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Digital output feedback event-based stabilization of nonlinear systems with state delays☆

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ABSTRACT

In this paper, the stabilization problem of nonlinear time-delay systems by means of digital dynamic output feedback event-triggered controllers is addressed. In particular, for the class of control-affine nonlinear systems with state delays, a methodology for the design of quantized sampled-data observer-based event-triggered (QSOE) stabilizers is provided. As a first step, the notion of Dynamic Output Steepest Descent Feedback (DOSDF), induced by a class of Lyapunov–Krasovskii functionals, is suitably revised in order to cope with the design of QSOE stabilizers. Then, the stabilization in the sample-and-hold sense theory is used as a tool to prove the existence of a suitably fast sampling and of an accurate quantization of the input/output channels such that: the digital implementation of DOSDFs, updated through a proposed event-based mechanism, ensures the semi-global practical stability property of the related closed-loop system with arbitrarily small final target ball of the origin. In the theory here developed, aperiodic sampling and the non-uniform quantization of the input/output channels are taken into account. Possible discontinuities in the functions describing the DOSDF at hand are also managed enlarging the possibilities to successfully designing QSOE stabilizers. Moreover, the proposed QSOE stabilizer is described by easily implementable difference equations avoiding the necessity to solve differential equations for the correct application of the controller at hand. Nonlinear delay-free systems are addressed as a special case. The proposed results are validated through practical examples concerning a Glucose-Insulin system and a Continuous Stirred Tank Reactor system.

1. Introduction

In the last twenty years, the study of quantized sampled-data control systems has attracted the attention of more and more researchers due to the growing utilization, in the context of many engineering applications, of digital devices for the practical implementation of controllers. Many approaches have been proposed in the literature for the study of sampled-data control systems with and without quantization (Di Ferdinando, Pepe, & Borri, 2021; Di Ferdinando, Pepe, & Di Gennaro, 2022; Fridman, 2010; Fridman, Seuret, & Richard, 2004; Hetel, Fiter, Omran, Seuret, Fridman, Richard, & Niculescu, 2017; Liberzon, 2006; Liu, Lin, Zhao, & Hu, 2021; Mattioni, Monaco, & Normand-Cyrot, 2017; Monaco, Normand-Cyrot, & Mattioni, 2017; Pepe, 2014). An interesting and popular framework for the design of stabilizers is the one based on the event-triggered control. Such an approach is very helpful to properly managing shared computation and communication resources in the digital world (Heemels, Johansson, & Tabuada, 2012; Tabuada, 2007) because control updates are sent to the system only when really necessary. Differently from the continuous event-triggered control (CETC) in which triggering conditions are evaluated at all times, in the sampled-data framework, the evaluation of the triggering laws is performed only at sampling instants (which are not necessarily uniformly distributed) leading to the so called periodic event-triggered control (PETC) (Abdelrahim, Postoyan, Daafouz, & Nesic, 2016; Dou & Ling, 2021; Postoyan, Tabuada, Nešić, & Anta, 2014; Seuret, Prieur, & Marchand, 2014; Sun, Yang, Zheng, & Li, 2022; Wang, Postoyan, Nešić, & Heemels, 2019). As far as the stabilization problem via periodic event-triggered controllers is concerned, many results have been provided in the literature for various classes of linear/nonlinear delay-free systems in presence of both

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sampling and quantization in the control scheme (see, among the others, Abdelrahim, Postoyan, Daafouz, and Nesic (2017), Borri, Di Ferdinando, and Pepe (2024), Fu and Qiao (2022), Liu and Jiang (2019), Scheres, Postoyan, and Heemels (2024), Zhao, Zheng, Ahn, Zong, Zhang, and Chen (2021) and the references therein). As far as nonlinear systems with state delays are concerned, in the literature, the stabilization problem via event-triggered controllers in presence of sampling and/or quantization has been mainly studied in the case of static state feedbacks (see, for instance, Borri and Pepe (2021), Di Ferdinando, Di Gennaro, Borri, Pola, and Pepe (2024), Pepe (2016), Zhang, Gharesifard, and Braverman (2022), Zhang, Liu, and Jiang (2020)). On the other hand, in the case of event-based dynamic output feedback controllers, the results provided in the literature are very few (Choi & Yoo, 2019; Di Ferdinando, Borri, Di Gennaro, & Pepe, 2024). In Di Ferdinando, Borri, et al. (2024), results concerning quantized sampled-data event-triggered stabilizers exploiting continuous-time state observers have been provided for nonlinear systems with state-delays. The digital implementation of the dynamical part of the controller is not considered in Di Ferdinando, Borri, et al. (2024) and the knowledge of the continuous-time output measurements is required for the correct implementation of the proposed event-based control scheme. Moreover, possible discontinuities in the functions describing the dynamics of the controller at hand are not considered in Di Ferdinando, Borri, et al. (2024). Actually, to our best knowledge, in the context of the event-based dynamic output feedback control of nonlinear delay-free/time-delay systems, no result has been provided taking simultaneously into account: (i) the presence of both sampling and quantization in the input/output channels; (ii) the presence of possible discontinuities in the functions describing both feedback and dynamics of the controller at hand; (iii) the discrete-time implementation of the dynamical part related to the proposed controller. Such aspects are very relevant when the practical implementation of a proposed digital dynamic output feedback event-based control strategy is concerned and, consequently, should be simultaneously considered during the design procedure.

In this paper, we fill this gap, by providing, for a class of nonlinear systems with state-delays, a methodology for the design of QSOE stabilizers which takes simultaneously into account, for the first time in the literature, points (i)–(iii). As a first step, the notion of Dynamic Output Steepest Descent Feedback (DOSDF) introduced in Di Ferdinando et al. (2022) is suitably reformulated in order to cope with the design of QSOE stabilizers. Then, the stabilization in the sample-and-hold sense theory (Clarke, 2010; Clarke, Ledyaev, Sontag, & Subbotin, 1997; Pepe, 2014) is used as a tool in order to prove the existence of a suitably fast sampling and of an accurate quantization of the input/output channels such that the digital implementation of DOSDFs (continuous or not), updated through a proposed event-triggered mechanism, ensures the semi-global practical stability property, with arbitrarily small final target ball of the origin, for the related closed-loop system. In the theory here developed: (i) discontinuities in the functions describing both feedback and dynamics of the controller at hand are allowed; (ii) the case of aperiodic sampling and the case of non-uniform quantization are taken into account; (iii) the proposed digital event-based dynamic output feedback controller is described by easily implementable difference equations. In the provided results, the case of nonlinear delay-free systems is included as a special case and relaxed conditions are provided for the design of the controller. The proposed methodology is validated through applications concerning: a particular class of nonlinear systems with state-delays with a related example regarding the plasma glucose regulation problem in Type 2 Diabetic patients via artificial pancreas; the temperature control problem of a Continuous Stirred Tank Reactor (CSTR).

Notation $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^\star$ denotes the extended real line $[-\infty, +\infty]$, $\mathbb{R}^+$ denotes the set of nonnegative reals $[0, +\infty)$. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. For a given positive integer n and for a symmetric, positive definite matrix $P \in \mathbb{R}^{n\times n}$, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and the minimum eigenvalue of P, respectively. For a given positive integer n and a given positive real H, the symbol $\mathcal{B}_H^n$ denotes the subset $\{x \in \mathbb{R}^n \mid |x| \le H\}$. The essential supremum norm of an essentially bounded function is indicated with the symbol $\|\cdot\|_\infty$. For a positive integer n, for a positive real Δ (maximum involved time-delay): $\mathcal{C}^n$ and $W_n^{1,\infty}$ denote the space of the continuous functions mapping $[-\Delta, 0]$ into $\mathbb{R}^n$ and the space of the absolutely continuous functions, with essentially bounded derivative, mapping $[-\Delta, 0]$ into $\mathbb{R}^n$, respectively; $\mathcal{Q}^n$ denotes the space of bounded, right-continuous functions, with possibly a finite number of points with jump-type discontinuity, mapping $[-\Delta, 0)$ into $\mathbb{R}^n$ and with finite left-hand limit at 0. For $\phi \in \mathcal{C}^n$, $\phi_{[-\Delta,0)}$ is the function in $\mathcal{Q}^n$ defined, for $\tau \in [-\Delta, 0)$, as $\phi_{[-\Delta,0)}(\tau) = \phi(\tau)$. Notice that, when $\Delta = 0$, the spaces $\mathcal{C}^n$ and $\mathbb{R}^n$ are isomorphic and, for any $\phi \in \mathcal{C}^n$, $\|\phi\|_\infty = |\phi(0)|$. For a positive real H, for $\phi \in \mathcal{C}^n$, $\mathcal{C}_H^n(\phi) = \{\psi \in \mathcal{C}^n \mid \|\psi - \phi\|_\infty \le H\}$. The symbol $\mathcal{C}_H^n$ denotes $\mathcal{C}_H^n(0)$. For a continuous function $x: [-\Delta, c) \to \mathbb{R}^n$, with $0 < c \le +\infty$, for any real $t \in [0, c)$, x_t is the function in $\mathcal{C}^n$ defined as $x_t(\tau) = x(t+\tau)$, $\tau \in [-\Delta, 0]$. For a positive integer n, for $\mathbb{S} = \mathbb{R}^n$ (or $\mathbb{R}^+$), $C^1(\mathbb{S}; \mathbb{R}^+)$ denotes the space of the continuous functions from $\mathbb{S}$ to $\mathbb{R}^+$, admitting continuous (partial) derivatives; $C_L^1(\mathbb{S}; \mathbb{R}^+)$ denotes the subset of the functions in $C^1(\mathbb{S}; \mathbb{R}^+)$ admitting locally Lipschitz (partial) derivatives. A continuous function $\gamma: \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{P}_0$ if $\gamma(0) = 0$; of class $\mathcal{P}$ if it is of class $\mathcal{P}_0$ and $\gamma(s) > 0$, $s > 0$; of class $\mathcal{K}$ if it is of class $\mathcal{P}$ and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. For positive integers n, m, for functions $f_e: \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, $f_{\hat{x}}: \mathcal{C}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $F: \mathcal{C}^{2n} \times \mathbb{R}^{m+n} \to \mathbb{R}^{2n}$, and for locally Lipschitz functionals $V_e: \mathcal{C}^n \to \mathbb{R}^+$, $V_{\hat{x}}: \mathcal{C}^n \to \mathbb{R}^+$, $V: \mathcal{C}^{2n} \to \mathbb{R}^+$, the derivatives (upper right-hand Dini directional derivatives in the case $\Delta = 0$, and derivatives in Driver's form in the case $\Delta > 0$, see Pepe (2007) and the references therein) $D^+V_e: \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^\star$, $D^+V_{\hat{x}}: \mathcal{C}^n \times \mathbb{R}^n \to \mathbb{R}^\star$ and $D^+V: \mathcal{C}^{2n} \times \mathbb{R}^{m+n} \to \mathbb{R}^\star$ of the functional V_e, $V_{\hat{x}}$ and V are defined, for $\phi_\chi \in \mathcal{C}^{2n}$, $\phi_e \in \mathcal{C}^n$, $\hat{\phi} \in \mathcal{C}^n$, $u \in \mathbb{R}^m$, $\tilde{u} \in \mathbb{R}^{m+n}$ and $v \in \mathbb{R}^n$ as

$$\begin{aligned} D^+V_e(\phi_e, \hat{\phi}, u, v) &= \limsup_{h\to 0^+} \frac{V(\phi^e_{h,\hat{\phi},u,v}) - V(\phi_e)}{h}, \\ D^+V_{\hat{x}}(\hat{\phi}, v) &= \limsup_{h\to 0^+} \frac{V(\hat{\phi}_{h,v}) - V(\hat{\phi})}{h}, \\ D^+V(\phi_\chi, \tilde{u}) &= \limsup_{h\to 0^+} \frac{V(\phi^\chi_{h,\tilde{u}}) - V(\phi_\chi)}{h}, \end{aligned} \tag{1}$$

where, in the case $\Delta > 0$, for $0 \le h < \Delta$, $\phi^e_{h,\hat{\phi},u,v}$, $\hat{\phi}_{h,v} \in \mathcal{C}^n$ and $\phi^\chi_{h,\tilde{u}} \in \mathcal{C}^{2n}$ are defined, for $s \in [-\Delta, 0]$, as

$$\phi^e_{h,\hat{\phi},u,v}(s) = \begin{cases} \phi_e(s+h) & s \in [-\Delta, -h) \\ \phi_e(0) + (s+h) f_e(\phi_e, \hat{\phi}, u, v) & s \in [-h, 0], \end{cases}$$

$$\hat{\phi}_{h,v}(s) = \begin{cases} \hat{\phi}(s+h) & s \in [-\Delta, -h) \\ \hat{\phi}(0) + (s+h) f_{\hat{x}}(\hat{\phi}, v) & s \in [-h, 0], \end{cases}$$

$$\phi^\chi_{h,\tilde{u}}(s) = \begin{cases} \phi_\chi(s+h) & s \in [-\Delta, -h) \\ \phi_\chi(0) + (s+h) F(\phi_\chi, \tilde{u}) & s \in [-h, 0], \end{cases}$$

and, for $\Delta = 0$ and $h \in (0, 1)$, as $\phi^e_{h,\hat{\phi},u,v}(0) = \phi_e(0) + h f_e(\phi_e, \hat{\phi}, u, v)$, $\hat{\phi}_{h,v}(0) = \hat{\phi}(0) + h f_{\hat{x}}(\hat{\phi}, v)$, $\phi^\chi_{h,\tilde{u}}(0) = \phi_\chi(0) + h F(\phi_\chi, \tilde{u})$.

2. Preliminaries and basic definitions

Let us consider a control-affine nonlinear system (the plant) described by the following retarded functional differential equation (RFDE) (Hale & Lunel, 1993; Kolmanovskii & Myshkis, 1999)

$$\begin{aligned}\dot{x}(t) &= f(x_t) + g(h(x_t))u(t), \qquad t \geq 0 \text{ a.e.} \\ y(t) &= h(x_t), \quad x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0]\end{aligned} \tag{2}$$

where: $x(t) \in \mathbb{R}^n$, $x_0, x_t \in \mathcal{C}^n$; $\Delta \geq 0$ is the maximum involved time delay, assumed to be known; $u(t) \in \mathbb{R}^m$ is the input signal, Lebesgue measurable and locally essentially bounded; $y(t) \in \mathbb{R}^p$ is the output signal; $g : \mathbb{R}^p \to \mathbb{R}^{n\times m}$, $f : \mathcal{C}^n \to \mathbb{R}^n$ and $h : \mathcal{C}^n \to \mathbb{R}^p$ are functions, Lipschitz on bounded subsets of $\mathbb{R}^p$ and $\mathcal{C}^n$, respectively; n, m, p are positive integers. It is assumed that $f(0) = 0$ and $h(0) = 0$. In the case $\Delta > 0$ it is assumed that the initial state $x_0 \in W_n^{1,\infty}$ (see Pepe (2014, 2016) and Remark 6 in Pepe (2017)). Notice that, multiple arbitrary discrete and distributed (i.e., involving a finite window of integration of the solution) time-delays can appear in (2).

Remark 1. We highlight here that, in (2), the term affine to the control input (i.e., $g(\cdot)$) is a function of the output signal $y(t) = h(x_t)$. Such a requirement is very helpful in the context of the design of observer-based controllers. See, for instance, Germani, Manes, and Pepe (2012) for a similar situation in the framework of observer-based continuous-time stabilizers for nonlinear globally Lipschitz retarded systems. We highlight also that there exist many physical systems in the form (2) which are commonly exploited in the engineering practice at the aim of control design. Some examples of practical systems in the form (2) are the Glucose-Insulin system (see, for instance, Palumbo, Panunzi, and Gaetano (2007), Panunzi, Gaetano, and Mingrone (2010), Panunzi, Palumbo, and Gaetano (2007)) and the CSTR system (see, for instance, Pepe (2015)) studied in the forthcoming Section 5.

For the reader's convenience, we recall here classes of Lyapunov–Krasovskii functionals very helpful in the context of the digital stabilization of time-delay systems. In particular, we recall the definition of smoothly separable functionals (Pepe, 2014, 2016) and of invariantly differentiable functionals (Kim, 1997; Pepe & Ito, 2012).

Definition 2. For a given positive integer ν, a functional $V : \mathcal{C}^\nu \to \mathbb{R}^+$ is said to be smoothly separable if there exist a function $V_1 \in C_L^1(\mathbb{R}^\nu; \mathbb{R}^+)$, a locally Lipschitz functional $V_2 : \mathcal{C}^\nu \to \mathbb{R}^+$, functions $\beta_i \in \mathcal{K}_\infty$, $i = 1, 2$, such that, for any $\phi \in \mathcal{C}^\nu$, the following hold

$$\begin{aligned}&V(\phi) = V_1(\phi(0)) + V_2(\phi) \\ &\beta_1(|\phi(0)|) \leq V_1(\phi(0)) \leq \beta_2(|\phi(0)|).\end{aligned} \tag{3}$$

As in Pepe and Ito (2012), the formalism used in the classical definition of invariantly differentiable functional (see Kim (1997)), is here suitably modified for the purpose of formalism uniformity over the paper. For a given positive integer ν, for any given $x \in \mathbb{R}^\nu$, $\phi \in \mathcal{Q}^\nu$ and for any given continuous function $\mathcal{Y} : [0, \Delta] \to \mathbb{R}^\nu$ with $\mathcal{Y}(0) = x$, let $\psi_h^{(x,\phi,\mathcal{Y})} \in \mathcal{Q}^\nu$, $h \in [0, \Delta)$, be defined as $\psi_0^{(x,\phi,\mathcal{Y})} = \phi$ and, for $h > 0$, $\psi_h^{(x,\phi,\mathcal{Y})}(s) = \begin{cases}\phi(s+h), & s \in [-\Delta, -h), \\ \mathcal{Y}(s+h), & s \in [-h, 0).\end{cases}$

Definition 3. A functional $V : \mathbb{R}^\nu \times \mathcal{Q}^\nu \to \mathbb{R}^+$ is said to be invariantly differentiable if, at any point $(x, \phi) \in \mathbb{R}^\nu \times \mathcal{Q}^\nu$:

- for any continuous function $\mathcal{Y} : [0, \Delta] \to \mathbb{R}^\nu$ with $\mathcal{Y}(0) = x$, there exists the right-hand derivative $\left.\frac{\partial V\left(x, \psi_h^{(x,\phi,\mathcal{Y})}\right)}{\partial h}\right|_{h=0}$ and such derivative is invariant with respect to the function $\mathcal{Y}$;
- there exists the derivative $\frac{\partial V(x,\phi)}{\partial x}$;
- for any continuous function $\mathcal{Y} : [0, \Delta] \to \mathbb{R}^\nu$ with $\mathcal{Y}(0) = x$, the following equality holds for any $z \in \mathbb{R}^\nu$, for any $h \in [0, \Delta)$, $V\left(x+z, \psi_h^{(x,\phi,\mathcal{Y})}\right) - V(x, \phi) = \frac{\partial V(x,\phi)}{\partial x} z + \left.\frac{\partial V\left(x, \psi_l^{(x,\phi,\mathcal{Y})}\right)}{\partial l}\right|_{l=0} h + o(\sqrt{|z|^2 + h^2})$, with $\lim_{s\to 0^+} \frac{o(\sqrt{s})}{\sqrt{s}} = 0$.

