging to the space $\Gamma(\mathbb{R})$. So (F_2) is true. We are in position to apply our Theorem 2.1 which guarantees the existence of a local (A.C.)-solution $x : [0, T] \to \mathbb{R}$ for problem (SLC). Now, this solution x satisfies

$$\begin{aligned} x''(t) \in -\frac{p'(t)}{p(t)}x'(t) + \frac{q(t)}{p(t)}x(t) - \frac{\lambda}{p(t)}H(x(t), x'(t)) \\ -\lambda V(x(t), x'(t)) - \frac{1}{p(t)}\bar{u}(x(t), x'(t)). \end{aligned}$$

This concludes our proof since we have shown the existence of a pair $\{x, \bar{u}\}$ satisfying $(SLP)^*$ such that $x : [0, T] \to \mathbb{R}$ is absolutely continuous with absolutely continuous derivative and $\bar{u} : \Omega \to \mathbb{R}$ is $B(\Omega)$-measurable, i.e. $\{x, u\}$ is a local admissible pair for problem $(SLP)^*$ and, as a consequence, for (SLP). □

From Theorem 3.1 can be obviously derived the result referring to the case of lack of control which it is worth to enunciate.

Theorem 3.2. *Under the assumptions* (p), (q), (V_1), (V_2) *and* (H) *there exists a local (A.C.)-solution for the Cauchy problem*

$$(SLP) \begin{cases} -(p(t)x'(t))' + q(t)x(t) \in \lambda(H(x(t), x'(t)) - p(t)V(t, x(t), x'(t))) \\ x(0) = x_0 \\ x'(0) = y_0 \end{cases}.$$

Furthermore, if the multimap V *is also bounded, every local (A.C.)-solution of* (SLP) *belongs to the Sobolev space* $W^{2,\infty}([0, T], \mathbb{R})$.

Proof. Trivially, Theorem 3.1 guarantees the existence of a local (A.C.)-solution $x : [0, T] \to \mathbb{R}$ for the Cauchy problem (SLP). We claim that $x \in W^{2,\infty}([0, T], \mathbb{R})$. Indeed, being x, x' absolutely continuous in the compact interval $[0, T]$, from Weierstrass Theorem we can say that $x, x' \in L^\infty([0, T], \mathbb{R})$. That is $x \in W^{1,\infty}([0, T], \mathbb{R})$. On the other hand, we know that x'' is measurable. Therefore, by using the boundness of H and V, we can conclude that $x'' \in L^\infty([0, T], \mathbb{R})$, i.e. $x \in W^{2,\infty}([0, T], \mathbb{R})$. □

Remark 3.1. We have seen that, by requiring the further hypothesis of boundness on the multimap V, Theorem 3.2 guarantees the existence of a solution $x : [0, T] \to \mathbb{R}$ in $W^{2,\infty}([0, T], \mathbb{R})$ for the Cauchy problem (SLP) governed by a Sturm–Liouville differential inclusion. The study about solutions of this regularity has been recently tackled in the case $V \equiv \{0\}$ in [1], [2]. It goes without saying that the authors of the aforementioned works require assumptions on the involved maps that are different from those in Theorem 3.2.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

4. Appendix

In this paper we shall denote with $\langle .,. \rangle$ the scalar product on $\mathbb{R}^n$ and with $\|.\|$ the norm on $\mathbb{R}^n$. Moreover, $B_{\mathbb{R}^n}(x,r)$ will refer to the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with radius $r > 0$. Given a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the *effective domain* of f is the so defined set

$$dom f = \{x \in \mathbb{R}^n : f(x) < +\infty\}.$$

Definition 4.1 ([15, §1.3]). Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. For every $x \in dom f$, the set

$$\partial f(x) = \{\xi \in \mathbb{R}^n : f(y) - f(x) \geq \langle \xi, y - x \rangle, \forall y \in \mathbb{R}^n\}$$

is called the *subdifferential* of f at x.

Definition 4.2 ([9, §2.1]). Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. The *Clarke subdifferential* of f at $x \in dom f$ is the subset $\partial^C f(x)$ of $\mathbb{R}^n$ defined by

$$\partial^C f(x) = \{\xi \in \mathbb{R}^n : \langle \xi, y \rangle \leq f^\uparrow(x; y), \forall y \in \mathbb{R}^n\}, \tag{15}$$

where $f^\uparrow(x; y)$ is the *generalized Rockafellar directional derivative* of f at x with respect to the vector y given by

$$f^\uparrow(x; y) = \limsup_{\substack{h \to x \\ t \downarrow 0}} \frac{f(h + ty) - f(h)}{t}. \tag{16}$$

Definition 4.3 ([9, §2.1]). Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. The *proximal subdifferential* of f at $x \in dom f$ is the subset $\partial^P f(x)$ of $\mathbb{R}^n$ consisting of all the vectors $\xi \in \mathbb{R}^n$ such that there exist $\sigma, \eta > 0$ for which

$$\langle \xi, h - x \rangle \leq f(h) - f(x) + \sigma \|h - x\|^2, \quad \forall h \in B_{\mathbb{R}^n}(x, \eta). \tag{17}$$

Definition 4.4 ([10, Definition 2.1]). Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let Ω be a nonempty open subset of $\mathbb{R}^n$ with the property that $\Omega \subseteq dom f$. The function f is said to be *β-uniformly regular* over Ω if there exists $\beta \geq 0$ such that for every $x \in \Omega$ and for all $\xi \in \partial^P f(x)$ one has

$$\langle \xi, y - x \rangle \leq f(y) - f(x) + \beta \|y - x\|^2, \quad \forall y \in \Omega.$$

