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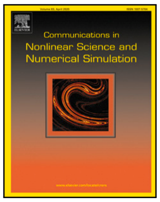

Research paper

Random periodic solutions of SDEs: Existence, uniqueness and numerical issues

Afsaneh Moradi, Raffaele D'Ambrosio *

Department of Information Engineering and Computer Science and Mathematics, University of L'Aquila, Italy

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ABSTRACT

The aim of this paper is to discuss existence and uniqueness of random periodic solutions to stochastic differential equations (SDEs) with multiplicative noise under a one-sided Lipschitz condition, as well as on their numerical approximation via two classes of stochastic θ-methods, i.e., θ-Maruyama methods with $\theta \in [1/2, 1]$ and θ-Milstein ones with $\theta \in [0, 1]$. The existence of the random periodic solutions as the limit of the pull-back flows of the discretized SDEs and the strong convergence rate of the aforementioned methods are also investigated. Selected numerical experiments confirming the theoretical analysis are also given.

1. Introduction

When faced with uncertainties or random effects, random dynamical systems are extensively discussed, used and analyzed in a vast number of problems discussing natural phenomena in fields such as Physics, Biology, Climatology, Finance and so on. The study of random dynamical systems was first proposed by Ulam and Neumann [1] and then extensively analyzed, for instance, in [2–7] and references therein.

A key aspect in these dynamical systems is the so-called random periodicity visible, for instance, in the oscillations characterizing economical and financial systems. For long time, the study of periodic motion has attracted the attention of authors interested in studying deterministic nonlinear dynamical systems and related features. However, since real problems are usually affected by noise, it is worth studying pathwise random periodic solutions. In [8], the definition of random periodic solutions was provided for a C^1-cocycle and later their existence for semi-flows obtained by non-autonomous SDEs and stochastic partial differential equations (SPDEs) with additive noise was studied in [9,10], which used an approach raised for coupled infinite horizon forward–backward integral equations. Let Q be a Banach space and $(\Omega, \mathscr{F}, \mathbb{P}, (\upsilon_s)_{s\in\mathbb{R}})$ a metric dynamical system, where $\upsilon_s : \Omega \to \Omega$ is supposed to be invertible for all $s \in \mathbb{R}$. Consider a stochastic periodic semi-flow $u : \Delta \times \Omega \times Q \to Q$ of period τ and with $\Delta := \{(t, s) \in \mathbb{R}^2, s \leq t\}$ satisfying the following semi-flow relation

$$u(t, r, \omega) = u(t, s, \omega) \circ u(s, r, \omega), \tag{1.1}$$

for all $r \leq s \leq t$. First, we give the definition of the random periodic solution.

Definition 1. A random periodic solution of period τ of a semi-flow $u : \Delta \times \Omega \times Q \to Q$ is an $\mathscr{F}$-measurable map $Y : \mathbb{R} \times \Omega \to Q$ such that

$$u(t + \tau, s + \tau, \omega) = u(t, s, \theta_\tau \omega), \tag{1.2}$$

* Corresponding author.
E-mail addresses: afsaneh.moradi@univaq.it (A. Moradi), raffaele.dambrosio@univaq.it (R. D'Ambrosio).

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for any $t \in \mathbb{R}$ and $\omega \in \Omega$.

SDEs with time-dependent coefficients which are periodic in time generate periodic semi-flows satisfying (1.1) and (1.2). A definition of random periodic path can be given as follows [9,10].

Definition 2. A random periodic path of period τ of the semi-flow $u : \Delta \times \Omega \times Q \to Q$ is an $\mathscr{F}$-measurable map $Y : \mathbb{R} \times \Omega \to Q$ such that

$$u(t, s, Y(s, \omega), \omega) = Y(t, \omega), \quad Y(s + \tau, \omega) = y(s, \upsilon_s \omega), \quad \forall (t, s) \in \Delta.$$

As a follow-up to this research area, a great deal of interest has raised on understanding random periodicity of coupled stochastic systems (see, for instance, [11–15] and references therein). In general, random periodic solutions, cannot be explicitly computed, so numerical approximations play an important role in this fields. Feng at el. in [16] used classical numerical methods (including Euler–Maruyama method and a modified Milstein method) to obtain random periodic solutions of a dissipative system with global Lipschitz condition. Wu studied in [17] random periodic solutions of SDEs with weaker conditions on the drift term and obtained their solutions using backward Euler–Maruyama method. In this paper, we study the random periodic solutions of SDEs with multiplicative noise and with weaker conditions on the drift term compared to [16]. We simulate random periodic solutions via two classes of theta methods: θ-Maruyama and θ-Milstein methods. For these schemes, drift coefficients are handled implicitly while the diffusion terms always appear explicitly, see for instance [18–31].

Consider the following m-dimensional SDE

$$\begin{cases} \mathrm{d}X_t^{t_0} = \left[-AX_t^{t_0} + f(t, X_t^{t_0})\right]\mathrm{d}t + g(t, X_t^{t_0})\mathrm{d}B_t, & \text{for } t \in (t_0, T], \\ X_{t_0}^{t_0} = \xi, \end{cases} \tag{1.3}$$

where ξ is a $\mathscr{F}^{t_0}$-measurable, and B_t is a standard Wiener process on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with the filtration defined by $\mathscr{F}_s^t = \sigma\{B_u - B_v : s \le v \le u \le t\}$ and $\mathscr{F}^t = \mathscr{F}_\infty^t = \bigvee_{s \le t} \mathscr{F}_s^t$. The solution of (1.3) can be determined by the variation of constants formula

$$X_t^{t_0}(\xi) = e^{-A(t-t_0)}\xi + \int_{t_0}^t e^{-A(t-s)} f(s, X_s^{t_0})\mathrm{d}s + \int_{t_0}^t e^{-A(t-s)} g(s, X_s^{t_0})\mathrm{d}B_s. \tag{1.4}$$

Denoting the standard ergodic Wiener shift by $\upsilon : \mathbb{R} \times \Omega \to \omega$, $\upsilon_t(s, \omega) := B(t + s) - B(t)$, $t, s \in \mathbb{R}$, we will indicate that the pull-back solution $X_t^{-k\tau}(\xi)$ starting from $-k\tau$ has a limit X_r^* in $L^2(\Omega)$ as $k \to \infty$ and X_r^* is the random periodic solution of SDE (1.3), satisfying the infinite horizon stochastic integral equation

$$X_r^* = \int_{-\infty}^r e^{-A(r-s)} f(s, X_s^*)\mathrm{d}s + \int_{-\infty}^r e^{-A(r-s)} g(s, X_s^*)\mathrm{d}B_s,$$

which is obtained by separating the linear term AX from the nonlinear term in (1.3) [9,10].

The paper is organized as follows. In Section 2, useful mathematical tools for later use and required assumptions on the matrix A and the coefficients f, g and ξ are given. Section 3 is dedicated to the existence and uniqueness of the random periodic solutions to the SDE (1.3) under the one-sided Lipschitz condition on the drift coefficient. In Section 4, we study the θ-Maruyama method when $\theta \in [1/2, 1]$, while in Section 5 θ-Milstein method when $\theta \in [0, 1]$ is considered to obtain random periodic solutions of SDE (1.3) in infinite horizon. Some numerical simulations to sample paths of random periodic solutions are presented in Section 6. Conclusions and open problems are object of Section 7.

2. Assumptions and preliminaries

Some useful mathematical tools for our later analysis are presented in this section. Throughout this work, $|\cdot|$ stands for the Euclidean norm, and the notations $a \vee b$ and $a \wedge b$ indicate the maximum and minimum between a and b, respectively. We consider the following assumptions on A, f and g in (1.3).

Assumption 1. A is a symmetric and positive definite $m \times m$ matrix whose spectrum $\{\lambda_j, j = 1, 2, \ldots, m\}$, satisfies $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$.

Assumption 2. The function $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous and τ-periodic in time. Also, assume that there exist constants $\mu < 0$ and $a > 0$ such that

$$\left\langle u_1 - u_2, f(t, u_1) - f(t, u_2)\right\rangle \le \mu |u_1 - u_2|^2, \quad \left\langle u, f(t, u)\right\rangle \le \mu |u|^2 + a,$$

for any $u, u_1, u_2 \in \mathbb{R}^m$ and $t \in [0, \tau)$.

Assumption 3. Concerning Assumption 2, we suppose that $\mu < \lambda_1$.

Assumption 4. The diffusion coefficient $g : \mathbb{R} \to \mathbb{R}$ is continuous and τ-periodic in time. Moreover, for all $u \in \mathbb{R}$ and $t \in [0, \tau)$ there exist constants $\sigma > 0$ and $b > 0$ such that

$$|g(t,u)|^2 \le \sigma |u|^2 + b,$$

with

$$2(\mu - \lambda_1) + \sigma < 0.$$

Assumption 5. There exists a constant C_ξ such that $\mathbb{E}|\xi| < C_\xi$.

Assumption 6. Assume that for any $u, u_1, u_2 \in \mathbb{R}^m$ and $t \in [0, \tau)$ there exists a positive constant K_1 such that

$$|f(t,u_1) - f(t,u_2)|^2 \vee |g(t,u_1) - g(t,u_2)|^2 \vee |Lg(t,u_1) - Lg(t,u_2)|^2 \le K_1 |u_1 - u_2|^2,$$

where

$$Lg(t,u) := g(t,u)\frac{\partial g(t,u)}{\partial u^l}, \quad l = 1, 2, \dots, m$$

and

$$2(\mu - \lambda_1) + K_1 < 0.$$

Assumption 7. For any $t_1, t_2 \in [0, \tau)$ and $u_1, u_2 \in \mathbb{R}^m$, there exists a positive constant L such that

$$\left|f(t_1,u_1) - f(t_2,u_2)\right| \vee \left|g(t_1,u_1) - g(t_2,u_2)\right| \le L(1 + |u_1| + |u_2|)|t_1 - t_2|.$$

Assumption 8. Assume that there exists a positive constant γ such that for any $u \in \mathbb{R}^m$, $t \in [0, \tau)$ and $l = 1, 2, \dots, m$,

$$\left|\frac{\partial g(t,u)}{\partial u^l}\right| \le \gamma.$$

The following useful results are given (see [32]), giving a continuous and a discrete Grönwall inequality.

Lemma 1. *Let a, b and u be real-valued functions defined on $I = [0, t]$. Suppose that a and b are continuous and that the negative part of a is integrable on every closed and bounded subinterval of I. Then if b is non-negative and if u satisfies the following inequality*

$$u(t) \le a(t) + \int_I b(s)u(s)\mathrm{d}s,$$

then

$$u(t) \le a(t) + \int_I a(s)b(s)\exp\left(\int_I b(r)dr\right)\mathrm{d}s. \tag{2.5}$$

In addition, if the function a is non-decreasing we have

$$u(t) \le a(t)\exp\left(\int_I b(r)\mathrm{d}r\right). \tag{2.6}$$

We state a simple version of the discrete-type Gronwall inequality in the next Lemma (see, for example, [33]).

Lemma 2. *Let u_N, $N = nk$ for some $n \in \mathbb{N}$, and D_1, D_2 be nonnegative. Assume that*

$$u_N \le D_1 + \frac{D_2}{n}\sum_{i=1}^{N-1} u_i,$$

holds true. Then

$$u_N \le D_1 \exp(kD_2).$$

Following the ideas in [34], to prove our theoretical results in the following sections we will use the two next lemmas from [35].

Lemma 3. *Let Assumption 2 hold true, then for any β_1, $\beta_2 \in \mathbb{R}$*

$$\left|u - \beta_1 f(t,u)\right|^2 + 2\beta_1 a \le \frac{1 - \mu\beta_1}{1 - \mu\beta_2}(\left|u - \beta_2 f(t,u)\right|^2 + 2\beta_2 a),$$

with $\beta_2 \ge \beta_1 > 0$.

Lemma 4. *Let Assumption 2 hold true, then the inequality*

$$\left|u_1 - u_2 - \alpha_1\left(f(t,u_1) - f(t,u_2)\right)\right| \le \frac{1 - \mu\alpha_1}{1 - \mu\alpha_2}\left|u_1 - u_2 - \alpha_2\left(f(t,u_1) - f(t,u_2)\right)\right|,$$

holds true for any α_1, $\alpha_2 \in \mathbb{R}$ with $\alpha_2 \ge \alpha_1 > 0$.

3. Existence and uniqueness of the random periodic solution

We now aim to prove the existence and uniqueness of a random periodic solution to SDEs (1.3). It is worth observing that existing results on the same topic (as highlighted in Section 1) rely on the assumption of global Lipschitz continuity of the drift. Here, we make of a weaker hypothesis, i.e., on its one-sided Lipschitz continuity (see Assumption 2). We also observe that the hypothesis of one-sided Lipschitz continuity of the drift is at the basis of the theory of stochastic dissipativity (see, for instance, [26,29,36–38]).

We first analyze the boundedness of the second moment of its solution under above required assumptions.

Lemma 5. *Assume that Assumptions 1–5 are given. Then we have*

$$\sup_{k\in\mathbb{N}}\sup_{t>-k\tau}\mathbb{E}\left|X_t^{-k\tau}(\xi)\right|^2 \le C_\xi^2 + \frac{L_2(2\lambda_1+\sigma)}{2(\lambda_1-\mu)-\sigma}, \tag{3.7}$$

where $L_2 := \dfrac{2a+b}{2\lambda_1}$.

