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Small Inertia Limit for Coupled Kinetic Swarming Models

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Abstract
We investigate various versions of multi-dimensional systems involving many species, modeling aggregation phenomena through nonlocal interaction terms. We establish a rigorous connection between kinetic and macroscopic descriptions by considering the small inertia limit at the kinetic level. The results are proved either under smoothness assumptions on all interaction kernels or under singular assumptions for *self-interaction* potentials. Utilizing different techniques in the two cases, we demonstrate the existence of a solution to the kinetic system, provide uniform estimates with respect to the inertia parameter, and show convergence toward the corresponding macroscopic system as the inertia approaches zero.

Keywords Kinetic equations · Aggregation phenomena · Singular potentials · Small inertia limit · Measure solutions

Mathematics Subject Classification 35A01 · 35A21 · 82C40 · 35Q70

Contents

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1 Introduction

This paper aims to investigate the connections among different descriptions of multi-species aggregation phenomena, focusing specifically on nonlocal systems of partial differential equations. The main objective is to study the following first-order macroscopic system

$$\begin{cases} \partial_t \rho_i + \nabla \cdot (\rho_i u_i) = 0, \\ u_i = -\sum_{j=1}^N \nabla K_{ij} * \rho_j, \end{cases} \tag{1}$$

for $i = 1, \ldots, N$, where N is the number of species, $\rho_i(t, x)$ is a function modeling the i-th species density, and K_{ij} are given space-dependent interaction potentials modeling interaction between species. Interactions between agents of the same species are modeled by the K_{ii} potentials that are called *self-interaction* kernels, whereas *cross-interaction* kernels K_{ij} describe the interactions of individuals of different species.

System (1) admits a discrete counterpart constructed as follows: consider M particles for each species and let x_i^k, $k = 1, \ldots, M$, be the locations of M particles of the i-th species, for $i = 1, \ldots, N$. Denoting by v_i^k the velocities of x_i^k, the dynamics of x_i^k is determined by the first-order ODE system

$$\begin{cases} \frac{\mathrm{d}x_i^k}{\mathrm{d}t} = v_i^k, \\ v_i^k = -\frac{1}{M} \sum_{j=1}^N \sum_{h=1}^M \nabla K_{ij}(x_i^k - x_j^h), \end{cases} \tag{2}$$

for $i = 1, \ldots, N, k = 1, \ldots, M$, where K_{ij} are the same kernels as in (1). System (2) can be also derived as *small inertia limit* of the second-order system

$$
\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_i^k = v_i^k, \\ \varepsilon \frac{\mathrm{d}}{\mathrm{d}t} v_i^k = -v_i^k - \frac{1}{M} \sum_{j=1}^N \sum_{h=1}^M \nabla K_{ij}(x_i^k - x_j^h), \end{cases} \tag{3}
$$

for $i = 1, \ldots, N$, $k = 1, \ldots, M$, with $\varepsilon > 0$ representing the *inertia* parameter, see Bodnar and Velazquez (2005). System (2) can be justified by formally sending $\varepsilon \to 0$ in (3) by assuming that inertia terms are negligible. However, this choice is quite restrictive in many cases since in this way a "reaction" time is not taking into account and velocities change instantaneously.

Taking the formal limit as the number of particles increases to infinity, namely $M \to \infty$, we can associate with (3) the kinetic system

$$
\partial_t f_i + v \cdot \nabla_x f_i = \frac{1}{\varepsilon} \nabla_v \cdot (v f_i) + \frac{1}{\varepsilon} \nabla_v \cdot \left(\left(\sum_{j=1}^N \nabla K_{ij} * \rho_j \right) f_i \right), \tag{4}
$$

for $i = 1, \ldots, N$, where $f_i(t, x, v)$ is the mesoscopic density of the i-th species at position $x \in \mathbb{R}^d$ with velocity $v \in \mathbb{R}^d$ and $\rho_i(t, x)$ is the associated macroscopic population density, i.e.,

$$
\rho_i(t, x) = \int_{\mathbb{R}^d} f_i(t, x, v) \, \mathrm{d}v.
$$

The main goal of the present paper is to investigate the small inertia limit at the continuum level. In particular, we want to study the $\varepsilon \to 0$ limit in (4) and prove that it converges toward the first-order PDEs model (1).

