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Perturbations of positive semigroups factorized via AM- and AL-spaces

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Abstract. In this note we extend perturbation results for positive C_0-semigroups on AM- and AL-spaces generalizing them to arbitrary Banach lattices in case the perturbation can be factorized appropriately. The abstract results are applied to domain perturbations of generators, a heat equation with boundary feedback and perturbations of the first derivative.

1. Introduction

Many systems evolving in time can be described by an Abstract Cauchy Problem of the form

$$\begin{cases} \frac{d}{dt} x(t) = Gx(t), \quad t \geq 0, \\ x(0) = x_0 \end{cases} \tag{ACP}$$

where G is an unbounded linear (e.g., differential) operator on a Banach space X, cf. [18, Chap. VI]. As shown in [18, Sect. II.6], this problem is well-posed if and only if G generates a C_0-semigroup $(S(t))_{t\geq 0}$ on X. Moreover, in this case its unique solution is given by $x(t) = S(t)x_0$.

For this reason it's vital to have tools at hand which allow verifying the generator property of a given operator G. In the dissipative case the Lumer–Phillips theorem is such a result which provides a characterization of generators of contraction semigroups in terms of G itself and applies to many concrete examples. Its counterpart in the non-dissipative case is the famous Hille–Yosida theorem. This result, however, is not based directly on the given operator G but on growth estimates of *all* powers of its resolvent which frequently are impossible to verify.

In order to check well-posedness of (ACP) for (non-dissipative) operators G where explicit computations involving the resolvent are impossible to perform, one can try to split G into a sum "$G = A + P$" for a simpler generator A and a perturbation P and then use some kind of perturbation theorem to conclude that also G generates a C_0-semigroup on X.

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Two important results in this direction are the perturbation theorems of Desch–Schappacher and Miyadera–Voigt, cf. [18, Thms. III.3.1 and 3.14]. One drawback of these results is that they are formulated by means of admissibility conditions which require an explicit knowledge of the unperturbed semigroup $(T(t))_{t\geq 0}$ generated by A. As shown by Bátkai et al. [12] and Desch–Voigt [16,26], there exist versions of these perturbation theorems which substitute these admissibility assumptions by (frequently simpler to verify) spectral conditions if one adds the notion of positivity on special types of Banach lattices into the general context.

To explain this better we first introduce our general setting. We choose two Banach spaces X and U. On these spaces we take operators

- $A : D(A) \subset X \to X$ with non-empty resolvent set $\rho(A)$,
- $B : U \to X_{-1}$, and $C : Z \to U$.

Here, Z is a Banach space satisfying[1] $D(A) \subseteq Z \subseteq X$. Moreover, X_{-1} denotes the extrapolation space with respect to A, cf. [18, Chap. II.5]. Then we consider structured perturbations of the form $P = BC : Z \to X_{-1}$ and obtain the perturbed operator $G = A_{BC}$ where

$$A_{BC} := (A_{-1} + BC)|_X, \quad D(A_{BC}) := \big\{x \in Z : (A_{-1} + BC)x \in X\big\}, \tag{1.1}$$

and $A_{-1} : X \subset X_{-1} \to X_{-1}$ is the extension of A to X.

This particular choice of perturbations $P = BC : Z \to X_{-1}$ is justified by its great flexibility and by many applications, see e.g. Sect. 4, [1–3] and [11, Sect. 4].

The above-mentioned simplified versions due to Desch–Voigt and Bátkai et al. fit in this framework if we choose $U = X$, assume A to generate a positive semigroup $(T(t))_{t\geq 0}$ on the Banach lattice X, and

- $C = \mathrm{Id}_X$, $P = B : X \to X_{-1}$ is positive where X is an AM-space, see [12], or
- $B = \mathrm{Id}_X$, $P = C : D(A) \to X$ is positive where X is an AL-space, see [16,26],

respectively. These assumptions allow rephrasing the admissibility conditions in terms of the resolvents $R(\lambda, A)$ of the unperturbed generator which in general are much easier to verify than those involving the operators $T(t)$.

The aim of this note is to generalize these positive perturbation results to operators $P = BC$ where in contrast to the previous setting we do *not* assume $U = X$ but only impose that

- $B : U \to X_{-1}$ and $C : X \to U$ are positive and U is an AM-space, or
- $B : U \to X$ and $C : D(A) \to U$ are positive and U is an AL-space,

respectively. Here in both cases X can be an arbitrary Banach lattice, while the hypotheses of Desch–Voigt and Bátkai et al. exclude, e.g., a priori applications on reflexive spaces X.

In the forthcoming paper [11] we will combine these results to obtain an analogous simplified version of the Weiss–Staffans perturbation result [1, Thm. 10] for generators

[1] For the Desch–Schappacher theorem $Z = X$ while for the Miyadera–Voigt theorem one has $Z = D(A)$.

of positive semigroups (which includes the Desch–Schappacher and Miyadera–Voigt theorems as special cases) in case $U = \mathbb{R}^N$ (i.e., U is simultaneously of type AL and AM) and

- $B : U \to X_{-1}$ and $C : Z \to U$ are positive,

where Z is a vector space satisfying $D(A) \subseteq Z \subseteq X$.

This paper is organized as follows. In Sect. 2 we introduce our general setup and prove an admissibility result for the control operator B. In Sect. 3 we present the main results of this note:

- Theorem 3.1 on positive structured perturbations factorized via AM-spaces,
- Theorem 3.7 on positive structured perturbations factorized via AL-spaces,
- Theorem 3.17 on structured perturbation via domination.

In Sect. 4 we demonstrate the power of our approach and apply these results to boundary perturbations of domains of generators, a heat equation with boundary feedback and perturbations of the first derivative. These examples show in particular the great advantage it makes assuming that U instead of X has to be an AM- or AL-space as needed in the previous works [12,16,26]. Finally, in Appendix A we briefly introduce some basic notions concerning Banach lattices and positive operators.

For related results on perturbation of generators of positive C_0-semigroups we refer to [5,7–10,17,20].

2. The general setup

Throughout this paper we impose the following standing assumption where we introduce the main objects of our investigations. Even if in the sequel we use some common terminology from systems theory we do not make use of any results of it.

Assumption 2.1. We consider

(i) a Banach lattice X called the *state space*;
(ii) a Banach lattice U called the *control/observation space*;
(iii) a *state operator* $A : D(A) \subset X \to X$ generating a positive C_0-semigroup $(T(t))_{t\geq 0}$ on X;
(iv) a *control operator* $B : U \to X_{-1}$;
(v) an *observation operator* $C : Z \to U$, where $D(A) \subseteq Z \subseteq X$.

Here the extrapolation space X_{-1} of X associated to the operator A is given by the completion $(X, \|\cdot\|_{-1})^{\sim}$ where $\|x\|_{-1} := \|R(\lambda, A)x\|$ for some fixed $\lambda \in \rho(A)$. Then for any $t \geq 0$ the operator $T(t)$ possesses a unique bounded extension $T_{-1}(t) \in \mathcal{L}(X_{-1})$ and $(T_{-1}(t))_{t\geq 0}$ is a C_0-semigroup on X_{-1} with generator A_{-1} having domain $D(A_{-1}) = X$. This definition does not depend on the particular choice of $\lambda \in \rho(A)$ since the norms $\|R(\lambda, A)\cdot\|$ are equivalent for any $\lambda \in \rho(A)$. For a detailed treatment of these facts we refer to [18, Chap. II.5].

In our case X is a Banach lattice, and it is a priori not clear how to extend the concept of positivity to X_{-1}. If A generates a positive C_0-semigroup on X, we follow [12, Def. 2.1] (for a more detailed analysis see [9]) and define the positive cone in X_{-1} as the closure

$$X_{-1,+} := \overline{X_+}^{\|\cdot\|_{-1}},$$

where X_+ denotes the positive cone of X (see Appendix A). It is immediate that $X_+ \subset X_{-1,+}$. If X is a real Banach lattice, then by [12, Prop. 2.3] we also have $X_+ = X_{-1,+} \cap X$. Moreover, by [12, Rem. 2.2] the following holds.

Lemma 2.2. *The operator $B : U \to X_{-1}$ is positive, i.e. $BU_+ \subseteq X_{-1,+}$, if and only if $R(\lambda, A_{-1})B : U \to X$ is positive for all $\lambda > \mathrm{s}(A)$.*

2.1. Admissibility

In this subsection we briefly recall some standard notions from linear systems theory, cf. [24, Chap. 10] and the references therein, which are essential for our approach.

Definition 2.3. The operator $B : U \to X_{-1}$ is said to be a 1-*admissible control operator* if there exists $t > 0$ such that

$$\int_0^t T_{-1}(t-s)Bu(s)\,ds \in X \quad \text{for all } u \in \mathrm{L}^1([0,t],U). \tag{2.1}$$

This means that the *controllability map* $\mathcal{B}_t : \mathrm{L}^1([0,t],U) \to X_{-1}$ given by

$$\mathcal{B}_t u := \int_0^t T_{-1}(t-s)Bu(s)ds, \quad u \in \mathrm{L}^1([0,t],U) \tag{2.2}$$

has range $\mathrm{rg}(\mathcal{B}_t) \subseteq X$. Since the operator $\mathcal{B}_t : \mathrm{L}^1([0,t],U) \to X_{-1}$ is bounded and X is continuously embedded in X_{-1}, by the Closed Graph Theorem it follows that $\mathcal{B}_t : \mathrm{L}^1([0,t],U) \to X$ is bounded (see [18, Cor. B.7]).

By changing the space of input functions u we obtain the following definition.

Definition 2.4. The operator $B : U \to X_{-1}$ is said to be an ∞-*admissible control operator* if there exists $t > 0$ such that

$$\int_0^t T_{-1}(t-s)Bu(s)ds \in X \quad \text{for all } u \in \mathrm{C}([0,t],U). \tag{2.3}$$

By the same reasoning as before, this definition means that the controllability map $\mathcal{B}_t : \mathrm{C}([0,t],U) \to X_{-1}$ given by

$$\mathcal{B}_t u := \int_0^t T_{-1}(t-s)Bu(s)ds, \quad u \in \mathrm{C}([0,t],U) \tag{2.4}$$

defines a bounded operator $\mathcal{B}_t : \mathrm{C}([0,t],U) \to X$. In Subsect. 3.1 we will verify the ∞-admissibility of a control operator B by using the following result.

Proposition 2.5. *The operator* $B : U \to X_{-1}$ *is an* ∞-admissible control operator *if there exists* $t > 0$*, a subspace* $\mathcal{D} \subset \mathrm{L}^\infty([0,t],U)$ *and* $M \geq 0$ *such that* $\mathrm{C}([0,t],U) \subseteq \overline{\mathcal{D}}$ *and*

$$(i) \quad \int_0^t T_{-1}(t-s)Bu(s)\,ds \in X \qquad \textit{for all } u \in \mathcal{D},$$
$$(ii) \quad \left\| \int_0^t T_{-1}(t-s)Bu(s)\,ds \right\|_X \leq M\|u\|_\infty \qquad \textit{for all } u \in \mathcal{D}.$$

Moreover, in this case $\|\mathcal{B}_t\|_{\mathcal{L}(\mathrm{C}([0,t],U),X)} \leq M$.

Proof. In any case, we can define the bounded operator $\mathcal{B}_t : \mathrm{L}^\infty([0,t],U) \to X_{-1}$ as in (2.4) for $u \in \mathrm{L}^\infty([0,t],U)$. Now assumptions (i) and (ii) imply that the restriction $\mathcal{B}_t|_{\mathcal{D}} : \mathcal{D} \to X$ is bounded of bound M, hence possesses a unique bounded extension $\mathcal{R} : \overline{\mathcal{D}} \to X$ of the same bound. Since by assumption $\mathrm{C}([0,t],U) \subseteq \overline{\mathcal{D}}$, for every $u \in \mathrm{C}([0,t],U)$ there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ converging to u in $\mathrm{L}^\infty([0,t],U)$. Then $\mathcal{B}_t u_n \to \mathcal{B}_t u$ in X_{-1} and $\mathcal{B}_t u_n = \mathcal{R}u_n \to \mathcal{R}u$ in X as $n \to +\infty$. This implies $\mathcal{B}_t u = \mathcal{R}u \in X$ for all $u \in \mathrm{C}([0,t],U)$, i.e., B is an ∞-admissible control operator. □

Remark 2.6. Condition (2.3) is weaker than (2.1). Moreover, a bounded operator $B : U \to X$ is always a 1-admissible control operator, hence also ∞-admissible.

