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Contributions sur le déclenchement d'événements contrôle pour les systèmes sur arbitraire domaines temporels

JURY

Rapporteur	POSTOYAN, Romain	CNRS Researcher, Université de Lorraine.
Rapporteur	ESPINOSA, Manuel Mazo	Associate Professeur. TU Delft.
Président & Examinateur	DE SANTIS, Elena	Associate Professeur. University of L'Aquila.
Examinatrice	AGGOUNE, Woihida	MCF. ENSEA.
Co-directeur	DJEMAI, Mohamed	Professeur. UPHF, INSA Hauts-de-France.
Co-directeur	DI GENNARO, Stefano	Professeur. University of L'Aquila.
Co-encadrant	DEFOORT, Michaël	Professeur. UPHF, INSA Hauts-de-France.

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Contributions on event–triggering control for systems on arbitrary time domains

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Subham Dey

Defence Date : 4/1/2023

Ph.D. Defence Chair
Prof. Elena De Santis (University of L'Aquila)

Ph.D. Coordinator
Prof. Vittorio Cortellessa (U. L'Aquila)

PhD. Supervisors
Prof. Mohamed Djemai (UPHF)
Prof. Stefano Di Gennaro (U. L'Aquila)
Prof. Michael Defoort (UPHF)

Rapporteurs
Prof. Romain Postoyan (CNRS researcher in control engineering at CRAN, Université de Lorraine (Nancy, France))
Prof. Manuel Mazo Espinosa (Associate Professor at TU Delft)

Examinateurs
Prof. Elena De Santis (University of L'Aquila)
Prof. Woihida Aggoune (MCF, ENSEA)

A.Y. 2022/2023

DECLARATION

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Abstract

This thesis deals with designing control algorithms for systems on any arbitrary time domain using the calculus of measure chain theory thereby keeping in mind the issues related to interruption of communications, availability of information, energy efficiency and safety–critical operation. The calculus of measure chain provides us with the opportunity to model systems on any arbitrary time domain be it continuous ($\mathbb{R}$), discrete ($\mathbb{Z}$) or continuous–discrete ($\mathbb{P}_{a,b}$) in a unified formulation. In this thesis, we first address the problem of observer based leader–follower bipartite consensus on a bipartite graph under some classes of intermittent failures in the communication channels by using the switched system formulation on a specific continuous–discrete measure chain called the $\mathbb{P}_{a,b}$ measure chain. Now, to overcome the problem related to energy consumption which had appeared when we addressed the problem of consensus, we looked in the direction of event–triggering. In this thesis, we have investigated some classical event–triggering problems for systems on measure chains using emulation based methods. Finally, we have addressed the safety–critical event–triggered control for nonlinear systems on measure chains using the notion of barrier functions on measure chains.

Keywords: Measure chain theory, Bipartite consensus, Event–triggering, Barrier functions.

Résumé

Cette thèse traite de la conception d'algorithmes de contrôle pour des systèmes sur n'importe quel domaine temporel arbitraire en utilisant le calcul de la théorie de la chaîne de mesure, gardant ainsi à l'esprit les problèmes liés à l'interruption des communications, à la disponibilité des informations, à l'efficacité énergétique et au fonctionnement critique pour la sécurité. Le calcul de la chaîne de la chaîne de mesure nous offre la possibilité de modéliser des systèmes sur n'importe quel domaine temporel arbitraire, qu'il soit continu (R), discret (Z) ou continu – discret ($P_{a,b}$) dans une formulation unifiée. Dans cette thèse, nous abordons d'abord le problème du consensus bipartite leader-suiveur basé sur un observateur pour un graphe bipartite sous certaines classes de défaillances intermittentes dans les canaux de communication en utilisant la formulation du système commuté sur une chaîne de mesure continue-discrète spécifique appelée $P_{a,b}$ chaîne de mesure. Pour pallier le problème lié à la consommation d'énergie qui était apparu lorsque l'on abordait le problème du consensus, on s'est tourné vers la commande par événement (event-triggered control en anglais). Dans cette thèse, nous avons étudié quelques problèmes classiques de commande par événement pour les systèmes sur chaînes de mesure en utilisant des méthodes basées sur l'émulation. Enfin, nous avons abordé commande par événement en considérant les aspects critiques liés à la sécurité pour les systèmes non linéaires sur des chaînes de mesure en utilisant la notion de fonctions barrières sur des chaînes de mesure.

Mots clés: Théorie de la chaîne de mesure, Consensus bipartite, Commande par événement, Fonctions barrières.

Nous présentons ici le thème central de cette thèse qui est de concevoir des algorithmes pour des systèmes sur des domaines temporels arbitraires tout en gardant à l'esprit les contraintes de certaines classes de défaillances et de sécurité. Les principales contributions de cette thèse sont énumérées ci-dessous :

Nous avons conçu un observateur distribué à rétroaction de sortie qui résout le problème du consensus bipartite dans un cadre leader-suiveur sous des communications intermittentes entre les canaux de communication à l'aide de la formulation des systèmes commutés sur une chaîne de mesure donnée et la notion de fonction de Lyapunov commune. La principale motivation pour résoudre ce problème est qu'une grande majorité des problèmes de consensus impliquent une topologie de graphe qui ne contient pas d'interactions antagonistes. Cependant, dans les applications physiques du monde réel, les interactions peuvent être antagonistes, le meilleur exemple étant les scénarios d'élections politiques ou les réseaux sociaux. En outre, des défaillances apparaissent également dans les canaux de communication et les outils des fonctionnelles de Lyapunov-Krasovskii ou des théorèmes de Lyapunov Razumikhin peuvent être utilisés pour résoudre le problème du consensus en cas de défaillance. Cependant, il n'est pas toujours facile de trouver une fonctionnelle de Lyapunov-Krasovskii appropriée. Ainsi, en supposant que les défaillances des canaux de communication sont limitées, on peut résoudre le problème du consensus en utilisant les méthodologies et les outils fournis par la théorie de la chaîne de mesure. Les limites de la conception d'algorithmes classiques déclenchés par le temps sont qu'ils sont mis à jour en permanence, indépendamment du fait qu'un système soit ou non en état de marche sans considération qu'un système soit dans son état transitoire initial ou dans son dans son état stable final. Ceci nous a incités à nous tourner vers le déclenchement d'événements et à justifier

notre logique derrière derrière l'exploration du sujet du commande par événements (event-triggered control) sur les chaînes de mesure.

Nous avons conçu des contrôleurs à déclenchement événementiel pour des systèmes sur des chaînes de mesure quelconques en utilisant le concept de fonctions de Lyapunov sur des chaînes de mesure. La principale motivation derrière la conception de contrôleurs déclenchés par des événements pour des systèmes sur des chaînes de mesure est non seulement d'unifier les procédures de conception de déclenchement d'événements existantes pour les domaines de temps continu ou discret, mais aussi de les étendre à tout domaine de temps arbitraire. L'intérêt principal de la conception de contrôleurs déclenchés par des événements est qu'ils peuvent être efficaces sur le plan énergétique. Les contrôleurs déclenchés par des événements conçus dans cette thèse appartiennent à la grande classe des contrôleurs déclenchés par des événements continus dans lesquels la condition de déclenchement est vérifiée à chaque point de la chaîne de mesure. En outre, les constantes des gains sur la commande ainsi que les constantes sur le mécanisme de déclenchement sont calculées séparément à l'aide de la procédure de conception basée sur l'émulation. Les contrôleurs à déclenchement par événement susmentionnés peuvent être vulnérables au phénomène de Zénon si la condition de déclenchement n'est pas conçue de manière appropriée pour certaines classes de chaînes de mesure, ce qui nous a obligés à nous pencher sur la direction du déclenchement par événement périodique dans la thèse.

L'estimation d'état est un domaine de recherche actif car, dans diverses applications d'intérêt scientifique et technique, il n'est pas possible ou il est coûteux de

mesurer toutes les variables décrivant la dynamique d'un processus. Les observateurs sont d'excellents outils pour surmonter cette difficulté. Depuis les travaux précurseurs de Luenberger dans les années 1960, de nombreuses recherches ont porté sur cette question et ont abouti à la conception d'observateurs pour les systèmes linéaires et non linéaires. Il convient de noter que ces résultats supposent une mesure continue de la sortie. En revanche, l'utilisation de dispositifs numériques implique naturellement que, dans de nombreuses applications pratiques, les seules mesures disponibles se font à des instants discrets. Cet aspect doit être pris en compte de manière appropriée lors de la conception d'un observateur. Un observateur à temps discret, conçu à l'aide du système de données échantillonnées, peut assurer la stabilité de l'erreur d'estimation dans les instants d'échantillonnage, pour des temps d'échantillonnage suffisamment petits, mais il ne garantit pas que l'erreur d'estimation convergera entre les instants d'échantillonnage. Pour faire face à cette situation, les observateurs dits impulsifs, qui ont une nature échantillonnée inhérente, ont été introduits récemment pour concevoir des observateurs garantissant une performance souhaitée également entre les instants d'échantillonnage. En outre, divers observateurs impulsifs ont été proposés pour les systèmes à temps continu dans la littérature. D'autre part, en utilisant des fonctions de Lyapunov discontinues, des conditions suffisantes pour l'existence de ces observateurs peuvent être dérivées en termes d'inégalités matricielles linéaires (LMI). Cependant, la plupart des travaux relatifs aux observateurs impulsifs n'ont considéré que des systèmes sur une chaîne de mesure continue alors que dans cette thèse, nous concevons un observateur impulsif déclenché par un événement périodique pour des systèmes sur n'importe quel domaine temporel arbitraire, pas nécessairement continu. Nous avons conçu des contrôleurs déclenchés par des événements périodiques et des contrôleurs de retour de sortie déclenchés par des événements périodiques utilisant des observateurs impulsifs pour des systèmes sur

n'importe quelle chaîne de mesure. Les résultats de stabilité proposés garantissent la stabilité asymptotique du système en boucle fermée sur n'importe quel domaine temporel, contrairement au concept de stabilité pratique que l'on peut trouver dans la littérature. En outre, le contrôle déclenché par des événements périodiques évite intrinsèquement le phénomène de Zeno. Comme pour le mécanisme de conception précédent, dans cette thèse, les constantes de gain et les paramètres de conception de la condition de déclenchement par événement sont évalués à l'aide de la méthode basée sur l'émulation.

Nous avons conçu des contrôleurs déclenchés par des événements critiques pour les systèmes non linéaires sur n'importe quelle chaîne de mesure en utilisant le concept de fonctions de barrière de sécurité de l'entrée à l'état sur les chaînes de mesure. Les résultats précédents sur le contrôle déclenché par un événement critique de sécurité se sont uniquement concentrés sur les domaines à temps continu, mais les résultats présentés dans cette thèse peuvent être appliqués à des systèmes sur n'importe quel domaine temporel arbitraire sans aucune restriction sur la non-linéarité.

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Symbols

$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{Z}$	Set of integers
$\mathbb{H}$	Set of harmonic numbers
$\mathbb{R}^{0+}$	Set of positive real numbers including zero
$\mathbb{Z}^{0+}$	Set of positive integers including zero
$\mathbb{N}^{0+}$	Set of natural numbers including zero
$\lambda_{min}(.)$	Minimum eigen–value of a square matrix
$\lambda_{max}(.)$	Maximum eigen–value of a square matrix
$A \otimes B$	Kronecker product between two matrices A and B
$\mathcal{A}_s$	Adjacency matrix
$\mathcal{L}$	Laplacian matrix
$\mathcal{D}$	Pinning matrix
A^T	Transpose of a matrix A
$Q > 0$	Q is a positive definite matrix
$Q \geq 0$	Q is a positive semi–definite matrix
$\|(.)\|$	Euclidean norm of a vector (.)
$(*)$	Conjugate transpose of a matrix

Chapter 1

Introduction

The famous mathematician Eric Temple Bell (see Fig. 1.1) made the following quote in the first half of the 20^{th} century in his book **Men of Mathematics**, "*A major task of mathematics today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both.*" Thus, measure chain theory provides us tools to perform calculus not only on continuous, discrete but on any non–uniform time domains. The concept of



FIGURE 1.1: *Portrait of John Taine (Eric Temple Bell).*



FIGURE 1.2: *Portrait of Stefan Hilger.*

measure chain theory was first proposed by German mathematician Stefan Hilger (see Fig. 1.2) in his PhD thesis. Some of the examples of systems wherein the dynamics evolve on non–uniform time domains are predator–prey systems, inventory models, economic models, epidemic growth models, non–electronic impulsive models or population growth models (Atici and Uysal (2008), Kumar et al. (2021), Ferreira and Silva (2018)). In (Russo and Wirth (2020)) the $SIQR$ epidemic model on measure chain is explained in which the set of uninfected individuals is taken as $S(t)$, the set of infected individuals who are not in quarantine is taken as $I(t)$, the set of infected individuals who are in quarantine is taken as $Q(t)$ and lastly the set of recovered individuals is taken as $R(t)$. Furthermore, a theorem is also stated which ensures that the solutions of the $SIQR$ model reaches to a disease free state on any measure chain whether it is continuous–discrete, discrete or continuous under certain conditions (see Fig. 1.3 for an example on $\mathbb{P}_{a,b}$ measure chain). Some results related to controller design for systems on measure chains wherein the classical consensus problem in a leader–follower setting is solved under some failure constraints have appeared in the literature (Taousser et al. (2015)) but it is practically hard to design state–feedback distributed controllers because all the state variables cannot be obtained at each period of time. Hence, we have explored the direction of designing distributed output feedback controllers for solving the leader–follower consensus problem wherein antagonistic interactions exist in the graph topology under the influence of some failure constraints. This forms one of the domains of our work. Secondly, event–triggering is a way to implement controllers, observers or output–feedback controllers which may result in an energy efficient form of implementation contrary to their classical time–triggered form of representation which has gained a huge popularity among the control system

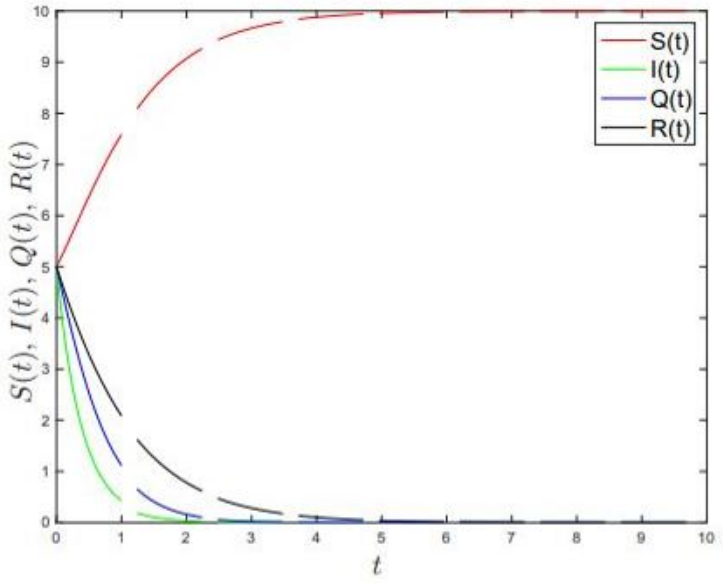


FIGURE 1.3: *Trajectories of various variables in the measure chain epidemic model on a continuous–discrete measure chain (Russo and Wirth (2020)).*

engineers since the last two decades, which forms another domain of our work. Lastly, we have also looked into how we can make our event–triggered controllers on arbitrary time domains safe because it forms the cornerstone of implementing any controllers for physical systems.

In this chapter, we recall some of the basics of measure chain theory, switched systems, event–triggered control and safety–critical control in brief which would be useful in explaining the concepts presented in the subsequent chapters. A set–theoretical representation of the topics explored in the thesis is given below in Fig. 1.4.

First, we will give some of the basic concepts of measure chain theory and its calculus which help us to study systems on any arbitrary time domains and form the backbone of the topics explored in this thesis.

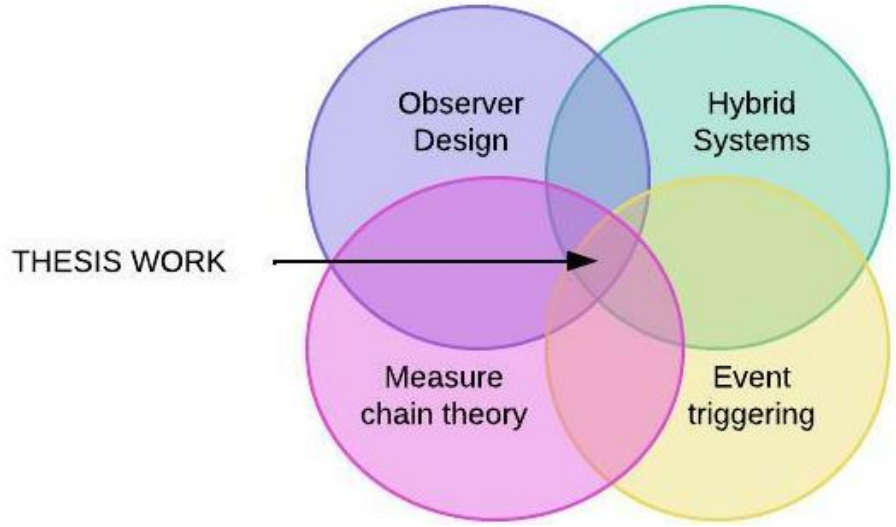


FIGURE 1.4: *Main topics related to thesis work.*

1.1 Measure chain theory

In this section, some preliminary notations related to measure chain theory will be explained. The notations are recalled from the books of (Martynyuk (2016)) and (Bohner and Peterson (2002)).

1.1.1 Calculus on measure chains

A measure chain $\mathbb{T}$ is a non–empty closed subset of the set of real numbers $\mathbb{R}$ $\Big((\mathbb{T} \subseteq \mathbb{R})$ and $(\mathbb{T} \neq \emptyset)\Big)$. Throughout the thesis we assume that the measure chain $\mathbb{T}$ has a topology which it inherits from the real numbers with standard topology. Various examples of measure chains which exist in the literature are given by

1. The set of real numbers $\mathbb{R}$.
2. The set of integer numbers $\mathbb{Z}$.
3. The set of harmonic numbers, $\mathbb{H} = \Big\{\sum_{k=1}^{n} \frac{1}{k} : n \in \mathbb{N}\Big\}$.
4. The set of multiples of integer numbers, $\mathrm{h}\mathbb{Z}\ \forall\ \mathrm{h}\ \in \mathbb{R}^{+}$.
5. The set of quantum numbers, $\mathbb{T} = a^{\mathbb{N}_0} = \{a^n : n \in \mathbb{N}_0\} \forall\ a > 1$.

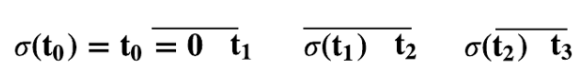


FIGURE 1.5: *Pictorial representation of a $\mathbb{P}_{a,b}$ measure chain.*

6. The measure chain $\overline{a^{\mathbb{Z}}} = \left\{a^k : k \in \mathbb{Z}\right\} \cup \{0\}, \quad$ where $\quad a > 0.$

The pictorial representation of $\mathbb{P}_{a,b}$ measure chain is given in Fig 1.5 which is a combination of $\mathbb{R}$ and $\mathbb{Z}$ measure chains. In the measure chain theory literature, there are several definitions which are used to perform any algebraic calculations. Now, we recall some of the definitions of measure chain theory. The definitions are given by,

Definition 1: (Bohner and Peterson (2002), Martynyuk (2016)) For any given measure chain, the expression of *forward jump operator* is given by

$$\sigma(t) := \inf\{w \in \mathbb{T} : w > t\{\}, \ \forall \ t \in \mathbb{T}. \blacksquare \tag{1.1}$$

Definition 2: (Bohner and Peterson (2002), Martynyuk (2016)) For any given measure chain, the expression of *backward jump operator* is given by

$$\rho(t) := \sup\{w \in \mathbb{T} : w < t\{\}, \ \forall \ t \in \mathbb{T}. \blacksquare \tag{1.2}$$

Definition 3: (Bohner and Peterson (2002), Martynyuk (2016)) For any given measure chain, the expressions of *graininess function* $\mu(t)$ is given by

$$\mu(t) := \sigma(t) - t, \ \forall \ t \in \mathbb{T}. \blacksquare \tag{1.3}$$

Definition 4: (Bohner and Peterson (2002), Martynyuk (2016)) For any given measure chain, the expression of *backward graininess function* is given by

$$\nu(t) := t - \rho(t), \ \forall \ t \in \mathbb{T}. \blacksquare \tag{1.4}$$

The values of $\sigma(t)$, $\rho(t)$ and $\mu(t)$ for various measure chain $\mathbb{T}$ are given below

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$h\mathbb{Z}$	$a^{\mathbb{N}}$	$2^{\mathbb{N}}$	$\mathbb{N}_0^2$
$\sigma(t)$	t	$t+1$	$t+h$	a^t	$2t$	$(\sqrt{t}+1)^2$
$\rho(t)$	t	$t-1$	$t-h$	t/a	$t/2$	$(\sqrt{t}-1)^2$
$\mu(t)$	0	1	h	$(a-1)t$	t	$2\sqrt{t}+1$

Any point $t \in \mathbb{T}$ can be classified as whether it is left–scattered, left–dense, right–scattered and right–dense based on the definitions of forward jump operators, backward jump operators, graininess function and backward graininess function. Thus,

Definition 5: (Bohner and Peterson (2002), Martynyuk (2016)) Any point on the measure chain $t \in \mathbb{T}$ is said to be *right–scattered* if $\sigma(t) > t$ or in other words $\mu(t) = \sigma(t) - t > 0$. ■

Definition 6: (Bohner and Peterson (2002), Martynyuk (2016)) Any point on the measure chain $t \in \mathbb{T}$ is said to be *right–dense* if $\sigma(t) = t$ or in other words $\mu(t) = \sigma(t) - t = 0$. ■

Definition 7: (Bohner and Peterson (2002), Martynyuk (2016)) Any point on the measure chain $t \in \mathbb{T}$ is said to be *left–scattered* if $\rho(t) < t$ or in other words $\nu(t) > 0$. ■

Definition 8: (Bohner and Peterson (2002), Martynyuk (2016)) Any point on the measure chain $t \in \mathbb{T}$ is said to be *left–dense* if $\rho(t) = t$ or in other words $\nu(t) = 0$. ■

Definition 9: (Bohner and Peterson (2002), Martynyuk (2016)) If a point on the measure chain is both left–scattered and right–scattered then the point is said to be *isolated*. ■

Definition 10: (Bohner and Peterson (2002), Martynyuk (2016)) If a point on the measure chain is both right–dense and left–dense then the point is said to be *dense*. ■

A pictorial representation explaining the concepts of left–scattered, left–dense, right–scattered, right–dense, isolated and dense is given below in Fig. (1.6)

In order to explain the concept of differentiation on measure chain $\mathbb{T}$, we need to define the set $\mathbb{T}^r$ which is given by

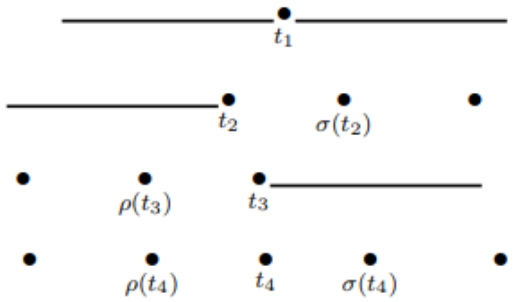


FIGURE 1.6: *t_1 is a dense point on measure chain $\mathbb{T}$, t_2 is a left dense and right scattered point on measure chain $\mathbb{T}$, t_3 is a right dense and left scattered point on measure chain $\mathbb{T}$, t_4 is an isolated point on measure chain $\mathbb{T}$.*

$$\mathbb{T}^r = \begin{cases} \mathbb{T} \setminus (\rho(\sup\mathbb{T}), \sup\mathbb{T}] & \text{if } \sup\mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup\mathbb{T} = \infty \end{cases}$$

The expression of delta–differentiation for a delta–differentiable function $\varsigma : \mathbb{T} \to \mathbb{R}$ on the set $\mathbb{T}^r$ is given by

$$\overset{\Delta}{\varsigma}(t) := \lim_{w \to t} \frac{\varsigma(\sigma(t)) - \varsigma(w)}{\sigma(t) - w} \tag{1.5}$$

Definition 11: (Bohner and Peterson (2002), Martynyuk (2016)) Assume $\varsigma : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^r$. Then we define $\overset{\Delta}{\varsigma}(t)$ to be the number (provided it exists) with the property that given any $a > 0$, there is a neighbourhood B of t ($(t-b, t+b \cap \mathbb{T})$ for a positive constant $b > 0$), such that the following inequality

$$|(\varsigma(\sigma(t)) - \varsigma(u)) - \overset{\Delta}{\varsigma}(\sigma(t) - u)| \leq a|\sigma(t) - u| \ \forall \ u \in (t - b, t + b) \cap \mathbb{T} \tag{1.6}$$

is satisfied then $\overset{\Delta}{\varsigma}(t)$ is called the delta–derivative of the function ς at the point t. ■

From the expression of delta–differentiation given in Eq. (1.5), one can easily see when $\mathbb{T} = \mathbb{R}$, the delta–derivative converts to classical differentiation

$$\dot{\varsigma}(t) = \lim_{w \to t} \frac{\varsigma(t) - \varsigma(w)}{t - w} \ , \forall \ t \ \in \ \mathbb{R} \tag{1.7}$$

Similarly, when $\mathbb{T} = \mathbb{Z}$ the expression of delta–derivative converts to classical difference equation

$$\varsigma(t+1) - \varsigma(t) = \frac{\varsigma(t+1) - \varsigma(t)}{t+1-t} \ , \forall \ t \ \in \ \mathbb{Z} \tag{1.8}$$

Some examples related to delta–differentiation on measure chains are given below,

1. Consider a measure chain $\mathbb{T} = 2^{\mathbb{N}}$ and a function $\varsigma(t) = t^2, \ \forall \ t \in \mathbb{T}$. Then the delta–derivative of ς on the considered measure chain is given by

$$\begin{aligned} \overset{\triangle}{\varsigma}(t) &= \lim_{w \to t} \frac{\varsigma(\sigma(t)) - \varsigma(w)}{\sigma(t) - w} \\ &= \lim_{w \to t} \frac{4t^2 - w^2}{2t - w} = 3t, \ \forall \ t \ \in \ \mathbb{T} \end{aligned} \tag{1.9}$$

2. Consider a measure chain $\mathbb{T} = h\mathbb{Z}$, and the function in this case is $\varsigma(t) = t \ \forall \ t \in \mathbb{T}$. The delta–derivative of ς on the measure chain $\mathbb{T}$ is

$$\begin{aligned} \overset{\triangle}{\varsigma}(t) &= \lim_{w \to t} \frac{\varsigma(\sigma(t)) - \varsigma(w)}{\sigma(t) - w} \\ &= \lim_{w \to t} \frac{t + h - w}{t + h - w} = 1, \ \forall \ t \ \in \ \mathbb{T} \end{aligned} \tag{1.10}$$

Now, we are in a position to recall some properties of delta–derivatives for functions on measure chains. Considering two delta–differentiable functions $s : \mathbb{T} \to \mathbb{R}$ and $q : \mathbb{T} \to \mathbb{R}$.

Theorem 1: (Bohner and Peterson (2002), Martynyuk (2016)) The combined function $r = s + q$, is delta–differentiable and the expression is given by

$$\overset{\triangle}{r} = \overset{\triangle}{s} + \overset{\triangle}{q} \quad \blacksquare \tag{1.11}$$

Theorem 2: (Bohner and Peterson (2002), Martynyuk (2016)) The delta–differentiation of any function $r = \beta q$ where β is any scalar constant is given by

$$\overset{\triangle}{r} = \beta \overset{\triangle}{q} \quad \blacksquare \tag{1.12}$$

Theorem 3: (Bohner and Peterson (2002), Martynuk (2016)) If the function $r = sq$ is delta–differentiable on measure chain $\mathbb{T}$, then the expression of delta–differentiation is given by

$$\begin{aligned} \overset{\triangle}{r} &= \overset{\triangle}{s} q + s^{\sigma} \overset{\triangle}{q} \\ &= s \overset{\triangle}{q} + \overset{\triangle}{s} q^{\sigma} \quad \blacksquare \end{aligned} \tag{1.13}$$

Theorem 4: (Bohner and Peterson (2002), Martynuk (2016)) The delta–differentiation of function $r = \frac{1}{s}$ exists provided $ss^{\sigma} \neq 0$ and its expression is given by

$$\overset{\triangle}{r} = -\frac{\overset{\triangle}{s}}{ss^{\sigma}} \quad \blacksquare \tag{1.14}$$

Theorem 5: (Bohner and Peterson (2002), Martynuk (2016)) The delta–differentiation of function $r = \frac{q}{s}$ exists when $ss^{\sigma} \neq 0$ and is given by

$$\overset{\triangle}{r} = \frac{\overset{\triangle}{q} s - q \overset{\triangle}{s}}{ss^{\sigma}} \quad \blacksquare \tag{1.15}$$

Theorem 6: (Bohner and Peterson (2002), Martynuk (2016)) Considering $a : \mathbb{R} \to \mathbb{R}$, then the delta–differentiation of the composition function $r = (a \circ s)$ is given by

$$\overset{\triangle}{r} = \left\{ \int_0^t \dot{a} \left(s + h\mu \overset{\triangle}{s} \right) dh \right\} \overset{\triangle}{s} \quad \blacksquare \tag{1.16}$$

1.1.2 Integration on measure chains

In this subsection, we recall some basics of integration on measure chain $\mathbb{T}$ (Bohner and Peterson (2002), Bohner and Guseinov (2006), Bohner and Georgiev (2016a), Guseinov (2003), Guseinov (2003)).

Definition 12: (Bohner and Peterson (2002), Martynuk (2016)) A function $s : \mathbb{T} \to \mathbb{R}$ is said to be *rd–continuous* (right–dense continuous) if the function is continuous at each and every right dense points on the measure chain as well as the left–hand limit exists at all left–dense points on the measure chain. The set

of all *rd–continuous* functions is denoted by $\mathcal{C}_{rd}(\mathbb{T},\mathbb{R})$. ■

Definition 13: (Bohner and Peterson (2002), Martynyuk (2016)) A function $s:\mathbb{T}\to\mathbb{R}$ on measure chain is said to have an anti–derivative $s_1:\mathbb{T}\to\mathbb{R}$, if the following inequality is satisfied $\forall\ t\ \in\mathbb{T}^k$

$$\overset{\Delta}{s}_1(t)=s(t) \quad \blacksquare \tag{1.17}$$

Theorem 7: (Bohner and Peterson (2002), Martynyuk (2016)) Any *rd–continuous* function always has an anti–derivative associated with it. ■

Definition 14: (Bohner and Peterson (2002), Martynyuk (2016)) Provided an anti–derivative $s_1:\mathbb{T}\to\mathbb{R}$ exists for a delta–differentiable function $s:\mathbb{T}\to\mathbb{R}$, then the expression of *Cauchy's integral* of $s(t)$ is given by

$$\int_{a_1}^{a_2} s(t)\Delta t=s_1(a_1)-s_1(a_2), \quad \text{where} \quad a_1,a_2\in\mathbb{T} \ \blacksquare \tag{1.18}$$

Now, we tabulate the expression of Cauchy's integral for two widely used measure chains $\mathbb{Z}$ and $\mathbb{R}$.

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$
$\sigma(t)$	t	$t+1$
$\rho(t)$	t	$t-1$
$\mu(t)$	0	1
$\overset{\Delta}{s}(t)$	$\dot{s}(t)$	$\Delta s(t)$
$\int_{a_1}^{a_2} s(t)\Delta t$	$\int_{a_1}^{a_2} s(t)dt$	$\sum_{t=a_1}^{a_2-1} s(t\Big)$
rd-continuity	continuity f	any f

Theorem 8: We hereby state some of the widely used formulas on Cauchy's integral for systems on measure chain $\mathbb{T}$

- (Bohner and Peterson (2002), Martynyuk (2016)): For any two rd–continuous functions s and r on $\mathbb{T}$, the expression of Cauchy's integral for the function $b=s+r$ is given by

$$\int_{a_1}^{a_2}[s(t)+r(t)]\Delta t=\int_{a_1}^{a_2} s(t)\Delta t+\int_{a_1}^{a_2} r(t)\Delta t \tag{1.19}$$

- (Bohner and Peterson (2002), Martynyuk (2016)): For any scalar variable $w \in \mathbb{R}$, the expression of Cauchy's integral for the scaled function $b = ws$ is given by

$$\int_{a_1}^{a_2} ws(t)\Delta t = w \int_{a_1}^{a_2} s(t)\Delta t \tag{1.20}$$

- (Bohner and Peterson (2002), Martynyuk (2016)): The expression of Cauchy's integral for any function $b = s(\sigma(t))\overset{\triangle}{r}$ is given by

$$\int_{a_1}^{a_2} s(\sigma(t))\overset{\triangle}{r}(t)\Delta t = s(a_2)r(a_2) - s(a_1)r(a_1) - \int_{a_1}^{a_2} \overset{\triangle}{s}(t)r(t)\Delta t \ \blacksquare \tag{1.21}$$

1.1.3 Exponential function on measure chains

In this subsection, we recall the definition of exponential function on measure chains and some definitions.

Definition 15: (Bohner and Peterson (2002), Martynyuk (2016)) Any function $s : \mathbb{T} \to \mathbb{R}$ which is rd–continuous on a given measure chain $\mathbb{T}$ is said to be *regressive* on the same measure chain $\mathbb{T}$ if the following property is satisfied,

$$1 + \mu(t)s(t) \neq 0 \ \forall \ t \ \in \mathbb{T}^k \tag{1.22}$$

The set of all regressive functions is denoted by $\mathcal{R}$. $\blacksquare$

Definition 16: (Bohner and Peterson (2002), Martynyuk (2016)) Any function $s : \mathbb{T} \to \mathbb{R}$ which is rd–continuous on a given measure chain $\mathbb{T}$ is said to be *positive–regressive* on the same measure chain $\mathbb{T}$ if the following property is satisfied,

$$1 + \mu(t)s(t) > 0 \ \forall \ t \ \in \mathbb{T}^k \tag{1.23}$$

The set of all positive–regressive functions is denoted by $\mathcal{R}^+$. $\blacksquare$

The definitions of regressivity and positive regressivity is indeed unique to the theory of measure chains because unlike continuous measure chains where the

value of exponential function is always positive irrespective of the exponent value, in case of general measure chains the value of exponential function can be negative. Thus regressivity helps us to link the concepts of graininess function of a measure chain and the positivity of exponential functions.

Definition 17: (Bohner and Peterson (2002), Martynyuk (2016)) In case of any measure chain $\mathbb{T}$ there exist operators $\oplus$ and $\ominus$. For any regressive functions $r, s \in \mathcal{R}$, the operator $\oplus$ is mathematically represented as

$$(r \oplus s)(t) = r(t) + s(t) + \mu(t)s(t)r(t),\ t\ \in\ \mathbb{T} \tag{1.24}$$

For the operator $\ominus$, the expression is given by

$$(r \ominus s)(t) = \frac{r(t) - s(t)}{1 + \mu(t)s(t)},\ t\ \in\ \mathbb{T}^k\ \blacksquare \tag{1.25}$$

Definition 18: (Bohner and Peterson (2002), Martynyuk (2016)) The set of regressive functions $\mathcal{R}$ and the operator $\oplus$ forms an Abelian group. The inverse element of any member $r \in \mathcal{R}$, of the Abelian group is given by

$$\mathcal{R} \ni \ominus r(t) = -\frac{r(t)}{1 + \mu(t)r(t)}\ \blacksquare \tag{1.26}$$

Theorem 9:(Bohner and Peterson (2002), Martynyuk (2016)) For any two regressive functions r and s,

$$r\ \oplus\ s,\ r\ \ominus\ s,\ r \oplus (\ominus s)\ \in \mathcal{R}\ \blacksquare \tag{1.27}$$

In order to introduce the formal definition of an exponential function on measure chain $\mathbb{T}$, we recall some concepts related to *Hilger complex plane*.

Definition 19: (Bohner and Peterson (2002), Martynyuk (2016)) The horizontal strip of the complex plane for any positive constant $a > 0$ is given by

$$\mathbb{Z}_a := \left\{ z \in \mathbb{C} : -\frac{\pi}{a} < \mathrm{Im}(z) \leq \frac{\pi}{a} \right\} \tag{1.28}$$

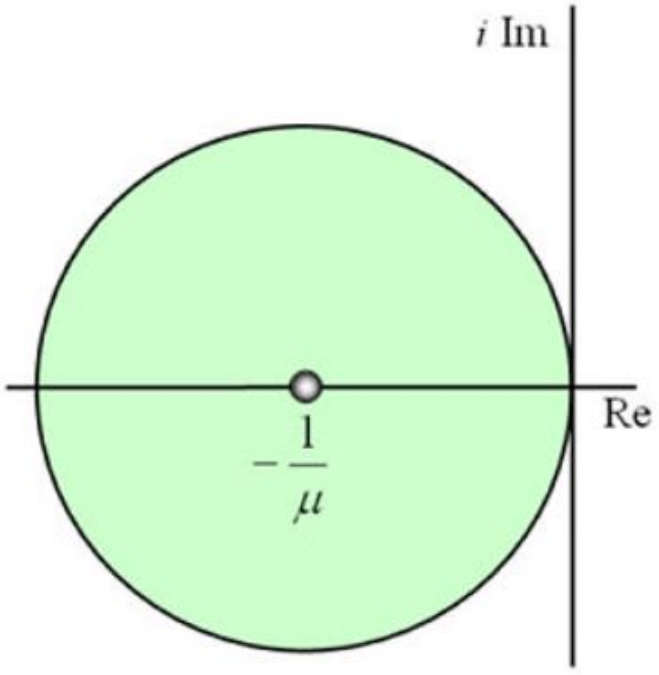


FIGURE 1.7: *A pictorial representation of the Hilger circle.*

The expression of the *Hilger complex plane* is given by

$$\mathbb{C}_a := \left\{ z \in \mathbb{C} : z \neq -\frac{1}{a} \right\} \tag{1.29}$$

The expression of the real part of the Hilger complex plane is given by

$$\mathbb{R}_a := \left\{ z \in \mathbb{R} : z > -\frac{1}{a} \right\} \tag{1.30}$$

The expression of the *Hilger circle* is given by

$$\mathbb{B}_a := \left\{ z \in \mathbb{C}_a : |z + \frac{1}{a}| = \frac{1}{a}, \forall\, a \neq 0 \right\} \blacksquare \tag{1.31}$$

The pictorial representation of the Hilger circle is given in Fig. 1.7. A pictorial representation of Hilger complex plane can be found in Fig. 1.8. In measure chain theory, the concept of *Hilger stability* can be explained with the help of Hilger circle. A constant regressive matrix $\mathcal{A}$ is Hilger stable if the spectrum of eigen–values of matrix $\mathcal{A}$ lies within the Hilger circle of minimum graininess function value. In measure chain theory, the concept of *cylindrical transformation* exists

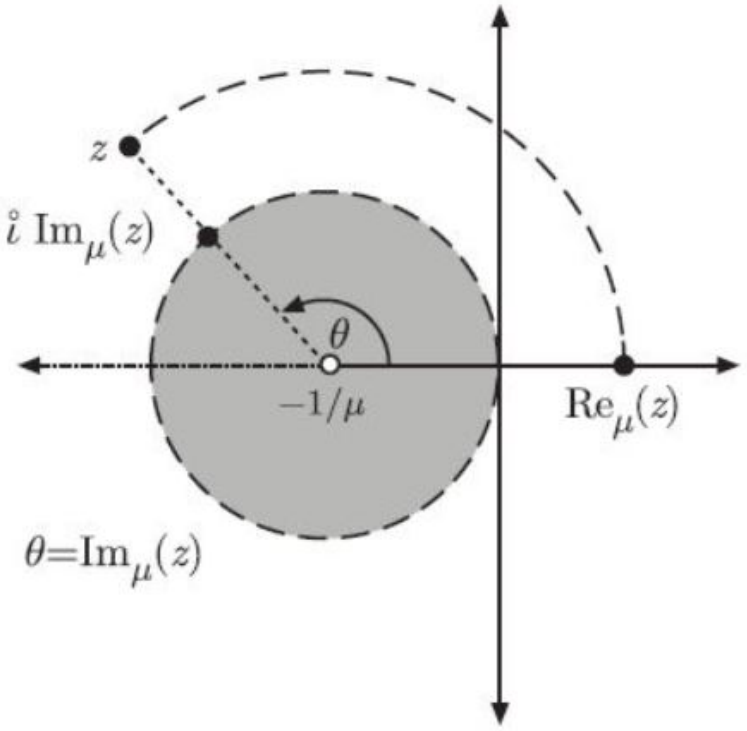


FIGURE 1.8: *A pictorial representation of Hilger complex plane.*

which is given by, $\mathcal{P}_a : \mathbb{C}_a \to \mathbb{Z}_a$

$$\mathcal{P}_a(z) = \begin{cases} \frac{1}{a}\log(1+za), & \text{if } a > 0 \\ z, & \text{if } a = 0 \end{cases} \tag{1.32}$$

where log represents the natural logarithm. Now, with the help of cylindrical transformation we can give the definition of exponential function on measure chains.

Definition 20: (Bohner and Peterson (2001b),Bodine and Lutz (2003), Agarwal et al. (2002), Gard and Hoffacker (2003)) The exponential function on measure chain $\mathbb{T}$ is given by the expression,

$$e_a(b_1, b_2) := \exp\left(\int_{b_2}^{b_1} \mathcal{P}_{\mu(t)}(a(\varphi))\Delta\varphi\right), \quad b_1, b_2 \in \mathbb{T} \tag{1.33}$$

where $a \in \mathcal{R}$ is a regressive function and $\mathcal{P}$ is the cylindrical transformation expression which is given in Eq. (1.32). ■

Now, we will mention some of the properties of exponential functions on measure

chain $\mathbb{T}$.

Theorem 10: (Bohner and Peterson (2002), Martynyuk (2016)) For any regressive functions $a_1, a_2 \in \mathcal{R}$ and $t_1, t_2, t_3 \in \mathbb{T}$ one has

1. $e_0(t_1, t_2) = 1$ also $e_{a_1}(t_1, t_1) = 1$.
2. $e_{a_1}(t_1, t_2) = \frac{1}{e_{a_1}(t_2, t_1)} = e_{\ominus a_1}(t_2, t_1)$.
3. $e_{a_1}(t_1, t_2) e_{a_1}(t_2, t_3) = e_{a_1}(t_1, t_3)$.
4. $e_{a_1}(t_1, t_2) e_{a_2}(t_1, t_2) = e_{a_1 \oplus a_2}(t_1, t_2)$.
5. $\frac{e_{a_1}(t_1, t_2)}{e_{a_2}(t_1, t_2)} = e_{a_1 \ominus a_2}(t_1, t_2)$.
6. $e_{a_1}(\sigma(t_1), t_2) = (1 + \mu(t) a_1(t)) e_{a_1}(t_1, t_2)$.
7. $e_{a_1}(t_1, \sigma(t_2)) = \frac{e_{a_1}(t_1, t_2)}{1 + \mu(t_2) a_1(t_2)}$
8. $\overset{\triangle}{e}_{a_1}(., t_2) = a_1 e_{a_1}(., t_2)$.
9. $\overset{\triangle}{e}_{a_1}(t_1, .) = (\ominus a_1) e_{a_1}(t_1, .)$.
10. $e_{a_1}(t_1, t_2) > 0$ only if a_1 is positive regressive function (i.e. $a_1 \in \mathcal{R}^+$). ■

Some numerical examples of exponential functions on measure chains are given by

1. When the measure chain is a continuous measure chain $\mathbb{T} = \mathbb{R}$, then the expression of exponential function is given by

$$e_{a_1}(t_1, t_2) = \exp^{\int_{t_2}^{t_1} a_1(\vartheta) d\vartheta} \tag{1.34}$$

   where a_1 is a regressive function on $\mathbb{R}$.

2. When the measure chain is a discrete measure chain $\mathbb{T} = \mathbb{Z}$, then the expression of exponential function is given by

$$e_{a_1}(t_1, t_2) = \prod_{\vartheta = t_2}^{t_1 - 1} (1 + a_1(\vartheta)) \tag{1.35}$$

   where a_1 is a regressive function on measure chain $\mathbb{Z}$.