We recall also the notion of partition of $[0, +\infty)$ (Clarke et al., 1997; Pepe, 2014) and of quantizer (Liberzon, 2006).

Definition 4. A partition $\pi = \{t_j,\ j = 0, 1, \ldots\}$ of $[0, +\infty)$ is a countable, strictly increasing sequence t_j, with $t_0 = 0$, such that $t_j \to +\infty$ as $j \to +\infty$. The diameter diam(π) of π is defined as $\sup_{j\geq 0} t_{j+1} - t_j$. The dwell time dwell$(\pi)$ of π, is defined as $\inf_{j\geq 0}(t_{j+1} - t_j)$. For any positive real $a \in (0, 1]$, $\delta > 0$, $\pi_{a,\delta}$ is any partition π with $a\delta \leq \text{dwell}(\pi) \leq \text{diam}(\pi) \leq \delta$.

For a given positive integer ν and for $z \in \mathbb{R}^\nu$, a quantizer is a piece-wise constant function $q_z : \mathbb{R}^\nu \to \mathcal{Q}_z^\nu$, with $\mathcal{Q}_z^\nu$ a suitable finite subset of $\mathbb{R}^\nu$, characterized, for some given positive reals E_z (range of the quantizer) and μ_z (quantization error), by the following implications (Liberzon, 2006)

$$|z| \leq E_z \quad \Rightarrow \quad |q_z(z) - z| \leq \mu_z. \tag{4}$$

3. Design of QSOE stabilizers

In this section, the methodology proposed for the design of quantized sampled-data observer-based event-triggered (QSOE) controllers is presented. In particular, a new approach for the design of QSOE stabilizers is proposed for the class of systems described by (2). To such an aim, let us consider functions $f_e : \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ and $f_{\hat{x}} : \mathcal{C}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined, for any $\hat{\phi}$, $\phi_e \in \mathcal{C}^n$, for any $u \in \mathbb{R}^m$ and for any $v \in \mathbb{R}^n$, as follows

$$\begin{aligned}&f_e(\phi_e, \hat{\phi}, u, v) = f(\hat{\phi} + \phi_e) + g(h(\hat{\phi} + \phi_e))u - v, \\ &f_{\hat{x}}(\hat{\phi}, v) = v\end{aligned} \tag{5}$$

where f, g and h are the functions in (2). Notice that, $f_e(0, 0, 0, 0) = f_{\hat{x}}(0, 0) = 0$.

Remark 5. The function f_e aims to describe the dynamics of an open-loop estimation error system characterized by the difference between the dynamics of the original system (2) and the ones of a proposed observer here mimicked by the new control input $v \in \mathbb{R}^n$ (see (A.3) in the proof of Theorem 12). Indeed, the functions $\hat{\phi}$, ϕ_e aim to mimic the state of a proposed observer and the related estimation error with the original system (2), respectively.

In the following, the notion of Dynamic Output Steepest Descent Feedback (DOSDF) introduced in Di Ferdinando et al. (2022) is suitably reformulated in order to cope with the design of QSOE stabilizers. Such a notion is inspired by the well-known definition of Steepest Descent Feedback induced by control Lyapunov–Krasovskii functionals (Clarke, 2010; Clarke et al., 1997; Di Ferdinando et al., 2022; Pepe, 2014). As a first step, the sets of candidate Lypaunov–Krasovskii functional exploited in the paper for the design of QSOE stabilizers is introduced. Let $V_{e,1} : \mathbb{R}^n \to \mathbb{R}^+$, $V_{\hat{x},1} : \mathbb{R}^n \to \mathbb{R}^+$, $V_{e,2} : \mathcal{Q}^n \to \mathbb{R}^+$ and $V_{\hat{x},2} : \mathcal{C}^n \to \mathbb{R}^+$ be Lipschitz on bounded subsets functions with $V_{e,2} = V_{\hat{x},2} = 0$ in the case $\Delta = 0$. Let $\mathcal{V}_e$ and $\mathcal{V}_{\hat{x}}$ be the sets of candidate Lyapunov–Krasovskii functionals $V_e : \mathcal{C}^n \to \mathbb{R}^+$ and $V_{\hat{x}} : \mathcal{C}^n \to \mathbb{R}^+$ satisfying the following properties:

(a) the functionals $V_e : \mathcal{C}^n \to \mathbb{R}^+$ and $V_{\hat{x}} : \mathcal{C}^n \to \mathbb{R}^+$ defined, for $\phi_e \in \mathcal{C}^n$ and $\hat{\phi} \in \mathcal{C}^n$, as

$$\begin{aligned} V_e(\phi_e) &= V_{e,1}(\phi_e(0)) + \widetilde{V}_{e,2}(\phi_e), \\ V_{\hat{x}}(\hat{\phi}) &= V_{\hat{x},1}(\hat{\phi}(0)) + V_{\hat{x},2}(\hat{\phi}), \end{aligned} \tag{6}$$

where $\widetilde{V}_{e,2} : \mathcal{C}^n \to \mathbb{R}^+$ is the functional defined for $\phi_e \in \mathcal{C}^n$ as $\widetilde{V}_{e,2}(\phi_e) = V_{e,2}(\phi_{e[-\Delta,0)})$, are smoothly separable with functions $\beta_{\hat{x},i}$, $\beta_{e,i}$, $i = 1, 2$, as in (3);

(b) the functions $(\phi_e, \hat{\phi}, u, v) \to D^+\widetilde{V}_{e,2}(\phi_e, \hat{\phi}, u, v)$ and $(\hat{\phi}, v) \to D^+V_{\hat{x},2}(\hat{\phi}, v)$, $\hat{\phi}$, $\phi_e \in \mathcal{C}^n$, $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, with the derivative in Driver's form (see (1)) of the functionals $\widetilde{V}_{e,2}$ and $V_{\hat{x},2}$ computed with respect to the functions f_e and $f_{\hat{x}}$ in (5), are Lipschitz on bounded subsets of $\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{R}^m \times \mathbb{R}^n$ and $\mathcal{C}^n \times \mathbb{R}^n$, respectively;

(c) the functional $\bar{V}_e : \mathbb{R}^n \times \mathcal{Q}^n \to \mathbb{R}^+$ defined, for $e \in \mathbb{R}^n$, $\phi_e \in \mathcal{Q}^n$, as $\bar{V}_e(e, \phi_e) = V_{e,1}(e) + V_{e,2}(\phi_e)$ is invariantly differentiable;

(d) there exist functions $\gamma_{e,i}$, $\gamma_{\hat{x},i}$, $i = 1, 2$, of class $\mathcal{K}_\infty$, such that, for any ϕ_e, $\hat{\phi} \in \mathcal{C}^n$,

$$\begin{aligned} &\gamma_{e,1}(|\phi_e(0)|) \le V_e(\phi_e) \le \gamma_{e,2}(\|\phi_e\|_\infty), \\ &\gamma_{\hat{x},1}(|\hat{\phi}(0)|) \le V_{\hat{x}}(\hat{\phi}) \le \gamma_{\hat{x},2}(\|\hat{\phi}\|_\infty). \end{aligned} \tag{7}$$

Remark 6. The items **(a)**, **(b)** and **(d)** are satisfied by a very large class of Lyapunov–Krasovskii functionals, including standard complete quadratic ones (see Niculescu (2001)). The invariant differentiability property of the functional V_e, as here connected with the smooth separability one (see item **(c)**), has been very used for several purposes in the context of nonlinear system with state delays (see, for instance, Di Ferdinando, Di Gennaro, and Pepe (2023), Di Ferdinando and Pepe (2017) and the references therein). In this paper, the invariant differentiability property and the smooth separability property will be very helpful for the design of QSOE controllers. Notice that a very large subset of the class of standard complete quadratic Lyapunov–Krasovskii functionals (see Gu, Kharitonov, and Chen (2003), Kharitonov (2013)), such as the subset used in Esfanjani and Nikravesh (2009), Jankovic (2000) (see also Pepe and Ito (2012)), fulfills items **(a)**–**(d)**. For instance, the following standard quadratic functional

$$\begin{aligned} V_e(\phi_e) &= \phi_e^T(0)P_e\phi_e(0) + \textstyle\int_{-\Delta}^0 \phi_e^T(\tau)Q_e\phi_e(\tau)d\tau, \ \ \phi_e \in \mathcal{C}^n, \\ V_{\hat{x}}(\hat{\phi}) &= \hat{\phi}^T(0)P_{\hat{x}}\hat{\phi}(0) + \textstyle\int_{-\Delta}^0 \hat{\phi}^T(\tau)Q_{\hat{x}}\hat{\phi}(\tau)d\tau, \ \ \hat{\phi} \in \mathcal{C}^n, \end{aligned}$$

fulfills items **(a)**–**(d)** with functions $\beta_{e,1}(s) = \gamma_{e,1}(s) = \lambda_{\min}(P_e)s^2$, $\beta_{\hat{x},1}(s) = \gamma_{\hat{x},1}(s) = \lambda_{\min}(P_{\hat{x}})s^2$, $\beta_{e,2}(s) = \lambda_{\max}(P_e)s^2$, $\beta_{\hat{x},2}(s) = \lambda_{\max}(P_{\hat{x}})s^2$, $\gamma_{e,2}(s) = (\lambda_{\max}(P_e)+\Delta\lambda_{\max}(Q_e))s^2$, $\gamma_{\hat{x},2}(s) = (\lambda_{\max}(P_{\hat{x}})+\Delta\lambda_{\max}(Q_{\hat{x}}))s^2$, where P_e, $P_{\hat{x}}$, Q_e and $Q_{\hat{x}}$ are symmetric and positive definite matrices. We highlight also that, in the case $\Delta = 0$, only item **(d)** has to be considered since $V_{e,2} = V_{\hat{x},2} = 0$. In Section 5.1, a standard quadratic Lyapunov–Krasovskii functional will be successfully used for the application of the proposed results to a particular class of control-affine nonlinear systems with state-delays including a Glucose-Insulin one (Palumbo et al., 2007; Panunzi et al., 2010, 2007) very used at the aim of control design in the context of the plasma glucose regulation problem in Type-2 Diabetic patients by means of Artificial Pancreas (see, for instance, Di Ferdinando, Pepe, Di Gennaro, Borri, and Palumbo (2021), Di Ferdinando, Pepe, Palumbo, Panunzi, and De Gaetano (2017), Di Ferdinando, Pepe, Palumbo, Panunzi, and Gaetano (2020)).

In order to properly reformulate the definition of DOSDF provided in Di Ferdinando et al. (2022) to cope with the design of QSOE stabilizers, in the following definition, we will denote with k the function (continuous or not) from $\mathcal{C}^n \times \mathbb{R}^p \times \mathbb{R}^m$ to $\mathbb{R}^{m+n}$, of the form

$$\begin{aligned} k(\hat{\phi}, y, u) &= \begin{pmatrix} u \\ k_O(\hat{\phi}, y, u) \end{pmatrix} = \begin{pmatrix} u \\ \hat{f}(\hat{\phi}, y) + g(y)u \end{pmatrix}, \\ u &= k_F(\hat{\phi}, y), \ \ \hat{\phi} \in \mathcal{C}^n, \ \ y \in \mathbb{R}^p, \ \ u \in \mathbb{R}^m, \end{aligned} \tag{8}$$

where: g is the function in (2); $\hat{f} : \mathcal{C}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a function Lipschitz on bounded subsets of $\mathbb{R}^p$, uniformly with respect to the first argument in bounded subsets of $\mathcal{C}^n$; $k_O : \mathcal{C}^n \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^n$ is the function readily defined in (8); $k_F : \mathcal{C}^n \times \mathbb{R}^p \to \mathbb{R}^m$ is a function Lipschitz on bounded subsets of $\mathbb{R}^p$, uniformly with respect to the first argument in bounded subsets of $\mathcal{C}^n$.

Definition 7. Let $V_e \in \mathcal{V}_e$ and $V_{\hat{x}} \in \mathcal{V}_{\hat{x}}$. A function k (continuous or not) in the form (8) is said to be a DOSDF for the system described by (2), induced by V_e and $V_{\hat{x}}$, if there exist positive reals η, μ such that, for any $\hat{\phi}$, $\phi_e \in \mathcal{C}^n$, the following inequalities hold

- $\Delta > 0$,

$$\begin{aligned} &D^+V_e(\phi_e, \hat{\phi}, k_F, k_O(\hat{\phi}, h(\hat{\phi} + \phi_e), k_F)) \\ &+ \eta \max\{0, D^+V_{e,1}(\phi_e, \hat{\phi}, k_F, k_O(\hat{\phi}, h(\hat{\phi} + \phi_e), k_F)) \\ &\qquad + \mu V_{e,1}(\phi_e(0))\} \le 0, \\ &D^+V_{\hat{x}}(\hat{\phi}, k_O(\hat{\phi}, h(\hat{\phi} + \phi_e), k_F)) \\ &+ \eta \max\{0, D^+V_{\hat{x},1}(\hat{\phi}, k_O(\hat{\phi}, h(\hat{\phi} + \phi_e), k_F)) \\ &\qquad + \mu V_{\hat{x},1}(\hat{\phi}(0))\} \le 0, \end{aligned} \tag{9}$$

- $\Delta = 0$,

$$\begin{aligned} &D^+V_e(\phi_e(0), \hat{\phi}(0), k_F, k_O(\hat{\phi}(0), h(\hat{\phi}(0) + \phi_e(0)), k_F)) \\ &\qquad \le -\gamma_{e,3}(|\phi_e(0)|), \\ &D^+V_{\hat{x}}(\hat{\phi}(0), k_O(\hat{\phi}(0), h(\hat{\phi}(0) + \phi_e(0)), k_F)) \le -\gamma_{\hat{x},3}(|\hat{\phi}(0)|), \end{aligned}$$

where: in the case $\Delta = 0$, $\gamma_{e,3}$, $\gamma_{\hat{x},3}$ are functions of class $\mathcal{K}_\infty$; $k_F = k_F(\hat{\phi}, h(\hat{\phi} + \phi_e))$; h is the function in (2); the derivatives in Driver's form (1) of the functionals V_e and $V_{\hat{x}}$ are computed with respect to the functions f_e and $f_{\hat{x}}$ in (5) with $u = k_F$ and $v = k_O(\hat{\phi}, h(\hat{\phi} + \phi_e), k_F)$.

Assumption 8. There exist functionals $V_e \in \mathcal{V}_e$, $V_{\hat{x}} \in \mathcal{V}_{\hat{x}}$ and a related DOSDF k for the system described by (2) according to Definition 7.

Remark 9. Notice that, in Definition 7, the inequalities in (9) concern robustness of negative definiteness, with respect to small perturbation terms, of the functionals derivatives V_e and $V_{\hat{x}}$ (see Remark 2 in Pepe (2014) for a detailed discussion in the case of static state feedbacks).

In order to introduce the proposed QSOE controller, under Assumption 8 (see also Definition 7), let:

(1a.) $V_{\hat{x},3} : \mathcal{C}^n \to \mathbb{R}^+$ be the functional defined, for $\hat{\phi} \in \mathcal{C}^n$, as $V_{\hat{x},3}(\hat{\phi}) = \sup_{\theta \in [-\Delta,0]} e^{\mu\theta} V_{\hat{x},1}(\hat{\phi}(\theta))$;

(2a.) $\mathcal{D}_{\hat{x},\infty} : \mathcal{C}^n \times \mathbb{R}^n \to \mathbb{R}$ be the functional defined, for $\hat{\phi} \in \mathcal{C}^n$, $v \in R^n$, as follows

$$\begin{aligned} \mathcal{D}_{\hat{x},\infty}(\hat{\phi}, v) &= D^+V_{\hat{x}}(\hat{\phi}, v) - \eta\mu V_{\hat{x},3}(\hat{\phi}) \\ &\quad + \eta \max\{0, D^+V_{\hat{x},1}(\hat{\phi}, v) + \mu V_{\hat{x},1}(\hat{\phi}(0))\}. \end{aligned} \tag{10}$$

Under Assumption 8, for a given positive real $\sigma \in (0, 1)$, for a given partition $\pi_{a,\delta}$, for a given output quantizer $q_y : \mathbb{R}^p \to \mathcal{Q}_y^p$ and for a given input quantizer $q_u : \mathbb{R}^m \to \mathcal{Q}_u^m$, the proposed

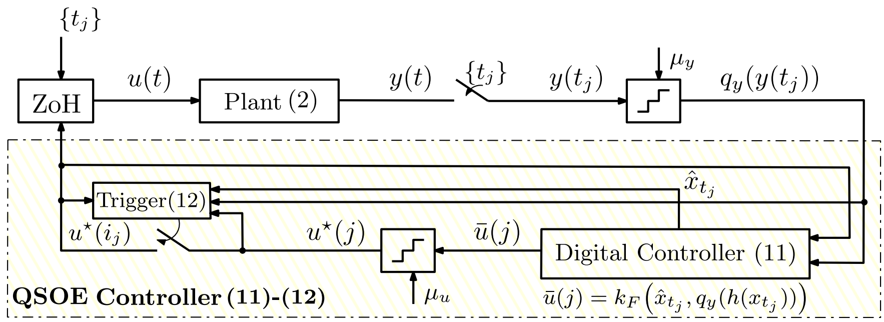

Fig. 1. Digital event-triggered control scheme.