For observation operators we recall an analogous notion

Definition 2.7. The operator $C : D(A) \to U$ is said to be a 1-*admissible observation operator* if there exist $t > 0$ and $M \geq 0$ such that

$$\int_0^t \left\|CT(s)x\right\|_U ds \leq M \cdot \|x\|_X \quad \text{for all } x \in D(A). \tag{2.5}$$

Since $D(A)$ is dense in X, the previous definition implies the existence of a bounded *observability map* $\mathcal{C}_t : X \to \mathrm{L}^1([0,t],U)$ satisfying $\|\mathcal{C}_t\| \leq M$ and

$$(\mathcal{C}_t x)(s) := CT(s)x, \quad x \in D(A),\ s \in [0,t]. \tag{2.6}$$

Remark 2.8. For an operator $C : D(A) \to U$ one could also introduce the concept of ∞-admissibility, i.e., ask for the existence of $t > 0$ and $M \geq 0$ satisfying

$$\sup_{s\in[0,t]} \left\|CT(s)x\right\|_U \leq M \cdot \|x\|_X \quad \text{for all } x \in D(A).$$

However, it is easy to see that C is ∞-admissible in this sense if and only if $C \in \mathcal{L}(X,U)$. Clearly, in this case the above estimate extends by density to all $x \in X$.

3. The main results

3.1. Positive structured perturbations factorized via AM-spaces

In this section we give a generalization of a positive perturbation theorem due to Bátkai et al. in [12], cf. Remark 3.2.(ii). With respect to our setting introduced in Sect. 2 we assume throughout this subsection that $Z = X$ which by [13, Thm. 10.20] implies that any positive $C : X \to U$ is bounded.

Theorem 3.1. *Let $A : D(A) \subseteq X \to X$ be the generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach lattice X. Assume that $B : U \to X_{-1}$ and $C : X \to U$ are positive linear operators, where U is an AM-space. If the spectral radius $\mathrm{r}(CR(\lambda, A_{-1})B) < 1$ for some $\lambda > \omega_0(A)$, then the operator*

$$\begin{aligned} A_{BC} &:= (A_{-1} + BC)|_X, \\ D(A_{BC}) &:= \{x \in X : (A_{-1} + BC)x \in X\} \end{aligned} \tag{3.1}$$

generates a positive C_0-semigroup $(S(t))_{t\geq 0}$ on X satisfying the variation of parameters formula

$$S(t)x := T(t)x + \int_0^t T_{-1}(t-s) \cdot BC \cdot S(s)x\,ds, \quad x \in D(A_{BC}),\ t \geq 0. \tag{3.2}$$

Remark 3.2. (i) If U is an AM-space with order unit e_U the assumption on the spectral radius can be dropped. Indeed, in this case by [4, Cor. 4.4] the space U can be equivalently normed such that $|u| \leq e_U$ for all $u \in U$ satisfying $\|u\|_U \leq 1$. Using this norm on U, for a positive operator $T \in \mathcal{L}(U)$ it holds $\|T\|_{\mathcal{L}(U)} = \|Te_U\|_U$. Hence, [18, Lem. II.3.4] implies

$$\begin{aligned} \mathrm{r}\big(CR(\lambda, A_{-1})B\big) &\leq \big\|CR(\lambda, A_{-1})B\big\|_{\mathcal{L}(U)} = \big\|CR(\lambda, A_{-1})A_{-1} \cdot A_{-1}^{-1}Be_U\big\|_U \\ &\leq \|C\|_{\mathcal{L}(X,U)} \cdot \big\|\big(\lambda R(\lambda, A) - \mathrm{Id}_U\big)A_{-1}^{-1}Be_U\big\|_X \\ &\to 0 \quad \text{as } \lambda \to +\infty. \end{aligned}$$

This is in accordance with [8, Cor. 5.1 and Prop. A.1] where the authors state and prove a related result in ordered Banach spaces without assuming that U is an AM-space.

(ii) As already mentioned, this result generalizes [12, Thm. 1.2] where it is assumed that $X = U$ and $C = \mathrm{Id}$. In particular, in [12] one needs X to be an AM-space while in Theorem 3.1 only the boundary space U has to be of type AM while X is allowed to be an arbitrary Banach lattice. See Subsect. 4.2 for an example where this generalization is crucial.

In order to prove this theorem we need some preparation. First, by rescaling and in virtue of [18, Sect. II.2.2] we can in the sequel always assume that $\omega_0(A) < 0$ and take $\lambda = 0$.

Next, under the assumptions of Theorem 3.1 the operator $C : X \to U$ is bounded, hence an ∞-admissible observation operator for all $t > 0$, cf. Remark 2.8. The following result shows that the operator $B : U \to X_{-1}$ is an ∞-admissible control operator as well. To verify this claim we use the space $\mathrm{T}([0,t],U)$ of all U-valued step functions defined on $[0,t]$ for some $t > 0$. Note that this is a normed sublattice of $\mathrm{L}^\infty([0,t],U)$, and its closure in $\mathrm{L}^\infty([0,t],U)$ contains $\mathrm{C}([0,t],U)$.

Lemma 3.3. *If U is an AM-space, then every positive $B : U \to X_{-1}$ is an ∞-admissible control operator. Moreover, for any $t > 0$ the controllability map $\mathcal{B}_t \in \mathcal{L}(\mathrm{C}([0,t],U),X)$ given by (2.4) is well-defined, positive and satisfies*

$$\|\mathcal{B}_t\|_{\mathcal{L}(\mathrm{C}([0,t],U),X)} \le \left\|A_{-1}^{-1}B\right\|_{\mathcal{L}(U,X)}. \tag{3.3}$$

Proof. This proof follows in part the ones of [12, Prop. 4.2 and Lem.4.3.(ii)], see also [27, Thm. 4.1.19]. We verify the conditions (i) and (ii) of Proposition 2.5 for $\mathcal{D} = \mathrm{T}([0,t],U)$. Fix $t > 0$ and take $u \in \mathrm{T}([0,t],U)$, i.e.,

$$u(s) := \sum_{n=1}^{N} \mathbb{1}_{I_n}(s)\, u_n, \quad s \in [0,t],$$

where $u_1,\ldots,u_N \in U$, $I_1,\ldots,I_N \subseteq [0,t]$ are pairwise disjoint intervals such that $\dot{\bigcup}_{n=1}^{N} I_n = [0,t]$ and $\mathbb{1}_{I_n}$ is the characteristic function of I_n. Then

$$\int_0^t T_{-1}(t-s)Bu(s)\,ds = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} T_{-1}(t-s)Bu_n\,ds,$$

where $t_{n-1} \le t_n$ denote the endpoints of I_n. Since

$$\int_{t_{n-1}}^{t_n} T_{-1}(t-s)Bu_n\,ds = \big(T(t-t_{n-1}) - T(t-t_n)\big)A_{-1}^{-1}Bu_n \in X \tag{3.4}$$

we obtain (i), i.e., $\mathcal{B}_t : \mathrm{T}([0,t],U) \to X$ is well-defined.

In order to prove condition (ii), we first assume that $0 \le u \in \mathrm{T}([0,t],U)$, i.e., $u_n \ge 0$ for all $n = 1,\ldots,N$. Next define

$$\bar{u} := \sup_{n=1,\ldots,N} u_n \ge 0$$

which satisfies $0 \le u(s) \le \bar{u}$ for all $s \in [0,t]$. Using the positivity of $(T(t))_{t\ge 0}$ and $\lambda R(\lambda, A_{-1})B$ for all $\lambda > 0$ (see Lemma 2.2) we conclude

$$\begin{aligned} 0 &\le \int_0^t T(t-s)\lambda R(\lambda, A_{-1})Bu(s)\,ds \le \int_0^{+\infty} T(s)\lambda R(\lambda, A_{-1})B\bar{u}\,ds \\ &= -\lambda R(\lambda, A)A_{-1}^{-1}B\bar{u} \end{aligned}$$

for all $\lambda > 0$. Since $\lambda R(\lambda, A) \to \mathrm{Id}$ in X and, by [18, Prop. A.3],

$$\lambda R(\lambda, A_{-1})T_{-1}(t-s)Bu(s) \to T_{-1}(t-s)Bu(s)$$

uniformly for $s \in [0,t]$ in X_{-1} as $\lambda \to +\infty$ this implies

$$0 \le \mathcal{B}_t u = \int_0^t T_{-1}(t-s)Bu(s)\,ds \le -A_{-1}^{-1}B\bar{u}. \tag{3.5}$$

Hence,

$$\begin{aligned}\|\mathcal{B}_t u\|_X &\le \left\|A_{-1}^{-1}B\bar{u}\right\|_X \le \left\|A_{-1}^{-1}B\right\|_{\mathcal{L}(U,X)} \cdot \left\|\sup_{n=1,\dots,N} u_n\right\|_U \\ &= \left\|A_{-1}^{-1}B\right\|_{\mathcal{L}(U,X)} \cdot \sup_{n=1,\dots,N}\|u_n\|_U = \left\|A_{-1}^{-1}B\right\|_{\mathcal{L}(U,X)} \cdot \|u\|_\infty,\end{aligned}$$

where in the second line we used that U is an AM-space. Now take $u \in \mathrm{T}([0,t],U)$ arbitrarily then $|u| \in \mathrm{T}([0,t],U)$ as well, and using (3.5) we conclude

$$|\mathcal{B}_t u| \le \mathcal{B}_t|u|.$$

Hence, for every $u \in \mathrm{T}([0,t],U)$ it follows that

$$\|\mathcal{B}_t u\| = \left\||\mathcal{B}_t u|\right\| \le \left\|\mathcal{B}_t|u|\right\| \le \left\|A_{-1}^{-1}B\right\|_{\mathcal{L}(U,X)} \cdot \left\||u|\right\|_\infty = \left\|A_{-1}^{-1}B\right\|_{\mathcal{L}(U,X)} \cdot \|u\|_\infty$$

proving condition (ii). Since the closure of $\mathrm{T}([0,t],U)$ in $\mathrm{L}^\infty([0,t],U)$ contains $\mathrm{C}([0,t],U)$, by Proposition 2.5 B is an ∞-admissible control operator and the estimate (3.3) holds. Finally, positivity of $\mathcal{B}_t$ follows from (3.5) and its proof. □

The next preliminary result extends the operator $\mathcal{B}_t$ given by (2.4) from the interval $[0,t]$ to $\mathbb{R}^+$. To this end we introduce the space $\mathrm{C}_0([0,+\infty),U)$ of all continuous U-valued functions vanishing at infinity endowed by the ∞-norm. Then for $u \in \mathrm{C}_0([0,+\infty),U)$ we define $\mathcal{B}_0 := 0$ and for $t > 0$ (with a little abuse of notation) $\mathcal{B}_t u := \mathcal{B}_t(u|_{[0,t]})$. In this way we can consider $\mathcal{B}_t : \mathrm{C}_0([0,+\infty),U) \to X$. Recall that without loss of generality we still assume that $\omega_0(A) < 0$.