In recent decades, systems such as (1) and (4) have been extensively employed to provide a biologically relevant representation of aggregative phenomena in population dynamics, particularly in the context of *swarming* phenomena (see Boi et al. 2000; Mogilner and Edelstein-Keshet 1999; Okubo and Levin 2001; Topaz and Bertozzi 2004). Common interaction potentials in these scenarios include the attractive *Morse* potential $G(x) = -e^{-|x|}$, attractive–repulsive Morse potentials $G(x) = -C_a e^{-|x|/l_a} + C_r e^{-|x|/l_r}$ (where l_a and l_r represent scales for the "attractive range" and the "repulsive range," respectively), combinations of Gaussian potentials $G(x) = -C_a e^{-|x|^2/l_a} + C_r e^{-|x|^2/l_r}$, or characteristic functions of a set $G(x) = \alpha \chi_A(x)$. Considerable attention is directed toward aggregation systems with singular kernels, particularly at the mesoscopic level.

A notable mathematical characteristic of these models that has drawn attention is the finite-time *blow-up* of solutions. Numerous contributions have been made in the literature for the one-species version of (1), as seen in Bertozzi and Brandman (2010), Bertozzi et al. (2009), Bertozzi and Laurent (2009), Bertozzi et al. (2011), Burger and Di Francesco (2008), Carrillo et al. (2011), Choi and Jeong (2021), Li and Toscani (2004). Notably, in Choi and Jeong (2021) the one-species case of (1) with

$$v = -\nabla K * \rho = c_K \Lambda^{\alpha-d} \nabla \rho,$$

where $c_K \in \mathbb{R}$ and $-2 \le \alpha - d \le 0$ are parameters and Λ^s is the $s-$fractional power of $\Lambda:=(-\Delta)^{\frac{1}{2}}$, to be defined precisely below, was studied establishing the local-in-time existence and uniqueness of classical solutions.

Inspired by the results in the single-species case, an existence theory was developed for the system (1) in Di Francesco and Fagioli (2013). In the case when the system presents a *symmetry*, namely $K_{ij} = K_{ji}$ for $i \neq j$ and $i, j = 1, \ldots, N$, then system (1) exhibits a formal gradient flow structure:

$$\partial_t \rho_i = \nabla \cdot \left(\rho_i \nabla \frac{\delta \mathcal{F}(\rho)}{\delta \rho_i} \right),$$

for $i = 1, \ldots, N$, where $\mathcal{F}(\rho)$ is a free energy given by

$$\mathcal{F}(\rho) = \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \rho_i K_{ij} * \rho_j \, \mathrm{d}x.$$

In Di Francesco and Fagioli (2013), the authors proved that the theory of gradient flows in Wasserstein spaces developed in Ambrosio et al. (2008), Carrillo et al. (2011) can be extended to systems under mildly singular assumptions on all kernels K_{ij}, i.e., Morse-type singularity. When the symmetry property is lost, the existence of weak-measure solutions is developed in Di Francesco and Fagioli (2013) using a semi-implicit version of the *JKO-scheme*, originally introduced in Jordan et al. (1998). Existence can be proved under mildly singular assumptions on the self-interaction potentials and smoothness assumptions on the cross-interaction potentials. To the best of the authors' knowledge, no results in the literature cover the case of singular non-symmetric cross-interaction kernels or self-interaction kernels under more singular assumptions, similar to the ones in Choi and Jeong (2021).

