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Mutual and total mutual visibility in hypercube-like graphs ☆

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ABSTRACT

Let G be a graph and $X \subseteq V(G)$. Then, vertices x and y of G are X-visible if there exists a shortest x,y-path where no internal vertices belong to X. The set X is a mutual-visibility set of G if every two vertices of X are X-visible, while X is a total mutual-visibility set if any two vertices from $V(G)$ are X-visible. The cardinality of a largest mutual-visibility set (resp. total mutual-visibility set) is the mutual-visibility number (resp. total mutual-visibility number) $\mu(G)$ (resp. $\mu_t(G)$) of G. It is known that computing $\mu(G)$ is an NP-complete problem, as well as $\mu_t(G)$. In this paper, we study the (total) mutual-visibility in hypercube-like networks (namely, hypercubes, Fibonacci cubes, cube-connected cycles, and butterflies). Concerning computing $\mu(G)$, we provide approximation algorithms for hypercubes, Fibonacci cubes and cube-connected cycles, while we give an exact formula for butterflies. Concerning computing $\mu_t(G)$ (in the literature, already studied in hypercubes), whereas we obtain exact formulae for both cube-connected cycles and butterflies.

1. Introduction

Problems about sets of points in the Euclidean plane and their mutual visibility have been investigated for a long time. For example, in [18] Dudeney posed the famous *no-three-in-line* problem: finding the maximum number of points that can be placed in an $n \times n$ grid such that there are no three points on a line. Beyond the theoretical interest, solutions to these types of geometric/combinatorial problems have proved useful in recent times in the context of swarm robotics. The requirement is to define algorithms that allow autonomous mobile robots to change in a finite time their configuration in the plane so they can see each other (see, e.g., [16,30,32]).

Mutual visibility in graphs with respect to a set of vertices has been recently introduced and studied in [17] in terms of the existence of a shortest path between two vertices not containing a third vertex from such a set. The visibility property is then understood as a kind of non-existence of "obstacles" between the two vertices in the mentioned shortest path, which makes them "visible" to each other. Formally, let G be a connected and undirected graph, and $X \subseteq V(G)$ a subset of the vertices of G. Two vertices $x, y \in V(G)$ are

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X-*visible* if there exists a shortest x, y-path where no internal vertex belongs to X. X is a *mutual-visibility set* if its vertices are pairwise X-visible. The cardinality of a largest mutual-visibility set is the *mutual-visibility number* of G, and it is denoted by $\mu(G)$. Computing one of such largest sets, referred to as a μ-*set* of G, solves the so-called MUTUAL-VISIBILITY problem. In [17], it is also shown that computing $\mu(G)$ is an NP-complete problem, but there exist exact formulae for the mutual-visibility number of special graph classes like paths, cycles, blocks, cographs, grids, and in [11] also for distance-hereditary graphs. In the subsequent works [13] and [14], exact formulae have been derived also for both the Cartesian and the strong product of graphs. The contributions presented in those initial works showed several interesting connections with other mathematical contexts like the relationship existing between such a visibility problem and an instance of the very well-known Zarankiewicz (see [13, Corollary 3.7]) and Turán (see [15, Theorems 3.7 and 3.9]) problems. The mutual-visibility is also related to the general position problem in graphs (see, e.g., [23,27,31] and references therein). It is worth noting that this new definition of mutual visibility on graphs has already been applied in the context of swarm robotics where robots operate on environments modeled by graphs and where visibility is verified along shortest paths (cf. [3,7–9]).

In [14], a natural extension of the mutual-visibility has been proposed. Formally, $X \subseteq V(G)$ is a *total mutual-visibility set* of G if every two vertices x and y of G are X-visible. A largest total mutual-visibility set of G is a μ_t-set of G, and its cardinality is the *total mutual-visibility number* of G denoted as $\mu_t(G)$. Of course, $\mu_t(G) \leq \mu(G)$. In [12] it is shown that also computing $\mu_t(G)$ is an NP-complete problem.

The most recent works in the literature on mutual-visibility concern total mutual-visibility. This setting is clearly more restrictive, but it surprisingly turns out to become very useful when considering networks having some Cartesian properties in the vertex set, namely, those ones of product-like structures (cf. [14]). In [24], in fact, the total mutual-visibility number of Cartesian products is bounded and several exact results proved. Furthermore, a sufficient and necessary condition is provided for asserting when $\mu_t(G) = 0$. In [25], the authors gave several bounds for $\mu_t(G)$ in terms of the diameter, order and/or connected domination number of G. They also determined the exact value of the total mutual-visibility number of lexicographic products. The total mutual-visibility of Hamming graphs (and, as a byproduct, for hypercubes) is studied in [6]. Finally, the mutual and the total visibility problems are studied for graphs of diameter two in [15] showing bounds and/or closed formulae.

1.1. Results overview

In this work, we study the mutual-visibility and the total mutual-visibility of hypercube-like graphs. These graphs usually model interconnection networks which are widely used in parallel and distributed computing systems. They offer efficient communication and connectivity patterns that make them suitable for various applications, including parallel processing, supercomputing, and designing efficient interconnection topologies for computer systems. In these systems, two nodes that are X-visible can communicate in an efficient way, that is through shortest paths, and their messages can be maintained confidential: the exchanged messages do not pass through nodes in X.

A *hypercube* of dimension d is denoted as Q_d and consists of 2^d vertices, each labeled by a binary string with d bits. Two vertices are adjacent if and only if the two binary strings differ by exactly one bit.

A *Fibonacci cube* of dimension d is denoted as Γ_d and can be defined as the subgraph of Q_d obtained by removing all vertices whose labels contain two consecutive 1's. It consists of F_{d+2} vertices, with F_i being the i-th Fibonacci number. It models networks that maintain some of the properties of the hypercube with the advantage of sensibly decreasing the number of vertices. Furthermore, since they have close relationships with Fibonacci numbers, interesting properties arise.

A *cube-connected cycle* of dimension d is denoted as CCC_d and consists of $d \cdot 2^d$ vertices. It models networks that combine the properties of both hypercube graphs and cycle graphs. Informally, CCC_d can be obtained from Q_d by replacing each vertex by a cycle with d vertices.

A *butterfly* of dimension d is denoted as $BF(d)$ and consists of $(d+1) \cdot 2^d$ vertices. It is composed of interconnected stages of switches that resemble the wings of a butterfly. Each stage connects nodes in a structured way, allowing efficient communication between nodes at different levels. Butterfly networks are known for their logarithmic diameter and suitability for sorting and data exchange operations. For these classes of graphs, we provide the following results:

- For hypercubes, we first prove the bounds $\mu(Q_d) \geq \binom{d}{\lfloor \frac{d}{2} \rfloor} + \binom{d}{\lfloor \frac{d}{2} \rfloor + 3}$ and $\mu(Q_d) \leq 2^{d-1}$, and then we exploit such results to get an $O(\sqrt{d})$-approximation algorithm for computing $\mu(Q_d)$. Alternatively, for an n-vertex hypercube, the approximation ratio can be expressed as $O(\sqrt{\log n})$.
- For Fibonacci cubes, we first provide optimal solutions for $\mu(\Gamma_d)$ when $d \leq 6$, whereas for larger values of d, we prove $\mu(\Gamma_d) \leq F_d + F_{d-2}$ and $\mu(\Gamma_d) \geq \max_{0 \leq k \leq \lceil \frac{d}{2} \rceil} \binom{d-k+1}{k}$. Then, we obtain a $\frac{1}{\varphi}\left(\lceil \frac{d}{2} \rceil + 1\right)$-approximation algorithm (whose ratio can be expressed as $O(\log n)$ for an n-vertex Fibonacci cube).[1]
- For cube-connected cycles, the provided bounds are $\mu(CCC_d) \leq 3 \cdot 2^{d-2}$ and $\mu(CCC_d) \geq 2^{\lceil \frac{d}{2} \rceil}$. They give rise to a $3 \cdot 2^{\lfloor \frac{d}{2} \rfloor - 2}$-approximation algorithm (whose ratio can be expressed as $O(\sqrt{n})$ for an n-vertex cube-connected cycle). Concerning the total mutual-visibility, we prove that $\mu_t(CCC_d) = 0$.
- For butterflies, we provide exact formulae: $\mu(BF(d)) = 2^{d+1} - 2$ and $\mu_t(BF(d)) = 2^d$.

[1] With φ being the well-known golden ratio.

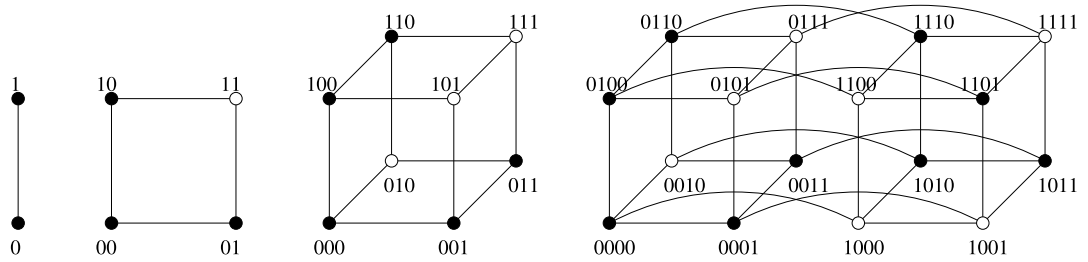


Fig. 1. A representation of the hypercube Q_d for $d = 1, 2, 3$ and 4. In each graph, the black vertices form a μ-set.