Lemma 3.4. *Let U be an AM-space and $B : U \to X_{-1}$ be positive. Then, the operator family $(\mathcal{B}_t)_{t\ge0} \subset \mathcal{L}(\mathrm{C}_0([0,+\infty),U),X)$ is positive, strongly continuous and uniformly bounded of bound $\|A_{-1}^{-1}B\|_{\mathcal{L}(U,X)}$.*

Proof. By Lemma 3.3, the operator B is an ∞-admissible control operator for all $t > 0$. Moreover, $\mathcal{B}_t$ is positive, and the estimate (3.3) holds for every $t \ge 0$. This implies that the family $(\mathcal{B}_t)_{t\ge0}$ is uniformly bounded. To show its strong continuity we define for $u \in \mathrm{C}_0([0,+\infty),U)$ and $0 \le r < t$ the translated function $u_{t-r} \in \mathrm{C}_0([0,+\infty),U)$ by

$$u_{t-r}(s) := \begin{cases} u(0) & \text{if } 0 \le s < t-r, \\ u(s-t+r) & \text{if } s \ge t-r. \end{cases}$$

Then

$$\begin{aligned}\mathcal{B}_t u - \mathcal{B}_r u &= \int_0^t T_{-1}(t-s)Bu(s)\,ds - \int_0^r T_{-1}(r-s)Bu(s)\,ds\\ &= \int_0^t T_{-1}(t-s)Bu(s)\,ds - \int_{t-r}^t T_{-1}(t-s)Bu(s-t+r)\,ds\\ &= \mathcal{B}_t u - \mathcal{B}_t u_{t-r} + \int_0^{t-r} T_{-1}(t-s)Bu(0)\,ds\\ &= \mathcal{B}_t(u-u_{t-r}) + \int_r^t T_{-1}(s)Bu(0)\,ds\\ &= \mathcal{B}_t(u-u_{t-r}) + \big(T(t)-T(r)\big)A_{-1}^{-1}Bu(0).\end{aligned}$$

Clearly, $(T(r)-T(t))A_{-1}^{-1}Bu(0) \to 0$ as $t-r \to 0$. Moreover,

$$\|\mathcal{B}_t(u-u_{t-r})\|_X \leq \left\|A_{-1}^{-1}B\right\|_{\mathcal{L}(U,X)} \cdot \|u-u_{t-r}\|_\infty.$$

Since $u \in \mathrm{C}_0([0,+\infty),U)$ is uniformly continuous, we conclude

$$\lim_{t-r\to 0}\|u-u_{t-r}\|_\infty = 0$$

which implies strong continuity of the family $(\mathcal{B}_t)_{t\geq 0}$. □

In the sequel we denote by $\mathrm{T}_c([0,+\infty),U)$ the space of all compactly supported U-valued step functions on $[0,+\infty)$, i.e., satisfying $\mathrm{supp}(u) \subset [0,b]$ for some $0 \leq b < +\infty$. In particular, this implies $u(s) = 0$ for all $s \geq b$. Moreover, the closure of $\mathrm{T}_c([0,+\infty),U)$ in $\mathrm{L}^\infty([0,+\infty),U)$ contains $\mathrm{C}_0([0,+\infty),U)$.

Lemma 3.5. *Let $B : U \to X_{-1}$ and $C : X \to U$ be positive operators where U is an AM-space. Then*

$$\big(\mathcal{F}_\infty u\big)(t) := C\mathcal{B}_t u \quad \textit{for } u \in \mathrm{C}_0([0,+\infty),U) \textit{ and } t \geq 0$$

defines a positive operator $\mathcal{F}_\infty \in \mathcal{L}(\mathrm{C}_0([0,+\infty),U))$ satisfying the estimates

$$\left\|\mathcal{F}_\infty^n\right\|_{\mathcal{L}(\mathrm{C}_0([0,+\infty),U))} \leq \left\|\big(CA_{-1}^{-1}B\big)^n\right\|_{\mathcal{L}(U)} \quad \textit{for every } n \in \mathbb{N}. \tag{3.6}$$

Proof. Let $u \in \mathrm{C}_0([0,+\infty),U)$. Then by Lemma 3.4 and the boundedness of $C : X \to U$ it is clear that $\mathcal{F}_\infty u : [0,+\infty) \to U$ is a continuous and bounded function. Hence, in any case $\mathcal{F}_\infty : \mathrm{C}_0([0,+\infty),U) \to \mathrm{C}_b([0,+\infty),U)$ is a positive and bounded operator, where $\mathrm{C}_b([0,+\infty),U)$ denotes the Banach lattice of all continuous, bounded U-valued functions on $[0,+\infty)$ equipped with the sup-norm $\|\cdot\|_\infty$. Now assume in addition that u has compact support contained in the interval $[0,b]$. Since $\omega_0(A) < 0$ we obtain for $t > b$

$$\begin{aligned}\big(\mathcal{F}_\infty u\big)(t) &= CT(t-b)\int_0^b T_{-1}(b-s)Bu(s)\,ds\\ &= CT(t-b)\,\mathcal{B}_b u \to 0 \quad \text{as } t\to+\infty,\end{aligned}$$

i.e., $\mathcal{F}_\infty u \in \mathrm{C}_0([0,+\infty),U)$. Since the functions in $\mathrm{C}_0([0,+\infty),U)$ having compact support are dense in $\mathrm{C}_0([0,+\infty),U)$ this implies $\mathrm{rg}(\mathcal{F}_\infty) \subseteq \mathrm{C}_0([0,+\infty),U)$, hence $0 \le \mathcal{F}_\infty \in \mathcal{L}(\mathrm{C}_0([0,+\infty),U))$ as claimed.

In order to show the estimates (3.6), we first observe that by (3.4) it follows that for every $u \in \mathrm{T}_c([0,+\infty),U)$ the function $0 \le t \mapsto r(t) := C\mathcal{B}_t u \in U$ is continuous and $r(t) \to 0$ as $t \to +\infty$. Hence, $r \in \mathrm{C}_0([0,+\infty),U)$ which implies that $\mathcal{F}_\infty^n u = \mathcal{F}_\infty^{n-1} r$ is well-defined for all $u \in \mathrm{T}_c([0,+\infty),U)$ and $n \in \mathbb{N}$.

Next we prove by induction that for any $0 \le u \in \mathrm{T}_c([0,+\infty),U)$ and $n \in \mathbb{N}$ it holds

$$0 \le \big(\mathcal{F}_\infty^n u\big)(t) \le \big(-CA_{-1}^{-1}B\big)^n \bar{u} \quad \text{for all } t \ge 0, \tag{3.7}$$

where for $u(s) := \sum_{n=1}^N u_n \mathbb{1}_{I_n}(s)$, $s \in [0,+\infty)$, with pairwise disjoint intervals I_n, we define as above $\bar{u} = \sup_{n=1,\dots,N} u_n \in U_+$.

For $n = 1$ this estimate follows immediately by multiplying (3.5) from the left by $C \ge 0$. Now assume that (3.7) holds for some fixed $n \ge 1$. Then using that $-A_{-1}^{-1}B \ge 0$ we obtain

$$\begin{aligned} 0 \le \big(\mathcal{F}_\infty^{n+1} u\big)(t) &= C\mathcal{B}_t\big(\mathcal{F}_\infty^n u\big) \le C\mathcal{B}_t\big(-CA_{-1}^{-1}B\big)^n \bar{u} \\ &= C\int_0^t T_{-1}(s)B\big(-CA_{-1}^{-1}B\big)^n \bar{u}\, ds \\ &= -CA_{-1}^{-1}B\big(-CA_{-1}^{-1}B\big)^n \bar{u} - CT(t)\big(-A_{-1}^{-1}B\big)\big(-CA_{-1}^{-1}B\big)^n \bar{u} \\ &\le \big(-CA_{-1}^{-1}B\big)^{n+1} \bar{u}, \end{aligned}$$

which proves (3.7) by induction. Now take an arbitrary $u \in \mathrm{T}_c([0,+\infty),U)$. Then $|u| \in \mathrm{T}_c([0,+\infty),U)$ as well, and we conclude

$$|\mathcal{F}_\infty^n u| = \big|\mathcal{F}_\infty^n(u^+ - u^-)\big| \le \mathcal{F}_\infty^n u^+ + \mathcal{F}_\infty^n u^- = \mathcal{F}_\infty^n |u|,$$

i.e., $|\mathcal{F}_\infty^n u|(t) \le (\mathcal{F}_\infty^n |u|)(t)$ for all $t \ge 0$. Using (3.7) it follows that

$$\begin{aligned} \big\|(\mathcal{F}_\infty^n u)(t)\big\|_U &= \big\||\mathcal{F}_\infty^n u|(t)\big\|_U \le \big\|\big(\mathcal{F}_\infty^n |u|\big)(t)\big\|_U \\ &\le \big\|\big(CA_{-1}^{-1}B\big)^n\big\|_{\mathcal{L}(U)} \cdot \|\bar{u}\|_U = \big\|\big(CA_{-1}^{-1}B\big)^n\big\|_{\mathcal{L}(U)} \cdot \|u\|_\infty, \end{aligned} \tag{3.8}$$

where in the last equality we used the AM-property of U. This shows that the operator $\mathcal{F}_\infty^n : \mathrm{T}_c([0,+\infty),U) \subset \mathrm{L}^\infty([0,+\infty),U) \to \mathrm{C}_0([0,+\infty),U)$ is bounded of bound $\|(CA_{-1}^{-1}B)^n\|_{\mathcal{L}(U)}$. Since $\mathrm{C}_0([0,+\infty),U)$ is complete, $\mathcal{F}_\infty^n$ has a unique bounded extension $\mathcal{R}_n = \mathcal{F}_\infty^{n-1} \cdot \mathcal{R}_1$ to the closure of the space $\mathrm{T}_c([0,+\infty),U)$ in $\mathrm{L}^\infty([0,+\infty),U)$ having the same bound $\|(CA_{-1}^{-1}B)^n\|_{\mathcal{L}(U)}$. Now as in the proof of Proposition 2.5 it follows that $\mathcal{R}_1|_{\mathrm{C}_0([0,+\infty),U)} = \mathcal{F}_\infty$, hence we obtain that $\mathcal{R}_n|_{\mathrm{C}_0([0,+\infty),U)} = \mathcal{F}_\infty^n$ which implies (3.6). □

Our last preliminary result deals with the invertibility of $\mathrm{Id} - \mathcal{F}_\infty$ and the Laplace transform of its inverse. The statement heavily depends on the spectral assumption $\mathrm{r}(CR(\lambda, A_{-1})B) < 1$. In the sequel $\mathscr{L}(\cdot)$ denotes the Laplace transform as introduced in [6, Sect. 1.4]. Recall that, as always, we assume $\omega_0(A) < 0$.

Lemma 3.6. *Let $B : U \to X_{-1}$ and $C : X \to U$ be positive where U is an AM-space. If $\mathrm{r}(CA_{-1}^{-1}B) < 1$, then $1 \in \rho(\mathcal{F}_\infty) \cap \rho(CR(\lambda, A_{-1})B)$ for any $\lambda > 0$ and*

$$\mathscr{L}\big((\mathrm{Id} - \mathcal{F}_\infty)^{-1}u(\cdot)\big)(\lambda) = \big(\mathrm{Id} - CR(\lambda, A_{-1})B\big)^{-1} \cdot \hat{u}(\lambda) \tag{3.9}$$

for all $u \in \mathrm{C}_0([0, +\infty), U)$, where $\hat{u}(\cdot)$ denotes the Laplace transform of $u(\cdot)$.

Proof. We start by showing that $1 \in \rho(CR(\lambda, A_{-1})B)$ for all $\lambda > 0$. By Lemma 2.2 we have $R(\lambda, A_{-1})B \geq 0$ for any $\lambda > \mathrm{s}(A)$. Since $\mathrm{s}(A) \leq \omega_0(A) < 0$, the resolvent identity implies

$$0 \leq CR(\lambda, A_{-1})B = CR(0, A_{-1})B - C\lambda R(\lambda, A)R(0, A_{-1})B \leq -CA_{-1}^{-1}B$$

for all $\lambda > 0$. Hence, by Proposition A.1 we conclude that for all $\lambda > 0$

$$\mathrm{r}\big(CR(\lambda, A_{-1})B\big) \leq \mathrm{r}\big(-CA_{-1}^{-1}B\big) = \mathrm{r}\big(CA_{-1}^{-1}B\big) < 1, \tag{3.10}$$

In particular this implies $1 \in \rho(CR(\lambda, A_{-1})B)$ for all $\lambda > 0$ as claimed.

Next observe that by (3.6) we have

$$\begin{aligned} \mathrm{r}(\mathcal{F}_\infty) &= \lim_{n\to+\infty} \left\| \mathcal{F}_\infty^n \right\|_{\mathcal{L}(\mathrm{C}_0([0,+\infty),U))}^{1/n} \leq \lim_{n\to+\infty} \left\| (CA_{-1}^{-1}B)^n \right\|_{\mathcal{L}(U)}^{1/n} \\ &= \mathrm{r}(CA_{-1}^{-1}B) < 1. \end{aligned} \tag{3.11}$$

Hence, $\mathrm{Id} - \mathcal{F}_\infty$ is invertible and its inverse is given by the Neumann series

$$(\mathrm{Id} - \mathcal{F}_\infty)^{-1} = \sum_{n=0}^{\infty} \mathcal{F}_\infty^n \tag{3.12}$$

which converges in $\mathcal{L}(\mathrm{C}_0([0, +\infty), U))$. It only remains to show (3.9).