Proposition 4.1 *([10, Proposition 2.4], (ii)). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function and let Ω be a nonempty open set in $\mathbb{R}^n$. If f is uniformly regular over Ω, then the proximal subdifferential of f at x, $\partial^P f(x)$, coincides with the Clarke subdifferential of f at x, $\partial^C f(x)$, at any point $x \in \mathbb{R}^n$.*

Now, let X be a topological space and Y be a linear topological space. In the sequel we will use the following notations:

$$\mathcal{P}(Y) = \{H \subseteq Y : H \neq \emptyset\},$$
$$\mathcal{P}_c(Y) = \{H \in \mathcal{P}(Y) : H\, convex\},$$
$$\mathcal{P}_f(Y) = \{H \in \mathcal{P}(Y) : H\, closed\},$$
$$\mathcal{P}_k(Y) = \{H \in \mathcal{P}(Y) : H\, compact\},$$
$$\mathcal{P}_{fc}(Y) = \mathcal{P}_f(Y) \cap \mathcal{P}_c(Y).$$

Definition 4.5 ([12, §1.2]). A multimap $F : X \to \mathcal{P}(Y)$ is said to be *lower semicontinuous* at $x_0 \in X$ if, for every $\Omega = \mathring{\Omega} \subseteq Y$ with $F(x_0) \cap \Omega \neq \emptyset$, there exists a neighbourhood V of x_0 such that $F(x) \cap \Omega \neq \emptyset,\ \forall x \in V$.

If $Y = (Y, d)$ is a metric space, then we can define the concept of ball centered at $S \subseteq Y$ with radius ϵ as

$$B(S, \epsilon) = \{y \in Y : \delta(y, S) < \epsilon\},$$

where

$$\delta(y, S) = \inf\{d(y, s) : s \in S\}. \tag{18}$$

Definition 4.6 ([12, §1.2]). A multimap $F : X \to \mathcal{P}(Y)$ is said to be *H-upper semicontinuous* in $x_0 \in X$ if $\forall\, \epsilon > 0$ there exists a neighbourhood V of x_0 such that $F(x) \subseteq B(F(x_0), \epsilon),\ \forall x \in V$.

Definition 4.7 (Definition 1.1.42 in [12]). Let $Y = (Y, d)$ be a metric space. Let $(A_n)_n$ be a sequence in $\mathcal{P}_f(Y)$. The *Kuratowski limit inferior* of the sequence $(A_n)_n$ is the set defined by

$$\underline{\lim} A_n = \{y \in Y : \forall r > 0\ \exists \bar{n} = \bar{n}(r) \in \mathbb{N} : B(y, r) \cap A_n \neq \emptyset,\ \forall n \geq \bar{n}\}.$$

Remark 4.1. (Remark 1.1.43 in [12]) The Kuratowski limit inferior of a sequence $(A_n)_n \subseteq \mathcal{P}_f(Y)$ has also this expression

$$\underline{\lim} A_n = \{y \in Y : y = \lim_{n \to +\infty} a_n,\ a_n \in A_n\ \forall n \in \mathbb{N}\}.$$

Definition 4.8 (see [5]). If (T, τ, μ) is a finite measure space and X, Y are topological spaces, a multimap $F : T \times X \to \mathcal{P}(Y)$ satisfies the *lower Scorza–Dragoni property* if (SD) for every $\epsilon > 0$ there exists a compact $K_\epsilon \subseteq T$ such that $\mu(T \backslash K_\epsilon) < \epsilon$ and F restricted to $K_\epsilon \times X$ is lower semicontinuous.

Definition 4.9 (see [5]). If (T, τ, μ) is a finite measure space, a multifunction and X, Y are topological spaces, a multimap $F : T \times X \to \mathcal{P}(Y)$ is said to be a *Michael map* if it satisfies the property (M) , i.e. for every closed set $Z \subseteq T \times X$ such that F restricted to Z is lower semicontinuous, there exists a continuous selection of F on Z (i.e. there exists a continuous function $f : Z \to Y$ such that $f(t, x) \in F(t, x),\ \forall (t, x) \in Z$).

Theorem 4.2 (Theorem 3.1 in [5]). *Let T, X and Y be Hausdorff topological spaces and let μ be a Radon measure on T such that $\mu(T) < +\infty$. If $F : T \times X \to \mathcal{P}(Y)$ is a multimap satisfying properties (SD) and (M), then F has a Carathéodory selection, i.e. there exists a function $f : T \times X \to Y$ such that:*

1. *for every $t \in T$, $f(t, \cdot)$ is continuous on X;*
2. *for every $x \in X$, $f(\cdot, x)$ is measurable;*
3. *for μ-a.e. $t \in T$ and for every $x \in X$, $f(t, x) \in F(t, x)$.*

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Tiziana Cardinali
Department of Mathematics and Computer Science
University of Perugia
1, Via Vanvitelli
Perugia 06132
Italy
e-mail: tiziana.cardinali@unipg.it

Elisa Continelli
Department of Information Engineering, Computer Science and Mathematics
University of L'Aquila
1, Via Vetoio
L'Aquila 67100
Italy
e-mail: elisacontinelli@hotmail.it

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