QSOE controller for the system (2) is described by (see Fig. 1)

$$
\begin{aligned}
& u(t) = u^\star(i_j) = q_u\left(k_F(\hat{x}_{t_{i_j}}, q_y(h(x_{t_{i_j}}))) \right), \\
& \quad t \in [t_j, t_{j+1}),\ t_j \in \pi_{a,\delta},\ \delta_j = t_{j+1} - t_j,\ j = 0, 1, \dots, \\
& \quad (\mathbf{a})\ \min\{\Delta,\ \delta_j\} = \delta_j, \\
& \hat{x}_{t_{j+1}}(\theta) = \begin{cases} \hat{x}_{t_j}(\theta + \delta_j), & \theta \in [-\Delta, -\delta_j) \\ \hat{x}_{t_j}(0) + (\theta + \delta_j) k_O(\hat{x}_{t_j}, q_y(h(x_{t_j})), u^\star(i_j)) & \\ & \theta \in [-\delta_j, 0] \end{cases} \\
& \quad (\mathbf{b})\ \min\{\Delta,\ \delta_j\} = \Delta,\ \theta \in [-\Delta, 0], \\
& \hat{x}_{t_{j+1}}(\theta) = \hat{x}_{t_j}(0) + (\theta + \delta_j) k_O(\hat{x}_{t_j}, q_y(h(x_{t_j})), u^\star(i_j)), \\
& \quad (\mathbf{c})\ \Delta = 0, \\
& \hat{x}_{t_{j+1}}(0) = \hat{x}_{t_j}(0) + \delta_j k_O\left(\hat{x}_{t_j}(0), q_y\left(h(x_{t_j}(0)) \right), u^\star(i_j) \right),
\end{aligned}
\tag{11}
$$

where: $\hat{x}_{t_j}$, $\hat{x}_0 \in \mathcal{C}^n$; $u(t) \in \mathbb{R}^m$ is the input signal in (2) which is here implemented via zero-order-hold devices; k_F and k_O are the functions in Definition 7; the sequence i_j is defined recursively as $i_0 = 0$ and, for $j \geq 1$, $i_j = j$ in the event that

$$
\begin{aligned}
& \bullet\ \ \Delta > 0, \\
& -\ \mathcal{D}_{\hat{x},\infty}(\hat{x}_{t_j}, k_O(\hat{x}_{t_j}, q_y(h(x_{t_j})), u^\star(i_{j-1}))) \\
& \quad + \sigma \mathcal{D}_{\hat{x},\infty}(\hat{x}_{t_j}, k_O(\hat{x}_{t_j}, q_y(h(x_{t_j})), u^\star(j))) \leq 0, \\
& \bullet\ \ \Delta = 0, \\
& -\ D^+ V_{\hat{x}}(\hat{x}_{t_j}(0), k_O(\hat{x}_{t_j}(0), q_y\left(h(x_{t_j}(0)) \right), u^\star(i_{j-1}))) \\
& \quad + \sigma D^+ V_{\hat{x}}(\hat{x}_{t_j}(0), k_O(\hat{x}_{t_j}(0), q_y\left(h(x_{t_j}(0)) \right), u^\star(j))) \leq 0,
\end{aligned}
\tag{12}
$$

and $i_j = i_{j-1}$, otherwise.

Remark 10. Notice that, in the considered digital framework, the notion of partition (see Definition 4) is introduced to describe the sequence of sampling instants t_j, $j = 0, 1, \dots$, in which the output of the system (2) is measured and sent to the controller (i.e., the times in which a communication from the system to the controller occurs) for the computation of the new control value $u^\star(j)$ and the check of the triggering condition (see (12)) managing the input signal updates to the system (see Fig. 1). The sampling intervals related to the partition are thus necessarily upper bounded by a finite positive real number δ according to Definition 4. Theorem 12 provides results concerning the existence of such upper bound ensuring the semi-global practical stability property of the related digital closed-loop system (2), (11) (see also Fig. 1). Moreover, taking into account that in Definition 4 the positive real a, characterizing the dwell time and allowing the consideration of aperiodic sampling, cannot be equal to zero, the notion of partition does not include the case of zero dwell-time. We highlight also that, for the practical implementation of the QSOE controller proposed in (11) the knowledge of the sampling times t_j, $j = 0, 1, \dots$, is not needed. On the other hand, for the computation of the estimated state $\hat{x}_{t_{j+1}}$, $j = 0, 1, \dots$, the knowledge of the time elapsed between the latest two sampling times (i.e., $\delta_j = t_{j+1} - t_j$) is needed. We highlight here that the knowledge of the sampling period δ_j is not required at the sampling instant t_j. Indeed, $u(t)$ (see (11)), $t_j \leq t < t_{j+1}$, does not depend on t_{j+1} and, consequently, the computation of $\hat{x}_{t_{j+1}}$ in (11) can be performed at time t_{j+1} when the term $(t_{j+1} - t_j)$ is known. We highlight also that, the proposed triggering condition (12) is checked just at times t_j, $j = 0, 1, \dots$, guaranteeing a minimum dwell-time $a\delta$ between two consecutive sampling instants (see Definition 4). Hence, no continuous-time monitoring of the state variables is needed and possible Zeno behaviors are avoided by the presence of sampling with dwell-time.

Remark 11. Notice that the main purpose of the proposed event-based mechanism is to reduce the communications from the controller to the system as much as possible. This is achieved by checking a proposed condition (see (12)), mainly related to the stability property of the digital closed-loop control scheme (see Fig. 1) and which establishes if, at the time t_j, $j = 0, 1, \dots$, the updated control signal has to be sent to the system or not. In particular, the proposed event-based mechanism exploits a proper function derived from the Lyapunov–Krasovskii functional $V_{\hat{x}}$ at hand (see Definition 7, Assumption 8, **(2a.)** and (12)) to compare, at each sampling instant t_j, $j = 0, 1, \dots$, the behavior of the system, from the derivative of the Lyapunov functional point of view, in the case of control input update (i.e. in the case that the updated control input $u^\star(j)$ is sent to the system at time t_j) with the non-updated one (i.e. in the case that the updated control input $u^\star(j)$ is not sent to the system at time t_j). The times t_{i_j} in (11) belong to the partition $\pi_{a,\delta}$ and characterize the instants of control input update via the sequence i_j, $j = 0, 1, \dots$. Such a sequence is updated at each sampling instant t_j, $j = 0, 1, \dots$, through the proposed triggering condition (12), by setting $i_j = j$, in the case that the updated value of the control signal is sent to the system and, $i_j = i_{j-1}$, otherwise. From a practical point of view, the benefits to consider an event-based strategy (see (12)) for reducing communications is shown in forthcoming Section 5 where the proposed design methodology is applied to the plasma glucose regulation problem in Type-2 Diabetic Patients and the temperature control problem of a CSTR. In particular, in Section 5, a comparison concerning the performances of the proposed controllers in the case of time-triggered solution and in the case of event-triggered solution will be provided showing the capability of the proposed event-based strategy in reducing control updates.

4. Main results

In this section, the main results of the paper are provided with a related discussion concerning the proposed design methodology (see Theorem 12 and Remarks 13–17). In particular, in the forthcoming Theorem 12, it is shown the existence of a suitably fast sampling δ and of an accurate quantization of the input/output channels (i.e., of quantizers ranges E_1, U and of quantization error bounds μ_y, μ_u) such that the semi-global practical stability property, with arbitrarily small final target ball of the origin, is ensured for the system described by (2) in closed-loop with the proposed QSOE controller (11).

Theorem 12. *Let Assumption 8 hold. Let $\sigma \in (0, 1)$ and $a \in (0, 1]$ be arbitrarily chosen. Then, for any positive reals q, r, R with $0 < r < R$, there exist positive reals δ, T, E, E_1, U, μ_y and μ_u such that: for any partition $\pi_{a,\delta}$, for any output quantizer q_y with range E_1 and quantization error μ_y, for any input quantizer q_u with range U and quantization error μ_u, for any initial state $\begin{pmatrix} x_0 \\ \hat{x}_0 \end{pmatrix} \in \mathcal{C}_R^{2n} \cap W_{2n}^{1,\infty}$ and, in the case $\Delta > 0$, satisfying $\text{esssup}_{\theta\in[-\Delta,0]} \left| \begin{pmatrix} \frac{dx_0(\theta)}{d\theta} \\ \frac{d\hat{x}_0(\theta)}{d\theta} \end{pmatrix} \right| \leq q$, the corresponding solution of the closed-loop system described by (2)–(11) exists for all $t \in \mathbb{R}^+$, and, furthermore, the following inequalities hold*

$$\|x_t\|_\infty \leq 2E, \ \forall t \in \mathbb{R}^+, \quad \|x_t\|_\infty \leq 2r, \ \forall t \geq T. \tag{13}$$

Proof. See the Appendix. ◇

Remark 13. Notice that Theorem 12 provides semi-global practical stability results concerning the digital closed-loop system described by (2)–(11) ensuring that for any large ball and small ball of the origin, there exist a finite time T, a suitably fast sampling (also non-uniform) and an accurate (also non-uniform) quantization of the input/output channels such that: the corresponding solution starting in any point of the large ball, exists, is uniformly bounded, is driven into the small ball of the origin in a finite time T and, moreover, is kept therein thereafter. The term semi-global refers to the fact that the radius of the ball related to the initial states R can be chosen arbitrarily large. In particular, for the digital closed-loop system described by (2)–(11), chosen arbitrarily the radius of the ball of initial states R, the radius of the final target ball $2r$, the positive reals $a \in (0, 1]$ (related to the definition of partition), $\sigma \in (0, 1)$ (related to the proposed triggering condition (12)) and $q \in \mathbb{R}^+$, Theorem 12 ensures the existence of: (i) an upper bound for the sampling period δ; (ii) upper bounds for the input and output quantization errors μ_y and μ_u; (iii) ranges E_1 and U_1 related to the input and output signal quantizers; (iv) an overshoot E for the solution (see (13)); (v) a settling time T such that for any $t \geq T$, the solution is kept within the final target ball with arbitrarily small radius $2r$ (see (13)).

Remark 14. We highlight that differently from the results provided in the literature concerning the event-based dynamic output feedback control of nonlinear time-delay/delay-free systems, where proposed digital control strategies are commonly based on the dynamical part of the controllers evolving in a continuous-time basis (see, for instance, Abdelrahim et al. (2017), Borri et al. (2024), Choi and Yoo (2019), Di Ferdinando, Borri, et al. (2024), Fu and Qiao (2022), Scheres et al. (2024), Zhao et al. (2021)), in the methodology here provided: (i) the proposed event-triggered control strategy fully evolves in a discrete-time basis avoiding the necessity to solve differential equations for the correct implementation of the controller (see, for instance, (10) in Di Ferdinando, Borri, et al. (2024) and (11)); (ii) discontinuities in both functions k_F and k_O describing the feedback and the dynamics of the controller at hand, respectively, are allowed (see Definition 7); (iii) the presence of aperiodic sampling and of the non-uniform quantization in both input/output channels is considered. To our best knowledge, it is the first time in the literature that results concerning quantized sampled-data dynamic output feedback controllers exploiting event-based mechanisms for the update of the control law are provided by taking simultaneously into account points (i)-(iii).

Remark 15. We highlight that, from a practical implementation point of view, the proof of Theorem 12 provides a methodology for the computation of upper bounds for the sampling period δ and quantization errors μ_y, μ_u, of quantizer ranges E_1 and U, of a settling time T, and of an overshoot E (see steps **(1)**–**(15)** soon after Lemma 18). According to steps **(1)**–**(15)**, the computation of such parameters requires just the choice of: (i) the radius related to the ball of initial states R; (ii) the radius related to the final target ball $2r$; (iii) the parameters $q \in \mathbb{R}^+$, $a \in (0, 1]$ and $\sigma \in (0, 1)$ (see Theorem 12 and steps **(1)**–**(15)**). We highlight also that, to our experience, steps **(1)**–**(15)** may well provide conservative upper bounds for sampling period and quantization errors. The source of such conservatism may be the use of Lipschitz constants of the involved functions, as well as lower and upper bounds of the involved Lyapunov functionals and derivatives. On the other hand, the results provided in Theorem 12 are of existence type, and the study of the conservativeness of the sampling frequency as well as of the quantization errors is beyond the aim of this work, and is left for future investigations.

Remark 16. The methodology proposed in this paper is based on the Artstein's approach (Artstein, 1983) exploiting candidate Lyapunov/Lyapunov–Krasovskii functionals for the design of controllers (see, for instance, Artstein (1983), Clarke (2010), Clarke et al. (1997), Di Ferdinando et al. (2022), Pepe (2014)) which is here extended, for the first time in the literature, to the design of QSOE stabilizers. Indeed, the digital stabilizer in (11) is built up by starting from two candidate Lyapunov–Krasovskii functionals chosen in the sets $\mathcal{V}_e$ and $\mathcal{V}_{\hat{x}}$, respectively and, trying to find the related functions k_F and k_O (see (8)) satisfying the inequalities in (9). In the following, a steps procedure summarizing the proposed design methodology is provided:

(1) choose candidate Lyapunov–Krasovskii functionals (Lyapunov functions in the case $\Delta = 0$) $V_e \in \mathcal{V}_e$ and $V_{\hat{x}} \in \mathcal{V}_{\hat{x}}$ (see points **(a)**–**(d)** and Remark 6);
(2) taking into account the functions f_e and $f_{\hat{x}}$ in (5), by the use of the Lyapunov–Krasovskii functionals (Lyapunov functions in the case $\Delta = 0$) in step (1), try to find functions k_F and k_O (see (8)) satisfying (9), i.e. a DOSDF (see Definition 7);
(3) implement the QSOE controller as in (11) by using the DOSDF in step (2) according to Theorem 12 (see also Remark 15).

Notice that, steps (1)–(3) provide a procedure on how Assumption 8 can be verified. The conditions introduced throughout the paper for the design of QSOE stabilizers (see Definition 7, Assumption 8 and, in particular, (9)), even if apparently restrictive, can be satisfied by many physical systems as, for instance, the Glucose-Insulin system and the CSTR system studied in the forthcoming Section 5. In particular, forthcoming Section 5 shows how DOSDFs can be designed in practice by exploiting standard quadratic Lyapunov/Lyapunov–Krasovskii functionals and the reasoning proposed in steps (1)–(3) above. We highlight also that, the positive reals η and μ in Definition 7 can be regarded as further degrees of freedom which can be suitably selected in order to try to satisfy (9). Moreover, there exist many Lyapunov based approaches in the literature which can be suitably exploited in order to find the functions k_F and k_O (i.e., DOSDFs). For instance, the methodology proposed in Di Ferdinando et al. (2023), concerning the design of observer-based controllers via the well-known Sontag's universal formula can be suitably exploited here for the design of the function k_F. We highlight also that one of the advantages of the proposed design methodology is the possibility to allow (partial) discontinuities in the functions describing the DOSDF at hand (see (8)), enlarging the chances of successfully designing QSOE stabilizers for nonlinear systems with and without state delays, since the continuity constraint is mitigated. Indeed, according to Definition 7, discontinuities with respect to the first argument in the functions describing the DOSDF at hand (see (8)) are here allowed. Moreover, the results provided in Theorem 12 still hold in the case of DOSDFs with discontinuities in both arguments (see (8)) at the price of assuming a perfect knowledge of the system output at each sampling instant (i.e. $h(x_{t_j})$), not considering the output quantization.

Remark 17. In the proof of Theorem 12, it will be proved that the solution $\begin{pmatrix} e_t \\ \hat{x}_t \end{pmatrix} = \begin{pmatrix} x_t - \hat{x}_t \\ \hat{x}_t \end{pmatrix}$ of the extended quantized sampled-data event-based closed-loop system characterized by the estimation error and the controller state variables (see (A.3), (A.4), (A.10), (A.22)) exists for any $t \geq 0$ and, furthermore, belongs to $\mathcal{C}_r^{2n}$, $\forall t \geq T$. From such a result, it follows that the proposed QSOE controller (11) is a state observer for the system (2) in a semiglobal practical sense. It is here highlighted that, differently from Di Ferdinando et al. (2022) where quantization effects and the event-based implementation of the stabilizer are not taken into account, here theoretical results are provided concerning the capability of the proposed controller to estimate the state of the closed-loop system at hand.

5. Applications

5.1. Application to particular classes of control-affine nonlinear systems

In this subsection, the methodology proposed for the design of QSOE stabilizers will be applied to particular classes of control-affine nonlinear systems. In particular, we will make use of steps (1)–(3) provided in Remark 16 to design QSOE controllers based on DOSDFs.

5.1.1. Example 1

Let us consider a time-delay system described by

$$\begin{aligned}
&\dot{x}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = f(x_t) + g(h(x_t))u(t) = \\
&\begin{pmatrix} -\alpha x_1(t) + f_1(x_1(t), x_1(t-\Delta))x_2(t) \\ -\beta x_2(t) + f_2(x_1(t), x_1(t-\Delta))x_2(t) \\ +f_3(x_1(t), x_1(t-\Delta)) + \bar{\beta} x_2(t-\Delta) \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma \end{pmatrix} u(t), \qquad (14)\\
&y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = h(x_t) = \begin{pmatrix} x_1(t) \\ x_1(t-\Delta) \end{pmatrix}, \\
&x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0],
\end{aligned}$$

where: $x_t = \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} \in \mathcal{C}^2$, $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \mathbb{R}^2$ is the state; $x_0 \in \mathcal{C}^2$ is the initial state; $u(t) \in \mathbb{R}$ is the input; $\Delta > 0$ is the involved time delay; $y(t) \in \mathbb{R}^2$ is the output; $f_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$ are Lipschitz on bounded subsets functions; α, β and γ are positive real parameters; $\bar{\beta}$ is a real number satisfying $2|\bar{\beta}| < \beta$. It is assumed that $f_2(\psi(0), \psi(-\Delta)) \leq 0$, $\forall \psi \in \mathcal{C}$. We highlight that system (14) is in the form (2). In this case, f_e and $f_{\hat{x}}$ in (5) are the functions from $\mathcal{C}^2 \times \mathcal{C}^2 \times \mathbb{R} \times \mathbb{R}^2$ to $\mathbb{R}^2$ and from $\mathcal{C}^2 \times \mathbb{R}$ to $\mathbb{R}^2$, respectively, defined, for any $\hat{\phi} = \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix} \in \mathcal{C}^2$, $\phi_e = \begin{pmatrix} \phi_{e_1} \\ \phi_{e_2} \end{pmatrix} \in \mathcal{C}^2$, $\hat{\phi}_i, \phi_{e_i} \in \mathcal{C}$, $i = 1, 2$, for any $u \in \mathbb{R}$ and for any $v = (v_1 \quad v_2)^T \in \mathbb{R}^2$, $v_i \in \mathbb{R}$, $i = 1, 2$, as follows

$$\begin{aligned}
&f_e(\phi_e, \hat{\phi}, u, v) = \left(f_{e,1}(\phi_e, \hat{\phi}, u, v) \quad f_{e,2}(\phi_e, \hat{\phi}, u, v) \right)^T, \\
&\qquad f_{\hat{x}}(\hat{\phi}, v) = (v_1 \quad v_2)^T \\
&f_{e,1}(\phi_e, \hat{\phi}, u, v) = -\alpha(\hat{\phi}_1(0) + \phi_{e_1}(0)) - v_1 \\
&\qquad + f_1^{\hat{\phi}_1+\phi_{e,1}}(\hat{\phi}_2(0) + \phi_{e_2}(0)) \qquad (15)\\
&f_{e,2}(\phi_e, \hat{\phi}, u, v) = -\beta(\hat{\phi}_2(0) + \phi_{e_2}(0)) + \gamma u - v_2 \\
&\qquad + f_2^{\hat{\phi}_1+\phi_{e,1}}(\hat{\phi}_2(0) + \phi_{e_2}(0)) + f_3^{\hat{\phi}_1+\phi_{e,1}} \\
&\qquad + \bar{\beta}(\hat{\phi}_2(-\Delta) + \phi_{e_2}(-\Delta)),
\end{aligned}$$

where $f_i^{\hat{\phi}_1+\phi_{e,1}} = f_i(\hat{\phi}_1(0) + \phi_{e_1}(0), \hat{\phi}_1(-\Delta) + \phi_{e_1}(-\Delta))$, $i = 1, 2, 3$. According to the design procedure proposed in Section 3 (see also steps (1)–(3) in Remark 16), let $V_{e,1} : \mathbb{R}^2 \to \mathbb{R}^+$ be the function defined for any $e \in \mathbb{R}^2$, as $V_{e,1}(e) = e^T P_e e$, where $P_e = \mathrm{diag}\{0.5p_1, 0.5p_2\}$, with $p_i > 0$, $i = 1, 2$. Let $\widetilde{V}_{e,2} : \mathcal{C}^2 \to \mathbb{R}^+$ be the functional defined for any $\phi_e \in \mathcal{C}^2$, as $\widetilde{V}_{e,2}(\phi_e) = \int_{-\Delta}^{0} \phi_e(\tau)^T Q_e \phi_e(\tau) d\tau$, where $Q_e = \mathrm{diag}\{0.5q_1, 0.5q_2\}$, with $q_i > 0$, $i = 1, 2$. Let $V_e : \mathcal{C}^2 \to \mathbb{R}^+$ be the functional defined, for any $\phi_e \in \mathcal{C}^2$, as $V_e(\phi_e) = V_{e,1}(\phi_e(0)) + \widetilde{V}_{e,2}(\phi_e)$. Let $V_{\hat{x},1} : \mathbb{R}^2 \to \mathbb{R}^+$ be the function defined, for $\hat{x} \in \mathbb{R}^2$, as $V_{\hat{x},1}(\hat{x}) = \hat{x}^T P_{\hat{x}} \hat{x}$, $P_{\hat{x}} = P_e$ and $V_{\hat{x},2} : \mathcal{C}^2 \to \mathbb{R}^+$ be the functional defined, for $\hat{\phi} \in \mathcal{C}^2$, as $V_{\hat{x},2}(\hat{\phi}) = 0$. Let $V_{\hat{x}} : \mathcal{C}^2 \to \mathbb{R}^+$ be the functional defined, for any $\hat{\phi} \in \mathcal{C}^2$, as $V_{\hat{x}}(\hat{\phi}) = V_{\hat{x},1}(\hat{\phi}(0)) + V_{\hat{x},2}(\hat{\phi})$. Notice that $V_e \in \mathcal{V}_e$ and $V_{\hat{x}} \in \mathcal{V}_{\hat{x}}$ (see Remark 6).