To this end we first verify that for all $u \in \mathrm{C}_0([0, +\infty), U)$ and $n \in \mathbb{N}$ we have

$$\mathscr{L}(\mathcal{F}_\infty^n u)(\lambda) = \big(C\, R(\lambda, A_{-1})\, B\big)^n \cdot \hat{u}(\lambda), \quad \lambda > 0. \tag{3.13}$$

Indeed, for $n = 1$, recalling that C is bounded, for $\lambda > 0$ and $u \in \mathrm{C}_0([0, +\infty), U)$ we obtain

$$\begin{aligned} \mathscr{L}(\mathcal{F}_\infty u)(\lambda) &= \int_0^{+\infty} e^{-\lambda t} (\mathcal{F}_\infty u)(t)\, dt \\ &= C \int_0^{+\infty} e^{-\lambda t} \left(\int_0^t T_{-1}(t-s)Bu(s)\, ds \right) dt \\ &= C \int_0^{+\infty} e^{-\lambda t} \big(T_{-1}(\cdot)B \star u(\cdot)\big)(t)\, dt \\ &= C \cdot \mathscr{L}\big(T_{-1}(\cdot)B \star u(\cdot)\big)(\lambda) = CR(\lambda, A_{-1})B \cdot \hat{u}(\lambda), \end{aligned}$$

where in the last line we have used the convolution theorem for Laplace transform (see, e.g., [14, Lemma 3.12]). If (3.13) holds for some $n \geq 1$, then for $u \in \mathrm{C}_0([0,+\infty),U)$ and $\lambda > 0$ we have

$$\begin{aligned}\mathscr{L}\big(\mathcal{F}_\infty^{n+1}u\big)(\lambda) &= \int_0^{+\infty} e^{-\lambda t}\Big(C\int_0^t T_{-1}(t-s)B(\mathcal{F}_\infty^n u)(s)\,ds\Big)\,dt \\ &= C\int_0^{+\infty} e^{-\lambda t}\Big(T_{-1}(\cdot)B \star \mathcal{F}_\infty^n u(\cdot)\Big)(t)\,dt \\ &= C\cdot\mathscr{L}\Big(T_{-1}(\cdot)B\star\mathcal{F}_\infty^n u(\cdot)\Big)(\lambda) \\ &= CR(\lambda,A_{-1})B\cdot\mathscr{L}(\mathcal{F}_\infty^n u)(\lambda) = \big(C\,R(\lambda,A_{-1})\,B\big)^{n+1}\cdot\hat{u}(\lambda),\end{aligned}$$

which implies (3.13) by induction. Finally, we use (3.13) to prove that (3.9) holds for all $u \in \mathrm{C}_0([0,+\infty),U)$ and $\lambda > 0$. To his aim, we define the sequence of continuous functions $f_n \in \mathrm{C}_0([0,+\infty),U)$, $n \in \mathbb{N}$ by

$$f_n := \sum_{k=0}^{n} \mathcal{F}_\infty^k u(\cdot).$$

Using (3.13) we immediately obtain that

$$\hat{f}_n(\lambda) = \sum_{k=0}^{n}\big(CR(\lambda,A_{-1})B\big)^k\cdot\hat{u}(\lambda) \quad \text{for all } \lambda > 0.$$

Moreover, norm convergence of the Neumann series (3.12) yields the uniform convergence of $(f_n)_{n\in\mathbb{N}}$ to $f := (\mathrm{Id}-\mathcal{F}_\infty)^{-1}u(\cdot)$. By [6, Thm. 1.7.5] this implies

$$\begin{aligned}&\mathscr{L}\big((\mathrm{Id}-\mathcal{F}_\infty)^{-1}u(\cdot)\big)(\lambda) = \hat{f}(\lambda) = \lim_{n\to+\infty}\hat{f}_n(\lambda) \\ &= \lim_{n\to+\infty}\sum_{k=0}^{n}\big(CR(\lambda,A_{-1})B\big)^k\cdot\hat{u}(\lambda) = \big(\mathrm{Id}-CR(\lambda,A_{-1})B\big)^{-1}\cdot\hat{u}(\lambda)\end{aligned}$$

for all $u \in \mathrm{C}_0([0,+\infty),U)$ and $\lambda > 0$, i.e., (3.9). □

We have now all the necessary tools to prove the main result of this section.

Proof (of Theorem 3.1). The idea is to define an operator family $(S(t))_{t\geq 0} \subset \mathcal{L}(X)$ and to prove that it is a C_0-semigroup with generator A_{BC} given by (3.1).

To this end we first introduce the positive bounded linear operator $\mathcal{C}_\infty : X \to \mathrm{C}_0([0,+\infty),U)$ by $(\mathcal{C}_\infty x)(t) := CT(t)x$. Then we use the invertibility of $\mathrm{Id}-\mathcal{F}_\infty$ shown in Lemma 3.6 to define positive operators

$$S(t) := T(t) + \mathcal{B}_t(\mathrm{Id}-\mathcal{F}_\infty)^{-1}\mathcal{C}_\infty \in \mathcal{L}(X) \quad \text{for } t \geq 0. \tag{3.14}$$

Now by Lemma 3.4, the operator family $(S(t))_{t\geq 0}$ is strongly continuous on X and uniformly bounded. At this point we can proceed as in [1, Proof of Thm. 10, (50)] and show that by the convolution theorem for the Laplace transform

$$\mathscr{L}\big(S(\cdot)x\big)(\lambda) = R(\lambda, A)x + R(\lambda, A_{-1})B \cdot \big(\mathrm{Id} - CR(\lambda, A_{-1}B)\big)^{-1} \cdot CR(\lambda, A)x$$
$$=: Q(\lambda)x \quad \text{for all } \lambda > 0 \text{ and } x \in X.$$

As in [1, (52), (53)] it then follows that $Q(\lambda)$ is a right- and left inverse of $\lambda - A_{BC}$, i.e., $Q(\lambda) = R(\lambda, A_{BC})$. Hence, by [6, Thm. 3.1.7] the operators defined by (3.14) form a positive C_0-semigroup on X with generator A_{BC}. Finally, the variation of parameters formula (3.2) follows as in [1, Thm. 10]. □

3.2. Positive structured perturbations factorized via AL-spaces

The following result generalizes a perturbation theorem for positive semigroups due to Desch [16] and Voigt [26], see Remark 3.8. With respect to our setting from Sect. 2 we assume throughout this subsection that $Z = D(A)$ and $\mathrm{rg}(B) \subseteq X$, which by [13, Thm. 10.20] implies that any positive $B : U \to X$ is bounded.

Theorem 3.7. *Let $A : D(A) \subseteq X \to X$ be the generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach lattice X. Assume $B : U \to X$ and $C : D(A) \to U$ to be positive linear operators, where U is an AL-space. If the spectral radius $\mathrm{r}(CR(\lambda, A)B) < 1$ for some $\lambda > \omega_0(A)$, then the operator*

$$A_{BC} := A + BC \quad \text{with domain } D(A_{BC}) = D(A) \tag{3.15}$$

generates a positive C_0-semigroup $(S(t))_{t\geq 0}$ on X satisfying the variation of parameters formula

$$S(t)x := T(t)x + \int_0^t T(t-s) \cdot BC \cdot S(s)x\, ds, \quad x \in D(A),\ t \geq 0. \tag{3.16}$$

Remark 3.8. As already mentioned, this result generalizes the main result in [16] and [26], see also [13, Sect. 13.3] or [10, Sect. 5.2.1], where it is assumed that $X = U$ and $B = \mathrm{Id}$. In particular, in these works one needs X to be an AL-space while in Theorem 3.7 only the boundary space U has to be of the type AL while X is allowed to be an arbitrary Banach lattice. See Subsect. 4.3 for an example where this generalization is crucial.

The proof of Theorem 3.7 is structured similarly as the one of Theorem 3.1. We note that by the rescaling argument from [18, Sect. II.2.2] we again assume $\omega_0(A) < 0$ and choose $\lambda = 0$ throughout the proof.

Under the assumptions of Theorem 3.7 the operator $B : U \to X$ is bounded and hence a 1-admissible control operator for all $t > 0$, cf. Remark 2.6. Next we show that the operator $C : D(A) \to U$ is 1-admissible too as observation operator.

Lemma 3.9. *If U is an AL-space, then every positive operator $C : D(A) \to U$ is a 1-admissible observation operator. Moreover, for any $t > 0$ the observability operator $\mathcal{C}_t : X \to \mathrm{L}^1([0,t], U)$ given by (2.6) is well-defined, positive and satisfies*

$$\|\mathcal{C}_t\|_{\mathcal{L}(X,\mathrm{L}^1([0,t],U))} \leq \big\|CA^{-1}\big\|_{\mathcal{L}(X,U)}. \tag{3.17}$$

Proof. We follow in part the proofs of [13, Prop. 13.7], see also [27, Thm.4.1.17]. First take $0 \le x \in D(A)$. Since U is an AL-space and $CT(s)x \ge 0$ for all $s \ge 0$ we conclude

$$\begin{aligned}\int_0^t \left\| CT(s)x \right\|_U ds = \left\| \int_0^t CT(s)x\, ds \right\|_U \le \left\| \int_0^{+\infty} CT(s)x\, ds \right\|_U \\ = \left\| -CA^{-1}x \right\|_U \le \left\| CA^{-1} \right\|_{\mathcal{L}(X,U)} \cdot \|x\|_X.\end{aligned} \tag{3.18}$$

If $x \in D(A)$ we decompose it as $x = x^+ - x^-$ and define for $\lambda > 0$

$$\begin{aligned}&x_\lambda := \lambda R(\lambda, A)x \in D(A),\\ &x_\lambda^P := \lambda R(\lambda, A)x^+ \in D(A)_+, \quad x_\lambda^N := \lambda R(\lambda, A)x^- \in D(A)_+.\end{aligned}$$

Then $x_\lambda \to x$, $x_\lambda^P \to x^+$ and $x_\lambda^N \to x^-$ in X as $\lambda \to +\infty$. Moreover, if $\|\cdot\|_{X_1}$ denotes the norm of $X_1 = D(A)$ given by $\|x\|_{X_1} = \|Ax\|_X$, then also

$$\|x_\lambda - x\|_{X_1} = \left\| \lambda R(\lambda, A)x - x \right\|_{X_1} = \left\| \lambda R(\lambda, A)Ax - Ax \right\|_X \to 0,$$

i.e., $x_\lambda \to x$ in X_1 as $\lambda \to +\infty$. Since for all $s \in [0, t]$ we have

$$\left| CT(s)x_\lambda \right| \le CT(s)\big(x_\lambda^P + x_\lambda^N\big)$$

using (3.18) we obtain

$$\begin{aligned}\int_0^t \left\| CT(s)x_\lambda \right\|_U ds \le \int_0^t \left\| CT(s)\big(x_\lambda^P + x_\lambda^N\big) \right\|_U ds \\ \le \left\| CA^{-1} \right\|_{\mathcal{L}(X,U)} \cdot \left\| x_\lambda^P + x_\lambda^N \right\|_X.\end{aligned} \tag{3.19}$$

Now choose $M \ge 1$ such that $\|T(s)\| \le M$ for all $s \ge 0$. Then

$$\begin{aligned}\left\| CT(s)x_\lambda - CT(s)x \right\|_U = \left\| CA^{-1}T(s)(Ax_\lambda - Ax) \right\|_U \\ \le M \cdot \left\| CA^{-1} \right\|_{\mathcal{L}(X,U)} \cdot \|x_\lambda - x\|_{X_1}.\end{aligned}$$

Hence, $CT(s)x_\lambda \to CT(s)x$ uniformly for $s \in [0, t]$ as $\lambda \to +\infty$ and (3.19) implies

$$\int_0^t \left\| CT(s)x \right\|_U ds \le \left\| CA^{-1} \right\|_{\mathcal{L}(X,U)} \cdot \|x^+ + x^-\|_X = \left\| CA^{-1} \right\|_{\mathcal{L}(X,U)} \cdot \|x\|_X.$$

This proves that C is a 1-admissible observation operator for all $t > 0$ and that the operator $\mathcal{C}_t$ defined in (2.6) satisfies the estimate (3.17). □

As in Subsect. 3.1, we need to extend the operators $\mathcal{B}_t$ and $\mathcal{C}_t$ defined in (2.2) and (2.6), respectively, on $\mathbb{R}^+$. To this end for $u \in \mathrm{L}^1([0, +\infty), U)$ we define again $\mathcal{B}_0 := 0$ and for $t > 0$ (with a little abuse of notation) $\mathcal{B}_t u := \mathcal{B}_t(u|_{[0,t]})$. In this way we can consider $\mathcal{B}_t : \mathrm{L}^1([0, +\infty), U) \to X$. As always, we assume that $\omega_0(A) < 0$.