3. Similarly for $\mathbb{T} = a\mathbb{Z}$, where a is a positive constant then the exponential function is given by

$$e_{a_1}(t_1, t_2) = \prod_{\vartheta=\frac{t_2}{a}}^{\frac{t_1}{a}-1} (1 + aa_1(\vartheta)) \tag{1.36}$$

4. The expression of exponential function for the set of Harmonic numbers for a given positive constant a is given by

$$e_a(\mathbb{H}_n, 0) = \frac{(n+a)...(1+a)}{n!} \tag{1.37}$$

The generalized exponential functions for various measure chains when $a(t) = a$ (with abuse of notation) is a constant value are given below

$\mathbb{T}$	$e_a(t, t_2)$
$\mathbb{R}$	$e^{a(t-t_2)}$
$\mathbb{Z}$	$(1+a)^{t-t_2}$
$\mu(t)\mathbb{Z}$ ($\mu(t) =$ constant)	$(1+a\mu(t))^{\frac{t-t_2}{\mu(t)}}$
$\frac{\mathbb{Z}}{n}$	$(1+\frac{a}{n})^{n(t-t_2)}$
$q^{\mathbb{N}_0},\ q > 1$	$\prod_{u\in[t_2,t)}[1+(q-1)au]$
$2^{\mathbb{N}_0}$	$\prod_{u\in[t_2,t)}[1+au]$
$\mathbb{H} = \left\{\sum_{k=1}^{n} \frac{1}{k} : n \in \mathbb{N}\right\}$	$\frac{(n+1)^{n-n_0}}{n^{n-n_0}}$ iff $t = \sum_{k=1}^{n} \frac{1}{k}$

The table of exponential functions for various measure chains and certain choice of initial conditions and $a(t)$ are given below,

$\mathbb{T}$	t_2	$a(t)$	$e_a(t,t_2)$
$\mathbb{R}$	0	1	e^t
$\mathbb{Z}$	0	t	2^t
$\mu(t)\mathbb{Z}(\mu(t)=\text{constant})$	0	1	$(1+\mu(t))^{\frac{t}{\mu(t)}}$
$\frac{\mathbb{Z}}{n}$	0	1	$[(1+\frac{1}{n})^n]^t$
$q^{\mathbb{N}_0},\ q>1$	1	1	$\prod_{u\in(0,t)}[1+(q-1)u]$
$q^{\mathbb{N}_0},\ q>1$	1	$\frac{1-t}{(q-1)t^2}$	$\sqrt{t}e^{-\frac{\ln^2(t)}{2\ln(q)}}$
$2^{\mathbb{N}_0}$	1	1	$\prod_{u\in(0,t)}[1+u]$
$2^{\mathbb{N}_0}$	1	$\frac{1-t}{t^2}$	$\sqrt{t}e^{-\frac{\ln^2(t)}{\ln(4)}}$
$\mathbb{N}_0^2$	0	1	$2^{\sqrt{t}}(\sqrt{t})!$
$\mathbb{H}=\left\{\sum_{k=1}^{n}\frac{1}{k}:n\in\mathbb{N}\right\}$	0	1	$n+1$ iff $t=\sum_{k=1}^{n}\frac{1}{k}$

1.1.4 Matrix exponential function on measure chains

Definition 21: (Bohner and Peterson (2002), Martynyuk (2016)) For any matrix valued function $\mathcal{A}$ having a dimension of $a_1 \times a_1$ such that each element of the matrix is rd–continuous and regressive on a given measure chain $\mathbb{T}$, then the matrix is said to be regressive and rd–continuous on the given measure chain $\mathbb{T}$. The expression of matrix exponential function for the aforementioned matrix $\mathcal{A}$ for any initial time t_1 is given by

$$e_{\mathcal{A}}(t,t_1) \quad \blacksquare \tag{1.38}$$

The matrix exponential function given in Definition 21, is used to find the solution of linear dynamic equation of the form $\overset{\triangle}{X} = \mathcal{A}X$ where the dimension of X is a_1.
Theorem 11: (Bohner and Peterson (2002), Martynyuk (2016)) For any two matrix valued regressive and rd–continuous functions A and B having dimensions $a_1 \times a_1$. The following properties are satisfied,

1. Consider $C = (A + B)$, we can then write $\overset{\triangle}{C} = \overset{\triangle}{A} + \overset{\triangle}{B}$.

2. Consider $C = (AB)$, we can then write $\overset{\triangle}{C} = \overset{\triangle}{A}B^{\sigma} + A\overset{\triangle}{B} = A^{\sigma}\overset{\triangle}{B} + \overset{\triangle}{A}B$.

3. Consider $C = A^{-1}$, and only when AA^{σ} is invertible then we can write $\overset{\triangle}{C} = -\left(A^{\sigma}\right)^{-1}\overset{\triangle}{A}A^{-1} = -A^{-1}\overset{\triangle}{A}\left(A^{\sigma}\right)^{-1}$.

4. Consider $C = AB^{-1}$ and only when BB^{σ} is invertible then we can write $\overset{\triangle}{C} = \left(\overset{\triangle}{A} - AB^{-1}\overset{\triangle}{B}\right)\left(B^{\sigma}\right)^{-1} = \left(\overset{\triangle}{A} - \left(AB^{-1}\right)^{\sigma}\overset{\triangle}{B}\right)B^{-1}$ ■.

Now, we will discuss about some properties of the exponential matrix valued functions on measure chains.

Theorem 12: (Bohner and Peterson (2002), Martynyuk (2016)) For any two regressive and rd–continuous matrices $\mathcal{A}$ and $\mathcal{B}$ as well as $t_1, t_2, t_3 \in \mathbb{T}$. Then,

1. $e_0(t_1, t_2) = I$ and $e_{\mathcal{A}}(t, t) = I$

2. $e_{\mathcal{A}}(t_1, t_2) = e_{\mathcal{A}}^{-1}(t_2, t_1) = e_{\ominus\mathcal{A}^*}^{*}(t_2, t_1)$

3. $e_{\mathcal{A}}(t_1, t_2)e_{\mathcal{A}}(t_2, t_3) = e_{\mathcal{A}}(t_1, t_3)$

4. $e_{\mathcal{A}}(t_1, t_2)e_{\mathcal{B}}(t_1, t_2) = e_{\mathcal{A}\oplus\mathcal{B}}(t_1, t_2)$

5. $e_{\mathcal{A}}(\sigma(t), t_1) = (I + \mu(t)A(t))e_{\mathcal{A}}(t, t_1)$

$(*)$ represents conjugate transpose of the matrix $\mathcal{A}$. ■

1.1.5 Initial value theorem on measure chains

In this subsection, the expression of initial value theorems on measure chains are stated.

Theorem 13 : (Akin-Bohner et al. (2005)) Considering a scalar dynamic equation on measure chain $\mathbb{T}$ along with disturbances,

$$\overset{\triangle}{z} = rz + f \tag{1.39}$$

where $r \in \mathcal{R}(\mathbb{T}, \mathbb{R})$ and $f \in \mathbb{C}_{rd}(\mathbb{T}, \mathbb{R})$. The unique solution of Eq. (1.39) can be written with the help of exponential functions on measure chains which is given by,

$$z(t) = e_{r(t)}(t, t_0)z(0) + \int_{t_0}^{t} e_{r(t)}(t, \sigma(\zeta))f(\zeta)\Delta\zeta, \ \forall \ t \in \ \mathbb{T} \tag{1.40}$$

where $z(t_0) = z(0)$ is the value of initial condition. ■

Theorem 14: (Bohner and Peterson (2002), Martynyuk (2016)) Considering a vector dynamic equation on measure chain $\mathbb{T}$ as

$$\overset{\Delta}{z} = Rz + f \tag{1.41}$$

where $R \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ and $f \in \mathbb{C}_{rd}(\mathbb{T}, \mathbb{R}^n)$. The expression of unique solution of Eq. (1.41) with the help of matrix valued exponential functions on measure chains is given by

$$z(t) = e_{R(t)}(t, t_0)z(0) + \int_{t_0}^{t} e_{R(t)}(t, \sigma(\zeta))f(\zeta)\Delta\zeta, \ \forall \ t \in \ \mathbb{T} \tag{1.42}$$

where $z(t_0) = z(0)$ is the value of initial condition. ■

After recalling the basics of measure chain theory, we are now in a position to recall some of the concepts related to switched systems on measure chains which would be helpful in solving the class of consensus problems which are addressed in Chapter 2 of this thesis under the assumption of bounded failures in network topology.

1.2 Switched systems

In this section, we will recall some of the basic concepts related to switched systems which are used in this thesis. Switched systems in layman's term are made up of a class of subsystems and a switching rule which defines on how the dynamics of the switched system is generated through appropriate switching between various subsystems which make up the switched system setup (Lin and Antsaklis (2009)).

The expression of continuous time–varying switched system dynamics is given by

$$\dot{z}(t) = \mathcal{F}_{\sigma(t)}(t, z(t), u(t)),\ \forall\ t\ \in\ \mathbb{R} \tag{1.43}$$

where the switching rule is defined by $\sigma(t)$, the state of the system is denoted by $z(t) \in \mathbb{R}^a$ and the control input is given by $u(t) \in \mathbb{R}^b$. The expression of discrete time–varying switched system dynamics is given by

$$z(r+1) = \mathcal{F}_{\sigma(r)}(r, z(r), u(r))\ \forall\ r\ \in\ \mathbb{Z} \tag{1.44}$$

where similarly to the continuous time case, the switching signal is denoted by $\sigma(r)$, whereas the state and control are denoted by $z(r) \in \mathbb{R}^d$ and $u(r) \in \mathbb{R}^e$.
The stability of switched systems does not only depend on the stability of the subsystems which make up the switched system framework but also on the switching signal. In the literature, it is shown that even if the subsystems are individually stable then also the resulting switched system may be unstable if the switching signal is not appropriately selected. Furthermore, a proper design of switching signal can also result in stabilising the switched system dynamics even when all the subsystems are unstable (Liberzon et al. (1999), Branicky (1998), Hespanha et al. (2005)). The autonomous time–invariant linear version of the switched system dynamics is the one which received extensive focus of research wherein people addressed problem statements related to stability, controllability, observability, accessibility, certain performance specifications etc (Asarin et al. (2000), Bemporad and Morari (1999), Bemporad et al. (2000), Johansson (1999), Liberzon and Morse (1999), Agrachev and Liberzon (2001), Cheng (2005), Ji et al. (2007b), Ji et al. (2007a), Sun et al. (2002), Sun and Ge (2005), Xu and Antsaklis (2004), Cassandras et al. (2001)). Some results can also be found in the literature dealing with stability analysis of nonlinear switched systems (Hu et al. (1999), Lee and Jiang (2008), MalmirChegini and Marvasti (2007), Bacciotti and Mazzi (2005)).

1.2.1 Switched systems on measure chains

In this subsection, some of the basics related to switched systems on measure chains and their corresponding stability analysis on measure chains are recalled.

In the first two decades of the present century, some results related to the stability analysis of linear systems on measure chains using the concept of classical exponential functions on measure chains had appeared in the literature (DaCunha (2005), Agarwal et al. (2002), Du et al. (2007)). Some results related to time–varying systems (Doan et al. (2010), DaCunha (2004)), systems with perturbations (Du et al. (2011)) and nonlinear systems (Bartosiewicz and Piotrowska (2013)) can also be found in the literature. In the field of switched systems on measure chains, the concept of common Lyapunov function on measure chains was used to prove the closed–loop stability analysis (Taousser et al. (2015)). A class of dynamical switched linear systems on a specific measure chain $\mathbb{P}_{a,b}$, which is pictorially given in Fig. (1.5) can be written as,

$$\overset{\Delta}{\zeta}(t) = \begin{cases} \mathcal{A}_1\zeta(t), \ t \in \bigcup_{u=0}^{\infty}[\sigma(t_u), t_{u+1}) \\ \mathcal{A}_2\zeta(t), \ t \in \bigcup_{u=0}^{\infty}\{t_{u+1}\} \end{cases} \tag{1.45}$$

where the matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ are constant regressive rd–continuous matrices over $\mathbb{P}_{a,b}$ measure chain and $\zeta(t) \in \mathbb{R}^n$ is used to denote the state dynamics.

Lemma 1: (Davis et al. (2010), Eisenbarth et al. (2014)) The equilibrium point of the system dynamics given in Eq. (1.45), is said to be asymptotically stable on the given measure chain provided there exists a common Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}^+$ which satisfies the following inequality

$$\overset{\Delta}{V}(\zeta(t)) < 0, \forall \ t \in \ \mathbb{P}_{a,b} \ \blacksquare \tag{1.46}$$

Theorem 15: (Taousser et al. (2015)) Considering the measure chain $\mathbb{P}_{a,b}$, to have a bounded graininess function which is taken as μ_m such that $0 < \mu(t) \leq \mu_m$, then a common Lyapunov function ($V = \zeta^T\mathcal{P}\zeta$) exists provided the following inequalities are satisfied on the measure chain $\mathbb{P}_{a,b}$.

1. $\mathcal{A}_1^T\mathcal{P} + \mathcal{P}\mathcal{A}_1 < 0$,
2. $\mathcal{A}_2^T\mathcal{P} + \mathcal{P}\mathcal{A}_2 + \mu_m\mathcal{A}_2^T\mathcal{P}\mathcal{A}_2 < 0$ $\blacksquare$

In Theorem 15 the condition 1 is checked when the system is traversing the continuous section of $\mathbb{P}_{a,b}$ measure chain whereas the condition 2 is checked when the system is traversing the discrete section of the $\mathbb{P}_{a,b}$ measure chain. Both the conditions can be obtained using a suitable choice of the graininess function which varies across the $\mathbb{P}_{a,b}$ measure chain and the theory of Lyapunov functions on measure chains.

A specific dynamical switched linear system with uncertainties on the measure chain $\mathbb{P}_{a,b}$ which is pictorially given in Fig. (1.5) can be written as

$$\overset{\vartriangle}{\zeta}(t) = \begin{cases} \mathcal{A}_1\zeta(t) + \mathcal{F}_c(t), \ t \in \bigcup_{u=0}^{\infty}[\sigma(t_u), t_{u+1}) \\ \mathcal{A}_2\zeta(t) + \mathcal{F}_d(t), \ t \in \bigcup_{u=0}^{\infty}\{t_{u+1}\} \end{cases} \tag{1.47}$$

where similar to Eq. (1.45), the matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ are constant regressive and rd–continuous matrices whereas $\mathcal{F}_c$ and $\mathcal{F}_d$ are rd–continuous matrices of appropriate dimensions.

Theorem 16: (Taousser et al. (2015)) The switched linear system given by Eq. (1.47) has a common Lyapunov function ($V = \zeta^T\mathcal{P}\zeta$) on a measure chain $\mathbb{P}_{a,b}$ whose graininess function is bounded ($0 < \mu(t) \leq \mu_m$) if the following conditions are satisfied simultaneously,

1. (a) $\mathcal{A}_1^T\mathcal{P} + \mathcal{P}\mathcal{A}_1 < -\mathcal{Q}_1$
   (b) $\mathcal{A}_2^T\mathcal{P} + \mathcal{P}\mathcal{A}_2 + \mu_m\mathcal{A}_2^T\mathcal{P}\mathcal{A}_2 < -\mathcal{Q}_2$

   for some positive definite matrices $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of appropriate dimensions.

2. The nonlinearities $\mathcal{F}_c(t)$ and $\mathcal{F}_d(t)$ satisfy

   (a) $\|\mathcal{F}_c(t)\| \leq \mathcal{L}_1\zeta(t)$

   (b) $\|\mathcal{F}_d(t)\| \leq \mathcal{L}_2\zeta(t)$ where the constants $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy the following inequalities

      i. $\mathcal{L}_1 < \frac{\lambda_{min}(\mathcal{Q}_1)}{2\lambda_{max}(\mathcal{P})}$

ii. $2\mathcal{L}_2(1+\mu_m\|\mathcal{A}_2\|)\lambda_{max}(\mathcal{P})+\mu_m\mathcal{L}_2^2\lambda_{max}(\mathcal{P}) < \lambda_{min}(\mathcal{Q}_2)$ where λ_{min} and λ_{max} is used to denote minimum and maximum eigen–values of the corresponding matrices $\mathcal{P}, \mathcal{Q}_1, \mathcal{Q}_2$. ■

While addressing the problem of Chapter 2, we were overwhelmed by the amount of control input updates which were needed to reach the final control goal objective. Hence, we looked into the control literature and the concept of event–triggering came up and therefore we addressed the problem of event–triggering on measure chains in Chapters 3, 4 and 5 respectively of our thesis. Now, we are in a position to recall some of the basics related to event–triggering which would be explored in details in the aforementioned chapters.

1.3 Event–triggered control

With the advent of digital technologies in the beginning of 21^{st} century, researchers began to research ways in order to implement control algorithms over networks which may result in an energy efficient form of implementation. This led to investigations along the direction of event–triggered control (Aström (2008), Åarzén (1999), Åström and Bernhardsson (1999), Heemels et al. (2008), Heemels et al. (2012b)) etc. Till the discovery of event–triggered control algorithms, classical time–triggered control algorithms were used in controlling physical systems. Such algorithms suffer from the limitations of periodic update of control law irrespective of whether the system is in transient phase or in the steady state phase. Furthermore, the time between two control updates need to be small to ensure stability at all periods of time. A pictorial representation of event–triggered control algorithm setup is given by Fig. 1.9.

The event–triggered control algorithm design process involves two distinct steps, first the design of a proper event–triggered condition and second the design of control gain constants. A fundamental issue to be addressed when one designs an event–triggered control law is the problem of Zeno phenomenon. It means if proper care is not taken into account while designing an event–triggering condition then

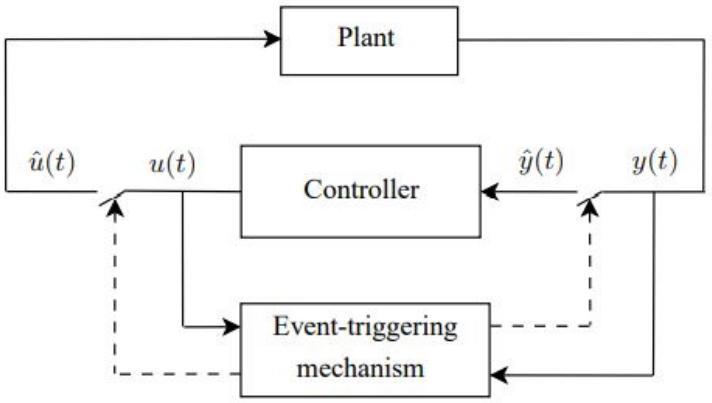


FIGURE 1.9: *A pictorial representation of the event–triggered control setup. (Donkers and Heemels (2011))*

infinite samplings can occur within a finite period of time. This renders the control algorithm useless for any physical applications as it would put high pressure on the control actuators which would in the long run break down.

Event–triggered control algorithms have also been known in the literature by various other names depending on the type of applications they are employed. For the field of sensor networks, event–triggering is also denoted by *send–in–delta* (Miskowicz (2006), Suh (2007), Plonnigs et al. (2004)). In case of signal processing the terminology of *level–crossing–sampling* is popular to denote event–triggering (MalmirChegini and Marvasti (2007), Guan et al. (2008), Akopyan et al. (2006)). Whereas for the control systems literature, the terminology of *Lebesgue–sampling* (Åström and Bernhardsson (2003), Astrom and Bernhardsson (2002)) or *deadband –sampling* (Otanez et al. (2002), Vasyutynskyy and Kabitzsch (2007)) are used to alternatively denote the concept of event–triggered control.

In recent years, the hybrid system methodology which was proposed in (Goebel et al. (2009), Forni et al. (2010)(for linear systems)) has been used to solve the problem of event–triggering as formulated in (Postoyan et al. (2014)) because it provides compact representation. In the next subsection, the basics of event–triggering is explained through the lens of hybrid system methodology.

1.3.1 Event–triggering through hybrid system formalism

The expression of nonlinear plant dynamics along with the dynamic controller dynamics is given by (Postoyan et al. (2013)),

$$\dot{z}_p = \mathcal{F}_p(z_p, u) \tag{1.48}$$

$$y = \mathcal{G}_p(z_p) \tag{1.49}$$

where the plant state is denoted by $z_p \in \mathbb{R}^a$, controller input by $u \in \mathbb{R}^b$ and output by $y \in \mathbb{R}^c$,

$$\dot{z}_c = \mathcal{F}_c(z_c, y) \tag{1.50}$$

$$u = \mathcal{G}_c(z_c, y) \tag{1.51}$$

where the controller state is given by $z_c \in \mathbb{R}^d$ and the output is given by $y \in \mathbb{R}^e$. There are mainly two distinct ways of designing the dynamic controller gain variables namely emulation based method and co–design based method which is explained in brief a few pages later. Considering $t_p,\ \forall\ p \in \mathbb{Z}$ as the time instances where the control input is sent through the communication channels then avoiding any delay in transmission one can write

$$\begin{aligned}
\dot{z}_p &= \mathcal{F}_p\left(z_p, \hat{u}\right) && t \in [t_p, t_{p+1}] \\
\dot{z}_c &= \mathcal{F}_c\left(z_c, \hat{y}\right) && t \in [t_p, t_{p+1}] \\
u &= \mathcal{G}_c\left(z_c, \hat{y}\right) \\
\dot{\hat{y}} &= 0 && t \in [t_p, t_{p+1}] \\
\dot{\hat{u}} &= 0 && t \in [t_p, t_{p+1}] \\
\hat{y}\left(t_p^+\right) &= y\left(t_p\right) \\
\hat{u}\left(t_p^+\right) &= u\left(t_p\right)
\end{aligned} \tag{1.52}$$

The values $\hat{u}$ and $\hat{y}$ are the last updated values of control input and output respectively which remain constant within an interval $\begin{bmatrix} t_p & t_{p+1} \end{bmatrix}$. Here zero order hold is used to generate the samplings, however various other generalized sampling methods are also available in the literature (Lunze and Lehmann (2010)). The expression of error ($e = \begin{bmatrix} e_o^T & e_e^T \end{bmatrix}^T$) due to the presence of network can be

written as

$$e_o = \hat{y} - y \tag{1.53}$$

$$e_e = \hat{u} - u \tag{1.54}$$

Since the resulting system is a hybrid system we recall the formalism proposed in the book (Goebel et al. (2012)) which provides extensive tools for stability analysis,

$$\begin{bmatrix} \dot{z} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} \mathcal{F}(z,e) \\ \mathcal{G}(z,e) \end{bmatrix}, \ (z,e) \in \mathcal{J}_c \tag{1.55}$$

$$\begin{bmatrix} z^+ \\ e^+ \end{bmatrix} = \begin{bmatrix} z \\ 0 \end{bmatrix}, (z,e) \in \mathcal{J}_d \tag{1.56}$$

wherein $z = \begin{bmatrix} z_p^T & z_c^T \end{bmatrix}^T$. The sets $\mathcal{J}_c$ and $\mathcal{J}_d$ are the flow and jump sets respectively. The values $\mathcal{F}(z,e)$ and $\mathcal{G}(z,e)$ are given by

$$\mathcal{F}(z,e) = \begin{bmatrix} \mathcal{F}_p(z_p, \mathcal{G}_c(z_c, y + e_o) + e_e) \\ \mathcal{F}_c(z_c, y + e_o) \end{bmatrix} \tag{1.57}$$

$$\mathcal{G}(z,e) = \begin{bmatrix} -\frac{\partial}{\partial z_p}\mathcal{G}_p(z_p)\mathcal{F}_p(z_p, \mathcal{G}_c(z_c, y + e_o) + e_e) \\ -\frac{\partial}{\partial z_c}\mathcal{G}_c(z_c, y + e_o)\mathcal{F}_c(z_c, y + e_o) \end{bmatrix} \tag{1.58}$$

A subset $\mathbb{P} \subset \mathbb{R}^{0+} \times \mathbb{Z}^{0+}$ is a hybrid time domain if for all $(\mathbb{E}_a, \mathbb{E}_b) \in \mathbb{P}, \mathbb{P} \cap ([0, \mathbb{E}_a] \times \{0, .., \mathbb{E}_b\}) = \cup_{p \in \{0,1,...,q-1\}}([t_p, t_{p+1}], p)$ with some finite sequence of times $0 = t_0 < t_1 < ... < t_q$. In case of designing event–triggered control using the hybrid system methodology apart from the dynamic controller design variables there are other design variables like the flow set $\mathcal{J}_c$ and jump set $\mathcal{J}_d$. Now, some widely investigated ways of designing the flow and jump set are explained assuming the design method to be an emulation based design method wherein the gain constants and event–triggering variables (in this case the flow and jump sets) are designed in a decoupled fashion,

1. **Static absolute event–triggered design:** In case of static event–triggered design process, the flow and jump sets are designed as follows

$$\mathcal{J}_c = \{(z, e) : \|e\| \leq D\} \tag{1.59}$$
$$\mathcal{J}_d = \{(z, e) : \|e\| \geq D\} \tag{1.60}$$

where D is a constant.

2. **Static relative event–triggering condition:** The main limitations of the static event–triggering condition (i.e. it does not ensure asymptotic stability). In this method the threshold on the norm of the network-induced error is adapted to the state of the plant. (Wang and Lemmon (2008), Mazo and Tabuada (2008), Heemels et al. (2008), Liu et al. (2014), Lemmon (2010)). The expression of flow and jump sets for state based event–triggering condition is given by

$$\mathcal{J}_c = \{(z, e) : \vartheta\|e\| \leq \theta\nu(\|z\|)\} \tag{1.61}$$
$$\mathcal{J}_d = \{(z, e) : \vartheta\|e\| \geq \theta\nu(\|z\|)\} \tag{1.62}$$

where $\theta \in (0, 1)$ is a constant. ν and ϑ are comparison functions of class $\mathcal{K}_\infty$. State–dependent event triggering mechanisms if not designed appropriately become prone to Zeno phenomenon. One alternative solution to avoid the Zeno phenomenon is by using time regularization method (Johansson et al. (1999) ,Mazo and Tabuada (2011), Wang et al. (2012), Forni et al. (2014)).

3. **Dynamic event–triggered condition:** Dynamic event–triggering is one of the ways through which the control design engineer can improve the inter event–time without affecting the stability conditions of the closed–loop system. In case of dynamic event–triggering some extra variables are added in the event–triggering condition (Girard (2014), Postoyan et al. (2011), Liu et al. (2020b)). The expression of a prospective dynamic event–triggering variable dynamics is given by

$$\dot{s} = \mathbb{A}(z, e, s) \ , \ (z, e, s) \ \in \mathcal{J}_c \tag{1.63}$$
$$s^+ = \mathbb{B}(z, e, s) \ , \ (z, e, s) \ \in \mathcal{J}_d \tag{1.64}$$

The functions $\mathbb{A}$ and $\mathbb{B}$ are some continuous functions. A prospective formulation of flow and jump sets under certain assumptions on functions on $\mathbb{A}$ and $\mathbb{B}$ can be written as (Postoyan et al. (2014))

$$\mathcal{J}_c = \{(z, e, s) : s \in \begin{bmatrix} c^- & c^+ \end{bmatrix}\} \tag{1.65}$$

$$\mathcal{J}_d = \{(z, e, s) : s = c^-\} \tag{1.66}$$

whereas the values c^- and c^+ are some constant values which satisfy the condition $0 < c^- < c^+$.

4. **Periodic event–triggering condition:** In case of periodic event–triggered condition, the event condition is not checked at every instances of time rather it is checked at certain periodic instances of time. In (Wang et al. (2019)), some results related to periodic event–triggering can be found.

Based on design procedures, event–triggering methods can be classified as follows,

1. **Emulation based method:** In case of emulation based method, the event condition and the gain constants (observer or controller gains) are designed independently of each other. It is a conservative way of design procedure (Abdelrahim et al. (2015)).

2. **Codesign based method:** In case of codesign based method, the event condition and the gain constants are designed simultaneously. It is less conservative in comparison to emulation based design method (Abdelrahim et al. (2019)).

In case of self–triggering based method, one obtains the next time instant where the triggering condition will be checked from the current triggering time instant instead of checking the triggering condition continuously or periodically as we have seen for continuous event–triggered or periodic event–triggered control (Anta and Tabuada (2010), Delimpaltadakis and Mazo (2020), Delimpaltadakis and Mazo (2021),Bahavarnia et al. (2018)).

As we all know for a networked control system, the problems related to quantization, packet loss and time delays are inevitable. In the past few years, event–triggered control algorithms have appeared in the literature (Abdelrahim et al. (2019), Dolk and Heemels (2017), Borgers et al. (2017), Borri and Pepe (2020), Zhang et al. (2021), Zhang and Gharesifard (2021)). Some results have also appeared in the literature using switched systems approach (Selivanov and Fridman (2015)). The concept of positive systems approach have also been explored for designing event triggered controllers for systems with or without delays (Mazenc et al. (2021), Mazenc et al. (2022)).

While addressing the problems of event–triggering in Chapters 3 and 4 respectively it struck our mind that to implement the designed event–triggering algorithms for any physical system on measure chain we need to ensure that our event–triggering algorithms are safe. So, when we did a literature survey we came to know about the field of safety-critical control where the designed control algorithms ensure safe operations of the system for which it is designed. Taking inspiration from the safety–critical control field we studied the concept of safety–critical control for systems on measure chains in Chapter 5 of our thesis. Now, we are in a position to recall some of the basics of safety–critical control which forms the basis of our Chapter 5.

1.4 Safety critical control

We are all aware that any engineering system needs to be safe for all periods of time. 'Safety' notion first appeared in the literature through the works of American computer scientist Leslie Lamport who introduced it in the year of 1977 through his paper (Lamport (1977)). Few years later, the concept of 'liveliness' was proposed in the paper (Alpern and Schneider (1985)). The main difference between the concepts of 'Safety' and 'Liveliness' is, 'Safety' means no unsafe or bad things happen to a system which is categorised as safe system whereas 'Liveliness' means good things will happen in the long run. A practical example of 'Liveliness' is the one in which asymptotic stability is achieved for a system whereas for 'Safety' the practical example is set invariance with respect to a given set which means that

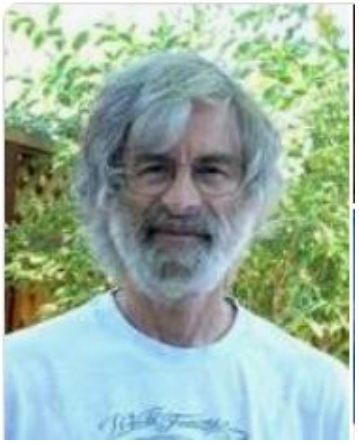


FIGURE 1.10: *Portrait of Leslie Lamport.*

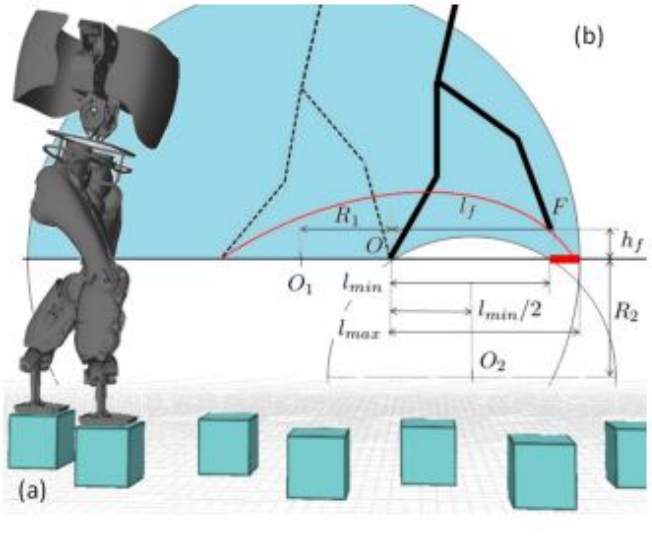


FIGURE 1.11: *a) Three dimensional dynamic walking problem.*
b) Geometric representation of foot length. (Nguyen and Sreenath (2015))

system trajectories starting from a given set at any initial time will remain in the set for all future periods of time. From the literature review, one can conclude that the amount of interest which 'Liveliness' received for several decades is primarily due to extensive research in the field of Lyapunov and its associated stability theorems. The field of 'Safety' has received extensive attention in the past decade because of the surging interest in various autonomous system research like bipedal robots (Nguyen and Sreenath (2015)) (see Fig. 1.11), unmanned aerial vehicles, spacecraft systems etc.



FIGURE 1.12: *Portrait of Mitio Nagumo.*

1.4.1 Barrier functions

The concept of safety with respect to set–invariance can be linked to the works of Japanese scientist Mitio Nagumo (Nagumo (1942)). A nice in depth review of the concepts related to set invariance can be found in the paper (Blanchini (1999)) as well as the recent manuscript (Abraham et al. (2012)) contains an advanced new proof of set invariance.

Definition 22: Consider system $\dot{z} = \mathcal{F}(z)$ where $z \in \mathbb{R}^n$, a safe set $\mathcal{J} \subset \mathbb{R}^n$ is the one which is the superlevel set of a function $\mathcal{P}$ defined as

$$\mathcal{P} : \mathbb{R}^n \to \mathbb{R} \tag{1.67}$$

The expression of the safe set can be written as

$$\mathcal{J} = \{z \in \mathbb{R}^n : \mathcal{P}(z) \geq 0\} \tag{1.68}$$

Also, if the partial derivative of the function $\mathcal{P}$ along the directions of system trajectories is not equal to zero when $\mathcal{P}(z) = 0$, then according to Nagumo's theorem, the condition for set invariance can be written as

$$\mathcal{J} \text{ is invariant for system, } \Leftrightarrow \frac{\partial P(z)}{\partial z}\mathcal{F}(z) \geq 0 \ \forall z \in \ \partial\mathcal{J} \tag{1.69}$$

where $\partial\mathcal{J}$ represents the boundary of the safe set. ■

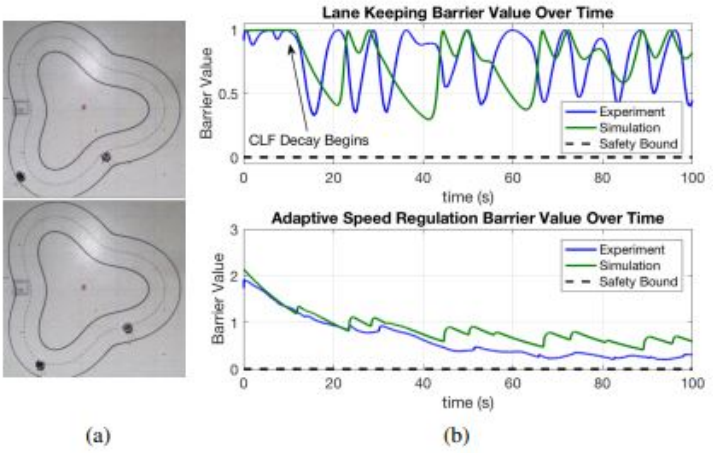


FIGURE 1.13: *Control barrier function for automotive system applications. a) Robot is forced to be in a lane due to a control barrier function known as lane keeping control barrier function and follows a different robot due to adaptive control barrier function which regulates speed. b) Simulation and experimental values of two different control barrier functions mentioned in a). (Ames et al. (2014b))*

The papers (Brezis (1970)) and (Bony (1969)) contain some of the initial results related to Definition 22. In the first half of 21^{st} century, one can see the terminologies related to barrier certificates appearing in the literature of nonlinear and hybrid systems in order to prove safety thereby rediscovering the concepts postulated by Mitio Nagumo (Prajna and Jadbabaie (2004), Prajna (2006)).

Definition 23: Considering two different sets, one the set of initial conditions $\mathcal{J}_0$ and other the set which is not safe $\mathcal{J}_{nc}$. Furthermore their exists a function $\mathcal{P} : \mathbb{R}^n \to \mathbb{R}$ which satisfies the following conditions

1. $\mathcal{P}(z) \leq 0 \ \forall \ z \in \mathcal{J}_0$.
2. $\mathcal{P}(z) > 0 \ \forall \ z \in \mathcal{J}_{nc}$.

The function $\mathcal{P}$ is said to be barrier certificate iff

$$\frac{\partial \mathcal{P}(z)}{\partial z} \mathcal{F}(z) \leq 0 \ \forall \ z \in \mathcal{J} \tag{1.70}$$

where $\mathcal{J}$ is the complement of the unsafe set $\mathcal{J}_{nc}$. ■

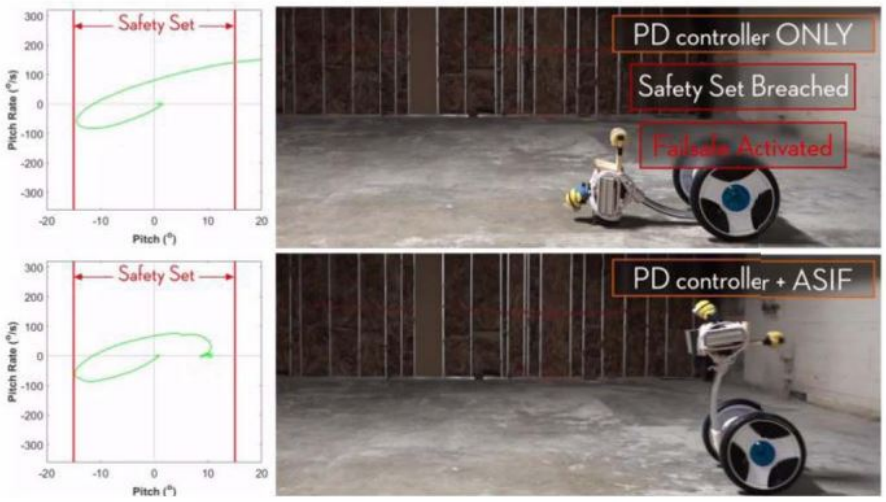


FIGURE 1.14: *Control barrier functions applied to an experimental robotic platform*
Plots of angle without control barrier function (no active set invariance filter (ASIF)) and with control barrier function (with active set invariance filter (ASIF)) plus experimental snapshot with and without control barrier function. (Gurriet et al. (2018)).

Some results in the stochastic time domain can be found in (Prajna et al. (2007)) and the necessity of existence of the barrier certificates can be found in (Prajna and Rantzer (2005)).

Due to the popularity of Lyapunov based methods for 'Liveliness', researchers also began to search for whether or not their exist some equivalent Lyapunov like representations for safety thereby proposing the new concept of 'Barrier Lyapunov functions' Tee et al. (2009). Even though 'Barrier Lyapunov functions' ensure safety it suffers from the limitation of conservatism. As research in the field of viability theory progressed, papers appeared in the literature (Aubin (1990), Aubin and Frankowska (2009), Aubin et al. (2011)), so the results of safety which were previously applicable only to systems without control input was extended to control affine systems of the form, $\dot{z} = \mathcal{F}(z) + \mathcal{G}(z)u$.

Definition 24: (Ames et al. (2014b), Ames et al. (2016)) A function $\mathcal{P}$ is called a control barrier function if for a control input u and a safe set $\mathcal{J}$, it satisfies the

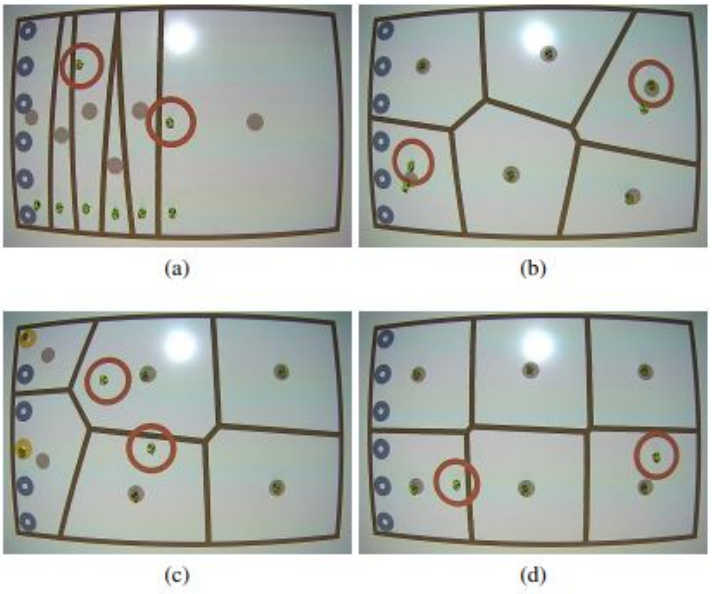


FIGURE 1.15: *A hexad group of robots performing the task of coverage control while avoiding two robots using the aid of control barrier function (Ames et al. (2019)).*

following condition

$$\exists\ u \text{ s.t. } \frac{\partial P(z)}{\partial z}(\mathcal{F}(z) + \mathcal{G}(z)u) \geq -\alpha\mathcal{P}(z) \tag{1.71}$$

where α belongs to the class of comparison functions of extended type $\mathcal{K}_\infty$ which finds extensive mention in the control theoretic literature through the classical input–to–state stability theorems (Sontag (2008)). ■

Definition 24 of control barrier function has been extensively used to design safe controllers for a large class of real world systems like quadrotors (Wu and Sreenath (2016), Wang et al. (2017a)), automotive systems (Xu et al. (2015), Xu et al. (2017b), Xu et al. (2017a)), robotic systems (Ames et al. (2014b) (see Fig. 1.13), Nguyen et al. (2016). Also in, Gurriet et al. (2018)) (see Fig. 1.14) the control barrier functions are used as safety filters on a ninebot segway which is not statically stable and mimics a two wheeled inverted pendulum wherein the safety objective is to avoid the robot from tipping over. Some results can also be found on multi–robot systems (Borrmann et al. (2015), Glotfelter et al. (2017), Wang et al. (2017b), Pickem et al. (2017)) to name a few. In (Ames et al. (2019)) (see

Fig. 1.15) a hexad group of robots monitor a region of robotarium (Pickem et al. (2017)) thereby performing coverage control using the help of control barrier functions where the Voronoi pattern is depicted by dark lines in each case. Two robots are also in the robotarium as dynamic obstacles which are denoted by red color and finally there are six charging stations for each of the hexad robots which are in blue and turn yellow as and when the corresponding robot is charging. Definition 24 of control barrier functions also allows easy unification of safety as well stability of closed–loop systems through the use of control Lyapunov functions.

Recently, the concept of input–to–state barrier function (Romdlony and Jayawardhana (2016), Romdlony and Jayawardhana (2017)) has also appeared in the literature which is the safety equivalent of the input–to–state stability concept. The concept of input–to–state stability is a well defined theory that proves stability of systems with disturbances (Sontag and Wang (1995)). The input–to–state safe barrier function definition helps one to design safety critical event–triggered controllers as because in event–triggered systems the closed–loop system stability in classical settings is guaranteed using the input–to–state stability method.

Definition 25 : (Kolathaya and Ames (2018), Taylor et al. (2020)) Considering a nonlinear closed–loop system of the form given by $\dot{z} = \mathcal{F}(z, k(z+e))$ where $z \in \mathbb{R}^n$ is the state variable, $\mathcal{F}(.)$ is a locally Lipschitz function, e is the error due to sampling, $k(.)$ is the feedback gain function, a safe set $\mathcal{J}$ as well as a function $\mathcal{P} : \mathbb{R}^n \to \mathbb{R}$ which is an input–to–state barrier function if the following condition is satisfied

$$\frac{\partial \mathcal{P}(z)}{\partial z}(\mathcal{F}(z, k(z+e)) \geq -\alpha(\mathcal{P}(z)) - \alpha_r(\|e\|) \tag{1.72}$$

where α and α_r are comparison functions belonging to the extended class of $\mathcal{K}_\infty$ and class $\mathcal{K}_\infty$ respectively. ■

Example 1: Now, through an example of nonlinear control affine system, $\dot{z} = \mathcal{F}(z) + \mathcal{G}(z)u$, where $z \in \mathbb{R}^n$ represents the system state, $u \in \mathbb{R}^m$ represents the control input, $\mathcal{F}(.)$ and $\mathcal{G}(.)$ are Lipschitz functions, various control barrier functions definitions are clarified. Considering $\mathcal{J}$ as the safe set and a function

$\mathcal{P} : \mathbb{R}^n \to \mathbb{R}$ as the barrier function such that the relation between the set and the barrier function is given by

$$\begin{aligned} \mathcal{J} &\triangleq \{z \in \mathbb{R}^n : \mathcal{P}(z) \geq 0\} \\ \partial\mathcal{J} &\triangleq \{z \in \mathbb{R}^n : \mathcal{P}(z) = 0\} \\ \text{Int}(\mathcal{J}) &\triangleq \{z \in \mathbb{R}^n : \mathcal{P}(z) > 0\} \end{aligned} \tag{1.73}$$

Now, the barrier function is a control barrier function if

$$\sup_u [\mathcal{L}_\mathcal{F}(\mathcal{P}(z)) + \mathcal{L}_\mathcal{G}(\mathcal{P}(z))u] \geq -\alpha(\mathcal{P}(z)) \tag{1.74}$$

where α is an extended comparison function of class $\mathcal{K}_\infty$. $\mathcal{L}_F(\mathcal{P}(z))$ and $\mathcal{L}_G(\mathcal{P}(z))$ are Lie directional derivative of the candidate control barrier function along the directions of vector fields $\mathcal{F}$ and $\mathcal{G}$ respectively.

Definition 26: A reciprocal control barrier function (Ames et al. (2014b)) is the one which blows up at the boundary

$$\inf_{z \in \text{Int}\mathcal{J}} \mathcal{P}(z) \geq 0, \quad \lim_{z \to \partial\mathcal{J}} \mathcal{P}(z) = \infty \tag{1.75}$$

The reciprocal control barrier function expression is given by,

$$\inf_u [\mathcal{L}_\mathcal{F}(\mathcal{P}(z)) + \mathcal{L}_\mathcal{G}(\mathcal{P}(z))u] \leq \alpha(\frac{1}{\mathcal{P}(z)}) \tag{1.76}$$

However in comparison to Eq. (1.76), the definition of control barrier function given in Eq. (1.74) is more widely applicable because the barrier function is well defined not only inside the safe set $\mathcal{J}$ but also outside the safe set $\mathcal{J}$.

The next problem to address when one is designing a safety critical controller for a nonlinear control affine system considered above is how to efficiently synthesize the safety critical controllers. The synthesis problem is addressed using optimization based methods. The quadratic program based control barrier function formulation is given by

$$u(z) = \underset{u\in\mathbb{R}^m}{\arg\min} \frac{\|u - k(z)\|^2}{2}$$
$$\text{s.t. } [\mathcal{L}_{\mathcal{F}}(\mathcal{P}(z)) + \mathcal{L}_{\mathcal{G}}(\mathcal{P}(z))u] \geq -\alpha(\mathcal{P}(z) \tag{1.77}$$

The above representation of optimization based controllers was first utilized in the paper (Ames et al. (2014a)).