1.2. Outline

Concerning the organization of the paper, in the next section we provide all the necessary notation and preliminary concepts. The subsequent Sections 3, 4, 5, and 6 are specialized for presenting - in order - results about mutual visibility and total mutual visibility for hypercubes, Fibonacci cubes, cube-connected cycles, and butterflies. Concluding remarks are provided in Section 7.

2. Notation and preliminaries

In this work, we consider undirected and connected graphs. We use standard terminologies from [4], some of which are briefly reviewed here.

Given a graph G, $V(G)$ and $E(G)$ are used to denote its vertex set and its edge set, resp., and $n(G)$ is used to represent the size of $V(G)$. Whenever G is clearly known by the context, we denote $n(G)$ simply by n. If $u, v \in V(G)$ are adjacent, (u, v) represents the corresponding edge. If $X \subseteq V(G)$, then $G[X]$ denotes the subgraph of G *induced* by X, that is the maximal subgraph of G with vertex set X.

The usual notation for representing special graphs is adopted. K_n is the *complete graph* with n vertices, $n \geq 1$, that is the graph where each pair of distinct vertices are adjacent. P_n represents any *path* $(v_1, v_2, \ldots, v_n)$ with $n \geq 2$ distinct vertices where v_i is adjacent to v_j if $|i - j| = 1$. Vertices $v_2, \ldots, v_{n-1}$ are all *internal* of P_n. A *cycle* C_n, $n \geq 3$, is obtained from a path $(v_0, v_1, \ldots, v_{n-1})$ by adding an edge (v_0, v_{n-1}). Two vertices v_i and v_j are *consecutive* in C_n if $j = (i + 1) \bmod n$ or $i = (j + 1) \bmod n$. The *distance* function on a graph G is the usual shortest path distance.

Two graphs G and H are *isomorphic* if there exists a bijection $\varphi : V(G) \to V(H)$ such that $(u, v) \in E(G) \Leftrightarrow (\varphi(u), \varphi(v)) \in E(H)$ for all $u, v \in V(G)$. Such a bijection φ is called *isomorphism*. Given G and H, the *Cartesian product* $G \square H$ has the vertex set $V(G) \times V(H)$ and the edge set $E(G \square H) = \{((g, h), (g', h')) : (g, g') \in E(G)$ and $h = h'$, or, $g = g'$ and $(h, h') \in E(H)\}$.

Given a subgraph H of G, H is said to be *convex* if all shortest paths in G between vertices of H actually belong to H. Concerning convex subgraphs, we recall the following useful statement:

Lemma 1. [17] *Let H be a convex subgraph of a graph G, and X be a mutual-visibility set of G. Then, $X \cap V(H)$ is a mutual-visibility set of H.*

Finally, note that binary strings are used as vertex labels or components of vertex labels for the classes of graphs studied in this paper (hypercubes, Fibonacci cubes, cube-connected cycles, and butterflies). For a binary string $x = x_0 x_1 \cdots x_i \cdots x_{d-1}$ with d bits, position 0 corresponds to the leftmost bit, and position $d - 1$ to the rightmost bit. Sometimes, we interpret these strings as (binary) numbers. We use $x(i)$ to denote the binary string obtained from x by complementing the bit in position i. We will abbreviate "the vertex with label x" to "vertex x".

3. Hypercube

The d-dimensional hypercube Q_d is an undirected graph with vertex set $V(Q_d) = \{0, 1\}^d$, and two vertices are adjacent if and only if the two binary strings differ by exactly one bit, that is, the Hamming distance[2] of the two binary strings is 1. Hypercube Q_d can also be recursively defined in terms of the Cartesian product of two graphs as follows:

- $Q_1 = K_2$,
- $Q_d = Q_{d-1} \square K_2$, for $d \geq 2$.

This makes clear that Q_d can be also seen as formed by two subgraphs both isomorphic to Q_{d-1} (cf. Fig. 1).

[2] The Hamming distance between two strings of equal length is the number of positions at which the corresponding symbols are different.

Table 1
On the size of a mutual-visibility set X for Q_d as d varies. We found μ-sets (i.e., optimal solutions) for $d \leq 5$ and an upper bound for $\mu(Q_d)$ when $d \geq 6$. Note that the μ-set for Q_d, $d < 6$, is unique up to isomorphisms.

d	$n(Q_d)$	$\mu(Q_d)$
1	2	$\mu(Q_d) = 2$
2	4	$\mu(Q_d) = 3$
3	8	$\mu(Q_d) = 5$
4	16	$\mu(Q_d) = 9$
5	32	$\mu(Q_d) = 16$
$d \geq 6$	2^d	$\mu(Q_d) \leq 2^{d-1}$

3.1. An upper bound for $\mu(Q_d)$

Here, we first analyze specific optimal solutions for the MUTUAL-VISIBILITY problem in hypercubes of small dimension $d \leq 5$ and then we provide an upper bound holding for any $d \geq 6$.

First of all, we can claim the following lemma.

Lemma 2. *Each subgraph Q' of Q_d that is isomorphic to $Q_{d'}$, $d' < d$, is convex.*

Proof. Consider a hypercube Q' subgraph of Q_d that is isomorphic to $Q_{d'}$ for some $d' < d$. By considering two vertices x and y of Q', we have to show that there does not exist a shortest path between x and y passing through a vertex $z \in V(Q_d) \setminus V(Q')$. Since the distance between two vertices in a hypercube is governed by the Hamming distance, the bits that differ between the labels associated with x and y concern only Q'. In fact, there cannot be a shortest path that makes use of a vertex $z \in V(Q_d) \setminus V(Q')$ as, by construction, the corresponding bit leading to (the dimension of) z is different from the one in the labels of both x and y. □

Corollary 1. $\mu(Q_d) \leq 2\mu(Q_{d-1})$, $d \geq 2$.

Proof. The proof simply follows by recalling that Q_d can be obtained by the Cartesian product of $Q_{d-1} \square K_2$, for $d \geq 2$, i.e., by suitably connecting two hypercubes of dimension $d - 1$. Therefore, by Lemmas 1 and 2 the claim holds. □

We now consider all the dimensions $d \leq 5$, one by one, for which we can provide optimal solutions.

From results provided in [17], we know that $\mu(Q_1) = \mu(K_2) = 2$ and that $\mu(Q_2) = \mu(C_4) = 3$. For Q_3 (that contains 8 vertices) there exists exactly one μ-set of size 5, up to isomorphisms. By referring to Fig. 1, the optimal solution is provided by the set {000, 001, 100, 110, 011}. It can be observed that there does not exist any solution with more than 5 vertices. By contradiction, assume that X, with $|X| \geq 6$, is a μ-set for Q_3. Then, for each subcube Q' of Q_3 isomorphic to Q_2 we have $|V(Q') \cap X| < 4$ (in fact, Q' is a convex subgraph isomorphic to a cycle). Hence, the two copies of Q_2 must have 3 elements each in X. To avoid having a cycle C_4 whose elements are all in X, the two vertices not in X must be antipodal; but, this implies that there exists another pair of antipodal vertices that are in X which are not mutually visible.

For Q_4 (that contains 16 vertices), Corollary 1 implies that each μ-set can have at most 10 elements, i.e., 5 vertices per each subcube of dimension 3. Since the μ-set for Q_3 is unique up to isomorphisms, we have few possibilities to combine the two subcubes. By a computer-assisted exhaustive search, we have that there does not exist a solution with 10 vertices selected, whereas we can provide a mutual-visibility set with 9 vertices selected. By referring to Fig. 1, the optimal solution is provided by the set {0000, 0001, 0100, 0110, 0011, 1101, 1010, 1011, 1110}. Furthermore, the obtained solution is unique up to isomorphisms.

Concerning Q_5, since $\mu(Q_4) = 9$, by Corollary 1 we can obtain a mutual-visibility set with at most 18 vertices. Again, by means of a computer-assisted case-by-case analysis, we have that the optimal and unique (up to isomorphisms) solution is of size 16. The optimal solution is provided by the set {00000, 00001, 00100, 00110, 00011, 01101, 01010, 01011, 01110, 10101, 10111, 11000, 11001, 11100, 11110, 11011}.

For $d \geq 6$, we can state the next corollary that provides an upper bound to $\mu(Q_d)$. In particular, since we have shown that for Q_5 any μ-set contains 2^4 elements, i.e., exactly half of the vertices, then the next claim simply follows by Corollary 1:

Corollary 2. $\mu(Q_d) \leq 2^{d-1}$, $d \geq 5$.

Summarizing, in Table 1 we report all the obtained results concerning the upper bounds for $\mu(Q_d)$.

3.2. Lower bound and an approximation algorithm

Here, we first provide a lower bound for $\mu(Q_d)$. Then, we derive an approximation algorithm for the mutual-visibility problem in hypercubes.

Theorem 1. $\mu(Q_d) \geq \binom{d}{\lfloor \frac{d}{2} \rfloor} + \binom{d}{\lfloor \frac{d}{2} \rfloor + 3}$, $d \geq 1$.

Proof. Let X_p be the subset of $V(Q_d)$ containing all the vertices that are at distance p from the vertex labeled with all zeroes. By construction, and by the Hamming distance, the elements of X_p are all the vertices whose labels contain exactly p 1's. Hence, it is easy to find a shortest path between two of the selected vertices, say x and y that are at a distance j from each other. It suffices to detect the j differences among the labels associated with the two vertices. Then, by first replacing (one per step) the 1's present in x but not in y with 0's and then similarly replacing the 0's with 1's, equals to determine a shortest path between x and y. Such a path is shortest because at each step makes a change in the direction of the destination, i.e., its length is exactly j. Furthermore, along the chosen path, there are no vertices in X_p as the number of 1's is always smaller than p until the destination.