Taking into account Remark 16, in the following, the introduced candidate Lyapunov–Krasovskii functionals V_e and $V_{\hat{x}}$ are exploited for trying to find a DOSDF for the system (14) according to Definition 7. As far as the first inequality in Definition 7 is concerned (see (9)), taking into account (15) and step (2) in Remark 16, by exploiting the proposed candidate Lyapunov–Krasovskii functional V_e, in this case, we obtain

$$\begin{aligned}
&D^+V_e(\phi_e, \hat{\phi}, u, v) = 2\phi_e(0)^T P_e f_e(\phi_e, \hat{\phi}, u, v) + 0.5q_1\phi_{e_1}^2(0) \\
&+ 0.5q_2\phi_{e_2}^2(0) - 0.5q_1\phi_{e_1}^2(-\Delta) - 0.5q_2\phi_{e_2}^2(-\Delta) = p_1\phi_{e_1}(0)\times \\
&\left(-\alpha(\hat{\phi}_1(0) + \phi_{e_1}(0)) - v_1 + f_1^{\hat{\phi}_1+\phi_{e,1}}(\hat{\phi}_2(0) + \phi_{e_2}(0)) \right) + \\
&p_2\phi_{e_2}(0) \left(-\beta(\hat{\phi}_2(0) + \phi_{e_2}(0)) + \gamma u - v_2 + f_3^{\hat{\phi}_1+\phi_{e,1}} + \right. \\
&\left. f_2^{\hat{\phi}_1+\phi_{e,1}}(\hat{\phi}_2(0) + \phi_{e_2}(0)) + \bar{\beta}(\hat{\phi}_2(-\Delta) + \phi_{e_2}(-\Delta)) \right) \\
&+ 0.5q_1\phi_{e_1}^2(0) + 0.5q_2\phi_{e_2}^2(0) - 0.5q_1\phi_{e_1}^2(-\Delta) - 0.5q_2\phi_{e_2}^2(-\Delta).
\end{aligned} \qquad (16)$$

Taking into account (8) and (16), let $\hat{f} : \mathcal{C}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, $g : \mathbb{R}^2 \to \mathbb{R}^2$ and $k_O : \mathcal{C}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ be the functions defined for any $u \in \mathbb{R}$, $\hat{\phi} = \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix} \in \mathcal{C}^2$, $\hat{\phi}_i \in \mathcal{C}$, $i = 1, 2$, $y = (y_1 \quad y_2)^T \in \mathbb{R}^2$, and $u \in \mathbb{R}$ as follows

$$\begin{aligned}
&\hat{f}(\hat{\phi}, y) = \left(\hat{f}_1(\hat{\phi}, y) \quad \hat{f}_2(\hat{\phi}, y) \right)^T, \quad g(y) = (0 \quad \gamma)^T, \\
&k_O(\hat{\phi}, y, u) = \hat{f}(\hat{\phi}, y) + g(y)u, \\
&\hat{f}_1(\hat{\phi}, y) = -\alpha\hat{\phi}_1(0) + f_1(y_1, y_2)\hat{\phi}_2(0) + G_1\hat{\phi}_2^2(0)(y_1 - \hat{\phi}_1(0)) \qquad (17)\\
&\hat{f}_2(\hat{\phi}, y) = -\beta\hat{\phi}_2(0) + f_2(y_1, y_2)\hat{\phi}_2(0) + f_3(y_1, y_2) \\
&\qquad + \tfrac{f_1(y_1, y_2)}{\rho}(y_1 - \hat{\phi}_1(0)) + \bar{\beta}\hat{\phi}_2(-\Delta),
\end{aligned}$$

where G_1 and ρ are positive tuning parameters to be selected. From (16), taking into account (17), we obtain

$$\begin{aligned} &D^+V_e(\phi_e,\hat{\phi},u,k_O(\hat{\phi},h(\hat{\phi}+\phi_e),u)) = p_1\phi_{e_1}(0)\Big(-\alpha\phi_{e_1}(0) \\ &+ f_1^{\hat{\phi}_1+\phi_{e,1}}\phi_{e_2}(0)\Big) + p_2\phi_{e_2}(0)\Big(-\beta\phi_{e_2}(0) + f_2^{\hat{\phi}_1+\phi_{e,1}}\phi_{e_2}(0) \\ &- \frac{f_1^{\hat{\phi}_1+\phi_{e,1}}}{\rho}\phi_{e_1}(0) + \bar{\beta}\phi_{e_2}(-\Delta)\Big) + 0.5q_1\phi_{e_1}^2(0) + 0.5q_2\phi_{e_2}^2(0) \\ &- 0.5q_1\phi_{e_1}^2(-\Delta) - 0.5q_2\phi_{e_2}^2(-\Delta). \end{aligned} \tag{18}$$

By choosing $p_2 = p_1\rho$, $q_1 < 2p_1\alpha$ in the case $\bar{\beta} \neq 0$ and $q_1 = 0$ in the case $\bar{\beta} = 0$ and $q_2 = 2\rho p_1|\bar{\beta}|$, taking into account that $2|\bar{\beta}| < \beta$ and that $f_2(\psi(0),\psi(-\Delta)) \leq 0$, $\forall\psi \in \mathcal{C}$, from (18), the following inequalities hold

$$\begin{aligned} &D^+V_e(\phi_e,\hat{\phi},u,k_O(\hat{\phi},h(\hat{\phi}+\phi_e),u)) \leq p_1(-\alpha\phi_{e_1}^2(0) - \rho\beta\phi_{e_2}^2(0) \\ &+ \rho\bar{\beta}\phi_{e_2}(0)\phi_{e_2}(-\Delta) + \rho|\bar{\beta}|\phi_{e_2}^2(0) - \rho|\bar{\beta}|\phi_{e_2}^2(-\Delta)) \\ &+ 0.5q_1(\phi_{e_1}^2(0) - \phi_{e_1}^2(-\Delta)) \leq -0.5\rho p_1|\bar{\beta}|\phi_{e_2}(-\Delta)^2 \\ &\quad - \min\{(p_1\alpha - 0.5q_1), 0.5\rho p_1|\bar{\beta}|\}|\phi_e(0)|^2. \end{aligned} \tag{19}$$

Taking into account (19), it follows that, for any $u \in \mathbb{R}$, $\hat{\phi} = \begin{pmatrix}\hat{\phi}_1\\ \hat{\phi}_2\end{pmatrix} \in \mathcal{C}^2$ and $\phi_e = \begin{pmatrix}\phi_{e,1}\\ \phi_{e,2}\end{pmatrix} \in \mathcal{C}^2$, $\phi_{e,i}$, $\hat{\phi}_i \in \mathcal{C}$, $i = 1,2$, the following inequalities hold

$$\begin{aligned} &D^+V_e(\phi_e,\hat{\phi},u,k_O(\hat{\phi},h(\hat{\phi}+\phi_e),u)) \\ &+ \eta\max\{0, D^+V_{e,1}(\phi_e,\hat{\phi},u,k_O(\hat{\phi},h(\hat{\phi}+\phi_e),u)) \\ &\qquad + \mu V_{e,1}(\phi_e(0))\} \leq -0.5\rho p|\bar{\beta}|\phi_{e_2}(-\Delta)^2 \\ &- \min\{(p_1\alpha - 0.5q_1), 0.5\rho p_1|\bar{\beta}|\}|\phi_e(0)|^2 + \\ &\eta\max\{0, -p_1\alpha\phi_{e_1}^2(0) - p_1\rho\beta\phi_{e_2}^2(0) \\ &+ p_1\rho\bar{\beta}\phi_{e_2}(0)\phi_{e_2}(-\Delta) + \mu\lambda_{\max}(P_e)|\phi_e(0)|^2\}. \end{aligned} \tag{20}$$

As far as the second inequality in Definition 7 is concerned (see (9)), taking into account (15), (17), step (2) in Remark 16 and that $P_{\hat{x}} = P_e$, by exploiting the proposed candidate Lyapunov–Krasovskii functional $V_{\hat{x}}$, in this case, we obtain

$$\begin{aligned} &D^+V_{\hat{x}}(\hat{\phi},k_O(\hat{\phi},h(\hat{\phi}+\phi_e),u)) = \\ &2\hat{\phi}(0)^T P_{\hat{x}} f_{\hat{x}}(\hat{\phi},k_O(\hat{\phi},h(\hat{\phi}+\phi_e),u)) = \\ &p_1\hat{\phi}_1(0)\Big(-\alpha\hat{\phi}_1(0) + f_1^{\hat{\phi}_1+\phi_{e,1}}\hat{\phi}_2(0) + G_1\hat{\phi}_2^2(0)\phi_{e,1}(0)\Big) \\ &+ p_2\hat{\phi}_2(0)\Big(-\beta\hat{\phi}_2(0) + f_2^{\hat{\phi}_1+\phi_{e,1}}\hat{\phi}_2(0) + f_3^{\hat{\phi}_1+\phi_{e,1}} \\ &+ \frac{f_1^{\hat{\phi}_1+\phi_{e,1}}}{\rho}\phi_{e_1}(0) + \bar{\beta}\hat{\phi}_2(-\Delta) + \gamma u\Big). \end{aligned} \tag{21}$$

Taking into account (8) and (21), let $k_F : \mathcal{C}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the function defined for any $\hat{\phi} = \begin{pmatrix}\hat{\phi}_1\\ \hat{\phi}_2\end{pmatrix} \in \mathcal{C}^2$, $\hat{\phi}_i \in \mathcal{C}$, $i = 1,2$, $y = \begin{pmatrix}y_1\\ y_2\end{pmatrix} \in \mathbb{R}^2$ as follows

$$\begin{aligned} &k_F(\hat{\phi},y) = \tfrac{1}{\gamma}\Big((\beta - G_2)\hat{\phi}_2(0) - f_3(y_1,y_2) \\ &- \tfrac{f_1(y_1,y_2)}{\rho}y_1 - f_2(y_1,y_2)\hat{\phi}_2(0) - G_3\mathrm{sign}(\hat{\phi}_2(0)) \\ &- \bar{\beta}\hat{\phi}_2(-\Delta) - \tfrac{G_1}{\rho}\hat{\phi}_1(0)\hat{\phi}_2(0)(y_1 - \hat{\phi}_1(0))\Big), \end{aligned} \tag{22}$$

where G_2 and G_3 are positive control parameters to be tuned. From (21), taking into account (22), we obtain

$$\begin{aligned} &D^+V_{\hat{x}}(\hat{\phi},k_O(\hat{\phi},h(\hat{\phi}+\phi_e),k_F)) = p_1\hat{\phi}_1(0)\Big(-\alpha\hat{\phi}_1(0) \\ &+ f_1^{\hat{\phi}_1+\phi_{e,1}}\hat{\phi}_2(0) + G_1\hat{\phi}_2^2(0)\phi_{e,1}(0)\Big) + p_2\hat{\phi}_2(0)\Big(-G_2\phi_2(0) \\ &- \frac{f_1^{\hat{\phi}_1+\phi_{e,1}}}{\rho}\hat{\phi}_1(0) - G_3\mathrm{sign}(\hat{\phi}_2(0)) - \tfrac{G_1}{\rho}\hat{\phi}_1(0)\hat{\phi}_2(0)\phi_{e,1}(0)\Big). \end{aligned} \tag{23}$$

From (23), taking into account that $p_2 = p_1\rho$, the following inequality holds

$$D^+V_{\hat{x}}(\hat{\phi},k_O(\hat{\phi},h(\hat{\phi}+\phi_e),k_F)) \leq -p_1\min\{\alpha,\rho G_2\}|\hat{\phi}(0)|^2. \tag{24}$$

Taking into account (24) and that $P_{\hat{x}} = P_e$, it follows that, for any $\hat{\phi}$, $\phi_e \in \mathcal{C}^2$, the following inequalities hold

$$\begin{aligned} &D^+V_{\hat{x}}(\hat{\phi},k_O(\hat{\phi},h(\hat{\phi}+\phi_e),k_F)) + \\ &\eta\max\{0, D^+V_{\hat{x},1}(\hat{\phi},k_O(\hat{\phi},h(\hat{\phi}+\phi_e),k_F)) + \mu V_{\hat{x},1}(\hat{\phi}(0))\} \leq \\ &- p_1\min\{\alpha,\rho G_2\}|\hat{\phi}(0)|^2 + \\ &\eta\max\{0, -p_1\min\{\alpha,\rho G_2\}|\hat{\phi}(0)|^2 + \mu\lambda_{\max}(P_e)|\hat{\phi}(0)|^2\}. \end{aligned} \tag{25}$$

Taking into account (20) and (25), it follows that by choosing, for instance, $\mu \leq \min\left\{\frac{p_1\min\{\alpha,1.5\rho|\bar{\beta}|\}}{\lambda_{\max}(P_e)}, \frac{p_1\min\{\alpha,\rho G_2\}}{\lambda_{\max}(P_e)}\right\}$ and $0 < \eta < 1$, the inequalities in (9) are here satisfied. Consequently, taking into account k_O in (17), the function $k : \mathcal{C}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3$ defined for any $\hat{\phi} \in \mathcal{C}^2$, $y \in \mathbb{R}^2$ and $u \in \mathbb{R}$ as follows

$$k(\hat{\phi},y,u) = \begin{pmatrix} u \\ \hat{f}_1(\hat{\phi},y) \\ \hat{f}_2(\hat{\phi},y) + \gamma u \end{pmatrix}, \quad u = k_F(\hat{\phi},y), \tag{26}$$

with $\hat{f}_1$, $\hat{f}_2$ and k_F provided in (17), (22), is a DOSDF for the system described by (14) (see Definition 7). Thus Assumption 8 is satisfied and we can apply Theorem 12. We highlight here that the DOSDF in (26) is described by discontinuous functions.

Practical Application: *The Plasma Glucose Regulation Problem in Type 2 Diabetic Patients.*

In the following, the DOSDF proposed in (26) (see also (17), (22)) is applied in the context of the plasma glucose regulation problem in Type 2 Diabetic Patients (T2DM). See, for instance, Ogurtsova, da Rocha fernandes, Huang, Linnenkamp, Guariguata, Cho, Cavan, Shaw, and Makaroff (2017) and the references therein. The considered nonlinear time-delay model is described by Palumbo et al. (2007), Panunzi et al. (2010, 2007)

$$\begin{aligned} &\dot{G}(t) = -K_{xgi}G(t)I(t) + \tfrac{T_{gh}}{V_G} \\ &\dot{I}(t) = -K_{xi}I(t) + \tfrac{T_{iG,\max}}{V_I}\varphi\left(G\left(t-\tau_g\right)\right) + \tfrac{v(t)}{V_I} \\ &\bar{y}(t) = (\bar{y}_1(t) \;\; \bar{y}_2(t))^T = (G(t) \;\; G(t-\tau_g))^T, \end{aligned} \tag{27}$$

where: $G(t)$ [mmol/L] is the plasma glucose concentrations; $I(t)$ [pmol/L] is the insulin concentrations; $\varphi(\cdot)$ is a sigmoidal function modeling the endogenous pancreatic insulin delivery rate according to $\varphi\left(G\left(t-\tau_g\right)\right) = (\frac{G(t-\tau_g)}{G^*})^\gamma(1+(\frac{G(t-\tau_g)}{G^*})^\gamma)^{-1}$; $v(t)$ [(pmol/kgBW)/min] is the exogenous intra-venous insulin delivery rate (i.e. the control input); τ_g is the apparent delay with which the pancreas varies its secondary insulin release in response to varying plasma glucose concentrations. Please see Palumbo et al. (2007), Panunzi et al. (2010, 2007) for more details concerning the model (27) and related parameters. From the model (27), chosen a desired glucose concentration G_{ref}, we obtain the corresponding desired insulin concentration I_{ref} and insulin infusion rate v_{ref}. As a preliminary step, it is useful to rewrite system (27) with respect to the displacement $x(t) = (x_1(t) \;\; x_2(t))^T = (G(t) - G_{\mathrm{ref}} \;\; I(t) - I_{\mathrm{ref}})^T$, with the new control input $u(t) = v(t) - v_{\mathrm{ref}}$ and the new output signal $y(t) = (y_1(t) \;\; y_2(t))^T = (x_1(t) \;\; x_1(t-\tau_g))^T$. In particular, from (27) we obtain

$$\begin{aligned} &\dot{x}_1(t) = -K_{xgi}x_1(t)x_2(t) - K_{xgi}G_{\mathrm{ref}}x_2(t) - K_{xgi}I_{\mathrm{ref}}x_1(t) \\ &\dot{x}_2(t) = -K_{xi}(x_2(t) + I_{\mathrm{ref}}) \\ &\qquad + \tfrac{T_{iG,\max}}{V_I}\varphi\left(x_1(t-\tau_g) + G_{\mathrm{ref}}\right) + \tfrac{v_{\mathrm{ref}}+u(t)}{V_I}, \\ &y(t) = (y_1(t) \;\; y_2(t))^T = (x_1(t) \;\; x_1(t-\tau_g))^T, \end{aligned} \tag{28}$$

Table 1
System parameters.