Proof. Using Itô formula for $e^{2\lambda_1 t}\left|X_t^{-k\tau}(\xi)\right|^2$ and taking expectation result in

$$\begin{aligned} e^{2\lambda_1 t}\mathbb{E}\left|X_t^{-k\tau}(\xi)\right|^2 =& e^{-2\lambda_1 k\tau}\mathbb{E}\left|\xi\right|^2 + 2\lambda_1\int_{-k\tau}^t e^{2\lambda_1 s}\mathbb{E}\left|X_s^{-k\tau}\right|^2\mathrm{d}s \\ &- 2\int_{-k\tau}^t e^{2\lambda_1 s}\mathbb{E}\left\langle X_s^{-k\tau}, AX_s^{-k\tau}\right\rangle\mathrm{d}s \\ &+ 2\int_{-k\tau}^t e^{2\lambda_1 s}\mathbb{E}\left\langle X_s^{-k\tau}, f(s,X_s^{-k\tau})\right\rangle\mathrm{d}s \\ &+ \int_{-k\tau}^t e^{2\lambda_1 s}\mathbb{E}\left|g(s,X_s^{-k\tau})\right|^2\mathrm{d}s. \end{aligned} \tag{3.8}$$

Using this fact that $2(\lambda_1 I - A)$ is non-positive definite and considering Assumptions 2 and 4 we have

$$\begin{aligned} e^{2\lambda_1 t}\mathbb{E}\left|X_t^{-k\tau}(\xi)\right|^2 \le& e^{-2\lambda_1 k\tau}\mathbb{E}\left|\xi\right|^2 + (2\mu+\sigma)\int_{-k\tau}^t e^{2\lambda_1 s}\mathbb{E}\left|X_s^{-k\tau}\right|^2\mathrm{d}s \\ &+ (2a+b)\int_{-k\tau}^t e^{2\lambda_1 s}\mathrm{d}s \\ \le& e^{-2\lambda_1 k\tau}\mathbb{E}\left|\xi\right|^2 + \frac{2a+b}{2\lambda_1}\left(e^{2\lambda_1 t} - e^{-2\lambda_1 k\tau}\right) \\ &+ (2\mu+\sigma)\int_{-k\tau}^t e^{2\lambda_1 s}\mathbb{E}\left|X_s^{-k\tau}\right|^2\mathrm{d}s. \end{aligned}$$

Set $L_1 := e^{-2\lambda_1 k\tau}\left(\mathbb{E}\left|\xi\right|^2 - \dfrac{2a+b}{2\lambda_1}\right)$, $L_2 := \dfrac{2a+b}{2\lambda_1}$ and $L_3 := 2\mu+\sigma$. By the Grönwall inequality we deduce

$$\begin{aligned} e^{2\lambda_1 t}\mathbb{E}\left|X_t^{-k\tau}(\xi)\right|^2 \le& L_1 + L_2 e^{2\lambda_1 t} + \int_{-k\tau}^t (L_1 + L_2 e^{2\lambda_1 s})L_3 e^{L_3(t-s)}\mathrm{d}s \\ \le& L_1 e^{L_3(k\tau+t)} + L_2 e^{2\lambda_1 t} + \frac{L_2L_3}{2\lambda_1 - L_3}(e^{2\lambda_1 t} - e^{-2\lambda_1 k\tau}) \\ \le& (L_1 e^{2\lambda_1 k\tau} + L_2)e^{2\lambda_1 t} + \frac{L_2L_3}{2\lambda_1 - L_3}e^{2\lambda_1 t}. \end{aligned}$$

By Assumptions 3 and 5 and since $L_1 e^{2\lambda_1 k\tau} + L_2 = \mathbb{E}\left|\xi\right|^2$ we have

$$\mathbb{E}|X_t^{-k\tau}|^2 \le \mathbb{E}\left|\xi\right|^2 + \frac{L_2L_3}{2\lambda_1 - L_3} \le C_\xi^2 + \frac{L_2(2\lambda_1+\sigma)}{2(\lambda_1-\mu)-\sigma}. \quad \square$$

Next lemma shows the continuous dependence of the solution on initial conditions.

Lemma 6. *Let Assumptions 1–3 and 6 hold and $X_t^{-k\tau}$ and $Y_t^{-k\tau}$ are two solutions of SDE (1.3) with different initial values ξ and η, respectively. Then*

$$\mathbb{E}\left|X_t^{-k\tau} - Y_t^{-k\tau}\right|^2 \le e^{(2(\mu-\lambda_1)+K_1)(t+k\tau)}\mathbb{E}\left|\xi-\eta\right|^2.$$

Proof. Denote by $D_t^{-k\tau} := X_t^{-k\tau} - Y_t^{-k\tau}$. It follows from (1.4) that

$$\begin{aligned} D_t^{-k\tau} =& e^{-A(t-t_0)}(\xi-\eta) + \int_{-k\tau}^t e^{-A(t-s)}\left(f(s,X_s^{-k\tau}) - f(s,Y_s^{-k\tau})\right)\mathrm{d}s \\ &+ \int_{-k\tau}^t e^{-A(t-s)}\left(g(s,X_s^{-k\tau}) - g(s,Y_s^{-k\tau})\right)\mathrm{d}B_s. \end{aligned}$$

Imposing Itô lemma to $e^{2\lambda_1 t}|D_t^{-k\tau}|^2$, taking expectation and making use of Assumptions 2 and 6, we have

$$\begin{aligned} e^{2\lambda_1 t}\mathbb{E}\left|D_t^{-k\tau}\right|^2 \le & e^{-2\lambda_1 k\tau}\mathbb{E}\left|\xi-\eta\right|^2 + 2\int_{-k\tau}^{t} e^{2\lambda_1 s}\mathbb{E}\left\langle D_s^{-k\tau}, f(s,X_s^{-k\tau}) - f(s,Y_s^{-k\tau})\right\rangle \mathrm{d}s \\ & + \int_{-k\tau}^{t} e^{2\lambda_1 s}\mathbb{E}\left|g(s,X_s^{-k\tau}) - g(s,Y_s^{-k\tau})\right|\mathrm{d}s, \\ \le & e^{-2\lambda_1 k\tau}\mathbb{E}\left|\xi-\eta\right|^2 + (2\mu+K_1)\int_{-k\tau}^{t} e^{2\lambda_1 s}\mathbb{E}\left|D_s^{-k\tau}\right|^2\mathrm{d}s. \end{aligned}$$

Applying the Grönwall inequality (2.6) leads to the desired result. □

Theorem 1. *Let Assumptions 1–6 hold, then there is a unique random periodic solution $X_t^*(\cdot)\in L^2(\Omega)$ such that the solution of SDE (1.3) satisfies*

$$\lim_{k\to\infty}\mathbb{E}\left|X_t^{-k\tau}(\xi) - X_t^*\right| = 0. \tag{3.9}$$

Proof. By Assumption 3 and Lemma 6, it is easy to verify that for every $\epsilon>0$ there is a $t\ge -k\tau$ such that

$$\mathbb{E}\left|X_{\tilde t}^{-k\tau} - Y_{\tilde t}^{-k\tau}\right|^2 < \epsilon, \quad \text{for } \tilde t \ge t.$$

Making use of this and Lemma 5 and following the same argument in the proof of Theorem 2.4 in [16], the main result can be obtained. □

To estimate the error of the numerical approximation obtained by θ-Maruyama method in Section 4, we will need a bound on the $\mathbb{E}\left|X_{t_1}^{-k\tau} - X_{t_2}^{-k\tau}\right|$ for any fixed time t_1 and t_2. This bound can be easily obtained following a similar discussion as in Propositions 5.4 and 5.5 in [39].

Proposition 1. *Let Assumptions 1–7 hold, then for all $t_1, t_2 \ge -k\tau$ there exists a positive constant C such that*

$$\mathbb{E}\left|X_{t_1}^{-k\tau} - X_{t_2}^{-k\tau}\right| \le C\left(1+\sup_{k\in\mathbb{N}}\sup_{t\ge -k\tau}\mathbb{E}\left|X_t^{-k\tau}\right|^2\right)\left|t_2-t_1\right|^{1/2}.$$

Also for all $t_3, t_4 \in [t_1, t_2]$

$$\begin{aligned} &\int_{t_1}^{t_2}\mathbb{E}\left| -A\left(X_s^{-k\tau} - X_{t_4}^{-k\tau}\right) + f(s,X_s^{-k\tau}) - f(t_3,X_{t_4}^{-k\tau}) + g(s,X_s^{-k\tau}) - g(t_3,X_{t_4}^{-k\tau})\right|\mathrm{d}s \\ &\qquad \le C\left(1+\sup_{k\in\mathbb{N}}\sup_{t\ge -k\tau}\mathbb{E}\left|X_t^{-k\tau}\right|^2\right)\left|t_2-t_1\right|^{3/2}. \end{aligned}$$

4. Random periodic θ-Maruyama solutions

The aim of this section is to introduce the θ-Maruyama method to approximate the solution on infinite horizon. To do this, consider an equidistant partition $\mathscr{I} := \{j\Delta t,\ j\in\mathbb{Z}\}$ with sufficiently small stepsize Δt. To simulate the periodic solution of SDEs (1.3) starting at $-k\tau$, the θ-Maruyama method on $\mathscr{I}$ is determined by the recursion

$$\begin{aligned} \widehat{X}_{-k\tau+(j+1)\Delta t}^{-k\tau} = & \widehat{X}_{-k\tau+j\Delta t}^{-k\tau} + (1-\theta)\left(-A\widehat{X}_{-k\tau+j\Delta t}^{-k\tau} + f\left(j\Delta t, \widehat{X}_{-k\tau+j\Delta t}^{-k\tau}\right)\right)\Delta t \\ & + \theta\left(-A\widehat{X}_{-k\tau+(j+1)\Delta t}^{-k\tau} + f\left((j+1)\Delta t, \widehat{X}_{-k\tau+(j+1)\Delta t}^{-k\tau}\right)\right)\Delta t \\ & + g\left(j\Delta t, \widehat{X}_{-k\tau+j\Delta t}^{-k\tau}\right)\Delta B_{-k\tau+j\Delta t}, \end{aligned} \tag{4.10}$$

for $\theta\in[0,1]$. Here, $j\in\mathbb{N}$, ξ is the initial value $\widehat{X}_{-k\tau}^{-k\tau}$ and $\Delta B_{-k\tau+j\Delta t} := B_{-k\tau+(j+1)\Delta t} - B_{-k\tau+j\Delta t}$.

4.1. Existence of random periodic solution

We aim to prove that the θ-Maruyama method (4.10) for $\theta\in[1/2,1]$ generates a pathwise unique discretized random periodic solution. With the aim of proving the convergence of the considered θ-method, we first need some similar estimates as in Lemmas 5 and 6. The following further lemma shows the boundedness of second moment of the numerical solution obtained by θ-Maruyama method (4.10) under some of the aforementioned assumptions.

Lemma 7. *Let Assumptions 1 to 3 be satisfied, then for $\Delta t\in(0,\Delta t^*)$ the numerical solution generated by θ-Maruyama method (4.10) obeys*

$$\mathbb{E}\left|\widehat{X}_{-k\tau+j\Delta t}^{-k\tau}\right|^2 \le C_1, \tag{4.11}$$

where C_1 is a constant that does not rely on j.