The number of vertices with a fixed number p of 1's is $\binom{d}{p}$, which is maximized for $p = \lfloor \frac{d}{2} \rfloor$.

For $d < 5$, the size of $X_{\lfloor \frac{d}{2} \rfloor}$ is $\binom{d}{\lfloor \frac{d}{2} \rfloor}$ and assumes the values $1, 2, 3, 6$ for $d = 1, 2, 3, 4$. These values are less than the corresponding optimal values of $\mu(Q_d)$ shown in Table 1. As in these cases $\binom{d}{\lfloor \frac{d}{2} \rfloor + 3} = 0$, the statement holds.

For $d \geq 5$, we can choose a larger set of vertices, still guaranteeing mutual visibility. Given a hypercube Q_d, we define a set X as follows:

- Let $p = \lfloor \frac{d}{2} \rfloor$;
- $X = X_p \cup X_{p+3}$.

Let $x \in X_p$ and $y \in X_{p+3}$. There are shortest paths between x and y whose internal vertices are labeled with $p+1$ and $p+2$ 1's only. Indeed, a shortest path from a vertex labeled with p 1's to one labeled with $p+3$ 1's can involve only internal vertices obtained by first changing two 0's in 1's and then alternating one 1 in 0 and one 0 in 1 until the last step. By the number of 1's in each label of the internal vertices we are guaranteed that x and y are mutually visible. If $x, y \in X_p$, to show the mutual visibility we can find a shortest path with vertices not in X as described above, whereas if $x, y \in X_{p+3}$ the vertices not in X in a shortest path between x and y, can be obtained by first replacing (one per step) the 0's present in x but not in y with 1's and then similarly replacing the 1's with 0's. □

An immediate consequence of the above theorem is the existence of an approximation algorithm for the mutual-visibility problem in the context of hypercubes, as stated by the next two corollaries.

Corollary 3. *There exists an algorithm for the* MUTUAL-VISIBILITY *problem on a hypercube Q_d providing a solution of size greater than* $\frac{2^d}{\sqrt{\frac{\pi}{2} d}}$.

Proof. For $d \leq 5$, we have already shown we can provide the optimal solution. For $d \geq 6$, we show that the procedure defined in the proof of Theorem 1 (for defining the μ-set X for Q_d) provides the requested algorithm.

The cardinality of X is equal to

$$|X| = \binom{d}{p} + \binom{d}{p+3} = \binom{d}{\lfloor \frac{d}{2} \rfloor} + \binom{d}{\lfloor \frac{d}{2} \rfloor + 3}.$$

By the Stirling's approximation [33], we have that, for *sufficiently large* values of d,

$$\binom{d}{\lfloor \frac{d}{2} \rfloor} \sim \frac{2^d}{\sqrt{\frac{\pi}{2} d}} \qquad \text{and consequently} \qquad |X| \sim \frac{2^{d+1}}{\sqrt{\frac{\pi}{2} d}}.$$

However, for $d \geq 6$, we can easily verify that:

$$|X| > \frac{2^d}{\sqrt{\frac{\pi}{2} d}},$$

since the inequality holds for $d = 6$, and from there on the gap between the two terms monotonically increases, then the statement holds. □

From Corollaries 2 and 3, we obtain

Corollary 4. *There exists an $O(\sqrt{d})$-approximation algorithm for the* MUTUAL-VISIBILITY *problem on a hypercube Q_d.*

Proof. By comparing the upper bound provided by Corollary 2 with the lower bound provided by Corollary 3, we have an approximation ratio of:

$$\frac{\mu(Q_d)}{|X|} \leq \frac{2^{d-1}\sqrt{\frac{\pi}{2}d}}{2^d} = O(\sqrt{d}). \quad \square$$

We remind that in a hypercube Q_d of n vertices, $n = 2^d$, i.e., $d = \log n$. Consequently, the approximation provided by the above theorem for an n-vertex hypercube can be expressed as $O(\sqrt{\log n})$.

3.3. Total mutual-visibility

In [6, Theorem 3], authors provide a lower bound for total mutual-visibility number of $K_s^{\square,r}$, where $K_s^{\square,r}$ denotes the Cartesian product of r copies of K_s. Their proof is based on a probabilistic approach similar to the idea used in the proof in [5, Section 4] for a famous hypergraph Turán-problem of Brown, Erdős, and Sós. The provided lower bound is

$$\mu_t(K_s^{\square,r}) \geq \frac{s^{r-2}}{r(r+1)}. \tag{1}$$

Since for $s = 2$ and $r = d$ the graph $K_s^{\square,r}$ corresponds to Q_d, the above result reduces to the following bound for hypercubes: $\mu_t(Q_d) \geq \frac{2^{d-2}}{d(d+1)}$. It is worth remarking that this bound is asymptotically much smaller than the bound $\mu(Q_d) > \frac{2^d}{\sqrt{\frac{\pi}{2}d}}$ provided in the proof of Corollary 3. In [6], the bound in Equation (1) is used to prove that $\mu_t(K_s^{\square,r}) = \Theta(s^{r-2})$.

4. Fibonacci cubes

Fibonacci cubes (denoted as Γ_d) have been introduced by Hsu in the seminal paper [20] in 1993 as a model for interconnection networks. It combines some properties of the Fibonacci numbers with those of the hypercube.

We provide an upper and a lower bound with respect to $\mu(\Gamma_d)$, and we propose an approximated solution. Before providing the formal definition of the Fibonacci cubes, we better revise about Fibonacci numbers.

4.1. Fibonacci numbers

Fibonacci numbers govern many aspects of certain natural growth phenomena as well as applications in mathematical sciences. Their definition has been provided in 1202 by Leonardo Pisano (alias Fibonacci) with the aim to describe "*How Many Pairs of Rabbits Are Created by One Pair in One Year?*". The sequence has been recursively defined as follows:

$$F_0 = 0; \quad F_1 = 1; \quad F_i = F_{i-1} + F_{i-2}, \tag{2}$$

and it goes like: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584,...

Fibonacci numbers show interesting relationships to other families of well-known sequences, such as the binomial coefficients (see Fig. 2.left and [2,28]):

$$F_i = \sum_{k=0}^{\lceil \frac{i}{2} \rceil} \binom{i-k-1}{k}, \tag{3}$$

or even to the well-known golden ratio (see Fig. 2.right):

$$\varphi = \frac{1+\sqrt{5}}{2} = \lim_{i\to\infty} \frac{F_{i+1}}{F_i}. \tag{4}$$

The golden ratio can also be used to define Fibonacci numbers according to the Binet's formula (see, e.g., [21]):

$$F_i = \frac{1}{\sqrt{5}}(\varphi^i - (-\varphi)^{-i}). \tag{5}$$

A natural question in the fields of parallel and distributed computing is whether it is possible to exploit Fibonacci properties when designing computer network architectures.

4.2. Definition of Fibonacci cubes

A Fibonacci cube Γ_d of dimension d can be defined as the subgraph of a hypercube Q_d obtained by removing from Q_d all the vertices associated with a label containing the substring '11'. This simple definition actually hides why Fibonacci cubes have such a name. To this respect, a more explicit definition may help, see Figs. 3 and 4.

A Fibonacci cube Γ_1 of dimension $d = 1$ is just a pair of vertices labeled '0' and '1', respectively, connected by an edge.

A Fibonacci cube Γ_2 of dimension $d = 2$ is obtained from Γ_1 by prefixing one further bit at '0' to its labels and adding one vertex labeled '10' connected to '00'.

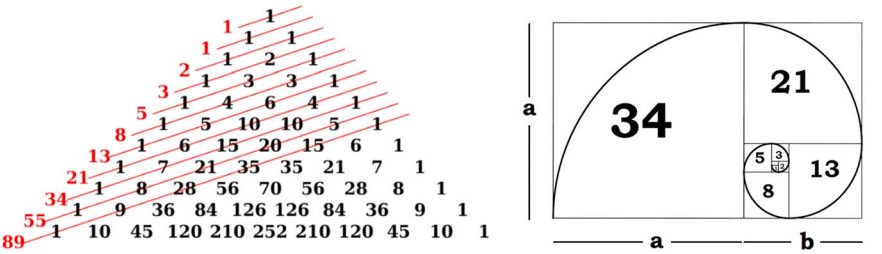


Fig. 2. (*left*): Fibonacci numbers calculated from the diagonals of the Tartaglia's triangle, i.e., the binomial coefficient; (*right*): The relationship between Fibonacci numbers and the golden ratio $\varphi = \frac{a}{b} = \frac{a+b}{a}$.

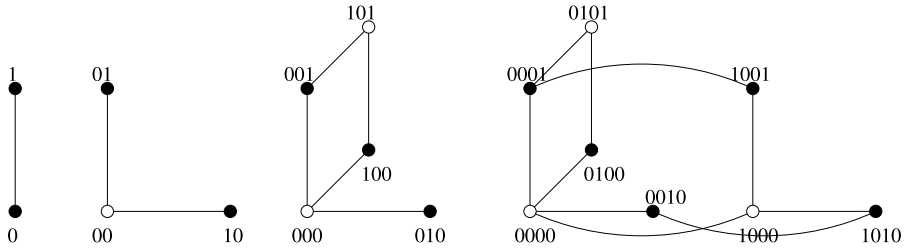


Fig. 3. A representation of the Fibonacci cubes Γ_d for $d = 1$, 2, 3 and 4. In each graph, the black vertices form a μ-set.