$K_{\text{xgi}} = 3.15 \times 10^{-5}$ [min^{-1}(pmol/L)$^{-1}$]
$T_{\text{gh}} = 0.0023$ [min^{-1}(mmol/kgBW)]
$V_G = 0.18$ $\quad V_I = 0.25$ [L/kgBW], $\quad K_{\text{xi}} = 0.038$ [min^{-1}]
$T_{iG,\max} = 1.695$ [min^{-1} (pmol/kgBW)]
$\gamma = 15.92$ $\quad G^* = 9$ [mmol/L] $\quad \tau_g = 6.5$ [min]

with $x(t) \in \mathbb{R}^2$, $x_0 \in \mathcal{C}^2$, $u(t) \in \mathbb{R}$. Notice that system (28) is in the form (14) where: (i) the functions $f_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$ are defined, for any $y_1, y_2 \in \mathbb{R}$, as follows

$$f_1(y_1, y_2) = -K_{\text{xgi}} y_1 - K_{\text{xgi}} G_{\text{ref}}, \; f_2(y_1, y_2) = 0,$$
$$f_3(y_1, y_2) = -K_{\text{xi}} I_{\text{ref}} + \frac{T_{iG,\max}}{V_I} \varphi\,(y_2 + G_{\text{ref}}) + \frac{v_{\text{ref}}}{V_I}; \quad (29)$$

(ii) $\alpha = K_{\text{xgi}} I_{\text{ref}}$, $\beta = K_{\text{xi}}$, $\gamma = \frac{1}{V_I}$ and $\bar{\beta} = 0$. In this case, the QSOE controller is described by (11) where k_O and k_F are provided in (17), (22).

In the performed simulations, the following choices have been taken: the model parameters as in Di Ferdinando, Pepe, Di Gennaro, et al. (2021) (see Table 1); $G_{\text{ref}} = 5$; the initial state $x_0(\tau) = (3.45 \quad -33.2787)^T$, $\tau \in [-\tau_g, 0]$ (see Di Ferdinando, Pepe, Di Gennaro, et al. (2021)); the initial state of the QSOE controller $\hat{x}_0(\tau) = (1.5 \quad -13)^T$, $\tau \in [-\tau_g, 0]$; the control parameters $\rho = 3 \times 10^{-5}$, $G_1 = 4 \times 10^{-5}$, $G_2 = 0.04$ and $G_3 = 9 \times 10^{-4}$ (see (17), (22)); the parameter related to the Lyapunov–Krasovskii functionals V_e and $V_{\hat{x}}$, $p_1 = 10^{-6}$. Taking into account Remark 15, by choosing $R = 60$, $r = 5$, $a = 1$, $\sigma = 0.5$ and $q = 0$, as expected, the use of steps **(1)–(15)** in the proof of Theorem 12, leads to conservative upper bounds for the sampling period $\delta < 10^{-6}$ and for the quantization errors $\mu_y < 10^{-7}$ and $\mu_u < 10^{-6}$. On the other hand, taking into account that the results in Theorem 12 are of existence type (see Remark 15), a campaign of simulations has been performed with sampling periods and quantization errors greater than the ones computed with steps **(1)–(15)** in the proof of Theorem 12. In particular, simulations have been carried out by varying the sampling period δ, the quantization errors μ_y, μ_u and the parameter σ related to triggering condition (12). Very good performances of the proposed digital control scheme have been observed also in the case of sampling periods and quantization errors greater than the ones computed with steps **(1)–(15)**. Fig. 2 reports the simulation results in the case of a uniform (i.e., $a = 1$) sampling period $\delta = 10$[min] and quantization of the input/output channels based on the round to nearest method with parameters $\mathcal{Q}_y^2 = \{y = [y_1\, y_2]^T \in \mathbb{R}^2 | \, y_i = \pm 0.1j,\, i = 1, 2,\, j = 0, 1, \ldots, 3 \times 10^2\}$ and $\mathcal{Q}_u = \{u \in \mathbb{R} | \, u = \pm 0.1j,\, j = 0, 1, \ldots, 10^3\}$. In particular, in Fig. 2, the system variables $G(t) = x_1(t) + G_{\text{ref}}$, $I(t) = x_2(t) + I_{\text{ref}}$ and the observer variables $\hat{G}(t) = \hat{x}_1(t) + G_{\text{ref}}$, $\hat{I}(t) = \hat{x}_2(t) + I_{\text{ref}}$ are reported in the case of time-triggered controller and in the case of event-based controller with $\sigma = 0.5$. The event-triggered solution achieves very good performances similar to the ones of the time-triggered solution, in spite of the much lower average frequency of control updates (9.8% of the sampling intervals). Simulations fully validate the theoretical results.

5.1.2. Example 2

Let us consider a time-delay system described by

$$\begin{aligned}\dot{x}(t) &= f(x_t) + g(h(x_t))u(t) \\ &= Ax(t) + \bar{f}(Cx(t), Cx(t-\Delta)) + Bu(t), \\ y(t) &= h(x_t) = \begin{pmatrix} Cx(t) \\ Cx(t-\Delta) \end{pmatrix}, \;\; x(\tau) = x_0(\tau), \;\; \tau \in [-\Delta, 0],\end{aligned} \quad (30)$$

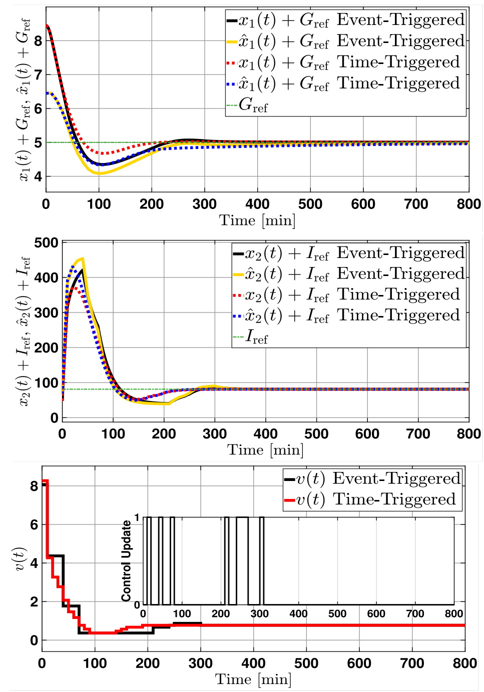

Fig. 2. In the firsts two panels the glucose $G(t)$ and the insulin $I(t)$ concentrations with their estimations $\hat{G}(t)$, $\hat{I}(t)$ (linear interpolation of discrete-time available values $\hat{G}(j\delta)$, $\hat{I}(j\delta)$, $j = 0, 1, \ldots$) are reported. In the third panel the infused insulin concentration $v(t)$ is reported.

where: $x_t \in \mathcal{C}^n$, $x(t) \in \mathbb{R}^n$ is the state; $x_0 \in \mathcal{C}^n$ is the initial state; $u(t) \in \mathbb{R}$ is the input; $\Delta > 0$ is the involved time delay; $y(t) \in \mathbb{R}^2$ is the output; $A \in \mathbb{R}^{n \times n}$, $B = (0 \;\; \cdots \;\; 0 \;\; 1)^T \in \mathbb{R}^n$, $C^T \in \mathbb{R}^n$; $\bar{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is the function defined for any $y_1, y_2 \in \mathbb{R}$ as $f(y_1, y_2) = \big(0 \;\; \cdots \;\; 0 \;\; \tilde{f}(y_1, y_2)\big)^T$ with $\tilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a function Lipschitz on bounded subsets of $\mathbb{R} \times \mathbb{R}$. It is assumed that the couple (A, B) is stabilizable and that there exists $\tilde{g} \in \mathbb{R}$ such that the matrix $(A - GC)$ is Hurwitz where $G = (0 \;\; \cdots \;\; 0 \;\; \tilde{g})^T \in \mathbb{R}^n$. We highlight that system (14) is in the form (2). In this case, f_e and $f_{\hat{x}}$ in (5) are the functions from $\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$ and from $\mathcal{C}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$, respectively, defined, for any $\hat{\phi}$, $\phi_e \in \mathcal{C}^n$, for any $u \in \mathbb{R}$ and for any $v \in \mathbb{R}^n$, as follows

$$\begin{aligned} f_e(\phi_e, \hat{\phi}, u, v) &= A(\hat{\phi}(0) + \phi_e(0)) + f^{\hat{\phi}+\phi_e} + Bu, \\ f_{\hat{x}}(\hat{\phi}, v) &= v, \end{aligned} \quad (31)$$

where $f^{\hat{\phi}+\phi_e} = f(C(\hat{\phi}(0) + \phi_e(0)), C(\hat{\phi}(-\Delta) + \phi_e(-\Delta)))$. According to the design procedure proposed in Section 3 (see also steps (1)–(3) in Remark 16), taking into account that $(A - GC)$ is a Hurwitz matrix, let $V_{e,1} : \mathbb{R}^n \to \mathbb{R}^+$ be the function defined for any $e \in \mathbb{R}^n$, as $V_{e,1}(e) = e^T P_e e$, where P_e is a symmetric positive definite matrix satisfying $(A - GC)^T P_e + P_e(A - GC) = -Q_e$ with Q_e a symmetric positive definite matrix arbitrarily chosen. Let $\widetilde{V}_{e,2} : \mathcal{C}^n \to \mathbb{R}^+$ be the functional defined for any $\phi_e \in \mathcal{C}^n$, as $\widetilde{V}_{e,2}(\phi_e) = 0$. Let $V_e : \mathcal{C}^n \to \mathbb{R}^+$ be the functional defined, for any $\phi_e \in \mathcal{C}^n$, as $V_e(\phi_e) = V_{e,1}(\phi_e(0)) + \widetilde{V}_{e,2}(\phi_e)$. Let $K \in \mathbb{R}^n$ be a vector such that the matrix $(A + BK)$ is Hurwitz. Let $V_{\hat{x},1} : \mathbb{R}^n \to \mathbb{R}^+$ be the function defined, for $\hat{x} \in \mathbb{R}^n$, as $V_{\hat{x},1}(\hat{x}) = \hat{x}^T P_{\hat{x}} \hat{x}$, where $P_{\hat{x}}$ is a symmetric positive definite matrix satisfying $(A + BK)^T P_{\hat{x}} + P_{\hat{x}}(A + BK) = -Q_{\hat{x}}$ with $Q_{\hat{x}}$ a symmetric positive definite matrix arbitrarily chosen. According to the proposed design procedure, taking into account Remark 16, in the following, the introduced Lyapunov–Krasovskii functionals V_e and $V_{\hat{x}}$ are exploited for trying to find a DOSDF for the system (30) according to Definition 7. As far as the first inequality in Definition 7 is concerned (see (9)), taking into account (31) and step (2) in Remark 16, by exploiting the proposed candidate Lyapunov–Krasovskii functional V_e, in this case, we obtain

$$D^+V_e(\phi_e,\hat{\phi},u,v)=2\phi_e(0)^TP_ef_e(\phi_e,\hat{\phi},u,v)= \\ 2\phi_e(0)^TP_e\left(A(\hat{\phi}(0)+\phi_e(0))+Bu+f^{\hat{\phi}+\phi_e}-v\right). \tag{32}$$

Taking into account (8) and (32), let $\hat{f}:\mathcal{C}^n\times\mathbb{R}^2\to\mathbb{R}^n$, $g:\mathbb{R}^2\to\mathbb{R}^n$ and $k_O:\mathcal{C}^n\times\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^n$ be the functions defined for any $\hat{\phi}\in\mathcal{C}^n$, $y=(y_1\ \ y_2)^T\in\mathbb{R}^2$ and $u\in\mathbb{R}$ as follows

$$\hat{f}(\hat{\phi},y)=A\hat{\phi}(0)+f(y_1,y_2)+G(y_1-C\hat{\phi}(0)), \\ g(y)=B,\quad k_O(\hat{\phi},y,u)=\hat{f}(\hat{\phi},y)+Bu, \tag{33}$$

where G is the vector such that $(A-GC)$ is Hurwitz. From (32), taking into account (33), we obtain

$$D^+V_e(\phi_e,\hat{\phi},u,k_O(\hat{\phi},h(\hat{\phi}+\phi_e),u))= \\ 2\phi_e(0)^TP_e(A-GC)\phi_e(0)\le-\lambda_{\min}(Q_e)|\phi_e(0)|^2. \tag{34}$$

Taking into account (34), it follows that, for any $u\in\mathbb{R}$, ϕ_e, $\hat{\phi}\in\mathcal{C}^n$, the following inequalities hold

$$D^+V_e(\phi_e,\hat{\phi},u,k_O(\hat{\phi},h(\hat{\phi}+\phi_e),u)) \\ +\eta\max\{0,D^+V_{e,1}(\phi_e,\hat{\phi},u,k_O(\hat{\phi},h(\hat{\phi}+\phi_e),u)) \\ +\mu V_{e,1}(\phi_e(0))\}\le \\ -\lambda_{\min}(Q_e)|\phi_e(0)|^2+\eta\max\{0,-\lambda_{\min}(Q_e)|\phi_e(0)|^2 \\ +\mu\lambda_{\max}(P_e)|\phi_e(0)|^2\}. \tag{35}$$

As far as the second inequality in Definition 7 is concerned (see (9)), taking into account (31), (33) and step (2) in Remark 16, by exploiting the proposed candidate Lyapunov–Krasovskii functional $V_{\hat{x}}$, in this case, we obtain

$$D^+V_{\hat{x}}(\hat{\phi},k_O(\hat{\phi},h(\hat{\phi}+\phi_e),u))= \\ 2\hat{\phi}(0)^TP_{\hat{x}}f_{\hat{x}}(\hat{\phi},k_O(\hat{\phi},h(\hat{\phi}+\phi_e),u))= \\ 2\hat{\phi}(0)^TP_{\hat{x}}\left(A\hat{\phi}(0)+f(y_1,y_2)+G(y_1-C\hat{\phi}(0))+Bu\right). \tag{36}$$

Taking into account (8) and (36), let $k_F:\mathcal{C}^n\times\mathbb{R}^2\to\mathbb{R}$ be the function defined for any $\hat{\phi}\in\mathcal{C}^n$, $y=(y_1\ \ y_2)^T\in\mathbb{R}^2$ as follows

$$k_F(\hat{\phi},y)=-\tilde{f}(y_1,y_2)-\tilde{g}(y_1-C\hat{\phi}(0))+K\hat{\phi}(0), \tag{37}$$

where K is the vector such that the matrix $(A+BK)$ is Hurwitz. From (36), taking into account (37), we obtain

$$D^+V_{\hat{x}}(\hat{\phi},k_O(\hat{\phi},h(\hat{\phi}+\phi_e),k_F))= \\ 2\hat{\phi}(0)^TP_{\hat{x}}(A+BK)\hat{\phi}(0)\le-\lambda_{\min}(Q_{\hat{x}})|\hat{\phi}(0)|^2. \tag{38}$$

Taking into account (38), it follows that, for any ϕ_e, $\hat{\phi}\in\mathcal{C}^n$, the following inequalities hold

$$D^+V_{\hat{x}}(\hat{\phi},k_O(\hat{\phi},h(\hat{\phi}+\phi_e),k_F))+ \\ \eta\max\{0,D^+V_{\hat{x},1}(\hat{\phi},k_O(\hat{\phi},h(\hat{\phi}+\phi_e),k_F))+\mu V_{\hat{x},1}(\hat{\phi}(0))\}\le \\ -\lambda_{\min}(Q_{\hat{x}})|\hat{\phi}(0)|^2+\eta\max\{0,-\lambda_{\min}(Q_{\hat{x}})|\hat{\phi}(0)|^2 \\ +\mu\lambda_{\max}(P_{\hat{x}})|\hat{\phi}(0)|^2\}. \tag{39}$$

Taking into account (35) and (39), it follows that by choosing, for instance, $\mu\le\min\left\{\frac{\lambda_{\min}(Q_e)}{\lambda_{\max}(P_e)},\frac{\lambda_{\min}(Q_{\hat{x}})}{\lambda_{\max}(P_{\hat{x}})}\right\}$ and $\eta>0$, the inequalities in (9) are here satisfied. Consequently, the function $k:\mathcal{C}^n\times\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^{n+1}$ defined for any $\hat{\phi}\in\mathcal{C}^n$, $y\in\mathbb{R}^2$ and $u\in\mathbb{R}$ as follows

$$k(\hat{\phi},y,u)=\begin{pmatrix}u\\ \hat{f}(\hat{\phi},y)+Bu\end{pmatrix},\ u=k_F(\hat{\phi},y), \tag{40}$$

with $\hat{f}$ and k_F provided in (33), (37), is a DOSDF for the system described by (30) (see Definition 7). Thus Assumption 8 is satisfied and we can apply Theorem 12.

Numerical Example: Let us consider the nonlinear system described by (30) with $A=\begin{pmatrix}0.1&0.5&0\\-0.5&-1&1\\0&0&1\end{pmatrix}$, $B=(0\ \ 0\ \ 1)^T$, $C=(0\ \ 0\ \ 1)$, $\tilde{f}(y_1,y_2)=y_1^2y_2+y_2^2$. Notice that, the couple (A,B) is stabilizable and there exists $\tilde{g}$ such that the matrix $A-GC$ is Hurwitz. In this case, the QSOE controller is described by (11) where k_O and k_F are the functions proposed in (33), (37).