Proof. Set $\rho := 1 + \frac{\sigma}{2(\mu-\lambda_1)} \wedge (2\theta - 1)$. By Assumptions 1–3 we have

$$\left|(I + \Delta t A\theta)\hat{X}^{-k\tau}_{-k\tau+(j+1)\Delta t} - \Delta t\theta f\left((j+1)\Delta t, \hat{X}^{-k\tau}_{-k\tau+(j+1)\Delta t}\right)\right|^2 + 2a\Delta t\theta(1+\Delta t\lambda_1\theta) \leq$$
$$\left|(I + \Delta t A(1-\theta))\hat{X}^{-k\tau}_{-k\tau+j\Delta t} - \Delta t(1-\theta) f\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\right|^2$$
$$+ 2a\Delta t(1-\theta)(1+\Delta t\lambda_1(1-\theta)) + [4\Delta t(1-\theta)(\mu-\lambda_1)+\sigma]\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right|^2$$
$$+ \left(2a((1-\theta)(1-\lambda_1\Delta t)+\Delta t\lambda_1\theta)+b\right)\Delta t + G_j,$$

where

$$G_j := 2\left\langle \hat{X}^{-k\tau}_{-k\tau+j\Delta t} + \Delta t(1-\theta)(-A\hat{X}^{-k\tau}_{-k\tau+j\Delta t} + f(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t})), g(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t})\Delta B_{-k\tau+j\Delta t} \right\rangle$$
$$+ \left|g(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t})\right|^2(\Delta B^2_{-k\tau+j\Delta t} - \Delta t).$$

Using Lemma 3 with $\beta_1 = \frac{\Delta t(1-\theta)}{1+\Delta t\lambda_1(1-\theta)}$ and $\beta_2 = \frac{\Delta t\theta}{1+\Delta t\lambda_1\theta}$ results in

$$\left|\hat{X}^{-k\tau}_{-k\tau+(j+1)\Delta t} - \frac{\Delta t\theta}{1+\Delta t\lambda_1\theta} f\left((j+1)\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\right|^2 + \frac{2a\Delta t\theta}{1+\Delta t\lambda_1\theta} \leq$$
$$\frac{(1+\Delta t(1-\theta)\lambda_1)(1-\Delta t(1-\theta)\mu)}{(1+\Delta t\lambda_1\theta)(1-\Delta t\theta\mu)}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t} - \frac{\Delta t\theta}{1+\Delta t\lambda_1\theta} f\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\right|^2$$
$$+ \frac{2a\Delta t\theta}{1+\Delta t\lambda_1\theta} + \frac{4\Delta t(1-\theta)(\mu-\lambda_1)+\sigma\Delta t}{(1+\Delta t\lambda_1\theta)^2}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right|^2$$
$$+ \frac{\left[2a((1-\theta)(1-\lambda_1\Delta t)+\lambda\Delta t\theta)+b\right]\Delta t}{(1+\Delta t\lambda_1\theta)^2} + \frac{G_j}{(1+\Delta t\lambda_1\theta)^2}.$$

We next denote

$$F_j = \mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t} - \beta_2 f\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\right|^2 + 2a\beta_2,$$

and

$$N_{\Delta t} = \frac{(1+\Delta t(1-\theta)\lambda_1)(1-\Delta t(1-\theta)\mu)}{(1+\Delta t\lambda_1\theta)(1-\Delta t\theta\mu)},$$
$$C_{\Delta t} = \frac{\left(2a((1-\theta)(1-\lambda_1\Delta t)+\lambda\Delta t\theta)+b\right)\Delta t}{(1+\Delta t\lambda_1\theta)^2}.$$

Taking expectation on both sides and using this fact that $\mathbb{E}G_j = 0$, lead to

$$F_{j+1} \leq N_{\Delta t}F_j + \frac{4\Delta t(1-\theta)(\mu-\lambda_1)+\sigma\Delta t}{(1+\Delta t\lambda_1\theta)^2}\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right|^2 + C_{\Delta t}.$$

Then, we obtain by iteration that

$$F_{j+1} \leq N^{j+1}_{\Delta t}[F_0 + 2(2\theta-1-\rho)\mu\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau}\right|^2] + \phi_\rho(\theta)\Delta t\sum_{i=0}^{j} N^{j-i}_{\Delta t}\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right|^2 + C_{\Delta t}\sum_{i=0}^{j} N^i_{\Delta t},$$

where

$$\phi_\rho(\theta) = \frac{4\Delta t(1-\theta)(\mu-\lambda_1)+\sigma\Delta t}{(1+\Delta t\lambda_1\theta)^2} - 2(2\theta-1-\rho)\mu N_h.$$

For $\theta \in (\theta^*, 1]$ where $\theta^* = 1 + \dfrac{\sigma}{4(\mu-\lambda_1)}$, we have $\phi_\rho(\theta) < 0$. Moreover, since $\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right|^2 \leq F_j$ and $N_{\Delta t} \in (0,1)$ for $\Delta t < \Delta t^* := \frac{-(1-2\theta)(2\lambda_1-\mu)}{2\lambda_1\theta(\lambda_1-\mu)}$, we obtain

$$\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right|^2 \leq N^j_h\left[F_0 + 2(2\theta-1-\rho)\mu\Delta t\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau}\right|^2\right] + \frac{C_{\Delta t}}{1-N_{\Delta t}} \leq C_1,$$

where

$$C_1 := \left[F_0 + 2(2\theta-1-\rho)\mu\Delta t\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau}\right|^2\right] + \frac{C_{\Delta t}}{1-N_{\Delta t}}.$$

For $\theta \in [1/2, \theta^*]$, choosing $\rho' < \rho$ sufficiently small such that $\phi_{\rho'}(\theta) < 0$ for any $\Delta t > 0$, and following the similar arguments as above give the desired result. □

Next lemma indicates that numerical solutions under various initial conditions are capable of being close after enough iterations.

Lemma 8. *Suppose that Assumptions 1–6 are given. Let us respectively denote by $\hat{X}^{-k\tau}_{-k\tau+j\Delta t}$ and $\hat{Y}^{-k\tau}_{-k\tau+j\Delta t}$ θ-Maruyama numerical solutions with initial values ξ and η. Then, there exists a constant $C_2^j > 0$ such that*

$$\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t} - \hat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right|^2 \le C_2,$$

where $\lim_{j\to\infty} C_2^j = 0$.

Proof. Following a similar argument as in the proof of Lemma 7, set $\rho = 1 + \frac{K_1}{2(\mu-\lambda_1)} \wedge (2\theta - 1)$. Denote $D_j := \hat{X}^{-k\tau}_{-k\tau+j\Delta t} - \hat{Y}^{-k\tau}_{-k\tau+j\Delta t}$. When $\theta^* := 1 + \frac{K1}{4(\mu-\lambda_1)} < \theta \le 1$, using Lemma 3 with $\alpha_1 = \frac{\Delta t(1-\theta)}{1+\Delta t\lambda_1(1-\theta)}$ and $\alpha_2 = \frac{\Delta t\theta}{1+\Delta t\lambda_1\theta}$, we have

$$\begin{aligned}
&\left|D_{j+1} - \frac{\Delta t\theta}{1+\Delta t\lambda_1\theta}\left[f\left((j+1)\Delta t, \hat{X}^{-k\tau}_{-k\tau+(j+1)\Delta t}\right) - f\left((j+1)\Delta t, \hat{Y}^{-k\tau}_{-k\tau+(j+1)\Delta t}\right)\right]\right|^2 \\
&\le \left(\frac{1+\Delta t\lambda_1(1-\theta)}{1+\Delta t\lambda_1\theta}\right)^2 \left|D_j - \frac{\Delta t(1-\theta)}{1+\Delta t\lambda_1(1-\theta)}\left[f\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right) - f\left(j\Delta t, \hat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right)\right]\right|^2 \\
&\quad + \frac{4\Delta t(1-\theta)}{(1+\Delta t\lambda_1\theta)^2}\left\langle D_j, f\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right) - f\left(j\Delta t, \hat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right)\right\rangle \\
&\quad + \Delta t\left|g(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}) - g(j\Delta t, \hat{Y}^{-k\tau}_{-k\tau+j\Delta t})\right|^2 + \frac{M_j}{(1+\Delta t\lambda_1\theta)^2} \\
&\le \left(\frac{1+\Delta t(1-\theta)(\lambda_1-\mu)}{1+\Delta t\theta(\lambda_1-\mu)}\right)^2 \left|D_j - \frac{\Delta t\theta}{1+\Delta t\lambda_1\theta}\left[f\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right) - f\left(j\Delta t, \hat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right)\right]\right|^2 \\
&\quad + \frac{4(1-\theta)(\mu-\lambda_1)+K_1}{(1+\Delta t\lambda_1\theta)^2}\Delta t\left|D_j\right|^2 + \frac{M_j}{(1+\Delta t\lambda_1\theta)^2},
\end{aligned}$$

where

$$\begin{aligned}
M_j :=& 2\left\langle (1-\Delta t\lambda_1(1-\theta))D_j + \Delta t(1-\theta)f\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right) - f\left(j\Delta t, \hat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right),\right. \\
&\left.\left(g(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}) - g(j\Delta t, \hat{Y}^{-k\tau}_{-k\tau+j\Delta t})\right)\Delta B_{-k\tau+j\Delta t}\right\rangle \\
&+ \left|g(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}) - g(j\Delta t, \hat{Y}^{-k\tau}_{-k\tau+j\Delta t})\right|^2(\Delta B^2_{-k\tau+j\Delta t} - \Delta t),
\end{aligned}$$

Denote

$$W_j := \mathbb{E}\left|D_j - \frac{\Delta t(1-\theta)}{1+\Delta t\lambda_1(1-\theta)}\left[f\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right) - f\left(j\Delta t, \hat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right)\right]\right|^2,$$

and

$$L_{\Delta t} := \left(\frac{1+\Delta t(1-\theta)(\lambda_1-\mu)}{1+\Delta t\theta(\lambda_1-\mu)}\right)^2.$$

Since $\mathbb{E}M_j = 0$, taking expectation on both sides we have

$$\begin{aligned}
W_{j+1} \le& L_{\Delta t}W_j + \frac{4(1-\theta)(\mu-\lambda_1)+K_1}{(1+\Delta t\lambda_1\theta)^2}\Delta t\left|D_j\right|^2 \\
\le& L^{j+1}_{\Delta t}\left[W_0 + 2(2\theta-1-\rho)\mu\Delta t\mathbb{E}\left|\xi-\eta\right|^2\right] + \phi_\rho(\theta)\Delta t\sum_{i=0}^{j} L^{j-i}_{\Delta t}\mathbb{E}\left|D_i\right|^2,
\end{aligned}$$

where

$$\phi_\rho(\theta) = \frac{4(1-\theta)(\mu-\lambda_1)+K_1}{(1+\Delta t\lambda_1\theta)^2} - 2(2\theta-1-\rho)\mu L_{\Delta t}.$$

For $\theta \in (\theta^*, 1]$, we have $\phi_\rho(\theta) < 0$. Since $\mathbb{E}\left|D_j\right|^2 \le W_j$ and $L_{\Delta t} \in (0,1)$, we get

$$\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t} - \hat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right|^2 \le C_2^j,$$

where

$$C_2^j := L^j_{\Delta t}\left[W_0 + 2(2\theta-1-\rho)\mu\Delta t\mathbb{E}\left|\xi-\eta\right|^2\right].$$

Regarding the case $\theta \in [1/2, \theta^*]$, we can choose a small enough $\rho' < \rho$ such that $\phi_{\rho'}(\theta) < 0$ for any $\Delta t > 0$, and the desired result can be proved using the similar arguments as above. □

In the following theorem, we explore the random periodicity of numerical solutions obtained by θ-Maruyama method.

Theorem 2. *Let Assumptions 1–3 hold. Then, for any $\Delta t \in (0,1)$ with $\tau = n\Delta t$, $n \in \mathbb{N}$, the θ-Maruyama method (4.10) generates a random periodic solution on $\mathscr{I}$, i.e., there exists $\widehat{X}_r^* \in L^2(\Omega)$ such that*

$$\lim_{k\to\infty} \mathbb{E}|\widehat{X}_r^{-k\tau}(\xi) - \widehat{X}_r^*| = 0. \tag{4.12}$$

Proof. Taking Lemma 8 into account, the proof can be easily shown by following a similar arguments as in the proof of Theorem 8 in [17]. □

4.2. Error analysis

In the previous subsection, we presented the existence of random periodic solutions of SDEs (1.3) obtained by the θ-Maruyama method (4.10) as the limit of semi-flows when the starting times were pushed to $-\infty$. To conclude this section, we need to provide an error analysis associated to this scheme. To do this, consider the exact solution at time $-k\tau + N\Delta t$, as follows

$$\begin{aligned} X^{-k\tau}_{-k\tau+N\Delta t} =& X^{-k\tau}_{-k\tau+(N-1)\Delta t} + \Delta t(1-\theta)\left(-AX^{-k\tau}_{-k\tau+(N-1)\Delta t} + f\left((N-1)\Delta t, X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right) \\ &+ \Delta t\theta\left(-AX^{-k\tau}_{-k\tau+N\Delta t} + f(N\Delta t, X^{-k\tau}_{-k\tau+N\Delta t})\right) + g\left((N-1)\Delta t, X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right) \\ &\times \Delta B_{-k\tau+(N-1)\Delta t} + R_N, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} R_N =& \theta \int_{-k\tau+(N-1)\Delta t}^{-k\tau+N\Delta t} \left[-A\left(X_s^{-k\tau} - X^{-k\tau}_{-k\tau+N\Delta t}\right) + f\left(s, X_s^{-k\tau}\right) - f\left(N\Delta, X_{-k\tau+N\Delta t}\right)\right] \mathrm{d}s \\ &+ (1-\theta)\int_{-k\tau+(N-1)\Delta t}^{-k\tau+N\Delta t} \Big[-A\left(X_s^{-k\tau} - X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right) \\ &+ f\left(s, X_s^{-k\tau}\right) - f\left((N-1)\Delta, X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\Big] \mathrm{d}s \\ &+ \int_{-k\tau+(N-1)\Delta t}^{-k\tau+N\Delta t} \left[g\left(s, X_s^{-k\tau}\right) - g\left((N-1)\Delta, X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right] \mathrm{d}B_s. \end{aligned}$$

The following result holds true.