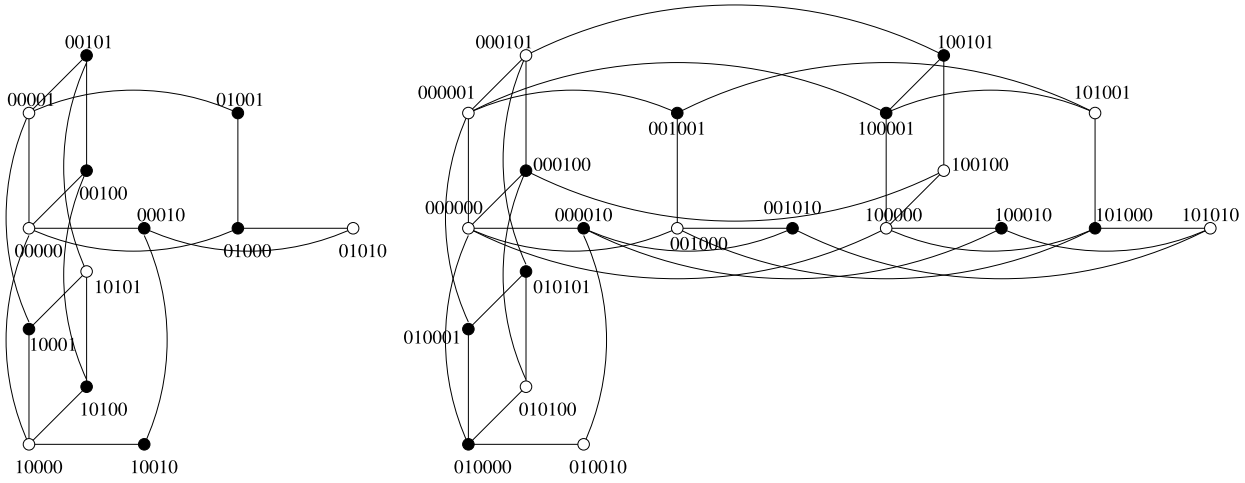


Fig. 4. A representation of the Fibonacci cubes Γ_d for $d = 5$ and 6. In each graph, the black vertices form a μ-set.

A Fibonacci cube Γ_d of dimension $d > 2$ is obtained by considering two Fibonacci cubes, Γ_{d-1} and Γ_{d-2} where the labels of Γ_{d-2} have been modified by prefixing one further bit, set to '0'; then, vertices with the same labels among Γ_{d-1} and Γ_{d-2} are connected; finally, the label of each vertex in Γ_d is modified by prefixing the labels of the vertices composing the subgraph Γ_{d-1} (resp. Γ_{d-2}) with a bit '0' (resp., '1').

By construction, each label of a Fibonacci cube Γ_d, $d > 0$, is composed by d bits. As the number of vertices composing Γ_1 is 2 and that for Γ_2 is 3, then $|V(\Gamma_d)|$, $d > 2$, is F_{d+2}, whereas $|E(\Gamma_d)| = \frac{1}{5}(dF_{d+1} + 2(d+1)F_d)$, see [19].

As shown in [22], Fibonacci cubes preserve the Hamming distance property among vertices as for the hypercubes but significantly reduce the number of vertices to that respect.

4.3. An upper bound for $\mu(\Gamma_d)$

Here, we first analyze specific optimal solutions for the MUTUAL-VISIBILITY problem in Fibonacci cubes of small dimension $d \leq 6$ and then we provide an upper bound holding for any $d \geq 6$.

First of all, we can claim the following lemma.

Lemma 3. $\mu(\Gamma_d) \leq \mu(\Gamma_{d-1}) + \mu(\Gamma_{d-2})$, $d > 2$.

Table 2
On the size of a mutual-visibility set X for Γ_d as d varies. We found μ-sets (i.e., optimal solutions) for $d \leq 6$ and an upper bound for $\mu(\Gamma_d)$ when $d \geq 7$.

d	$n(\Gamma_d)$	$\mu(\Gamma_d)$
1	2	2
2	3	2
3	5	3
4	8	5
5	13	8
6	21	11
$d \geq 7$	F_{d+2}	$\mu(\Gamma_d) \leq F_d + F_{d-2}$

Proof. By construction, we have that a Fibonacci cube Γ_d of dimension d can be obtained by suitably connecting two Fibonacci cubes Γ'_{d-2} and Γ''_{d-1} of dimensions $d-2$ and $d-1$, respectively. Without loss of generality, we consider two vertices x and y belonging to Γ'.[3] We have to show that there does not exist a shortest path between x and y passing through a vertex $z \in V(\Gamma_d) \setminus V(\Gamma'')$.

Since the distance between two vertices in a Fibonacci cube is governed by the Hamming distance, the bits that differ between the labels associated with x and y concern only Γ'. In fact, there cannot be a shortest path that makes use of a vertex $z \in V(\Gamma_d) \setminus V(\Gamma'')$ as, by construction, the corresponding bit leading to (the dimension of) z is different from the one in the labels of both x and y. □

We now consider all the dimensions $d \leq 6$, one by one, for which we can provide optimal solutions, see Figs. 3 and 4.

From results provided in [17], we know that $\mu(\Gamma_1) = \mu(K_2) = 2$ and that $\mu(\Gamma_2) = \mu(P_3) = 2$. For Γ_3 (that contains 5 vertices) there exist μ-sets of size 3. By referring to Fig. 3, the optimal solution is provided, for instance, by the set {010, 001, 100}.

For Γ_4 (that contains 8 vertices), Lemma 3 implies that $\mu(\Gamma_4) \leq 5$. Actually we can provide a mutual-visibility set with 5 vertices selected which is then optimal. By referring to Fig. 3, the optimal solution is provided by the set {0001, 0001, 0100, 1010, 1001}.

Concerning Γ_5 (that contains 13 vertices), since $\mu(\Gamma_4) = 5$ $\mu(\Gamma_3) = 3$, by Lemma 3 we may obtain a mutual-visibility set with at most 8 vertices. By means of a computer-assisted case-by-case analysis, we have that the optimal solution is of size 8, and it can be provided by the set {00010, 00100, 00101, 01000, 01001, 10001, 10010, 10100}

Also for Γ_6 (that contains 21 vertices) we can obtain an optimal mutual-visibility set. This is composed of 11 vertices: {000010, 000100, 001001, 001010, 010000, 010001, 010101, 100001, 100010, 100101, 101000}. It is worth noting that $\mu(\Gamma_6)$ turns out to be smaller than $\mu(\Gamma_5) + \mu(\Gamma_4) = 13$.

For $d \geq 7$, we can state the next corollary that provides an upper bound to $\mu(\Gamma_d)$.

Corollary 5. $\mu(\Gamma_d) \leq F_d + F_{d-2}$, $d \geq 6$.

Proof. We prove the statement by induction on d. For $d = 6$ we know that $\mu(\Gamma_6) = 11 \leq F_6 + F_4$. Assume the claim holds for any $d' < d$, we show it for d. By construction and by Lemma 3, $\mu(\Gamma_d) \leq \mu(\Gamma_{d-1}) + \mu(\Gamma_{d-2})$. By induction hypothesis, $\mu(\Gamma_{d-1}) + \mu(\Gamma_{d-2}) \leq F_{d-1} + F_{d-3} + F_{d-2} + F_{d-4} = F_d + F_{d-2}$, hence the claim holds. □

Summarizing, in Table 2 we report all the obtained results concerning the optimal solutions and the upper bounds for $\mu(\Gamma_d)$.

4.4. Lower bound and approximation algorithm

Here, we first provide a lower bound for $\mu(\Gamma_d)$ and then we derive an approximation algorithm for the mutual-visibility problem in the class of Fibonacci cubes.

Theorem 2. $\mu(\Gamma_d) \geq \max_{0 \leq k \leq \lceil \frac{d}{2} \rceil} \binom{d-k+1}{k}$, $d \geq 1$.

Proof. Let X_k be the subset of $V(\Gamma_d)$ containing all the vertices that are at distance k from the vertex labeled with all zeroes. By construction, and by the Hamming distance, the elements of X_k are all the vertices whose labels contain exactly k 1's. Hence, it is easy to find a shortest path between two of the selected vertices, say x and y that are at a distance j from each other. It suffices to detect the j differences among the labels associated with the two vertices. Then, by first replacing (one per step) the 1's present in x but not in y with 0's and then similarly replacing the 0's with 1's, equals to determine a shortest path between x and y. Such a path is shortest because at each step makes a change in the direction of the destination, i.e., its length is exactly j (the Hamming distance). Furthermore, along the chosen path, there are no vertices in X_k as the number of 1's is always smaller than k until the destination.

The number of vertices with a fixed number k of 1's in a Fibonacci cube of dimension d is $\binom{d-k+1}{k}$ (see [2]). Since we are looking for a lower bound of $\mu(d)$, we can maximize the obtained binomial coefficient, hence obtaining the claim. □

[3] The case where x and y belong both to Γ'' would be similar.

An immediate consequence of the above theorem is the existence of an approximation algorithm for the mutual-visibility problem in the context of Fibonacci cubes, as stated by the next two corollaries.

From Corollary 5 and Theorem 2, we obtain

Corollary 6. *There exists an* $\frac{1}{\varphi}\left(\lceil\frac{d}{2}\rceil+1\right)$*-approximation algorithm for the* MUTUAL-VISIBILITY *problem on a Fibonacci cube* Γ_d*, for any* $d \geq 1$.