Lemma 3.10. *Let $B : U \to X$ and $C : D(A) \to U$ be positive operators where U is an AL-space. Then,* *(1)* *B is a 1-admissible control operator for every $t > 0$, i.e., for every $t > 0$ it holds $\mathcal{B}_t \in \mathcal{L}(\mathrm{L}^1([0, +\infty), U), X)$ where*

$$\mathcal{B}_t u := \int_0^t T(t-s)Bu(s)\,ds \quad \textit{for } u \in \mathrm{L}^1([0, +\infty), U). \tag{3.20}$$

Moreover, the family $(\mathcal{B}_t)_{t\geq 0}$ is positive, strongly continuous and uniformly bounded.

(2) *C is a 1-admissible observation operator for all $t \geq 0$, i.e., there exists a bounded operator $\mathcal{C}_\infty : X \to \mathrm{L}^1([0, +\infty), U)$ such that*

$$\big(\mathcal{C}_\infty x\big)(s) = CT(s)x \quad \textit{for } x \in D(A),\ s \geq 0.$$

Proof. (i). Since $B : U \to X$ is positive it is bounded by [13, Thm. 10.20]. The 1-admissibility of B for every $t > 0$ as well as positivity then follow immediately. Using the assumption $\omega_0(A) < 0$, Young's inequality implies uniform boundedness. Strong continuity for $u \in \mathrm{C}_0([0, +\infty), U) \cap \mathrm{L}^1([0, +\infty), U)$ can be shown as in the proof of Lemma 3.4. Since $\mathrm{C}_0([0, +\infty), U) \cap \mathrm{L}^1([0, +\infty), U)$ is dense in $\mathrm{L}^1([0, +\infty), U)$ and $\mathcal{B}_t$ is uniformly bounded for $t \geq 0$, strong continuity then holds for all $u \in \mathrm{L}^1([0, +\infty), U)$, see [18, Lem. I.5.2]. Finally, assertion (ii) is clear by (3.17) since $(\mathcal{C}_t x)(s) = (\mathcal{C}_r x)(s)$ for all $x \in X$ and $0 \leq s \leq r \leq t$. □

Next we use again the space of all compactly supported U-valued step functions in $[0, +\infty)$, denoted as before by $\mathrm{T}_c([0, +\infty), U)$ which forms a dense sublattice of $\mathrm{L}^1([0, +\infty), U)$.

Lemma 3.11. *Let $B : U \to X$ and $C : D(A) \to U$ be positive operators where U is an AL-space. Then*

$$(\mathcal{F}_\infty u)(t) := C \int_0^t T(t-s)Bu(s)\,ds, \quad t \geq 0 \tag{3.21}$$

is well-defined for all $u \in \mathrm{T}_c([0, +\infty), U)$. Moreover, it possesses a (unique) positive bounded extension (still denoted by $\mathcal{F}_\infty$) $\mathcal{F}_\infty \in \mathcal{L}(\mathrm{L}^1([0, +\infty), U))$ satisfying the estimates

$$\big\|\mathcal{F}_\infty^n\big\|_{\mathcal{L}(\mathrm{L}^1([0,+\infty),U))} \leq \big\|(CA^{-1}B)^n\big\|_{\mathcal{L}(U)} \quad \textit{for every } n \in \mathbb{N}. \tag{3.22}$$

Proof. We verify that for all $u \in \mathrm{T}_c([0, +\infty), U)$ we have $\mathcal{F}_\infty u \in \mathrm{L}^1([0, +\infty), U)$ and that the following two conditions hold:

$$(i) \quad \int_0^t T(t-s)Bu(s)\,ds \in D(A) \text{ for any } t > 0,$$

$$(ii) \quad \|\mathcal{F}_\infty u\|_1 \leq \big\|CA^{-1}B\big\|_{\mathcal{L}(U)} \cdot \|u\|_1.$$

Condition (i) follows by the same reasoning as in the proof of Lemma 3.3, bearing in mind that for every $n = 1, \ldots, N$ the term Bu_n in (3.4) now belongs to X, hence the right-hand side belongs to $D(A)$. Moreover, from (3.21) it immediately follows that $\mathcal{F}_\infty$ is positive.

To prove condition (ii), fix $u \in \mathrm{T}_c([0,+\infty),U)$. Then $|u| \in \mathrm{T}_c([0,+\infty),U)$ as well and since $\mathcal{F}_\infty$ is positive we conclude

$$\big|(\mathcal{F}_\infty u)(t)\big| \le (\mathcal{F}_\infty u^+)(t) + (\mathcal{F}_\infty u^-)(t) = \big(\mathcal{F}_\infty |u|\big)(t) \quad \text{for all } t \ge 0.$$

Since U is an AL-space and using Fubini's Theorem and (i), we obtain

$$\begin{aligned}
\|\mathcal{F}_\infty u\|_1 &\le \int_0^{+\infty} \big\|\big(\mathcal{F}_\infty |u|\big)(t)\big\|_U \, dt \\
&= \int_0^{+\infty} \left\| C \int_0^t T(t-s)B|u(s)|\, ds \right\|_U dt \\
&= \left\| \int_0^{+\infty} CA^{-1} \int_0^t A_{-1}T(t-s)B|u(s)|\, ds\, dt \right\|_U \\
&= \left\| CA^{-1} \int_0^{+\infty} \left(\int_s^{+\infty} A_{-1}T(t-s)B|u(s)|\, dt \right) ds \right\|_U \\
&= \left\| CA^{-1} \int_0^{+\infty} \left(\int_0^{+\infty} A_{-1}T(r)B|u(s)|\, dr \right) ds \right\|_U \\
&= \left\| -CA^{-1}B \int_0^{+\infty} |u(s)|\, ds \right\|_U \le \big\|CA^{-1}B\big\|_{\mathcal{L}(U)} \cdot \|u\|_1. \qquad (3.23)
\end{aligned}$$

Therefore, $(\mathcal{F}_\infty u)(t)$ given by (3.21) defines a positive linear operator

$$\mathcal{F}_\infty : \mathrm{T}_c([0,+\infty),U) \subset \mathrm{L}^1([0,+\infty),U) \to \mathrm{L}^1([0,+\infty),U)$$

satisfying condition (ii) for all $u \in \mathrm{T}_c([0,+\infty),U)$. Hence, $\mathcal{F}_\infty$ possesses a unique bounded positive extension $\mathcal{F}_\infty \in \mathcal{L}(\mathrm{L}^1([0,+\infty),U))$ satisfying the norm estimate (3.22) for $n = 1$.

To prove (3.22) for arbitrary $n \in \mathbb{N}$ we first verify by induction that for any $u \in \mathrm{L}^1([0,+\infty),U)$ and $t \ge 0$ we have

$$\int_0^{+\infty} \big|(\mathcal{F}_\infty^n u)(t)\big|\, dt \le \big(-CA^{-1}B\big)^n \int_0^{+\infty} \big|u(t)\big|\, dt \quad \text{for all } n \in \mathbb{N}. \qquad (3.24)$$

Using Fubini's Theorem like in (3.23) and the positivity of $\mathcal{F}_\infty$ the case $n = 1$ follows, since

$$\int_0^{+\infty} \big|(\mathcal{F}_\infty u)(t)\big|\, dt \le \int_0^{+\infty} \big(\mathcal{F}_\infty |u|\big)(t)\, dt = -CA^{-1}B \int_0^{+\infty} |u(t)|\, dt.$$

Now assume (3.24) holds for some $n \geq 1$. Then, again by the positivity of $\mathcal{F}_\infty^n$ we obtain

$$\begin{aligned}\int_0^{+\infty} \left|(\mathcal{F}_\infty^{n+1}u)(t)\right| dt &= \int_0^{+\infty} \left|\big(\mathcal{F}_\infty^n(\mathcal{F}_\infty u)\big)(t)\right| dt \\ &\leq \int_0^{+\infty} \big(\mathcal{F}_\infty^n \left|\mathcal{F}_\infty u\right|\big)(t)\, dt \\ &\leq \big(-CA^{-1}B\big)^n \int_0^{+\infty} \left|(\mathcal{F}_\infty u)(t)\right| dt \\ &\leq \big(-CA^{-1}B\big)^{n+1} \int_0^{+\infty} \left|u(t)\right| dt, \end{aligned} \tag{3.25}$$

which proves (3.24). Keeping in mind that U is an AL-space, (3.24) implies for arbitrary $u \in \mathrm{L}^1([0,+\infty), U)$, $t \geq 0$ and $n \in \mathbb{N}$

$$\begin{aligned}\left\|\mathcal{F}_\infty^n u\right\|_1 &= \int_0^{+\infty} \left\|(\mathcal{F}_\infty^n u)(t)\right\|_U dt = \left\|\int_0^{+\infty} \left|(\mathcal{F}_\infty^n u)(t)\right| dt\right\|_U \\ &\leq \left\|\big(-CA^{-1}B\big)^n \int_0^{+\infty} \left|u(t)\right| dt\right\|_U \leq \left\|\big(CA^{-1}B\big)^n\right\|_{\mathcal{L}(U)} \cdot \|u\|_1. \end{aligned} \tag{3.26}$$

This completes the proof of (3.22). □

At this point we need a result similar to Lemma 3.6 concerning the invertibility of $\mathrm{Id} - \mathcal{F}_\infty$ and the Laplace transform of its inverse. Again, for this the spectral assumption $\mathrm{r}(CR(\lambda, A)B) < 1$ plays a crucial role. Recall that, as always, we assume $\omega_0(A) < 0$.

Lemma 3.12. *Let $B : U \to X$ and $C : D(A) \to U$ be positive where U is an AL-space. If $\mathrm{r}(CA^{-1}B) < 1$ then $1 \in \rho(\mathcal{F}_\infty) \cap \rho(CR(\lambda, A)B)$ for any $\lambda > 0$ and*

$$\mathscr{L}\big((\mathrm{Id} - \mathcal{F}_\infty)^{-1}u\big)(\lambda) = \big(\mathrm{Id} - CR(\lambda, A)B\big)^{-1} \cdot \hat{u}(\lambda), \tag{3.27}$$

for all $u \in \mathrm{L}^1([0,+\infty), U)$, where $\hat{u}(\cdot)$ denotes the Laplace transform of $u(\cdot)$.

Proof. The claim $1 \in \rho(CR(\lambda, A)B)$ follows like in the proof of Lemma 3.6 from (3.10). Moreover, (3.22) implies $\mathrm{r}(\mathcal{F}_\infty) \leq \mathrm{r}(CA^{-1}B) < 1$ as in (3.11) and hence $1 \in \rho(\mathcal{F}_\infty)$. The identity (3.27) then follows by [1, Lem. 13]. □

We have now all the necessary tools to prove the main result of this section.

Proof (of Theorem 3.7). The proof is very similar to the one of Theorem 3.1: Using Lemma 3.12 we define positive operators

$$S(t) := T(t) + \mathcal{B}_t(\mathrm{Id} - \mathcal{F}_\infty)^{-1}\mathcal{C}_\infty \in \mathcal{L}(X) \quad \text{for } t \geq 0, \tag{3.28}$$

where $\mathcal{B}_t$ is given by (3.20). Then Lemma 3.10 implies that $(S(t))_{t\geq 0}$ is strongly continuous on X and uniformly bounded. The same reasoning as in the proof of Theorem 3.1 then implies that $(S(t))_{t\geq 0}$ defines a positive C_0-semigroup with generator A_{BC} satisfying the variation of parameters formula (3.16). □

3.3. Structured perturbation via domination

In this subsection we relax the positivity assumptions on the operators $B : U \to X_{-1}$ and $C : Z \to U$ where $D(A) \subseteq Z \subseteq X$, and only suppose that they are dominated by positive operators $\widetilde{B} : U \to X_{-1}$ and $\widetilde{C} : Z \to U$ such that $A_{\widetilde{B}\widetilde{C}}$ is a generator. More precisely, we impose the following conditions. Note that in this subsection we do not need X or U to be AM- or AL-spaces.

Assumption 3.13. In the situation of Assumption 2.1, we suppose that there exist positive operators $B_+, B_- : U \to X_{-1}$ and $\widetilde{C} : Z \to U$ such that

(i) $|Cx| \le \widetilde{C}x$ for all $x \in Z_+ = Z \cap X_+$,

(ii) $B = B_+ - B_-$, i.e., B is regular.

Moreover, we define the positive operator $\widetilde{B} := B_+ + B_- : U \to X_{-1}$.

Remark 3.14. If $Z = X$, Assumption 3.13.(i) can be equivalently reformulated as

$$|Cx| \le \widetilde{C}|x| \quad \text{for all } x \in X.$$

Another notion we need in the sequel is the *compatibility* of the triple (A, B, C).