Focusing on the system (4), the kinetic approach is widely employed in investigating aggregation phenomena. In Cañizo et al. (2009), the single-species version of equation (4) is examined, considering both a self-propulsion term and a friction term. The former influences individuals independently of others, while the latter introduces a velocity-averaging effect, compelling agents to adjust their velocities based on nearby agents. The paper presents results on well-posedness, existence, uniqueness, and continuous dependence in the space of probability measures $\mathcal{P}_1(\mathbb{R}^d)$ equipped with the Monge–Kantorovich–Rubinstein distance. Additionally, the corresponding microscopic system is explored, and a convergence result from the particle system to the kinetic equation is established. The existence of smooth solutions is addressed using the classical framework for Vlasov-type equations, as outlined in Glassey (1996). We refer to Cesbron and Iacobelli (2023) and references therein for recent treatments of the Vlasov–Poisson equation. Let us also mention the contribution in Carrillo et al. (2022) to the theory of the Vlasov–Poisson–Fokker–Planck system.

In Fetecau and Sun (2015), equation (4) is further investigated in the one-species case in a multi-dimensional space, accounting for inertial effects. Assuming smooth conditions on the kernel and applying the theory developed in Cañizo et al. (2009), the paper proves existence and uniqueness results in the sense of measures. A small inertia limit is also examined, providing convergence to the corresponding macroscopic system.

Starting from the seminal work (Kramers 1940), several contributions have appeared in the literature in the study of the limit from (4) to (1) in its one-species version; see Freidlin (2004), Hottovy et al. (2012), Narita (1994). In Duong et al. (2018), Duong et al. (2017), variational techniques were introduced to study the rigorous limit from the Vlasov–Fokker–Planck equation to the corresponding macroscopic equation under suitable regularity and integrability assumptions on the interaction potential. Finally, in Goudon (2005), Poupaud and Soler (2000), qualitative analysis of the *overdamped* limit from the Vlasov–Poisson–Fokker–Planck system toward the drift diffusion equation, with interaction kernels given as attractive or repulsive Coulomb potential, has been conducted. More recently, a rigorous *quantified overdamped limit* for the Vlasov–Fokker–Planck equation with smooth and singular nonlocal forces has been established in Röckner et al. (2021) and Choi and Tse (2022), respectively.

The rest of this paper is organized as follows. In Sect. 2, we present the collection of assumptions and the statements of the main theorems of the paper, namely the small inertia limits in the smooth case in Theorem 2.1 and for singular self-interaction kernels in Theorem 2.2. Section 3 is devoted to the proof of Theorem 2.1. The existence of solutions to system (4) is first proved by applying the method of characteristic. Uniform in ε estimates and convergence to solutions to (1) is then proved by adapting to systems the results in Fetecau and Sun (2015). We then provide the proof of the main result of the paper that is Theorem 2.2 in Sect. 4. A regularization procedure yields the existence of solutions to system (4) in the case of singular self-interaction kernels. Suitable a priori estimates allow us to prove the existence and uniqueness of classical solution to (1) under the singular setting by extending the one-species result in Choi and Jeong (2021). Finally, Theorem 2.2 is proved by using the modulated energy estimate technique.

2 Preliminaries and Main Results

2.1 Preliminaries and Assumptions

Let $\mathcal{P}_1(\mathbb{R}^d)$ be the set of probability measures with finite first moment, i.e.,

$$\mathcal{P}_1(\mathbb{R}^d) = \left\{ f \in \mathcal{P}(\mathbb{R}^d) \ : \ \int_{\mathbb{R}^d} |x| f(x)\, \mathrm{d}x < \infty \right\}.$$

We equip $\mathcal{P}_1(\mathbb{R}^d)$ with the 1-Wasserstein distance defined as (cf. Villani (2003))

$$W_1(\mu, \nu) = \inf_{\gamma \in \Pi(\mu,\nu)} \left\{ \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| \, \mathrm{d}\gamma(x, y) \right\}$$

for all $\mu,\ \nu \in \mathcal{P}_1(\mathbb{R}^d)$, where $\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ is a probability measure on the product space $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and ν, respectively, i.e.,

$$\Pi(\mu, \nu) := \{\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)\ \ \pi^1_\# \gamma = \mu, \quad \pi^2_\# \gamma = \nu\},$$

where $\pi^i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, 2$, is the projection operator on the i-th component of the product space $\mathbb{R}^d \times \mathbb{R}^d$. We also introduce the set of probability measures with compact support, that is,

$$\mathcal{P}_c(\mathbb{R}^d) = \Big\{ f \in \mathcal{P}(\mathbb{R}^d)\ :\ f \text{ has compact support} \Big\}.$$