Some papers are available in the literature to prove the safety and stability of the closed–loop system in a unified manner wherein stability refers to asymptotic stability of the equilibrium point of the closed–loop system with the help of control Lyapunov functions (Galloway et al. (2015), Ames and Powell (2013)). Thus taking inspirations from Eq. (1.77), we can write the expression of quadratic program based safety critical stable controller as,

$$u(z) = \underset{(u,\theta)\in\mathbb{R}^{m+1}}{\arg\min} \frac{1}{2}(u)^T\mathcal{H}(z)(u) + a\theta^2$$
$$\text{s.t. } [\mathcal{L}_{\mathcal{F}}(\mathcal{P}(z)) + \mathcal{L}_{\mathcal{G}}(\mathcal{P}(z))u] \geq -\alpha(\mathcal{P}(z))$$
$$[\mathcal{L}_{\mathcal{F}}(\mathcal{V}(z))) + \mathcal{L}_{\mathcal{G}}(\mathcal{V}(z))u] \leq -b(\mathcal{V}(z)) + \theta \tag{1.78}$$

where $\mathcal{V}(z)$ is the classical control Lyapunov function (Sontag (1983), Sontag (1989), Artstein (1983)), $\mathcal{H}(z)$ is a matrix which is positive definite whereas the variable which ensures solvability of the quadratic program is taken as θ with penalty variable a. The functions α and b are extended comparison function of class $\mathcal{K}_\infty$ and comparison function of class $\mathcal{K}$ respectively.

Papers can also be found in the literature which take into account the actuator constraints and design control barrier functions (Squires et al. (2018), Wang et al. (2018b)).

The major limitations of control barrier functions mentioned in the example and Definition 25 above is that it deals with systems which only have a relative degree of one. In order to deal with higher relative degree systems, the concept of exponential control barrier function was proposed. In (Wu and Sreenath (2015)), the

concept was extended to systems with relative degree equal to two. However in (Hsu et al. (2015)), it was applied to any arbitrary relative degree systems with the help of backstepping based method. As the backstepping based barrier function method was challenging, so to enforce higher relative degree safety constraints the concept of exponential control barrier function was given in (Nguyen and Sreenath (2016)). The expression of safe and stable control input obtained with the help of exponential control barrier function for systems having a relative degree of β and optimization based method is given by

$$\begin{aligned} u(z) = \underset{(u,\theta,\nu)\in\mathbb{R}^{m+2}}{\arg\min} \; & \frac{1}{2}(u)^T\mathcal{H}(z)(u) + a\theta^2 \\ \text{s.t. } & [\mathcal{L}_{\mathcal{F}}^{\beta}(\mathcal{P}(z)) + \mathcal{L}_{\mathcal{G}}(\mathcal{P}(z))\mathcal{L}_{\mathcal{F}}^{\beta-1}(\mathcal{P}(z))u] = \nu \\ & [\mathcal{L}_{\mathcal{F}}(\mathcal{V}(z))) + \mathcal{L}_{\mathcal{G}}(\mathcal{V}(z))u(t)] \leq -b(\mathcal{V}(z)) + \theta \\ & \nu \geq -K_\alpha Y_{\mathcal{P}(z)} \end{aligned} \tag{1.79}$$

where $Y_{\mathcal{P}(z)}$ is given by $\begin{bmatrix}\mathcal{P}(z) & \mathcal{L}_{\mathcal{F}}(\mathcal{P}(z)) & \mathcal{L}_{\mathcal{F}}^2(\mathcal{P}(z)) & .... & \mathcal{L}_{\mathcal{F}}^{\beta-1}(\mathcal{P}(z))\end{bmatrix}^T$ and K_α is a feedback gain matrix.

1.5 Contributions

In this last section of Chapter 1 we enumerate the major contributions of this thesis in ascending order. We portray the central theme of this thesis which is to design algorithms for systems on any arbitrary time domains keeping in mind the constraints of certain class of failures and safety. The main contributions of the thesis are listed below:

1. In Chapter 2, we have designed a distributed output feedback observer which solves the bipartite consensus problem in a leader follower setting under intermittent communications between the communication channels using the help of switched systems formulation on a given measure chain and the notion of common Lyapunov function. The main motivation behind solving the

problem is that a large majority of consensus problems involve graph topology which does not contain any antagonistic interactions. However, in real world physical applications, the interactions can be antagonistic, the best example of which are political election scenario or social networks (Easley and Kleinberg (2010), Proskurnikov and Tempo (2017), Wasserman et al. (1994)). Furthermore, failures also appear in the communication channels and the tools of Lyapunov–Krasovskii functionals or Lyapunov Razumikhin theorems can be used to solve the consensus problem under failures. However, finding an appropriate Lyapunov–Krasovskii functional is not always straightforward. So, under the assumption that the failures in communication channels are bounded one can solve the problem of consensus using methodologies and tools provided by measure chain theory. The limitations of designing classical time triggered algorithms are that they are updated constantly irrespective of whether a system is in its initial transient state or in the final steady state which inspired us to look in the direction of event–triggering and justify our logic behind exploring the topic of event–triggering on measure chains in the next chapter.

2. In Chapter 3, we have designed event–triggered controllers for systems on any measure chains using the concept of input–to–state stable Lyapunov functions on measure chains. The main motivation behind designing event–triggered controllers for systems on measure chains is to not only unify existing event–triggering design procedures for continuous or discrete time domains but also extend it to any arbitrary time domains. The main interest behind designing event–triggered controllers is that they may be energy efficient in nature. The event–triggered controllers designed in this chapter belong to the broad class of continuous event–triggered controllers wherein the triggering condition is checked at every point on the measure chain. Furthermore, the design constants as well as the gain constants are calculated separately using the emulation based design procedure. The event–triggering controllers explored in this chapter can be vulnerable to Zeno phenomenon if the event condition is not designed appropriately for certain classes of measure chains which forced us to look into the direction of periodic event–triggering in our next chapter.

3. In Chapter 4, we have designed periodic event–triggered controller and periodic event–triggered output feedback controllers using impulsive observers for systems on any measure chains. The stability results proposed in this chapter ensure asymptotic stability of the closed–loop system on any time domains unlike the practical stability concept which can be found in the literature (Etienne et al. (2017)). Furthermore, periodic event–triggered control inherently avoids Zeno phenomenon. Similar to Chapter 2, in this chapter also the gain constants and design parameters of the event–triggering condition are evaluated using the emulation based method.

4. In Chapter 5, we have designed safety critical event–triggered controllers for nonlinear systems on any measure chain using the concept of input–to–state safe barrier functions on measure chains. Previous results on safety critical event–triggered control had only focused on continuous time domains (Taylor et al. (2020)), however the results presented in this chapter can be applied to systems on any arbitrary time domains with no restrictions on nonlinearity.

The thesis concludes with Chapter 6 which provides us certain future research directions.

Chapter 2

Leader–follower bipartite consensus under failures

Consensus problems have been widely studied in the literature due to its wide range of applications in various fields, like coordination control of robots/satellites/UAV/sensor networks (Ren et al. (2007), Haghighi and Cheah (2012), Erdong et al. (2008), Ogren et al. (2004)), flocking in a group of animals (Motsch and Tadmor (2014))(see Fig. 2.1), to name a few. A large part of the literature focuses on consensus problems, in which the communication topology among agents is represented by a graph with non-negative weight for the edges (Olfati-Saber et al. (2007)). This is not always realistic for some scenarios. In fact, the generalized communication topology can be represented by a signed graph that has both positive and negative edge weights, wherein the positive weights reflect cooperative interactions, whereas the negative weights represent antagonistic interactions between agents. Some applications of consensus on signed network are social networks, markets, and competitions (Easley and Kleinberg (2010), Valcher and Misra (2014)). Consensus on signed graphs and the property of the structural balance of a signed graph have been investigated in (Altafini (2012)). If the signed graph representing the communication topology is structurally balanced, then the achieved consensus is termed as bipartite consensus.



FIGURE 2.1: *Flock of wild geese on a migration trail.*

In the case of bipartite consensus, the majority of works focus on undirected signed graphs, and only a few on directed signed graphs, for agents having dynamics ranging from first–integrator, second–integrator to higher–order linear dynamics (Zhu et al. (2018), Guo et al. (2018), Zhang and Chen (2017), Zhu et al. (2018), Miao and Su (2021), Meng et al. (2016), Meng et al. (2015), Wang et al. (2018a), Zhao et al. (2017)). Furthermore, they usually focus on leaderless bipartite consensus, with few exceptions in the field of leader–follower consensus (Bhowmick and Panja (2018), Meng (2017), Meng et al. (2019)). It is worth noting that in practical scenarios the existence of a leader introduces the presence of a reference trajectory that the followers have to track to achieve consensus.

In any communication network, the problem of communication failures exist which leads to intermittent information transmission. The consensus problem under intermittent failures can be viewed as a stabilization problem on non–uniform time domains. It can be solved by modeling the system as a time delay system and using the tools which are available in the literature (Lyapunov–Razumukhin or Lyapunov–Krasovskii theorems) (Hespanha et al. (2007), Liu et al. (2019)). Nevertheless, an appropriate choice of Lyapunov–Krasovskii functional must be

done to derive appropriate stability conditions. In this chapter, we choose another way to solve this problem, leading to relatively simple conditions. To relax the conservatism related to complex communication time rate conditions, it was shown in (Taousser et al. (2016)) that using measure chains theory, which unifies the continuous—time and discrete—time dynamics, the problem of communication losses can be converted into an asymptotic stabilization problem of a switched system on a particular non–uniform time–domain formed by the union of disjoint intervals with variable lengths and variable gaps. Indeed, the closed–loop system is modeled as a switched system between continuous–time dynamics with variable intervals length (when the communication is perfect), and discrete-time dynamics, with variable step sizes (when the communication fails). Thus, it is of interest to mix the continuous—time and discrete–time dynamics under a unified framework (Bohner and Peterson (2001a)), Martynyuk (2016)).

Systems on non–uniform time domain can reflect hybrid systems in general (Lakshmikantham and Vatsala (2002)). The consensus problem on measure chains has been investigated in (Xiao and Huang (2016), Babenko et al. (2018)). In (Taousser et al. (2016), Taousser et al. (2019)), the leader–follower consensus problem under intermittent information transmission was studied with a distributed full state–feedback controller protocol, by converting the tracking error dynamics to a switched system stabilization problem using the concept of common Lyapunov function (CLF), and eigenvalues approach. However, for real engineering MAS applications, it is sometimes difficult to measure and transmit all states or even not desirable due to related communication cost and compact size of the agents. Observer–based consensus protocol under intermittent communications was studied in (Qin et al. (2015)) and recently event–triggered output feedback based protocol was used to study the leader–follower bipartite consensus problem (Cai et al. (2020)). The field of discrete–time bipartite consensus is also relatively unexplored (Meng et al. (2016)). To the best of our knowledge, the problem of observer–based leader–follower bipartite consensus in the presence of intermittent failures in the measurement and control input channels has not been addressed in the literature.

Thus, the main contributions of this chapter are two–fold. First, since all state

variables are not always available for full state feedback control protocol design, a distributed observer–based output feedback protocol is introduced in this chapter by considering failures in the measurement and control input channels which is then formulated as the stabilization problem of a switched system evolving on a non–homogeneous measure chain. Secondly, due to the presence of non–uniform time domain, the observation error and tracking error cannot be addressed separately and thus sufficient conditions are provided using a common Lyapunov function (CLF) for this particular switched system (where the state variable is made up of observation and tracking error) by introducing the measure chain theory. The chapter is organised as follows, some preliminaries related to graph theory is stated in first section. In second section, the problem formulation addressed in this chapter is stated. Whereas, third section contains some numerical simulations to support the problem statement addressed in second section and the chapter ends with fourth section giving some conclusion statements.

2.1 Graph theory

The communication topology between agents (leader and followers) is modeled by a graph (directed/undirected) where the nodes represent the agents and the edges represent the interaction between the corresponding agents. From the concept of graph theory, a graph is given by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ where $\mathcal{V} = \{v_1, v_2..., v_N\}$ is the set of nodes and the edges are given by $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The weight assigned to each edge of the graph is denoted by a_{ij} wherein when $a_{ij} \neq 0$ there is a connection from node j to node i, otherwise $a_{ij} = 0$. When $a_{ij} > 0$, it reflects collaborative interaction between agents whereas when $a_{ij} < 0$, it reflects antagonistic interactions. Furthermore there is no self–loop on the graph (i.e. $a_{ii} = 0$). The neighbour set $\mathcal{N}_i = \{v_j : (v_j, v_i) \in \mathcal{E}\}$ represents the nodes which are neighbours of agent i. The adjacency matrix for a *signed graph* is given by

$$\mathcal{A}_s = (a_{ij})_{N \times N} \tag{2.1}$$

The Laplacian matrix for a signed graph is given by

$$\mathcal{L} = \text{diag}\{\sum_{j=1}^{N} |a_{1j}|, \cdots, \sum_{j=1}^{N} |a_{Nj}|\} - \mathcal{A}_s \tag{2.2}$$

The pinning matrix $D = \text{diag}\{f_1, f_2, \cdots, f_N\}$ is used to represent whether or not there is a connection between the leader and followers, where $f_i = 0$, $\forall\, i \in N$, indicates there is no existence of a connection whereas $f_i = 1$ indicates there is a connection. The modified Laplacian matrix is given by, $H = \mathcal{L} + D$. The concept of *structurally balanced signed graph* is of utmost importance for bipartite consensus (Altafini (2012)). If the node set $\mathcal{V}$ of a signed graph can be partitioned into two sets $\mathcal{V}_1$, $\mathcal{V}_2$, such that $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ wherein $a_{ij} \leq 0$, $\forall\, v_i \in \mathcal{V}_{a_1}, \forall v_j \in \mathcal{V}_{a_2}$, such that $a_1, a_2 \in \{1, 2\}$, and $a_1 \neq a_2$, then the graph is structurally balanced. The signature matrix is given by $M = \text{diag}\{m_1, m_2, \cdots, m_N\}$ where $m_i = 1$, $\forall\, v_i \in \mathcal{V}_{a_1}$, and $m_i = -1$, $\forall\, v_i \in \mathcal{V}_{a_2}$, with $a_1 \neq a_2$. Two examples of bipartite graph topology through which we try to explain the concept of structurally balanced graph are given in Figs. 2.2 and 2.3 respectively. One can obtain bipartite consensus in Fig. 2.2 because it is structurally balanced graph wherein the two subgroups of the node set are given by $\mathcal{V}_1 = \{v_3, v_4\}$ and $\mathcal{V}_2 = \{v_1, v_2, v_5, v_6\}$ respectively. In Fig. 2.3, we can only obtain trivial consensus as because the graph is unbalanced.

Remark 1: An advanced formulation of bipartite consensus, is the phenomenon of *matrix–weighted bipartite consensus* phenomenon. In matrix–weighted consensus each node of the graph topology under consideration is now not a scalar variable but a matrix in itself (Pan et al. (2018), Su et al. (2019), Pan et al. (2021a), Pan et al. (2020), Van Tran et al. (2021), Su and Miao (2022), Miao and Su (2022), Miao (2021), Trinh et al. (2022), Pan et al. (2022), Pan et al. (2021b), Liu et al. (2020a)). In matrix–weighted bipartite consensus we need the concept of matrix sign function for a symmetric matrix $\mathcal{A} \in \mathbb{R}^{e\times e}$ which is given by, $sign(\mathcal{A}) = 1$, when $\mathcal{A} > 0$ or $\mathcal{A} \geq 0$, also $sign(\mathcal{A}) = 0$, when $\mathcal{A} = 0$ and finally $sign(\mathcal{A}) = -1$, when $\mathcal{A} < 0$ or $\mathcal{A} \leq 0$. The matrix–weighted undirected graph is considered as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}_s)$, where the node set is given by $\mathcal{V} = [v_1, v_2, ..., v_n]$ is used to represent the number of matrices in the network topology. The matrix–weighted adjacency matrix is given by $\mathcal{A}_s = (\mathcal{A}_{ij}) \in \mathbb{R}^{ne\times ne}$. Also, when $(i, j) \in \mathcal{E}$,

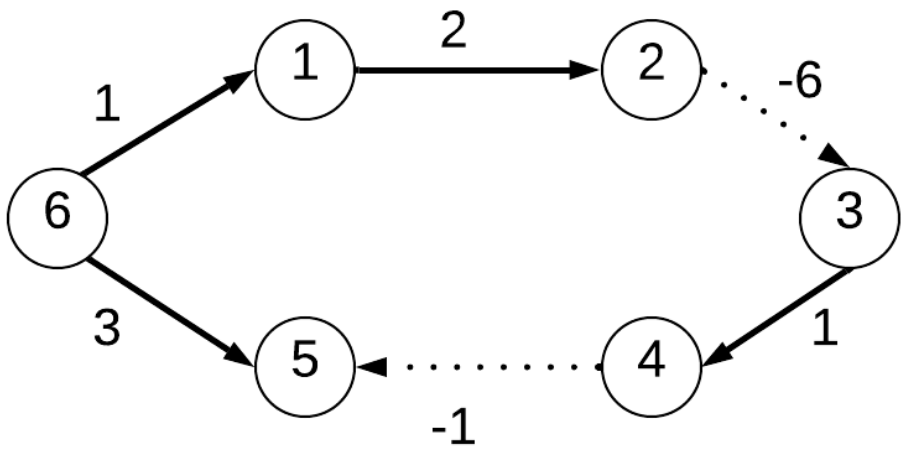


FIGURE 2.2: *A graph topology which has a spanning tree and is structurally balanced. (Zhang and Chen (2014))*

we have $sign(\mathcal{A}_{ij})\mathcal{A}_{ij} > 0$ or $sign(\mathcal{A}_{ij})\mathcal{A}_{ij} \geq 0$ and if $(i,j) \notin \mathcal{E}$ then $\mathcal{A}_{ij} = 0$. The expression of matrix–weighted degree matrix is given by, $\mathcal{D} = blkdig(\mathcal{D}_i)$ where $\mathcal{D}_i$ is $\mathcal{D}_i = \sum_{j \in \mathcal{N}_i} \text{sign}(\mathcal{A}_{ij})\mathcal{A}_{ij}$. The expression of matrix–weighted Laplacian matrix is given by $\mathcal{L} = \mathcal{D} - \mathcal{A}_s$.

Remark 2: Recently from the field of bipartite consensus some works have also appeared to study tripartite consensus wherein the network topology is as such that the interactions between agents belonging to the same group are cooperative whereas the interaction between agents belonging to different groups are antagonistic in nature (De Pasquale and Valcher (2020)) which was further extended to the concept of k–partite consensus wherein the number of groups can be an arbitrary number k (De Pasquale and Valcher (2022)), which stems its roots to the field of cluster consensus.

Lemma 1: (Altafini (2012)) If a directed signed graph with a spanning tree is structurally balanced then 0 is one of the eigenvalues of the Laplacian matrix and the other eigenvalues have a positive real part.

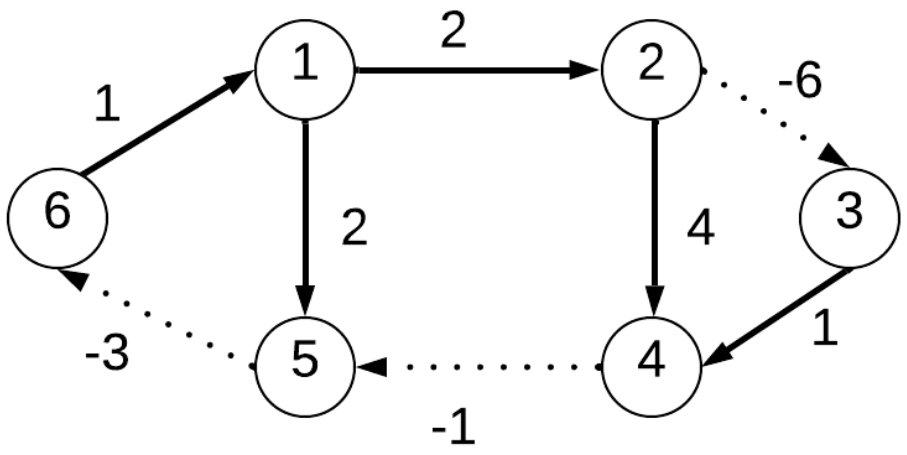


FIGURE 2.3: *A graph topology which is strongly connected and structurally unbalanced. (Zhang and Chen (2014))*

2.2 Problem Formulation

Consider a team of $N+1$ agents with N followers and one leader. The dynamical equation representing the dynamics of the follower agents $i \in \mathcal{I} = \{1, \cdots, N\}$ is given by

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + B_u u_i(t) \\ y_i(t) &= Cx_i(t) \end{aligned} \tag{2.3}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$ and $y_i(t) \in \mathbb{R}^p$ are the state, control input, and output measurement of the i^{th} agent respectively. The matrices $A \in \mathbb{R}^{n\times n}$, $B_u \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are assumed to be constant. The dynamical equation of the leader is given by

$$\begin{aligned} \dot{x}_0(t) &= Ax_0(t) \\ y_0(t) &= Cx_0(t) \end{aligned} \tag{2.4}$$

where $x_0(t) \in \mathbb{R}^n$ and $y_0(t) \in \mathbb{R}^p$ denotes the state and the output of the leader, respectively. The following assumptions will be used throughout the chapter.

Assumption 1. (A, B_u) is stabilizable.

Assumption 2. (C, A) is detectable.

Note that y_i are transmitted between agents according to the communication topology, which satisfies the following assumption.

Assumption 3. The combined graph topology containing the leader and followers contains a directed spanning tree, where the leader is the root node, and is structurally balanced.

The aim of this chapter is to design, using locally available information, an observer based controller in the presence of intermittent failures in the measurement and control input channels which solves the *bipartite consensus*, such that

$$\lim_{t\to\infty} \|x_i(t) - m_i x_0(t)\| = 0, \quad \forall i \in \mathcal{I} \tag{2.5}$$

Eq. (2.5) is valid for any initial condition. Different cases may appear: among them, Case a: There is a failure only in the control input channel (i.e. the control input will not be updated); Case b: There is a failure only in the measurement channel (i.e. the output measurement will not be updated); Case c: both Case a and Case b happens simultaneously. In this chapter, we will investigate Case c since the two other cases can be easily deduced from the latter. Furthermore, we consider the following assumption.

Assumption 4. The duration of an intermittent failure is bounded. Here, in this chapter intermittent failures are used to represent the situation wherein failures occur across all communication channels simultaneously wherein the values of control input, output and estimated output are kept constant to their values before intermittent failures and they are allowed to change only when the failure has disappeared. Further, when the intermittent fault disappears, there is a time $T_{min} > 0$ when no failure happens.

2.2.1 Observer Based Output Consensus under Intermittent Failures

To solve the leader-follower bipartite consensus problem despite the failures in the control and measurement channels, a distributed output feedback observer

based protocol is investigated. Using the theory of measure chains, the problem of intermittent failures can be converted into an asymptotic stabilization problem of a switched system on a particular non–uniform time domain formed by the union of disjoint intervals with variable lengths and variable gaps.

The closed–loop system will evolve during some continuous–time intervals when there is no failure. When an intermittent fault occurs due to, for example, sensor failures, actuator failure, obstacles etc., the control input and the output measurements will not evolve, but will be held at their last value, and will be updated after some time, i.e. when the failure disappears or the obstacle is removed. The problem can be converted to a switched system on the measure chain $\mathbb{T} = \bigcup_{b=0}^{\infty}[\sigma(t_b), t_{b+1}]$, with $\sigma(t_0) = t_0$, between continuous–time subsystem with variable interval lengths (when there is no failure) and a discrete-time subsystem with variable step sizes (when a failure occurs). The strictly increasing sequence $\{t_1, t_2, \ldots\}$ characterizes the time when a failure happens. A representation of time-scale $\mathbb{T}$ is given in Fig. 2.4.

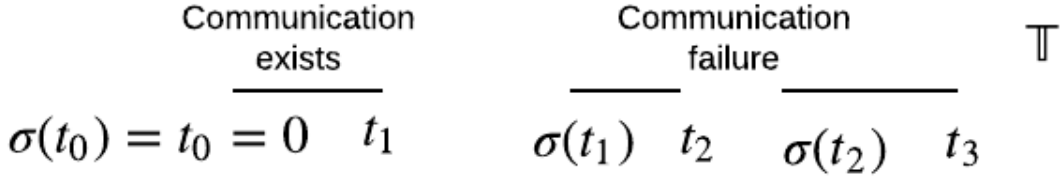


FIGURE 2.4: *Measure–chain representation of intermittent failures*

Under Assumptions 1-4 and using local available information from the agent itself and its neighbors (including leader and other follower agents) a distributed observer for an agent is constructed along the same lines as it is done for leaderless consensus and bereft of intermittent failures (Zhang and Chen (2017)),

$$\begin{aligned}\dot{\hat{x}}_i(t) = A\hat{x}_i(t) + B_u u_i(t) + K_0\Big[\sum_{j\in\mathcal{N}_i}|a_{ij}|\Big((y_i(t) \\ - \operatorname{sign}(a_{ij})y_j(t)) - (\hat{y}_i(t) - \operatorname{sign}(a_{ij})\hat{y}_j(t))\Big) \\ + f_i(y_i(t) - \hat{y}_i(t))\Big],\ \forall i \in \mathcal{I}\end{aligned} \tag{2.6}$$

where $\hat{x}_i(t)$ is the estimated state of agent i, K_0 is the observer gain to be designed. Due to failures in the measurement channel and $\forall i \in \mathcal{I}$, the output measurement does not involve on $\bigcup_{b=0}^{\infty}[t_{b+1}, \sigma(t_{b+1}))$ and can be formulated as

$$y_i(t) = \begin{cases} Cx_i(t), \ t \in \bigcup_{b=0}^{\infty}[\sigma(t_b), t_{b+1}) \\ Cx_i(t_{b+1}), \ t \in \bigcup_{b=0}^{\infty}[t_{b+1}, \sigma(t_{b+1})) \end{cases} \tag{2.7}$$

Similarly, the estimated output $\forall i \in \mathcal{I}$ becomes

$$\hat{y}_i(t) = \begin{cases} C\hat{x}_i(t), \ t \in \bigcup_{b=0}^{\infty}[\sigma(t_b), t_{b+1}) \\ C\hat{x}_i(t_{b+1}), \ t \in \bigcup_{b=0}^{\infty}[t_{b+1}, \sigma(t_{b+1})) \end{cases} \tag{2.8}$$

Based on the estimated state of the agent $\forall i \in \mathcal{I}$ and its neighbors, the proposed output feedback leader-follower bipartite consensus protocol can be formulated as the following switched control law due to possible intermittent failures in the control input channel

$$u_i(t) = \begin{cases} K(\sum_{j\in\mathcal{N}_i} |a_{ij}|(\hat{x}_i(t) - \text{sign}(a_{ij})\hat{x}_j(t)) + f_i(\hat{x}_i(t) - m_i x_0(t))), \\ \qquad t \in \bigcup_{b=0}^{\infty}[\sigma(t_b), t_{b+1}) \\ K(\sum_{j\in\mathcal{N}_i} |a_{ij}|(\hat{x}_i(t_{b+1}) - \text{sign}(a_{ij})\hat{x}_j(t_{b+1})) + f_i(\hat{x}_i(t_{b+1}) - m_i x_0(t_{b+1}))) \\ \qquad t \in \bigcup_{b=0}^{\infty} \in [t_{b+1}, \sigma(t_{b+1})) \end{cases} \tag{2.9}$$

wherein on $\bigcup_{b=0}^{\infty}[t_{b+1}, \sigma(t_{b+1}))$, the control input remains constant (i.e. there is no update due to control input channel failure). Let the bipartite-consensus error be

$$\epsilon_i(t) = x_i(t) - m_i x_0(t), \ \forall i \in \mathcal{I} \tag{2.10}$$

and the observer error be

$$e_i(t) = x_i(t) - \hat{x}_i(t), \ \forall i \in \mathcal{I} \tag{2.11}$$

Using Eqs. (2.3)-(2.4) and Eqs. (2.6)-(2.9), the bipartite consensus error and the observer error dynamics can be written in a compact form with $\epsilon(t) = [\epsilon_1(t)^T, .., \epsilon_N(t)^T]^T$, $e(t) = [e_1(t)^T, .., e_N(t)^T]^T$, I_N is the $N \times N$ identity matrix

and $\otimes$ is the Kronecker product, and by considering $\zeta(t) = [e(t)^T,\ \epsilon(t)^T]^T$ as,

$$\dot{\zeta}(t) = \begin{cases} \begin{bmatrix} A_{11} & 0_{Nn} \\ -H \otimes B_u K & A_{22} \end{bmatrix} \zeta(t),\ t \in \bigcup_{k=0}^{\infty}[\sigma(t_b), t_{b+1}) \\ \begin{bmatrix} I_N \otimes A & 0_{Nn} \\ 0_{Nn} & I_N \otimes A \end{bmatrix} \zeta(t) + \\ \begin{bmatrix} -H \otimes K_0 C & 0_{Nn} \\ -H \otimes B_u K & H \otimes B_u K \end{bmatrix} \zeta(t_{b+1}), \\ t \in \bigcup_{b=0}^{\infty}[t_{b+1}, \sigma(t_{b+1})) \end{cases} \tag{2.12}$$

where $A_{11} = [(I_N \otimes A) - (H \otimes K_0 C)]$ and $A_{22} = [(I_N \otimes A) + (H \otimes B_u K)]$. For simplicity, let
$A_1 = \begin{bmatrix} A_{11} & 0_{Nn} \\ -H \otimes B_u K & A_{22} \end{bmatrix}$, $A_2 = \begin{bmatrix} I_N \otimes A & 0_{Nn} \\ 0_{Nn} & I_N \otimes A \end{bmatrix}$ and $B = \begin{bmatrix} -H \otimes K_0 C & 0_{Nn} \\ -H \otimes B_u K & H \otimes B_u K \end{bmatrix}$.
The dynamical system given by Eq. (2.12) becomes

$$\dot{\zeta}(t) = \begin{cases} A_1 \zeta(t),\ t \in \bigcup_{b=0}^{\infty}[\sigma(t_b), t_{b+1}) \\ A_2 \zeta(t) + B\zeta(t_{b+1}),\ t \in \bigcup_{b=0}^{\infty}[t_{b+1}, \sigma(t_{b+1})) \end{cases} \tag{2.13}$$

To simplify the design analysis, discretization of the second equation in Eq. (2.13) is performed at discrete time instants $\{t_{b+1}\}$ and $\{\sigma(t_{b+1})\}$.
For $t \in [t_{b+1}, \sigma(t_{b+1}))$, we have

$$\zeta(t) = e^{A_2(t - t_{b+1})}\zeta(t_{b+1}) + w(t)B\zeta(t_{b+1}) \tag{2.14}$$

where $w(t) = \int_{t_{b+1}}^{t} e^{A_2(t-s)}ds$. For $t = \sigma(t_{b+1})$, we have

$$\zeta(\sigma(t_{b+1})) = e^{A_2\mu(t_{b+1})}\zeta(t_{b+1}) + w(\sigma(t_{b+1}))B\zeta(t_{b+1}) \tag{2.15}$$

For $t = t_{b+1}$, the Δ-derivative of $\zeta(t)$ is given by

$$\overset{\Delta}{\zeta}(t_{k+1}) = \frac{\zeta(\sigma(t_{b+1})) - \zeta(t_{b+1}))}{\mu(t_{b+1})} = A_{s2}\zeta(t_{b+1}) \tag{2.16}$$

where $A_{s2} = (\mu(t_{b+1}))^{-1}[(e^{A_2\mu(t_{b+1})} - I_{2Nn}) + w(\sigma(t_{b+1}))B]$.

Finally, we get the following switched system on $\mathbb{T} = \bigcup_{k=0}^{\infty}[\sigma(t_b), t_{b+1}]$ between continuous–time and discrete–time dynamics

$$\overset{\Delta}{\zeta}(t) = \begin{cases} A_1\zeta(t), \ t \in \bigcup_{b=0}^{\infty}[\sigma(t_b), t_{b+1}) \\ A_{s2}\zeta(t), \ t \in \bigcup_{b=0}^{\infty}\{t_{b+1}\} \end{cases} \tag{2.17}$$

Remark 3: If A_2 is non–singular then $w(t) = [e^{A_2(t-t_{b+1})} - I_{2Nn}]A_2^{-1}$, $w(\sigma(t_{b+1})) = [e^{A_2\mu(t_{b+1})} - I_{2Nn}]A_2^{-1}$ and $A_{s2} = \left(\dfrac{e^{A_2\mu(t_{b+1})} - I_{2Nn}}{\mu(t_{b+1})}\right)[I_{2Nn} + A_2^{-1}B]$. If A_2 is singular then one can use the power series formulation for finding the expression of $w(t)$. In the next subsection, we will analyse the stability of the switched system (2.17).

2.2.2 Stability Analysis

Taking inspirations from Theorem 17 given in the Introduction chapter of the thesis which was used to solve the consensus problem under full state feedback in presence of intermittent failures as reported in (Taousser et al. (2015)), in this subsection a control Lyapunov function based approach is used to prove the exponential stability. Considering the switched system (2.17), we will design the gain matrices K and K_0 such that A_1 and A_{s2} be Hilger stable and regressive with respect to the measure chain $\mathbb{T} = \bigcup_{b=0}^{\infty}[\sigma(t_b), t_{b+1}]$.

Theorem 1: Considering the MAS given by Eqs. (2.3), (2.4) under the observer based output feedback consensus described in the previous subsection and assuming that Assumptions 1–4 hold. If there exist positive definite symmetric matrices $P = P^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$ and A_{s2} is regressive such that the following inequalities are satisfied

$$A_1^T P + PA_1 < -Q_1 \ \forall \ t \ \in \mathbb{T} \tag{2.18}$$

$$A_{s2}^T P + PA_{s2} + \mu(t)A_{s2}^T PA_{s2} < -Q_2 \ \forall \ t \ \in \mathbb{T} \tag{2.19}$$

$$1 - \mu(t)\frac{\lambda_{min}(Q_2)}{\lambda_{max}(P)} \neq 0 \ \forall \ t \ \in \mathbb{T} \tag{2.20}$$

Then, the observer based bipartite consensus problem under intermittent failures is solved. ■

Proof: Let the Lyapunov function for the switched system (2.17), $V : \mathbb{R}^{2Nn} \to \mathbb{R}$, defined by

$$V(\zeta(t)) = \zeta^T(t)P\zeta(t),\ \forall t \in \mathbb{T} \tag{2.21}$$

which satisfies

$$\lambda_{\min}(P)\|\zeta(t)\|^2 \leq V(\zeta(t)) \leq \lambda_{\max}(P)\|\zeta(t)\|^2,\ \forall t \in \mathbb{T} \tag{2.22}$$

where $\lambda_{\min}(.)$ and $\lambda_{\max}(.)$ is used to denote the minimum and maximum eigenvalues of a matrix respectively. From conditions (2.18), (2.19) and (2.22), we have the following.

The Δ–derivative of V along the trajectories of the continuous–time subsystem of (2.17) is given by

$$\begin{aligned} \overset{\Delta}{V}(\zeta(t)) &= (\overset{\Delta}{\zeta})^T(t)P\zeta(\sigma(t)) + \zeta^T(t)P\overset{\Delta}{\zeta}(t) \\ &\leq \frac{-\lambda_{\min}(Q_1)}{\lambda_{\max}(P)}V(\zeta(t)) \end{aligned} \tag{2.23}$$

The Δ–derivative of V along the trajectories of the discrete time subsystem of (2.17) is given by

$$\begin{aligned} \overset{\Delta}{V}(\zeta(t)) &= \zeta^T(t)(A_{s2}^TP + PA_{s2} + \mu(t)A_{s2}^TPA_{s2})\zeta(t) \\ &\leq \frac{-\lambda_{\min}(Q_2)}{\lambda_{\max}(P)}V(\zeta(t)) \end{aligned} \tag{2.24}$$

Let $\delta_1 = \dfrac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)} > 0$ and $\delta_2 = \dfrac{\lambda_{\min}(Q_2)}{\lambda_{\max}(P)} > 0$. Thus,

$$\overset{\Delta}{V}(\zeta(t)) \leq \begin{cases} -\delta_1 V(\zeta(t)),\ t \in \bigcup_{b=0}^{\infty}[\sigma(t_b), t_{b+1}) \\ \\ -\delta_2 V(\zeta(t)),\ t \in \bigcup_{b=0}^{\infty}\{t_{b+1}\} \end{cases} \tag{2.25}$$

For $t \in \bigcup_{b=0}^{\infty}[\sigma(t_b), t_{b+1})$, we have

$$V(\zeta(t)) \leq e^{-\delta_1(t-\sigma(t_b))}V(\zeta(\sigma(t_b)))$$

For $t \in \bigcup_{b=0}^{\infty}\{t_{b+1}\}$, we have

$$V(\zeta(\sigma(t))) \quad \leq e_{-\delta_2}(\sigma(t), t)V(\zeta(t)) = (1-\mu(t)\delta_2)V(\zeta(t))$$

(see Bohner and Peterson (2001a) for more details). Notice that since the discrete-time subsystem is considered to be Hilger stable, so $|1-\mu(t)\delta_2| < 1$, $\forall t \in \bigcup_{b=0}^{\infty}\{t_{b+1}\}$. By induction, we get
For $\sigma(t_0) = t_0 \leq t \leq t_1$,

$$V(\zeta(t)) \leq e^{-\delta_1(t-\sigma(t_0))}V(\zeta(\sigma(t_0))) = e^{-\delta_1(t-t_0)}V(\zeta(t_0)).$$

For $t = t_1$, $\quad V(\zeta(t_1)) \leq e^{-\delta_1(t_1-t_0)}V(\zeta(t_0))$.
For $t = \sigma(t_1)$,

$$V(\zeta(\sigma(t_1))) \quad \leq (1-\mu(t_1)\delta_2)e^{-\delta_1(t_1-t_0)}V(\zeta(t_0)) \tag{2.26}$$

For $\sigma(t_1) \leq t \leq t_2$,

$$V(\zeta(t)) \quad \leq e^{-\delta_1(t-\sigma(t_1))}(1-\mu(t_1)\delta_2)e^{-\delta_1(t_1-t_0)}V(\zeta(t_0)) \tag{2.27}$$

and finally, for $\sigma(t_b) \leq t \leq t_{b+1}$, $\forall b \in \mathbb{N}$, we get

$$V(\zeta(t)) \leq e^{-\delta_1(t-t_0-\sum_{i=1}^{b}\mu_i)}\prod_{i=1}^{b}(1-\mu(t_i)\delta_2)V(\zeta(t_0)) \tag{2.28}$$

From condition (2.22), we derive

$$\|\zeta(t)\| \leq \xi\ e^{-\frac{1}{2}\delta_1(t-t_0-\sum_{i=1}^{b}\mu_i)}\prod_{i=1}^{b}|1-\mu(t_i)\delta_2|^{\frac{1}{2}}\zeta(t_0) \tag{2.29}$$

with $\xi = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$. One can get the upper bound,

$$\|\zeta(t)\| \leq \xi \; e^{\delta t} \; \|\zeta_0\|, \; \forall t \in \mathbb{T} \tag{2.30}$$

where $\delta = \max\left\{-\delta_1, \frac{\log(\max_i |1-\mu(t_i)\delta_2|)}{\mu_{\max}}\right\} < 0$, represents the rate of decrease of the Lyapunov function and the state along the trajectories of the switched system. Therefore, the existence of a common Lyapunov function V for the switched system (2.17) ensures the bipartite consensus even in the presence of intermittent failures and partial measurement. ■

Remark 4: In this chapter, we have considered that the duration of an intermittent failure does not violate the stability of the discrete-time subsystem and for that the control Lyapunov function has been introduced in the study of the consensus problem. An extension could be later investigated for the design of observer based consensus protocols with failures by considering that unstable modes may exist. If Hilger stability condition for the subsystems does not hold true then the concept of multiple Lyapunov functions or parameter dependent Lyapunov function can be used to formulate new stability conditions.

Remark 5: The gain matrices K and K_0 are chosen in such a way that A_1 and A_{s2} are Hilger stable and regressive with respect to the measure chain $\mathbb{T} = \bigcup_{b=0}^{\infty}[\sigma(t_b), t_{b+1}]$. Furthermore, inequalities (2.18)-(2.20) given in Theorem 1 should be checked simultaneously for the given measure chain. If no feasible common Lyapunov function matrix is found to satisfy inequalities (2.18)-(2.20), then further tuning of controller and observer gain matrices are done keeping in mind the Hilger stability and regressivity criterion of the subsystems.

Remark 6: The algorithm proposed in this chapter assumes that failures happen in the communication channels simultaneously which is a conservative assumption. Furthermore, the dynamics of the agents are assumed to be homogeneous which is tough to hold in realistic scenarios.

Remark 7: The model uncertainties and external disturbances are not considered in the agent dynamics. The authors of this chapter believe that the stability proof of the closed–loop switched system on measure chains considering uncertainties

and external disturbances is not trivial. Such a problem can be investigated by extending the current proposed theorem with some additional constraints.

2.3 Numerical Simulations

The multi–agent system under consideration in this section consists of four followers and one leader agent. The representation of the graph topology is given in Fig. 2.5. Antagonistic interactions are represented by dotted line whereas cooperative interactions are represented by solid lines. The graph is structurally balanced, which means that it can be broken into $v_1 = \{1, 2\}$ and $v_2 = \{3, 4\}$. The signed digraph also has a spanning tree. Thus for the given graph topology, Lemma 1 is satisfied. The state, control and output matrices for the agents whose dynamics represent modified simple harmonic oscillators (see Liu et al. (2017)) are given by

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{2.31}$$

The modified Laplacian matrix H is given by

$$H = \begin{bmatrix} 4 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & -2 \\ 3 & 0 & -3 & 6 \end{bmatrix} \tag{2.32}$$

In the following example, the intermittent failures are represented by the measure chain, $\mathbb{T} = \bigcup_{b=0}^{\infty}[\sigma(t_b), t_{b+1}]$, such that

$$\begin{cases} \sigma(t_0) = t_0 = 0, \;\; \sigma(t_b) = t_b + c \\ \sigma(t_b) = 0.12(2b-1) + 0.5\log(b) + c, \;\; \forall b \in \mathbb{N} \end{cases} \tag{2.33}$$

where $t_b, b \in N$ represents the time instants when a failure occurs. The graininess function value is bounded by constant $c = 0.12$. The above choice of measure chain ensures that the failure duration lies within the discrete set of values $\{0s, 0.12s\}$. The choice of controller and observer matrices are $K = \begin{bmatrix} -1 & -1 \end{bmatrix}$ and $K_0 =$

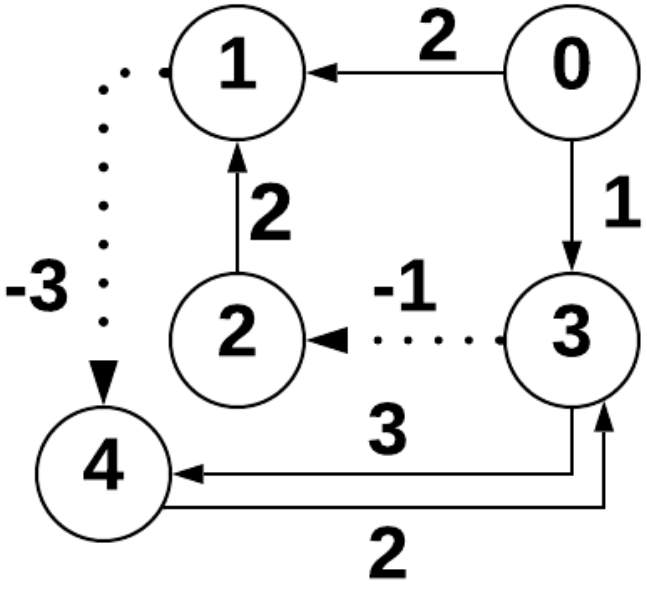


FIGURE 2.5: *Graph topology representing interactions between leader and followers.*

$\begin{bmatrix}1 & -0.9\end{bmatrix}^T$ respectively, which ensures the Hilger stability and regressivity for A_1 and A_{s2} with respect to the measure chain $\mathbb{T}$.

Using the value of $Q_1 = 0.8I_{16}$ and $Q_2 = 0.1I_{16}$, the matrix P can be numerically calculated with the help of convex optimization routine (cvx) of Matlab. The initial values of state variables and its estimates are taken to be random values. The trajectories of the observer errors and the consensus errors are given by Fig. 2.6 and Fig. 2.7 respectively. From these figures, it can be said that the designed distributed control algorithm performs satisfactorily as both bipartite consensus error and observer error converges to zero even if there are intermittent failures.

Remark 8: If the duration of failure is large then the value of c must be taken large for the $\mathbb{P}_{a,b}$ measure chain, in which case the value of common Lyapunov function will change and the gain constants need to be recalculated keeping in mind the condition of Hilger stability for both the subsystems of the switched system formulation.