5.2. Application to a continuous stirred tank reactor

In the following, the methodology proposed in this paper is applied to a CSTR system described by the following ODEs (see Pepe (2015) and references therein for more details on the model and related parameters):

$$\dot{x}(t)=\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix}=f(x(t))+g(h(x(t)))u(t)= \\ \begin{pmatrix}p_1(C_{A_0}-x_1(t)-C_{A_{eq}}) \\ -(x_1(t)+C_{A_{eq}})k_0\varphi(x_2(t)+T_{R_{eq}}) \\ \\ p_1(T_0-x_2(t)-T_{R_{eq}}) \\ +\bar{g}(T_{J_{eq}}-x_2(t)-T_{R_{eq}}) \\ +p_2k_0(x_1(t)+C_{A_{eq}})\varphi(x_2(t)+T_{R_{eq}})\end{pmatrix}+\begin{pmatrix}0\\ \bar{g}\end{pmatrix}u(t), \\ y(t)=h(x(t))=x_2(t), \tag{41}$$

where: $x(t)\in\mathbb{R}^2$ is the deviation of reactant concentration and reactor temperature from the chosen operating points $C_{A_{eq}}$ and $T_{R_{eq}}$ (Pepe, 2015); $u(t)\in\mathbb{R}$ is the control input; $\varphi:\mathbb{R}\to(0,1]$ is the function defined, for any $s\in\mathbb{R}$, as $\varphi(s)=e^{-\frac{\bar{E}}{\bar{R}\max(1,s)}}$; p_1, p_2, $\bar{g}$, $\bar{E}$ and $\bar{R}$ are positive reals (Pepe, 2015). Notice that φ is a globally Lipschitz function with constant L_φ. According to the design procedure proposed in Section 3 (see also steps (1)–(3) in Remark 16), let $V_e:\mathbb{R}^2\to\mathbb{R}^+$ and $V_{\hat{x}}:\mathbb{R}^2\to\mathbb{R}^+$ be the functions defined for any $e,\hat{x}\in\mathbb{R}^2$, as $V_e(e)=0.5e^Te$, $V_{\hat{x}}(\hat{x})=0.5\hat{x}^T\hat{x}$. Notice that, $V_e\in\mathcal{V}_e$ and $V_{\hat{x}}\in\mathcal{V}_{\hat{x}}$. By exploiting the same approach used in the previous applications (see Section 5.1 and Remark 16) and taking into account Definition 7 (see, in particular, (9)), we derive, from the introduced Lyapunov functions V_e and $V_{\hat{x}}$, the function $k:\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^3$ defined for any $\hat{x}=(\hat{x}_1\ \ \hat{x}_2)^T\in\mathbb{R}^2$, $\hat{x}_i\in\mathbb{R}$, $i=1,2$, $y\in\mathbb{R}$ and $u\in\mathbb{R}$ as follows

$$k(\hat{x},y,u)=\begin{pmatrix}u\\ \hat{f}_1(\hat{x},y)\\ \hat{f}_2(\hat{x},y)+g(y)u\end{pmatrix},\quad u=k_F(\hat{x},y), \\ \hat{f}_1(\hat{x},y)=p_1(C_{A_0}-\hat{x}_1-C_{A_{eq}}) \\ -(\hat{x}_1+C_{A_{eq}})k_0\varphi(y+T_{R_{eq}})-C_{A_{eq}}k_0L_\varphi\mathrm{sign}(\hat{x}_1)|y-\hat{x}_2|, \\ \hat{f}_2(\hat{x},y)=p_1(T_0-y-T_{R_{eq}})+\bar{g}(T_{J_{eq}}-y-T_{R_{eq}}) \\ +p_2k_0(\hat{x}_1+C_{A_{eq}})\varphi(y+T_{R_{eq}})+G(y-\hat{x}_2), \\ k_F(\hat{x},y)=-\frac{1}{\bar{g}}\left(Gy+K\hat{x}_2+p_1(T_0-y-T_{R_{eq}})\right. \\ +\bar{g}(T_{J_{eq}}-y-T_{R_{eq}})+C_{A_{eq}}k_0L_\varphi\mathrm{sign}(\hat{x}_2)|\hat{x}_1| \\ \left.+p_2k_0(\hat{x}_1+C_{A_{eq}})\varphi(y+T_{R_{eq}})\right), \tag{42}$$

where K is a positive control parameter, $c_1=C_{A_{eq}}k_0L_\varphi+p_2k_0$, $c_2=\frac{c_1}{1.8p_1}$, $G=\frac{c_1c_2}{2}+\tilde{G}$ with $\tilde{G}$ a positive control parameter. It is easy to show that the function k in (42) is a DOSDF for the system described by (41) according to Definition 7. In particular, the inequalities in (9) are here satisfied by choosing, for instance, $\gamma_{e,3}(s)=\min\{0.1p_1,\tilde{G}\}s^2$ and $\gamma_{\hat{x},3}(s)=\min\{p_1,K\}s^2$, $s\in\mathbb{R}^+$. In the performed simulations, the following choice have been taken: the model parameters as in Pepe (2015); the initial state $x_0=(2\ \ 3)^T$; the initial state of the QSOE controller $\hat{x}_0=(0\ \ 0)^T$;

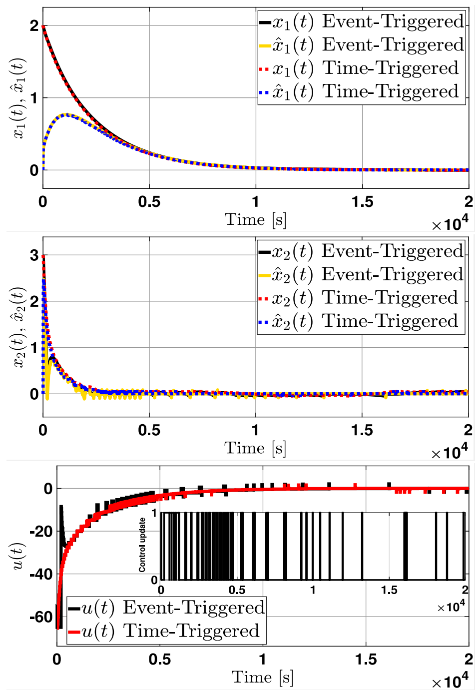

Fig. 3. In the firsts two panels the system variables $x_1(t), x_2(t)$ and the controller variables $\hat{x}_1(t), \hat{x}_2(t)$ (linear interpolation of discrete-time available values $\hat{x}_1(j\delta)$, $\hat{x}_2(j\delta)$, $j = 0, 1, \dots$) are reported. In the third panel the control input $u(t)$ is reported.

the control parameters $\tilde{G} = 0.008$ and $K = 0.002$ (see (42)). Taking into account Remark 15, by choosing $R = 6$, $r = 1$, $a = 1$ and $\sigma = 0.1$, as expected, the use of steps **(1)**–**(15)** (see the proof of Theorem 12), leads to conservative upper bounds for the sampling period $\delta < 10^{-8}$ and for the quantization errors $\mu_y < 10^{-7}$ and $\mu_u < 10^{-7}$. On the other hand, taking into account that the results in Theorem 12 are of existence type (see Remark 15), a campaign of simulations has been performed with sampling periods and quantization errors greater than the ones computed with steps **(1)**–**(15)**. In particular, simulations have been carried out by varying the sampling period δ, the quantization errors μ_y, μ_u and the parameter σ related to triggering condition (12). Very good performances of the proposed digital control scheme have been observed also in the case of sampling periods and quantization errors greater than the ones computed with steps **(1)**–**(15)**. Fig. 3 reports the simulation results in the case of a uniform (i.e., $a = 1$) sampling period $\delta = 10$[s] and quantization of the input/output channels based on the round to nearest method with parameters $\mathcal{Q}_y = \{y \in \mathbb{R} | \, y = \pm 0.1j, \, j = 0, 1, \dots, 10^3\}$ and $\mathcal{Q}_u = \{u \in \mathbb{R} | \, u = \pm 0.2j, \, j = 0, 1, \dots, 10^3\}$. In particular, in Fig. 3, the system variables $x_1(t), x_2(t)$ and the observer variables $\hat{x}_1(t), \hat{x}_2(t)$ are reported in the case of time-triggered controller and in the case of event-based controller with $\sigma = 0.1$. Also in this case, the event-triggered solution achieves very good performances similar to the ones of the time-triggered solution, in spite of the much lower average frequency of control updates (8.1% of the sampling intervals). Simulations fully validate the theoretical results.

6. Conclusions

In this paper, the stabilization problem of control-affine nonlinear time–delay systems via quantized sampled-data observer-based event-triggered (QSOE) controllers has been addressed. In particular, a methodology for the design of QSOE stabilizers has been proposed. Firstly, the notion of Dynamic Output Steepest Descent Feedback (DOSDF) has been suitably reformulated in order to cope with the design of QSOE controllers. Then, the stabilization in the sample-and-hold sense theory has been used as a tool to prove the existence of a suitably fast sampling and of an accurate quantization of the input/output channels such that the digital implementation of DOSDFs, updated through a proposed event-based mechanism, ensures the semi-global practical stability property of the related closed-loop system with arbitrarily small final target ball of the origin. Time-varying sampling intervals as well as the non-uniform quantization in both input/output channels have been taken into account. Possible discontinuities in the functions describing the DOSDF at hand have been also managed. The particular case of nonlinear delay-free systems has been also addressed. In this case, relaxed conditions have been provided for the design of QSOE stabilizers. In the proposed design methodology the requirement to solve differential equations for the correct application of the controller at hand is avoided because the provided QSOE stabilizer is described by easily implementable difference equations. The proposed results have been validated through applications concerning a Glucose-Insulin system and a CSTR.

Appendix. Proof of Theorem 12

As a first step, we introduce some useful results which will be used for proving Theorem 12. In particular, we introduce some key inequalities and properties for the functionals at hand (i.e., V_e and $V_{\hat{x}}$) which will be used as tools during the proof.

Lemma 18. *Let Assumption 8 hold. Let $\alpha_{\hat{x},i}$, $\alpha_{e,i}$, $i = 1, 2, 3$, be the functions of class $\mathcal{K}_\infty$, defined for $s \in \mathbb{R}^+$, as $\alpha_{e,1}(s) = \eta e^{-\mu\Delta}\beta_{e,1}(s)$, $\alpha_{\hat{x},1}(s) = \eta e^{-\mu\Delta}\beta_{\hat{x},1}(s)$, $\alpha_{e,2}(s) = \gamma_{e,2}(s) + \eta\beta_{e,2}(s)$, $\alpha_{\hat{x},2}(s) = \gamma_{\hat{x},2}(s) + \eta\beta_{\hat{x},2}(s)$, $\alpha_{e,3}(s) = \eta\mu e^{-\mu\Delta}\beta_{e,1}(s)$ and $\alpha_{\hat{x},3}(s) = \eta\mu e^{-\mu\Delta}\beta_{\hat{x},1}(s)$. Let $V_{\hat{x},3}$, $\mathcal{D}_{\hat{x},\infty}$ be the functionals in* **(1a.)**–**(2a.)**. *Let: $V_{\hat{x},\infty} : \mathcal{C}^n \to \mathbb{R}^+$ be the functional defined, for $\hat{\phi} \in \mathcal{C}^n$, as*

$$V_{\hat{x},\infty}(\hat{\phi}) = V_{\hat{x}}(\hat{\phi}) + \eta V_{\hat{x},3}(\hat{\phi}); \tag{A.1}$$

(1b.) *$V_{e,3} : \mathcal{C}^n \to \mathbb{R}^+$ be the functional defined, for $\phi_e \in \mathcal{C}^n$, as $V_{e,3}(\phi_e) = \sup_{\theta\in[-\Delta,0]} e^{\mu\theta}V_{e,1}(\phi_e(\theta))$;*

(2b.) *$V_{e,\infty} : \mathcal{C}^n \to \mathbb{R}^+$ be the functional defined, for $\phi_e \in \mathcal{C}^n$, as $V_{e,\infty}(\phi_e) = V_e(\phi_e) + \eta V_{e,3}(\phi_e)$;*

(3b.) *$\mathcal{D}_{e,\infty} : \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be the functional defined, for $\phi_e \in \mathcal{C}^n$, $\hat{\phi} \in \mathcal{C}^n$, $u \in \mathbb{R}^m$ and $v \in R^n$, as*

$$\begin{aligned}\mathcal{D}_{e,\infty}(\phi_e, \hat{\phi}, u, v) &= D^+V_e(\phi_e, \hat{\phi}, u, v) - \eta\mu V_{e,3}(\phi_e) \\ &+\eta \max\{0, D^+V_{e,1}(\phi_e, \hat{\phi}, u, v) + \mu V_{e,1}(\phi_e(0))\}.\end{aligned} \tag{A.2}$$

Then, the following conditions hold:

(i) $\alpha_{e,1}(\|\phi_e\|_\infty) \le V_{e,\infty}(\phi_e) \le \alpha_{e,2}(\|\phi_e\|_\infty)$, $\forall \phi_e \in \mathcal{C}^n$; $\alpha_{\hat{x},1}(\|\hat{\phi}\|_\infty) \le V_{\hat{x},\infty}(\hat{\phi}) \le \alpha_{\hat{x},2}(\|\hat{\phi}\|_\infty)$, $\forall \hat{\phi} \in \mathcal{C}^n$;

(ii) the function $(\hat{\phi}, v) \to \mathcal{D}_{\hat{x},\infty}(\hat{\phi}, v)$, $\hat{\phi} \in \mathcal{C}^n$, $v \in \mathbb{R}^n$, is Lipschitz on bounded subsets of $\mathcal{C}^n \times \mathbb{R}^n$; the function $(\phi_e, \hat{\phi}, u, v) \to \mathcal{D}_{e,\infty}(\phi_e, \hat{\phi}, u, v)$, ϕ_e, $\hat{\phi} \in \mathcal{C}^n$, $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, is Lipschitz on bounded subsets of $\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{R}^m \times \mathbb{R}^n$;

(iii) $D^+V_{\hat{x},\infty}(\hat{\phi}, v) \le \mathcal{D}_{\hat{x},\infty}(\hat{\phi}, v)$, $\forall \hat{\phi} \in \mathcal{C}^n$, $\forall v \in \mathbb{R}^n$; $D^+V_{e,\infty}(\phi_e, \hat{\phi}, u, v) \le \mathcal{D}_{e,\infty}(\phi_e, \hat{\phi}, u, v)$, $\forall \phi_e$, $\hat{\phi} \in \mathcal{C}^n$, $\forall u \in \mathbb{R}^m$, $\forall v \in \mathbb{R}^n$;

(iv) $\mathcal{D}_{e,\infty}(\phi_e, \hat{\phi}, k_F, k_O(\hat{\phi}, h(\hat{\phi} + \phi_e), k_F)) \le -\alpha_{e,3}(\|\phi_e\|_\infty)$, $\mathcal{D}_{\hat{x},\infty}(\hat{\phi}, k_O(\hat{\phi}, h(\hat{\phi} + \phi_e), k_F)) \le -\alpha_{\hat{x},3}(\|\hat{\phi}\|_\infty)$, $\forall \hat{\phi}$, $\phi_e \in \mathcal{C}^n$.

Proof. The same reasoning used in the proof of Theorem 3.5 in Pepe (2014) can be repeated here to prove points (i)–(iv). The steps are here omitted because are exactly the same used in the proof of Theorem 3.5 in Pepe (2014). ◇

Taking into account system (2), let us consider the open-loop system described by

$$
\begin{aligned}
&\dot{e}(t) = f(\hat{x}_t + e_t) + g(\hat{x}_t + e_t)u(t) - v(t),\ t \geq 0 \text{ a.e.},\\
&\dot{\hat{x}}(t) = v(t),\\
&y(t) = h(\hat{x}_t + e_t),\\
&e(\tau) = e_0(\tau), \quad \hat{x}(\tau) = \hat{x}_0(\tau), \quad \tau \in [-\Delta, 0],
\end{aligned}
\tag{A.3}
$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the observer state variable; $e(t) \in \mathbb{R}^n$ is the related estimation error defined as $e(t) = x(t) - \hat{x}(t)$, with $x(t)$ the state variable of the system (2); e_t, $\hat{x}_t \in \mathcal{C}^n$; $\Delta \geq 0$ is the maximum involved time delay in (2); $u(t) \in \mathbb{R}^m$ is the input in (2); $y(t) \in \mathbb{R}^p$ is the output in (2); f, g and h are the functions in (2); $v(t) \in \mathbb{R}^n$ is a new input signal Lebesgue measurable and locally essentially bounded. Let (as long as the solution of (A.3) exists) $\chi(t) = (\,e(t)^T \quad \hat{x}(t)^T\,)^T \in \mathbb{R}^{2n}$, $\chi_t = \begin{pmatrix} e_t \\ \hat{x}_t \end{pmatrix} \in \mathcal{C}^{2n}$, $\tilde{v}(t) = (\,u(t)^T \quad v(t)^T\,)^T \in \mathbb{R}^{m+n}$. Then, the open-loop system (A.3) can be rewritten as follows

$$
\begin{aligned}
\dot{\chi}(t) &= \begin{pmatrix} \dot{e}(t) \\ \dot{\hat{x}}(t) \end{pmatrix} = \begin{pmatrix} f_e(e_t, \hat{x}_t, u(t), v(t)) \\ f_{\hat{x}}(\hat{x}_t, v(t)) \end{pmatrix} = F(\chi_t, \tilde{v}(t))\\
&= \begin{pmatrix} f(\hat{x}_t + e_t) + g(\hat{x}_t + e_t)u(t) - v(t) \\ v(t) \end{pmatrix}\\
\chi(\tau) &= \chi_0(\tau) = (\,e_0(\tau)^T \quad \hat{x}_0(\tau)^T\,)^T, \quad \tau \in [-\Delta, 0]
\end{aligned}
\tag{A.4}
$$

where f_e and $f_{\hat{x}}$ are the functions in (5), $F: \mathcal{C}^{2n} \times \mathbb{R}^{m+n} \to \mathbb{R}^{2n}$ is the function readily defined in (A.4). The results provided in forthcoming Lemma 19 will be the key tools for proving Theorem 12.

Lemma 19. *Let Assumption 8 hold. Let $V_{\hat{x},\infty}$, $\mathcal{D}_{\hat{x},\infty}$ and $V_{e,\infty}$, $\mathcal{D}_{e,\infty}$ be the functionals defined in (A.1), **(2a.)**, **(2b.)** and **(3b.)**, respectively. Let $\alpha_{\hat{x},i}$, $\alpha_{e,i}$, $i = 1, 2, 3$, be the functions of class $\mathcal{K}_\infty$ in Lemma 18. Let α_i $i = 1, 2, 3$, be the functions of class $\mathcal{K}_\infty$ such that, for any $\phi_\chi = \begin{pmatrix} \phi_e \\ \hat{\phi} \end{pmatrix} \in \mathcal{C}^{2n}$, ϕ_e, $\hat{\phi} \in \mathcal{C}^n$, $\alpha_{e,1}(\|\phi_e\|_\infty) + \alpha_{\hat{x},1}(\|\hat{\phi}\|_\infty) \geq \alpha_1(\|\phi_\chi\|_\infty)$, $\alpha_{e,2}(\|\phi_e\|_\infty) + \alpha_{\hat{x},2}(\|\hat{\phi}\|_\infty) \leq \alpha_2(\|\phi_\chi\|_\infty)$, $\alpha_{e,3}(\|\phi_e\|_\infty) + \alpha_{\hat{x},3}(\|\hat{\phi}\|_\infty) \geq \alpha_3(\|\phi_\chi\|_\infty)$ (see Lemma 4.3 in Khalil (2000)). Let*

(1c.) *$V_\infty : \mathcal{C}^{2n} \to \mathbb{R}^+$ be the functional defined, for $\phi_\chi = \begin{pmatrix} \phi_e \\ \hat{\phi} \end{pmatrix} \in \mathcal{C}^{2n}$, ϕ_e, $\hat{\phi} \in \mathcal{C}^n$, as $V_\infty(\phi_\chi) = V_{e,\infty}(\phi_e) + V_{\hat{x},\infty}(\hat{\phi})$;*

(2c.) *$\mathcal{D}_\infty : \mathcal{C}^{2n} \times \mathbb{R}^{m+n} \to \mathbb{R}$ be the functional defined, for $\phi_\chi = \begin{pmatrix} \phi_e \\ \hat{\phi} \end{pmatrix} \in \mathcal{C}^{2n}$, ϕ_e, $\hat{\phi} \in \mathcal{C}^n$, $\tilde{v} = (\,u^T v^T\,)^T \in \mathbb{R}^{m+n}$, $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, as follows*

$$
\mathcal{D}_\infty(\phi_\chi, \tilde{v}) = \mathcal{D}_{e,\infty}(\phi_e, \hat{\phi}, u, v) + \mathcal{D}_{\hat{x},\infty}(\hat{\phi}, v). \tag{A.5}
$$

Then, the following conditions hold:

(p.1) $\alpha_1(\|\phi_\chi\|_\infty) \leq V_\infty(\phi_\chi) \leq \alpha_2(\|\phi_\chi\|_\infty)$, $\forall \phi_\chi \in \mathcal{C}^{2n}$;

(p.2) the function $(\phi_\chi, \tilde{v}) \to \mathcal{D}_\infty(\phi_\chi, \tilde{v})$, $\phi_\chi \in \mathcal{C}^{2n}$, $\tilde{v} \in \mathbb{R}^{m+n}$, is Lipschitz on bounded subsets of $\mathcal{C}^{2n} \times \mathbb{R}^{m+n}$;

(p.3) $D^+V_\infty(\phi_\chi, \tilde{v}) \leq \mathcal{D}_\infty(\phi_\chi, \tilde{v})$, $\forall \phi_\chi \in \mathcal{C}^{2n}$, $\forall \tilde{v} \in \mathbb{R}^{m+n}$;

(p.4) $\mathcal{D}_\infty(\phi_\chi, k(\hat{\phi}, h(\phi_e + \hat{\phi}), k_F)) \leq -\alpha_3(\|\phi_\chi\|_\infty)$, $\forall \phi_\chi = \begin{pmatrix} \phi_e \\ \hat{\phi} \end{pmatrix} \in \mathcal{C}^{2n}$, ϕ_e, $\hat{\phi} \in \mathcal{C}^n$.