Theorem 3. *Let Assumptions 1 to 7 be satisfied. Then, there exists a constant C_3 which depends on θ, A, f, g and m such that, for any $\Delta t < \frac{2(\lambda_1-\mu)}{(1-\theta)(\lambda_1^2+K_1)}$ with $\tau = n\Delta t$, $n \in \mathbb{N}$, and initial conditions $\widehat{X}^{-k\tau}_{-k\tau} = X^{-k\tau}_{-k\tau} = \xi$, we have*

$$\sup_{k,N} \mathbb{E}\left|X^{-k\tau}_{-k\tau+N\Delta t} - \widehat{X}^{-k\tau}_{-k\tau+N\Delta t}\right| \le C_3 \Delta t^{1/2}. \tag{4.14}$$

Proof. Set $e_{N+1} := X^{-k\tau}_{-k\tau+N\Delta t} - \widehat{X}^{-k\tau}_{-k\tau+N\Delta t}$. We have

$$\begin{aligned} \mathbb{E}\left|e_N\right|^2 =& \mathbb{E}\Big\langle e_{N-1} + \Delta t(1-\theta)\left(-Ae_{N-1} + f\left(X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right) - f\left(\widehat{X}^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right), e_N\Big\rangle \\ &+ \Delta t\theta\mathbb{E}\Big\langle -Ae_N + f\left(X^{-k\tau}_{-k\tau+N\Delta t}\right) - f\left(\widehat{X}^{-k\tau}_{-k\tau+N\Delta t}\right), e_N\Big\rangle \\ &+ \mathbb{E}\Big\langle g\left(X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right) - g\left(\widehat{X}^{-k\tau}_{-k\tau+(N-1)\Delta t}\right), e_N\Big\rangle \Delta B_{-k\tau+(N-1)\Delta t} + \mathbb{E}\Big\langle e_N, R_N\Big\rangle \\ \le& \frac{1}{2}(I + 2\Delta t\theta(\mu I - A))\mathbb{E}\left|e_N\right|^2 + \frac{1}{2}\mathbb{E}\Big|e_{N-1} \\ &+ \Delta t(1-\theta)\left(-Ae_{N-1} + f\left(X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right) - f\left(\widehat{X}^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right)\Big|^2 \\ &+ \mathbb{E}\Big\langle \theta\int_{-k\tau+(N-1)\Delta t}^{-k\tau+N\Delta t}\Big[-A\left(X_s^{-k\tau} - X^{-k\tau}_{-k\tau+N\Delta t}\right) \\ &+ f\left(s, X_s^{-k\tau}\right) - f\left(N\Delta, X^{-k\tau}_{-k\tau+N\Delta t}\right)\Big]\mathrm{d}s, e_N\Big\rangle \\ &+ \mathbb{E}\Big\langle (1-\theta)\int_{-k\tau+(N-1)\Delta t}^{-k\tau+N\Delta t}\Big[-A\left(X_s^{-k\tau} - X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right) \\ &+ f\left(s, X_s^{-k\tau}\right) - f\left((N-1)\Delta, X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\Big]\mathrm{d}s, e_N\Big\rangle \\ &+ \mathbb{E}\Big\langle \int_{-k\tau+(N-1)\Delta t}^{-k\tau+N\Delta t}\left[g\left(s, X_s^{-k\tau}\right) - g\left((N-1)\Delta, X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right]\mathrm{d}B_s, e_N\Big\rangle. \end{aligned}$$

Using Young inequality yields

$$2ab \le \epsilon^2 a^2 + \frac{b^2}{\epsilon^2}, \quad \forall a, b > 0$$

and Assumptions 2 and 6 allow us to choose $\epsilon_0^2 := \Delta t(\lambda_1 - \mu)/2$, thereby

$$\begin{aligned}\mathbb{E}\left|e_N\right|^2 \leq & \frac{1}{2}(I + 2\Delta t\theta(\mu I - A))\mathbb{E}\left|e_N\right|^2 \\ & + \frac{I + 2\Delta t(1-\theta)(\mu I - A) + \Delta t^2(1-\theta)^2(A^2 + K_1 I)}{2}\mathbb{E}\left|e_{N-1}\right|^2 \\ & + \frac{\theta}{2}\left(\epsilon_0^2\mathbb{E}\left|e_N\right|^2 + \frac{1}{\epsilon_0^2}\mathbb{E}\left|\int_{-k\tau+(N-1)\Delta t}^{-k\tau+N\Delta t}\left[-A\left(X_s^{-k\tau} - X_{-k\tau+(N-1)\Delta t}^{-k\tau}\right)\right.\right.\right. \\ & \left.\left.\left. + f\left(s, X_s^{-k\tau}\right) - f\left((N-1)\Delta, X_{-k\tau+(N-1)\Delta t}^{-k\tau}\right)\right] \mathrm{d}s\right|^2\right) \\ & + \frac{1-\theta}{2}\left(\epsilon_0^2\mathbb{E}\left|e_N\right|^2 + \frac{1}{\epsilon_0^2}\mathbb{E}\left|\int_{-k\tau+(N-1)\Delta t}^{-k\tau+N\Delta t}\left[-A\left(X_s^{-k\tau} - X_{-k\tau+N\Delta t}^{-k\tau}\right)\right.\right.\right. \\ & \left.\left.\left. + f\left(s, X_s^{-k\tau}\right) - f\left(N\Delta, X_{-k\tau+N\Delta t}^{-k\tau}\right)\right] \mathrm{d}s\right|^2\right) + \frac{1}{2}\left(\epsilon_0^2\left|e_N\right|^2\right. \\ & \left. + \frac{1}{\epsilon_0^2}\mathbb{E}\left|\int_{-k\tau+(N-1)\Delta t}^{-k\tau+N\Delta t}\left[g\left(s, X_s^{-k\tau}\right) - g\left((N-1)\Delta, X_{-k\tau+(N-1)\Delta t}^{-k\tau}\right)\right] \mathrm{d}B_s\right|^2\right).\end{aligned}$$

By Proposition 1, there exists a constant C depend on A, f and g such that

$$\begin{aligned}&(1-\theta)\mathbb{E}\left|\int_{-k\tau+(N-1)\Delta t}^{-k\tau+N\Delta t}\left[-A\left(X_s^{-k\tau} - X_{-k\tau+(N-1)\Delta t}^{-k\tau}\right)\right.\right. \\ &\quad \left.\left. + f\left(s, X_s^{-k\tau}\right) - f\left((N-1)\Delta, X_{-k\tau+(N-1)\Delta t}^{-k\tau}\right)\right] \mathrm{d}s\right|^2 \\ &\quad + \theta\mathbb{E}\left|\int_{-k\tau+(N-1)\Delta t}^{-k\tau+N\Delta t}\left[-A\left(X_s^{-k\tau} - X_{-k\tau+N\Delta t}^{-k\tau}\right) + f\left(s, X_s^{-k\tau}\right) - f\left(N\Delta, X_{-k\tau+N\Delta t}^{-k\tau}\right)\right] \mathrm{d}s\right|^2 \\ &\quad + \mathbb{E}\left|\int_{-k\tau+(N-1)\Delta t}^{-k\tau+N\Delta t}\left[g\left(s, X_s^{-k\tau}\right) - g\left((N-1)\Delta, X_{-k\tau+(N-1)\Delta t}^{-k\tau}\right)\right] \mathrm{d}B_s\right|^2 \\ &\quad \leq C\Delta t^3\left(1 + \sup_{k,N}\mathbb{E}\left|X_{-k\tau+N\Delta t}^{-k\tau}\right|^2\right) := \beta\Delta t^3.\end{aligned}$$

By Assumption 1 and the above mentioned estimate we have

$$(1 + \Delta t(2\theta - 1)(\lambda_1 - \mu))\mathbb{E}\left|e_N\right|^2 \leq \widetilde{C}\mathbb{E}\left|e_{N-1}\right|^2 + \frac{\beta\Delta t^3}{\epsilon_0^2},$$

where $\widetilde{C} := 1 + 2\Delta t(1-\theta)(\mu - \lambda_1) + \Delta t^2(1-\theta)^2(\lambda_1^2 + K_1)$. Since $\widetilde{C} < 1$ we have

$$(1 + \Delta t(2\theta - 1)(\lambda_1 - \mu))\mathbb{E}\left|e_N\right|^2 \leq \mathbb{E}\left|e_{N-1}\right|^2 + \frac{2\beta\Delta t^2}{\lambda_1 - \mu}.$$

Choosing $\widehat{\alpha} := \frac{2\beta\Delta t}{(2\theta-1)(\lambda_1-\mu)^2}$, above inequality can be rearranged as follows

$$(1 + \Delta t(2\theta - 1)(\lambda_1 - \mu))\left(\mathbb{E}\left|e_N\right|^2 - \widehat{\alpha}\right) \leq \mathbb{E}\left|e_{N-1}\right|^2 - \widehat{\alpha}.$$

By iteration and using $e_0 = 0$, we get

$$\left|e_N\right|^2 \leq \left(1 - \frac{1}{(1 + \Delta t(2\theta - 1)(\lambda_1 - \mu))^N}\right)\frac{2\beta\Delta t}{(2\theta - 1)(\lambda_1 - \mu)^2},$$

and finally by Assumption 3 one can conclude that $\left|e_N\right|^2 \leq \frac{2\beta\Delta t}{(2\theta-1)(\lambda_1-\mu)^2}$, which the desired result follows. □

Corollary 1. *Let Assumptions 1 to 7 be satisfied, then for any $\Delta t < 1$ and $\tau = n\Delta t$, $n \in \mathbb{N}$, there exists a constant C depending on A, f and g such that the exact solution given in Theorem 1 and the numerical random periodic solutions of (4.10) given in Theorem 2 satisfy*

$$\sup_{t\in\mathscr{I}}\mathbb{E}\left|X_t^* - \widehat{X}_t^*\right| \leq C\Delta t^{1/2}. \tag{4.15}$$

Proof. The result simply follows from

$$\mathbb{E}\left|X_t^* - \widehat{X}_t^*\right|^2 \leq \limsup_k\left[\mathbb{E}\left|X_t^* - \widehat{X}_t^{-k\tau}\right|^2 + \mathbb{E}\left|X_t^{-k\tau} - \widehat{X}_t^{-k\tau}\right|^2 + \mathbb{E}\left|\widehat{X}_t^{-k\tau} - \widehat{X}_t^*\right|^2\right]. \quad \square$$

5. The random periodic solution of the θ-Milstein method

In this section we are going to study the θ-Milstein methods to approximate the solutions of SDEs (1.3) on the infinite horizon. As in the previous section, we consider the equidistant partition $\mathscr{I} = \{j\Delta t,\ j \in \mathbb{Z}\}$ with sufficiently small stepsize Δt. The θ-Milstein method applied to (1.3) starting at $-k\tau$ is given by the recursion

$$\begin{aligned}\hat{X}^{-k\tau}_{-k\tau+(j+1)\Delta t} =&\hat{X}^{-k\tau}_{-k\tau+j\Delta t} + (1-\theta)\left(-A\hat{X}^{-k\tau}_{-k\tau+j\Delta t} + f\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\right)\Delta t\\ &+ \theta\left(-A\hat{X}^{-k\tau}_{-k\tau+(j+1)\Delta t} + f\left((j+1)\Delta t, \hat{X}^{-k\tau}_{-k\tau+(j+1)\Delta t}\right)\right)\Delta t\\ &+ g\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\Delta B_{-k\tau+j\Delta t} + \frac{1}{2}Lg\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\left[\Delta B^2_{-k\tau+j\Delta t} - \Delta t\right],\end{aligned} \tag{5.16}$$

for $\theta \in [0,1]$. Here, $j \in \mathbb{N}$, ξ is the initial value $\hat{X}^{-k\tau}_{-k\tau}$, $\Delta B_{-k\tau+j\Delta t} = B_{-k\tau+(j+1)\Delta t} - B_{-k\tau+j\Delta t}$ and

$$Lg\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right) := g\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)g'\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right).$$

5.1. *Existence of random periodic*

In this subsection we prove that the θ-Milstein method (5.16) with $\theta \in [0,1]$ generates a unique discretized random periodic solution. For didactic purpose, we divide this section in two parts: in the first part, we discuss the case in which $\theta \in [1/2,1]$; then, considering some additional assumptions on the drift coefficient, we discuss the case $\theta \in [0,1/2)$. Using the results of the next two parts, the convergence of the discretized semi-flow to a random periodic solution is then proved in Section 5.2.

5.1.1. *Case $\theta \in [1/2,1]$*

For $\theta \in [1/2,1]$ the proofs of the following results can be obtained by a similar arguments as in Section 4.1. Then, we omit their proofs for the sake of brevity.

Next lemma is related to the boundedness of the solution obtained by (5.16).

Lemma 9. *Under Assumptions 1 to 3 and 8, for any $\Delta t \in (0, \Delta t^* \wedge \Delta t^{**})$ where Δt^* is determined in Lemma 7 and $\Delta t^{**} := -2(4(1-\theta)(\mu-\lambda_1)+\sigma)/\sigma\gamma^2$, the numerical solution generated by θ-Milstein method (5.16) satisfies*

$$\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right|^2 \leq \tilde{C}_1,$$

where $\tilde{C}_1$ is a constant that does not rely on j.

The following lemma is focused on the difference of the two solutions with different initial values.

Lemma 10. *Suppose that $\hat{X}^{-k\tau}_{-k\tau+j\Delta t}$ and $\hat{Y}^{-k\tau}_{-k\tau+j\Delta t}$ are numerical solutions of θ-Milstein method with initial values ξ and η, respectively. Under Assumptions 1, 2, and 3 to 8, for any $\Delta t < -2(4(1-\theta)(\mu-\lambda_1)+K_1)/K1$, there exists a constant $\tilde{C}^j_2 > 0$ such that*

$$\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t} - \hat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right|^2 \leq \tilde{C}^j_2,$$

where $\lim_{j\to\infty}\tilde{C}^j_2 = 0$.