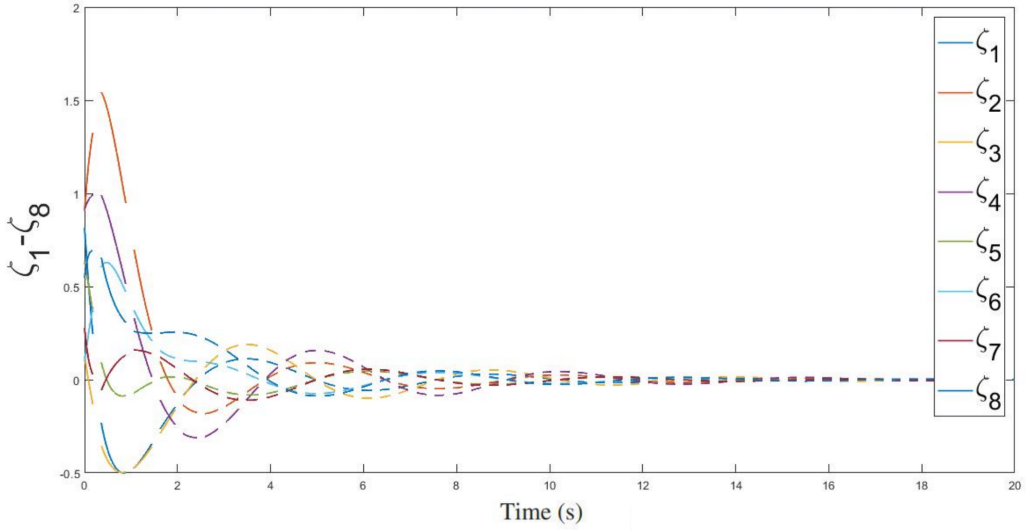


FIGURE 2.6: *Observer error for follower agents.*

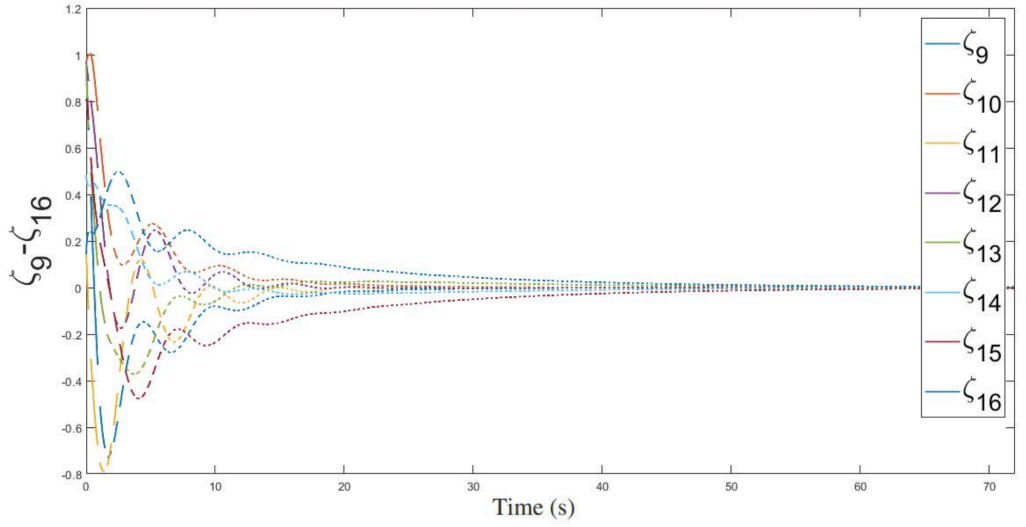


FIGURE 2.7: *Bipartite consensus error between leader and follower agents.*

2.4 Conclusion

In this chapter, an observer–based leader follower bipartite consensus protocol has been introduced for linear multi–agent systems under intermittent failures. Using the concept of measure chain theory, the closed–loop system dynamics has been converted to a switched system between continuous–time and discrete–time dynamics. Exponential stability of the closed–loop system has been proved using common Lyapunov function and measure chain theory. In this chapter, the condition of energy efficiency has not been taken into consideration while solving the consensus problem. Hence, in the next chapter we would try to explore event–triggered controllers which are energy efficient controllers.

Chapter 3

Event–triggered control for systems on non–uniform time domains using measure chain theory

During the past two decades, a lot of researches has been carried out in the field of event–triggered control as an efficient, cost-effective replacement to the classical time–triggered implementation of any system across digital platforms (Åarzén (1999)). In the case of event–triggered design procedure, the input or the output are not sampled periodically at certain specific sampling times but rather are sampled only when certain event–triggered conditions are violated. In the last few decades, various types of event–triggering conditions have been investigated, namely static event–triggering mechanisms (Etienne et al. (2017)), wherein the triggering conditions are made up of the state or output variables of the system which are implemented through digital channels or dynamic event–triggering mechanisms (Girard (2014)), wherein extra variables are introduced in the triggering condition which results in increasing the inter–event time without compromising certain performance metric. Along the design direction, there exist various methods like emulation based method (Trimpe (2017)) in which the gain constants of the event–triggered controller/observer/output feedback controllers are

designed through classical methods, like pole placement or algebraic Riccati equation (ARE) based methods (Borgers et al. (2017)), and the triggering conditions are designed separately. In the case of co–design based methods (Tarbouriech et al. (2016)), the gain constants, as well as the triggering conditions, are designed simultaneously using appropriate linear matrix inequalities (LMIs), which give less conservative results in comparison to the emulation based methods. One of the significant limitations of event–triggered controls is the occurrence of Zeno phenomenon, which leads to infinite sampling within a finite time duration. The above–mentioned problem of infinite sampling in a finite duration has a detrimental effect on the actuators of the plant, which may be alleviated using time regularization techniques in the event–triggering condition (Borgers et al. (2016)). Quantization effects, time delays, packet dropouts and variable transmission times across communication channels, where the event–triggered control is implemented, were also addressed in the literature (Dolk and Heemels (2017), Abdelrahim et al. (2017)). The main contributions of this chapter are to design event–triggered controllers using emulation based method for systems on any arbitrary time domains with the help of measure chain theory. This chapter is organized as follows. The problem formulation is given in the first section. Numerical simulations are given in second section and the chapter concludes with a conclusion section given in the third section.

3.1 Event–triggered controller design for systems on measure chains

In the literature concerning ISS Lyapunov function based design of event–triggered control, distinct results and stability theorems can be found for continuous–time systems (Heemels et al. (2012b)), or discrete–time systems (Eqtami et al. (2010)). Apart from the continuous and discrete–time systems, a broad class of systems exists, wherein the time domain is non–uniform such as economic models, inventory models, predator–prey systems, population growth models or nonelectronic impulsive models (Atici and Uysal (2008), Zhao (2007), Kumar et al. (2021)). For

such systems, the discrete and continuous event–triggered design procedure cannot be applied; thus, there comes the applicability of measure chain theory (Hilger (1990)) which allows us to represent systems on non–uniform time domains and also provides us tools to perform calculus on any non–uniform time domains. According to the author of this chapter, the problem of event–triggering for systems on non–uniform time domains have not received ample attention from the control system community. Thus, in this chapter, the issue of designing event–triggered controllers for a class of uncertain nonlinear systems on non–uniform time domains have been addressed using novel Lyapunov based input–to–state stability (ISS) conditions and the theory of measure chains. Thus, these results unify existing results for the continuous and discrete event–triggering and extend to event–triggering on non–uniform time domains.

Definition 1: The class of comparison functions considered in this chapter are $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}_\Delta$ which are defined on a measure chain $\mathbb{T}$. A function $\beta : \mathbb{T} \to \mathbb{R}_+$ belongs to $\mathcal{L}_\Delta$ if it is rd–continuous and $\lim_{\mathbb{T}\ni t\to\infty} \beta(t) = 0$. Furthermore, $\beta : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}_+$ belongs to $\mathcal{KL}_\Delta$ if it is rd–continuous and $\{\beta(\cdot, b) \in \mathcal{K}, \beta(a, \cdot) \in \mathcal{L}_\Delta\}$ ■.

3.1.1 Event–triggered control for nonlinear systems

In this subsection, an event–triggered controller for a general nonlinear system on an arbitrary measure chain with the property of bounded graininess function ($0 \leq \mu(t) \leq \mu_{\max}$) is designed. A nonlinear system on an arbitrary measure chain $\mathbb{T}$ is given by

$$\overset{\Delta}{z}(t) = \varepsilon(z(t), u(t)),\ z(t_0) = z_0 \quad \forall\, t \in \mathbb{T} \tag{3.1}$$

where the state vector and control input are given by $z(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ respectively. $\varepsilon : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a $\mathcal{R}$ and Lipschitz function. Assume that system given by (3.1) is stabilizable (Bartosiewicz and Piotrowska (2013)) by a state feedback controller of the form $u(t) = k(z(t))$, $\forall\ t\ \mathbb{T}$ with $k(.)$ being rd–continuous and Lipschitz. Now, consider the event–triggered setup for which the control input is only updated at the discrete time instants $t_i \in \mathbb{T}, \forall\, i \in \mathbb{Z}^{0+}$.

The controller designed by considering event–triggering mechanism is given by $u(t) = k(z(t_i)) = k(z(t) + e(t)), \quad \forall\, \mathbb{T} \ni t \in [t_i, t_{i+1})$, where the error due to event–triggering is $e(t) = z(t_i) - z(t)$. The resulting closed–loop system becomes

$$\overset{\Delta}{z}(t) = \varepsilon(z(t), k(z(t) + e(t))), \quad \forall\, t \in \mathbb{T} \tag{3.2}$$

The error dynamics can be written as $\overset{\Delta}{e}(t) = -\overset{\Delta}{z}(t)$, $\forall\, t \in \mathbb{T}$. Now, we will state an event–condition and a theorem which will prove asymptotic stability of the closed–loop system (3.2) with the help of input–to–state (ISS) stable Lyapunov functions on measure chains.

Theorem 1: Assume that there exists a Lyapunov like function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\alpha_1(\|z(t)\|) \leq V(z(t)) \leq \alpha_2(\|z(t)\|) \tag{3.3}$$

$$\overset{\Delta}{V}(z(t)) \leq -\alpha_3(\|z(t)\|) + \gamma^{\nu}(\|e(t)\|) \tag{3.4}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\gamma^{\nu} \in \mathcal{K}$. The proposed triggering mechanism is,

$$t_{i+1} = \inf\Big\{t > t_i \in \mathbb{T} : \gamma^{\nu}(\|e(t)\|) \geq \theta\alpha_3(\|z(t)\|)\Big\} \tag{3.5}$$

where $\theta \in (0, 1)$ is selected in such a way that $(1 + \mu(t)(\theta - 1))$ is positive across the entire measure chain. The event–triggering condition given by (3.5) ensures asymptotic stability of the system given by (3.2). ■

Proof: Considering the time interval $[t_0, t] \in \mathbb{T}$, and the case when the triggering condition is not violated. Thus, with the help of Definition 1 one can write

$$\overset{\Delta}{V}(z(t)) \leq (\theta - 1)\alpha_3(\alpha_1^{-1}(V(z(t))) \tag{3.6}$$

Since $(\theta - 1) \in \mathcal{R}^+$,

$$\|z(t)\| \leq \alpha_1^{-1}(e_{(\theta-1)}(t, t_0)\alpha_3(\alpha_1^{-1}(\alpha_2(\|z(t_0)\|)))) \tag{3.7}$$

The control input is updated at a time instant $\mathbb{T} \ni t_j \in ((t_0, t) \in \mathbb{T})$, when (3.5) is violated. Then, $\forall\, [\sigma(t_j), t] \ni t \in \mathbb{T}$ one gets,

$$\begin{aligned} \|z(t)\| \leq \alpha_1^{-1}(e_{(\theta-1)}(t, \sigma(t_j))\alpha_3(\alpha_1^{-1}(\alpha_2(\|z(t_j)\|) \\ + \mu(t_j)(-\alpha_3(\|z(t_j)\|) + \gamma''(\|e(t_j)\|))))) \end{aligned} \tag{3.8}$$

Thus, with the help of (3.7) and (3.8), asymptotic stability of system given by (3.2) with the event–triggering condition given in (3.5) is proved. ■

Remark 1: Theorem 1 unifies the design procedure of event–triggered controllers for nonlinear systems using ISS stable Lyapunov functions on discrete $\mathbb{T} = \mathbb{Z}$, continuous $\mathbb{T} = \mathbb{R}$ or any non–uniform time domains. The above theorem reduces to continuous event–triggered controller design when $\mu(t) = 0$ and discrete event–triggered controller design when $\mu(t) = 1$. The avoidance of Zeno phenomenon for nonlinear systems on continuous time domain can be carried out along the same lines as given in (Tabuada (2007)).

However, one must also note that finding an appropriate input–to–state Lyapunov function considered in Theorem 1 is not a trivial task. Hence, we have considered certain class of Lipschitz nonlinear systems and designed event–triggered controllers for those class of systems on measure chains.

3.1.2 Event–triggered control for nonlinear Lipschitz systems

In this subsection, using measure chain theory we have designed event–triggered controllers for a class of nonlinear Lipschitz systems. Consider a linear system with Lipschitz uncertainties,

$$\overset{\Delta}{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t) + f_1(z(t)),\ z(t_0) = z_0 \quad \forall\, t \in \mathbb{T} \tag{3.9}$$

with the state vector, control input, state matrix (constant and regressive) and input matrix are given by $z(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $\mathcal{A} \in \mathbb{R}^{n\times n}$ and $\mathcal{B} \in \mathbb{R}^{n\times m}$ respectively. The function $f_1(z(t))$ is rd–continuous and Lipschitz (i.e. $\|f_1(z(t))\| \leq$

$L\|z(t)\|$ where L is a positive constant). It is assumed that system (3.9) is stabilizable (Bartosiewicz and Piotrowska (2013)) by a state feedback controller $u(t) = Kz(t),\ \forall\ t \in \mathbb{T}$, where $K \in \mathbb{R}^{m\times n}$. The closed–loop system with $u(t) = Kz(t_i), \forall\ \mathbb{T} \ni t \in [t_i, t_{i+1})$ as control input, and $t_i,\ \forall\, i \in \mathbb{Z}^{0+}$ as triggering instants is

$$\overset{\Delta}{z}(t) = (\mathcal{A} + \mathcal{B}K)z(t) + \mathcal{B}Ke(t) + f_1(z(t)),\ \forall\ t \in \mathbb{T} \tag{3.10}$$

Now, we are in a position to state a theorem and an event condition which ensures asymptotic stability for the closed–loop system representation given in Eq. (3.10).

Theorem 2: Given the closed–loop system (3.10), with $f_1(z(t))$ an rd–continuous and Lipschitz function, with Lipschitz constant L, as well as matrices $P = P^T$ and $Q = Q^T$ which are positive definite such that

$$(\mathcal{A} + \mathcal{B}K)^T P + P(\mathcal{A} + \mathcal{B}K) + \mu(t)(\mathcal{A} + \mathcal{B}K)^T P(\mathcal{A} + \mathcal{B}K) \leq -Q \tag{3.11}$$

and

$$L \leq \frac{\lambda^Q_{\min}}{\lambda^P_{\max} 2d_2} \tag{3.12}$$

then the event–triggering condition

$$t_{i+1} = \inf\left\{t > t_i \in \mathbb{T} : \|e(t)\| \geq \theta\varrho\|z(t)\|\right\} \tag{3.13}$$

with

$$\varrho = \left(\frac{\lambda^Q_{\min} - c_1}{c_2}\right)^{1/2}, \qquad c_1 = \lambda^P_{\max} d_2 L + \frac{\lambda^Q_{\min}}{2}$$
$$c_2 = \lambda^P_{\max}\left(\frac{2\lambda^P_{\max} d_1^2 d_0}{\lambda^Q_{\min}} + \mu(t)d_0\right)d_0, \quad d_1 = 1 + \mu(t)L + \mu(t)\lambda^{\mathcal{A}+\mathcal{B}K}_{\max}$$
$$d_0 = \|\mathcal{B}K\|, \qquad d_2 = 2 + \mu(t)L + 2\mu(t)\lambda^{\mathcal{A}+\mathcal{B}K}_{\max}$$

ensures asymptotic stability for system (3.10) wherein θ is selected in a similar manner as given in Theorem 1. ■

Proof: Considering, $V(z(t)) = (z(t))^T P z(t),\ \forall\ t\ \in\ \mathbb{T}$ and delta–differentiating along the trajectories of (3.10), one gets

$$\overset{\Delta}{V}(z(t)) = \Big((\mathcal{A}+\mathcal{B}K)z(t) + \mathcal{B}Ke(t) + f_1(z(t))\Big)^T Pz(t) + \Big(z(t) + \mu(t)((\mathcal{A} + \mathcal{B}K)z(t) + \mathcal{B}Ke(t) + f_1(z(t)))\Big)^T P\Big((\mathcal{A}+\mathcal{B}K)z(t) + \mathcal{B}Ke(t) + f_1(z(t))\Big) \tag{3.14}$$

$$\leq -\lambda^Q_{\min}\|z(t)\|^2 + \lambda^P_{\max}\Big(d_2 L\|z(t)\|^2 + 2d_0 d_1\|z(t)\|\|e(t)\| + \mu(t)d_0^2\|e(t)\|^2\Big) \tag{3.15}$$

$$\leq -(\lambda^Q_{\min} - c_1)\|z(t)\|^2 + c_2\|e(t)\|^2 \tag{3.16}$$

$$\leq (\lambda^Q_{\min} - c_1)(\theta - 1)\|z(t)\|^2 \tag{3.17}$$

Thus, from the triggering condition (3.13) and using Theorem 1, the asymptotic stability for the nonlinear system under the designed event–triggering mechanism given in (3.13) is proved. ■

Remark 2: Only certain classes of systems are admissible where the nonlinearities satisfies the condition on L given in Theorem 2. Apart from the above mentioned Lipschitz nonlinearities, broad class of nonlinearities exists in the literature which includes cone bounded (Moreira et al. (2019)), slope restricted nonlinearities (Moreira et al. (2020)) etc.

Now, we state a corollary of Theorem 2 wherein the condition of event–triggering for a linear system on measure chains is given which ensures asymptotic stability of the closed–loop linear system.

Corollary 1: For a linear system in the absence of external disturbances and system uncertainties,

$$\overset{\Delta}{z}(t) = (\mathcal{A}+\mathcal{B}K)z(t) + \mathcal{B}Ke(t)\ \forall\ t\ \in\ \mathbb{T} \tag{3.18}$$

assuming matrices $P = P^T$ and $Q = Q^T$ exist which are positive definite such that (3.11) is satisfied, then the event–triggering condition

$$t_{i+1} = \inf\Big\{t > t_i \in \mathbb{T} : \|e(t)\| \geq \theta \varrho_\ell \|z(t)\|\Big\} \tag{3.19}$$

with

$$\varrho_\ell = \left(\frac{\lambda^Q_{\min} - c_{\ell 1}}{c_{\ell 2}}\right)^{1/2}, \qquad c_{\ell 1} = \frac{\lambda^Q_{\min}}{2}$$
$$c_{\ell 2} = \lambda^P_{\max}\left(\frac{2\lambda^P_{\max} d_0 (d_{\ell 1})^2}{\lambda^Q_{\min}} + \mu(t) d_0\right) d_0, \quad d_0 = \|\mathcal{B}K\|$$
$$d_{\ell 1} = 1 + \mu(t)\lambda^{\mathcal{A}+\mathcal{B}K}_{\max}$$

ensures asymptotic stability for the system given by (3.18) wherein the value of θ is similar to the value given in Theorem 1 . ■

Proof: Considering, $V(z(t)) = (z(t))^T P z(t)$, $\forall\, t \in \mathbb{T}$ and delta–differentiating along the trajectories of (3.18) we get

$$\overset{\Delta}{V}(z(t)) = \Big((\mathcal{A}+\mathcal{B}K)z(t) + \mathcal{B}Ke(t)\Big)^T P z(t) + \Big(z(t) + \mu(t)((\mathcal{A}+\mathcal{B}K)z(t) + \mathcal{B}Ke(t))\Big)^T P\Big((\mathcal{A}+\mathcal{B}K)z(t) + \mathcal{B}Ke(t)\Big) \tag{3.20}$$

$$\leq -\lambda^Q_{\min}\|z(t)\|^2 + \lambda^P_{\max}\Big(2d_0 d_{\ell 1}\|z(t)\|\|e(t)\| + \mu(t) d_0^2 \|e(t)\|^2\Big) \tag{3.21}$$

$$\leq -\lambda^Q_{\min}\|z(t)\|^2 + \frac{\lambda^Q_{\min}}{2}\|z(t)\|^2 + \lambda^P_{\max}\left(\frac{2\lambda^P_{\max} d_0 (d_{\ell 1})^2}{\lambda^Q_{\min}} + \mu(t) d_0\right) d_0 \|e(t)\|^2 \tag{3.22}$$

$$\leq -(\lambda^Q_{\min} - c_{\ell 1})\|z(t)\|^2 + c_{\ell 2}\|e(t)\|^2 \tag{3.23}$$

$$\leq (\lambda^Q_{\min} - c_{\ell 1})(\theta - 1)\|z(t)\|^2 \tag{3.24}$$

Then the proof proceeds from the triggering condition (3.19) using Theorem 1. ■

Remark 3: For homogeneous measure chains, like $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$ where the value $\mu(t)$ is constant, the expression in the triggering condition is constant, i.e. $t_{i+1} = \inf\{t > t_i \in \mathbb{T} : \|e\| \geq \varrho_c \|z\|\}$ where ϱ_c is a constant value. On the contrary,

for non–homogeneous measure chains the value of triggering condition is not constant and depends on $\mu(t)$, which varies across the bounded measure chain under consideration.

Proposition 1: The event–triggered rules (3.13) and (3.19) ensure strictly positive inter–event–times between two triggering instants. ■

Proof: Delta–differentiating $\delta = \left\|\frac{e(t)}{z(t)}\right\|^2$, $\forall\ t\ \in\ \mathbb{T}$ one gets

$$\overset{\Delta}{\delta} = \frac{\mu(\overset{\Delta}{z})^T \overset{\Delta}{z}(z^T z - e^T e) - 2(\overset{\Delta}{z})^T(ze^T e + ez^T z)}{z^T z(z^T z + 2\mu(\overset{\Delta}{z})^T z + \mu^2(\overset{\Delta}{z})^T \overset{\Delta}{z})} \tag{3.25}$$

Referring to the more general rule (3.13), in the following we discuss the two possible cases.

1. When $\mu = 0$, the Δ–derivative coincides with the classical derivative on continuous time scale, and (3.25) can be written as

$$\dot{\delta} = \frac{-2\dot{z}^T(ze^T e + ez^T z)}{z^T z z^T z} \tag{3.26}$$

$$\leq 2(\|\mathcal{A} + \mathcal{B}K\| + \|\mathcal{B}K\| + L\Big)\delta + 2\|\mathcal{B}K\|\delta^{1.5} + 2\Big(\|\mathcal{A} + \mathcal{B}K\| + L\Big)\delta^{0.5} \tag{3.27}$$

   Thus, between two triggering instants a minimum amount of time exist and the proof is straightforward from (Tabuada (2007)) thus it is omitted here.

2. When $\mu > 0$, (3.25) becomes

$$\overset{\Delta}{\delta} \leq \frac{\|\mu(\overset{\Delta}{z})^T \overset{\Delta}{z}\|(\|z\|^2 + \|e\|^2) + \|2\overset{\Delta^T}{z}(ze^T e + ez^T z)\|}{\|z^T z\|\|(4\mu(\overset{\Delta}{z})^T z)\|} \tag{3.28}$$

$$\leq \left\|\frac{\mathcal{A} + \mathcal{B}K}{4}\right\| + \frac{L}{4} + \left(\left\|\frac{\mathcal{B}K}{4}\right\| + \frac{1}{2\mu}\right)\delta^{0.5} + \left(\left\|\frac{\mathcal{A} + \mathcal{B}K}{4}\right\| + \frac{1}{2\mu} + \frac{L}{4}\right)\delta + \left\|\frac{\mathcal{B}K}{4}\right\|\delta^{1.5} \tag{3.29}$$

Thus, using the theory of integration on measure chains one can say that no Zeno phenomenon occurs even when $\mu(t) \neq 0$. ■

3.2 Numerical Simulations

To validate the design procedure proposed in Theorem 2 and Corollary 1, four different scenarios are considered. Let us consider system given by (3.9) with $\mathcal{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The initial conditions for all four cases are considered randomly. The triggering condition is given by (3.19) for the cases (a)–(c), whereas for the case (d) it is given by (3.13). The value of θ is taken by trial and error method. The values of K, P and Q are obtained using the emulation based design procedure and cvx routine of Matlab. The investigated scenarios are as follows

- Case a: The measure chain under consideration is $\mathbb{T} = \mathbb{R}$. The gain constant matrix is taken as $K = (-3 \quad -3)$. The values of P and Q are

$$P = \begin{bmatrix} 2.5 & -2.5 \\ -2.5 & 5 \end{bmatrix}, \; Q = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

  The value of θ is 0.2.

- Case b: The measure chain under consideration is a discrete measure chain with varying graininess function. The maximum graininess function $\mu_{\max}$ is 0.1320 s. The minimum graininess function $\mu_{\min}$ is 0.01 s. Thus $\mu(t)$ takes discrete values in the interval $\{0.01s, 0.1320s\}$. The gain constant matrix is taken as $K = (-2 \quad -5)$. The values of

$$Q = \begin{bmatrix} 4.1 & 0 \\ 0 & 4.1 \end{bmatrix} \; P = \begin{bmatrix} 3.7218 & 2.0458 \\ 2.0458 & 15.2699 \end{bmatrix}$$

  are obtained numerically. The value of θ is 0.8.

- Case c: The measure chain under consideration is a measure chain with random graininess function value,

$$\begin{cases} \sigma(t_0) = t_0 = 0, \sigma(t_c) = t_c + 0.18 + \alpha_{11}, \ \forall \ c \in \mathbb{N} \\ t_c = 0.18(2c-1) + 0.5\log(c) + \alpha_{12}, \ \forall \ c \in \mathbb{N} \end{cases} \tag{3.30}$$

  where α_{11} and α_{12} are random numbers. The expression of graininess function value is $\mu(t) = 0.18 + \alpha_{11}$. The value of $\mu_{\max}$ is 0.19 s. The gain constant matrix is taken as $K = (-5 \quad -4)$. The values

$$Q = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \ P = \begin{bmatrix} 2.3740 & 2.1605 \\ 2.1605 & 11.7267 \end{bmatrix}$$

  are obtained similarly as in Case a. The value of θ is 0.9.

- Case d: The measure chain under consideration is that of Case c with same gains but with uncertainties given by $f_1(z(t)) = 0.01 \sin z(t)$, $\forall\, t \ \in \mathbb{T}$ for which the value of Lipschitz constant L is 0.01.

Remark 4: There is no clear relation between the Lipschitz constant and the efficiency of the event–triggered control proposed in this chapter. Hence, the nonlinearities should have a Lipschitz constant which is sufficiently small and satisfies all conditions of Theorem 2.

From the plots of the inter–event times corresponding to the various cases under study, it can be noticed that the designed emulation based event–triggering conditions given by (3.13) and (3.19) for systems on non–uniform time domains are nontrivial because the total number of triggering instants are less than the classical time–triggered implementation. Also, from the plots of inter–event times we can conclude that Zeno phenomenon is avoided in the continuous portion of various measure chains be it homogeneous or non–homogeneous. Furthermore, from the plots of state trajectories corresponding to various cases, one can observe that the event–triggered controllers designed by (3.13) and (3.19) results in asymptotic stability. Thus, from Figs. 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7 and 3.8, one can claim

that the designed event–triggered controllers designed by (3.13) and (3.19) work satisfactorily for any measure chain $\mathbb{T}$.

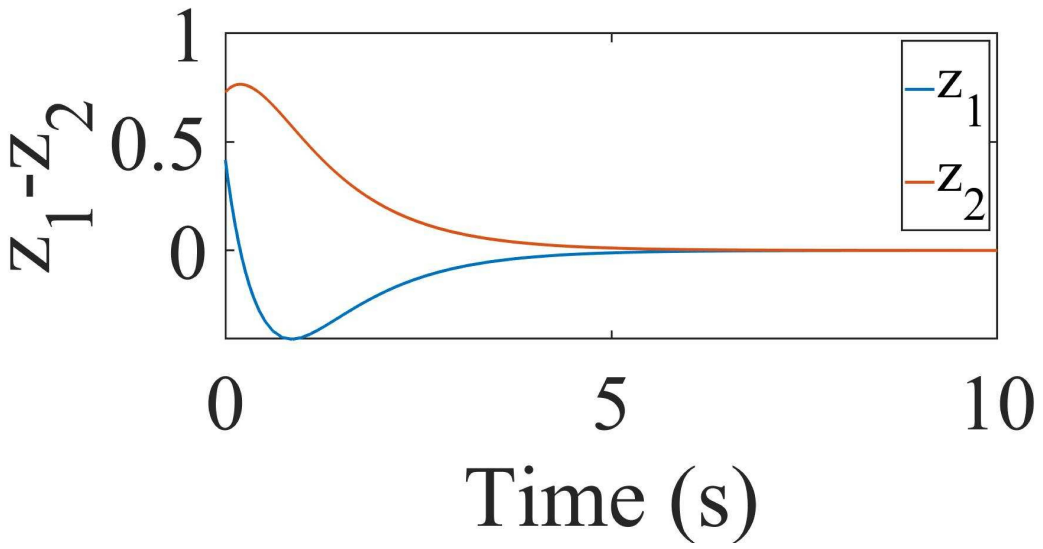


FIGURE 3.1: *State trajectories for Case–a*

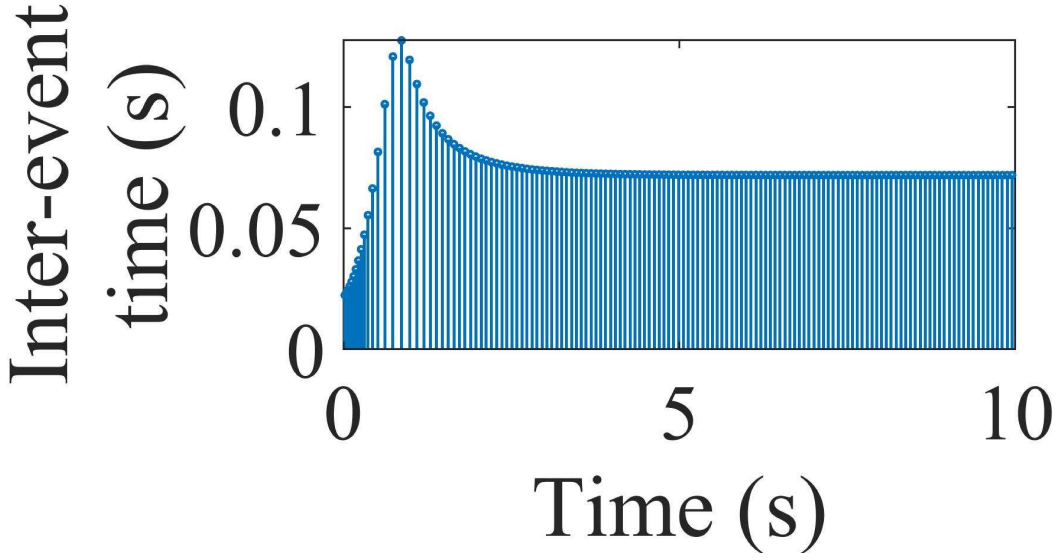


FIGURE 3.2: *Inter–event–time for Case–a*

3.3 Conclusion

This chapter exploits the concept of measure chains to design emulation based event–triggered controllers for systems on non–uniform time domains, using ISS

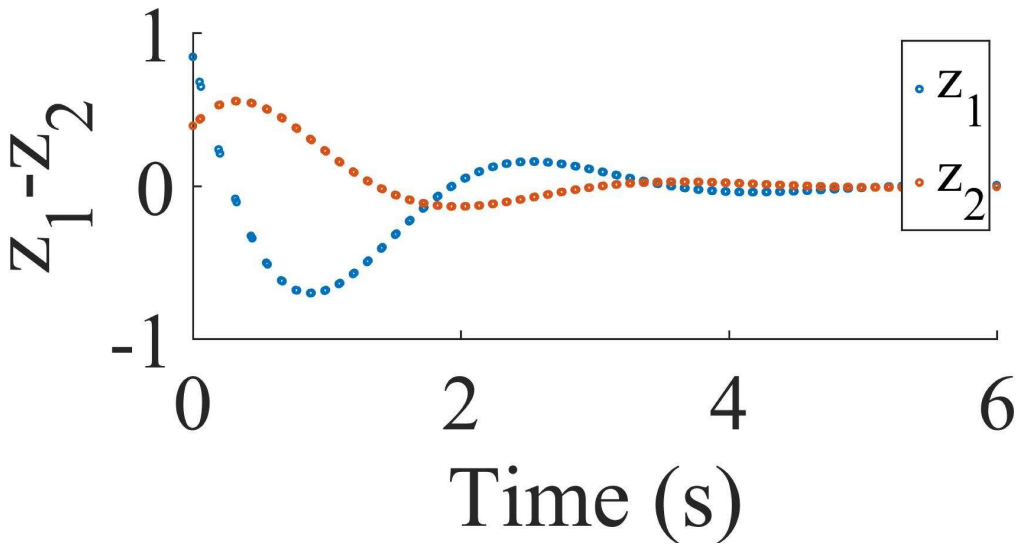


FIGURE 3.3: *State trajectories for Case–b*

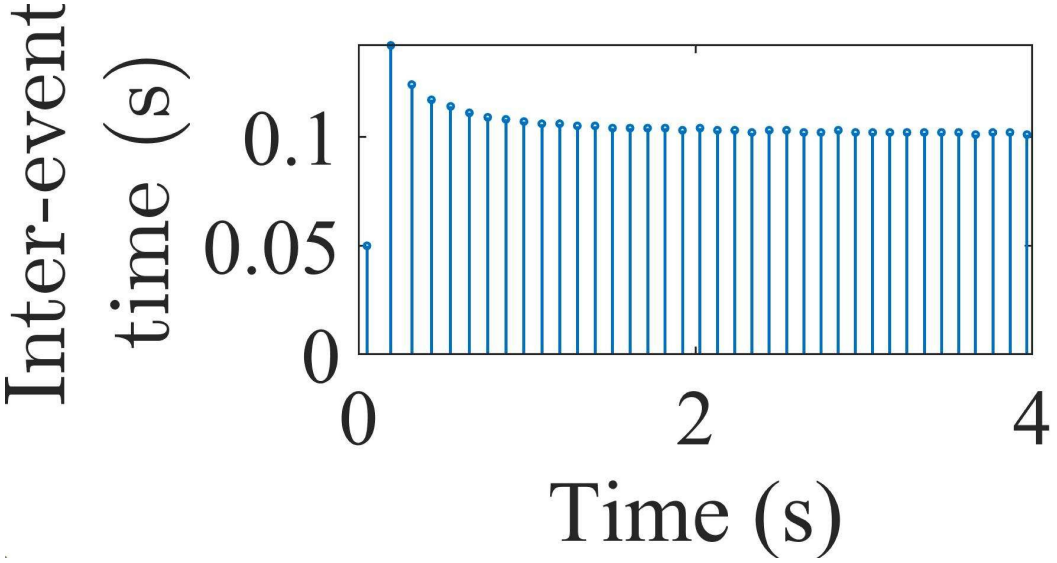


FIGURE 3.4: *Inter–event–time for Case–b*

stable Lyapunov functions on measure chains. Two theorems and a corollary are stated in this chapter from which we eventually obtain the expression of event–triggering which ensure asymptotic stability for various classes of systems be it nonlinear, Lipschitz nonlinear or linear systems. Numerical simulations for systems on various measure chains justify the design procedure elucidated in this chapter. As we have seen in this chapter that the event–triggered controller designed depends on the value of Lipschtiz constant. Future research can be carried

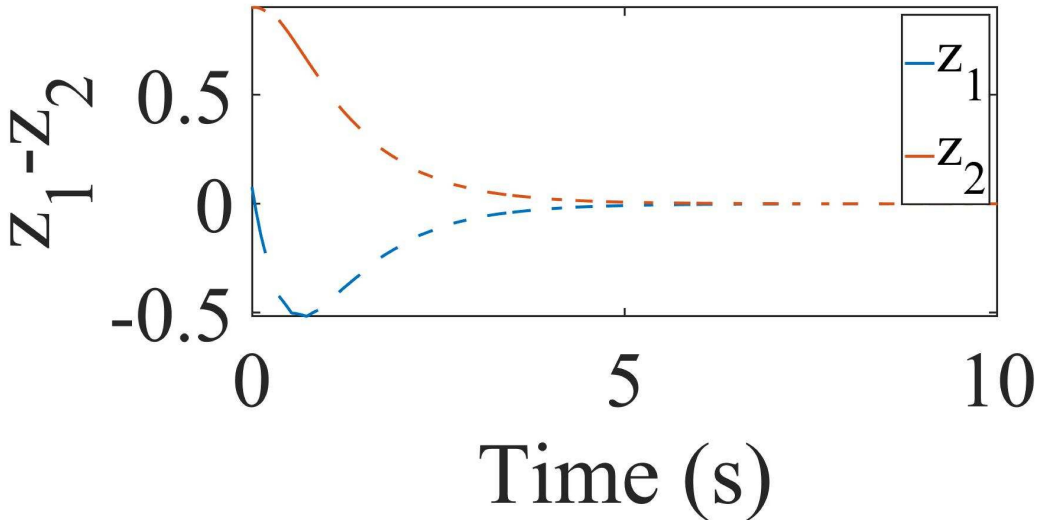


FIGURE 3.5: *State trajectories for Case–c*

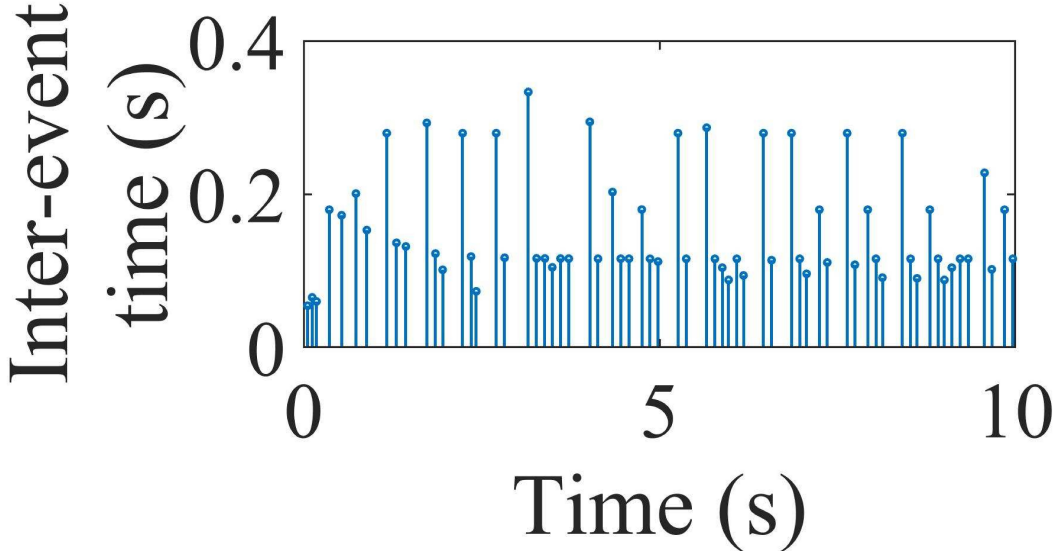


FIGURE 3.6: *Inter–event–time for Case–c*

out considering broad class of nonlinearities (see Remark 2). Future research will include extension of the results to observers and output feedback event–triggered controllers considering other triggering mechanisms like periodic event–triggering mechanism which is inherently devoid of Zeno phenomenon a section of which will be explained in the next chapter.

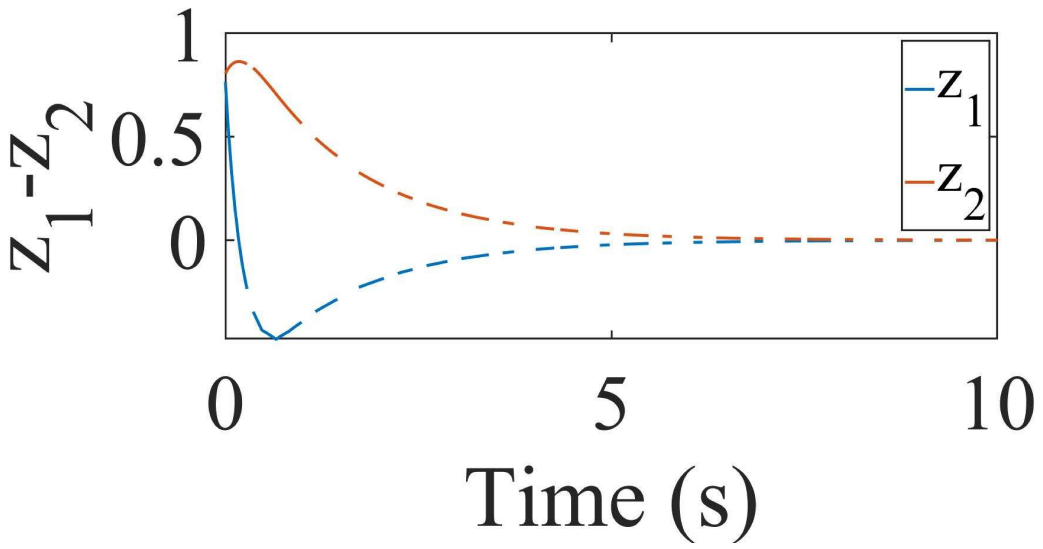


FIGURE 3.7: *State trajectories for Case–d*

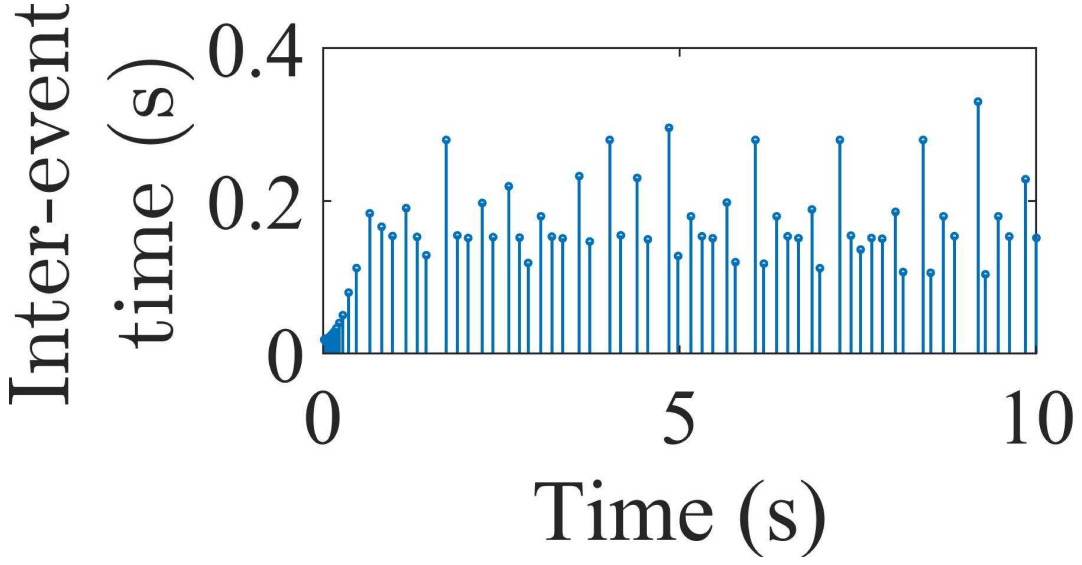


FIGURE 3.8: *Inter–event–time for Case–d*

Chapter 4

Periodic event–triggering for a class of systems on measure chains

In the previous chapter, we have addressed the problem of event–triggering on arbitrary time domains using a method wherein the event condition is checked at any time instant on the measure chain over which the system is defined. However, we know that such a design method can be susceptible to the problem of infinite sampling within a finite duration of time if the event condition is not designed with proper care in regions where the graininess function is zero (i.e. in homogeneous measure chain $\mathbb{R}$ and continuous portions of the non–homogeneous measure chains like $\mathbb{P}_{a,b}$).Some ways of avoiding the Zeno phenomenon are by using the time regularization method (Heemels et al. (2015a)) and the periodic event triggering method (Heemels et al. (2012a)). During the last decade, research in the domain of periodic event–triggering has gained prominence in the literature which is also easy to implement on digital hardware in comparison to its continuous counterpart Wang et al. (2019), Heemels et al. (2015b). Furthermore, in the past few decades, a lot of results have appeared in the field of impulsive observers (Ferrante et al. (2016), Etienne et al. (2017)). For linear systems in (Chen et al. (2014)), an output feedback controller is designed using impulsive observer with the help of time–varying Lyapunov functions. Whereas for nonlinear systems in (Jaramillo

et al. (2020)), an impulsive observer is designed for a class of Lipschitz nonlinear systems with time–varying uncertainties. Thus, in this chapter we address the problem of periodic event–triggered controller as well as output feedback periodic event–triggered controllers using impulsive observers. The problem statement is given in section 1 of this chapter, whereas the numerical simulations are given in section 2 of this chapter. The chapter concludes with the section 3.