Proof. Concerning $d \leq 6$, we have shown optimal solutions. For $d \geq 7$, we can compare the upper bound provided by Corollary 5 with the lower bound obtained by means of the set X_k provided in the proof of Theorem 2. By exploiting the relationship of Fibonacci numbers with the binomial coefficient as shown in Equation (3), we obtain an approximation algorithm that guarantees a ratio of:

$$\frac{\mu(\Gamma_d)}{|X_k|} \leq \frac{F_d + F_{d-2}}{\max\limits_{0 \leq k \leq \lceil\frac{d}{2}\rceil} \binom{d-k+1}{k}} = \frac{F_d + F_{d-2}}{F_{d+2}} \frac{F_{d+2}}{\max\limits_{0 \leq k \leq \lceil\frac{d}{2}\rceil} \binom{d-k+1}{k}} <$$

$$< \frac{1}{\varphi} \frac{\sum_{k=0}^{\lceil\frac{d}{2}\rceil} \binom{d-k+1}{k}}{\max\limits_{0 \leq k \leq \lceil\frac{d}{2}\rceil} \binom{d-k+1}{k}} \leq \frac{1}{\varphi}\left(\left\lceil\frac{d}{2}\right\rceil+1\right),$$

where the last step comes by the fact that the summation at the numerator is composed by $\left\lceil\frac{d}{2}\right\rceil+1$ elements, each of which admits a ratio with the maximum of at most 1. □

We remind that in a Fibonacci cube Γ_d of n vertices we have $n = F_{d+2}$. By the Binet's formula shown in Equation (5), we obtain $d = O(\log n)$. Consequently, the approximation provided by the above theorem for an n-vertex Fibonacci cube can be expressed as $O(\log n)$.

5. Cube-connected cycles

A cube-connected cycle of order $d \geq 3$ (denoted CCC_d) can be defined as an undirected graph formed by a set of $d \cdot 2^d$ vertices labeled $[\ell, x]$, where ℓ is an integer between 0 and $d-1$ and x is a binary string of length d.

Two vertices $[\ell, x]$ and $[\ell', x']$ are adjacent if and only if:

(i) either $x = x'$ and $(\ell - \ell') \equiv 1 \pmod d$,

(ii) or $\ell = \ell'$ and $x' = x(\ell)$. In this last case, x and x' differ exactly for the bit in position ℓ.

The edges satisfying condition (i) are referred to as *cycle edges*, those satisfying condition (ii) are instead referred to as the *hypercube edges*.

Note that, the removal of all hypercube edges produces a graph with 2^d components, each of which is isomorphic to a cycle C_d. For this reason, each cycle C_d in CCC_d which does not include any hypercube edge is referred to as a *supervertex* of the (embedded) hypercube.

Furthermore, contracting all the cycle edges in CCC_d will produce a graph with 2^d vertices isomorphic to a hypercube Q_d. Consequently, the smallest CCC_d is in fact defined for $d = 3$.

Natural routing in CCC_d Let $u = [\ell, x]$ and $v = [\ell', x']$ be two vertices of a CCC_d, and $H = \{p_0, \ldots, p_{h-1}\}$ be the positions in which the binary strings representing x and x' differ. By [34], the distance between u and v equals $h + k$, with k being the number of edges in the shortest walk on a cycle C_d starting at index ℓ and ending at index ℓ' which includes a visit to every vertex with an index in the set H. Then, a *natural routing* from u to v is to traverse the shortest walk while traversing the hypercube edges each time a vertex with an index in the set H is met for the first time.

Lemma 4. *Each subgraph* Q' *composed by the supervertices of a* CCC_d*, isomorphic to* $Q_{d'}$*,* $d' < d$*, induces a subgraph of* CCC_d *that is convex.*

Proof. By Lemma 1, we have that the shortest paths between vertices in $Q_{d'}$ are confined within $Q_{d'}$ itself. Consider now the subgraph obtained by restoring each supervertex with the corresponding cycle C_d along with its edges within $Q_{d'}$. By the natural routing, any shortest path among two vertices u and v of the defined subgraph makes use of the hypercube edges only from vertices with index in the set H. Since all such edges are included in the obtained subgraph, the claim holds. □

Lemma 5. $\mu(CCC_d) \leq 3 \cdot 2^{d-2}$*,* $d \geq 3$.

Proof. The proof follows by first observing that for $d = 3$, the optimal solution provides $\mu(CCC_3) = 3 \cdot 2^{3-2} = 6$. In fact, we have obtained this result by a computer-assisted exhaustive search, and the optimal solution found is shown in Fig. 5. Then, by Lemma 4, we have that, as d increases, $\mu(CCC_d)$ can double at most. □

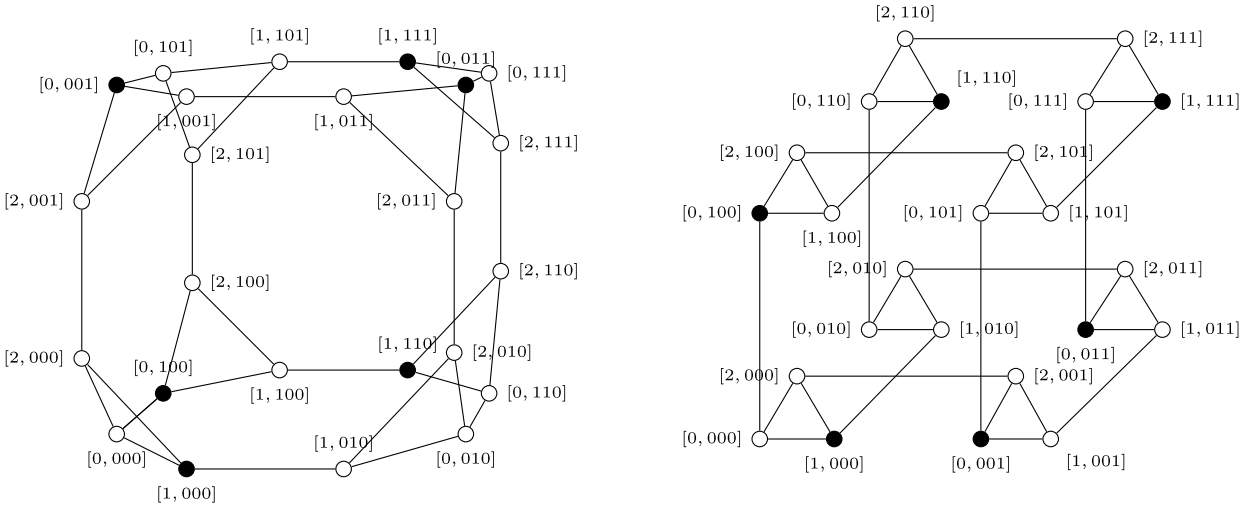


Fig. 5. *(left)* A representation of a CCC_3 graph, black vertices represent a μ-set for CCC_3. *(right)* An alternative representation of a CCC_3.

Theorem 3. $\mu(CCC_d) \geq 2^{\lceil \frac{d}{2} \rceil}$, $d \geq 3$.

Proof. Given a cube connected cycle CCC_d, $d \geq 3$, we define a set X according to the following procedure:

– Insert into X any vertex $v = [0, x]$ with a number of 1's bounded by $\lceil \frac{d}{2} \rceil$ in the $\lceil \frac{d}{2} \rceil$ most significative bits composing x.

In doing so, and reminding the natural routing in a CCC_d, we have that the shortest path between two vertices v_1 and v_2 belonging to X is well defined by considering the shortest walk in C_d that from 0 reaches p_{h-1} and then comes back to 0 either proceeding backward or forward, it depends on the distances. The choice of X guarantees to traverse at most d cycle edges, i.e., an entire C_d at most, or just a portion back and forth. Since all vertices in X have their first coordinate 0, all the traversed edges within any encountered C_d as well as all hypercube edges included in the shortest path never meet other vertices in X.

Summarizing, we have that any two vertices $[0, x]$ and $[0, y]$ in X are mutually visible since the shortest path forced by the chosen set X ensures to never encounter a vertex $[0, z]$ for any $z \notin \{x, y\}$. By construction, $|X| = 2^{\lceil \frac{d}{2} \rceil}$. □

From Lemma 5 and Theorem 3 we obtain

Corollary 7. *There exists a $3 \cdot 2^{\lfloor \frac{d}{2} \rfloor - 2}$-approximation algorithm for the MUTUAL-VISIBILITY problem on a cube-connected cycle CCC_d.*

Proof. By Lemma 5 and Theorem 3, we obtain the claimed approximation ratio for the algorithm provided in the proof of Lemma 3:

$$\frac{\mu(CCC_d)}{|X|} \leq \frac{3 \cdot 2^{d-2}}{2^{\lceil \frac{d}{2} \rceil}} = 3 \cdot 2^{\lfloor \frac{d}{2} \rfloor - 2}. \quad \square$$

We remind that in a cube-connected cycle CCC_d of n vertices, $n = d \cdot 2^d$, i.e., $d = \log n - \log d = \log n - o(\log n)$. Consequently, the approximation provided by the above theorem for an n-vertex cube-connected cycle can be expressed as $O(\sqrt{n})$.

5.1. Total mutual visibility

In [24], a characterization of all graphs having total mutual-visibility zero is provided. The characterization is based on the notion of *bypass vertex*: a vertex u of G is a bypass vertex if u is not the middle vertex of a convex P_3 of G. Let $\text{bp}(G)$ be the number of bypass vertices of G.

Theorem 4. [24, Theorem 3.3] *Let G be a graph with $n(G) \geq 2$. Then $\mu_t(G) = 0$ if and only if $\text{bp}(G) = 0$.*

This result directly provides the value of $\mu_t(CCC_d)$.

Corollary 8. $\mu_t(CCC_d) = 0$, $d \geq 3$.