Definition 3.15. The triple (A, B, C) of a state operator $A : D(A) \subseteq X \to X$, a control operator $B : U \to X_{-1}$ and an observation operator $C : Z \to U$ is said to be *compatible* if for some $\lambda \in \rho(A)$ one has $\operatorname{rg}(R(\lambda, A_{-1})B) \subseteq Z$.

If this range condition is satisfied for some $\lambda \in \rho(A)$, then by the resolvent identity it holds for all $\lambda \in \rho(A)$.

Remark 3.16. Any triple (A, B, C) as in Definition 3.15 satisfying Assumption 2.1 for $Z = X$ or $\operatorname{rg}(B) \subseteq X$ is compatible.

In the sequel we call a triple $(A, \widetilde{B}, \widetilde{C})$ positive if A generates a positive semigroup (which we assume throughout this paper) and $\widetilde{B}, \widetilde{C}$ are positive. Since any everywhere defined positive operator between Banach lattices is bounded (see, e.g., [13, Thm. 10.20]), Lemma 2.2 implies that for every positive compatible triple $(A, \widetilde{B}, \widetilde{C})$ we have $\widetilde{C}R(\lambda, A_{-1})\widetilde{B} \in \mathcal{L}(U)$ for all $\lambda > \omega_0(A)$

Now the following holds where we recall that $\widetilde{B} = B_+ + B_-$.

Theorem 3.17. *Let Assumptions 2.1 and 3.13 be satisfied and suppose that the triples (A, B, C) and $(A, \widetilde{B}, \widetilde{C})$ are compatible. If $A_{\widetilde{B}\widetilde{C}}$ generates a positive C_0-semigroup $(\widetilde{S}(t))_{t\ge0}$ on X and $\operatorname{r}(\widetilde{C}R(\lambda, A_{-1})\widetilde{B}) < 1$ for some $\lambda > \omega_0(A)$, then the operator*

$$\begin{aligned} A_{BC} &:= \big(A_{-1} + BC\big)|_X, \\ D(A_{BC}) &:= \big\{x \in Z : A_{-1}x + BCx \in X\big\} \end{aligned}$$

generates a C_0-semigroup $(S(t))_{t\ge0}$ satisfying $|S(t)x| \le \widetilde{S}(t)x$ for all $t \ge 0$, $x \ge 0$.

We refer to Subsects. 4.2 and 4.3 for concrete applications of this theorem.

For its proof we need some preparation. As usual, we might assume in the sequel that $\omega_0(A) < 0$ and $\mathrm{r}(\widetilde{C}A_{-1}^{-1}\widetilde{B}) < 1$.

Lemma 3.18. *If the triples* (A, B, C) *and* $(A, \widetilde{B}, \widetilde{C})$ *are compatible, then for all* $\lambda > 0$ *it holds*

(1) $CR(\lambda, A) \in \mathcal{L}(X, U)$ *and* $\|CR(\lambda, A)\|_{\mathcal{L}(X,U)} \le \|\widetilde{C}R(\lambda, A)\|_{\mathcal{L}(X,U)}$,
(2) $R(\lambda, A_{-1})B \in \mathcal{L}(U, X)$ *and* $\|R(\lambda, A_{-1})B\|_{\mathcal{L}(U,X)} \le \|R(\lambda, A_{-1})\widetilde{B}\|_{\mathcal{L}(U,X)}$,
(3) $CR(\lambda, A_{-1})B \in \mathcal{L}(U)$ *and* $\|(CR(\lambda, A_{-1})B)\|_{\mathcal{L}(U)} \le \|(\widetilde{C}R(\lambda, A_{-1})\widetilde{B})\|_{\mathcal{L}(U)}$,
(4) $\mathrm{r}(CR(\lambda, A_{-1})B) \le \mathrm{r}(\widetilde{C}R(\lambda, A)\widetilde{B}) \le \mathrm{r}(\widetilde{C}A_{-1}^{-1}\widetilde{B})$.

Proof. (i). Since $R(\lambda, A)$ is positive for $\lambda > 0$, using Assumption 3.13.(i) we obtain for all $x \in X_+$ and $\lambda > 0$

$$\left|CR(\lambda, A)x\right| \le \widetilde{C}R(\lambda, A)x.$$

Hence, for all $x \in X$ and $\lambda > 0$ we have

$$\begin{aligned} 0 \le \left|CR(\lambda, A)x\right| &\le \left|CR(\lambda, A)x^+\right| + \left|CR(\lambda, A)x^-\right| \\ &\le \widetilde{C}R(\lambda, A)x^+ + \widetilde{C}R(\lambda, A)x^- = \widetilde{C}R(\lambda, A)|x|. \end{aligned} \tag{3.29}$$

Applying the norm and taking the supremum in the unit ball of U gives (i).

(ii). By Assumption 3.13.(ii), $B_\pm$ are positive linear operators and satisfy $B = B_+ - B_-$ and $\widetilde{B} = B_+ + B_-$. Moreover, the triples $(A, B_\pm, C)$ and $(A, B_\pm, \widetilde{C})$ are compatible. Hence, for all $u \in U$ and $\lambda > 0$ using Lemma 2.2 and the positivity of $R(\lambda, A_{-1})B_\pm$ we conclude

$$\begin{aligned} 0 \le \left|R(\lambda, A_{-1})Bu\right| &= \left|R(\lambda, A_{-1})\,(B_+ - B_-)\,u\right| \\ &\le \left|R(\lambda, A_{-1})B_+u\right| + \left|R(\lambda, A_{-1})B_-u\right| \\ &\le R(\lambda, A_{-1})\big(B_+|u| + B_-|u|\big) = R(\lambda, A_{-1})\widetilde{B}|u|. \end{aligned}$$

Applying the norm and taking the supremum in the unit ball of U gives (ii).

(iii)–(iv). Using Assumption 3.13, Lemma 2.2 and the resolvent identity we obtain for all $u \in U$ and $\lambda > 0$

$$\begin{aligned} 0 &\le \left|CR(\lambda, A_{-1})Bu\right| = \left|CR(\lambda, A_{-1})\big(B_+u - B_-u\big)\right| \\ &\le \left|CR(\lambda, A_{-1})B_+u\right| + \left|CR(\lambda, A_{-1})B_-u\right| \\ &\le \left|CR(\lambda, A_{-1})B_+u^+\right| + \left|CR(\lambda, A_{-1})B_+u^-\right| \\ &\quad + \left|CR(\lambda, A_{-1})B_-u^+\right| + \left|CR(\lambda, A_{-1})B_-u^-\right| \\ &\le \widetilde{C}R(\lambda, A_{-1})\big(B_+u^+ + B_+u^- + B_-u^+ + B_-u^-\big) \\ &= \widetilde{C}R(\lambda, A_{-1})\big(B_+|u| + B_-|u|\big) = \widetilde{C}R(\lambda, A_{-1})\widetilde{B}|u| \\ &= \widetilde{C}R(0, A_{-1})\widetilde{B}|u| - \widetilde{C}\lambda R(\lambda, A)R(0, A_{-1})\widetilde{B}|u| \\ &\le -\widetilde{C}A_{-1}^{-1}\widetilde{B}|u|. \end{aligned} \tag{3.30}$$

Assertions (iii) and (iv) then follow from (3.30) by Proposition A.1. □

Lemma 3.19. *Let the triples* (A, B, C) *and* $(A, \widetilde{B}, \widetilde{C})$ *be compatible and* $A_{\widetilde{B}\widetilde{C}}$ *generate a positive semigroup on* X. *If* $\mathrm{r}(\widetilde{C}R(\lambda, A_{-1}\widetilde{B}) < 1$ *for some* $\lambda > 0$ *then*

(i) A_{BC} *is closed and densely defined,*
(ii) $\lambda \in \rho(A_{BC}) \cap \rho(A_{\widetilde{B}\widetilde{C}})$,
(iii) $\|R(\lambda, A_{BC})^n\|_{\mathcal{L}(X)} \leq \|R(\lambda, A_{\widetilde{B}\widetilde{C}})^n\|_{\mathcal{L}(X)}$ *for all* $n \in \mathbb{N}$.

Proof. By Lemma 3.18.(iv) the hypothesis $\mathrm{r}(\widetilde{C}R(\lambda, A_{-1}\widetilde{B}) < 1$ implies that $1 \in \rho(CR(\lambda, A_{-1}B)) \cap \rho(\widetilde{C}R(\lambda, A_{-1}\widetilde{B}))$. Hence, by [3, Thm. 2.3.(d)] we conclude $\lambda \in \rho(A_{BC}) \cap \rho(A_{\widetilde{B}\widetilde{C}})$ and

$$R(\lambda, A_{BC}) = R(\lambda, A) + R(\lambda, A_{-1})B\big(\mathrm{Id} - CR(\lambda, A_{-1})B\big)^{-1}CR(\lambda, A), \tag{3.31}$$

$$R(\lambda, A_{\widetilde{B}\widetilde{C}}) = R(\lambda, A) + R(\lambda, A_{-1})\widetilde{B}\big(\mathrm{Id} - \widetilde{C}R(\lambda, A_{-1})\widetilde{B}\big)^{-1}\widetilde{C}R(\lambda, A), \tag{3.32}$$

which proves (ii). Moreover, this implies that A_{BC} is closed. To show that $D(A_{BC})$ is dense in X we fix $x \in X$. Then $\lambda R(\lambda, A)x \to x$ and $\lambda R(\lambda, A_{\widetilde{B}\widetilde{C}})x \to x$ in X as $\lambda \to +\infty$. From (3.32) it follows

$$\lambda R(\lambda, A_{-1})\widetilde{B}\big(\mathrm{Id} - \widetilde{C}R(\lambda, A_{-1})\widetilde{B}\big)^{-1}\widetilde{C}R(\lambda, A)x \to 0 \quad \text{as } \lambda \to +\infty.$$

Using (3.29)–(3.30) and Proposition A.1.(i) this yields for arbitrary $x \in X$

$$\begin{aligned} 0 \leq \big|\lambda R(\lambda, A_{-1})B\big(\mathrm{Id} - CR(\lambda, A_{-1})B\big)^{-1}CR(\lambda, A)x\big| \\ &= \Big|\lambda R(\lambda, A_{-1})B \cdot \sum_{n=0}^{\infty}\big(CR(\lambda, A_{-1})B\big)^n \cdot CR(\lambda, A)x\Big| \\ &\leq \lambda R(\lambda, A_{-1})\widetilde{B} \cdot \sum_{n=0}^{\infty}\big(\widetilde{C}R(\lambda, A_{-1})\widetilde{B}\big)^n \cdot \widetilde{C}R(\lambda, A)|x| \\ &= \lambda R(\lambda, A_{-1})\widetilde{B}\big(\mathrm{Id} - \widetilde{C}R(\lambda, A_{-1})\widetilde{B}\big)^{-1}\widetilde{C}R(\lambda, A)|x| \\ &\to 0 \quad \text{as } \lambda \to +\infty. \end{aligned} \tag{3.33}$$

Hence, from (3.31) it follows $D(A_{BC}) \ni \lambda R(\lambda, A_{BC})x \to x$ as $\lambda \to +\infty$. Since $x \in X$ was arbitrary, this shows A_{BC} is densely defined as claimed. It only remains to show (iii). Using (3.31) and (3.32), the same reasoning as in (3.33) implies

$$\big|R(\lambda, A_{BC})x\big| \leq R(\lambda, A_{\widetilde{B}\widetilde{C}})x \tag{3.34}$$

for all $x \in X_+$. By Proposition A.1.(ii) this proves (iii). □

Proof (of Theorem 3.17). Without loss of generality we may assume $\omega_0(A) < 0$ and $\mathrm{r}(\widetilde{C}A_{-1}^{-1}\widetilde{B}) < 1$. Then by Lemma 3.19.(i) the operator A_{BC} is closed and densely defined. Moreover, by Lemma 3.18.(iv) and Lemma 3.19.(ii) we know that $\mathbb{R}_+ \subset \rho(A_{BC}) \cap \rho(A_{\widetilde{B}\widetilde{C}})$. Since by assumption $A_{\widetilde{B}\widetilde{C}}$ is a generator it verifies the Hille–Yosida estimates for the nth powers of its resolvent. By Lemma 3.19.(iii) the same is true for A_{BC} which implies that it is a generator as well. Finally, the domination $|S(t)x| \leq \widetilde{S}(t)x$ for all $t \geq 0, x \geq 0$ follows from (3.34) by the Post–Widder inversion formula, cf. [18, Cor. III.5.5]. □

4. Applications

4.1. Abstract domain perturbation of generators

In this subsection we show that boundary perturbations in the sense of Greiner [19] (see also [21]) give rise to operators of the form "$G = A + BC$". To this end we consider the following abstract framework.