5.1.2. *Case $\theta \in [0,1/2)$*

We start the analysis of the case $\theta \in [0,1/2)$ by assuming a linear growth condition on the drift coefficient.

Assumption 9. There exist positive constants ζ and c such that

$$|f(x)|^2 \leq \zeta|x|^2 + c,$$

for any $x \in \mathbb{R}^m$.

Lemma 11. *Under Assumptions 1–9, there exists a constant $\tilde{C}_3 > 0$ such that the solution generated by the θ-Milstein (5.16) satisfies*

$$\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right|^2 \leq \tilde{C}_3,$$

for any $\Delta t < \frac{-(2(\mu-\lambda_1)+\sigma)}{(1-\theta)^2(\lambda_1^2+2\lambda_1\mu+\zeta)+\frac{1}{2}\gamma^2\sigma}$.

Proof. By considered Assumptions we have

$$\begin{aligned}\left|\hat{X}^{-k\tau}_{-k\tau+(j+1)\Delta t}\right|^2 \leq& \frac{1}{2}(I-\Delta t\theta)\left|\hat{X}^{-k\tau}_{-k\tau+(j+1)\Delta t}\right|^2\\ &+ \Delta t\theta\left\langle \hat{X}^{-k\tau}_{-k\tau+(j+1)\Delta t}, f\left((j+1)\Delta t, \hat{X}^{-k\tau}_{-k\tau+(j+1)\Delta t}\right)\right\rangle\\ &+ \frac{1}{2}\left[(I-2\Delta t(1-\theta)A)\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right|^2\right.\\ &+2\Delta t(1-\theta)\left\langle \hat{X}^{-k\tau}_{-k\tau+j\Delta t}, f\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\right\rangle\\ &+ \Delta t^2(1-\theta)^2\left|-A\hat{X}^{-k\tau}_{-k\tau+j\Delta t} + f\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\right|^2 + \left|g\left(j\Delta t, \hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\right|^2\Delta t\end{aligned}$$

$$+\frac{1}{4}\left|Lg\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)(\Delta B^2_{-k\tau+j\Delta t}-\Delta t)\right|^2\Big]$$
$$+\left|g\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\right|^2(\left|\Delta B_{-k\tau+j\Delta t}\right|^2-\Delta t)$$
$$+2\left\langle \hat{X}^{-k\tau}_{-k\tau+j\Delta t}+\Delta t(1-\theta)\left[-A\hat{X}^{-k\tau}_{-k\tau+j\Delta t}+f\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\right],g\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\right.$$
$$\left.\times\,\Delta B_{-k\tau+j\Delta t}+\frac{1}{2}Lg\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)(\Delta B^2_{-k\tau+j\Delta t}-\Delta t)\right\rangle$$
$$+\left\langle g\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\Delta B_{-k\tau+j\Delta t},Lg\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)(\Delta B^2_{-k\tau+j\Delta t}-\Delta t)\right\rangle.$$

Using this fact that $\mathbb{E}(\Delta B_{-k\tau+j\Delta t})=0$ and $\mathbb{E}\left|\Delta B_{-k\tau+j\Delta t}\right|^2=\Delta t$, we can see

$$\mathbb{E}\left\langle \hat{X}^{-k\tau}_{-k\tau+j\Delta t}+\Delta t(1-\theta)\left[-A\hat{X}^{-k\tau}_{-k\tau+j\Delta t}+f\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\right],g\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\right.$$
$$\left.\times\,\Delta B_{-k\tau+j\Delta t}+\frac{1}{2}Lg\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)(\Delta B^2_{-k\tau+j\Delta t}-\Delta t)\right\rangle=0,$$

$$\mathbb{E}\left\langle g\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\Delta B_{-k\tau+j\Delta t},Lg\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)(\Delta B^2_{-k\tau+j\Delta t}-\Delta t)\right\rangle=0$$

and

$$\mathbb{E}\left(\left|g\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\right|^2(\left|\Delta B_{-k\tau+j\Delta t}\right|^2-\Delta t)\right)=0.$$

By Assumptions 4, 6 and 8 it can be seen that

$$\mathbb{E}\left|Lg\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\Delta B^2_{-k\tau+j\Delta t}-Lg\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)\Delta t\right|^2\le 2\Delta t^2\gamma^2\left(\sigma\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right|^2+b\right).$$

Then, side-by-side expectation yields

$$\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau+(j+1)\Delta t}\right|^2\le A_1\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right|^2+A_2,$$

where

$$A1:=\frac{1}{1+2\Delta t\theta(\lambda_1-\mu)}\left(1+2\Delta t(1-\theta)(\mu-\lambda_1)+\Delta t^2(1-\theta)^2(\lambda_1^2-2\lambda_1\mu+\zeta)+\sigma\Delta t+\frac{\Delta t^2}{2}\sigma\gamma^2\right)$$

and

$$A_2:=\frac{1}{1+2\Delta t\theta(\lambda_1-\mu)}\left(2a\Delta t(1-\theta)+\Delta t^2(1-\theta)^2(c-2\lambda_1 a)+b\Delta t+\frac{\Delta t^2}{2}b\gamma^2+2a\Delta t\theta\right).$$

Since $A_1\in(0,1)$ by iteration we get

$$\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right|\le A_1^j\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau}\right|^2+A_2\sum_{i=0}^{j}A_1^i\le\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau}\right|^2+\frac{A_2}{1-A_1}:=\tilde{C}_3,$$

leading to the thesis. □

Next lemma is concerned with the estimate of the difference of two solutions obtained by θ-Milstein method with distinct initial values.

Lemma 12. *Suppose that $\hat{X}^{-k\tau}_{-k\tau+j\Delta t}$ and $\hat{Y}^{-k\tau}_{-k\tau+j\Delta t}$ are numerical solutions computed by θ-Milstein method with initial values ξ and η, respectively. Under Assumptions 1–3 and 6–8, for any $\Delta t<-(2(\mu-\lambda)+K_1)/((1-\theta)^2(\lambda_1^2+K_1)+K_1/2)$, there exists a constant $\tilde{C}_4^j>0$ such that*

$$\mathbb{E}\left|\hat{X}^{-k\tau}_{-k\tau+j\Delta t}-\hat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right|^2\le\tilde{C}_4^j,$$

where $\lim_{j\to\infty}\tilde{C}_4^j=0$.

Proof. Denoting $H_j:=\hat{X}^{-k\tau}_{-k\tau+j\Delta t}-\hat{Y}^{-k\tau}_{-k\tau+j\Delta t}$ and following similar arguments as in the proof of Lemma 11, we have

$$\left|H_{j+1}\right|^2\le\frac{1}{2}(I-\Delta t\theta A)\left|H_{j+1}\right|^2+\Delta t\theta\left\langle H_{j+1},f\left((j+1)\Delta t,\hat{X}^{-k\tau}_{-k\tau+(j+1)\Delta t}\right)\right.$$
$$\left.-f\left((j+1)\Delta t,\hat{Y}^{-k\tau}_{-k\tau+(j+1)\Delta t}\right)\right\rangle+\frac{1}{2}\left[(I-2\Delta t(1-\theta)A)\left|H_j\right|^2\right.$$
$$+2\Delta t(1-\theta)\left\langle H_j,f\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)-f\left(j\Delta t,\hat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right)\right\rangle$$
$$+\Delta t^2(1-\theta)^2\left|-AH_j+f\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)-f\left(j\Delta t,\hat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right)\right|^2$$
$$+\frac{1}{4}\left|\left(Lg\left(j\Delta t,\hat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)-Lg\left(j\Delta t,\hat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right)\right)(\Delta B^2_{-k\tau+j\Delta t}-\Delta t)\right|^2$$

$$+\Delta t\left|g\left(j\Delta t,\widehat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)-g\left(j\Delta t,\widehat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right)\right|^2+Q_j\Big],$$

where

$$\begin{aligned}Q_j:=&\left|g\left(j\Delta t,\widehat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)-g\left(j\Delta t,\widehat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right)\right|^2(\Delta B^2_{-k\tau+j\Delta t}-\Delta t)\\&+2\left\langle H_j+\Delta t(1-\theta)\left(f\left(j\Delta t,\widehat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)-f\left(j\Delta t,\widehat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right)\right),\right.\\&\left(g\left(j\Delta t,\widehat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)-g\left(j\Delta t,\widehat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right)\right)\Delta B_{-k\tau+j\Delta t}\\&\left.+\frac{1}{2}\left[Lg\left(j\Delta t,\widehat{X}^{-k\tau}_{-k\tau+j\Delta t}\right)-Lg\left(j\Delta t,\widehat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right)\right]\left(\Delta B^2_{-k\tau+j\Delta t}-\Delta t\right)\right\rangle.\end{aligned}$$

Since $\mathbb{E}(Q_j)=0$, by taking expectation and making use of Assumptions 1, 2, 6 and 8, we obtain

$$\mathbb{E}\left|H_{j+1}\right|^2\le A_3\left|H_j\right|^2,$$

where

$$A_3:=\frac{1+2\Delta t(1-\theta)(\mu-\lambda_1)+K_1\Delta t+\Delta t^2(1-\theta)^2(\lambda_1^2+K_1)+\frac{K_1}{2}\Delta t^2}{1+2\Delta t\theta(\lambda_1-\mu)}.$$

For any $\Delta t<\dfrac{-(2(\mu-\lambda)+K_1)}{(1-\theta)^2(\lambda_1^2+K_1)+K_1/2}$, we have $A_3\in(0,1)$ and so by iteration

$$\mathbb{E}\left|\widehat{X}^{-k\tau}_{-k\tau+j\Delta t}-\widehat{Y}^{-k\tau}_{-k\tau+j\Delta t}\right|^2\le A_3^j\mathbb{E}\left|\xi-\eta\right|^2:=\widetilde{C}_4^j.\quad\square$$

The random periodicity of the numerical solutions obtained by θ-Milstein method (5.16) is shown in the following theorem.

Theorem 4. *Let Assumptions 1–3 hold. Then for any $\Delta t\in(0,1)$ with $\tau=n\Delta t$, $n\in\mathbb{N}$, the θ-Milstein method (5.16) generates a random periodic solution on $\mathscr{I}$, i.e., there exists $\widehat{X}_r^*\in L^2(\Omega)$ such that*

$$\lim_{k\to\infty}\mathbb{E}|\widehat{X}_r^{-k\tau}(\xi)-\widehat{X}_r^*|=0.\tag{5.17}$$

Proof. By Lemma 10 when $\theta\in[1/2,1]$ and Lemma 12 when $\theta\in[0,1/2)$, the proof is straightforward by following a similar argument in the proof of Theorem 8 in [17]. $\square$

5.2. Error analysis

The aim of this part is to provide a strong convergence result for the θ-Milstein method (5.16). To this end, we first consider the local error of the numerical approximation (5.16). Replacing $\widehat{X}^{-k\tau}_{-k\tau+(N-1)\Delta t}$ and $\widehat{X}^{-k\tau}_{-k\tau+N\Delta t}$ in (5.16) by exact solution $X^{-k\tau}_{-k\tau+(N-1)\Delta t}$ and $X^{-k\tau}_{-k\tau+N\Delta t}$, respectively, we define the local error term R as

$$\begin{aligned}\widetilde{R}_N:=&X^{-k\tau}_{-k\tau+N\Delta t}-X^{-k\tau}_{-k\tau+(N-1)\Delta t}\\&-\Delta t(1-\theta)\left[-AX^{-k\tau}_{-k\tau+(N-1)\Delta t}+f\left((N-1)\Delta t,X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right]\\&-\Delta t\theta\left[-AX^{-k\tau}_{-k\tau+N\Delta t}+f\left(N\Delta t,X^{-k\tau}_{-k\tau+N\Delta t}\right)\right]\\&-\Delta B_{-k\tau+(N-1)\Delta t}g\left((N-1)\Delta t,X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\\&-I_{(1,1)}L_1g\left((N-1)\Delta t,X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right),\end{aligned}\tag{5.18}$$

where $I_{(1,1)}=\frac{1}{2}(\Delta B^2_{-k\tau+(N-1)\Delta t}-\Delta t)$ and $L_1=g(x)\frac{\partial}{\partial x}$. Using Itô-Taylor expansion, we obtain

$$\begin{aligned}X^{-k\tau}_{-k\tau+N\Delta t}=&(I-\Delta tA)X^{-k\tau}_{-k\tau+(N-1)\Delta t}+\Delta tf\left((N-1)\Delta t,X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\\&+\Delta B_{-k\tau+(N-1)\Delta t}g\left((N-1),X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\\&+I_{(1,1)}L_1g\left((N-1)\Delta t,X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)+R_1,\end{aligned}\tag{5.19}$$

$$\begin{aligned}f\left(N\Delta t,X^{-k\tau}_{-k\tau+N\Delta t}\right)=&f\left((N-1)\Delta t,X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\\&+\Delta tL_0f\left((N-1)\Delta t,X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\\&+\Delta B_{-k\tau+(N-1)\Delta t}L_1f\left((N-1)\Delta t,X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)+R_2,\end{aligned}\tag{5.20}$$

where the operator L_0 is given by

$$L_0=f(x)\frac{\partial}{\partial x}+\frac{1}{2}g^2(x)\frac{\partial^2}{\partial x^2}$$

and the reminder terms R_1 and R_2 satisfy $\mathbb{E}R_i = O(\Delta t^2)$, $i = 1,2$. Then, inserting (5.19) and (5.20) in (5.18) gives the local error term by

$$\begin{aligned}\widetilde{R}_N = &-\Delta t^2\theta\left(L_0 f\left((N-1)\Delta t, X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right.\\ &\left.-A\left[-AX^{-k\tau}_{-k\tau+(N-1)\Delta t} + f\left((N-1)\Delta t, X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right]\right)\\ &+ R_1 + \Delta t\theta(AR_1 - R_2) + \Delta t\theta\Delta B_{-k\tau+(N-1)\Delta t}\left(Ag\left((N-1)\Delta t, X_{-k\tau+(N-1)\Delta t}\right)\right.\\ &\left.-f\left((N-1)\Delta t, X_{-k\tau+(N-1)\Delta t}\right)L_0 f\left((N-1)\Delta t, X_{-k\tau+(N-1)\Delta t}\right)\right)\\ &+ \Delta t\theta I_{(1,1)}L_1 g\left((N-1)\Delta t, X_{-k\tau+(N-1)\Delta t}\right).\end{aligned}$$

Since $\mathbb{E}I_{(1,1)} = 0$, it is easy to obtain that $\mathbb{E}\widetilde{R}_N = O(\Delta t^2)$.