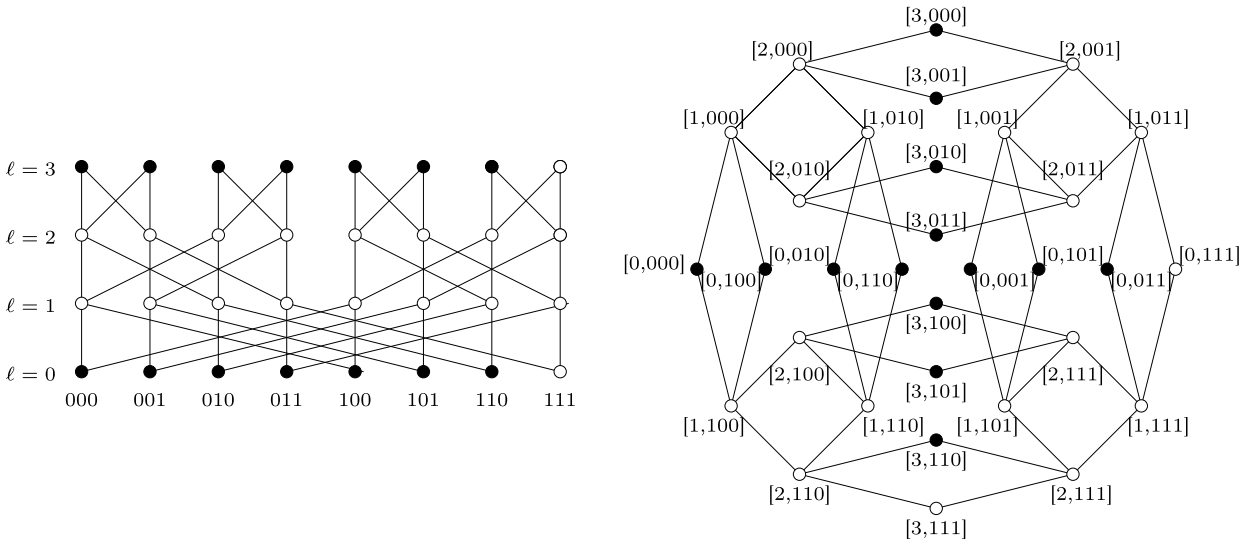


Fig. 6. Normal representation and diamond representation of $BF(3)$, black vertices represent a μ-set for $BF(3)$.

Proof. Let u be a vertex of CCC_d and consider a path P_3 given by (v', u, v''), where v' and v'' belong to different cycles of the graph. It can be observed that such a path is convex. Since it is well-known that CCC_d is a vertex-transitive graph, then each vertex u is not a bypass vertex and hence $\mathrm{bp}(CCC_d) = 0$. □

6. Butterfly

A d-dimensional butterfly $BF(d)$ is an undirected graph with vertices $[\ell, c]$, where $\ell \in \{0, 1, \ldots, d\}$ is the *level* and $c \in \{0,1\}^d$ is the *column*. The vertices $[\ell, c]$ and $[\ell', c']$ are adjacent if $|\ell - \ell'| = 1$, and either $c = c'$ or c and c' differ precisely in the ℓ'-th bit. $BF(d)$ has $d+1$ levels with 2^d vertices at each level, which gives $(d+1)\cdot 2^d$ vertices in total. The vertices at levels 0 and d have degree 2 whereas the rest have degree 4. $BF(d)$ has two standard graphical representations, namely normal and diamond representations (see Fig. 6). For further details one may refer to [26].

Some additional notation is required: let $A_i = \{[\ell, i] \mid \ell \in \{0, 1, \ldots, d\}\}$ be the vertex set forming the i-th column of the butterfly, and let $L_j = \{[j, c] \mid c \in \{0,1\}^d\}$, be the vertex set forming its j-th level. Note that, $BF(d)$ can be partitioned into two copies of $BF(d-1)$ (that we denote as $BF'(d-1)$ and $BF''(d-1)$) and L_0 (cf. Fig. 7). Notice that both $BF'(d-1)$ and $BF''(d-1)$ are convex subgraphs of $BF(d)$.

Natural routing in $BF(d)$ Consider the case in which starting from a vertex $[0, i]$, it is necessary to reach a vertex $[d, j]$ along a shortest path. A *natural routing* for such a task simply requires comparing the corresponding bits of i and j starting from the leftmost: if they coincide, the edge along the current column is traversed, otherwise, the edge for changing column is used. Symmetrically, a routing from $[d, i]$ toward $[0, j]$ can be obtained in a similar way but comparing the corresponding bits of i and j starting from the rightmost. This leads to the following useful property:

$\mathcal{P}$: Let $u = [\ell', i]$ and $v = [\ell'', j]$ be two vertices, $\ell_{\min} = \min\{\ell', \ell''\}$ and $\ell_{\max} = \max\{\ell', \ell''\}$. Each shortest path between u and v is comprised between levels $\min\{k' - 1, \ell_{\min}\}$ and $\max\{k'', \ell_{\max}\}$, where k' and k'' are the positions of the first and last bit, resp., in which i and j differ.

As special cases for this property we get: ($\mathcal{P}_1$) if from $[0, i]$ it is necessary to reach $[0, j]$, then a shortest path reaches level k iff the k-th bits of i and j differ; ($\mathcal{P}_2$) if from $[d, i]$ it is necessary to reach $[d, j]$, then a shortest path reaches level $d - k$ iff the $(d-k)$-th bits of i and j differ.

Lemma 6. *Let X be a mutual-visibility set of $BF(d)$. Then, $|X \cap A_i| \leq 2$ for each $i \in \{0, 1, \ldots, d\}$.*

Proof. According to Lemma 1, since A_i is a convex subgraph, A_i is isomorphic to a path graph, which mutual-visibility number is two. □

Lemma 7. *$X = (L_0 \cup L_d) \setminus \{[0, 111\ldots1], [d, 111\ldots1]\}$ is a mutual-visibility set of $BF(d)$.*

Proof. Consider $u = [0, i]$ and $v = [d, j]$. A shortest path that makes u and v X-visible is given by the natural routing from u to v.

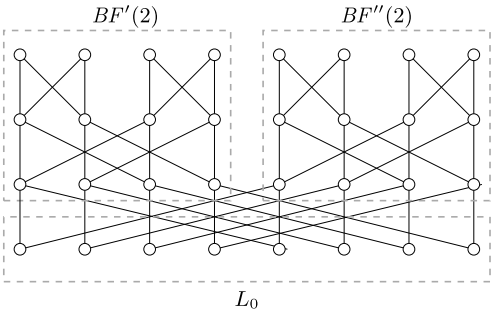


Fig. 7. Partition of $BF(3)$ into $BF'(2)$, $BF''(2)$, and L_0.

Consider now $u = [0, i]$ and $v = [0, j]$. According to Property $\mathcal{P}_1$, if the last bit of i and j does not differ, then there exists a shortest path whose interior vertices belong all at levels $1, 2, \ldots, d-1$, and this property trivially makes the two vertices X-visible. If the last bit of i and j differs, then their distance is $2d$ and a shortest path that makes the two vertices X-visible can be defined by composing the natural routing from u to $[d, 111\ldots1]$ with the natural routing from v to $[d, 111\ldots1]$.

Consider the last case in which $u = [d, i]$ and $v = [d, j]$. According to Property $\mathcal{P}_2$, if the first bit of i and j does not differ, then there exists a shortest path whose interior vertices belong all at levels $1, 2, \ldots, d-1$, and this property trivially makes the two vertices X-visible. If the first bit of i and j differs, then their distance is $2d$ and a shortest path that makes the two vertices X-visible can be defined by composing the natural routing from $[0, 111\ldots1]$ to u with that from $[0, 111\ldots1]$ to v. □

Lemma 8. $\mu(BF(d)) \leq 2^{d+1} - 2,\ d \geq 2.$

Proof. By contradiction, assume the statement is false, and let X be a mutual-visibility set of $BF(d)$. We can restrict the analysis to the case $|X| = 2^{d+1} - 1$ because, if the size is 2^{d+1}, then the removal of one element from X would still produce a mutual-visibility set. According to the cardinality of X and Lemma 6, we get

$$|X \cap A_\tau| = 1 \text{ for some } \tau \in \{0,1\}^d, \tag{6}$$

and

$$|X \cap A_i| = 2 \text{ for each } i \in \{0,1\}^d \setminus \{\tau\}. \tag{7}$$

If A_i shares two elements with X, we denote them as $[X_{\max}, i]$ and $[X_{\min}, i]$, where $[X_{\max}, i]$ ($[X_{\min}, i]$, resp.) is the element in $|X \cap A_i|$ at the largest (smallest, resp.) level. If A_i shares just one element with X, then $[X_{\min}, i] = [X_{\max}, i]$.

Case 1: $|X \cap L_0| \geq 2$**.** Let A_i be a column such that $[X_{\min}, i] = [0, i]$. Without loss of generality, assume that column A_i intersects $BF'(d-1)$. Let $[1, j]$ be the neighbor of $[0, i]$ belonging to $BF''(d-1)$. Notice that $|X \cap L_0| \geq 2$ allows us to assume $[X_{\min}, j] \neq [X_{\max}, j]$ (in fact, if $j = \tau$ we can start this analysis from another column $A_{i'}$ such that $[X_{\min}, i'] = [0, i']$).