Proof. Points (p.1)–(p.4) readily follow from the results in Lemma 18. ◇

Notice that, taking into account item **(c)** in Section 3, from the invariant differentiability property of the functional $\bar{V}_e$, it follows that, for any $\hat{\phi}$, $\phi_e \in \mathcal{C}^n$, for any $u \in \mathbb{R}^m$ and for any $v \in \mathbb{R}^n$,

$$
D^+\widetilde{V}_{e,2}(\phi_e, \hat{\phi}, u, v) = D^+\widetilde{V}_{e,2}(\phi_e, 0, 0, 0). \tag{A.6}
$$

Moreover, in the case $\Delta = 0$, the forthcoming reasoning is performed in a simplified fashion by considering that: (i) V_∞ (see Lemma 19) is the function from $\mathbb{R}^{2n}$ to $\mathbb{R}^+$ defined, for any $\chi = (\,e^T \quad \hat{x}^T\,)^T \in \mathbb{R}^{2n}$, e, $\hat{x} \in \mathbb{R}^n$, as $V_\infty(\chi) = V_e(e) + V_{\hat{x}}(\hat{x})$, where V_e and $V_{\hat{x}}$ are the Lyapunov functions related to the DOSDF at hand (see Definition 7 and Assumption 8); (ii) the functions α_i, $i = 1, 2, 3$, in Lemma 19 are such that, for any $\chi = (\,e^T \quad \hat{x}^T\,)^T \in \mathbb{R}^{2n}$, e, $\hat{x} \in \mathbb{R}^n$, $\gamma_{e,1}(|e|) + \gamma_{\hat{x},1}(|\hat{x}|) \geq \alpha_1(|\chi|)$, $\gamma_{e,2}(|e|) + \gamma_{\hat{x},2}(|\hat{x}|) \leq \alpha_2(|\chi|)$, $\gamma_{e,3}(|e|) + \gamma_{\hat{x},3}(|\hat{x}|) \geq \alpha_3(|\chi|)$ (see Lemma 4.3 in Khalil (2000)); (iii) the introduction of the function $\mathcal{D}_\infty$ is no more needed (see Lemma 19) because the following equality/inequalities hold:

$$
\begin{aligned}
&D^+V_\infty(\chi, k(\hat{x}, h(\hat{x} + e), k_F)) =\\
&D^+V_e(e, \hat{x}, k_F, k_O(\hat{x}, h(\hat{x} + e), k_F)) +\\
&D^+V_{\hat{x}}(\hat{x}, k_O(\hat{x}, h(\hat{x} + e), k_F)) \leq -\gamma_{e,3}(|e|) - \gamma_{\hat{x},3}(|\hat{x}|) \leq -\alpha_3(|\chi|).
\end{aligned}
\tag{A.7}
$$

Let: **(1)** V_∞ and $\mathcal{D}_\infty$ be the functionals in Lemma 19;
(2) α_i, $i = 1, 2, 3$, be the functions in Lemma 19;
(3) r, R, be any positive reals, $0 < r < R$;
(4) $\overline{R} = R\sqrt{5}$;
(5) e_1, e_2, E be positive reals satisfying $0 < e_2 < e_1 < r < \overline{R} < E$, $\alpha_1(r) > \alpha_2(e_1)$, $\alpha_1(E) > \alpha_2(\overline{R})$;
(6) $E_1 = \sup_{\chi \in \mathcal{C}_E^{2n}} |h(\phi_e + \hat{\phi})|$; $E_2 = E_1 + 1$;
$U = \sup_{\phi_\chi \in \mathcal{C}_E^{2n}, y \in \mathcal{B}_{E_2}^p} |k_F(\hat{\phi}, y)|$; $U_1 = U + 1$;
$U_O = \sup_{\phi_\chi \in \mathcal{C}_E^{2n}, y \in \mathcal{B}_{E_2}^p, u \in \mathcal{B}_{U_1}^m} |k_O(\hat{\phi}, y, u)|$;
$M_{\hat{f}} = \sup_{\phi_\chi \in \mathcal{C}_E^{2n}, y \in \mathcal{B}_{E_2}^p} |\hat{f}(\hat{\phi}, y)|$; $\widetilde{U} = |\,(\,U_1 \quad U_O\,)^T\,|$;
(7) M, $M_{V_{e,1}}$, $L_{\hat{f}}$, $M_{\hat{f}}$, L_{k_F}, L_{k_O}, L_g, L_h, $L_{V_{e,3}}$, $\bar{L}_{V_{e,1}}$, $L_{V_{e,1}}$, L_{f,V_e}, $L_{D^+\widetilde{V}_{e,2}}$, L_k, $L_{\mathcal{D}_\infty}$, $L_{\mathcal{D}_{\hat{x},\infty}}$ be positive reals such that the following conditions hold for any $\phi_{\chi_1} = \begin{pmatrix} \phi_{e_1} \\ \hat{\phi}_1 \end{pmatrix}$, $\phi_{\chi_2} = \begin{pmatrix} \phi_{e_2} \\ \hat{\phi}_2 \end{pmatrix} \in \mathcal{C}_E^{2n}$, y_1, $y_2 \in \mathcal{B}_{E_2}^p$ and $\tilde{v}_1 = (\,u_1^T \quad v_1^T\,)^T$, $\tilde{v}_2 = (\,u_2^T \quad v_2^T\,)^T \in \mathcal{B}_{\widetilde{U}}^{m+n}$, u_1, $u_2 \in \mathcal{B}_{U_1}^m$, v_1, $v_2 \in \mathcal{B}_{U_O}^n$,

$$
\begin{aligned}
&|F(\phi_{\chi_1}, \tilde{v}_1)| \leq M, \quad \left| \frac{\partial V_{e,1}(e)}{\partial e}\Big|_{e=\phi_{e_1}(0)} \right| \leq M_{V_{e,1}},\\
&|\hat{f}(\hat{\phi}_1, y_1) - \hat{f}(\hat{\phi}_2, y_2)| \leq L_{\hat{f}}|y_1 - y_2|, \quad |\hat{f}(\hat{\phi}_1, y_1)| \leq M_{\hat{f}},\\
&|k_F(\hat{\phi}_1, y_1) - k_F(\hat{\phi}_1, y_2)| \leq L_{k_F}|y_1 - y_2|,\\
&|k_O(\hat{\phi}_1, y_1, u_1) - k_O(\hat{\phi}_1, y_2, u_2)|\\
&\qquad \leq L_{k_O}(|y_1 - y_2| + |u_1 - u_2|),\\
&|g(y_1) - g(y_2)| \leq L_g|y_1 - y_2|,\\
&|h(\hat{\phi}_1 + \phi_{e_1}) - h(\hat{\phi}_2 + \phi_{e_2})| \leq L_h\|\phi_{\chi_1} - \phi_{\chi_2}\|_\infty,\\
&|V_{e,3}(\phi_{e_1}) - V_{e,3}(\phi_{e_2})| \leq L_{V_{e,3}}\|\phi_{\chi_1} - \phi_{\chi_2}\|_\infty,\\
&|V_{e,1}(\phi_{e_1}(0)) - V_{e,1}(\phi_{e_2}(0))| \leq \bar{L}_{V_{e,1}}\|\phi_{\chi_1} - \phi_{\chi_2}\|_\infty,\\
&\left| \frac{\partial V_{e,1}(e)}{\partial e}\Big|_{e=\phi_{e_1}(0)} - \frac{\partial V_{e,1}(e)}{\partial e}\Big|_{e=\phi_{e_2}(0)} \right| \leq L_{V_{e,1}}\|\phi_{\chi_1} - \phi_{\chi_2}\|_\infty,\\
&\left| \frac{\partial V_{e,1}(e)}{\partial e} + \Big|_{e=\phi_{e_1}(0)} f(\hat{\phi}_1 + \phi_{e_1}) - \frac{\partial V_{e,1}(e)}{\partial e}\Big|_{e=\phi_{e_2}(0)} f(\hat{\phi}_2 + \phi_{e_2}) \right|\\
&\qquad \leq L_{f,V_e}\|\phi_{\chi_1} - \phi_{\chi_2}\|_\infty,\\
&|D^+\widetilde{V}_{e,2}(\phi_{e_1}, 0, 0, 0) - D^+\widetilde{V}_{e,2}(\phi_{e_2}, 0, 0, 0)|\\
&\qquad \leq L_{D^+\widetilde{V}_{e,2}}\|\phi_{\chi_1} - \phi_{\chi_2}\|_\infty,\\
&|k(\hat{\phi}_1, y_1, u_1) - k(\hat{\phi}_1, y_2, u_2)| \leq L_k(|y_1 - y_2| + |u_1 - u_2|),\\
&|\mathcal{D}_\infty(\phi_{\chi_1}, \tilde{v}_1) - \mathcal{D}_\infty(\phi_{\chi_2}, \tilde{v}_2)|\\
&\qquad \leq L_{\mathcal{D}_\infty}(\|\phi_{\chi_1} - \phi_{\chi_2}\|_\infty + |\tilde{v}_1 - \tilde{v}_2|),\\
&|\mathcal{D}_{\hat{x},\infty}(\hat{\phi}_1, v_1) - \mathcal{D}_{\hat{x},\infty}(\hat{\phi}_2, v_2)|\\
&\qquad \leq L_{\mathcal{D}_{\hat{x},\infty}}(\|\phi_{\chi_1} - \phi_{\chi_2}\|_\infty + |v_1 - v_2|),
\end{aligned}
\tag{A.8}
$$

where F is the function in (A.4), $V_e = V_{e,1} + V_{e,2}$ and $V_{e,3}$ with their derivatives are the functions related to the DOSDF at hand (see Definition 7, **(1b.)** in Lemma 18 and (1)), k_F, k_O, k and $\hat{f}$ are the functions describing the DOSDF at hand (see (8) and Definition 7), f, g and h are the functions in (2), $\mathcal{D}_{\hat{x},\infty}$ and $\mathcal{D}_{\infty}$ are the functions in **(2a.)** and (A.5);

(8) $\sigma \in (0,1)$ be arbitrarily fixed and $\beta = \sigma\alpha_3(e_2)$;

(9) $a \in (0,1]$ be arbitrarily fixed;

(10) $T = \frac{3\alpha_2(\bar{R})}{\beta a} + 1$;

(11) q be any positive real;

(12) $\tilde{q} = \max\left\{\sqrt{5}q,\ M\right\}$;

(13) δ, μ_y and μ_u be positive reals such that

$$\begin{aligned}
&\delta < \min\{1, \Delta\}, \quad 0 < \mu_y \le 1, \quad 0 < \mu_u \le 1,\\
&e_2 + \delta M < e_1, \quad R + \delta M < E, \quad \alpha_1(r) > \alpha_2(e_1) + \tfrac{2}{3}\beta\delta,\\
&\tfrac{\beta}{3} > L_{\mathcal{D}_\infty}(2\tilde{q}\delta + L_k(\mu_y + \mu_u) + L_k L_{k_F}\mu_y)\\
&+ L_{\mathcal{D}_{\hat{x},\infty}}\left(2\tilde{q}\delta + (2+\sigma)L_{k_O}(\mu_y + \mu_u + L_{k_F}\mu_y)\right) + \gamma(\delta, \mu_y),
\end{aligned} \tag{A.9}$$

where $\gamma(\delta,\mu_y) = 2(2L_{f,V_e}\tilde{q}\delta + M_{V_{e,1}}U_1L_g(\mu_y + 2L_h\tilde{q}\delta) + 2M_{\hat{f}}L_{V_{e,1}}\tilde{q}\delta + M_{V_{e,1}}L_{\hat{f}}\mu_y) + 2L_{D^+\widetilde{V}_{e,2}}\tilde{q}\delta + 2\eta\mu L_{V_{e,3}}\tilde{q}\delta + 2\mu\bar{L}_{V_{e,1}}\tilde{q}\delta$.

(14) q_y, q_u be an output and input quantizer with ranges E_1, U and error bounds μ_y, μ_u, respectively (see (4));

(15) $\begin{pmatrix} x_0 \\ \hat{x}_0 \end{pmatrix} \in \mathcal{C}^{2n}_R \cap W^{1,\infty}_{2n}$ satisfying $\mathrm{esssup}_{\theta\in[-\Delta,0]}\left|\begin{pmatrix} \frac{dx_0(\theta)}{d\theta} \\ \frac{d\hat{x}_0(\theta)}{d\theta} \end{pmatrix}\right| \le q$.

By considering that $\begin{pmatrix} x_0 \\ \hat{x}_0 \end{pmatrix} \in \mathcal{C}^{2n}_R \cap W^{1,\infty}_{2n}$, $\mathrm{esssup}_{\theta\in[-\Delta,0]}\left|\begin{pmatrix} \frac{dx_0(\theta)}{d\theta} \\ \frac{d\hat{x}_0(\theta)}{d\theta} \end{pmatrix}\right| \le q$, and that $e(t) = x(t) - \hat{x}(t)$ (see (2), (A.3)), it follows that $\chi_0 = \begin{pmatrix} e_0 \\ \hat{x}_0 \end{pmatrix} = \begin{pmatrix} x_0 - \hat{x}_0 \\ \hat{x}_0 \end{pmatrix} \in \mathcal{C}^{2n}_R \cap W^{1,\infty}_{2n}$ and $\mathrm{esssup}_{\theta\in[-\Delta,0]}\left|\begin{pmatrix} \frac{de_0(\theta)}{d\theta} \\ \frac{d\hat{x}_0(\theta)}{d\theta} \end{pmatrix}\right| \le \sqrt{5}q$. Let us consider a partition $\pi_{a,\delta}$. Let $\sigma \in (0,1)$. Let us consider the system described by (A.4) with (as long as the related solution exists)

$$\begin{aligned}
\tilde{v}(t) &= \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} u^\star(i_j) \\ k_O(\hat{x}_{t_j}, q_y(h(\hat{x}_{t_j} + e_{t_j})), u^\star(i_j)) \end{pmatrix}\\
&= k(\hat{x}_{t_j}, q_y(h(\hat{x}_{t_j} + e_{t_j})), u^\star(i_j)) = k^\star(j, i_j),\\
&t_j \le t < t_{j+1},\ t_j \in \pi_{a,\delta},\ j = 0, 1, \ldots,
\end{aligned} \tag{A.10}$$

where i_j is the sequence defined in (12).

Let $\chi(t) = (\,e(t)^T \quad \hat{x}(t)^T\,)^T$ be the solution of the closed-loop system (A.4)–(A.10), in a maximal time interval $[0,b)$, $0 < b \le \infty$. In the following, for readability convenience, we will call with:

(i) $k^\star_O(j,i) = k_O(\hat{x}_{t_j}, q_y(h(\hat{x}_{t_j} + e_{t_j})), u^\star(i))$;

(ii) $\bar{u}(j) = k_F(\hat{x}_{t_j}, q_y(h(\hat{x}_{t_j} + e_{t_j})))$;

(iii) $k_F(j) = k_F(\hat{x}_{t_j}, h(\hat{x}_{t_j} + e_{t_j}))$;

(iv) $k_O(j,i) = k_O(\hat{x}_{t_j}, h(\hat{x}_{t_j} + e_{t_j}), k_F(i))$;

(v) $\mathcal{D}^{j,u}_\infty = \mathcal{D}_\infty(\chi_{t_j}, k(\hat{x}_{t_j}, h(\hat{x}_{t_j} + e_{t_j}), u))$;

(vi) $\mathcal{D}^{j,u}_{\hat{x},\infty} = \mathcal{D}_{\hat{x},\infty}(\hat{x}_{t_j}, k_O(\hat{x}_{t_j}, h(\hat{x}_{t_j} + e_{t_j}), u))$.