Theorem 5. *Let Assumptions 1, 2 and 6 be satisfied. Then, there exists a constant $\widetilde{C}_5$ such that for any $\Delta t = \tau/n$ for some $n \in \mathbb{N}$, with $N = kn$, and initial conditions $\widehat{X}^{-k\tau}_{-k\tau} = X^{-k\tau}_{-k\tau} = \xi$, we have*

$$\sup_{k,N} \mathbb{E}\left|X^{-k\tau}_{-k\tau+N\Delta} - \widehat{X}^{-k\tau}_{-k\tau+N\Delta}\right| \le \widetilde{C}_5\Delta t. \tag{5.21}$$

Proof. Subtracting (5.16) from (5.18), we derive

$$\begin{aligned}X^{-k\tau}_{-k\tau+N\Delta t} - \widehat{X}^{-k\tau}_{-k\tau+N\Delta t} =& X^{-k\tau}_{-k\tau+(N-1)\Delta t} - \widehat{X}^{-k\tau}_{-k\tau+(N-1)\Delta t}\\ &+ \Delta t(1-\theta)\left[-A(X^{-k\tau}_{-k\tau+(N-1)\Delta t} - \widehat{X}^{-k\tau}_{-k\tau+(N-1)\Delta t})\right.\\ &\left.+f\left((N-1)\Delta t, X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right) - f\left((N-1)\Delta t, \widehat{X}^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right]\\ &+ \Delta t\theta\left[-A(X^{-k\tau}_{-k\tau+N\Delta t} - \widehat{X}^{-k\tau}_{-k\tau+N\Delta t})\right.\\ &\left.+f\left(N\Delta t, X^{-k\tau}_{-k\tau+N\Delta t}\right) - f\left(N\Delta t, \widehat{X}^{-k\tau}_{-k\tau+N\Delta t}\right)\right]\\ &+ \Delta B_{-k\tau+(N-1)\Delta t}\left[g\left((N-1)\Delta t, X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right.\\ &\left.-g\left((N-1)\Delta t, \widehat{X}^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right]\\ &+ \left[L_1 g\left((N-1)\Delta t, X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right.\\ &\left.-L_1 g\left((N-1)\Delta t, \widehat{X}^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right] I_{(1,1)} + \widetilde{R}_N.\end{aligned}$$

Denoting $e_N = X^{-k\tau}_{-k\tau+(N-1)\Delta t} - \widehat{X}^{-k\tau}_{-k\tau+(N-1)\Delta t}$ we have the following recursive relation

$$e_N = e_{N-1} + \Delta t\Delta F_{N-1} + \Delta\Gamma_{N-1} + \widetilde{R}_N = e_0 + \Delta t\sum_{i=0}^{N-1}\Delta F_i + \sum_{i=0}^{N-1}\Delta\Gamma_i + \sum_{i=0}^{N-1}\widetilde{R}_i, \tag{5.22}$$

where

$$\begin{aligned}\Delta F_{N-1} :=&(1-\theta)\left[-Ae_{N-1} + f\left((N-1)\Delta t, X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right) - f\left((N-1)\Delta t, \widehat{X}^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right]\\ &+ \theta\left[-Ae_N + f\left(N\Delta t, X^{-k\tau}_{-k\tau+N\Delta t}\right) - f\left(N\Delta t, \widehat{X}^{-k\tau}_{-k\tau+N\Delta t}\right)\right],\end{aligned}$$

and

$$\begin{aligned}\Delta\Gamma_{N-1} :=&\Delta B_{-k\tau+(N-1)\Delta t}\left[g\left((N-1)\Delta t, X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right) - g\left((N-1)\Delta t, \widehat{X}^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right]\\ &+ \left[L_1 g\left((N-1)\Delta t, X^{-k\tau}_{-k\tau+(N-1)\Delta t}\right) - L_1 g\left((N-1)\Delta t, \widehat{X}^{-k\tau}_{-k\tau+(N-1)\Delta t}\right)\right] I_{(1,1)}.\end{aligned}$$

Taking expectation from both sides of (5.22) yields

$$\mathbb{E}\left|e_N\right|^2 \le 4\left[\mathbb{E}\left|e_0\right|^2 + \Delta t^2\mathbb{E}\left|\sum_{i=0}^{N-1}\Delta F_i\right|^2 + \mathbb{E}\left|\sum_{i=0}^{N-1}\Delta\Gamma_i\right|^2 + \mathbb{E}\left|\sum_{i=0}^{N-1}\widetilde{R}_i\right|^2\right]. \tag{5.23}$$

The expectation of the second term in (5.23) is estimated as

$$\begin{aligned}\Delta t^2\mathbb{E}\left|\sum_{i=0}^{N-1}\Delta F_i\right|^2 &\le nk\Delta t^2\sum_{i=0}^{N-1}\mathbb{E}\left|\Delta F_{N-1}\right|^2\\ &\le 2k\tau\Delta t(\lambda_1^2+\mu)\sum_{i=0}^{N-1}\left[(1-\theta)^2\mathbb{E}\left|e_{i-1}\right|^2 + \theta^2\sum_{i=0}^{N-1}\mathbb{E}\left|e_i\right|^2\right]\\ &\le 2k\tau\Delta t(\lambda_1^2+\mu)\left[\theta^2\sum_{i=0}^{N-1}\left|e_i\right|^2 + C_\theta\sum_{i=0}^{N-1}\mathbb{E}\left|e_{i-1}\right|^2\right],\end{aligned} \tag{5.24}$$

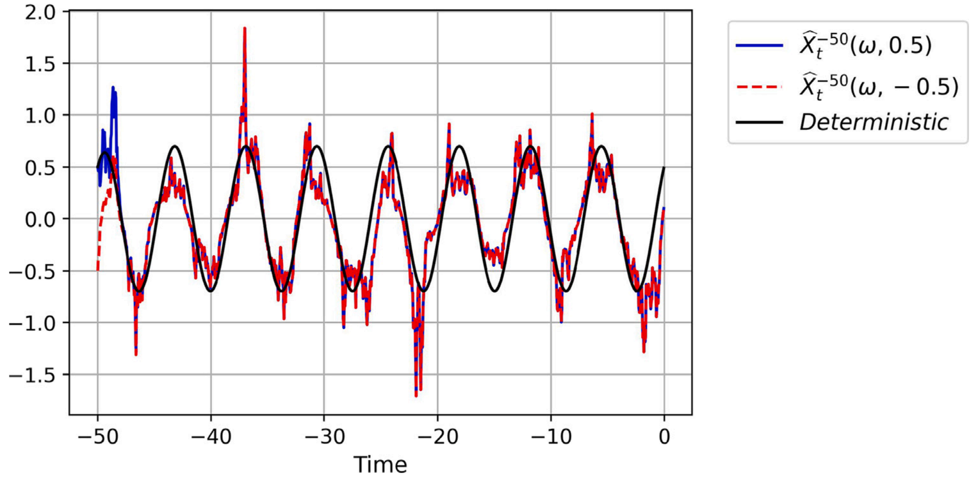

Fig. 6.1. Two paths generated by θ-Maruyama method when $\theta = 0.6$ from various initial values together with the periodic solution of deterministic ODE with $\sigma = 0$ for Eq. (6.28).

where C_θ : $2(\theta^2 \vee (1 - \theta)^2)$. Since $\mathbb{E}\Delta B_{-k\tau+(N-1)\Delta t} = 0$ and $\mathbb{E}_{(1,1)} = 0$ and since we know that the expectation of products of terms in $\mathbb{E}\left|\sum_{i=0}^{N-1} \Delta\Gamma_i\right|^2$ vanishes, we obtain

$$\mathbb{E}\left|\sum_{i=0}^{N-1} \Delta\Gamma_i\right|^2 \leq \sum_{i=0}^{N-1} \mathbb{E}\left|\Delta\Gamma_i\right|^2 \leq 2\Delta t(\lambda_1^2 + K_1) \sum_{i=0}^{N-1} \mathbb{E}\left|e_i\right|^2 \tag{5.25}$$

and similarly

$$\mathbb{E}\left|\sum_{i=0}^{N-1} \widetilde{R}_i\right|^2 \leq nk \sum_{i=0}^{N-1} \mathbb{E}\left|\widetilde{R}_i\right|^2 \leq O(\Delta t^2). \tag{5.26}$$

Substituting (5.24) and (5.25) into (5.23) and choosing Δt^* such that $4D_1\Delta t < \frac{1}{2}$ lead to

$$\mathbb{E}\left|e_N\right|^2 \leq 8\left[\mathbb{E}\left|e_0\right|^2 + \frac{\tau}{n}D_2 \sum_{i=0}^{N-1} \mathbb{E}\left|e_i\right|^2 + \mathbb{E}\left|\sum_{i=0}^{N-1} \widetilde{R}_i\right|^2\right],$$

where $D_1 := 2k\tau(\lambda_1^2 + \mu)\theta^2$ and $D_2 := 2k\tau(\lambda_1^2 + \mu)C_\theta + 2(\lambda_1^2 + K_1)$. Using (5.26) and Lemma 2 we get

$$\max\ \mathbb{E}\left|e_N\right|^2 \leq 8\exp(8\tau D_2)\left[\mathbb{E}\left|e_0\right|^2 + O(\Delta t^2)\right],$$

which taking the square root of both sides of the above-mentioned inequality results in the desired assertion. □

Corollary 2. *Let Assumptions 1–9 be satisfied, then for any $\Delta t \in (0,1)$ with $\tau = n\Delta t$, $n \in \mathbb{N}$, there exists a constant depending on A, f and g such that the exact solution given in Theorem 1 and the numerical random periodic solutions of (5.16) given in Theorem 4 satisfy*

$$\sup_{t\in\mathscr{I}} \mathbb{E}\left|X_t^* - \widehat{X}_t^*\right| \leq \widehat{C}\Delta t. \tag{5.27}$$

Proof. The result follows from

$$\mathbb{E}\left|X_t^* - \widehat{X}_t^*\right|^2 \leq \limsup_k \left[\mathbb{E}\left|X_t^* - \widehat{X}_t^{-k\tau}\right|^2 + \mathbb{E}\left|X_t^{-k\tau} - \widehat{X}_t^{-k\tau}\right|^2 + \mathbb{E}\left|\widehat{X}_t^{-k\tau} - \widehat{X}_t^*\right|^2\right]. \quad \square$$

6. Numerical experiments

In this section, some numerical examples are presented to support our theoretical results and show that the considered methods have random periodic solutions. To accomplish this, we consider two examples of SDEs with multiplicative noise and time periodic solutions.

6.1. Periodicity test

To indicate that the θ-Maruyama method generates random periodic solutions, consider the following equation

$$dX_t^{t_0} = \left[-X_t^{t_0} + \cos(t)\right]dt + \sigma X_t^{t_0}\mathrm{d}B_t. \tag{6.28}$$

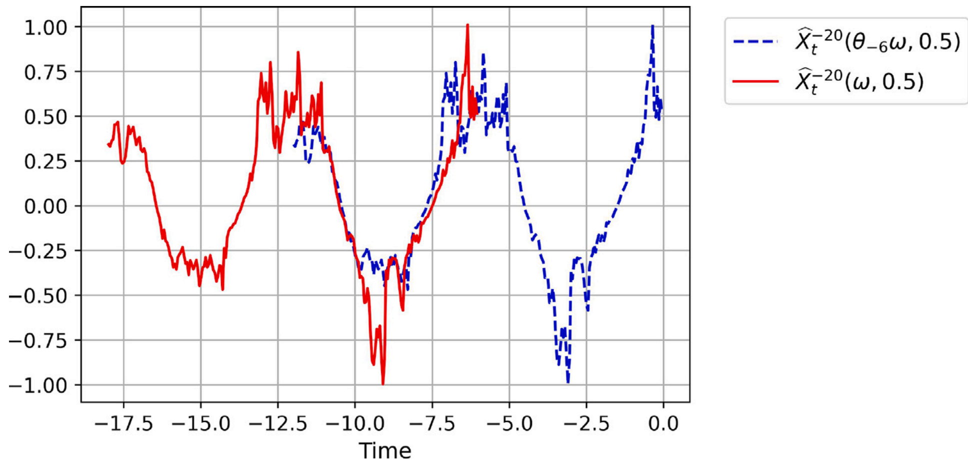

Fig. 6.2. Two paths generated by θ-Maruyama method for Eq. (6.28) when $\theta = 0.6$ on different realizations.