4.1 Problem Statement

The class of nonlinear systems under consideration on an arbitrary measure chain $\mathbb{T}$ is given by

$$\begin{aligned} \overset{\Delta}{x} &= \mathcal{A}x + \mathcal{B}u + \mathcal{D}\phi(\mathcal{H}x) \\ y &= \mathcal{C}x \end{aligned} \tag{4.1}$$

where the state matrix, control matrix and output matrix are given by $\mathcal{A} \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$, $\mathcal{B} \in \mathbb{R}^{n\times m}$ and $\mathcal{C} \in \mathbb{R}^{w\times n}$ respectively. The matrices $\mathcal{H} \in \mathbb{R}^{s\times n}$ and $\mathcal{D} \in \mathbb{R}^{n\times e}$ are used to formulate the nonlinearity present in Eq. (4.1). The state, control and output matrices are given by $x \in \mathbb{R}^n$, $u \in C_{rd}(\mathbb{T}, \mathbb{R}^m)$ and $y \in \mathbb{R}^w$ respectively. The nonlinear function $\phi : \mathbb{R}^s \to \mathbb{R}^e$ is assumed to be rd-continuous and Lipschitz thus satisfying the relation,

$$\|\phi(s_1) - \phi(s_2)\| \leq \mathcal{L}_\gamma \|s_1 - s_2\| \tag{4.2}$$

for a positive constant $\mathcal{L}_\gamma$. Also $\phi(0) = 0$. When $\mathbb{T} = \mathbb{R}$, Eq. (4.1) transforms to $\dot{x} = \mathcal{A}x + \mathcal{B}u + \mathcal{D}\phi(\mathcal{H}x)$, whereas when $\mathbb{T} = \mathbb{Z}$ it becomes $x(t+1) - x(t) = \mathcal{A}x(t) + \mathcal{B}u(t) + \mathcal{D}\phi(\mathcal{H}x(t))$ and for any other continuous–discrete measure chains, the system dynamics is a combination of aforementioned system dynamics.

Assumption 1: The nominal system $(\mathcal{A}, \mathcal{B})$ is assumed to be stabilizable.

Assumption 2: The nominal system $(\mathcal{A}, \mathcal{C})$ is assumed to be detectable.

The sensors and actuators may be updated periodically. The controller updating period is considered as $\varphi = t^{j+1} - t^j, \forall\ t^j \in \mathbb{T},\ j \in \mathbb{N}^{0+}$ whereas the observer updating period is considered as $\beta = t^{k+1} - t^k, \forall\ t^k \in \mathbb{T},\ k \in \mathbb{N}^{0+}$. A pictorial

representation of the closed–loop system with controller and observer updating instances are given in Fig. 4.1. The vertical dashed lines in Fig. 4.1(b), represents the instances t^j where the event–triggering condition is checked on the actuator side whereas t^k represents the instances where the event–condition is checked on the sensor side. Also, the measure chain is non–homogeneous so its graininess function $\mu(t)$ is varying all throughout the measure chain. The value of $\mu(t)$ in the discrete section is highlighted in the figure. In this chapter first a periodic event triggered controller on measure chains for system given by Eq. (4.1) is designed considering controller updates under consideration and avoiding updates in the observer direction. Then a periodic event–triggering impulsive observer is designed for the same system considering observer updates under consideration and finally a periodic event–triggering output feedback controller is designed by combining the design procedures for periodic event–triggering controller and impulsive observer thereby validating the separation principle for the class of nonlinear systems on measure chains which we study in this chapter.

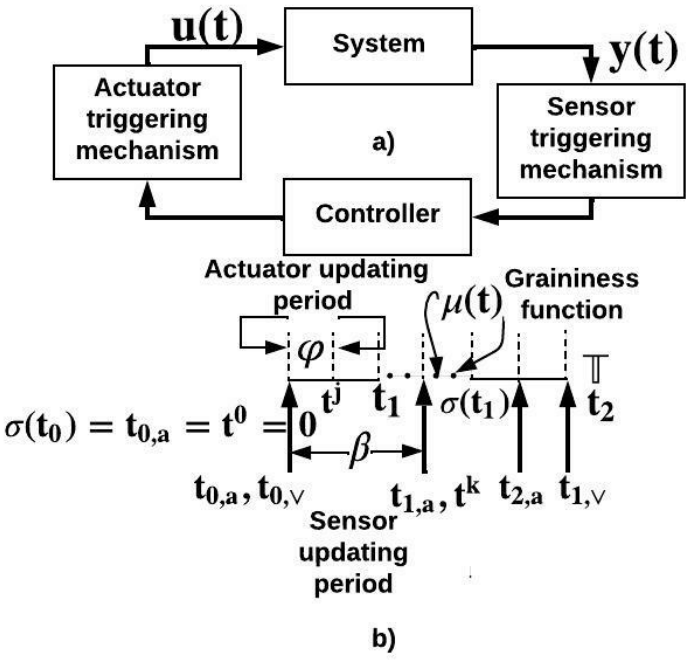


FIGURE 4.1: *a) Closed–loop representation with output feedback periodic event–triggering, b) Time stamps on a given measure chain $\mathbb{P}_{a,b} \bigcup \mathbb{T}_{iso}$ with the instances of event–triggering on actuator and sensor side.*

4.1.1 Periodic event–triggering controller on measure chains

In this section, a periodic event–triggering controller is designed for the system given in Eq. (4.1) considering controller updates only. The designed control input is updated at certain time instants $t_{m,a}$, which is a subset of the periodic time instants determined by a periodic event triggering rule,

$$u(t) = Kx(t_{m,a}), \forall\ t \in [t_{m,a}, t_{m+1,a}) \cap \mathbb{T},\ m \in \mathbb{N}^{0+} \tag{4.3}$$

with $K \in C_{rd}(\mathbb{T}, \mathbb{R}^{m\times n})$. K can be designed in such a way that the nominal closed–loop system is Hilger stabilizable. The periodic event–triggering condition considered in this chapter is

$$t_{m+1,a} = \inf\{j\varphi > t_{m,a} | \|Kx(j\varphi) - Kx(t_{m,a})\| \geq \varepsilon \|x(j\varphi)\|\} \tag{4.4}$$

where ε is a positive design constant. Thus, using the system representation given in Eq. (4.1), along with the control input given in Eq. (4.3) and the event–triggering condition given in Eq. (4.4) one can easily write the closed–loop system dynamics as,

$$\begin{aligned}
\overset{\Delta}{x}(t) &= \mathcal{A}x(t) + \mathcal{B}Kx(t_{m,a}) + \mathcal{D}\phi(\mathcal{H}x(t)),\ \forall\ t \in [j\varphi, (j+1)\varphi) \cap \mathbb{T} \\
&= \mathcal{A}x(t) + \mathcal{B}K(x(j\varphi)) + \mathcal{D}\phi(\mathcal{H}x(t)) + \mathcal{B}K(x(t_{m,a}) - x(j\varphi)) \\
&= \mathcal{A}x(t) + \mathcal{B}K(x(j\varphi)) + \mathcal{D}\phi(\mathcal{H}x(t)) + \mathcal{B}Ke(t) \\
&= (\mathcal{A} + \mathcal{B}K)x(t) + \mathcal{B}Kd_s(t) + \mathcal{D}\phi(\mathcal{H}x(t)) + \mathcal{B}Ke(t)
\end{aligned} \tag{4.5}$$

with $d_s(t) = x(j\varphi) - x(t)$ and $e(t) = x(t_{m,a}) - x(j\varphi)$. From, the closed–loop system dynamics given in Eq. (4.5), and the Lipschitz condition given in Eq. (4.2), one obtains with the help of norm on measure chains

$$\begin{aligned}
\|\overset{\Delta}{x}\| \leq& (\|\mathcal{A} + \mathcal{B}K\| + \mathcal{L}_\gamma \|\mathcal{D}\| \|\mathcal{H}\| + \varepsilon\|\mathcal{B}\|)\|x\| + (\|\mathcal{B}K\| + \varepsilon\|\mathcal{B}\|)\|d_s\| \\
&\leq \Pi(\|x\| + \|d_s\|)
\end{aligned} \tag{4.6}$$

where $\Pi = \max((\|\mathcal{A}+\mathcal{B}K\| + \mathcal{L}_\gamma\|\mathcal{D}\|\|\mathcal{H}\| + \varepsilon\|\mathcal{B}\|), (\|\mathcal{B}K\| + \varepsilon\|\mathcal{B}\|)$.

4.1.1.1 Asymptotic stability under periodic event–triggered control

In this section, a theorem is given which ensures global asymptotic stability of the closed–loop system on measure chains under the influence of the designed periodic event–triggered control law given in Eq. (4.3) and event condition given by Eq. (4.4).

Assumption 3: The measure chain $\mathbb{T}$ under consideration has a bounded maximum graininess function μ_m wherein for homogeneous measure chains (e.g. $\mathbb{T} = \mathbb{R}$, $\mathbb{Z}$ etc) we have $\mu(t) = \mu_m$ and for non–homogeneous measure chains (e.g. $\mathbb{T} = \mathbb{P}_{a,b}$, $\mathbb{P}_{a,b} \cup \mathbb{T}_{iso}$ etc) we obtain $\mu(t) \leq \mu_m$.

Theorem 1: The closed–loop system given in Eq. (4.5) satisfying Assumption 1, with the periodic event–triggering control law given in Eq. (4.3) and event condition given in Eq. (4.4) is asymptotically stable if there exist positive definite symmetric matrix $\mathcal{P} = \mathcal{P}^T$, an appropriately designed ε, as well as positive regressive constants $\varepsilon_{c,i}$ $(i = 1, 2, 3)$, such that for any given measure chain $\mathbb{T}$ satisfying Assumption 3, the following LMIs are satisfied simultaneously

Condition a:

$$\mathcal{T}_i = \begin{bmatrix} \mathcal{T}_{ai} & \mathcal{T}_{bi} \\ \mathcal{T}_{ci} & \mathcal{T}_{di} \end{bmatrix} \leq 0, \ \forall \ i = 1, 2, 3 \tag{4.7}$$

with $\mathcal{T}_{ai} = (1/3)((\mathcal{A}+\mathcal{B}K)^T\mathcal{P} + \mathcal{P}(\mathcal{A}+\mathcal{B}K) + \mu(t)(\mathcal{A}+\mathcal{B}K)^T\mathcal{P}(\mathcal{A}+\mathcal{B}K) + \mathcal{L}_\gamma^2\mathcal{H}^T\mathcal{H} + 2\mathcal{L}_\gamma^2\mu(t)^2\mathcal{H}^T\mathcal{H} + \mathcal{L}_\gamma^2\mu(t)^2\mathcal{H}^T\mathcal{D}^T\mathcal{D}\mathcal{H} + \mu(t)\mathcal{L}_\gamma^2\mathcal{H}^T\mathcal{D}^T\mathcal{P}\mathcal{D}\mathcal{H} + I + \mu(t)^2(\mathcal{A}+\mathcal{B}K)^T(\mathcal{A}+\mathcal{B}K) + \varepsilon_{c,1}\mathcal{P})$, $\mathcal{T}_{b1} = \mathcal{T}_{c1}^T = \mathcal{P}\mathcal{D}$, $\mathcal{T}_{b2} = \mathcal{T}_{c2}^T = (\mathcal{A}+\mathcal{B}K)^T\mathcal{P}\mathcal{D}$, $\mathcal{T}_{b3} = \mathcal{T}_{c3}^T = \mathcal{P} + \mu(t)\mathcal{P}(\mathcal{A}+\mathcal{B}K)$ and $\mathcal{T}_{di} = -I$.

Condition b:

$$\mathcal{Z}_i = \begin{bmatrix} \mathcal{Z}_{ai} & \mathcal{Z}_{bi} \\ \mathcal{Z}_{ci} & \mathcal{Z}_{di} \end{bmatrix} \leq 0, \ \forall \ i = 1, 2 \tag{4.8}$$

with $\mathcal{Z}_{ai} = (1/2)(\mu(t)K^T\mathcal{B}^T\mathcal{P}\mathcal{B}K - \varepsilon_{c,2}\mathcal{P})$, $\mathcal{Z}_{b1} = \mathcal{Z}_{c1}^T = (\mathcal{B}K)^T\mathcal{P}\mathcal{D}$, $\mathcal{Z}_{b2} = \mathcal{Z}_{c2}^T = \sqrt{3}K^T\mathcal{B}^T\mathcal{P}$, and $\mathcal{Z}_{di} = -I$.
Condition c:

$$\mathcal{U} = \begin{bmatrix} \mathcal{U}_a & \mathcal{U}_b \\ \mathcal{U}_c & \mathcal{U}_d \end{bmatrix} \leq 0, \tag{4.9}$$

with $\mathcal{U}_a = \mu(t)K^T\mathcal{B}^T\mathcal{P}\mathcal{B}K + (\mathcal{B}K)^T\mathcal{B}K + \mu(t)^2(\mathcal{B}K)^T\mathcal{B}K - \varepsilon_{c,3}\mathcal{P}$, $\mathcal{U}_b = \mathcal{U}_c^T = K^T\mathcal{B}^T\mathcal{P}$ and $\mathcal{U}_d = -I$. ■

The corresponding maximum value of the periodic controller updating period φ is denoted as φ_{max}. So as long as $\varphi \leq \varphi_{max}$ the closed–loop system is globally asymptotically stable. Three different cases are considered below which allows us to evaluate φ_{max} for any candidate measure chain $\mathbb{T}$. Considering $\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})\Lambda}{\varepsilon_{c,3}\lambda_{max}(\mathcal{P})} = \Gamma$,

1. Case a: When $\mu(t) = 0$,

$$\varphi \leq \varphi_{max}(\mu(t) = 0) = (1 - \frac{1}{1 + \Gamma^{0.5}})\frac{1}{\Pi} \tag{4.10}$$

2. Case b: When $\mu(t) \neq 0$, it is obtained by finding the maximum value of $t > t^j$ such that the following inequality is satisfied,

$$\sum_{s=t^j}^{t} \frac{(\frac{\pi}{4} + \frac{2+\Pi\mu(t)}{4\mu(t)}\eta(s)^{0.5} + \frac{\Pi}{4}\eta^{1.5}(s))}{(1 + \frac{\Pi\mu(t)+2}{4})e_{\frac{\Pi\mu(t)+2}{4\mu(t)}}(s,t)}\mu(t) \leq \Gamma \tag{4.11}$$

   It is given by $\varphi_{max}(\mu(t) \neq 0) = t - t^j$.

3. Case c: In case of measure chains which are both continuous and discrete in nature as pictorially represented in Fig. 4.1, the value of maximum actuator updating period φ_{max} is obtained by finding the minimum value of $t > t^j$,

where the following inequality is satisfied,

$$\sum_{y=1}^{n}\sum_{s=t_y}^{\sigma(t_y)}\frac{(\frac{\pi}{4}+\frac{2+\Pi\mu(t)}{4\mu(t)}\eta(s)^{0.5}+\frac{\Pi}{4}\eta^{1.5}(s))}{(1+\frac{\Pi\mu(t)+2}{4})e_{\frac{\Pi\mu(t)+2}{4\mu(t)}}(s,t)}\mu(t)+\sum_{y=1}^{n-1}\int_{s=\sigma(t_y)}^{t_{y+1}}2\Pi(\sqrt{\eta}+\eta)(1+$$
$$\sqrt{\eta})ds+\int_{s=\sigma(t_n)}^{t}2\Pi(\sqrt{\eta}+\eta)(1+\sqrt{\eta})ds+\int_{s=t^j}^{t_1}2\Pi(\sqrt{\eta}+\eta)(1+\sqrt{\eta})ds\leq\Gamma \tag{4.12}$$

Similar to Case b the expression of φ_{max} is given by $\varphi_{max}=t-t^j$. The ε is selected in such a way that $\mathbb{F}_{\mathbb{T}}<1$ for any given $\mathbb{T}$.

Proof: Let us consider the Lyapunov function candidate $V(t)=x(t)^T\mathcal{P}x(t),\ \forall\ t\in[j\varphi,(j+1)\varphi)\cap\mathbb{T}$ where $\mathcal{P}=\mathcal{P}^T>0$. Delta-differentiating the Lyapunov functional candidate and using the system dynamics given in Eq. (4.1) we get,

$$\begin{aligned}
\overset{\Delta}{V}&=(\overset{\Delta}{x})^T\mathcal{P}x+(x^T+\mu(t)(\overset{\Delta}{x})^T)\mathcal{P}\overset{\Delta}{x}\\
&=((\mathcal{A}+\mathcal{B}K)x+\mathcal{B}K(x(j\varphi)-x)+\mathcal{D}\phi(\mathcal{H}x)+\mathcal{B}Ke)^T\mathcal{P}x+x^T\mathcal{P}((\mathcal{A}+\mathcal{B}K)x\\
&+\mathcal{B}K(x(j\varphi)-x)+\mathcal{D}\phi(\mathcal{H}x)+\mathcal{B}Ke)+\mu(t)((\mathcal{A}+\mathcal{B}K)x+\mathcal{B}K(x(j\varphi)-x)+\\
&\mathcal{D}\phi(\mathcal{H}x)+\mathcal{B}Ke)^T\mathcal{P}((\mathcal{A}+\mathcal{B}K)x+\mathcal{B}K(x(j\varphi)-x)+\mathcal{D}\phi(\mathcal{H}x)+\mathcal{B}Ke)\\
&=x^T((\mathcal{A}+\mathcal{B}K)^T\mathcal{P}+\mathcal{P}(\mathcal{A}+\mathcal{B}K)+\mu(t)(\mathcal{A}+\mathcal{B}K)^T\mathcal{P}(\mathcal{A}+\mathcal{B}K))x++2(\phi(\mathcal{H}x))^T\\
&\mathcal{D}^T\mathcal{P}x+2x^T\mathcal{P}\mathcal{B}Ke+\mu(t)(\mathcal{D}\phi(\mathcal{H}x))^T\mathcal{P}(\mathcal{D}\phi(\mathcal{H}x))+\mu(t)(\mathcal{B}Ke)^T\mathcal{P}\mathcal{B}Ke+2\mu(t)\\
&(\phi(\mathcal{H}x))^T\mathcal{D}^T\mathcal{P}(\mathcal{A}+\mathcal{B}K)x+2\mu(t)x^T(\mathcal{A}+\mathcal{B}K)^T\mathcal{P}\mathcal{B}Ke+2\mu(t)(\phi(\mathcal{H}x))^TD^T\mathcal{P}\mathcal{B}Ke\\
&+d_s^TK^T\mathcal{B}^T(2\mathcal{P}+2\mu(t)\mathcal{P}(\mathcal{A}+\mathcal{B}K))x+\mu(t)d_s^T(\mathcal{B}K)^T\mathcal{P}\mathcal{B}Kd_s+2\mu(t)d_s^TK^T\mathcal{B}^T\mathcal{P}x\\
&\mathcal{D}\phi(\mathcal{H}x)+2\mu(t)d_s^TK^T\mathcal{B}^T\mathcal{P}\mathcal{B}Ke
\end{aligned}\tag{4.13}$$

With the help of Eq. (4.2), the following set of inequalities can be considered

$$2\phi(\mathcal{H}x)^T\mathcal{D}^T\mathcal{P}x\leq\mathcal{L}_\gamma^2x^T\mathcal{H}^T\mathcal{H}x+x^T\mathcal{P}\mathcal{D}\mathcal{D}^T\mathcal{P}x \tag{4.14}$$

$$\begin{aligned}
&2\mu(t)(\phi(\mathcal{H}x)^T\mathcal{D}^T\mathcal{P}\mathcal{B}Ke\leq\mathcal{L}_\gamma^2\mu(t)^2x^T\mathcal{H}^T\mathcal{H}x+e^TK^T\mathcal{B}^T\\
&\mathcal{P}\mathcal{D}\mathcal{D}^T\mathcal{P}\mathcal{B}Ke
\end{aligned}\tag{4.15}$$

$$\mu(t)(\mathcal{D}\phi(\mathcal{H}x))^T\mathcal{P}(\mathcal{D}\phi(\mathcal{H}x)) \leq \mu(t)\mathcal{L}_\gamma^2 x^T\mathcal{H}^T\mathcal{D}^T\mathcal{P}\mathcal{D}\mathcal{H}x \tag{4.16}$$

$$2\mu(t)x^T(\mathcal{A}+\mathcal{B}K)^T\mathcal{P}\mathcal{B}Ke \leq \mu(t)^2x^T(\mathcal{A}+\mathcal{B}K)^T(\mathcal{A}+\mathcal{B}K)x + e^TK^T\mathcal{B}^T\mathcal{P}\mathcal{P}\mathcal{B}Ke \tag{4.17}$$

$$2\mu(t)(\phi(\mathcal{H}x))^T\mathcal{D}^T\mathcal{P}(\mathcal{A}+\mathcal{B}K)x \leq \mathcal{L}_\gamma^2\mu^2(t)x^T\mathcal{H}^T\mathcal{H}x + x^T(\mathcal{A}+\mathcal{B}K)^T\mathcal{P}\mathcal{D}\mathcal{D}^T\mathcal{P}(\mathcal{A} + \mathcal{B}K)x \tag{4.18}$$

$$2\mu(t)(\phi(\mathcal{H}x)^T\mathcal{D}^T\mathcal{P}\mathcal{B}Kd_s \leq \mathcal{L}_\gamma^2\mu(t)^2x^T\mathcal{H}^T\mathcal{D}^T\mathcal{D}\mathcal{H}x + d_s^TK^T\mathcal{B}^T\mathcal{P}\mathcal{P}\mathcal{B}Kd_s \tag{4.19}$$

$$2x^T\mathcal{P}\mathcal{B}Ke \leq x^Tx + e^T(\mathcal{B}K)^T\mathcal{P}\mathcal{P}\mathcal{B}Ke \tag{4.20}$$

$$d_s^TK^T\mathcal{B}^T(2\mathcal{P}+2\mu(t)\mathcal{P}(\mathcal{A}+\mathcal{B}K))x \leq d_s^TK^T\mathcal{B}^TBKd_s + x^T(\mathcal{P}+\mu(t)\mathcal{P}(\mathcal{A}+\mathcal{B}K))^T (\mathcal{P}+\mu(t)\mathcal{P}(\mathcal{A}+\mathcal{B}K))x \tag{4.21}$$

$$2\mu(t)d_s^TK^T\mathcal{B}^T\mathcal{P}\mathcal{B}Ke \leq \mu(t)^2d_s^TK^T\mathcal{B}^T\mathcal{B}Kd_s + e^TK^T\mathcal{B}^T\mathcal{P}\mathcal{P}\mathcal{B}Ke \tag{4.22}$$

Therefore, with the help of conditions a-c of Theorem 1,

$$\overset{\Delta}{V} \leq -\varepsilon_{c,1}\lambda_{min}(\mathcal{P})\|x\|^2 + \varepsilon_{c,2}\lambda_{max}(\mathcal{P})\|e\|^2 + \varepsilon_{c,3}\lambda_{max}(\mathcal{P})\|d_s\|^2 \tag{4.23}$$

Therefore with the help of a constant $\Lambda \in (0,1)$ one can write,

$$\overset{\Delta}{V} \leq -\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)\|x\|^2 + \varepsilon_{c,2}\lambda_{max}(\mathcal{P})\|e\|^2 + (\varepsilon_{c,3}\lambda_{max}(\mathcal{P})\|d_s\|^2 - \varepsilon_{c,1}\lambda_{min}(\mathcal{P})\Lambda\|x\|^2) \tag{4.24}$$

Let us define,

$$\eta = \frac{\|d_s\|^2}{\|x\|^2} \leq \frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})\Lambda}{\varepsilon_{c,3}\lambda_{max}(\mathcal{P})} = \Gamma \tag{4.25}$$

There exist two cases, one when $\mu(t) = 0$ and other one when $\mu(t) \neq 0$.

1. Case a: When $\mu(t) = 0$, the delta differentiation of Eq. (4.25) yields

$$\begin{aligned}\dot{\eta} &= \frac{-2d_s^T\dot{x}x^Tx - 2d_s^Td_sx^T\dot{x}}{x^Txx^Tx} = -\frac{2d_s^T\dot{x}}{x^Tx} - \frac{2\eta x^T\dot{x}}{x^Tx} \\ &\leq 2\Pi(\sqrt{\eta}+\eta)(1+\sqrt{\eta}) \end{aligned} \tag{4.26}$$

   Thus, using the comparison lemma Tabuada (2007), one can obtain the minimum time bound by integrating Eq. (4.26) over the time interval t^j to t and using Eq. (4.25) as well as the condition $\eta(t^j) = 0$.

$$\begin{aligned}&\frac{1}{1-\Pi(t-t_j)} - 1 \leq \Gamma^{0.5} \\ &t \leq t^j + (1 - \frac{1}{1+\Gamma^{0.5}})\frac{1}{\Pi}\end{aligned} \tag{4.27}$$

2. Case b: When $\mu(t) \neq 0$, the delta differentiation of Eq. (4.25) with the help of Definition 5 yields,

$$\begin{aligned}\overset{\Delta}{\eta} &= \frac{-2(\overset{\Delta}{x})^T(d_s+\eta x) + \mu(t)(\overset{\Delta}{x})^T\overset{\Delta}{x}(1-\eta)}{(x+\mu(t)(\overset{\Delta}{x}))^2} \\ &\leq \frac{2\|\frac{\overset{\Delta}{x}}{x}\|(\|\frac{d_s}{x}\|+\eta) + \mu(t)\|\frac{\overset{\Delta}{x}}{x}\|\|\frac{\overset{\Delta}{x}}{x}\|(1+\eta)}{\|(1+\mu(t)\frac{\overset{\Delta}{x}}{x})^2\|} \\ &\leq (1+\sqrt{\eta})(2\sqrt{\eta}+\Pi\mu(t)(1+\eta))\frac{1}{4\mu(t)} \\ &\leq \frac{\Pi + (\frac{2}{\mu(t)}+\Pi)\sqrt{\eta} + (\Pi+\frac{2}{\mu(t)})\eta + \Pi\eta^{1.5}}{4}\end{aligned} \tag{4.28}$$

   Using Lemma 1 and definitions given in the mathematical preliminaries section, the closed–loop expression of solution to Eq. (4.28) can be written as,

$$\int_{t^j}^{t} \frac{(\frac{\pi}{4} + \frac{2+\Pi\mu(t)}{4\mu(t)}\eta(\tau)^{0.5} + \frac{\Pi}{4}\eta^{1.5}(\tau))}{(1+\frac{\Pi\mu(t)+2}{4})e_{\frac{\Pi\mu(t)+2}{4\mu(t)}}(\tau,t)}\Delta\tau \leq \Gamma \tag{4.29}$$

Thus, for discrete time scale $\mathbb{T} = \mu(t)\mathbb{Z}$, Eq. (4.29) translates to,

$$\sum_{s=t^j}^{t} \frac{(\frac{\pi}{4} + \frac{2+\Pi\mu(t)}{4\mu(t)}\eta(s)^{0.5} + \frac{\Pi}{4}\eta^{1.5}(s))}{(1 + \frac{\Pi\mu(t)+2}{4})e_{\frac{\Pi\mu(t)+2}{4\mu(t)}}(s,t)}\mu(t) \leq \Gamma \tag{4.30}$$

From, Eq. (4.30) one obtains the value of $\varphi_{max} = t - t^j$ when $\mu(t) \neq 0$ using the fact that $\eta(t^j) = 0$.

$$\overset{\Delta}{V} \leq -\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)\|x\|^2 + \frac{\varepsilon_{c,2}\varepsilon\lambda_{max}(\mathcal{P})}{K^TK}\|x(j\varphi)\|^2 \tag{4.31}$$

Thus using $\lambda_{min}(\mathcal{P})\|x\|^2 \leq V \leq \lambda_{max}(\mathcal{P})\|x\|^2$ one can write

$$\begin{aligned}
\overset{\Delta}{V} &\leq -\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}V + \frac{\varepsilon_{c,2}\varepsilon\lambda_{max}(\mathcal{P})}{K^TK\lambda_{min}(\mathcal{P})}V(j\varphi) \\
V((j+1)\varphi) &\leq \lim_{t\to(j+1)\varphi}\left(e_{-\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}}(t,j\varphi) + \frac{\varepsilon_{c,2}\varepsilon\lambda_{max}(\mathcal{P})}{K^TK\lambda_{min}(\mathcal{P})}\right. \\
&\quad \left.\int_{j\varphi}^{t}\frac{e_{-\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}}(t,\tau)}{1-\mu(t)\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}}\Delta\tau\right)V(j\varphi) \\
&\leq \mathbb{F}_{\mathbb{T}}V(j\varphi)
\end{aligned} \tag{4.32}$$

Assuming the choice of Λ ensures, $-\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})} \in \mathcal{R}(\mathbb{T},\mathbb{R})$. Furthermore if the value of ε is chosen in such a way that $\mathbb{F}_{\mathbb{T}} < 1$, then it ensures global asymptotic stability for closed–loop system on any candidate measure chain $\mathbb{T}$, assuming that the value of $\mu(t)$ is constant between two periodic instances where the event–triggered condition is checked. ■

As examples one can consider two cases, $\mathbb{T} = \mu(t)\mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$. When $\mathbb{T} = \mu(t)\mathbb{Z}$, Eq. (4.32) can be written as

$$V((j+1)\varphi) \leq \mathbb{F}_{\mathbb{T}=\mu(t)\mathbb{Z}}V(j\varphi) \tag{4.33}$$

where $\mathbb{F}_{\mathbb{T}=\mu(t)\mathbb{Z}} = \lim_{t\to(j+1)\varphi}((1-\mu(t)\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})})^{\frac{t-j\varphi}{\mu(t)}} + \frac{\varepsilon_{c,2}\varepsilon\lambda_{max}(\mathcal{P})}{K^TK\lambda_{min}(\mathcal{P})}\sum_{j\varphi}^{t}(1-\mu(t)\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})})^{\frac{t-j\varphi}{\mu(t)}-1}\mu(t))$. Whereas when $\mathbb{T} = \mathbb{R}$, Eq. (4.32) can be written

as

$$V((j+1)\varphi) \leq \mathbb{F}_{\mathbb{T}=\mathbb{R}} V(j\varphi) \tag{4.34}$$

where $\mathbb{F}_{\mathbb{T}=\mathbb{R}} = \lim_{t\to(j+1)\varphi}(e^{-\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}(t-j\varphi)} + \frac{\varepsilon_{c,2}\varepsilon\lambda_{max}(\mathcal{P})}{K^T K\lambda_{min}(\mathcal{P})} \frac{1-e^{-\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}(t-j\varphi)}}{\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}})$.

Remark 1: In order to find the maximum controller updating period φ_{max} for measure chain like $\mathbb{T} = \mathbb{P}_{a,b} \bigcup \mathbb{T}_{iso}$ which is given in Fig. 4.1(b), one uses the same method as adopted in Case c of Theorem 1 for $\mathbb{P}_{a,b}$ measure chain. Similarly in order to prove global asymptotic stability for measure chains $\mathbb{T} = \mathbb{P}_{a,b}$ or $\mathbb{T} = \mathbb{P}_{a,b} \bigcup \mathbb{T}_{iso}$, the respective limiting values of $\mathbb{F}_{\mathbb{T}=\mathbb{P}_{a,b}}$ or $\mathbb{F}_{\mathbb{T}=\mathbb{P}_{a,b} \bigcup \mathbb{T}_{iso}}$ can be easily calculated and the global asymptotic stability can be proved by ensuring that the values are less than one through proper choice of ε.

Remark 2: It is worth noting that the event condition proposed in this chapter ensures global asymptotic stability for $\mathbb{T} = \mathbb{R}$ in comparison to practical stability as proposed in (Etienne et al. (2017)). Furthermore, when $\mathbb{T} = \mathbb{R}$ ($\mu(t) = 0$) or $\mathbb{T} = \mathbb{Z}$ ($\mu(t) = 1$), we get the original system dynamics in continuous and discrete time domains according to measure chain theory thereby obtaining similar results which exist in the literature.

The pseudo code for designing periodic event–triggering controller is given below

- Design K using emulation based method such that the resulting system is Hilger stabilizable with respect to $\mathbb{T}$.
- Solve a system of matrix inequalities in order to calculate φ_{max}.
- The constant ε should take a value such that a condition

$$\lim_{t\to(j+1)\varphi} \left(e_{-\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}}(t, j\varphi) + \frac{\varepsilon_{c,2}\varepsilon\lambda_{max}(\mathcal{P})}{K^T K\lambda_{min}(\mathcal{P})} \int_{j\varphi}^{t} \frac{e_{-\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}}(t,\tau)}{1-\mu(t)\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}} \Delta\tau \right) < 1 \tag{4.35}$$

State estimation is an active field of research due to the fact that, in various applications of interest in science and engineering, it is not possible or it is expensive

to measure all the variables describing the dynamics of a process. Observers are excellent tools to overcome this difficulty. Since the seminal work of (Luenberger (1964)), many researches have investigated this issue leading to the design of observers both for linear and nonlinear systems. It has to be noted that these results assume a continuous measurement of the output. In contrast, the use of digital devices naturally implies that in many practical applications the only available measurements are at discrete—time instants. This aspect has to be appropriately taken into account when designing an observer. A discrete—time observer, designed using the sampled data system, may ensure stability of the estimation error in the sampling instants, for sufficiently small sampling times, but it does not guarantee that the estimation error will converge between sampling instants. To cope with this situation, recently the so called impulsive observers, which have an inherent sampled nature, have been introduced to design observers guaranteeing a desired performance also between sampling instants. Additionally, various impulsive observers have been proposed for continuous–time systems, see for instance (Andrieu and Nadri (2010), Dinh et al. (2014), Khaled et al. (2013), Farza et al. (2013)). On the other hand, in works as (Etienne et al. (2017), Chen et al. (2014),Raff and Allgower (2007)) using discontinuous Lyapunov functions, sufficient conditions for the existence of these observers can be derived in terms of Linear Matrix Inequalities (LMIs). However most of the works related to impulsive observers have only considered systems on continuous measure chain whereas in the next section we design a periodic event–triggered impulsive observer for systems on any arbitrary time domain not necessarily continuous time domain.

4.1.2 Periodic event–triggering impulsive observer on measure chains

In this section, a periodic event–triggering impulsive observer is designed for system whose dynamical representation is given in Eq. (4.1). The impulsive observer dynamics considered in this chapter is given by

$$\begin{aligned}\overset{\Delta}{\hat{z}}(t) &= \mathcal{A}z(t) + \mathcal{B}u(t) + \mathcal{D}\phi(\mathcal{H}z), \;\; \mathbb{T} \ni t \in (k\beta, (k+1)\beta] \\ z(k\beta^{+}) &= z(k\beta) + \beta\mathcal{G}(y(t_{l,\vee}) - \hat{y}(k\beta)), \;\; \mathbb{T} \ni t = k\beta,\, l \in \mathbb{N}^0\end{aligned}$$

$$\hat{y}(k\beta) = \mathcal{C}z(k\beta) \tag{4.36}$$

where $z \in \mathbb{R}^n$ and $\hat{y} \in \mathbb{R}^w$ are used to represent the observer state and estimated output respectively. The observer gain matrix is given by $\mathcal{G} \in \mathbb{R}^{n\times w}$. The event–triggering condition at the sensor end is given by

$$t_{l+1,\vee} = \inf\{k\beta > t_{l,\vee} | \|y(k\beta) - y(t_{l,\vee})\| \geq \varepsilon_\nu\} \tag{4.37}$$

where $t_{l,\vee} \forall\, l \in \mathbb{N}^{0+}$ represents the instances of periodic event–triggering in the sensor end and ε_ν is a positive design constant. The expression of observer error dynamics ($e_o = x - z$) is given by

$$\begin{aligned} \overset{\Delta}{e}_o(t) &= \mathcal{A}e_o(t) + \mathcal{D}(\phi(\mathcal{H}x(t)) - \phi(\mathcal{H}z(t))) \\ &\quad (t \in [t_{l,\vee}, t_{l+1,\vee}), t \neq k\beta, k = \mathbb{N}) \cap \mathbb{T} \end{aligned} \tag{4.38}$$

The impulsive error dynamics ($e_o(k\beta^+) = x(k\beta^+) - z(k\beta^+)$) is given by

$$e_o(k\beta^+) = (I - \beta\mathcal{G}\mathcal{C})e_o(k\beta) + \beta\mathcal{G}\mathcal{C}(x(k\beta) - x(t_{l,\vee})) \tag{4.39}$$

At the instances of triggering the error given in Eq. (4.39) reduces to $e_o(k\beta^+) = (I - \beta\mathcal{G}\mathcal{C})e_o(k\beta) = (I - \beta\mathcal{G}\mathcal{C})e_o(t_{l,\vee})$.

4.1.2.1 Practical exponential convergence of observer error dynamics under periodic event–triggered impulsive observer

In this section, a theorem is stated which ensures practical exponential convergence of the observer error obtained using the system dynamics given in Eq. (4.1), the impulsive observer dynamics given in Eq. (4.39) and event triggering rule in Eq. (4.37).

Theorem 2: For the system dynamics given in Eq. (4.1), and assuming Assumption 2 and 3 are satisfied, then there exist matrices $\mathcal{P}_i, \forall\, i = 1, 2, 3$ such that $\mathcal{P}_1 = \mathcal{P}_1^T$ and $\mathcal{P}_2 = \mathcal{P}_2^T$. The designed periodic event–triggered impulsive observer ensures closed–loop practical exponential convergence on a given measure chain $\mathbb{T}$ if the following set of LMI conditions are satisfied simultaneously

Condition a:

$$\mathcal{S}_i = \begin{bmatrix} \mathcal{X}_{ai} & \mathcal{X}_{bi} \\ \mathcal{X}_{ci} & \mathcal{X}_{di} \end{bmatrix} \leq -\varepsilon_u I,\ \forall\ i = 1, 2 \tag{4.40}$$

$$\mathcal{S}_3 = \begin{bmatrix} \mathcal{X}_e & \mathcal{X}_f \\ \mathcal{X}_g & \mathcal{X}_h \end{bmatrix} \leq 0 \tag{4.41}$$

where $\mathcal{X}_{ai}$, $\mathcal{X}_{bi}$, $\mathcal{X}_{ci}$, $\mathcal{X}_{di}$, $\mathcal{X}_e$, $\mathcal{X}_f$, $\mathcal{X}_g$ and $\mathcal{X}_h$ are given by

- $\mathcal{X}_{ai} = \mathcal{A}^T\mathcal{P}_i + \mathcal{P}_i\mathcal{A} + \mu(t)\mathcal{A}^T(\mathcal{P}_i + \mu(t)\frac{\mathcal{P}_2-\mathcal{P}_1}{\beta})\mathcal{A} + \frac{\mathcal{P}_2-\mathcal{P}_1}{\beta}(I + 2\mu(t)\mathcal{A}) + \mathcal{L}_\gamma^2\mathcal{H}^T\mathcal{H}$
- $\mathcal{X}_{ci} = (\mathcal{X}_{bi})^T = \mathcal{D}^T((\mathcal{P}_i + 0.5\mu(t)\mathcal{A}) + \mu(t)\frac{\mathcal{P}_2-\mathcal{P}_1}{\beta}(I + \mu(t)\mathcal{A}) + 0.5\mu(t)\mathcal{P}_i\mathcal{A})$
- $\mathcal{X}_{di} = \mu(t)\mathcal{D}^T(\mathcal{P}_i + \mu(t)\frac{(\mathcal{P}_2-\mathcal{P}_1)}{\beta})\mathcal{D} - I$
- $\mathcal{X}_e = -\mathcal{P}_2$
- $\mathcal{X}_f = (\mathcal{X}_g)^T = (\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2 - \mathcal{P}_1) - \beta\mathcal{P}_3C)$
- $\mathcal{X}_h = -(\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2 - \mathcal{P}_1))$

ε_u and $\varepsilon_{u,1}$ are constant positive regressive functions.

Condition b: Only those measure chains are admissible where the value of exponential function $(e_{-\varepsilon_{u,1}}(t, t_0))$ satisfies

$$0 < (e_{-\varepsilon_{u,1}}(t, t_0)) < 1, \forall\ \mathbb{T} \ni t, t_0 \in (k\beta, (k+1)\beta] \ \blacksquare \tag{4.42}$$

The expression of observer gain constant is given by $\mathcal{G} = (\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2 - \mathcal{P}_1))^{-1}\mathcal{P}_3$.

Proof: Using Eqs. (4.38) and (4.39) we can write

$$\begin{aligned} \overset{\Delta}{e}_o &= \mathcal{A}e_o + \mathcal{D}\mathcal{E}^1 \\ e_o(k\beta^+) &= (I - \beta\mathcal{G}\mathcal{C})e_o(k\beta) + \beta\mathcal{G}\mathcal{C}\mathcal{E}^2 \end{aligned} \tag{4.43}$$

where $(\phi(\mathcal{H}x) - \phi(\mathcal{H}z)) = \mathcal{E}^1$ and $(x(k\beta) - x(t_{l,\vee})) = \mathcal{E}^2$. Considering a time–varying Lyapunov function candidate on $\mathbb{T}$, $V_o = e_o^T\mathcal{P}(t)e_o$ where $\mathcal{P}(t)$ is given by

$$\mathcal{P}(t) = \mathcal{P}_1 + \frac{t - k\beta}{\beta}(\mathcal{P}_2 - \mathcal{P}_1)\ t \in (k\beta, (k+1)\beta] \cap \mathbb{T} \tag{4.44}$$

Also, the following property holds true, $\alpha_1(\|e_o\|) \leq V_o \leq \alpha_2(\|e_o\|)$ where α_1 and $\alpha_2 \in \mathcal{K}_\infty$. Delta-differentiating the Lyapunov function V_o we get,

$$\begin{aligned} \overset{\Delta}{V}_o &= (\mathcal{A}e_o + \mathcal{D}\mathcal{E}^1)^T\mathcal{P}(t)e_o + (e_o + \mu(t)(\mathcal{A}e_o + \mathcal{D}\mathcal{E}^1))^T(\frac{(\mathcal{P}_2 - \mathcal{P}_1)}{\beta}e_o + (\mathcal{P}(t) \\ &+ \mu(t)\frac{(\mathcal{P}_2 - \mathcal{P}_1)}{\beta})(\mathcal{A}e_o + \mathcal{D}\mathcal{E}^1)) \\ &= e_o^T(\mathcal{A}^T\mathcal{P}(t) + \mathcal{P}(t)\mathcal{A} + \mu(t)\mathcal{A}^T(\mathcal{P}(t) + \mu(t)\frac{\mathcal{P}_2 - \mathcal{P}_1}{\beta})\mathcal{A} + \frac{\mathcal{P}_2 - \mathcal{P}_1}{\beta}(I + 2\mu(t)\mathcal{A})) \\ &e_o + (\mathcal{E}^1)^T\mathcal{D}^T((2\mathcal{P}(t) + \mu(t)\mathcal{A}) + \mu(t)\frac{\mathcal{P}_2 - \mathcal{P}_1}{\beta}(2I + 2\mu(t)\mathcal{A}) + \mu(t)\mathcal{P}(t)\mathcal{A})e_o \\ &+ \mu(t)(\mathcal{E}^1)^T\mathcal{D}^T(\mathcal{P}(t) + \mu(t)\frac{(\mathcal{P}_2 - \mathcal{P}_1)}{\beta})\mathcal{D}\mathcal{E}^1 + (\mathcal{E}^1)^T\mathcal{E}^1 - (\mathcal{E}^1)^T\mathcal{E}^1 \\ &\leq e_o^T(\mathcal{A}^T\mathcal{P}(t) + \mathcal{P}(t)\mathcal{A} + \mu(t)\mathcal{A}^T(\mathcal{P}(t) + \mu(t)\frac{\mathcal{P}_2 - \mathcal{P}_1}{\beta})\mathcal{A} + \frac{\mathcal{P}_2 - \mathcal{P}_1}{\beta}(I + 2\mu(t)\mathcal{A}) \\ &+ \mathcal{L}_\gamma^2\mathcal{H}^T\mathcal{H})e_o + (\mathcal{E}^1)^T\mathcal{D}^T((2\mathcal{P}(t) + \mu(t)\mathcal{A}) + \mu(t)\frac{\mathcal{P}_2 - \mathcal{P}_1}{\beta}(2I + 2\mu(t)\mathcal{A}) + \mu(t)\mathcal{P}(t) \\ &\mathcal{A})e_o + (\mathcal{E}^1)^T(\mu(t)\mathcal{D}^T(\mathcal{P}(t) + \mu(t)\frac{(\mathcal{P}_2 - \mathcal{P}_1)}{\beta})\mathcal{D} - I)\mathcal{E}^1 \\ &\leq \mathcal{R}^T\mathcal{S}_1\mathcal{R} + \frac{t - k\beta}{\beta}\mathcal{R}^T(\mathcal{S}_2 - \mathcal{S}_1)\mathcal{R} \end{aligned} \tag{4.45}$$

where $\mathcal{R} = \begin{bmatrix} e_o^T & (\mathcal{E}^1)^T \end{bmatrix}^T$. If the LMI conditions given in Case a of Theorem 2 are valid then we can write, $\overset{\Delta}{V}_o \leq -\varepsilon_u\|\mathcal{R}\|^2 \leq -\varepsilon_u\|e_o\|^2 \leq -\varepsilon_{u,1}V_o$. Furthermore,

$$V_o \leq e_{-\varepsilon_{u,1}}(t, t_0)V_o(t_0) \tag{4.46}$$

$$\|e_o\| \leq \alpha_1^{-1}(e_{-\varepsilon_{u,1}}(t,t_0)\alpha_2(\|e_o(t_0)\|)) \tag{4.47}$$

Now, considering the impulsive error dynamics case

$$\begin{aligned}
&V_o(k\beta^+) - V_o(k\beta) = ((I-\beta\mathcal{GC})e_o(k\beta) + \beta\mathcal{GCE}^2)^T(\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2-\mathcal{P}_1))\\
&((I-\beta\mathcal{GC})e_o(k\beta) + \beta\mathcal{GCE}^2) - e_o^T(k\beta)\mathcal{P}_2 e_o(k\beta)\\
&\leq -(e_o(k\beta))^T(\mathcal{S}_3^a)(e_o(k\beta)) + \beta^2(\mathcal{CE}^2)^T\mathcal{G}^T(\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2-\mathcal{P}_1))\mathcal{G}(\mathcal{CE}) + 2\beta(\mathcal{E}^2)^T\\
&\mathcal{C}^T\mathcal{G}^T(\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2-\mathcal{P}_1))(I-\beta\mathcal{GC})e_o(k\beta)
\end{aligned} \tag{4.48}$$

where $\mathcal{G} = (\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2-\mathcal{P}_1))^{-1}\mathcal{P}_3$, $\mathcal{S}_3^a = (\mathcal{P}_2 - (\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2-\mathcal{P}_1) - \beta\mathcal{P}_3 C)^T(\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2-\mathcal{P}_1))^{-1}(\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2-\mathcal{P}_1) - \beta\mathcal{P}_3\mathcal{C}))$ is the schur complement of the matrix -$\mathcal{S}_3$.