It can be easily observed that if the level of $[X_{\min}, j]$ is greater or equal to 1, then $[X_{\min}, i]$ and $[X_{\max}, j]$ are not X-visible since they are obstructed by $[X_{\min}, j]$. As a consequence, we get that $[X_{\min}, j] = [0, j]$. By repeating the previous arguments, from $[X_{\min}, j] = [0, j]$ we get that there exists $[X_{\min}, k]$, for some $k \notin \{i, j\}$, that is located at level 0. Iteratively, we get that

$$[X_{\min}, i] = [0, i] \text{ for each } i \in \{0,1\}^d \setminus \{\tau\}. \tag{8}$$

If also $[X_{\min}, \tau] = [0, \tau]$ holds, then the entire level 0 of $BF(d)$ belongs to X. In this case, by selecting a vertex $[X_{\max}, i']$ from $BF'(d-1)$ and a vertex $[X_{\max}, i'']$ from $BF''(d-1)$ we easily get that each shortest path between them passes through a vertex at level 0. This implies that $[X_{\max}, i']$ and $[X_{\max}, i'']$ are not X-visible.

If $[X_{\min}, \tau] \neq [0, \tau]$, then, without loss of generality, we assume that A_τ intersects $BF''(d-1)$. In this case, in order to guarantee the mutual-visibility property of X, each shortest path from a vertex of X belonging to $BF''(d-1)$ to a vertex of X belonging to $BF'(d-1)$ must pass through the edge $([0, \tau], [1, t])$, where A_t is a column intersecting $BF'(d-1)$. In order to reach from such an edge each element of X located in $BF'(d-1)$, all such elements to be reached must necessarily be at level d.

Observe that $[X_{\min}, \tau] \neq [0, \tau]$ implies the membership of $[X_{\min}, \tau]$ to $BF''(d-1)$. Since $BF''(d-1)$ is a convex subgraph, it can be noticed that in order to guarantee the mutual-visibility property of X all the vertices of X located in $BF''(d-1)$ must be at level d. Hence we get

$$[X_{\max}, i] = [d, i] \text{ for each } i \in \{0,1\}^d. \tag{9}$$

According to Eq. (9), select now two elements of X that are both at level 0 and at adjacent columns. These vertices have distance $2d$ and each shortest path connecting them passes through a vertex at level d. Since by Eq. (9) each vertex at level d is in X, then we get that the two selected vertices at level 0 are not X-visible.

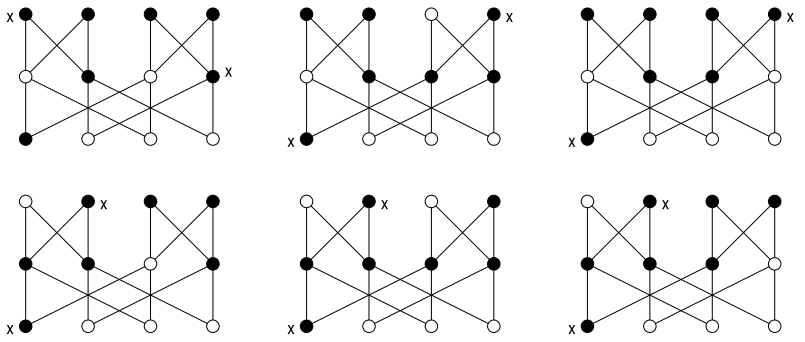


Fig. 8. Visualization of all the butterflies $BF(2)$, up to isomorphisms, defined in the proof of Lemma 8. For each graph, black vertices form a subset to be tested against mutual visibility, and the pair of vertices marked with 'x' are not mutually visible.

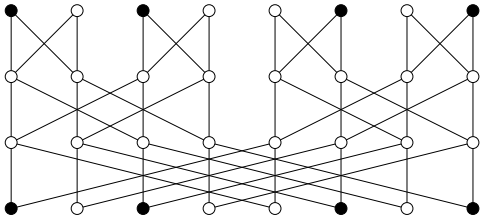


Fig. 9. Visualization of $L'_0 \cup L'_d$ as computed in $BF(3)$.

Case 2: $|X \cap L_0| \leq 1$**.** Let X' (X'', respectively) be the set containing all vertices of X that are contained in $BF'(d-1)$ ($BF''(d-1)$, respectively). According to Lemma 1 and since both $BF'(d-1)$ and $BF''(d-1)$ are convex subgraphs, then X' is a mutual-visibility set of $BF'(d-1)$ and X'' is a mutual-visibility set of $BF''(d-1)$. Since the cardinality of both X' and X'' is at least $2^d - 1$, then we can recursively apply this proof to both the subgraphs. Then, either Case 1 will eventually apply, or we end the recursion with the terminal situation given by a subgraph isomorphic to $BF(2)$ that shares with the original set X seven or eight elements, at most two of such elements per column, and at most one element at the lower level.

If $BF(2)$ shares 8 vertices with X, it is easy to observe that the presence of at most one vertex of X at the lower level implies the existence of a subgraph isomorphic to $BF(1)$ with its 4 vertices all belonging to X. Since $BF(1)$ is isomorphic to a cycle C_4 and $\mu(C_4) = 3$, then we get a contradiction for Lemma 1.

If $BF(2)$ shares 7 vertices with X, then there are only few possible configurations for $BF(2)$ that do not contain a subgraph $BF(1)$ with its 4 vertices all in X. Fig. 8 shows all the configurations that, up to isomorphisms, match all such conditions. In each case, we get that there are 2 vertices that are not in mutual visibility. Again, this contradicts Lemma 1. □

Since the mutual-visibility set provided by Lemma 7 contains $2^{d+1} - 2$ elements, then from Lemma 8 we have:

Theorem 5. $\mu(BF(d)) = 2^{d+1} - 2,\ d \geq 2.$

6.1. Total mutual-visibility

Given $BF(d)$, let L'_0 (L'_d, resp.) be the subset of L_0 (L_d, resp.) containing all vertices $[0,i]$ ($[d,i]$, resp.) fulfilling one of the following conditions, where i is interpreted as a binary number:

- i is odd and $i \leq 2^{d-1}$,
- i is even and $i > 2^{d-1}$.

Fig. 9 shows $L'_0 \cup L'_d$ in $BF(3)$.

Lemma 9. *$X = L'_0 \cup L'_d$ is a total mutual-visibility set of $BF(d)$, $d \geq 1$.*

Proof. Let u and v be two distinct vertices of $BF(d)$. From the proof of Lemma 7 we easily derive that u and v are X-visible in each of the following three cases: (1) $u = [0,i]$ and $v = [0,j]$, (2) $u = [0,i]$ and $v = [d,j]$, and (3) $u = [d,i]$ and $v = [d,j]$. Then, it remains to consider three additional situations.

(4) $u = [0, i]$ and $v = [\ell, j]$, with $\ell \notin \{0, d\}$. If $d(u, v) \leq d$, then u and v are X-visible. Otherwise there is a u, v-shortest path passing through $u' = [d, j]$. If u' is not occupied by a robot, then u and v are X-visible, otherwise there exists $u'' = [d, j']$, with $d(u', u'') = 2$, that is not occupied such that u'' belongs to a u, v-shortest path, therefore u and v are X-visible.
(5) $u = [d, i]$ and $v = [\ell, j]$, with $\ell \notin \{0, d\}$. It is symmetric to the previous case.
(6) $u = [\ell', i]$ and $v = [\ell'', j]$, with $\ell', \ell'' \notin \{0, d\}$. There are some subcases.
   (a) $i = j$. The statement trivially holds since a u, v-shortest path not passing through vertices of X is contained in column A_i.
   (b) $i \neq j$, and u and v are located on the same sub-butterfly. Without loss of generality, assume that both u and v are in $BF'(d-1)$ and $i < j$. Since u and v are in the same sub-butterfly, each u, v-shortest path does not touch level 0 of $BF(d)$.
   If i and j are both even, then, according to Property $\mathcal{P}$, we get that each u, v-shortest path does not touch level d of $BF(d)$ unless u or v are at that level. Since u and v are not at level d by hypothesis, it follows that u and v are X-visible. When both i and j are odd, their binary strings do not differ in the last position, and hence from Property $\mathcal{P}$ we still get that each u, v-shortest path does not touch level d; again, this guarantees the mutual-visibility between the two vertices.
   If i is odd and j is even, then, according to Property $\mathcal{P}$, we get that each u, v-shortest path must touch level d of $BF(d)$, regardless of the level of u an v; in this case, it can be observed that such a shortest path can pass through the vertex $[d, i]$. Conversely, if i is even and j is odd, the requested shortest path can pass through either $[d, i+1]$ or $[d, j]$.
   (c) $i \neq j$, and u and v are located on different sub-butterflies. Without loss of generality, assume u in $BF'(d-1)$ and v in $BF''(d-1)$. If i is odd, then it can be easily observed that a shortest path from u to v can be traversed by first reaching the vertex $[1, j]$ and then $[0, j']$ in the other sub-butterfly (note that this vertex is not in X by definition). Now, the remaining part of the u, v-shortest path can be identified as in the case (4) above. If i is even, the requested shortest path can be defined by the same strategy, but now it is necessary to reach the vertex $[0, j]$ first and then $[1, j']$ in the other sub-butterfly. The remaining part of the u, v-shortest path can be identified as in the case (6.a) or (6.b). □

Theorem 6. $\mu_t(BF(d)) = 2^d$, $d \geq 1$.

Proof. Let X be any total mutual-visibility set of $BF(d)$. If $u = [0, i]$ and $v = [d, j]$, according to the natural routing of the butterfly, there exists a unique u, v-shortest path in $BF(d)$. Consequently, each vertex in this u, v-shortest path cannot belong to X. In general, this leads to the property that each vertex $[\ell, i]$ such that $\ell \notin \{0, d\}$ does not belong to X.

Consider now two vertices $u = [d, i]$ and $v = [d, i+1]$, with i even. They belong to a convex subgraph of the butterfly isomorphic to a cycle C_4, and hence they cannot both belong to X otherwise the other pair of vertices in that cycle are not in mutual visibility. Symmetrically, vertices $u = [0, i]$ and $v = [0, i + 2^{d-1}]$, i even, cannot both belong to X as they belong to a convex subgraph C_4.