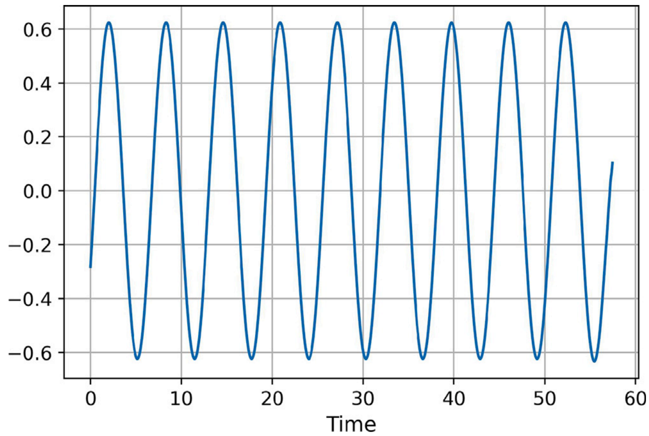

Fig. 6.3. The pull-back path $X^{-30}(t, v_{-t}\omega)$ generated by θ-Maruyama method for Ex. (6.28).

It can be easily verified the Assumptions 1 to 6 are fulfilled with $\lambda_1 = a = b = K_1 = 1$, $\mu = -1$, and $\sigma = 0.05$, and so that (6.28) has a random periodic solution according to Theorem 1 and its targeted θ simulations also admit a random periodic path. Let us first show that the mentioned schemes converge to random periodic paths despite various initial values and how the random periodic solutions oscillate around the deterministic periodic solution of the noiseless ODE when $\sigma = 0$. For this problem, we simulate two processes starting from $t_0 = -50$ to $T = 0$ with the stepsize $\Delta t = 0.05$ and two initial values 0.5 and −0.5. These two simulated paths are plotted in Fig. 6.1 by applying the θ-Maruyama method, when $\theta = 0.6$ and $\sigma = 1$, with the given initial values and with a shared Brownian realization. As displayed in Fig. 6.1, two paths match after a very short time.

Following the ideas in [16,17], to show the periodicity of the solutions we can use two approaches. In a first approach, we simulate the processes $\widehat{X}_t^*(\omega) = \widehat{X}_t^{-20}(\omega, 0.6)$ for $-18 \le t \le -6$ and $\widehat{X}_t^*(v_{-6}\omega) = X_t^{-20}(v_{-6}\omega, 0.6)$ for $-12 \le t \le 0$, with the same ω and stepsize $\Delta t = 0.05$. From Fig. 6.2 one can observe that the two simulated trajectories repeat each other with a time shift of period six.

As a second approach, in Fig. 6.3 we simulate $\widehat{X}_t^*(v_{-t}\omega)$ starting from $t_0 = 0$ to $T = 50$ for the same ω and stepsize as previous test. Indeed, one can see that $\widehat{X}_t^0(v_{-t}\omega)$ is periodic with period τ which shows a periodic pull-back path as expected.

Now, we aim to show that θ-Milstein method generates random periodic solution. To do this, we consider the following problem

$$dX_t^{t_0} = -\pi X_t^{t_0} dt + \sin(\pi t) dt + \sigma X_t^{t_0} \mathrm{d}B_t. \tag{6.29}$$

Following a similar way as for Eq. (6.28), we first show that the θ-Milstein method converges to its random periodic path despite various initial values and how the random periodic solutions oscillate around the deterministic periodic solution of the noiseless ODE when $\sigma = 0$. As before, we simulate two trajectories starting from $t_0 = -6$ to $T = 0$ with the stepsize $\Delta t = 0.05$ and two initial values 0.15 and −0.15. These two simulated paths are plotted in Fig. 6.4 by applying the θ-Milstein method, when $\theta = 0.6$ and $\sigma = 1$, with given initial values and shared Brownian realization. As shown in Fig. 6.4, two trajectories match after a short time.

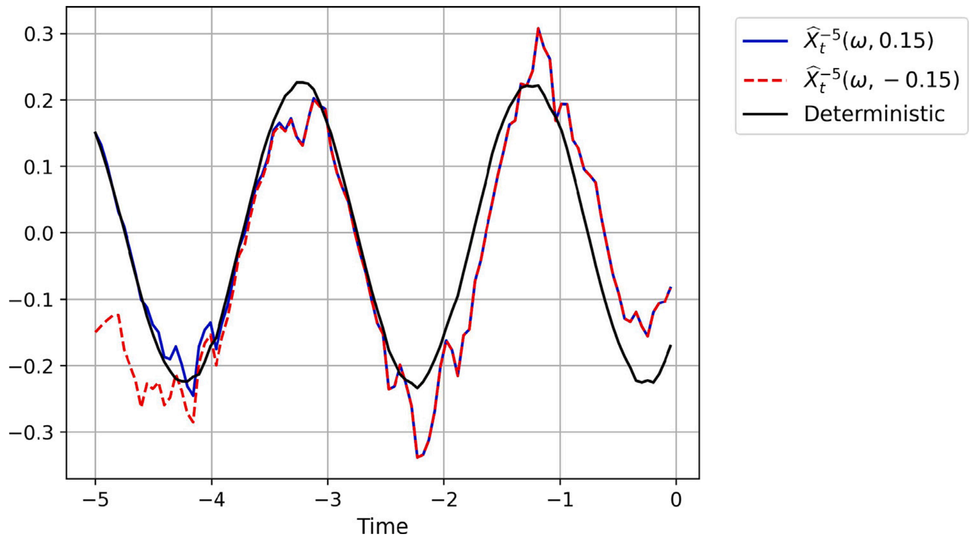

Fig. 6.4. Two paths generated by θ-Milstein method when $\theta = 0.6$ from various initial values together with the periodic solution of deterministic ODE with $\sigma = 0$ for Eq. (6.29).

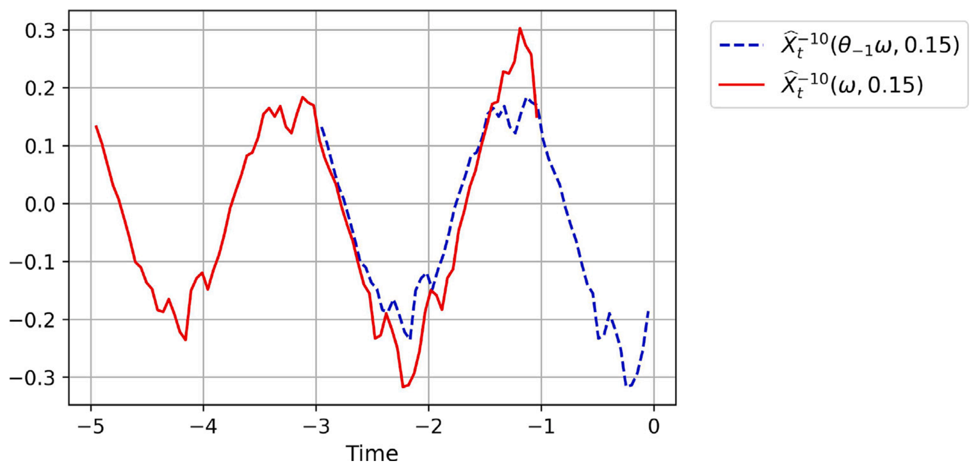

Fig. 6.5. Two paths generated by θ-Milstein method when $\theta = 0.6$ on different realizations for Eq. (6.29).

To show the periodicity of generated solutions obtained by θ-Milstein method, we first simulate two paths $\widehat{X}_t^*(\omega) = \widehat{X}_t^{-10}(\omega, 0.15)$ for $-5 \leq t \leq -1$ and $\widehat{X}_t^*(\upsilon_{-2}\omega) = X_t^{-10}(\upsilon_{-2}\omega, 0.15)$ for $-3 \leq t \leq 0$, with the same ω and stepsize $\Delta t = 0.05$. The results are plotted in Fig. 6.5 which show that the two simulated trajectories repeat each other with a time shift of period two. As the final test, in Fig. 6.6 we simulate $\widehat{X}_t^*(\upsilon_{-t}\omega)$ starting from $t_0 = 0$ to $T = 6$ for the same ω and stepsize as previous test. From this figure, we can observe that $\widehat{X}_t^0(\upsilon_{-t}\omega)$ is periodic with period τ which shows a periodic pull-back path as expected.

6.2. Convergence study

We conclude our selected experiments by testing the order of convergence of the considered schemes and comparing their performance with those of the backward Euler–Maruyama method. To attain this, we first choose the stepsize $\Delta t_{\text{ref}} = 2^{-15}$ to obtained a reference solution and then compare the numerical solutions generated by the mentioned θ-methods with larger stepsizes.

After simulating the random periodic solutions of Eqs. (6.28) and (6.29) over 500 different trajectories by both θ-Maruyama and θ-Milstein methods, we compute Monte Carlo estimates of the root-mean-squared errors between the reference solution and the individual methods with 5 different stepsizes $\Delta t = 2^{-i}$, $i = 4, 5, 6, 7, 8$. The results are presented in Figs. 6.7 and 6.8. In these figures we have also plotted the errors corresponding to the backward Euler–Maruyama and Implicit Milstein methods (when $\theta = 1$) with the same stepsizes. As we can see for both examples, the θ-Milstein method with $\theta = 0.6$ attains the expected order of convergence and, for Eq. (6.29), the order of convergence for θ-Maruyama methods looks even greater than $1/2$, which is beyond the theoretical order of convergence.

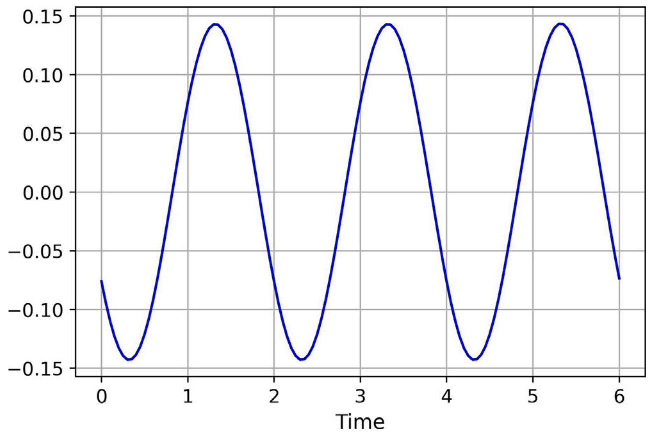

Fig. 6.6. The pull-back path $X^{-5}(t, v_{-t}\omega)$ generated by θ-Milstein method for Eq. (6.29).

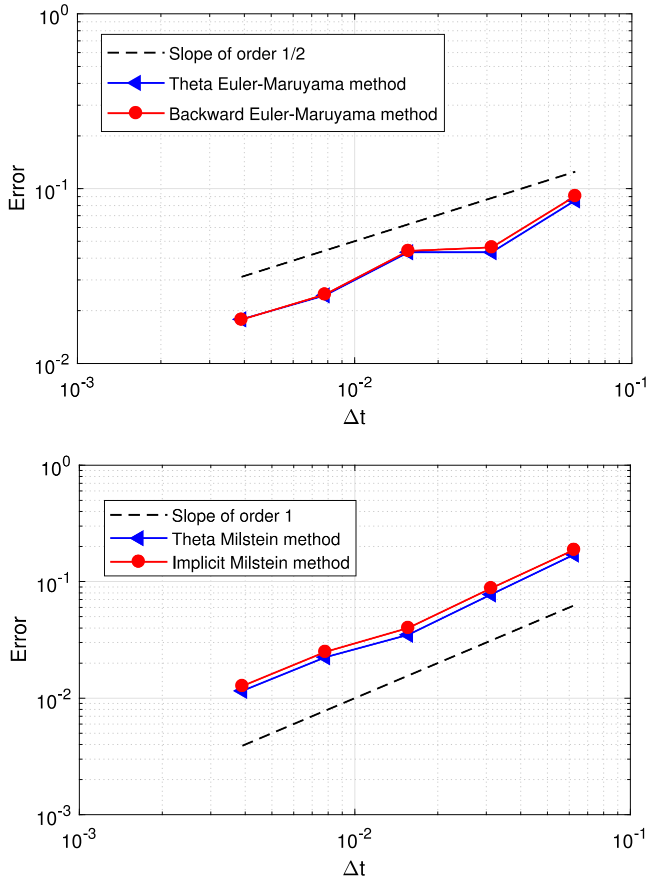

Fig. 6.7. Root-mean-square error vs. stepsize Δt as log–log plot for Eq. (6.28).