Now, considering the third and second term of Eq. (4.48) and proceeding in the similar manner as given in (Etienne et al. (2017)) one can write

$$\begin{aligned}
&2\beta(\mathcal{E}^2)^T\mathcal{C}^T\mathcal{G}^T(\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2-\mathcal{P}_1))(I-\beta\mathcal{GC})e_o(k\beta) \leq 2\beta\sqrt{\lambda_{max}(\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2-\mathcal{P}_1))}\\
&\|\mathcal{G}\|\varepsilon_v\sqrt{V_o(k\beta)}
\end{aligned} \tag{4.49}$$

$$\beta^2(\mathcal{CE}^2)^T\mathcal{G}^T(\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2-\mathcal{P}_1))\mathcal{G}(\mathcal{CE}) \leq \lambda_{max}(\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2-\mathcal{P}_1))\beta^2\|\mathcal{G}\|^2\varepsilon_v^2 \tag{4.50}$$

Replacing Eqs. (4.49) and (4.50) in Eq. (4.48) one can write

$$V_o(k\beta^+) \leq (\sqrt{V_o(k\beta)} + \theta)^2 \leq (\theta_a\sqrt{V_o((k-1)\beta^+)} + \theta)^2 \tag{4.51}$$

where θ_a and θ are given by $e_{-\varepsilon_{u,1}}(k\beta,(k-1)\beta^+)$ and $\sqrt{\lambda_{max}(\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2-\mathcal{P}_1))}\beta\|G\|\varepsilon_v$ respectively. Furthermore the solution of Eq. (4.51) can be written as,

$$\sqrt{V_o(k\beta^+)} \leq \theta_a^k\sqrt{V_o(0+)} + \theta\sum_{i=0}^{k-1}\theta_a^i \tag{4.52}$$

Thus, from Eqs. (4.47) and (4.52) one can conclude that the impulsive error dynamics exponentially converges to a set $U = \{\|e_o\| \leq \frac{\theta}{\lambda_{min}(\mathcal{P}_1,\mathcal{P}_2)(1-\theta_a)}\}$. ■

Remark 3: In case of homogeneous measure chains like $\mathbb{R}$ or $\mathbb{Z}$, the value of θ is constant throughout the measure chain whereas for non–homogeneous measure chains like $\mathbb{P}_{a,b}$ or $\mathbb{P}_{a,b} \cup \mathbb{T}_{iso}$, the value of θ varies across the measure chain and set U is obtained as the intersection of the various sets U obtained from the corresponding value of graininess function $\mu(t)$.
The pseudo code for designing periodic event–triggered impulsive observer is given below

- We solve a system of matrix inequalities for a constant value of $\mu(t)$ and a value of β.
- We calculate the value of observer gain matrix $\mathcal{G}= (\mathcal{P}_1 + \frac{\mu(t)}{\beta}(\mathcal{P}_2 - \mathcal{P}_1))^{-1}\mathcal{P}_3$.

4.1.3 Periodic event–triggered impulsive observer based controller

In the previous sections, periodic event triggered controller and periodic event triggered impulsive observer have been designed for systems on measure chains. In this section an output feedback periodic event triggered controller is designed for systems on measure chains. The expression of control input is given by $u = Kz(t_{n,a})$. The expression of closed–loop system along with the observer error dynamics Eq. (4.43) is,

$$\begin{aligned}\overset{\Delta}{x}(t) &= \mathcal{A}x(t) + \mathcal{B}Kz(t_{n,a}) + \mathcal{D}\phi(\mathcal{H}x(t)) \;\forall\; \mathbb{T} \ni t \in [t_{n,a}, t_{n+1,a}) \\ &= \mathcal{A}x(t) + \mathcal{B}Kx(j\varphi) + \mathcal{D}\phi(\mathcal{H}x(t)) + \mathcal{B}K\mathcal{E}^3 \end{aligned} \tag{4.53}$$

where $\mathcal{E}^3$ is $(z(j\varphi) - x(j\varphi)) + (z(t_{n,a}) - z(j\varphi))$. The closed–loop system given in Eq. (4.53) can also be rewritten as

$$\overset{\Delta}{x} = (\mathcal{A} + \mathcal{B}K)x + \mathcal{D}\phi(\mathcal{H}x) + \mathcal{B}K(d_s + \mathcal{E}^3) \tag{4.54}$$

4.1.3.1 Asymptotic stability under periodic event–triggered impulsive observer based controller

In this section a theorem is given which ensures asymptotic stability of the system given in Eq. (4.1) with event triggering condition in the actuator and sensor side given by Eq. (4.3) and Eq. (4.37) respectively. The impulsive observer which is used to estimate the state–variables is given in Section 2.

Theorem 3: Assuming Assumptions 1–3 are satisfied as well as the conditions given Theorem 1 and Theorem 2 hold true except the one on $\mathbb{F}_{\mathbb{T}}$ (in case of Theorem 1) , then the closed–loop system is asymptotically stable provided the value of ε is chosen in such a way that $\mathbb{E}_{\mathbb{T}} < 1$. ■

Proof: Taking inspirations from the proof given in Theorem 1, one can write the delta–derivative of Lyapunov function $V = x(t)^T \mathcal{P} x(t),\ \forall\ \mathbb{T} \ni t \in [j\varphi, (j+1)\varphi)$ where $\mathcal{P} = \mathcal{P}^T > 0$, along the directions of Eq. (4.54) as

$$\overset{\Delta}{V} \leq -\varepsilon_{c,1}\lambda_{min}(\mathcal{P})\|x\|^2 + \varepsilon_{c,2}\lambda_{max}(\mathcal{P})\|e\|^2 + \varepsilon_{c,2}\lambda_{max}(\mathcal{P})\|e_o\|^2 + \varepsilon_{c,3}\lambda_{max}(\mathcal{P})\|d_s\|^2 \tag{4.55}$$

Thus, using the same expression of maximum actuator updating period as given in Theorem 1 and also the observer error expression given in Eq. (4.47) , one can also write

$$\overset{\Delta}{V} \leq -\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)\|x\|^2 + \frac{\varepsilon_{c,2}\varepsilon\lambda_{max}(\mathcal{P})}{K^T K}\|x(j\varphi)\|^2 + \varepsilon_{c,2}\lambda_{max}(\mathcal{P}) \\ (\alpha_1^{-1}(e_{-\varepsilon_{u,1}}(t,t_0)\alpha_2(\|e_o(t_0)\|)))^2 \tag{4.56}$$

Similarly as in the proof of Theorem 1 and also with the help of $\alpha_1(\|e_o\|) = \lambda_{min}(\mathcal{P}_1,\mathcal{P}_2)(\|e_o\|)^2$ and $\alpha_2(\|e_o\|) = \lambda_{max}(\mathcal{P}_1,\mathcal{P}_2)(\|e_o\|)^2$ as well as $V_o(j\varphi) < V(j\varphi)$,

Eq. (4.56) can be written as

$$\overset{\Delta}{V} \leq -\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}V + \frac{\varepsilon_{c,2}\varepsilon\lambda_{max}(\mathcal{P})}{K^T K\lambda_{min}(\mathcal{P})}V(j\varphi)+$$
$$\varepsilon_{c,2}\lambda_{max}(\mathcal{P})\left(\frac{e_{-\varepsilon_{u,1}}(t,j\varphi)V(j\varphi)}{\lambda_{min}(\mathcal{P}_1,\mathcal{P}_2)}\right)$$
$$V((j+1)\varphi) \leq lim_{t\to(j+1)\varphi}\Bigg(e_{-\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}}(t,j\varphi)+$$
$$\Bigg(\frac{\varepsilon_{c,2}\varepsilon\lambda_{max}(\mathcal{P})}{K^T K\lambda_{min}(\mathcal{P})}\int_{j\varphi}^{t}\frac{e_{-\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}}(t,\tau)}{1-\mu(t)\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}}\Delta\tau + \frac{\varepsilon_{c,2}\lambda_{max}(\mathcal{P})}{\lambda_{min}(\mathcal{P}_1,\mathcal{P}_2)}$$
$$\int_{j\varphi}^{t}\frac{e_{-\varepsilon_{u,1}}(\tau,j\varphi)e_{-\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}}(t,\tau)}{1-\mu(t)\frac{\varepsilon_{c,1}\lambda_{min}(\mathcal{P})(1-\Lambda)}{\lambda_{max}(\mathcal{P})}}\Delta\tau\Bigg)V(j\varphi)\Bigg) \tag{4.57}$$
$$\leq \mathbb{E}_{\mathbb{T}}V(j\varphi) \tag{4.58}$$

The value of ε is chosen in such a way that $\mathbb{E}_{\mathbb{T}} < 1$. Thus, asymptotic stability for the closed–loop system is proved for any given measure chain $\mathbb{T}$. ■

Remark 4: Similar to the proof of Theorem 1, the exact expression of $\mathbb{E}_{\mathbb{T}}$ can be easily calculated for various measure chains like $\mathbb{T} = \mathbb{R}, \mathbb{Z}, \mathbb{P}_{a,b}, \mathbb{P}_{a,b} \bigcup \mathbb{T}_{iso}$.

4.2 Numerical Simulations

The numerical simulations section of this chapter is divided in two parts. In the first part of this section results related to periodic event–triggered control on measure chains are reported whereas in the second part results related to periodic event–triggered impulsive observer on measure chains are stated.

4.2.1 Periodic event–triggered controller on measure chains

In the numerical simulations section we will first consider the design of periodic event–triggered controller on measure chain. The system matrix numerical values

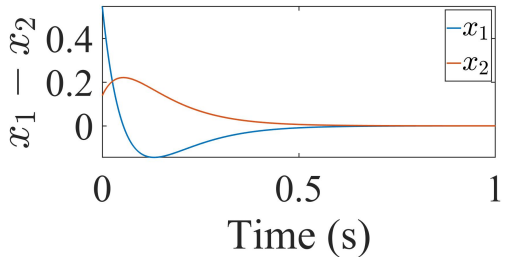


FIGURE 4.2: *State–trajectories for Case 1.*

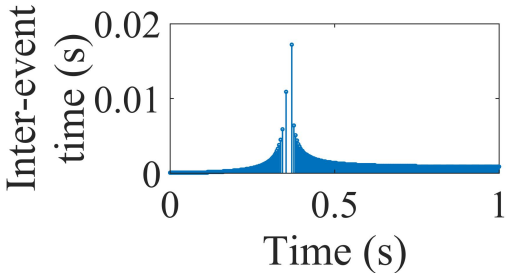


FIGURE 4.3: *Inter–event time for Case 1.*

considered for simulations for the system given in Eq. (4.1) are given by

$$\mathcal{A} = \begin{bmatrix} 0 & 10 \\ 7 & 0 \end{bmatrix} \quad \mathcal{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathcal{H} = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \mathcal{D} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \tag{4.59}$$

The nominal system given by Eq. (4.1) satisfies Assumption 1. The value of $\mathcal{L}_\gamma$ is 0.01 and $\phi = 0.01sin(x)$. Three different measure chains are considered in this chapter, $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mu(t)\mathbb{Z}$ and $\mathbb{T} = \mathbb{P}_{a,b} = \bigcup_{p=0}^{\infty}[(2p) \times 10^{-5}, (2p+1) \times 10^{-5}]$. The system given by Eq. (4.1) is Hilger stabilizable with respect to all the three measure chains, which implies that the gain constant matrix can be designed by ensuring the eigenvalues of the nominal system lies within the Hilger circle of maximum graininess function value. The value of ε is selected in a manner that it satisfies the condition given in Theorem 1. The initial values for each simulation cases are considered randomly. The value of K is taken as $K = \begin{bmatrix} -26 & -32.8571 \end{bmatrix}$.

1. **Case 1:** When measure chain is $\mathbb{T} = \mathbb{R}$, the value of $\mu(t)$ is 0 s which satisfies Assumption 2. The value of φ_{max} calculated with the help of formulae given in Case a of Theorem 1 is 4.13×10^{-4} s. The plots of state trajectories considering periodic sampling of $\varphi = 1 \times 10^{-4}$ s are given in Fig. 4.2 whereas the plot of inter–event time is given in Fig. 4.3. The value of ε is 0.02. The minimum value of inter–event time is 1×10^{-4} s.

2. **Case 2:** When measure chain is $\mathbb{T} = \mu(t)\mathbb{Z}$, with $\mu(t) = 1 \times 10^{-8}$ s which is constant throughout $\mathbb{T}$ and satisfies Assumption 2 . The value of φ_{max} is numerically calculated as 3×10^{-8} s from Case b of Theorem 1. The plots of zoomed in state trajectories with controller periodic sampling $\varphi = 2 \times 10^{-8}$

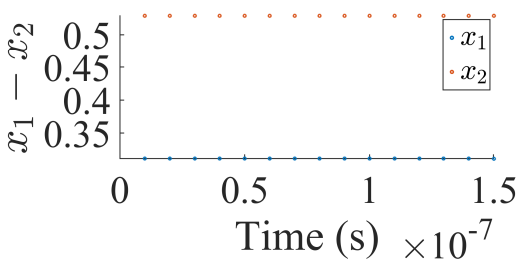


FIGURE 4.4: *State–trajectories for Case 2.*

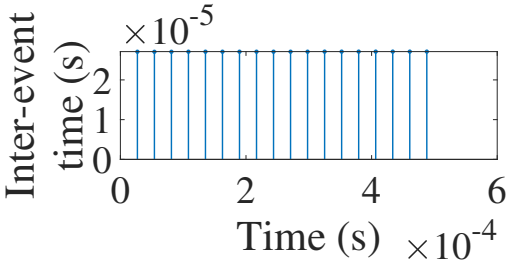


FIGURE 4.5: *Inter–event time for Case 2.*

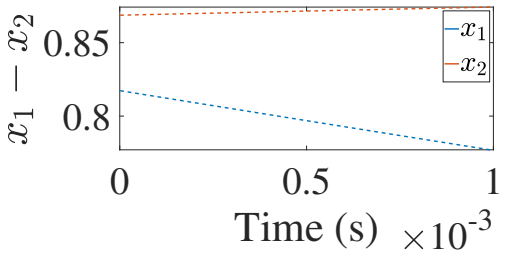


FIGURE 4.6: *Zoomed in state–trajectories for Case 3.*

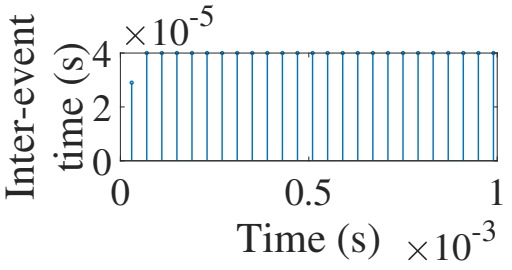


FIGURE 4.7: *Zoomed in inter–event time for Case 3.*

s are given in Fig. 4.4 and the zoomed in inter–event time is given in Fig. 4.5. The minimum value of the inter–event time is 2×10^{-5} s which is significantly higher than the controller periodic updating period φ, thus ensuring non–triviality of the design procedure. The value of ε is 0.02.

3. **Case 3:** The measure chain under consideration is a $\mathbb{P}_{a,b}$ measure chain which is depicted in Fig. 1.7. The value of $\mu(t)$ is varying between 0 s (continuous part of $\mathbb{T} = \mathbb{P}_{a,b}$) and 1×10^{-5} s (discrete part of $\mathbb{T} = \mathbb{P}_{a,b}$) which satisfies Assumption 2. The value of φ_{max} for the considered measure chain $\mathbb{P}_{a,b}$ can be obtained using Case c of Theorem 1 which is numerically calculated as 1×10^{-5} s. Thus the value controller periodic updating time φ is taken as 1×10^{-5} s. The value of ε is 0.02. The plots of state trajectories are given in Fig. 4.6 whereas the inter–event time plot is given in Fig. 4.7. The minimum value of inter–event time is given by 3×10^{-5} s.

Thus, from the plots of inter–event time and system trajectories given in case 1-3, one can conclude that the periodic event–triggered controller proposed in Eq.

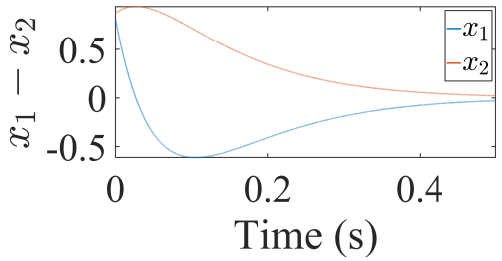


FIGURE 4.8: *State–trajectories for Case 3.*

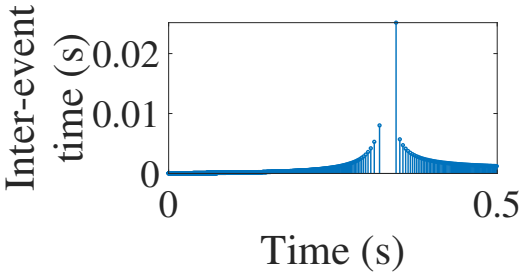


FIGURE 4.9: *Inter–event time for Case 3.*

(4.3), ensures global asymptotic stability and nontrivial inter–event time. Periodic event–triggering proposed in this paper also gives lower transmission instants in comparison to the classical event–triggering proposed in (Dey et al. (2021)), because considering the discrete–time case, the minimum inter–event time is 5 times the minimum graininess function value as mentioned in (Dey et al. (2021)) whereas in this paper the minimum inter–event time is 1000 times the graininess function value.

4.2.2 Periodic event–triggered impulsive observer on measure chains

Now, we will give some numerical simulation result for periodic event–triggered impulsive observer for various measure chains. The values of system matrix dimensions for simulation purposes are given in Eq. (4.59). The value of $\mathcal{C}$ is taken as

$$\mathcal{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.60}$$

The nominal system given by Eq. (4.1) satisfies Assumption 2. The value of $\mathcal{L}_\gamma$ is 0.01 and $\phi = 0.01\sin(x)$. Three different measure chains are considered, $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mu(t)\mathbb{Z}$ and $\mathbb{T} = \mathbb{P}_{a,b} = \bigcup_{p=0}^{\infty}[(2p) \times 10^{-2}, (2p+1) \times 10^{-2}]$. The original system for which we are designing periodic event–triggered impulsive observers are assumed to be Hilger stabilizable. The value of ε_v is chosen in such a way that

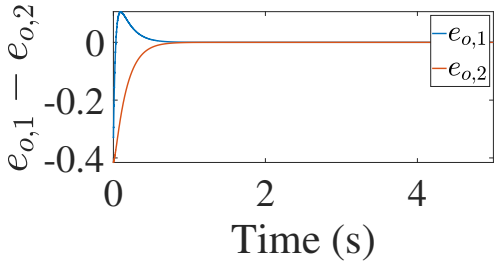


FIGURE 4.10: *Error trajectories for Case 1.*

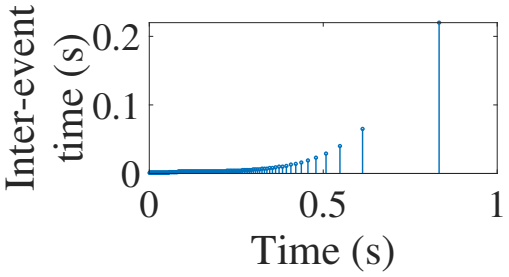


FIGURE 4.11: *Inter–event time for Case 1.*

the conditions stated in Theorem 2 are satisfied. The initial condition values are separate for each simulation case.

- **Case 1:** When the measure chain is $\mathbb{T} = \mathbb{R}$, we have the value of $\mu(t) = 0$ s. The value of periodic sampling β is taken as 0.001 s. The value of ε_v is taken as 0.005. The value of observer gain matrix $\mathcal{G}$ obtained from Theorem 2 is given by

$$\mathcal{G} = \begin{bmatrix} 39.05 & 24.88 \\ 1.235 & 4.782 \end{bmatrix} \tag{4.61}$$

  The minimum value of inter–event time is given by 1×10^{-3} s. The plots of observer error and inter–event time are given by Figs. (4.10) and (4.11) respectively.

- **Case 2:** When the measure chain is $\mathbb{T} = \mu(t)\mathbb{Z}$. The value of $\mu(t)$ is taken as 0.01 s. The value of ε_v is taken as 0.05. The value of β is taken as 0.02 s. The value of observer gain matrix $\mathcal{G}$ obtained from Theorem 2 is given by

$$\mathcal{G} = \begin{bmatrix} 5.72 & 3.22 \\ 2.96 & 5.085 \end{bmatrix} \tag{4.62}$$

  The minimum value of inter–event time is given by 0.01 s. The plots of observer error and inter–event time are given in Figs. (4.12) and (4.13) respectively.

- **Case 3:** When the measure chain is a $\mathbb{P}_{a,b}$ measure chain. The measure chain under consideration has a graininess function $\mu(t)$ which vary across

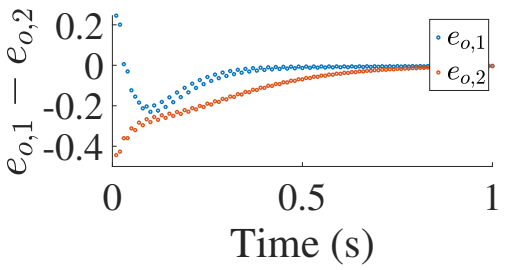


FIGURE 4.12: *Error trajectories for Case 2.*

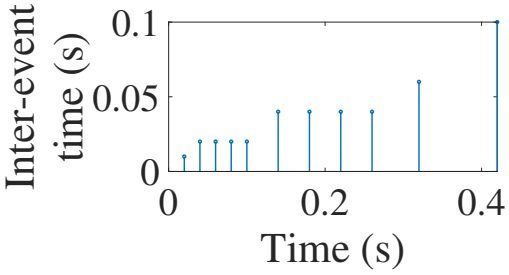


FIGURE 4.13: *Inter–event time for Case 2.*

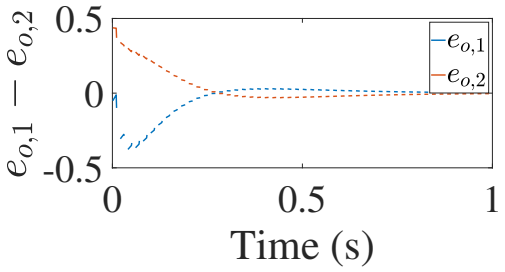


FIGURE 4.14: *Error trajectories for Case 3.*

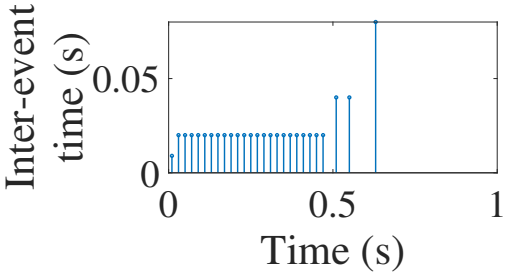


FIGURE 4.15: *Inter–event time for Case 3.*

the measure chain. The value of graininess function $\mu(t)$ is zero in the continuous part of the measure chain whereas it is 0.01 s in the discrete part of the measure chain. The value of design constant ε_v is 0.005 in the continous portion and 0.05 in the discrete portion of the measure chain. The value of β is 0.01. The values of observer gain $\mathcal{G}$ varies across the measure chain wherein

$$\mathcal{G} = \begin{bmatrix} 2.19 & 0.18 \\ 0.18 & 2.34 \end{bmatrix} \text{ when } \mu(t) = 0.01;\ \mathcal{G} = \begin{bmatrix} 19.51 & 16.75 \\ 15.47 & 15.6 \end{bmatrix} \text{ when } \mu(t) = 0 \tag{4.63}$$

The plots of state–trajectories and inter–event time is given by (4.14) and (4.15) respectively.

Thus, from Case 1–3 we can say that the designed periodic event–triggered impulsive observer on measure chains works satisfactorily. Similar simulations can be performed to realize the periodic event–triggered output feedback impulsive observer on measure chains.

4.3 Conclusion

In this chapter, we have designed periodic event–triggered controller and periodic event–triggered impulsive observer for a class of systems on measure chains. We have also proposed a way to design output feedback periodic event–triggered controllers for systems on measure chains with the help of periodic event–triggered impulsive observer and periodic event–triggered controller which were designed independently in their respective subsections. The results reported for the periodic event–triggered controller in case of systems on purely discrete measure chain give conservative results which do not exist when we design periodic event–triggered impulsive observers for such a system. Similar conservative results for continuous time systems can be found in the literature (Etienne et al. (2017)). The author believes that the results in (Wang et al. (2019)) are less conservative when one considers designing periodic event–triggered controllers for nonlinear systems on continuous measure chains with the help of hybrid systems formalism, the extension of which to systems on arbitrary measure chains is not trivial and will be a focus of future research.

Chapter 5

Safety–critical event–triggered control on measure chains

The main objective of safety–critical control is to ensure that the system trajectories of the closed–loop system remain within a certain safe set for every period of time starting from the initial time. The research in the field of safety–critical control has gained wide scale prominence in the past few decades due to its wide range of applications to physical real time practical applications ranging from humanoid robotics to autonomous vehicles (Cortez et al. (2019), Ames et al. (2016)). The concept of input–to–state safe barrier functions has been defined only for systems on continuous time domains (Kolathaya and Ames (2018)) which was further extended to design event–triggered safety–critical controllers with the aid of input–to–state (ISS) barrier functions and different event triggering conditions on continuous time domains $\mathbb{R}$ (Taylor et al. (2020), Long and Wang (2022)). In (Dey et al. (2021)), initial results related to the continuous event–triggered control design for a class of nonlinear systems was solved with the help of ISS–stable Lyapunov functions on measure chains. It is worth noting that the concept of the safety–critical event–triggered controllers for nonlinear systems on measure chains is yet to be addressed in the literature. The research on safety–critical control is of utmost importance if we wish to apply the control algorithm for realistic applications. Thus, the main contributions of the chapter are listed below

1. The concept of input–to–state safe barrier functions on measure chains is introduced in this chapter.

2. A dynamic safety–critical event–triggered control is designed on measure chains which improves the mixed triggering condition proposed for measure chain $\mathbb{R}$ in (Taylor et al. (2020)).

3. The designed safety–critical event–triggering condition acts as an unifier (i.e. it can be applied to any time domain be it $\mathbb{R}, \mathbb{Z}$, $h\mathbb{Z}$, $\mathbb{H}$, $\mathbb{P}_{a,b}$ etc).

The chapter is organized as follows. Section 1 explains the problem formulation and main results which are addressed in this chapter. Section 2 contains numerical simulations with different measure chains whereas the chapter concludes with Section 3.

5.1 Problem Formulation

A nonlinear dynamical system on an arbitrary measure chain $\mathbb{T}$ is given by

$$\overset{\Delta}{z}(t) = f(z(t), u(t)),\ \forall\ t \in \mathbb{T} \tag{5.1}$$

where $z \in \mathbb{R}^a$ and $u \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}^b)$ represents the state vector and control input respectively. The nonlinear function $f(.)$ is rd–continuous and locally Lipschitz. The control input is considered as

$$u = R(z) \tag{5.2}$$

where $R : \mathbb{R}^a \to \mathbb{R}^b$. In case of event triggered control, the control law given in Eq. (5.2) can be rewritten as $u(t) = R(z(t_i)),\ \forall\, t \in [t_i, t_{i+1}) \cap \mathbb{T},\ i \in \mathbb{N}^0$ with the instances of event–triggering being denoted by t_i. The expression of sampling error is taken as, $e(t) = z(t_i) - z(t)$. With the event triggered control law under consideration, the expression of the nonlinear system given in Eq. (5.1) can be written as,

$$\overset{\Delta}{z}(t) = f(z(t), R(e(t) + z(t))),\ \forall\, t \in [t_i, t_{i+1}) \cap \mathbb{T},\ i \in \mathbb{N}^0$$

$$= -\overset{\triangle}{e}(t),\ \forall\ t\ \in\ \mathbb{T} \tag{5.3}$$

Taking motivations from the work (Kolathaya and Ames (2018)), a delta–differentiable function $r : \mathbb{R}^a \to \mathbb{R}$ is defined as

$$\mathbb{C} \triangleq \{z \in \mathbb{R}^a : r(z) \geq 0\} \tag{5.4}$$

$$\partial\mathbb{C} \triangleq \{z \in \mathbb{R}^a : r(z) = 0\} \tag{5.5}$$

$$\text{Int}(\mathbb{C}) \triangleq \{z \in \mathbb{R}^a : r(z) > 0\} \tag{5.6}$$

The set $\mathbb{C}$ is termed as the safe set if it is forward invariant and nonempty.

Definition 1 (Barrier function on measure chains): For the system given in Eq. (5.2), $r : \mathbb{R}^a \to \mathbb{R}$ is a barrier function on the set $\mathbb{C}$, if the directional delta derivative of $r(z)$ along the directions of Eq. (5.2) satisfies the following inequality

$$\overset{\triangle}{r}(z) \geq -\varepsilon(r(z)),\ \forall\ z \in\ \mathbb{R}^a \tag{5.7}$$

where $\varepsilon \in \mathcal{K}_{\infty,e}$.

Definition 2 (Input–to–state safe Barrier function on measure chains): A candidate Barrier function $r : \mathbb{R}^a \to \mathbb{R}$ is termed as Input-to-state safe Barrier function on measure chains if the directional delta derivative of $r(z)$ along the directions of Eq. (5.3) satisfies

$$\overset{\triangle}{r}(z) \geq -\varepsilon(r(z)) - \vartheta(\|e\|),\ \forall\ (z, e)\ \in \mathbb{R}^a \tag{5.8}$$

where $\varepsilon \in \mathcal{K}_{\infty,e}$ and $\vartheta \in \mathcal{K}_{\infty}$.

Definition 3 (Input–to–state Lyapunov function on measure chains): A candidate Lyapunov function $V : \mathbb{R}^a \to \mathbb{R}^{0+}$ is an input–to–state Lyapunov function (ISS-LF) on measure chains if the following conditions are satisfied

1.

$$\vartheta_1(\|z\|) \leq V(z) \leq \vartheta_2(\|z\|),\ \forall\ z \in\ \mathbb{R}^a \tag{5.9}$$

where $\vartheta_1 \in \mathcal{K}_{\infty}$ and $\vartheta_2 \in \mathcal{K}_{\infty}$.

2. The directional delta derivative of the candidate Lyapunov function along the directions of Eq. (5.3) satisfies

$$\overset{\Delta}{V}(z) \leq -\vartheta_3(\|z\|) + \vartheta(\|e\|) \ \forall \ z, e \in \ \mathbb{R}^a \tag{5.10}$$

where $\vartheta_3 \in \mathcal{K}_\infty$ and $\vartheta \in \mathcal{K}$.

The expression of event–triggering condition considered in this chapter inspired from the work (Girard (2014)) is given by

$$t_{i+1} = \inf\{\mathbb{T} \ni t > t_i | \zeta + \eta(p\varepsilon(r(z)) - \vartheta(\|e\|)) \leq 0\}$$
$$\forall \ i \in \mathbb{N}^{0+} \tag{5.11}$$

The dynamics of the dynamic variable on the same measure chain $\mathbb{T}$ as the system dynamics is given by

$$\overset{\Delta}{\zeta} = -q\zeta + p\varepsilon(r(z)) - \vartheta(\|e\|)$$
$$\zeta(t_0) = r(z(t_0)) \geq 0 \tag{5.12}$$

with parameters $\eta \in \mathbb{R}^{0+}$, $q \in \mathbb{R}^{0+}$ and $p \in \mathbb{R}^{0+} \cap (0, 1)$ are constant regressive design variables.

Lemma 1:(Gage and Hamilton (1986)) For any two positive numbers a_1 and a_2 the following inequality is satisfied for any given positive constant $s > 0$,

$$2a_1a_2 \leq \frac{a_1^2}{s} + sa_2^2 \tag{5.13}$$

Now, we are in position to state Theorem 1 which will provide us certain constraints on the design variables q and η in order to ensure that the dynamic event–variable ζ decreases.

Theorem 1: For the event condition proposed in Eq. (5.12), the dynamic event variable $\zeta(t) \geq 0,\ \forall\ t \in \mathbb{T}$, with $t > t_0$ and $z(t_0) \in \mathbb{C}$ as well as strictly decreasing provided the design variables η, q and p satisfy the following conditions

1. $(e_{-q-\frac{1}{\eta}}(t, t_0) \geq 0\ \forall\ t \in \mathbb{T})$ and $(\eta \neq 0)$
2. $e_{-q-\frac{1}{\eta}}(t, t_0)$ is strictly decreasing $\forall\ t \in \mathbb{T}$.

provided $(1 + \mu(-q - \frac{1}{\eta})) > 0$ with respect to the measure chain $\mathbb{T}$. ■

Proof: From the event condition given in Eq. (5.11), we can write

$$\zeta(t) + \eta(p\varepsilon(r(z(t))) - \vartheta(\|e(t)\|)) \geq 0\ \forall\ t \in \mathbb{T} \text{ and } t > t_0 \tag{5.14}$$

There are two cases under consideration

1. Case 1: When $\eta = 0$, $\zeta(t) \geq 0,\ \forall\ t \in \mathbb{T} \cap t > t_0$ as from Eq. (5.14).
2. Case 2: When $\eta \neq 0$

$$(p\varepsilon(r(z)) - \vartheta(\|e\|)) \geq -\frac{\zeta}{\eta} \tag{5.15}$$

From the dynamics of dynamic event–triggered variable on measure chains given in Eq. (5.12) one gets

$$\begin{aligned} \overset{\Delta}{\zeta}(t) &\geq -q\zeta(t) - \frac{\zeta(t)}{\eta} \\ \zeta(t) &\geq e_{-q-\frac{1}{\eta}}(t, t_0)\zeta(t_0) \end{aligned} \tag{5.16}$$

Thus if the conditions given in Theorem 1 holds true then $\zeta(t) \geq 0\ \forall\ t \in \mathbb{T}$. ■

Remark 1: From the proof of Theorem 1 and Eq. (5.5), it can be easily concluded that $\zeta(t) = 0$ iff $z(t) \in \partial\mathbb{C}$.

Remark 2: The expression of $e_{-q-\frac{1}{\eta}}(t, t_0)$ for various measure chains are given below,

1. When the measure chain is a continuous measure chain $\mathbb{T} = \mathbb{R}$, then the expression of exponential function is given by

$$e_{-q-\frac{1}{\eta}}(t, t_0) = \exp^{\int_{t_0}^{t}(-q-\frac{1}{\eta})d\vartheta} \tag{5.17}$$

   where $-q - \frac{1}{\eta}$ is regressive on measure chain $\mathbb{R}$.

2. When the measure chain is a discrete measure chain $\mathbb{T} = \mathbb{Z}$, then the expression of exponential function is given by

$$e_{-q-\frac{1}{\eta}}(t, t_0) = \prod_{\vartheta=t_0}^{t-1}(1 - q - \frac{1}{\eta}) \tag{5.18}$$

   where $-q - \frac{1}{\eta}$ is regressive on measure chain $\mathbb{Z}$.

3. Similarly for $\mathbb{T} = a\mathbb{Z}$, where a is a positive constant then the exponential function is given by

$$e_{-q-\frac{1}{\eta}}(t, t_0) = \prod_{\vartheta=\frac{t_0}{a}}^{\frac{t}{a}-1}(1 + a(-q - \frac{1}{\eta})) \tag{5.19}$$

   where $-q - \frac{1}{\eta}$ is regressive on measure chain $\mathbb{T} = a\mathbb{Z}$.

The main significance of the above remark is to draw the attention of authors who read this chapter towards the fact that exponential functions on measure chains are not the same as the classical exponential functions for systems on real measure chains. If the design variables are not chosen appropriately then there is a possibility that exponential functions on measure chains turn out to be negative.

Assumption 1: An ISS stable barrier function $r : \mathbb{R}^a \to \mathbb{R}$ exists for the nonlinear closed–loop system given in Eq. (5.3) which is defined by Eq. (5.4)–Eq. (5.6) such that ε is locally Lipschitz and its delta–derivative is bounded.

Theorem 2: The closed–loop system described by Eq. (5.3) is safe with respect to the set $\mathbb{C}$ for any given measure chain $\mathbb{T}$, under the event triggering condition given in Eq. (5.11) with no existence of Zeno phenomenon. ■

Proof: A delta–differentiable function is considered,

$$Z(r(z),\zeta) = r(z) - \zeta \tag{5.20}$$

Delta–differentiating Eq. (5.20) along the directions of system Eq. (5.3) as well as considering the dynamics of dynamic event triggered variable given in Eq. (5.12), one can write

$$\begin{aligned}\overset{\Delta}{Z}(r(z),\zeta) &= \overset{\Delta}{r}(z) - \overset{\Delta}{\zeta} \\ &\geq -\varepsilon(r(z)) - \vartheta(\|e\|) + q\zeta - p\varepsilon(r(z)) + \vartheta(\|e\|) \\ &\geq -(1+p)\varepsilon(r(z)) + q\zeta \end{aligned} \tag{5.21}$$

Using the help of mean value theorem on measure chains (Bohner and Georgiev (2016b)) and Assumption 1,

$$\begin{aligned}&\varepsilon(r(z(t))) - \varepsilon(r(z(t)) - \zeta) = \overset{\Delta}{\varepsilon}(o)\zeta(t) \\ &\forall\ t \in \mathbb{T} \text{ and } (z(t),\zeta(t)) \in (\mathbb{R}^a \times \mathbb{R})\end{aligned} \tag{5.22}$$

where $o \in [r(z(t)), r(z(t)) - \zeta(t)]$. Furthermore, if there exists a Lipschitz constant $\mathcal{L}_1$ and with the help of Assumption 1 we can write,

$$\begin{aligned}&\varepsilon(r(z(t))) - \varepsilon(r(z(t)) - \zeta(t)) \leq \mathcal{L}_1\zeta(t) \\ &\forall\ t \in \mathbb{T} \text{ and } (z,\zeta) \in (\mathbb{R}^a \times \mathbb{R}^{0+})\end{aligned} \tag{5.23}$$

Using a suitable choice of p and q, one can take the Lipschitz constant $\mathcal{L}_1$ as $\frac{q}{1+p}$. Hence with the above Lipschitz constant, Eq. (5.23) can be rewritten as

$$\begin{aligned}&-(1+p)\varepsilon(r(z(t))) + q\zeta \geq -(1+p)\varepsilon(r(z(t)) - \zeta(t)) \\ &\forall\ t \in \mathbb{T} \text{ and } (z(t),\zeta(t)) \in (\mathbb{R}^a \times \mathbb{R}^{0+})\end{aligned} \tag{5.24}$$

Thus, with the help of Eq. (5.21) and Eq. (5.24) one can write

$$\begin{aligned}\overset{\Delta}{Z}(r(z(t)),\zeta(t)) &\geq -(1+p)\varepsilon(r(z(t)) - \zeta(t)) \\ &= -\tilde{\varepsilon}(Z(r(z(t)),\zeta(t)))\end{aligned}$$

$$\forall\ t \in \mathbb{T} \text{ and } (z(t), \zeta(t)) \in (\mathbb{C} \times \mathbb{R}^{0+}) \tag{5.25}$$

with $\tilde{\varepsilon}(s) = (1+p)\varepsilon(s)$ and $\tilde{\varepsilon} \in \mathcal{K}_{\infty,e}$. Furthermore, it could be noted that $\partial(\mathbb{C} \times \mathbb{R}^{0+}) = (\partial\mathbb{C} \times \mathbb{R}^{0+}) \cup (\mathbb{C} \times \{0\})$. Similar to (Long and Wang (2022)), we have considered two distinct cases in this chapter for any given measure chain $\mathbb{T}$

1. Case 1: When $(z^T, \zeta) \in \partial\mathbb{C} \times \mathbb{R}^{0+}$ we have $r(z) = 0$ and $\zeta \geq 0$. Thus Eq. (5.25) can be written as

$$\overset{\Delta}{\tilde{Z}}(r(z), \zeta) \geq -\tilde{\varepsilon}(-\zeta) \geq 0 \tag{5.26}$$

2. Case 2: When $(z^T, \zeta) \in \mathbb{C} \times \{0\}$ we have $\zeta = 0$ and with the aid of Remark 1 one can write $z \in \partial\mathbb{C}$. Therefore Eq. (5.25) can be written as,

$$\overset{\Delta}{\tilde{Z}}(r(z), \zeta) \geq -\tilde{\varepsilon}(r(z)) = 0 \tag{5.27}$$

Clearly from Case 1 and 2, one can conclude $\overset{\Delta}{\tilde{Z}}(r(z(t)), \zeta(t)) \geq 0\ \forall\ t \in \mathbb{T}$ and $(z, \zeta) \in (\partial(\mathbb{C} \times \mathbb{R}^{0+}))$. With the help of Nagumo's theorem given in (Blanchini and Miani (2008)) one can conclude that the set $\mathbb{C} \times \mathbb{R}^{0+}$ is forward invariant. Thus $z \in \mathbb{C}$ and $\zeta \in \mathbb{R}^{0+}$ for any measure chain $\mathbb{T}$ and any initial condition belonging to the respective sets. Thus the closed–loop system given by Eq. (5.3) is safe with respect to the event condition in Eq. (5.11) and (5.12) respectively.

The absense of Zeno phenomenon can be proved by using the contradiction method. Assuming, the designed event triggered control ensures that Zeno phenomena exists then $\lim_{i\to\infty} t_i = \mathbb{E} < \infty$, for all positive number $\mathbb{E}$, furthermore the inter event–time expression is given by $\mathbb{Y}_i = t_{i+1} - t_i$ also converges to $\lim_{i\to\infty} \mathbb{Y}_i = 0$. For any positive value of y_o, there always exists an integer value depending on the positive value y_o which can be written as $\mathbb{N}(y_o)$. Thus $t_i,\ \forall\ i \in \mathbb{N}^{0+}$ can be written as

$$t_i \in [\mathbb{E} - y_o, \mathbb{E}],\ \forall\ i \geq \mathbb{N}(y_o)\ \forall\ t_i \in \mathbb{T} \tag{5.28}$$

Assuming the next triggering instant to be t_{i+1}, we know from the theory of event triggering that the error is reset to zero at the instant $\sigma(t_{i+1})$ but just before the

instant of event–triggering we can write

$$\begin{aligned}&\zeta(\rho(t_{i+1})) + \eta(p\varepsilon(r(z(\rho(t_{i+1})))) - \vartheta(\|e(\rho(t_{i+1}))\|)) \leq 0\\ &\vartheta(\|e(\rho(t_{i+1}))\|) - p\varepsilon(r(z(\rho(t_{i+1})))) \geq \frac{\zeta(\rho(t_{i+1}))}{\eta}\end{aligned} \tag{5.29}$$

Also, as the event triggering condition ensures safety of the closed–loop system one gets

$$\vartheta(\|e(\rho(t_{i+1}))\|) - p\varepsilon(r(z(\rho(t_{i+1}))))) \leq \vartheta(\|e(\rho(t_{i+1}))\|) \tag{5.30}$$

Using Eqs. (5.29) and (5.30) we can write

$$\vartheta(\|e(\rho(t_{i+1}))\|) \geq \frac{\zeta(\rho(t_{i+1}))}{\eta} \tag{5.31}$$

We know that the closed–loop system is safe with respect to a given set $\mathbb{C}$, ensuring that the delta derivative of the state z is also bounded (i.e. $\|\overset{\Delta}{z}(t)\| \leq \mathcal{L}_2 \ \forall \ t \in \mathbb{T}$). Thus,

$$\begin{aligned}\vartheta(\|e(t)\|) = \vartheta(\|z(t_i) - z(t)\|) \leq \vartheta((t - t_i)\mathcal{L}_2)\\ \forall\, t \in [t_i, t_{i+1}) \cap \mathbb{T},\ i \in \mathbb{N}^0\end{aligned} \tag{5.32}$$

From Eq. (5.28), one obtains $t_{i+1} - t_0 \leq \mathbb{E} - t_0$. With the help of Theorem 1, Eqs. (5.31)–(5.32) we can write

$$\begin{aligned}&\vartheta(\mathbb{Y}_i(\mathcal{L}_2)) = \vartheta((t_{i+1} - t_i)\mathcal{L}_2) \geq \vartheta(\|e(\rho(t_{i+1}))\|) \geq \frac{\zeta(\rho(t_{i+1}))}{\eta} \geq \frac{e_{-q-\frac{1}{\eta}}(t_{i+1}, t_0)\zeta(t_0)}{\eta}\\ &\geq \frac{e_{-q-\frac{1}{\eta}}(\mathbb{E}, t_0)\zeta(t_0)}{\eta}\end{aligned} \tag{5.33}$$

Therefore from Eq. (5.33),

$$\begin{aligned}\mathbb{Y}_i \geq \frac{1}{\mathcal{L}_2}\vartheta^{-1}\left(\frac{e_{-q-\frac{1}{\eta}}(\mathbb{E}, t_0)\zeta(t_0)}{\eta}\right) > 0,\ \forall\ i \geq \mathbb{N}(y_o),\\ z(t_0) \in\ \mathrm{Int}(\mathbb{C})\end{aligned} \tag{5.34}$$

Thus it can be clearly seen that as $\lim_{i\to\infty}\mathbb{Y}_i > 0$, which violates the assumption

made in the beginning of the proof or in other words the inter–event time for the designed safety–critical event triggered control can never be zero thereby avoiding the existence of Zeno phenomenon. ■

Remark 3: The avoidance of Zeno phenomenon is only significant for homogeneous measure chain $\mathbb{R}$ or non homogeneous measure chains like $\mathbb{P}_{a,b}$ as given in Fig. 1.5 which contains some continuous section in them. The use of measure chain theory tools allows us to prove avoidance of Zeno phenomena for non–homogeneous measure chains like $\mathbb{P}_{a,b}$ in a compact manner unlike (Long and Wang (2022)) where the proof was done only for continuous measure chain $\mathbb{R}$.