We start with a "maximal[2] operator" $A_m : D(A_m) \subseteq X \to X$ on a Banach space X. In order to single out a particular restriction A of A_m we take a Banach space ∂X, called "boundary space", and a linear "boundary (or trace) operator" $L : D(A_m) \to \partial X$ and define the operator A with "abstract Dirichlet boundary conditions" by

$$A \subset A_m, \quad D(A) := \ker(L) = \big\{x \in D(A_m) : Lx = 0\big\}. \tag{4.1}$$

Next we perturb the domain of A using an operator $\Phi : X \to \partial X$ and define

$$A^\Phi \subset A_m, \quad D(A^\Phi) := \ker(L - \Phi) = \big\{x \in D(A_m) : Lx = \Phi x\big\}. \tag{4.2}$$

Hence, A^Φ can be considered as domain perturbation of the operator A.

To proceed we need the following result. For a proof we refer to [3, Sect. 3].

Lemma 4.1. *Assume that $A : D(A) \subseteq X \to X$ given in (4.1) has non-empty resolvent set $\rho(A)$. Moreover, suppose that*

(i) A_m or $\binom{A_m}{L}$ is closed and $L : D(A_m) \to \partial X$ is surjective, or
(ii) for some $\mu \in \mathbb{C}$ the restriction

$$L|_{\ker(\mu - A_m)} : \ker(\mu - A_m) \to \partial X$$

is invertible with bounded inverse

$$L_\mu := \big(L|_{\ker(\mu - A_m)}\big)^{-1} \in \mathcal{L}(\partial X, X). \tag{4.3}$$

Then for all $\lambda \in \rho(A)$

$$L|_{\ker(\lambda - A_m)} : \ker(\lambda - A_m) \to \partial X$$

is invertible with bounded inverse given by

$$L_\lambda = (\mu - A)R(\lambda, A)L_\mu \in \mathcal{L}(\partial X, X). \tag{4.4}$$

As we see next the so-called abstract *Dirichlet operator*[3] L_μ defined in (4.3) plays a crucial role in the context of this generic example. Note that by (4.4) $L_\lambda = R(\lambda, A_{-1})(\mu - A_{-1})L_\mu$, hence the operator

$$L_A := (\mu - A_{-1})L_\mu = (\lambda - A_{-1})L_\lambda \in \mathcal{L}\big(\partial X, X_{-1}\big) \tag{4.5}$$

is independent of μ and $\lambda \in \rho(A)$. Using this operator we obtain the following representation of A^Φ from (4.2). For a proof we refer again to [3, Sect. 3].

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[2] "Maximal" in the sense of a "big" domain, e.g., a differential operator without boundary conditions.

[3] This notion is justified by the fact that $f = L_\mu x_0$ solves the abstract Dirichlet problem $(\mu - A_m)f = 0$, $Lf = x_0$.

Lemma 4.2. *If $L_\lambda : X \to \partial X$ exists for some $\lambda \in \rho(A)$, then*

$$A^\Phi = \big(A_{-1} + L_A \cdot \Phi\big)\big|_X. \tag{4.6}$$

Hence, A^Φ given in (4.2) can be represented as A_{BC} like in (1.1) for $U := \partial X$ and the operators

$$B := L_A : U \to X_{-1} \quad \text{and} \quad C := \Phi : X \to U. \tag{4.7}$$

Finally, we apply Theorem 3.1 to this situation where we assume X and ∂X to be Banach lattices. Since by (4.5) and (4.7) we have $CR(\lambda, A_{-1})B = \Phi L_\lambda$ we obtain the following result.

Corollary 4.3. *Let $A : D(A) \subseteq X \to X$ given in (4.1) be the generator of a positive C_0-semigroup. Assume that $L_\mu : \partial X \to X$ for $\mu > \omega_0(A)$ and $\Phi : X \to \partial X$ are positive linear operators, where ∂X is an AM-space. If*

(i) the spectral radius $\mathrm{r}(\Phi L_\lambda) < 1$ for some $\lambda > \omega_0(A)$, or

(ii) ∂X has an order unit,

then the operator A^Φ defined in (4.2) generates a positive C_0-semigroup on X.

4.2. Heat equation with boundary feedback on $\mathrm{L}^2(\Omega)$

We give an application of Corollary 4.3 and Theorem 3.17. To this end we consider the partial differential equation

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \Delta u(t,x), & x \in \Omega,\ t \geq 0, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,z) = \displaystyle\int_\Omega \varphi(z,x)\, u(t,x)\, dx, & z \in \partial\Omega,\ t \geq 0, \end{cases} \tag{4.8}$$

where

- Ω is a bounded domain in $\mathbb{R}^n$ with C^2-boundary $\partial\Omega$,
- Δ denotes the Laplace operator on Ω,
- $u_0 \in \mathrm{L}^2(\Omega)$, and $\varphi \in \mathrm{C}(\partial\Omega, \overline{\Omega})$.

The well-posedness of a related heat equation with dynamic boundary conditions was studied in [15, Sect. 6] using a matrix approach on the state space $\mathcal{X} = \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\partial\Omega)$. In contrast, here we consider (4.8) on the space $X = \mathrm{L}^2(\Omega)$ and show that the associated operator A^Φ generates a C_0-semigroup. To this end we introduce the following spaces and operators:

- $X := \mathrm{L}^2(\Omega)$, $\partial X := \mathrm{L}^2(\partial\Omega)$,
- $A_m := \Delta : D(A_m) \subset X \to X$ with domain $D(A_m) := \{f \in \mathrm{H}^{\frac{1}{2}}(\Omega) \cap \mathrm{H}^2_{\mathrm{loc}}(\Omega) : \Delta f \in X\}$,
- $L : D(A_m) \to \partial X$, $Lf := f|_{\partial\Omega}$ in the sense of traces, cf. [22, Chap. 2],
- $\Phi : X \to \partial X$, $(\Phi f)(z) := \int_\Omega \varphi(z,x)\, f(x)\, dx$, $z \in \partial\Omega$, and

- $A, A^\Phi \subset A_m$, $D(A) := \ker L$, $D(A^\Phi) := \ker(L - \Phi)$.

As shown in [15, Sects. 3&6], in this context the Dirichlet operators $L_\lambda : X \to \partial X$ introduced in Lemma 4.1 exist and are positive for all $\lambda \geq 0$. Moreover, by [25, Cap. 5 & App. A9] the operator A generates a positive analytic semigroup on X satisfying $\omega_0(A) < 0$. Hence, by the results of the previous subsection we can represent the operator A^Φ as in (4.6). However, written in this way we cannot apply any of our results since $\partial X = \mathrm{L}^2(\partial\Omega)$ is neither an AM- nor an AL-space.

To overcome this difficulty observe that $\varphi \in \mathrm{C}(\partial\Omega, \overline{\Omega})$ is uniformly continuous which implies that

$$\mathrm{rg}(\Phi) \subset \mathrm{C}(\partial\Omega) =: \widehat{\partial X}.$$

Hence, we can define $\widehat{\Phi} : X \to \widehat{\partial X}$, $\widehat{\Phi} f := \Phi f$ for $f \in X$ which is bounded by the closed graph theorem. Moreover, let $\widehat{L_\lambda} := L_\lambda|_{\widehat{\partial X}} \in \mathcal{L}(\widehat{\partial X}, X)$ and $\widehat{L_A} := -A_{-1}\widehat{L_0} : \widehat{\partial X} \to X_{-1}$. Then clearly we also have

$$A^\Phi = \big(A_{-1} + \widehat{L_A} \cdot \widehat{\Phi}\big)$$

where $\widehat{\Phi} \in \mathcal{L}(X, \widehat{\partial X})$ and $\widehat{L_A} \in \mathcal{L}(\widehat{\partial X}, X_{-1})$. Since $\widehat{\partial X}$ equipped with the sup-norm $\|\cdot\|_\infty$ is an AM-space with order unit we can now apply Corollary 4.3 to conclude that A^Φ generates a positive semigroup on X for all $0 \leq \varphi \in \mathrm{C}(\partial\Omega, \overline{\Omega})$.

If $\varphi \in \mathrm{C}(\partial\Omega, \Omega)$ is arbitrary we denote the integral operator associated to $0 \leq |\varphi| \in \mathrm{C}(\partial\Omega, \Omega)$ by $\widetilde{\Phi}$. Then obviously $|\widehat{\Phi} f| \leq \widetilde{\Phi}|f|$ for all $f \in X$. Hence, by Theorem 3.17 and Remark 3.14 we obtain that for arbitrary $\varphi \in \mathrm{C}(\partial\Omega, \Omega)$ the operator

$$A^\Phi := \Delta : D(A^\Phi) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega),$$

$$D(A^\Phi) := \left\{ f \in \mathrm{H}^{\frac{1}{2}}(\Omega) \cap \mathrm{H}^2_{\mathrm{loc}}(\Omega) \;\middle|\; \begin{array}{l} \Delta f \in \mathrm{L}^2(\Omega), \\ f(z) = \displaystyle\int_\Omega \varphi(z,x)\, f(x)\, dx,\ z \in \partial\Omega \end{array} \right\}$$

generates a C_0-semigroup on $\mathrm{L}^2(\Omega)$. Since (4.8) can be rewritten as (ACP) for $G := A^\Phi$, we conclude that Eq. (4.8) is well-posed.

Remark 4.4. Since $X = \mathrm{L}^2(\Omega)$ is not an AM-space, the before-mentioned results from [12] cannot be applied to this example. Hence, imposing that the observation/control space U is of type AM (or AL) is a real generalization of the hypothesis that the state space X itself has this property. Also, the results of [19] do not work in this example since they only work on non-reflexive Banach spaces X.

4.3. Unbounded perturbation of the first derivative on $\mathrm{C}_0(0, 1]$

We give an application of Theorems 3.7 and 3.17. To this end we choose

- $X := \mathrm{C}_0(0, 1] = \{f \in \mathrm{C}[0, 1] : f(0) = 0\}$, $U = \mathrm{L}^1[0, 1]$ which is an AL-space,

- $A := -\frac{d}{dx} : D(A) \subset X \to X$, $D(A) := \{f \in \mathrm{C}^1[0,1] : f(0) = f'(0) = 0\}$,
- $P : D(A) \subset X \to X$, $(Pf)(x) := \int_0^x \frac{b(x-r)}{r^\alpha} \cdot f(r)\,dr$ for $\alpha \in [1,2)$ and $b \in \mathrm{L}^\infty[0,1]$, and
- $G := A + P : D(A) \subset X \to X$.

We claim that G generates a C_0-semigroup which is positive in case $b \geq 0$.

For the proof we first observe that A generates the positive nilpotent right-shift semigroup on X, cf. [18, Sect. I.4.17]. Then we factorize $P = BC$ where

- $B : U \to X$, $(Bu)(x) := (b * u)(x)$, and
- $C : D(A) \subset X \to U$, $(Cf)(x) := \frac{f(x)}{x^\alpha}$.

Indeed, $\mathrm{rg}(B) \subset X$ follows by Young's inequality while the fact $\alpha - 1 < 1$ and

$$\lim_{x \to 0^+} \frac{\frac{f(x)}{x^\alpha}}{\frac{1}{x^{\alpha-1}}} = \lim_{x \to 0^+} \frac{f(x)}{x} = f'(x) = 0$$

for $f \in D(A)$ implies $\mathrm{rg}(C) \subset U$.