From the arguments above it follows that in X there cannot exist any vertex with degree four. Moreover, only half of the vertices at level d and at level 0 can belong to X. Hence, $\mu_t(BF(d)) \leq 2^d$. The statement follows by observing that the total mutual-visibility set provided by Lemma 9 contains exactly 2^d elements. □

7. Conclusions

In this paper, we have studied the mutual-visibility in hypercubes, Fibonacci cubes, cube-connected cycles, and butterflies. We have provided approximation algorithms for all the topologies but for butterflies for which we have obtained an exact formula.

Concerning the total mutual visibility we have studied cube-connected cycles and butterflies by providing exact formulae. About hypercubes, we have exploited the results from [6] about Hamming graphs, whereas we leave open the case of Fibonacci cubes.

It is worth noting that while we were in the process of submitting this paper, the results about hypercubes have been improved in [1] for both the mutual and the total mutual visibilities developing from some of the results appeared in the preliminary version of this article [10]. Further investigations on such topologies certainly deserve main attention.

Furthermore, we aim at studying distributed algorithms for swarm of robots to bring them in configurations where mutual visibility is satisfied. First results have been shown in [8] for robots moving on Trees and Square Grids. Hypercube-like graphs, represent a challenging environment as the high level of symmetry of the topologies does not provide useful means for the resolution of the problem.

Data availability

No data was used for the research described in the article.

References

[1] M. Axenovich, D. Liu, Visibility in hypercubes, ArXiv, https://doi.org/10.48550/arXiv.2402.04791, 2024.
[2] M.K. Azarian, Fibonacci identities as binomial sums, Int. J. Contemp. Math. Sci. 7 (38) (2012) 1871–1876.
[3] S. Badri, S. Cicerone, A. Di Fonso, G. Di Stefano, An optimal algorithm for geodesic mutual visibility on hexagonal grids, in: T. Masuzawa, Y. Katayama, H. Kakugawa, J. Nakamura, Y. Kim (Eds.), Stabilization, Safety, and Security of Distributed Systems - 26th International Symposium, SSS 2024, Nagoya, Japan, October 20-22, 2024, Proceedings, in: Lecture Notes in Computer Science, vol. 14931, Springer, 2024, pp. 161–176.
[4] A. Brandstädt, V.B. Le, J.P. Spinrad, Graph Classes: A Survey, SIAM, Philadelphia, PA, USA, 1999.
[5] W.G. Brown, P. Erdős, V.T. Sós, Some extremal problems on r-graphs, in: Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, 1971, Academic Press, 1973, pp. 53–63.

[6] C. Bujtás, S. Klavžar, J. Tian, Total mutual-visibility in Hamming graphs, ArXiv, arXiv:2307.05168 [abs], 2023.
[7] S. Cicerone, A. Di Fonso, G. Di Stefano, A. Navarra, The geodesic mutual visibility problem for oblivious robots: the case of trees, in: 24th Int. Conference on Distributed Computing and Networking, ICDCN 2023, ACM, 2023, pp. 150–159.
[8] S. Cicerone, A. Di Fonso, G. Di Stefano, A. Navarra, The geodesic mutual visibility problem: oblivious robots on grids and trees, Pervasive Mob. Comput. 95 (2023) 101842, https://doi.org/10.1016/j.pmcj.2023.101842.
[9] S. Cicerone, A. Di Fonso, G. Di Stefano, A. Navarra, Time-optimal geodesic mutual visibility of robots on grids within minimum area, in: S. Dolev, B. Schieber (Eds.), Stabilization, Safety, and Security of Distributed Systems, Springer Nature Switzerland, Cham, 2023, pp. 385–399.
[10] S. Cicerone, A. Di Fonso, G. Di Stefano, A. Navarra, F. Piselli, Mutual visibility in hypercube-like graphs, in: Y. Emek (Ed.), Structural Information and Communication Complexity - 31st International Colloquium, SIROCCO 2024, Vietri sul Mare, Italy, May 27-29, 2024, Proceedings, in: Lecture Notes in Computer Science, vol. 14662, Springer, 2024, pp. 192–207.
[11] S. Cicerone, G. Di Stefano, Mutual-visibility in distance-hereditary graphs: a linear-time algorithm, in: C.G. Fernandes, S. Rajsbaum (Eds.), Proceedings of the XII Latin-American Algorithms, Graphs and Optimization Symposium, LAGOS 2023, Huatulco, Mexico, September 18-22, 2023, in: Procedia Computer Science, vol. 223, Elsevier, 2023, pp. 104–111.
[12] S. Cicerone, G. Di Stefano, L. Drožđek, J. Hedžet, S. Klavžar, I.G. Yero, Variety of mutual-visibility problems in graphs, Theor. Comput. Sci. 974 (2023) 114096, https://doi.org/10.1016/j.tcs.2023.114096.
[13] S. Cicerone, G. Di Stefano, S. Klavžar, On the mutual visibility in Cartesian products and triangle-free graphs, Appl. Math. Comput. 438 (2023) 127619, https://doi.org/10.1016/j.amc.2022.127619.
[14] S. Cicerone, G. Di Stefano, S. Klavžar, I.G. Yero, Mutual-visibility in strong products of graphs via total mutual-visibility, Discrete Appl. Math. 358 (2024) 136–146, https://doi.org/10.1016/j.dam.2024.06.038.
[15] S. Cicerone, G. Di Stefano, S. Klavžar, I.G. Yero, Mutual-visibility problems on graphs of diameter two, Eur. J. Comb. 120 (2024) 103995, https://doi.org/10.1016/j.ejc.2024.103995.
[16] G.A. Di Luna, P. Flocchini, S.G. Chaudhuri, F. Poloni, N. Santoro, G. Viglietta, Mutual visibility by luminous robots without collisions, Inf. Comput. 254 (2017) 392–418, https://doi.org/10.1016/j.ic.2016.09.005.
[17] G. Di Stefano, Mutual visibility in graphs, Appl. Math. Comput. 419 (2022) 126850, https://doi.org/10.1016/j.amc.2021.126850.
[18] H.E. Dudeney, Amusements in Mathematics, Nelson, Edinburgh, 1917.
[19] Ö. Eğecioğlu, S. Klavžar, M. Mollard, Fibonacci Cubes with Applications and Variations, WorldScientific, 2023, https://doi.org/10.1142/13228.
[20] W. Hsu, Fibonacci cubes-a new interconnection technology, IEEE Trans. Parallel Distrib. Syst. 4 (1) (1993) 3–12, https://doi.org/10.1109/71.205649.
[21] E. Kiliç, The Binet formula, sums and representations of generalized Fibonacci p-numbers, Eur. J. Comb. 29 (3) (2008) 701–711, https://doi.org/10.1016/j.ejc.2007.03.004.
[22] S. Klavžar, Structure of Fibonacci cubes: a survey, J. Comb. Optim. 25 (4) (2013) 505–522, https://doi.org/10.1007/s10878-011-9433-z.
[23] S. Klavžar, P.K. Neethu, S.V.U. Chandran, The general position achievement game played on graphs, Discrete Appl. Math. 317 (2022) 109–116, https://doi.org/10.1016/j.dam.2022.04.019.
[24] S. Klavžar, J. Tian, Graphs with total mutual-visibility number zero and total mutual-visibility in Cartesian products, Discuss. Math., Graph Theory (2023), https://doi.org/10.7151/dmgt.2496.
[25] D. Kuziak, J.A. Rodríguez-Velázquez, Total mutual-visibility in graphs with emphasis on lexicographic and Cartesian products, Bull. Malays. Math. Sci. Soc. 46 (2023) 197.
[26] P.D. Manuel, M.I. Abd-El-Barr, I. Rajasingh, B. Rajan, An efficient representation of Benes networks and its applications, J. Discret. Algorithms 6 (1) (2008) 11–19, https://doi.org/10.1016/j.jda.2006.08.003.
[27] P.D. Manuel, S. Klavžar, The graph theory general position problem on some interconnection networks, Fundam. Inform. 163 (4) (2018) 339–350, https://doi.org/10.3233/FI-2018-1748.
[28] E. Munarini, N.Z. Salvi, Structural and enumerative properties of the Fibonacci cubes, Discrete Math. 255 (1–3) (2002) 317–324, https://doi.org/10.1016/S0012-365X(01)00407-1.
[29] A. Navarra, F. Piselli, Mutual visibility in Fibonacci cubes, in: Proc. 38th International Conference on Advanced Information Networking and Applications (AINA), AICS, Springer, 2024.
[30] P. Poudel, A. Aljohani, G. Sharma, Fault-tolerant complete visibility for asynchronous robots with lights under one-axis agreement, Theor. Comput. Sci. 850 (2021) 116–134, https://doi.org/10.1016/j.tcs.2020.10.033.
[31] R. Prabha, S.R. Devi, P. Manuel, General position problem of butterfly networks, ArXiv, https://arxiv.org/abs/2302.06154, 2023.
[32] G. Sharma, R. Vaidyanathan, J.L. Trahan, Optimal randomized complete visibility on a grid for asynchronous robots with lights, Int. J. Netw. Comput. 11 (1) (2021) 50–77.
[33] J. Stirling, F. Holliday, The Differential Method: Or, a Treatise Concerning Summation and Interpolation of Infinite Series, E. Cave, 1749.
[34] D. Van, W. Marilynn, L. Quentin, Q. Stout, Perfect dominating sets on cube-connected cycles, Congr. Numer. 97 (1993) 51–70.