When approximating the random periodic solutions of SDEs, different choices of method parameters, denoted as θ, can have impacts on the accuracy and efficiency of the approximation. It has been proved that the step size or time increment used in the numerical integration scheme is influenced by the choice of θ. Consequently, the choice of θ can significantly impact the accuracy of the approximation. When $\theta = 1$, the scheme becomes fully implicit, requiring the solution of a system of equations at each time step. Implicit schemes tend to be more stable and accurate but may impose stricter constraints on the time increment. For intermediate values of θ (e.g. $0 < \theta < 1$), the scheme becomes semi-implicit or partially implicit. These schemes strike a balance between accuracy and stability by blending explicit and implicit components. The choice of θ determines the weight given to the explicit and implicit

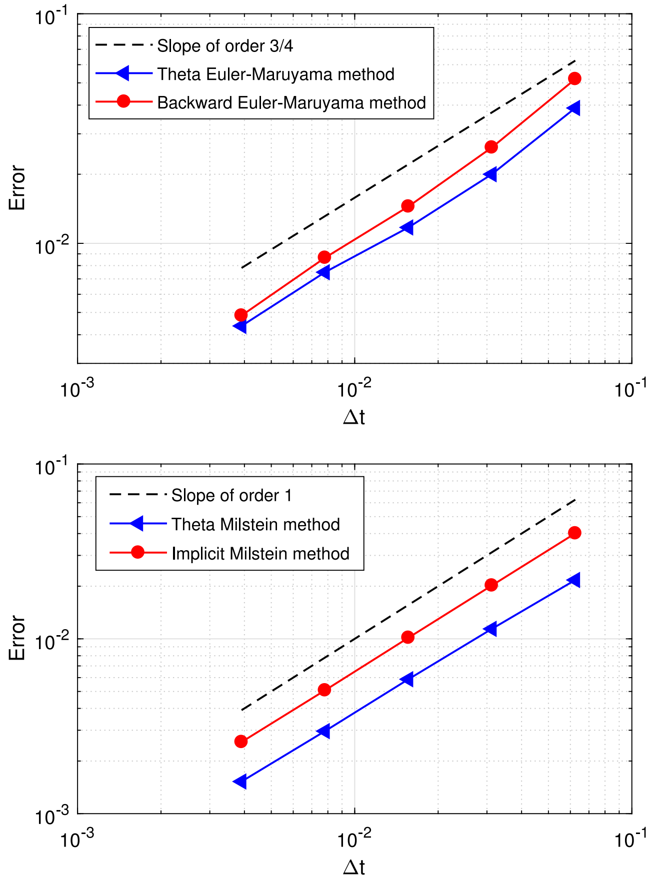

Fig. 6.8. Root-mean-square error vs. stepsize Δt as log–log plot for Eq. (6.29).

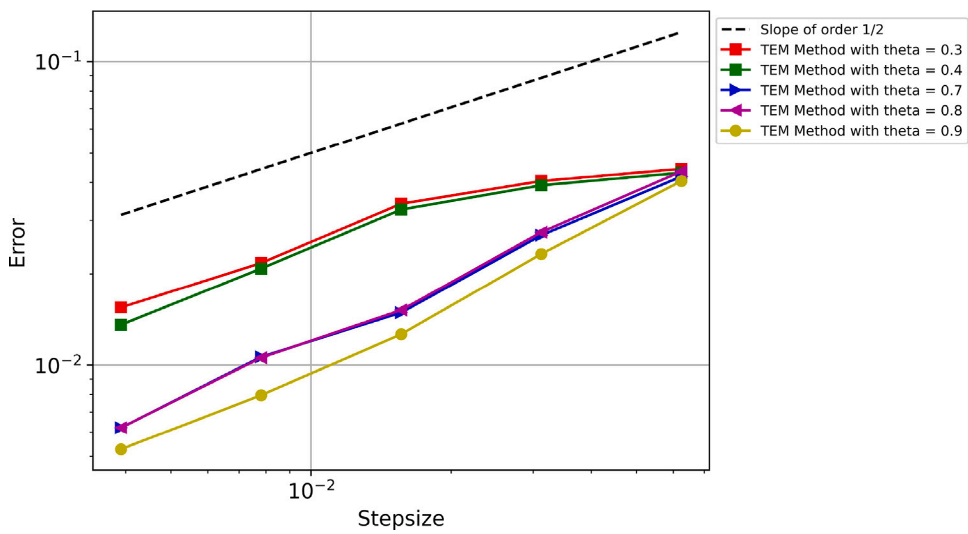

Fig. 6.9. Root-mean-square error obtained by θ-Euler–Maruyama (TEM) method vs. stepsize as log–log plot for Eq. (6.28) and different values of θ.

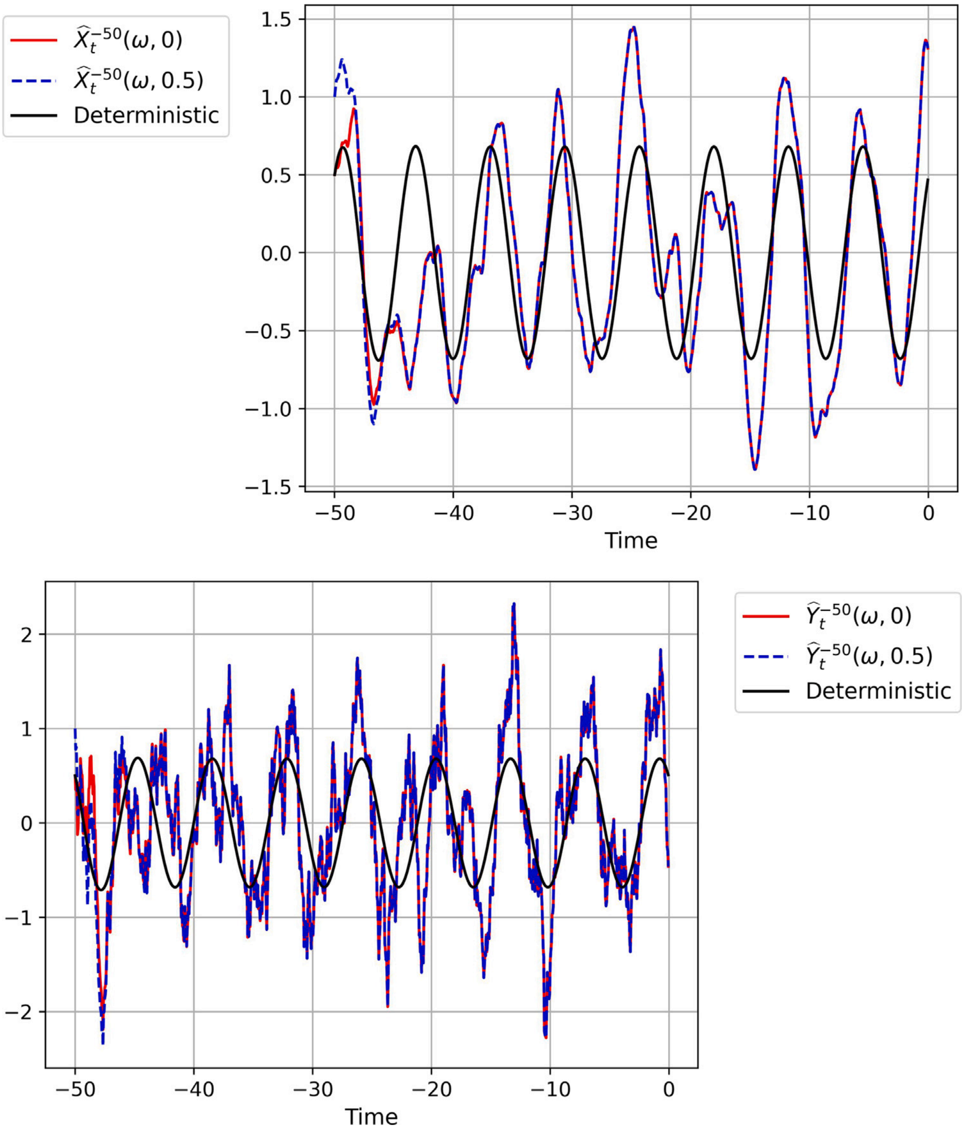

Fig. 7.10. Two paths generated by θ-Maruyama method when $\theta = 0.6$ from various initial values together with the periodic solution of deterministic ODE with $\sigma = 0$ for Eq. (7.30).

parts of the scheme, which influences the allowable time increment. To illustrate this impact, we examined the effect of different choices of θ on the Monte Carlo estimates of the root-mean-squared errors, as shown in Fig. 6.9, for problem (6.28).

7. Conclusions and open problems

In this paper, we have studied the existence and uniqueness of random periodic solutions to SDEs having the following form

$$\mathrm{d}X_t^{t_0} = \left[-AX_t^{t_0} + f(t, X_t^{t_0})\right]\mathrm{d}t + g(t, X_t^{t_0})\mathrm{d}B_t,$$

under a one-sided Lipschitz condition, and approximated their solutions via two classes of stochastic θ-methods. While SDEs with random periodic solutions have been studied for many years, there exists just a few effort in literature to approximate their solutions via numerical methods, by accurately retaining the periodic character. At the best of our knowledge, in the case of periodic phenomena described by second order problems, such as those described by Duffing equation, the numerical analysis of random periodicity remains an open problem. For example, consider a scalar periodic stochastic Duffing equation, that reduces to

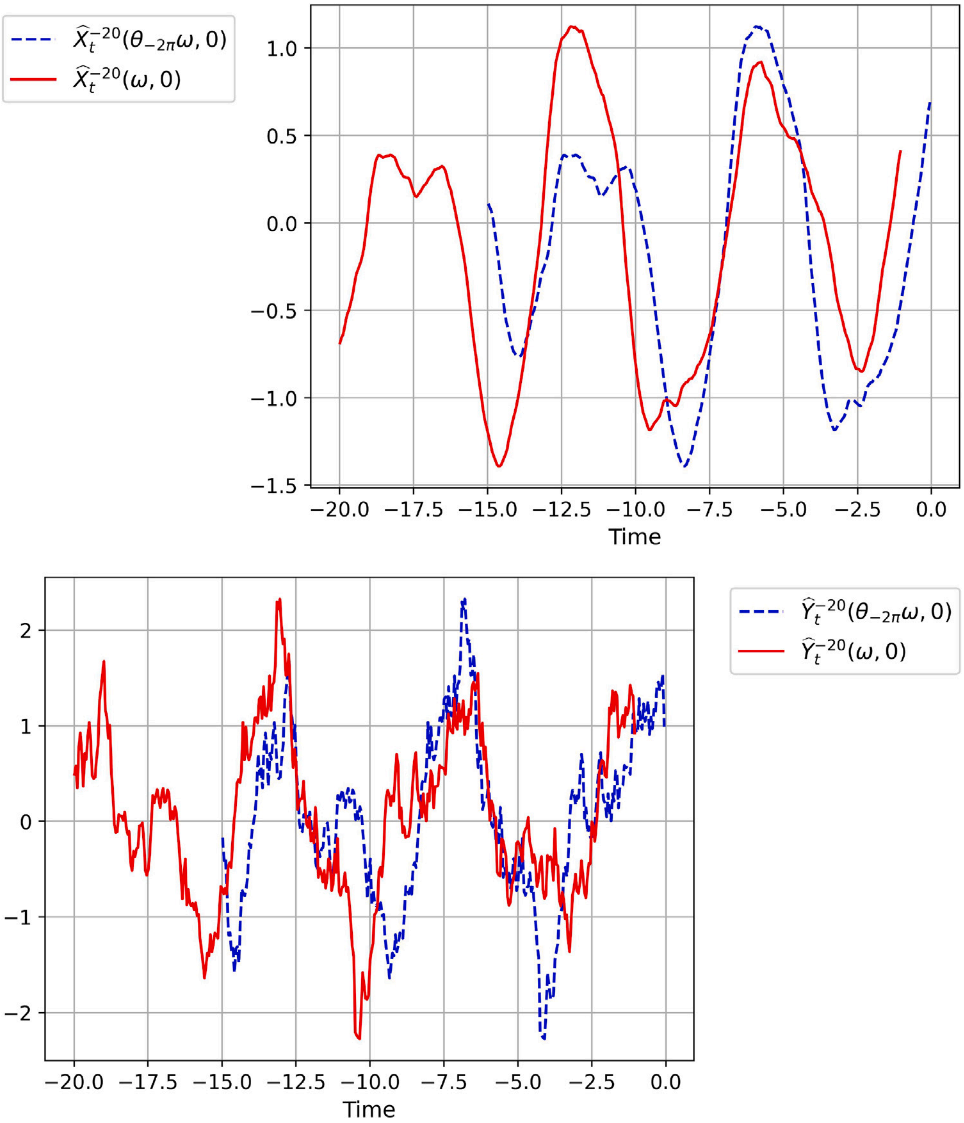

Fig. 7.11. Two paths generated by θ-Maruyama method when $\theta = 0.6$ on different realization.

the following system of first order SDEs

$$d\begin{pmatrix} X_t^{t_0} \\ Y_t^{t_0} \end{pmatrix} = \begin{pmatrix} Y_t^{t_0} \\ -2X_t^{t_0} - Y_t^{t_0} - \cos t \end{pmatrix} dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dB_t. \tag{7.30}$$

The existence of random periodic solutions is studied in [40]. From Figs. 7.10 and 7.11, we can guess that numerical solutions of (7.30) also admit periodic solutions in distribution and show a periodic pull-back path. However, our theoretical analysis does not cover this case. The existence and uniqueness of numerical solutions for this type of problem remains open.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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