In order to apply Theorem 3.7 we first assume that $b \geq 0$ which implies $B, C \geq 0$. Then to verify that $\mathrm{r}(CR(\lambda, A)B) < 1$ for $\lambda > 0$ sufficiently large we estimate for $g \in U$

$$\left\|CR(\lambda, A)Bu\right\|_1 \leq \left\|CR(\lambda, A)\right\|_{\mathcal{L}(\mathrm{C}_0(0,1],\mathrm{L}^1[0,1])} \cdot \left\|Bu\right\|_\infty. \tag{4.9}$$

Now a simple computation shows that for $f \in X$ and $\lambda > 0$ we have

$$\big(R(\lambda, A)f\big)(x) = \int_0^x e^{-\lambda(x-r)} f(r)\,dr, \;\; x \in [0,1].$$

Hence, for $f \in X$ satisfying $\|f\|_\infty \leq 1$ we obtain

$$\begin{aligned} \left\|CR(\lambda, A)f\right\|_1 &= \int_0^1 \left| x^{-\alpha} \int_0^x e^{-\lambda(x-r)} f(r)\,dr \right| dx \\ &\leq \int_0^1 x^{-\alpha} \int_0^x e^{-\lambda(x-r)}\,dr\,dx \\ &= \int_0^1 \lambda^{-1} x^{-\alpha} \big(1 - e^{-\lambda x}\big)\,dx =: \int_0^1 r_\lambda(x)\,dx. \end{aligned} \tag{4.10}$$

Since $1 - e^{-\lambda x} \leq \lambda x$ for all $x \in [0,1]$ and $\lambda > 0$ it follows that $r_\lambda(x) \leq x^{1-\alpha} =: m(x)$ for all $x \in [0,1]$. Moreover, $m \in \mathrm{L}^1[0,1]$ since $\alpha < 2$ and $\lim_{\lambda \to +\infty} r_\lambda(x) = 0$ for all $x \in (0,1]$. By Lebesgue's dominated convergence theorem it follows that

$$\int_0^1 r_\lambda(x)\,dx \to 0 \quad \text{as } \lambda \to +\infty.$$

By (4.10) this implies $\|CR(\lambda, A)\|_{\mathcal{L}(\mathrm{C}_0(0,1],\mathrm{L}^1[0,1])} \to 0$ as $\lambda \to +\infty$ and from (4.9) we finally obtain

$$\mathrm{r}\big(CR(\lambda, A)B\big) \leq \left\|CR(\lambda, A)B\right\|_{\mathcal{L}(U)} \to 0 \quad \text{as } \lambda \to +\infty.$$

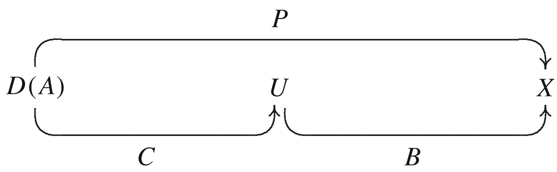
At this point we can apply Theorem 3.7 and conclude that G is a generator if $b \geq 0$. In case $b \in \mathrm{L}^{\infty}[0,1]$ is not positive, we define the positive convolution operators $B_{\pm} \in \mathcal{L}(U, X)$ by $(B_{\pm}u)(x) := (b_{\pm} * u)$ for which $B = B_{+} - B_{-}$ holds. The assertion then follows from Theorem 3.17.

Remark 4.5. In this example we applied Theorem 3.7 regarding AL-spaces to the perturbation of an operator A acting on an AM-space. We note that the results of [12] on AM-spaces are not applicable in this case since they only cover perturbations $P : X \to X_{-1}$. Moreover, the results of [16,26] are not applicable since X is not an AL-space. Finally, we mention that by essentially the same proof this example can be generalized substituting the convolution operator B by an arbitrary regular operator $B : U \to X$.

5. Conclusion

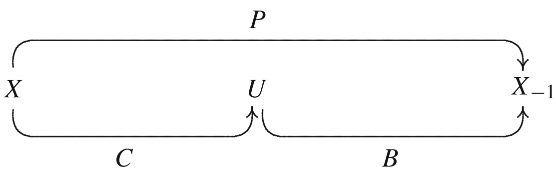
In this paper we showed how factorizing perturbations

- $P : D(A) \to X$ as $P = BC$ for operators $C : D(A) \to U$ and $B : U \to X$ via an AL-space U as in the diagram



or

- $P : X \to X_{-1}$ as $P = BC$ for operators $C : X \to U$ and $B : U \to X_{-1}$ via an AM-space U as in the diagram



allows generalizing significantly previous perturbation results for generators of positive semigroups on Banach lattices. More precisely, in [12,16,26] the state space X itself has to be an AM- or AL-space which, e.g., a priori excludes applications on (infinite dimensional) reflexive spaces. In contrast, our approach only needs U to be of type AM or AL which, as shown in Sect. 4, significantly widens the possible applications.

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Declarations

Conflict of interest The authors declare that they do not have any conflict of interest.

A: Appendix

In this appendix we briefly recall some notions and basic facts about Banach lattices and positive operators. For more details we refer to [23, Chap. C-I], [13, Chap. 10] or [10, Sect. 2.2].

A real vector space X equipped with a partial order $\leq$ is a *vector lattice* if for any pair $x, y \in X$ of elements the greatest lower bound $\inf\{x, y\} \in X$ and the least upper bound $\sup\{x, y\} \in X$ exist, and the order is compatible with the vector space structure, i.e.,

- $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in X$;
- $0 \leq x$ implies $0 \leq \alpha x$ for all $\alpha \in \mathbb{R}^+$.

For a vector lattice X we denote by $X_+ := \{x \in X : 0 \leq x\}$ its *positive cone*. Moreover, for $x \in X$ we define its *positive part* $x^+ = \sup\{x, 0\}$, its *negative part* $x^- = \sup\{-x, 0\}$ and its *absolute value* $|x| = \sup\{x, -x\}$. In this case we have $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

A norm $\|\cdot\|$ on a vector lattice X is called a *lattice norm* if

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\| \quad \text{for } x, y \in X. \tag{A.1}$$

Note that for a lattice norm we always have $\||x|\| = \|x\|$ for all $x \in X$. If a real vector lattice X endowed with a lattice norm is complete, then X is called a real *Banach lattice*. For a Banach lattice X, the operations $\sup\{\cdot, \cdot\}$ and $\inf\{\cdot, \cdot\}$ are continuous. This implies that the positive cone X_+ is closed (cf. [13, Prop. 10.8]). A Banach lattice X is said to be an

- *AL-space*, if $\|x + y\| = \|x\| + \|y\|$ for all $x, y \geq 0$,
- *AM-space*, if $\|\sup\{x, y\}\| = \sup\{\|x\|, \|y\|\}$ for all $x, y \geq 0$.

An element $e \in X_+$ is called an *order unit* of X if for each $x \in X$ there exists some $\lambda > 0$ such that $|x| \le \lambda e$. If X has an order unit, then it can be equivalently renormed in such a way that it becomes an AM-space having the order interval $[-e, e] := \{x \in X : -e \le x \le e\}$ as its closed unit ball, see [4, p. 195]. We remark that if for a Banach lattice X there exist two comparable norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ such that $(X, \|\cdot\|_1)$ is of type AL and $(X, \|\cdot\|_\infty)$ of type AM, then $X \simeq \mathbb{R}^N$ for some $N \in \mathbb{N}$.

A vector subspace $\mathcal{C}$ of a vector lattice X is called *vector sublattice* if for all $x \in \mathcal{C}$ one has $|x| \in \mathcal{C}$, hence also $x^+, x^- \in \mathcal{C}$.

Besides $\mathbb{R}^N$ the standard examples of Banach lattices are the sequence spaces l^p for $1 \le p \le +\infty$, c, c_0 and the function spaces $\mathrm{C}(K)$, $\mathrm{C}_0(\Omega)$ and $\mathrm{L}^p(\Omega; \mu)$ equipped with the natural order and the canonical norms. Here, cf. [13, Expl. 10.6],

- $(\mathbb{R}^N, |\cdot|_1)$, l^1 and $\mathrm{L}^1(\Omega; \mu)$ are AL-spaces, and
- $(\mathbb{R}^N, |\cdot|_\infty)$, c, l^∞, $\mathrm{L}^\infty(\Omega; \mu)$, $\mathrm{C}(K)$ are AM-spaces with order unit, c_0 and $\mathrm{C}_0(\Omega)$ are AM-spaces without order unit.

If X and Y are two real Banach lattices, an operator $T : X \to Y$ is said to be *positive* if $Tx \in Y_+$ for every $x \in X_+$. In this case we use the notation $T \ge 0$. Operators which can be written as the difference of two positive operators are called *regular*. We remark that any linear positive operator $T : X \to Y$ between Banach lattices is bounded, see [13, Thm. 10.20]. By [13, Lem. 10.18] for a linear operator $T : X \to Y$ between two real Banach lattices, the following are equivalent:

(a) $T \ge 0$;
(b) $(Tx)^+ \le Tx^+$ and $(Tx)^- \le Tx^-$ for all $x \in X$;
(c) $|Tx| \le T|x|$ for all $x \in X$.

We denote by $\mathcal{L}(X, Y)_+$ the set of positive linear operators from the Banach lattice X to the Banach lattice Y. For positive operators the operator norm is given by

$$\|T\|_{\mathcal{L}(X,Y)} := \sup\big\{\|Tx\|_Y : x \in X_+ \text{ and } \|x\|_X \le 1\big\}.$$

When there is no risk of ambiguity the operator norm is simply denoted by $\|\cdot\|$. Since the norm $\|\cdot\|_Y$ on the Banach lattice Y satisfies (A.1), the operator norm inherits the same property, i.e., if $S, T \in \mathcal{L}(X, Y)_+$ satisfy $S \le T$, then $\|S\| \le \|T\|$.

We recall that for an operator $T \in \mathcal{L}(X)$ on a Banach space X its *spectrum*

$$\sigma(T) := \big\{\lambda \in \mathbb{C} : \lambda - T \text{ is not bijective}\big\}$$

is a nonempty, compact subset of $\mathbb{C}$. Consequently, the *resolvent set* $\rho(A) := \mathbb{C}\setminus\sigma(A)$ is an open, nonempty subset of $\mathbb{C}$. The *spectral radius* of T is defined as

$$\mathrm{r}(T) := \sup\big\{|\lambda| : \lambda \in \sigma(T)\big\}$$

which satisfies $\mathrm{r}(T) \le \|T\|$ (see [18, Cor. IV.1.4]). The spectral radius can be calculated using *Gelfand's formula*.

$$\mathrm{r}(T) = \lim_{n\to\infty} \|T^n\|^{1/n}. \tag{A.2}$$

Moreover, if $|\lambda| > \mathrm{r}(T)$, then $R(\lambda, T)$ is given by the *Neumann series*

$$R(\lambda, T) = \sum_{n=0}^{+\infty} \lambda^{-(n+1)} \cdot T^n.$$

The identity (A.2) implies the monotonicity of the spectral radius as follows.

Proposition A.1. *Let $S, T : X \to X$ be linear operators on a Banach lattice X satisfying $|Sx| \leq Tx$ for every $x \geq 0$. Then*

(i) $|S^n x| \leq T^n |x|$ *for every $n \in \mathbb{N}$ and $x \in X$;*
(ii) $S, T \in \mathcal{L}(X)$ *and* $\|S^n\| \leq \|T^n\|$ *for every $n \in \mathbb{N}$;*
(iii) $\mathrm{r}(S) \leq \mathrm{r}(T)$.

Proof. We start by proving (i) by induction. For $n = 1$ we have for $x = x^+ - x^- \in X$

$$|Sx| = \left|S(x^+ - x^-)\right| \leq |Sx^+| + |Sx^-| \leq T(x^+ + x^-) = T|x|$$

as claimed. Now assume that (i) holds for some $n \in \mathbb{N}$. Then

$$\left|S^{n+1}x\right| = \left|S^n Sx\right| \leq T^n |Sx| \leq T^n T|x| = T^{n+1}|x|$$

which proves (i). To prove (ii) we first observe that by [13, Thm. 10.20] the operator T is bounded. Now take $x \in X$. Then from (i) we obtain

$$\|S^n x\| = \left\||S^n x|\right\| \leq \|T^n |x|\| \leq \|T^n\| \cdot \||x|\| = \|T^n\| \cdot \|x\|.$$

This implies (ii). Using (ii) we then conclude that

$$\mathrm{r}(S) = \lim_{n\to\infty} \|S^n\|^{1/n} \leq \lim_{n\to\infty} \|T^n\|^{1/n} = \mathrm{r}(T)$$

which proves (iii). □

Finally, for the generator A of a C_0-semigroup $(T(t))_{t\geq 0}$ we introduce its *spectral bound* $\mathrm{s}(A)$ and *growth bound* $\omega_0(A)$ by

$$\begin{aligned} \mathrm{s}(A) &:= \sup\big\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\big\}, \\ \omega_0(A) &:= \inf\{\omega \in \mathbb{R} : \exists M_\omega \geq 1 \text{ such that } \left\|T(t)\right\| \leq M_\omega \cdot e^{\omega t}\ \forall t \geq 0\big\}. \end{aligned}$$

Then by [18, Cor. II.1.13] we always have $-\infty \leq \mathrm{s}(A) \leq \omega_0(A) < +\infty$.

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