Notice that, taking into account (4) and steps **(6)**, **(13)**, **(14)**, for any $\phi_\chi \in \mathcal{C}^{2n}_E$, $\phi_e, \hat{\phi} \in \mathcal{C}^n$, the following hold: (i) $q_y(h(\hat{\phi} + \phi_e)) \in \mathcal{B}^p_{E_2}$; (ii) $q_u\left(k_F\left(\hat{\phi}, q_y(h(\hat{\phi} + \phi_e))\right)\right) \in \mathcal{B}^m_{U_1}$. We show first that the solution exists in $[0,t_1]$. Otherwise, by contradiction, if the solution blows up, there exists a time $\tau \in [0,t_1)$ such that $|\chi(t)| < E$, $t \in [0,\tau)$, and $|\chi(\tau)| = E$. But, from (A.8), (A.9), for $t \in [0,\tau]$, the inequalities hold:

$$|\chi(t)| \le |\chi_0(0)| + \int_0^t |F(\chi_\theta, k^\star(0,0))| d\theta \le R + \delta M < E. \tag{A.11}$$

Thus, taking $t = \tau$, the absurd inequality arises $E < E$. Therefore, the solution exists in $[0,t_1]$ and, by (A.11), it follows that $\chi_t \in \mathcal{C}^{2n}_E$, $t \in [0,t_1]$. Let $W(t) = V_\infty(\chi_t)$, $t \in [0,t_1]$, with $V_\infty : \mathcal{C}^{2n} \to \mathbb{R}^+$ provided in step **(1)**. Taking into account the simplified notation introduced in (i)–(vi), the Lipschitz on bounded sets property of the functional $\mathcal{D}_\infty$ (see point (p.2) in Lemma 19) and that $\chi_t \in \mathcal{C}^{2n}_E$, $t \in [0,t_1]$, by exploiting the mean value theorem, an adding and subtracting technique, points (p.3) (p.4) in Lemma 19 and steps **(7)**, **(8)**, **(12)**–**(14)**, for any fixed $t \in (0,t_1]$, for some $t^* \in [0,t]$, the following inequalities hold:

$$\begin{aligned}
&W(t) - W(0) \le \int_0^t D^+V_\infty(\chi_\tau, k(\hat{x}_0, q_y(h(\hat{x}_0 + e_0)), u^\star(0)))d\tau\\
&\le tD^+V_\infty(\chi_{t^*}, k(\hat{x}_0, q_y(h(\hat{x}_0 + e_0)), u^\star(0))) \le\\
&t\mathcal{D}_\infty(\chi^*_t, k(\hat{x}_0, q_y(h(\hat{x}_0 + e_0)), u^\star(0))) - t\mathcal{D}^{0,\bar{u}(0)}_\infty + t\mathcal{D}^{0,\bar{u}(0)}_\infty \le\\
&tL_{\mathcal{D}_\infty}(2\tilde{q}\delta + L_k(\mu_y + \mu_u)) + t\mathcal{D}^{0,\bar{u}(0)}_\infty - t\mathcal{D}^{0,k_F(0)}_\infty\\
&+ t\mathcal{D}^{0,k_F(0)}_\infty \le tL_{\mathcal{D}_\infty}(2\tilde{q}\delta + L_k(\mu_y + \mu_u) + L_kL_{k_F}\mu_y)\\
&+ t\mathcal{D}^{0,k_F(0)}_\infty - \sigma t\mathcal{D}^{0,k_F(0)}_\infty + \sigma t\mathcal{D}^{0,k_F(0)}_\infty \le\\
&tL_{\mathcal{D}_\infty}(2\tilde{q}\delta + L_k(\mu_y + \mu_u) + L_kL_{k_F}\mu_y) - t\sigma\alpha_3(\|\chi_0\|_\infty),
\end{aligned} \tag{A.12}$$

where, $\|\chi_{t^*} - \chi_0\|_\infty \le 2\tilde{q}\delta$ (see the similar reasoning in Pepe (2016)) and, taking into account (4), $|k(\hat{x}_0, q_y(h(\hat{x}_0 + e_0)), u^\star(0)) - k(\hat{x}_0, h(\hat{x}_0 + e_0), \bar{u}(0))| \le L_k(\mu_y + \mu_u)$ and $|k(\hat{x}_0, h(\hat{x}_0 + e_0), \bar{u}(0)) - k(\hat{x}_0, h(\hat{x}_0 + e_0), k_F(0))| \le L_kL_{k_F}\mu_y$. From (A.12) and taking into account (A.9), for $t \in [0,t_1]$, the following inequality holds $W(t) \le W(0) - t\sigma\alpha_3(\|\chi_0\|_\infty) + \frac{\beta}{3}t$. Let us now consider the following two cases: (1) $\|\chi_0\|_\infty \le e_2$; (2) $\|\chi_0\|_\infty > e_2$. As far as case (1) is concerned, from (A.9) and point (p.1) in Lemma 19, by using again the first inequality in (A.11), it follows $W(t) \le \alpha_2(e_1)$, $t \in [0,t_1]$. As far as case (2) is concerned, taking into account step **(8)** and (A.9), we have, for any $t \in [0,t_1]$, $W(t) \le W(0) - \frac{2}{3}\beta t$. Let us introduce the following claim, which will be proved later.

Claim 20. *The solution $\chi(t)$ exists in $[0,+\infty)$ and, furthermore, $\chi_t \in \mathcal{C}^{2n}_E$, $\forall t \ge 0$.*

Let $W(t) = V_\infty(\chi_t)$, $t \in \mathbb{R}^+$. Taking into account (10), (A.2) and (A.5), for any fixed $t \in (t_j, t_{j+1}]$, $j \ge 1$, for some $t^* \in [t_j, t]$, the following inequalities hold:

$$\begin{aligned}
&W(t) - W(t_j) \le (t - t_j)\mathcal{D}_\infty(\chi^*_t, k^\star(j, i_j)) = (t - t_j)\times\\
&\left(\mathcal{D}_{e,\infty}(e^*_t, \hat{x}_{t^*}, u^\star(i_j), k^\star_O(j, i_j)) + \mathcal{D}_{\hat{x},\infty}(\hat{x}^*_t, k^\star_O(j, i_j))\right).
\end{aligned} \tag{A.13}$$

Taking into account (A.2), (A.6) and that the function V_e is smoothly separable and invariantly differentiable, by exploiting an adding and subtracting technique, the following equalities hold:

$$\begin{aligned}
&\mathcal{D}_{e,\infty}(e^*_t, \hat{x}_{t^*}, u^\star(i_j), k^\star_O(j, i_j)) =\\
&D^+V_{e,1}(e_{t^*}, \hat{x}_{t^*}, u^\star(i_j), k^\star_O(j, i_j)) + D^+\widetilde{V}_{e,2}(e_{t^*}, 0, 0, 0)\\
&- \eta\mu V_{e,3}(e_{t^*}) + \eta\max\Big\{0,\\
&D^+V_{e,1}(e_{t^*}, \hat{x}_{t^*}, u^\star(i_j), k^\star_O(j, i_j)) + \mu V_{e,1}(e(t^*))\Big\} =\\
&\tfrac{\partial V_{e,1}(e)}{\partial e}\Big|_{e=e(t^*)}\left(f(\hat{x}_{t^*} + e_{t^*}) + g(h(\hat{x}_{t^*} + e_{t^*}))u^\star(i_j)\right.\\
&\left.- \hat{f}(\hat{x}_{t_j}, q_y(h(\hat{x}_{t_j} + e_{t_j}))) - g(q_y(h(\hat{x}_{t_j} + e_{t_j})))u^\star(i_j)\right)\\
&+ \tfrac{\partial V_{e,1}(e)}{\partial e}\Big|_{e=e(t_j)}\left(f(\hat{x}_{t_j} + e_{t_j}) - f(\hat{x}_{t_j} + e_{t_j})\right.\\
&+ g(h(\hat{x}_{t_j} + e_{t_j}))k_F(j) - g(h(\hat{x}_{t_j} + e_{t_j}))k_F(j)\\
&\left.+ \hat{f}(\hat{x}_{t_j}, h(\hat{x}_{t_j} + e_{t_j})) - \hat{f}(\hat{x}_{t_j}, h(\hat{x}_{t_j} + e_{t_j}))\right) +\\
&D^+\widetilde{V}_{e,2}(e_{t^*}, 0, 0, 0) - D^+\widetilde{V}_{e,2}(e_{t_j}, 0, 0, 0) + D^+\widetilde{V}_{e,2}(e_{t_j}, 0, 0, 0)\\
&- \eta\mu V_{e,3}(e_{t^*}) + \eta\mu V_{e,3}(e_{t_j}) - \eta\mu V_{e,3}(e_{t_j}) + \eta\max\Big\{0,\\
&\tfrac{\partial V_{e,1}(e)}{\partial e}\Big|_{e=e(t^*)}\left(f(\hat{x}_{t^*} + e_{t^*}) + g(h(\hat{x}_{t^*} + e_{t^*}))u^\star(i_j)\right.\\
&\left.- \hat{f}(\hat{x}_{t_j}, q_y(h(\hat{x}_{t_j} + e_{t_j}))) - g(q_y(h(\hat{x}_{t_j} + e_{t_j})))u^\star(i_j)\right)\\
&+ \tfrac{\partial V_{e,1}(e)}{\partial e}\Big|_{e=e(t_j)}\left(f(\hat{x}_{t_j} + e_{t_j}) - f(\hat{x}_{t_j} + e_{t_j})\right.\\
&+ g(h(\hat{x}_{t_j} + e_{t_j}))k_F(j) - g(h(\hat{x}_{t_j} + e_{t_j}))k_F(j)\\
&\left.+ \hat{f}(\hat{x}_{t_j}, h(\hat{x}_{t_j} + e_{t_j})) - \hat{f}(\hat{x}_{t_j}, h(\hat{x}_{t_j} + e_{t_j}))\right)\\
&+ \mu V_{e,1}(e(t^*)) - \mu V_{e,1}(e(t_j)) + \mu V_{e,1}(e(t_j))\Big\}.
\end{aligned} \tag{A.14}$$

From (A.14), taking into account that $\chi_t \in \mathcal{C}_E^{2n}$, $\forall t \geq 0$ (see Claim 20) and by exploiting steps **(6)**, **(7)**, **(13)** and **(14)**, we obtain

$$\begin{aligned}
&\mathcal{D}_{e,\infty}(e_t^*, \hat{x}_{t^*}, u^\star(i_j), k_O^\star(j, i_j)) \leq \\
&D^+V_{e,1}(e_{t_j}, \hat{x}_{t_j}, k_F(j), k_O(j,j)) + 2L_{f,V_e}\tilde{q}\delta \\
&+ M_{V_{e,1}}U_1L_g(\mu_y + 2L_h\tilde{q}\delta) + 2M_{\hat{f}}L_{V_{e,1}}\tilde{q}\delta + M_{V_{e,1}}L_{\hat{f}}\mu_y \\
&+ 2L_{D^+\widetilde{V}_{e,2}}\tilde{q}\delta + D^+\widetilde{V}_{e,2}(e_{t_j}, 0, 0, 0) - \eta\mu V_{e,3}(e_{t_j}) \\
&+ 2\eta\mu L_{V_{e,3}}\tilde{q}\delta + \eta \max\Big\{ 0, \\
&D^+V_{e,1}(e_{t_j}, \hat{x}_{t_j}, k_F(j), k_O(j,j)) + 2L_{f,V_e}\tilde{q}\delta \\
&+ M_{V_{e,1}}U_1L_g(\mu_y + 2L_h\tilde{q}\delta) + 2M_{\hat{f}}L_{V_{e,1}}\tilde{q}\delta + M_{V_{e,1}}L_{\hat{f}}\mu_y \\
&+ \mu V_{e,1}(e(t_j)) + 2\mu\bar{L}_{V_{e,1}}\tilde{q}\delta \Big\}\leq \\
&\mathcal{D}_{e,\infty}(e_{t_j}, \hat{x}_{t_j}, k_F(j), k_O(j,j)) + \gamma(\delta, \mu_y).
\end{aligned} \tag{A.15}$$

From (A.13), by taking into account (A.15), the following inequalities hold

$$\begin{aligned}
&W(t) - W(t_j) \leq (t-t_j)\Big(\mathcal{D}_{e,\infty}(e_t^*, \hat{x}_{t^*}, u^\star(i_j), k_O^\star(j, i_j)) \\
&+ \mathcal{D}_{\hat{x},\infty}(\hat{x}_t^*, k_O^\star(j, i_j)) \Big)\leq \\
&(t-t_j)\Big(\mathcal{D}_{e,\infty}(e_{t_j}, \hat{x}_{t_j}, k_F(j), k_O(j,j)) + \gamma(\delta, \mu_y) \Big) \\
&+ (t-t_j)\Big(\mathcal{D}_{\hat{x},\infty}(\hat{x}_t^*, k_O^\star(j, i_j)) - \mathcal{D}_{\hat{x},\infty}^{j,\bar{u}(i_j)} + \mathcal{D}_{\hat{x},\infty}^{j,\bar{u}(i_j)} \\
&- \mathcal{D}_{\hat{x},\infty}^{j,k_F(i_j)} + \mathcal{D}_{\hat{x},\infty}^{j,k_F(i_j)} - \sigma\mathcal{D}_{\hat{x},\infty}^{j,k_F(j)} + \sigma\mathcal{D}_{\hat{x},\infty}^{j,k_F(j)} \Big).
\end{aligned} \tag{A.16}$$

Then, taking into account (A.5), points (iv) and (p.4) in Lemmas 18, 19 and steps **(7)**, **(14)**, the following inequality holds

$$\begin{aligned}
&W(t) - W(t_j) \leq (t-t_j)(-\sigma\alpha_3(\|\chi_{t_j}\|_\infty) + \gamma(\delta, \mu_y) + \\
&L_{\mathcal{D}_{\hat{x},\infty}}\Big(2\tilde{q}\delta + L_{k_O}(\mu_y + \mu_u + L_{k_F}\mu_y) \Big) + \mathcal{D}_{\hat{x},\infty}^{j,k_F(i_j)} - \sigma\mathcal{D}_{\hat{x},\infty}^{j,k_F(j)}).
\end{aligned} \tag{A.17}$$

Taking into account (12), we have that

$$\mathcal{D}_{\hat{x},\infty}^{j,k_F(i_j)} - \sigma\mathcal{D}_{\hat{x},\infty}^{j,k_F(j)} = \begin{cases} (1-\sigma)\mathcal{D}_{\hat{x},\infty}^{j,k_F(j)} & i_j = j, \\ \mathcal{D}_{\hat{x},\infty}^{j,k_F(i_{j-1})} - \sigma\mathcal{D}_{\hat{x},\infty}^{j,k_F(j)} & i_j = i_{j-1}. \end{cases} \tag{A.18}$$

Taking into account point (iv) in Lemma 18, if $i_j = j$ (trigger), the following inequality holds: $(1-\sigma)\mathcal{D}_{\hat{x},\infty}^{j,k_F(j)} \leq 0$. In the case that $i_j = i_{j-1}$ (no trigger), taking into account (12), the following inequality holds $\mathcal{D}_\infty(\hat{x}_{t_j}, k_O^\star(j, i_{j-1})) - \sigma\mathcal{D}_\infty(\hat{x}_{t_j}, k_O^\star(j,j)) \leq 0$. Then, from (A.18), taking into account steps **(7)**, **(14)**, in the case of no trigger, the following inequalities hold:

$$\begin{aligned}
&\mathcal{D}_{\hat{x},\infty}^{j,k_F(i_{j-1})} - \sigma\mathcal{D}_{\hat{x},\infty}^{j,k_F(j)} \leq \mathcal{D}_{\hat{x},\infty}^{j,k_F(i_{j-1})} - \mathcal{D}_{\hat{x},\infty}(\hat{x}_{t_j}, k_O^\star(j, i_{j-1})) \\
&+ \sigma\mathcal{D}_{\hat{x},\infty}(\hat{x}_{t_j}, k_O^\star(j,j)) - \sigma\mathcal{D}_{\hat{x},\infty}^{j,k_F(j)} \\
&+ \mathcal{D}_{\hat{x},\infty}(\hat{x}_{t_j}, k_O^\star(j, i_{j-1})) - \sigma\mathcal{D}_{\hat{x},\infty}(\hat{x}_{t_j}, k_O^\star(j,j)) \leq \\
&(1+\sigma)L_{\mathcal{D}_{\hat{x},\infty}}L_{k_O}(\mu_y + \mu_u + L_{k_F}\mu_y).
\end{aligned} \tag{A.19}$$

Then, from (A.17), taking into account (A.18), (A.19) and step **(13)**, for $t \in [t_j, t_{j+1}]$, $j \geq 1$, the following inequality holds

$$\begin{aligned}
&W(t) - W(t_j) \leq -(t-t_j)\sigma\alpha_3(\|\chi_{t_j}\|_\infty) + (t-t_j)\times \\
&\Big(L_{\mathcal{D}_{\hat{x},\infty}}\Big(2\tilde{q}\delta + (2+\sigma)L_{k_O}(\mu_y + \mu_u + L_{k_F}\mu_y) \Big) \\
&+ \gamma(\delta, \mu_y) \Big)\leq (t-t_j)\tfrac{\beta}{3} - (t-t_j)\sigma\alpha_3(\|\chi_{t_j}\|_\infty).
\end{aligned} \tag{A.20}$$

Then, taking into account of both cases $\|\chi_{t_j}\|_\infty \leq e_2$ and $\|\chi_{t_j}\|_\infty > e_2$, for any $t \in [t_j, t_{j+1}]$, $j = 0, 1, \ldots$, we obtain:

$$\begin{aligned}
&W(t) \leq (W(t_j) - \tfrac{2}{3}\beta(t-t_j))H(\|\chi_{t_j}\|_\infty - e_2) \\
&+ \alpha_2(e_1)H_0(e_2 - \|\chi_{t_j}\|_\infty).
\end{aligned} \tag{A.21}$$

The symbols H_0 and H denote Heaviside functions defined, for $s \in \mathbb{R}$, as follows: $H_0(s) = 1$ if $s \geq 0$, $H_0(s) = 0$ if $s < 0$; $H(s) = 1$ if $s > 0$, $H(s) = 0$ if $s \leq 0$. From here on, by suitably exploiting (A.21), the same steps used in the proof of Theorem 5.3 in Pepe (2014) can be properly repeated, in order to prove that the solution $\chi(t)$ of the closed-loop system (A.4)–(A.10) exists for all $t \in \mathbb{R}^+$ and, furthermore, satisfies $\chi_t \in \mathcal{C}_E^{2n}$, $\forall t \in \mathbb{R}^+$ (i.e., Claim 20 holds true) and $\chi_t \in \mathcal{C}_r^{2n}$, $\forall t \geq T$. The reader can refer to steps from (5.15) to (5.23) in Pepe (2014) with $k_2 = [\frac{3\alpha_2(R)}{\beta a\delta}] + 1$. Now, from (A.4)–(A.10), it follows that $\chi_t = \binom{e_t}{\hat{x}_t}$ is the solution, for $t \in \mathbb{R}^+$, of the closed-loop system described by the equations

$$\begin{aligned}
&\dot{e}(t) = f(\hat{x}_t + e_t) + g(\hat{x}_t + e_t)u^\star(i_j) - k_O^\star(j, i_j) \\
&\dot{\hat{x}}(t) = k_O^\star(j, i_j), \qquad t \in \big[t_j, t_{j+1}\big), \ t_j \in \pi_{a,\delta}, \\
&e(\tau) = e_0(\tau), \ \hat{x}(\tau) = \hat{x}_0(\tau), \ \tau \in [-\Delta, 0].
\end{aligned} \tag{A.22}$$

Thus, taking into account $e(t) = x(t) - \hat{x}(t)$, it follows that the solution x_t of the closed-loop system described by (2)–(11) exists $\forall t \in \mathbb{R}^+$ and, furthermore, satisfies (13). The proof of the theorem is complete. ◇

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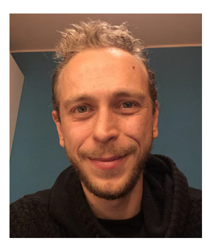

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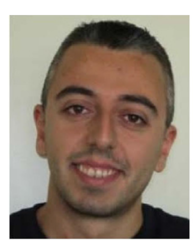

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