Now, we are in a position to state a theorem which ensures stability of the closed–loop system under the influence of designed safety–critical dynamic event–triggered control law.

Assumption 2: An ISS stable Lyapunov function exists for the closed–loop system given by Eq. (5.3).

Theorem 3: Let us consider the closed–loop system given by Eq. (5.3) satisfying Assumptions 1 and 2 respectively. Then the proposed event triggering condition given in Eqs. (5.11) and (5.12) ensures that the system trajectories of the closed–loop system on any given measure chain $\mathbb{T}$ converge to a set surrounding the origin with the help of specific choice of design parameters. ■

Proof: A Lyapunov function candidate $U : \mathbb{R}^a \times \mathbb{R}^{0+} \to \mathbb{R}^{0+}$ is considered for the combined system dynamics given by Eqs. (5.3) and (5.12) respectively

$$U(z(t), \zeta(t)) = V(z(t)) + \zeta(t) \ \forall \ t \in \mathbb{T} \tag{5.35}$$

Delta–differentiating Eq. (5.35) and using Eq. (5.12) we can write

$$\begin{aligned} \overset{\Delta}{U}(z, \zeta) &= \overset{\Delta}{V}(z) + \overset{\Delta}{\zeta} \\ &\leq -\vartheta_3(\|z\|) + \vartheta(\|e\|) - q\zeta + p\varepsilon(r(z)) - \vartheta(\|e\|) \\ &\leq -\vartheta_3(\|z\|) + p\varepsilon(r(z)) - q\zeta \end{aligned} \tag{5.36}$$

From the proof of Theorem 1, we can say that $\varepsilon(r(z))$ is bounded by a Lipschitz constant $\mathcal{L}_3$. Thus,

$$\overset{\Delta}{U}(z,\zeta) \leq -\vartheta_3(\|z\|) - q\zeta + p\mathcal{L}_3 \tag{5.37}$$

Thus the set where the trajectories of the closed–loop system converge is given by

$$\mathbb{C}_a = \{(z,\zeta) \in \mathbb{R}^a \times \mathbb{R}^{0+} | \vartheta_3(\|z\|) + q\zeta \leq p\mathcal{L}_3\} \tag{5.38}$$

One can easily see that through the proper choice of design parameters, the practical set of convergence $\mathbb{C}_a$ can always be made a subset of the safe set which is obtained by concatenation of set $\mathbb{C}$ and the set denoting the domain of dynamic event–triggered variable ζ. The choice of design parameters influences the size of practical set of convergence. ■

5.2 Numerical simulations

The nonlinear dynamical system on measure chain $\mathbb{T}$ considered for numerical simulations is given by

$$\begin{aligned} \overset{\Delta}{z}_1 &= z_2 - 1.5z_1 \\ \overset{\Delta}{z}_2 &= \sin(z_1) - z_2 + u \end{aligned} \tag{5.39}$$

The expression of event–triggered control input considered for numerical simulation purposes is taken as $u(t) = -\sin(z_1(t_i)) + 0.5z_1(t_i) - z_2(t_i), \forall\, t \in [t_i, t_{i+1}) \cap \mathbb{T},\ i \in \mathbb{N}^0$. The expression of closed–loop system is given by

$$\begin{aligned} \overset{\Delta}{z}_1(t) &= z_2(t) - 1.5z_1(t) \\ \overset{\Delta}{z}_2(t) &= \sin(z_1(t)) - z_2(t) - \sin(z_1(t_i)) + 0.5z_1(t_i) - z_2(t_i),\ \forall\, t\ \in\ \mathbb{T} \end{aligned} \tag{5.40}$$

The safe set constructed with the help of the barrier function r is taken as $\mathbb{C} = \{z \in \mathbb{R}^{0+} \times \mathbb{R}^{0+} | r(z) = 1 - z_1 - z_2 \geq 0\}$. Thus, the final goal is to ensure that the closed–loop system given by Eq. (5.39) on any measure chain $\mathbb{T}$ is always

within the safe set $\mathbb{C}$ with the aid of event–triggered control algorithm u proposed in this section. Delta–differentiating the barrier function along the directions of closed–loop system given by Eq. (5.40) one obtains

$$\begin{aligned}\overset{\Delta}{r}(z(t)) &= -\sin(z_1(t)) + \sin(z_1(t_i)) - 0.5e_1(t) + e_2(t) + 1 - r(z(t)) \\ &\geq -3\|e(t)\|^2 - r(z(t)),\ \forall\ t\ \in\ \mathbb{T}\end{aligned} \tag{5.41}$$

Thus, Assumption 1 holds true with $\varepsilon(a) = a$ and $\vartheta(a) = 3a^2$. The expression of event–triggering condition for simulation purposes,

$$\begin{aligned}t_{i+1} = \inf\{\mathbb{T} \ni t > t_i | \zeta(t) + \eta(p(r(z(t))) - 3(\|e(t)\|)^2) \leq 0\} \\ \forall\ i \in \mathbb{N}^{0+}\end{aligned} \tag{5.42}$$

The dynamics of the dynamic variable on the same measure chain $\mathbb{T}$ as the system dynamics is given by

$$\begin{aligned}\overset{\Delta}{\zeta}(t) &= -q\zeta(t) + p(r(z(t))) - 3(\|e(t)\|)^2,\ \forall\ t\ \in\ \mathbb{T} \\ \zeta(t_0) &= r(z(t_0)) \geq 0\end{aligned} \tag{5.43}$$

with parameters $\eta \in \mathbb{R}^{0+}$, $q \in \mathbb{R}^{0+}$ and $p \in \mathbb{R}^{0+} \cap (0, 1)$ respectively. In order to prove safety, we consider a delta–differentiable function $Z(r(z), \zeta) = r(z) - \zeta$. Delta–differentiating the function,

$$\begin{aligned}\overset{\Delta}{Z}(r(z), \zeta) &\geq -3\|e\|^2 - r(z) + q\zeta \\ &- p(r(z)) + 3(\|e\|)^2 \geq -(1 + p)(r(z)) + q\zeta\end{aligned} \tag{5.44}$$

Thus event–triggered safety and avoidance of Zeno phenomenon is proved with the help of Theorem 2 provided the gain constants are selected based on Theorem 1. The last step is to prove closed–loop stability for which we consider a delta–differentiable function $U(z, \zeta) = V(z) + \zeta = z_1^2 + z_2^2 + \zeta$. Delta–differentiating and using the system dynamics given in Eq. (5.40) we can write,

$$\begin{aligned}\overset{\Delta}{U}(z, \zeta) &= 2\overset{\Delta}{z}_1 z_1 + \mu\overset{\Delta}{z}_1\overset{\Delta}{z}_1 + 2\overset{\Delta}{z}_2 z_2 + \mu\overset{\Delta}{z}_2\overset{\Delta}{z}_2 + \overset{\Delta}{\zeta} \\ &= (2 - 3\mu)z_2 z_1 - 3z_1^2 + \mu z_2^2 + 2.25\mu z_1^2 + 2z_2 \sin z_1\end{aligned}$$

$$
\begin{aligned}
&- 2z_2\sin(z_1(t_i)) + z_2 z_1(t_i) - 2z_2^2 - 2z_2 z_2(t_i) + \mu\sin^2(z_1) \\
&+ \mu\sin^2(z_1(t_i)) + 0.25\mu z_1^2(t_i) + \mu z_2^2(t_i) - 2\mu\sin(z_1(t))z_2 \\
&- 2\mu\sin(z_1)\sin(z_1(t_i)) + \mu\sin(z_1)z_1(t_i) - 2\mu\sin(z_1)z_2(t_i) \\
&+ 2\mu z_2\sin(z_1(t_i)) + \mu z_2^2 - \mu z_2 z_1(t_i) + 2\mu z_2 z_2(t_i) \\
&- \mu\sin(z_1(t_i))z_1(t_i) + 2\mu\sin(z_1(t_i))z_2(t_i) - \mu z_1(t_i)z_2(t_i) - q\zeta \\
&+ p(r(z)) - 3(\|e\|)^2
\end{aligned} \tag{5.45}
$$

We have considered the following set of inequalities to simplify Eq. (5.45) with the help of Lemma 1,

$$
2z_2\sin(z_1) - 2z_2\sin(z_1(t_i)) \leq z_2^2 + \|e\|^2 \tag{5.46}
$$

$$
\begin{aligned}
&z_2 z_1(t_i) = z_2(e_1 + z_1) = z_2 e_1 + z_2 z_1 \\
&\leq \frac{\|e\|^2}{2a_1} + \frac{a_1}{2} z_2^2 + z_2 z_1
\end{aligned} \tag{5.47}
$$

$$
\begin{aligned}
&- 2z_2 z_2(t_i) = -2z_2(e_2 + z_2) = -2z_2 e_2 - 2z_2^2 \leq \frac{\|e\|^2}{a_2} + a_2 z_2^2 \\
&- 2z_2^2
\end{aligned} \tag{5.48}
$$

$$
\mu\sin^2(z_1) + \mu\sin^2(z_1(t_i)) \leq 2\mu \tag{5.49}
$$

$$
\begin{aligned}
&0.25\mu z_1^2(t_i) = 0.25\mu(e_1 + z_1)^2 \leq 0.25\mu e_1^2 + 0.25\mu z_1^2 + \frac{0.25\mu\|e\|^2}{a_3} \\
&+ \mu 0.25 a_3\|z\|^2
\end{aligned} \tag{5.50}
$$

$$
\mu z_2^2(t_i) \leq \mu e_2^2 + \mu z_2^2 + \frac{\mu\|e\|^2}{a_2} + \mu a_2 z_2^2 \tag{5.51}
$$

$$
- 2\mu\sin(z_1)z_2 \leq 2\mu\|z\|^2 \tag{5.52}
$$

$$-2\mu\sin(z_1)\sin(z_1(t_i)) \leq 2\mu \tag{5.53}$$

$$\mu\sin(z_1)z_1(t_i) = \mu\sin(z_1)(e_1+z_1) \leq \frac{\mu\|e\|^2}{2a_4} + \mu\frac{a_4}{2}\|z\|^2 + \mu\|z\|^2 \tag{5.54}$$

$$\begin{aligned} -2\mu\sin(z_1)z_2(t_i) &= -2\mu\sin(z_1)e_2 - 2\mu\sin(z_1)z_2 \\ &\leq \frac{\mu\|e\|^2}{a_4} + \mu a_4\|z\|^2 + \frac{\mu\|z\|^2}{a_5} + \mu a_5\|z\|^2 \end{aligned} \tag{5.55}$$

$$2\mu z_2\sin(z_1(t_i)) \leq 2\mu\|z\|^2 \tag{5.56}$$

$$\begin{aligned} -\mu z_2 z_1(t_i) &= -\mu z_2 e_1 - \mu z_2 z_1 \\ &\leq \frac{\mu\|e\|^2}{2a_1} + \mu\frac{a_1}{2}\|z\|^2 - \mu z_2 z_1 \end{aligned} \tag{5.57}$$

$$2\mu z_2 z_2(t_i) = 2\mu z_2 e_2 + 2\mu z_2^2 \leq 2\mu z_2^2 + \frac{\|e\|^2}{a_2} + a_2 z_2^2 \tag{5.58}$$

$$\begin{aligned} -\mu\sin(z_1(t_i))z_1(t_i) &= -\mu\sin(z_1(t_i))(e_1+z_1) \\ &\leq \frac{\mu}{2} + \frac{\mu}{2}\|e\|^2 + \frac{\mu}{2}\|z\|^2 + \frac{\mu\|e\|^2}{2a_3} + \frac{\mu a_3}{2}\|z\|^2 \end{aligned} \tag{5.59}$$

$$\begin{aligned} 2\mu\sin(z_1(t_i))z_2(t_i) &= 2\mu\sin(z_1(t_i))(e_2+z_2) \\ &\leq \mu + \mu\|z\|^2 + \mu\|e\|^2 + \frac{\mu\|e\|^2}{a_2} + \mu a_2 z_2^2 \end{aligned} \tag{5.60}$$

$$\begin{aligned} -\mu z_1(t_i)z_2(t_i) &= -\mu e_1 e_2 - \mu e_1 z_2 \\ -\mu z_1 e_2 - \mu z_1 z_2 &\leq \mu\|e\|^2 - \mu z_1 z_2 \end{aligned}$$

$$+\frac{\mu\|e\|^2}{2a_1}+\frac{\mu a_1}{2}\|z\|^2+\frac{\mu\|e\|^2}{2a_4}+\frac{\mu a_4}{2}\|z\|^2 \tag{5.61}$$

where a_1, a_2, a_3, a_4 and a_5 are positive constants. Thus, using the inequalities in Eq. (5.45) one obtains

$$\begin{aligned}
&\overset{\Delta}{U}(z,\zeta)\leq(3-5\mu)z_2z_1-z_1^2-2\|z\|^2\\
&+(1+2a_2+2\mu a_2+\frac{a_1}{2})z_2^2+(14\mu+\mu a_1\\
&+0.75\mu a_3+2\mu a_4+\mu a_5+\frac{\mu}{a_5})\|z\|^2+(1+\\
&\frac{1+2\mu}{2a_1}+\frac{2+2\mu}{a_2}+2.75\mu+\frac{1.75\mu}{a_3}+\frac{2\mu}{a_4})\\
&\|e\|^2+5.5\mu-q\zeta(t)+p(r(z))-3(\|e\|)^2
\end{aligned} \tag{5.62}$$

Thus, a way to select the positive constants is to ensure that the following equation is satisfied for any measure chain $\mathbb{T}$,

$$\frac{1+2\mu}{2a_1}+\frac{2+2\mu}{a_2}+2.75\mu+\frac{1.75\mu}{a_3}+\frac{2\mu}{a_4}\leq 2 \tag{5.63}$$

When Eq. (5.63) is satisfied then the expression of set $\mathbb{C}_a$ becomes

$$\begin{aligned}
&\mathbb{C}_a=\{(z,\zeta)\in(\mathbb{R}^{0+})^2\times\mathbb{R}^{0+}|-(3-5\mu)z_2z_1\\
&z_1^2+2\|z\|^2-(1+2a_2+2\mu a_2+\frac{a_1}{2})z_2^2-(14\mu\\
&+\mu)a_1+0.75\mu a_3+2\mu a_4+\mu a_5+\frac{\mu}{a_5})\\
&\|z\|^2+q\zeta-5.5\mu\leq p\mathcal{L}_3\}
\end{aligned} \tag{5.64}$$

This completes the proof of closed–loop stability for system given in Eq. (5.39) under the influence of designed safety–critical event triggered control law.
We now consider three different cases

1. **Case 1:** When the measure chain $\mathbb{T}=\mathbb{R}$ ($\mu=0$) is continuous.
2. **Case 2:** When the measure chain $\mathbb{T}=\mu\mathbb{Z}$ ($\mu>0$) is a discrete homogeneous measure chain.

3. **Case 3:** When the measure chain is a continuous–discrete measure chain where the graininess function varies as, $\mu \in \{0\ , 0.05\ , 0.01\}$. The non–homogeneous measure chain can be written as the union of six distinct homogeneous measure chains ($\mathbb{T} = \mathbb{T}_a \cup \mathbb{T}_b \cup \mathbb{T}_c \cup \mathbb{T}_d \cup \mathbb{T}_e \cup \mathbb{T}_f$) where $\mathbb{T}_{a,d,f}$ are continuous measure chains with $\mu = 0$, $\mathbb{T}_{c,e}$ are discrete measure chains with $\mu = 0.01$ and $\mathbb{T}_b$ is a measure chain with $\mu = 0.05$.

The initial conditions for all the three different cases are taken as $z(t_0) = \begin{bmatrix} 0 & 0.9 \end{bmatrix}^T$.

1. **Case 1:** The values of design constants are taken as $\eta = 2$, $q = 5$ and $p = 0.0001$ respectively keeping in mind the conditions stated in Theorem 1. Eq. (5.63), can be easily satisfied with a proper choice of a_1 and a_2 as $\mu = 0$. The choice of $a_1 \geq 0.75$ and $a_2 = 1.5$ defines the dimension of convergence set $\mathbb{C}_a$. The plots of state–trajectories along with safe–set, inter–event time and dynamic event–triggering variable are given in Figs. 5.1, 5.2 and 5.3 respectively.

2. **Case 2:** The discrete homogeneous measure chain has a graininess function value of $\mu = 0.05$. The values of design constants for this case are $\eta = 1.5$, $q = 2$ and $p = 0.0001$ respectively which ensures that the conditions given in Theorem 1 are satisfied. The design constants satisfy both the conditions given in Theorem 1. The values of $a_1 = 1$, $a_2 \geq 1.87$ $a_3 = 1$, $a_4 = 1$ are used to satisfy Eq. (5.63), which impact the size of the practical set of convergence $\mathbb{C}_a$. The plots of state–trajectories along with safe–set, inter–event time and dynamic event–triggering variable are given in Figs 5.4, 5.5 and 5.6 respectively.

3. **Case 3:** For the non–homogeneous measure chain of Case 3, the values of design constants are the similar to that of Case 1 when $\mu = 0$ and when $\mu = 0.01$ or $\mu = 0.05$ it is similar to Case 2, which automatically ensures that the conditions given in Theorem 1 are true for the entire measure chain. The practical set of convergence $\mathbb{C}_a$ for this Case can be obtained by intersection of the sets obtained with different values of μ (i.e. $\mu = 0$, $\mu = 0.05$ and $\mu = 0.01$ respectively). Similar to the previous cases, the

plots of state–trajectories along with safe–set, inter–event time and dynamic event–triggering variable are given in Figs. 5.7, 5.8 and 5.9 respectively.

When we look at the plots given in Figs. 5.2, 5.5 and 5.8 which correspond to three numerical simulation cases considered in this chapter, one can easily see that the proposed safety–critical event–triggered works satisfactorily for any given measure chain $\mathbb{T}$ as it avoids Zeno phenomena for the continuous time case (Case a) or continuous portion of the continuous–discrete case (Case c). Furthermore, for the purely discrete time case (Case b) it also ensures non–trivial implementation (i.e. the control algorithm is not updated at every time instant). Also, the system trajectories for any measure chain remain within the safe set under the influence of the designed safety–critical event–triggered control algorithm. In case of representation purposes only the absolute values of state variables are used in this chapter.

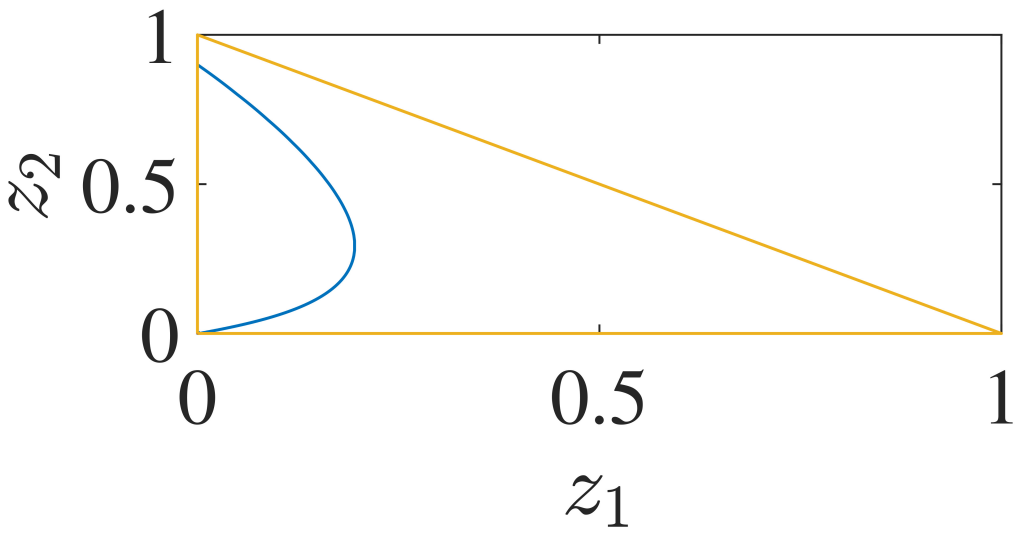


FIGURE 5.1: *The yellow line is used to represent the boundaries of the safe–set whereas blue line represents the system trajectories of system represented by Eq. 5.40 on* $\mathbb{T} = \mathbb{R}$.

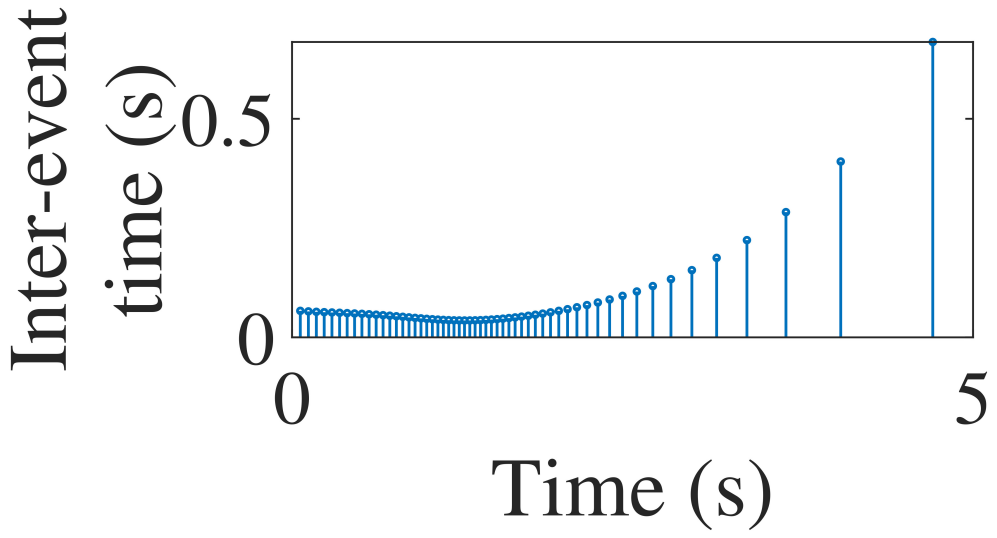


FIGURE 5.2: *Inter–event time for continuous measure chain* $\mathbb{R}$.

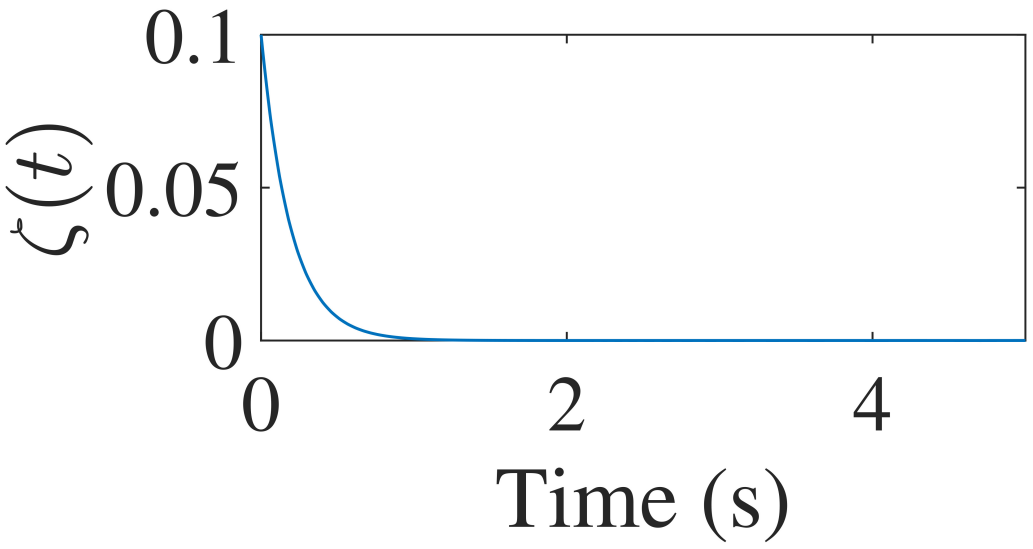


FIGURE 5.3: *The plot of dynamic event–triggering variable* $\zeta(t)$ *for continuous measure chain* $\mathbb{R}$.

5.3 Conclusion

In this chapter, a generalized safe event–triggering mechanism is proposed for nonlinear systems on any time domain or measure chains. Safety, avoidance of Zeno phenomenon as well as stability to a practical set of convergence for the closed–loop system under the influence of designed event–triggering algorithm is proved.

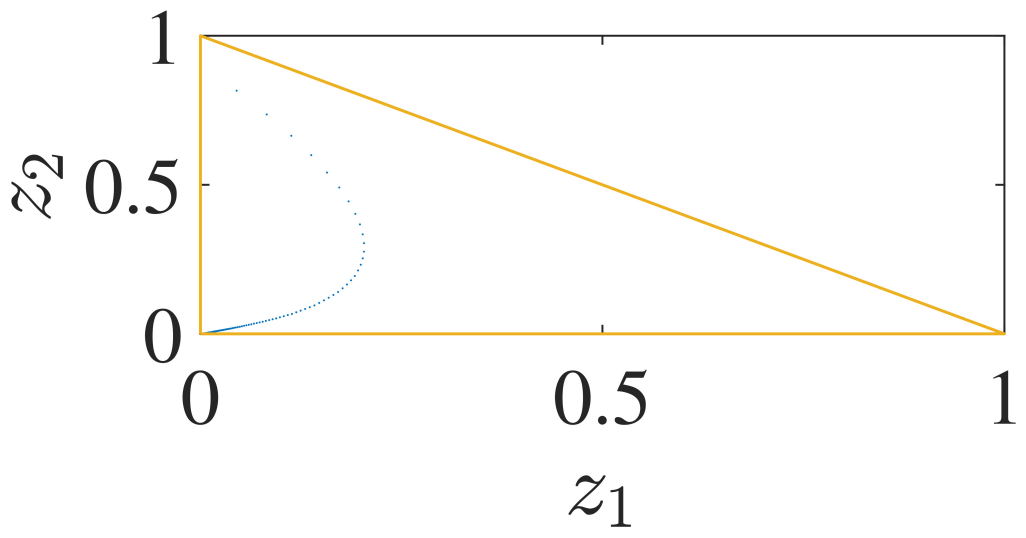


FIGURE 5.4: *The yellow line is used to represent the boundaries of the safe–set whereas blue line represents the system trajectories of system represented by Eq. 5.40 on* $\mathbb{T} = \mu\mathbb{Z}$.

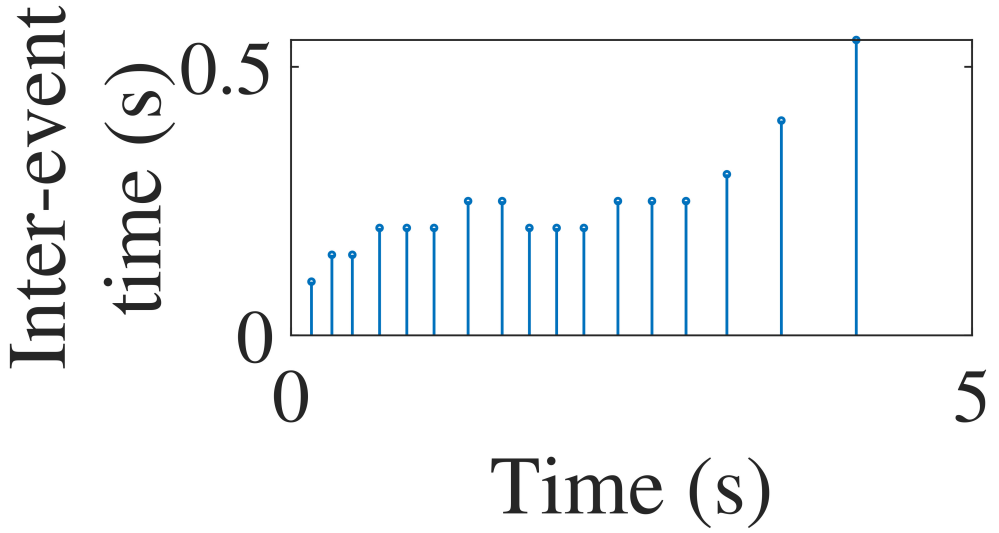


FIGURE 5.5: Inter–event time for homogeneous discrete measure chain $\mu\mathbb{Z}$.

Future research direction would involve considering the effect of time delays in designing safety–critical event–triggered control algorithms and also extending the results to periodic dynamic safety–critical event–triggered or self–triggered safety–critical control on any measure chain $\mathbb{T}$.

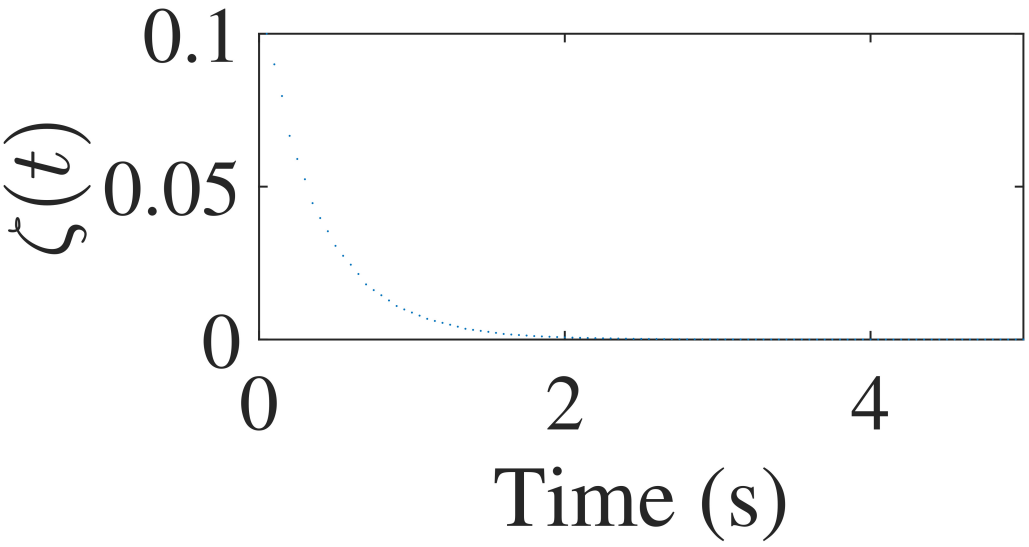


FIGURE 5.6: *The plot of dynamic event–triggering variable $\zeta(t)$ for discrete measure chain $\mu\mathbb{Z}$.*

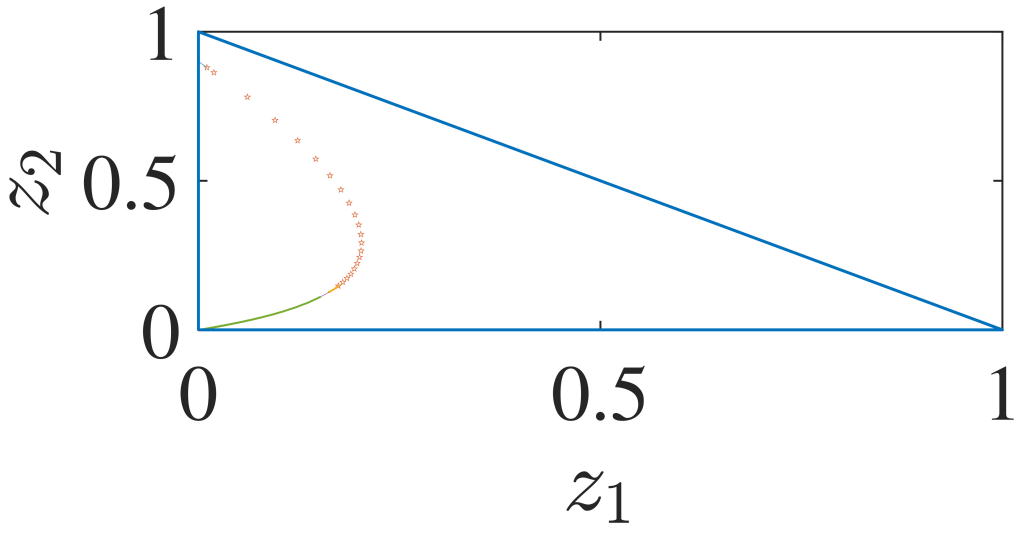


FIGURE 5.7: *The blue line here represents the boundaries of the safe–set whereas the system trajectories are represented by different colors each representing a different graininess function region of the non–homogeneous measure chain considered in Case c.*

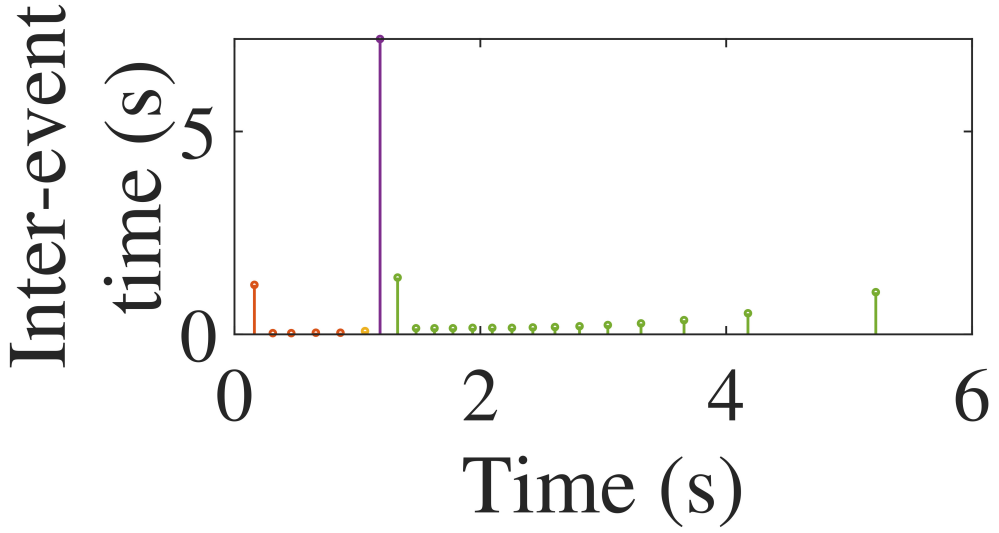


FIGURE 5.8: *Inter–event time of the non–homogeneous measure chain considered in Case c.*

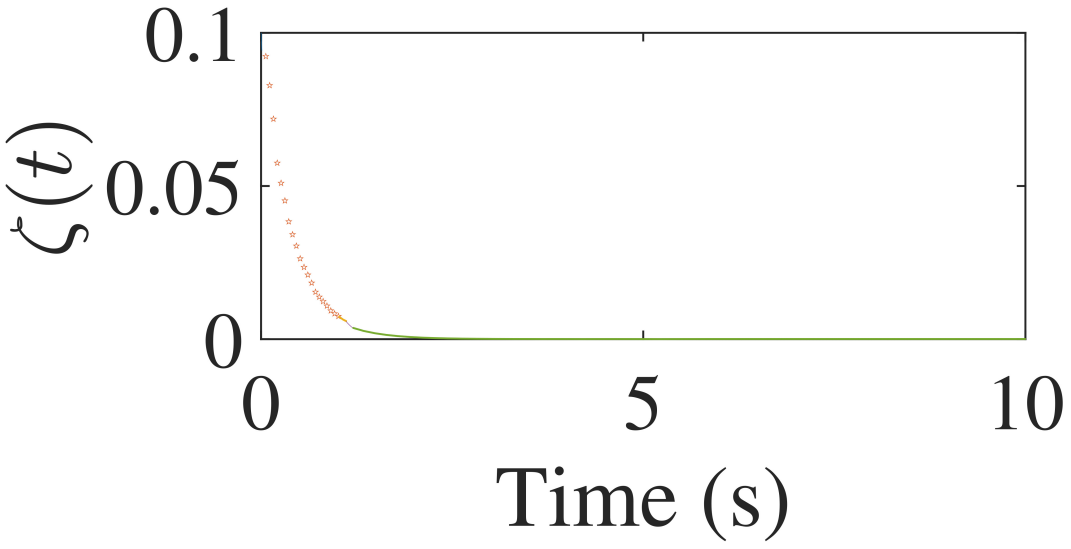


FIGURE 5.9: *The plot of dynamic event–triggered variable for the non–homogeneous measure chain considered in Case c.*

Chapter 6

General Conclusions and Future Perspectives

This thesis deals with designing control algorithms for systems on any arbitrary time domain using the theory of measure chains.

- In Chapter 1 of my thesis, I have introduced some mathematical preliminaries which are needed to understand the problems solved in later parts of the thesis. Also in this chapter I have stated the general motivation of my work. A brief literature review of the major topics explored in this thesis can also be found in Chapter 1.

- In Chapter 2 of my thesis, I have addressed the problem of observer based leader–follower bipartite consensus under intermittent failures in the communication channels by using the concept of switched systems on a specific $\mathbb{P}_{a,b}$ measure chain. A theorem is proposed in the chapter which helps us to design the observer and controller gain constants thereby ensuring the stability of closed loop switched system on measure chain.

- In Chapter 3 of my thesis, I have addressed the problem of event–triggered control for systems on arbitrary time domain using emulation based method and the designed event–triggered control algorithm belongs to the broad class of continuous event–triggered control algorithm wherein the event condition

is evaluated at every instant of time on the measure chain. The event–triggered design method proposed in this chapter not only unifies existing design methods for continuous or discrete systems but also extends them to systems on any time domain.

- In Chapter 4 of my thesis, I have addressed the problem of periodic event–triggered control for a class of nonlinear systems on measure chain. I have also designed a periodic event–triggered impulsive observer for the similar class of systems on measure chains. Finally, an output feedback controller is designed with the help of aforementioned periodic event–triggered impulsive observer.

- In Chapter 5 of my thesis, I have investigated the safety–critical event–triggered problem for nonlinear systems on measure chains using a dynamic event–triggered formalism and novel safe barrier functions on measure chains. Furthermore, no assumption on nonlinearity is considered in this work and we believe such a work is of utmost significance if we want to apply event–triggered control algorithms on measure chains for practical real life applications.

Now, we will state some future research directions

- In case of the problem statement addressed in Chapter 2, a possible future research direction would involve exploring the use of multiple Lyapunov functions or parameter dependent Lyapunov functions in order to prove stability of the switched system on $\mathbb{P}_{a,b}$ measure chain. Future research can also be performed for solving matrix–weighted bipartite consensus problem instead of scalar bipartite consensus problem addressed in chapter 2. Recently, there is also some focus on open multi–agent systems wherein agents can enter, leave or replace other agents in the network structure (Hendrickx and Martin (2016)). One can also consider faults in the network topology of the open multi–agent systems and solve the problem using the method of switched systems on measure chain theory explained in chapter 2 (Xue et al. (2020)).

- The results for periodic event–triggered control proposed in Chapter 4 gives conservative results because of the stability method used to prove closed loop

stability of the system. Future research can involve solving the same problem by taking inspirations from the stability tools reviewed in (Hetel et al. (2017)) and extending them to solve the aforementioned event–triggered problem for systems on any measure chain.

- The results presented in Chapter 5 of this thesis assumes the existence of an ISS stable barrier function on measure chain which is a significant limitation of the control design procedure explored in the corresponding chapter. Future research can involve designing safety–critical event–triggered control for systems with unknown dynamics on measure chains. An introductory text along the direction will be (Xiao et al. (2022)) where the results are reported for systems on continuous measure chain $\mathbb{R}$. Research can also be carried out in the direction of learning the control barrier functions from experimental data as given in (Lindemann et al. (2021), Robey et al. (2020), Dai et al. (2022)).

- There are some recent results in the literature where the concept of positivity for systems on measure chains have been addressed (Xiao et al. (2020), Xiao et al. (2021)). Furthermore, event–triggered control using positive systems approach have been explored in some of the recent papers (Mazenc et al. (2021), Mazenc et al. (2022)). Thus, a possible future research direction will be to design event–triggered controllers for systems on measure chains using the positive systems method.

- There are some results related to solving time delayed systems on measure chains using Halanay inequalities as given in (Adıvar and Bohner (2011)). The research on event–triggered control of time delayed systems has received some extensive attention in the past few years (Zhang et al. (2021), Zhang et al. (2022b), Zhang et al. (2022a)). Solving the problem of event–triggered control of time delayed system on measure chains with the aid of Halanay inequalities can be an interesting direction of future research.

- Self–triggered control is an area which is also untouched in this thesis. Research can also be carried out in the direction which addresses of self–triggered control for systems on measure chain.

- There are some interesting research work in the field of reduced order event–triggered controllers or interval observer based event–triggered controllers which can be extended to systems on measure chains (Rabehi et al. (2020)).

- Some results for stochastic systems on measure chains can be found in (Poulsen et al. (2017), Poulsen et al. (2013), Poulsen et al. (2019), Poulsen and Wintz (2019), Poulsen et al. (2012), Poulsen et al. (2011)). Event–triggering for stochastic systems on measure chains is an interesting concept which can be explored in the future.

- Recently, people have been working on the analysis of inter event–time and its various properties. Some similar research can also be carried out for systems on measure chains (Rajan and Tallapragada (2020), Rajan and Tallapragada (2022), Postoyan et al. (2019)).

- Research can also be carried out in the direction of formal analysis of event–triggered control algorithms or their corresponding inter–event time analysis for systems on measure chains (Kolarijani and Mazo (2016), Fu and Mazo (2018), Hashimoto et al. (2019)).

Appendix A

Comparison Functions

Let $C(X, Y)$ be used to define the set of continuous functions from set X to set Y. The classes of comparison functions are given by

- $\mathcal{P} := \{\gamma \in C(\mathbb{R}_+, \mathbb{R}_+) : \gamma(0) = 0 \text{ and } \gamma(r) > 0 \text{ for } r > 0\}$.
- $\mathcal{K} := \{\gamma \in \mathcal{P} : \gamma \text{ is strictly increasing}\}$.
- $\mathcal{K}_\infty := \{\gamma \in \mathcal{K} : \gamma \text{ is unbounded}\}$.
- $\mathcal{L} := \{\gamma \in C(\mathbb{R}_+, \mathbb{R}_+) : \gamma \text{ is strictly decreasing with } \lim_{t\to\infty} \gamma(t) = 0\}$.
- $\mathcal{KL} := \{\beta \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+) : \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K},\ \forall t \geq 0,\ \beta(r, \cdot) \in \mathcal{L},\ \forall r > 0\}$.
- A continuous function $\gamma : (-b, a) \to \mathbb{R}$ with $a, b > 0$, is extended class $\mathcal{K}(\alpha \in \mathcal{K}_e)$ if $\gamma(0) = 0$ and γ is strictly monotonically increasing.
- An extended class $\mathcal{K}$ function if $\gamma(0) = 0$ and γ is strictly monotonically increasing. If $a, b = \infty$, $\lim_{r\to\infty}\gamma(r) = \infty$, and $\lim_{r\to-\infty}\gamma(r) = -\infty$, then γ is an extended class $\mathcal{K}_\infty(\gamma \in \mathcal{K}_{\infty,e})$.

Appendix B

Input–to–state stability and Lie derivatives

Consider a nonlinear system of the form (Sontag (2008))

$$\dot{x} = f(x, u) \tag{B.1}$$

where $x \in \mathbb{R}^{\kappa}$. The input $u \in \mathbb{R}^m$ is measurably locally essentially bounded. The map $f : \mathbb{R}^n \times \mathbb{R}^m$ is locally Lipschitz considering $f(0, 0) = 0$. The system given by Eq. (B.1) is said to be input–to–state stable for any initial condition $x(0)$, control input $u(.)$ and $t \geq 0$,

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|u\|)_{\infty} \tag{B.2}$$

where $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_{\infty}$. In absence of any input $u(t)$, the definition of ISS stability converts to the concept of "0-GAS" or more commonly the GAS.
Consider a nonlinear control affine system of the form

$$\dot{x} = f(x) + g(x)u \tag{B.3}$$

$$y = h(x) \tag{B.4}$$

where f, g and h are sufficiently smooth functions in a domain D. $f : D \to \mathbb{R}^n$ and $g : D \to \mathbb{R}^n$ are vector fields on D. Then,

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] = \mathcal{L}_f h(x) + \mathcal{L}_g h(x)u \tag{B.5}$$

Thus, $\mathcal{L}_f h(x) = \frac{\partial h}{\partial x} f(x)$ is know as the Lie derivative of h with respect to f or along the direction f. Some other Lie derivatives are also stated below

$$\mathcal{L}_g \mathcal{L}_f h(x) = \frac{\partial (\mathcal{L}_f h)}{\partial x} g(x) \tag{B.6}$$

$$\mathcal{L}_f^2 h(x) = \mathcal{L}_f \mathcal{L}_f h(x) = \frac{\partial (\mathcal{L}_f h)}{\partial x} f(x) \tag{B.7}$$

$$\mathcal{L}_f^k h(x) = \mathcal{L}_f \mathcal{L}_f^{k-1} h(x) = \frac{\partial (\mathcal{L}_f^{k-1} h)}{\partial x} f(x) \tag{B.8}$$

$$\mathcal{L}_f^0 h(x) = h(x) \tag{B.9}$$

Appendix C

Publications

Journals:

- Dey, S., Taousser, F.Z., Djemai, M., Defoort, M. and Di Gennaro, S., 2020. Observer based leader–follower bipartite consensus with intermittent failures using Lyapunov functions and time scale theory. IEEE Control Systems Letters, 5(6), pp.1904-1909.
- Dey, S., Defoort, M., Djemai, M. and Di Gennaro, S., 2021. Event–Triggered Control for Systems on Non–Uniform Time Domains Using Measure Chain Theory. IEEE Transactions on Circuits and Systems II: Express Briefs, 69(3), pp.1198-1202.
- "Safety–critical event–triggered control on measure chains" under submission in a journal.
- "Periodic event–triggered control on measure chains" under submission in a journal.

Conferences:

- Dey, S., Taousser, F.Z., Djemai, M., Defoort, M. and Di Gennaro, S., Observer based leader–follower bipartite consensus with intermittent failures using Lyapunov functions and time scale theory. (ACC 2021)

Bibliography

Åarzén, K.-E. (1999). A simple event-based pid controller. *IFAC Proceedings Volumes*, 32(2):8687–8692.

Abdelrahim, M., Dolk, V. S., and Heemels, W. (2019). Event-triggered quantized control for input-to-state stabilization of linear systems with distributed output sensors. *IEEE Transactions on Automatic Control*, 64(12):4952–4967.

Abdelrahim, M., Postoyan, R., Daafouz, J., and Nešić, D. (2015). Stabilization of nonlinear systems using event-triggered output feedback controllers. *IEEE Transactions on Automatic Control*, 61(9):2682–2687.

Abdelrahim, M., Postoyan, R., Daafouz, J., and Nešić, D. (2017). Robust event-triggered output feedback controllers for nonlinear systems. *Automatica*, 75:96–108.

Abraham, R., Marsden, J. E., and Ratiu, T. (2012). *Manifolds, tensor analysis, and applications*, volume 75. Springer Science & Business Media.

Adıvar, M. and Bohner, E. A. (2011). Halanay type inequalities on time scales with applications. *Nonlinear Analysis: Theory, Methods & Applications*, 74(18):7519–7531.

Agarwal, R., Bohner, M., o'Regan, D., and Peterson, A. (2002). Dynamic equations on time scales: a survey. *Journal of Computational and Applied Mathematics*, 141(1-2):1–26.

Agrachev, A. A. and Liberzon, D. (2001). Lie-algebraic stability criteria for switched systems. *SIAM Journal on Control and Optimization*, 40(1):253–269.

Akin-Bohner, E., Bohner, M., and Akin, F. (2005). Pachpatte inequalities on time scales. *Journal of Inequalities in Pure and Applied Mathematics*, 6(1):1–23.

Akopyan, F., Manohar, R., and Apsel, A. B. (2006). A level-crossing flash asynchronous analog-to-digital converter. In *12th IEEE International Symposium on Asynchronous Circuits and Systems (ASYNC'06)*, pages 11–pp. IEEE.

Alpern, B. and Schneider, F. B. (1985). Defining liveness. *Information processing letters*, 21(4):181–185.

Altafini, C. (2012). Consensus problems on networks with antagonistic interactions. *IEEE transactions on automatic control*, 58(4):935–946.

Ames, A. D., Coogan, S., Egerstedt, M., Notomista, G., Sreenath, K., and Tabuada, P. (2019). Control barrier functions: Theory and applications. In *2019 18th European control conference (ECC)*, pages 3420–3431. IEEE.

Ames, A. D., Galloway, K., Sreenath, K., and Grizzle, J. W. (2014a). Rapidly exponentially stabilizing control lyapunov functions and hybrid zero dynamics. *IEEE Transactions on Automatic Control*, 59(4):876–891.

Ames, A. D., Grizzle, J. W., and Tabuada, P. (2014b). Control barrier function based quadratic programs with application to adaptive cruise control. In *53rd IEEE Conference on Decision and Control*, pages 6271–6278. IEEE.

Ames, A. D. and Powell, M. (2013). Towards the unification of locomotion and manipulation through control lyapunov functions and quadratic programs. In *Control of Cyber-Physical Systems*, pages 219–240. Springer.

Ames, A. D., Xu, X., Grizzle, J. W., and Tabuada, P. (2016). Control barrier function based quadratic programs for safety critical systems. *IEEE Transactions on Automatic Control*, 62(8):3861–3876.

Andrieu, V. and Nadri, M. (2010). Observer design for lipschitz systems with discrete-time measurements. In *49th IEEE Conference on Decision and Control (CDC)*, pages 6522–6527. IEEE.

Anta, A. and Tabuada, P. (2010). To sample or not to sample: Self-triggered control for nonlinear systems. *IEEE Transactions on automatic control*, 55(9):2030–2042.

Artstein, Z. (1983). Stabilization with relaxed controls. *Nonlinear Analysis: Theory, Methods & Applications*, 7(11):1163–1173.

Asarin, E., Bournez, O., Dang, T., Maler, O., and Pnueli, A. (2000). Effective synthesis of switching controllers for linear systems. *Proceedings of the IEEE*, 88(7):1011–1025.

Aström, K. J. (2008). Event based control. In *Analysis and design of nonlinear control systems*, pages 127–147. Springer.

Åström, K. J. and Bernhardsson, B. (1999). Comparison of periodic and event based sampling for first-order stochastic systems. *IFAC Proceedings Volumes*, 32(2):5006–5011.

Åström, K. J. and Bernhardsson, B. (2003). Systems with lebesgue sampling. In *Directions in mathematical systems theory and optimization*, pages 1–13. Springer.

Astrom, K. J. and Bernhardsson, B. M. (2002). Comparison of riemann and lebesgue sampling for first order stochastic systems. In *Proceedings of the 41st IEEE Conference on Decision and Control, 2002.*, volume 2, pages 2011–2016. IEEE.

Atici, F. M. and Uysal, F. (2008). A production–inventory model of hmms on time scales. *Applied Mathematics Letters*, 21(3):236–243.

Aubin, J.-P. (1990). A survey of viability theory. *SIAM Journal on Control and Optimization*, 28(4):749–788.

Aubin, J.-P., Bayen, A. M., and Saint-Pierre, P. (2011). *Viability theory: new directions*. Springer Science & Business Media.

Aubin, J.-P. and Frankowska, H. (2009). *Set-valued analysis*. Springer Science & Business Media.

Babenko, S., Defoort, M., Djemai, M., and Nicaise, S. (2018). On the consensus tracking investigation for multi-agent systems on time scale via matrix-valued lyapunov functions. *Automatica*, 97:316–326.

Bacciotti, A. and Mazzi, L. (2005). An invariance principle for nonlinear switched systems. *Systems & Control Letters*, 54(11):1109–1119.

Bahavarnia, M., Mousavi, H. K., and Motee, N. (2018). Sparse sensing, communication, and actuation via self-triggered control algorithms. *arXiv preprint arXiv:1812.08964*.

Bartosiewicz, Z. and Piotrowska, E. (2013). On stabilisability of nonlinear systems on time scales. *International Journal of Control*, 86(1):139–145.

Bemporad, A., Ferrari-Trecate, G., and Morari, M. (2000). Observability and controllability of piecewise affine and hybrid systems. *IEEE transactions on automatic control*, 45(10):1864–1876.

Bemporad, A. and Morari, M. (1999). Control of systems integrating logic, dynamics, and constraints. *Automatica*, 35(3):407–427.

Bhowmick, S. and Panja, S. (2018). Leader–follower bipartite consensus of linear multiagent systems over a signed directed graph. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 66(8):1436–1440.

Blanchini, F. (1999). Set invariance in control. *Automatica*, 35(11):1747–1767.

Blanchini, F. and Miani, S. (2008). *Set-theoretic methods in control*, volume 78. Springer.

Bodine, S. and Lutz, D. (2003). Exponential functions on time scales: their asymptotic behavior and calculation. *Dynamic Systems and Applications*, 12(1/2):23–44.

Bohner, M. and Georgiev, S. G. (2016a). Multiple integration on time scales. In *Multivariable Dynamic Calculus on Time Scales*, pages 449–515. Springer.

Bohner, M. and Georgiev, S. G. (2016b). Partial differentiation on time scales. In *Multivariable Dynamic Calculus on Time Scales*, pages 303–447. Springer.

Bohner, M. and Guseinov, G. S. (2006). Multiple lebesgue integration on time scales. *Advances in Difference Equations*, 2006:1–12.

Bohner, M. and Peterson, A. (2001a). *Dynamic equations on time scales: An introduction with applications*. Springer Science & Business Media.

Bohner, M. and Peterson, A. (2001b). A survey of exponential functions on time scales. *Cubo Mat. Educ*, 3(2):285–301.

Bohner, M. and Peterson, A. C. (2002). *Advances in dynamic equations on time scales*. Springer Science & Business Media.

Bony, J.-M. (1969). Principe du maximum, inégalité de harnack et unicité du probleme de cauchy pour les opérateurs elliptiques dégénérés. In *Annales de l'institut Fourier*, volume 19, pages 277–304.

Borgers, D. P., Dolk, V., and Heemels, W. (2016). Dynamic event-triggered control with time regularization for linear systems. In *2016 IEEE 55th Conference on Decision and Control (CDC)*, pages 1352–1357. IEEE.

Borgers, D. P., Dolk, V. S., and Heemels, W. (2017). Riccati-based design of event-triggered controllers for linear systems with delays. *IEEE Transactions on Automatic Control*, 63(1):174–188.

Borri, A. and Pepe, P. (2020). Event-triggered control of nonlinear systems with time-varying state delays. *IEEE Transactions on Automatic Control*, 66(6):2846–2853.

Borrmann, U., Wang, L., Ames, A. D., and Egerstedt, M. (2015). Control barrier certificates for safe swarm behavior. *IFAC-PapersOnLine*, 48(27):68–73.

Branicky, M. S. (1998). Multiple lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on automatic control*, 43(4):475–482.

Brezis, H. (1970). On a characterization of flow-invariant sets. *Communications on Pure and Applied Mathematics*, 23(2):261–263.

Cai, Y., Zhang, H., Duan, J., and Zhang, J. (2020). Distributed bipartite consensus of linear multiagent systems based on event-triggered output feedback control scheme. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 51(11):6743–6756.

Cassandras, C. G., Pepyne, D. L., and Wardi, Y. (2001). Optimal control of a class of hybrid systems. *IEEE Transactions on Automatic Control*, 46(3):398–415.

Chen, W.-H., Yang, W., and Lu, X. (2014). Impulsive observer-based stabilisation of uncertain linear systems. *IET Control Theory & Applications*, 8(3):149–159.

Cheng, D. (2005). Controllability of switched bilinear systems. *IEEE Transactions on Automatic Control*, 50(4):511–515.

Cortez, W. S., Oetomo, D., Manzie, C., and Choong, P. (2019). Control barrier functions for mechanical systems: Theory and application to robotic grasping. *IEEE Transactions on Control Systems Technology*, 29(2):530–545.

DaCunha, J. J. (2004). *Lyapunov stability and Floquet theory for nonautonomous linear dynamic systems on time scales*. Baylor University.

DaCunha, J. J. (2005). Stability for time varying linear dynamic systems on time scales. *Journal of Computational and Applied Mathematics*, 176(2):381–410.

Dai, B., Krishnamurthy, P., and Khorrami, F. (2022). Learning a better control barrier function. *arXiv preprint arXiv:2205.05429*.

Davis, J. M., Gravagne, I. A., Marks, R. J., Miller, J. E., and Ramos, A. A. (2010). Stability of switched linear systems on non-uniform time domains. In *2010 42nd Southeastern Symposium on System Theory (SSST)*, pages 127–132. IEEE.

De Pasquale, G. and Valcher, M. E. (2020). Consensus problems on clustered networks. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 3675–3680. IEEE.

De Pasquale, G. and Valcher, M. E. (2022). Consensus for clusters of agents with cooperative and antagonistic relationships. *Automatica*, 135:110002.

Delimpaltadakis, G. and Mazo, M. (2020). Isochronous partitions for region-based self-triggered control. *IEEE Transactions on Automatic Control*, 66(3):1160–1173.

Delimpaltadakis, G. and Mazo, M. (2021). Region-based self-triggered control for perturbed and uncertain nonlinear systems. *IEEE Transactions on Control of Network Systems*, 8(2):757–768.

Dey, S., Defoort, M., Djemai, M., and Di Gennaro, S. (2021). Event–triggered control for systems on non–uniform time domains using measure chain theory. *IEEE Transactions on Circuits and Systems II: Express Briefs*.

Dinh, T. N., Andrieu, V., Nadri, M., and Serres, U. (2014). Continuous-discrete time observer design for lipschitz systems with sampled measurements. *IEEE Transactions on Automatic Control*, 60(3):787–792.

Doan, T. S., Kalauch, A., Siegmund, S., and Wirth, F. (2010). Stability radii for positive linear time-invariant systems on time scales. *Systems & Control Letters*, 59(3-4):173–179.

Dolk, V. and Heemels, M. (2017). Event-triggered control systems under packet losses. *Automatica*, 80:143–155.

Donkers, M. and Heemels, W. (2011). Output-based event-triggered control with guaranteed $\mathcal{L}_\infty$ -gain and improved and decentralized event-triggering. *IEEE Transactions on Automatic Control*, 57(6):1362–1376.

Du, N. H. et al. (2007). On the exponential stability of dynamic equations on time scales. *Journal of Mathematical Analysis and Applications*, 331(2):1159–1174.

Du, N. H., Liem, N. C., et al. (2011). Stability radius of implicit dynamic equations with constant coefficients on time scales. *Systems & control letters*, 60(8):596–603.

Easley, D. and Kleinberg, J. (2010). *Networks, crowds, and markets: Reasoning about a highly connected world*. Cambridge university press.

Eisenbarth, G., Davis, J. M., and Gravagne, I. (2014). Stability of simultaneously triangularizable switched systems on hybrid domains. *Electronic Journal of Differential Equations*, 63:1–19.

Eqtami, A., Dimarogonas, D. V., and Kyriakopoulos, K. J. (2010). Event-triggered control for discrete-time systems. In *Proceedings of the 2010 american control conference*, pages 4719–4724. IEEE.

Erdong, J., Xiaolei, J., and Zhaowei, S. (2008). Robust decentralized attitude coordination control of spacecraft formation. *Systems & Control Letters*, 57(7):567–577.

Etienne, L., Di Gennaro, S., and Barbot, J.-P. (2017). Periodic event-triggered observation and control for nonlinear lipschitz systems using impulsive observers. *International Journal of Robust and Nonlinear Control*, 27(18):4363–4380.

Farza, M., M'Saad, M., Fall, M. L., Pigeon, E., Gehan, O., and Busawon, K. (2013). Continuous-discrete time observers for a class of mimo nonlinear systems. *IEEE Transactions on Automatic Control*, 59(4):1060–1065.

Ferrante, F., Gouaisbaut, F., Sanfelice, R. G., and Tarbouriech, S. (2016). State estimation of linear systems in the presence of sporadic measurements. *Automatica*, 73:101–109.

Ferreira, R. A. and Silva, C. M. (2018). A nonautonomous epidemic model on time scales. *Journal of Difference Equations and Applications*, 24(8):1295–1317.

Forni, F., Galeani, S., Nešić, D., and Zaccarian, L. (2010). Lazy sensors for the scheduling of measurement samples transmission in linear closed loops over networks. In *49th IEEE conference on decision and control (CDC)*, pages 6469–6474. IEEE.

Forni, F., Galeani, S., Nešić, D., and Zaccarian, L. (2014). Event-triggered transmission for linear control over communication channels. *Automatica*, 50(2):490–498.

Fu, A. and Mazo, M. (2018). Traffic models of periodic event-triggered control systems. *IEEE Transactions on Automatic Control*, 64(8):3453–3460.

Gage, M. and Hamilton, R. S. (1986). The heat equation shrinking convex plane curves. *Journal of Differential Geometry*, 23(1):69–96.

Galloway, K., Sreenath, K., Ames, A. D., and Grizzle, J. W. (2015). Torque saturation in bipedal robotic walking through control lyapunov function-based quadratic programs. *IEEE Access*, 3:323–332.

Gard, T. and Hoffacker, J. (2003). Asymptotic behavior of natural growth on time scales. *Dynamic Systems and Applications*, 12(1/2):131–148.

Girard, A. (2014). Dynamic triggering mechanisms for event-triggered control. *IEEE Transactions on Automatic Control*, 60(7):1992–1997.

Glotfelter, P., Cortés, J., and Egerstedt, M. (2017). Nonsmooth barrier functions with applications to multi-robot systems. *IEEE control systems letters*, 1(2):310–315.

Goebel, R., Sanfelice, R. G., and Teel, A. R. (2009). Hybrid dynamical systems. *IEEE control systems magazine*, 29(2):28–93.

Goebel, R., Sanfelice, R. G., and Teel, A. R. (2012). Hybrid dynamical systems: modeling stability, and robustness.

Guan, K. M., Kozat, S. S., and Singer, A. C. (2008). Adaptive reference levels in a level-crossing analog-to-digital converter. *EURASIP Journal on Advances in Signal Processing*, 2008:1–11.

Guo, X., Liang, J., and Lu, J. (2018). Asymmetric bipartite consensus over directed networks with antagonistic interactions. *IET Control Theory & Applications*, 12(17):2295–2301.

Gurriet, T., Singletary, A., Reher, J., Ciarletta, L., Feron, E., and Ames, A. (2018). Towards a framework for realizable safety critical control through active set invariance. In *2018 ACM/IEEE 9th International Conference on Cyber-Physical Systems (ICCPS)*, pages 98–106. IEEE.

Guseinov, G. S. (2003). Integration on time scales. *Journal of Mathematical Analysis and Applications*, 285(1):107–127.

Haghighi, R. and Cheah, C.-C. (2012). Multi-group coordination control for robot swarms. *Automatica*, 48(10):2526–2534.

Hashimoto, K., Saoud, A., Kishida, M., Ushio, T., and Dimarogonas, D. V. (2019). A symbolic approach to the self-triggered design for networked control systems. *IEEE Control Systems Letters*, 3(4):1050–1055.

Heemels, W., Dullerud, G., and Teel, A. (2015a). $\mathcal{L}_2$-gain analysis for a class of hybrid systems with applications to reset and event-triggered control: A lifting approach. *IEEE Transactions on Automatic Control*, 61(10):2766–2781.

Heemels, W., Postoyan, R., Donkers, M., Teel, A. R., Anta, A., Tabuada, P., and Nešic, D. (2015b). Periodic event-triggered control. *Event-based control and signal processing*, pages 105–119.

Heemels, W., Sandee, J., and Van Den Bosch, P. (2008). Analysis of event-driven controllers for linear systems. *International journal of control*, 81(4):571–590.

Heemels, W. H., Donkers, M., and Teel, A. R. (2012a). Periodic event-triggered control for linear systems. *IEEE Transactions on automatic control*, 58(4):847–861.

Heemels, W. P., Johansson, K. H., and Tabuada, P. (2012b). An introduction to event-triggered and self-triggered control. In *2012 ieee 51st ieee conference on decision and control (cdc)*, pages 3270–3285. IEEE.

Hendrickx, J. M. and Martin, S. (2016). Open multi-agent systems: Gossiping with deterministic arrivals and departures. In *2016 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 1094–1101. IEEE.

Hespanha, J. P., Liberzon, D., Angeli, D., and Sontag, E. D. (2005). Nonlinear norm-observability notions and stability of switched systems. *IEEE Transactions on Automatic Control*, 50(2):154–168.

Hespanha, J. P., Naghshtabrizi, P., and Xu, Y. (2007). A survey of recent results in networked control systems. *Proceedings of the IEEE*, 95(1):138–162.

Hetel, L., Fiter, C., Omran, H., Seuret, A., Fridman, E., Richard, J.-P., and Niculescu, S. I. (2017). Recent developments on the stability of systems with aperiodic sampling: An overview. *Automatica*, 76:309–335.

Hilger, S. (1990). Analysis on measure chains—a unified approach to continuous and discrete calculus. *Results in Mathematics*, 18(1):18–56.

Hsu, S.-C., Xu, X., and Ames, A. D. (2015). Control barrier function based quadratic programs with application to bipedal robotic walking. In *2015 American Control Conference (ACC)*, pages 4542–4548. IEEE.

Hu, B., Xu, X., Michel, A. N., and Antsaklis, P. J. (1999). Stability analysis for a class of nonlinear switched systems. In *Proceedings of the 38th IEEE Conference on Decision and Control (Cat. No. 99CH36304)*, volume 5, pages 4374–4379. IEEE.

Jaramillo, O., Castillo-Toledo, B., and Di Gennaro, S. (2020). Impulsive observer–based stabilization for a class of lipschitz nonlinear systems with time–varying uncertainties. *Journal of the Franklin Institute*, 357(17):12518–12537.

Ji, Z., Feng, G., and Guo, X. (2007a). A constructive approach to reachability realization of discrete-time switched linear systems. *Systems & control letters*, 56(11-12):669–677.

Ji, Z., Wang, L., and Guo, X. (2007b). Design of switching sequences for controllability realization of switched linear systems. *Automatica*, 43(4):662–668.

Johansson, K. H., Lygeros, J., Sastry, S., and Egerstedt, M. (1999). Simulation of zeno hybrid automata. In *Proceedings of the 38th IEEE Conference on Decision and Control (Cat. No. 99CH36304)*, volume 4, pages 3538–3543. IEEE.

Johansson, M. (1999). *Piecewise linear control systems*. PhD thesis, Ph. D. Thesis, Lund Institute of Technology, Sweden.

Khaled, Y., Barbot, J.-P., Busawon, K., and Benmerzouk, D. (2013). Impulsive observer-based control for linear systems using irregularly sampled measurements. In *2013 Africon*, pages 1–5. IEEE.

Kolarijani, A. S. and Mazo, M. (2016). Formal traffic characterization of lti event-triggered control systems. *IEEE Transactions on Control of Network Systems*, 5(1):274–283.

Kolathaya, S. and Ames, A. D. (2018). Input-to-state safety with control barrier functions. *IEEE control systems letters*, 3(1):108–113.

Kumar, V., Djemai, M., Defoort, M., and Malik, M. (2021). Finite-time stability and stabilization results for switched impulsive dynamical systems on time scales. *Journal of the Franklin Institute*, 358(1):674–698.

Lakshmikantham, V. and Vatsala, A. (2002). Hybrid systems on time scales. *Journal of Computational and Applied Mathematics*, 141(1-2):227–235.

Lamport, L. (1977). Proving the correctness of multiprocess programs. *IEEE transactions on software engineering*, (2):125–143.

Lee, T.-C. and Jiang, Z.-P. (2008). Uniform asymptotic stability of nonlinear switched systems with an application to mobile robots. *IEEE Transactions on Automatic Control*, 53(5):1235–1252.

Lemmon, M. (2010). Event-triggered feedback in control, estimation, and optimization. *Networked control systems*, pages 293–358.

Liberzon, D., Hespanha, J. P., and Morse, A. S. (1999). Stability of switched systems: a lie-algebraic condition. *Systems & Control Letters*, 37(3):117–122.

Liberzon, D. and Morse, A. S. (1999). Basic problems in stability and design of switched systems. *IEEE control systems magazine*, 19(5):59–70.

Lin, H. and Antsaklis, P. J. (2009). Stability and stabilizability of switched linear systems: a survey of recent results. *IEEE Transactions on Automatic control*, 54(2):308–322.

Lindemann, L., Hu, H., Robey, A., Zhang, H., Dimarogonas, D., Tu, S., and Matni, N. (2021). Learning hybrid control barrier functions from data. In *Conference on Robot Learning*, pages 1351–1370. PMLR.

Liu, J., Li, H., Ji, J., and Luo, J. (2020a). Group-bipartite consensus in the networks with cooperative-competitive interactions. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 67(12):3292–3296.

Liu, K., Selivanov, A., and Fridman, E. (2019). Survey on time-delay approach to networked control. *Annual Reviews in Control*, 48:57–79.

Liu, K.-Z., Teel, A. R., Sun, X.-M., and Wang, X.-F. (2020b). Model-based dynamic event-triggered control for systems with uncertainty: a hybrid system approach. *IEEE Transactions on Automatic Control*, 66(1):444–451.

Liu, Q., Wang, Z., He, X., and Zhou, D. (2014). A survey of event-based strategies on control and estimation. *Systems Science & Control Engineering: An Open Access Journal*, 2(1):90–97.

Liu, Y., Zhao, Y., and Shi, Z. (2017). Sampled-data based consensus for multiple harmonic oscillators with directed switching topology. *Journal of the Franklin Institute*, 354(8):3519–3539.

Long, L. and Wang, J. (2022). Safety-critical dynamic event-triggered control of nonlinear systems. *Systems & Control Letters*, 162:105176.

Luenberger, D. G. (1964). Observing the state of a linear system. *IEEE transactions on military electronics*, 8(2):74–80.

Lunze, J. and Lehmann, D. (2010). A state-feedback approach to event-based control. *Automatica*, 46(1):211–215.

MalmirChegini, M. and Marvasti, F. (2007). Performance improvement of level-crossing a/d converters. In *2007 IEEE International Conference on Telecommunications and Malaysia International Conference on Communications*, pages 438–441. IEEE.

Martynyuk, A. A. (2016). *Stability theory for dynamic equations on time scales*. Springer.

Mazenc, F., Malisoff, M., Barbalata, C., and Jiang, Z.-P. (2021). Event-triggered control using a positive systems approach. *European Journal of Control*, 62:63–68.

Mazenc, F., Malisoff, M., Barbalata, C., and Jiang, Z.-P. (2022). Event-triggered control for linear time-varying systems using a positive systems approach. *Systems & Control Letters*, 161:105131.

Mazo, M. and Tabuada, P. (2008). On event-triggered and self-triggered control over sensor/actuator networks. In *2008 47th IEEE conference on decision and control*, pages 435–440. IEEE.

Mazo, M. and Tabuada, P. (2011). Decentralized event-triggered control over wireless sensor/actuator networks. *IEEE Transactions on Automatic Control*, 56(10):2456–2461.

Meng, D. (2017). Bipartite containment tracking of signed networks. *Automatica*, 79:282–289.

Meng, D., Du, M., and Jia, Y. (2016). Interval bipartite consensus of networked agents associated with signed digraphs. *IEEE Transactions on Automatic Control*, 61(12):3755–3770.

Meng, D., Du, M., and Wu, Y. (2019). Extended structural balance theory and method for cooperative–antagonistic networks. *IEEE Transactions on Automatic Control*, 65(5):2147–2154.

Meng, D., Jia, Y., and Du, J. (2015). Finite-time consensus for multiagent systems with cooperative and antagonistic interactions. *IEEE transactions on neural networks and learning systems*, 27(4):762–770.

Miao, S. (2021). Matrix-weighted consensus for multi-agent systems with sampled-data. *Journal of the Franklin Institute*.

Miao, S. and Su, H. (2021). Bipartite consensus for second-order multiagent systems with matrix-weighted signed network. *IEEE Transactions on Cybernetics*.

Miao, S. and Su, H. (2022). Consensus of matrix-weighted hybrid multiagent systems. *IEEE Transactions on Cybernetics*.

Miskowicz, M. (2006). Send-on-delta concept: An event-based data reporting strategy. *sensors*, 6(1):49–63.

Moreira, L., Tarbouriech, S., Seuret, A., and da Silva Jr, J. G. (2019). Observer-based event-triggered control in the presence of cone-bounded nonlinear inputs. *Nonlinear Analysis: Hybrid Systems*, 33:17–32.

Moreira, L. G., Gomes da Silva Jr, J. M., Tarbouriech, S., and Seuret, A. (2020). Observer-based event-triggered control for systems with slope-restricted nonlinearities. *International Journal of Robust and Nonlinear Control*, 30(17):7409–7428.

Motsch, S. and Tadmor, E. (2014). Heterophilious dynamics enhances consensus. *SIAM review*, 56(4):577–621.

Nagumo, M. (1942). Über die lage der integralkurven gewöhnlicher differentialgleichungen. *Proceedings of the Physico-Mathematical Society of Japan. 3rd Series*, 24:551–559.

Nguyen, Q., Hereid, A., Grizzle, J. W., Ames, A. D., and Sreenath, K. (2016). 3d dynamic walking on stepping stones with control barrier functions. In *2016 IEEE 55th Conference on Decision and Control (CDC)*, pages 827–834. IEEE.

Nguyen, Q. and Sreenath, K. (2015). Safety-critical control for dynamical bipedal walking with precise footstep placement. *IFAC-PapersOnLine*, 48(27):147–154.

Nguyen, Q. and Sreenath, K. (2016). Exponential control barrier functions for enforcing high relative-degree safety-critical constraints. In *2016 American Control Conference (ACC)*, pages 322–328. IEEE.

Ogren, P., Fiorelli, E., and Leonard, N. E. (2004). Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment. *IEEE Transactions on Automatic control*, 49(8):1292–1302.

Olfati-Saber, R., Fax, J. A., and Murray, R. M. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233.

Otanez, P. G., Moyne, J. R., and Tilbury, D. M. (2002). Using deadbands to reduce communication in networked control systems. In *Proceedings of the 2002 American Control Conference (IEEE Cat. No. CH37301)*, volume 4, pages 3015–3020. IEEE.

Pan, L., Shao, H., Mesbahi, M., Li, D., and Xi, Y. (2022). Cluster consensus on matrix-weighted switching networks. *Automatica*, 141:110308.

Pan, L., Shao, H., Mesbahi, M., Xi, Y., and Li, D. (2018). Bipartite consensus on matrix-valued weighted networks. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 66(8):1441–1445.

Pan, L., Shao, H., Mesbahi, M., Xi, Y., and Li, D. (2020). Consensus on matrix-weighted time-varying networks. *arXiv preprint arXiv:2001.11179*.

Pan, L., Shao, H., Xi, Y., and Li, D. (2021a). Bipartite consensus problem on matrix-valued weighted directed networks. *Science China Information Sciences*, 64(4):1–3.

Pan, L., Wang, C., Shao, H., Li, D., and Xi, Y. (2021b). Event-triggered consensus on matrix-weighted networks. In *2021 China Automation Congress (CAC)*, pages 7521–7525. IEEE.

Pickem, D., Glotfelter, P., Wang, L., Mote, M., Ames, A., Feron, E., and Egerstedt, M. (2017). The robotarium: A remotely accessible swarm robotics research testbed. In *2017 IEEE International Conference on Robotics and Automation (ICRA)*, pages 1699–1706. IEEE.

Plonnigs, J., Neugebauer, M., and Kabitzsch, K. (2004). A traffic model for networked devices in the building automation. In *IEEE International Workshop on Factory Communication Systems, 2004. Proceedings.*, pages 137–145. IEEE.

Postoyan, R., Anta, A., Heemels, W., Tabuada, P., and Nešić, D. (2013). Periodic event-triggered control for nonlinear systems. In *52nd IEEE conference on decision and control*, pages 7397–7402. IEEE.

Postoyan, R., Sanfelice, R. G., and Heemels, W. M. H. (2019). Inter-event times analysis for planar linear event-triggered controlled systems. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 1662–1667. IEEE.

Postoyan, R., Tabuada, P., Nešić, D., and Anta, A. (2011). Event-triggered and self-triggered stabilization of distributed networked control systems. In *2011 50th IEEE Conference on Decision and Control and European Control Conference*, pages 2565–2570. IEEE.

Postoyan, R., Tabuada, P., Nešić, D., and Anta, A. (2014). A framework for the event-triggered stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, 60(4):982–996.

Poulsen, D., Defoort, M., and Djemai, M. (2019). Mean square consensus of double-integrator multi-agent systems under intermittent control: A stochastic time scale approach. *Journal of the Franklin Institute*, 356(16):9076–9094.

Poulsen, D. and Wintz, N. (2019). The kalman filter on stochastic time scales. *Nonlinear Analysis: Hybrid Systems*, 33:151–161.

Poulsen, D. R., Davis, J. M., and Gravagne, I. A. (2013). Stochastic time scales: quadratic lyapunov functions and probabilistic regions of stability. In *45th Southeastern Symposium on System Theory*, pages 98–103. IEEE.

Poulsen, D. R., Davis, J. M., and Gravagne, I. A. (2017). Optimal control on stochastic time scales. *IFAC-PapersOnLine*, 50(1):14861–14866.

Poulsen, D. R., Davis, J. M., Gravagne, I. A., and Marks, R. J. (2012). On the stability of mu-varying dynamic equations on stochastically generated time scales. In *Proceedings of the 2012 44th Southeastern Symposium on System Theory (SSST)*, pages 18–23. IEEE.

Poulsen, D. R., Spivey, M. Z., and Marks, R. J. (2011). The poisson process and associated probability distributions on time scales. In *2011 IEEE 43rd Southeastern Symposium on System Theory*, pages 49–54. IEEE.

Prajna, S. (2006). Barrier certificates for nonlinear model validation. *Automatica*, 42(1):117–126.

Prajna, S. and Jadbabaie, A. (2004). Safety verification of hybrid systems using barrier certificates. In *International Workshop on Hybrid Systems: Computation and Control*, pages 477–492. Springer.

Prajna, S., Jadbabaie, A., and Pappas, G. J. (2007). A framework for worst-case and stochastic safety verification using barrier certificates. *IEEE Transactions on Automatic Control*, 52(8):1415–1428.

Prajna, S. and Rantzer, A. (2005). On the necessity of barrier certificates. *IFAC Proceedings Volumes*, 38(1):526–531.

Proskurnikov, A. V. and Tempo, R. (2017). A tutorial on modeling and analysis of dynamic social networks. part i. *Annual Reviews in Control*, 43:65–79.

Qin, W., Liu, Z.-x., and Chen, Z.-q. (2015). Observer-based consensus for nonlinear multi-agent systems with intermittent communication. *Neurocomputing*, 154:230–238.

Rabehi, D., Meslem, N., El Amraoui, A., and Ramdani, N. (2020). Interval impulsive observer for linear systems with aperiodic discrete measurements. *IEEE Transactions on Automatic Control*, 66(11):5407–5413.

Raff, T. and Allgower, F. (2007). Observers with impulsive dynamical behavior for linear and nonlinear continuous-time systems. In *2007 46th IEEE conference on decision and control*, pages 4287–4292. IEEE.

Rajan, A. and Tallapragada, P. (2020). Analysis of inter-event times for planar linear systems under a general class of event triggering rules. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 5206–5211. IEEE.

Rajan, A. and Tallapragada, P. (2022). Analysis of inter-event times in linear systems under event-triggered or self-triggered control. *arXiv preprint arXiv:2201.02090*.

Ren, W., Beard, R. W., and Atkins, E. M. (2007). Information consensus in multivehicle cooperative control. *IEEE Control systems magazine*, 27(2):71–82.

Robey, A., Hu, H., Lindemann, L., Zhang, H., Dimarogonas, D. V., Tu, S., and Matni, N. (2020). Learning control barrier functions from expert demonstrations. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 3717–3724. IEEE.

Romdlony, M. Z. and Jayawardhana, B. (2016). On the new notion of input-to-state safety. In *2016 IEEE 55th conference on decision and control (CDC)*, pages 6403–6409. IEEE.

Romdlony, M. Z. and Jayawardhana, B. (2017). On the sufficient conditions for input-to-state safety. In *2017 13th IEEE International Conference on Control & Automation (ICCA)*, pages 170–173. IEEE.

Russo, G. and Wirth, F. (2020). Matrix measures, stability and contraction theory for dynamical systems on time scales. *arXiv preprint arXiv:2007.08879*.

Selivanov, A. and Fridman, E. (2015). Event-triggered $\mathcal{H}_\infty$ control: A switching approach. *IEEE Transactions on Automatic Control*, 61(10):3221–3226.

Sontag, E. D. (1983). A lyapunov-like characterization of asymptotic controllability. *SIAM journal on control and optimization*, 21(3):462–471.

Sontag, E. D. (1989). A 'universal'construction of artstein's theorem on nonlinear stabilization. *Systems & control letters*, 13(2):117–123.

Sontag, E. D. (2008). Input to state stability: Basic concepts and results. In *Nonlinear and optimal control theory*, pages 163–220. Springer.

Sontag, E. D. and Wang, Y. (1995). On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24(5):351–359.

Squires, E., Pierpaoli, P., and Egerstedt, M. (2018). Constructive barrier certificates with applications to fixed-wing aircraft collision avoidance. In *2018 IEEE Conference on Control Technology and Applications (CCTA)*, pages 1656–1661. IEEE.

Su, H., Chen, J., Yang, Y., and Rong, Z. (2019). The bipartite consensus for multi-agent systems with matrix-weight-based signed network. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 67(10).

Su, H. and Miao, S. (2022). Consensus on directed matrix-weighted networks. *IEEE Transactions on Automatic Control*.

Suh, Y. S. (2007). Send-on-delta sensor data transmission with a linear predictor. *Sensors*, 7(4):537–547.

Sun, Z. and Ge, S. S. (2005). Analysis and synthesis of switched linear control systems. *Automatica*, 41(2):181–195.

Sun, Z., Ge, S. S., and Lee, T. H. (2002). Controllability and reachability criteria for switched linear systems. *Automatica*, 38(5):775–786.

Tabuada, P. (2007). Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52(9):1680–1685.

Taousser, F., Defoort, M., and Djemai, M. (2016). Consensus for linear multi-agent system with intermittent information transmissions using the time-scale theory. *International journal of Control*, 89(1):210–220.

Taousser, F. Z., Defoort, M., and Djemai, M. (2015). Stability analysis of a class of uncertain switched systems on time scale using lyapunov functions. *Nonlinear Analysis: Hybrid Systems*, 16:13–23.

Taousser, F. Z., Defoort, M., Djemai, M., Djouadi, S. M., and Tomsovic, K. (2019). Stability analysis of a class of switched nonlinear systems using the time scale theory. *Nonlinear Analysis: Hybrid Systems*, 33:195–210.

Tarbouriech, S., Seuret, A., da Silva Jr, J. M. G., and Sbarbaro, D. (2016). Observer-based event-triggered control co-design for linear systems. *IET Control Theory & Applications*, 10(18):2466–2473.

Taylor, A. J., Ong, P., Cortés, J., and Ames, A. D. (2020). Safety-critical event triggered control via input-to-state safe barrier functions. *IEEE Control Systems Letters*, 5(3):749–754.

Tee, K. P., Ge, S. S., and Tay, E. H. (2009). Barrier lyapunov functions for the control of output-constrained nonlinear systems. *Automatica*, 45(4):918–927.

Trimpe, S. (2017). Event-based state estimation: An emulation-based approach. *IET Control Theory & Applications*, 11(11):1684–1693.

Trinh, M. H., Nguyen, H. M., Van Tran, Q., and Ahn, H.-S. (2022). Matrix-weighted consensus algorithms for euler-lagrange systems. In *2022 13th Asian Control Conference (ASCC)*, pages 1190–1195. IEEE.

Valcher, M. E. and Misra, P. (2014). On the consensus and bipartite consensus in high-order multi-agent dynamical systems with antagonistic interactions. *Systems & Control Letters*, 66:94–103.

Van Tran, Q., Trinh, M. H., and Ahn, H.-S. (2021). Discrete-time matrix-weighted consensus. *IEEE Transactions on Control of Network Systems*, 8(4):1568–1578.

Vasyutynskyy, V. and Kabitzsch, K. (2007). Simple pid control algorithm adapted to deadband sampling. In *2007 IEEE Conference on Emerging Technologies and Factory Automation (EFTA 2007)*, pages 932–940. IEEE.

Wang, H., Yu, W., Wen, G., and Chen, G. (2018a). Finite-time bipartite consensus for multi-agent systems on directed signed networks. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 65(12):4336–4348.

Wang, L., Ames, A. D., and Egerstedt, M. (2017a). Safe certificate-based maneuvers for teams of quadrotors using differential flatness. In *2017 IEEE international conference on robotics and automation (ICRA)*, pages 3293–3298. IEEE.

Wang, L., Ames, A. D., and Egerstedt, M. (2017b). Safety barrier certificates for collisions-free multirobot systems. *IEEE Transactions on Robotics*, 33(3):661–674.

Wang, L., Han, D., and Egerstedt, M. (2018b). Permissive barrier certificates for safe stabilization using sum-of-squares. In *2018 annual American control conference (ACC)*, pages 585–590. IEEE.

Wang, W., Postoyan, R., Nešić, D., and Heemels, W. (2019). Periodic event-triggered control for nonlinear networked control systems. *IEEE Transactions on Automatic Control*, 65(2):620–635.

Wang, X. and Lemmon, M. D. (2008). Event design in event-triggered feedback control systems. In *2008 47th IEEE Conference on Decision and Control*, pages 2105–2110. IEEE.

Wang, X., Sun, Y., and Hovakimyan, N. (2012). Asynchronous task execution in networked control systems using decentralized event-triggering. *Systems & Control Letters*, 61(9):936–944.

Wasserman, S., Faust, K., et al. (1994). Social network analysis: Methods and applications.

Wu, G. and Sreenath, K. (2015). Safety-critical and constrained geometric control synthesis using control lyapunov and control barrier functions for systems evolving on manifolds. In *2015 American Control Conference (ACC)*, pages 2038–2044. IEEE.

Wu, G. and Sreenath, K. (2016). Safety-critical control of a planar quadrotor. In *2016 American control conference (ACC)*, pages 2252–2258. IEEE.

Xiao, Q. and Huang, Z. (2016). Consensus of multi-agent system with distributed control on time scales. *Applied Mathematics and Computation*, 277:54–71.

Xiao, Q., Zeng, Z., Huang, T., and Lewis, F. L. (2020). Positivity and stability of delayed timescale-type differential-difference equations. *IEEE Transactions on Automatic Control*, 66(7):3221–3226.

Xiao, Q., Zeng, Z., Huang, T., and Lewis, F. L. (2021). Positivity and stability of coupled differential–difference equations with time-varying delay on time scales. *Automatica*, 131:109774.

Xiao, W., Belta, C., and Cassandras, C. G. (2022). Event-triggered control for safety-critical systems with unknown dynamics. *IEEE Transactions on Automatic Control*.

Xu, X. and Antsaklis, P. J. (2004). Optimal control of switched systems based on parameterization of the switching instants. *IEEE transactions on automatic control*, 49(1):2–16.

Xu, X., Grizzle, J. W., Tabuada, P., and Ames, A. D. (2017a). Correctness guarantees for the composition of lane keeping and adaptive cruise control. *IEEE Transactions on Automation Science and Engineering*, 15(3):1216–1229.

Xu, X., Tabuada, P., Grizzle, J. W., and Ames, A. D. (2015). Robustness of control barrier functions for safety critical control. *IFAC-PapersOnLine*, 48(27):54–61.

Xu, X., Waters, T., Pickem, D., Glotfelter, P., Egerstedt, M., Tabuada, P., Grizzle, J. W., and Ames, A. D. (2017b). Realizing simultaneous lane keeping and adaptive speed regulation on accessible mobile robot testbeds. In *2017 IEEE Conference on Control Technology and Applications (CCTA)*, pages 1769–1775. IEEE.

Xue, M., Tang, Y., Ren, W., and Qian, F. (2020). Stability of multi-dimensional switched systems with an application to open multi-agent systems. *arXiv preprint arXiv:2001.00435*.

Zhang, H. and Chen, J. (2014). Bipartite consensus of general linear multi-agent systems. In *2014 American Control Conference*, pages 808–812. IEEE.

Zhang, H. and Chen, J. (2017). Bipartite consensus of multi-agent systems over signed graphs: state feedback and output feedback control approaches. *International Journal of Robust and Nonlinear Control*, 27(1):3–14.

Zhang, K., Braverman, E., and Gharesifard, B. (2022a). Event-triggered control for discrete-time delay systems. *arXiv preprint arXiv:2209.07560*.

Zhang, K. and Gharesifard, B. (2021). Hybrid event-triggered and impulsive control for time-delay systems. *Nonlinear Analysis: Hybrid Systems*, 43:101109.

Zhang, K., Gharesifard, B., and Braverman, E. (2021). Event-triggered control for nonlinear time-delay systems. *IEEE Transactions on Automatic Control*, 67(2):1031–1037.

Zhang, K., Gharesifard, B., and Braverman, E. (2022b). A note on stability of event-triggered control systems with time delays. *arXiv preprint arXiv:2207.03566*.

Zhao, C.-j. (2007). On a periodic predator–prey system with time delays. *Journal of mathematical analysis and applications*, 331(2):978–985.

Zhao, L., Jia, Y., and Yu, J. (2017). Adaptive finite-time bipartite consensus for second-order multi-agent systems with antagonistic interactions. *Systems & Control Letters*, 102:22–31.

Zhu, Y., Li, S., Ma, J., and Zheng, Y. (2018). Bipartite consensus in networks of agents with antagonistic interactions and quantization. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 65(12):2012–2016.