Since we deal with N interacting species, the measure space we consider is $(\mathcal{P}_1(\mathbb{R}^d)^N, \mathcal{W}_1)$, where $\mathcal{W}_1$ is the 1-Wasserstein distance on $\mathcal{P}_1(\mathbb{R}^d)^N$ defined below. In order to fix the notation, we write

$$\mathbf{f} = (f_i)_{i=1}^N \in \mathcal{P}_1(\mathbb{R}^d)^N$$

to denote a N-tuple of probability measures in the product space $\mathcal{P}_1(\mathbb{R}^d)^N$.

Definition 2.1 (*1-Wasserstein distance*) Let $T > 0$ be fixed. Consider $f_i, g_i : [0, T] \to \mathcal{P}_1(\mathbb{R}^d)$, for $i = 1, \dots, N$. Introduce $\mathbf{f} = (f_i)_{i=1}^N$, $\mathbf{g} = (g_i)_{i=1}^N$. We define the 1-*Wasserstein distance between* $\mathbf{f}$ *and* $\mathbf{g}$ as

$$\mathcal{W}_1(\mathbf{f}, \mathbf{g}) := \sup_{t \in [0,T]} \Big[W_1(f_1, g_1) + \cdots + W_1(f_N, g_N) \Big].$$

In the following, we will denote with B_R a ball in $\mathbb{R}^{2d}$ with radius $R > 0$, with B^1_R the ball in $\mathbb{R}^d$, namely, with respect to the $x-$variable.

Given Q a generic potential involved in systems (1) and (4), we will call the potential *smooth* if it satisfies the following assumption

$$Q \in \mathcal{C}^2(\mathbb{R}^d) \quad \text{and} \quad \nabla Q \in W^{1,\infty}(\mathbb{R}^d). \tag{\textbf{Pot}}$$

Remark 2.1 *(Lipschitz constant)* We denote by $Lip_R(Q)$ the Lipschitz constant of Q in the ball $B^1_R \subset \mathbb{R}^d$. If Q depends also on time, i.e., $Q : [0, T] \times \mathbb{R}^d \to \mathbb{R}^n$, $Q = Q(t, x)$, we write $Lip_R(Q)$ to denote the Lipschitz constant of Q with respect to x in the ball $B^1_R \subset \mathbb{R}^d$ that is the smallest constant such that

$$|Q(t, x) - Q(t, y)| \leq Lip_R(Q)|x - y|,$$

for all $x, y \in B^1_R$ and for all $t \in [0, T]$.

In the proposition below, we state a property on the convergence of measures, see Villani (2003) for more details.

Proposition 2.1 *Let (X, d) be a complete and separable metric space. Let μ_n be a sequence of probability measures in $\mathcal{P}_1(X)$, and let $\mu \in \mathcal{P}_1(X)$. Then, the following are equivalent:*

1. *$W_1(\mu_n, \mu) \to 0$ as $n \to \infty$.*
2. *$\mu_n \to \mu$ in the weak sense as $n \to \infty$ and the following tightness condition holds: for any $x_0 \in X$,*
$$\lim_{R\to\infty} \limsup_{k\to\infty} \int_{d(x_0,x)\geq R} d(x_0, x) \mathrm{d}\mu_n = 0.$$
3. *$\mu_n \to \mu$ in the weak sense as $n \to \infty$, and there is a convergence of the moment of first order, i.e., for any $x_0 \in X$,*
$$\int_X d(x_0, x)\, \mathrm{d}\mu_n(x) \to \int_X d(x_0, x)\, \mathrm{d}\mu(x),$$
as $n \to \infty$.

2.2 Main Results

In this subsection, we state our main results on the small inertia limits for the kinetic system (4) and its convergence to (1). We consider two cases: smooth and singular self-interactions potentials. In order to emphasize our results on the small inertia limit of the kinetic system (4) toward the first-order macroscopic system (1), here we only state the theorems on that. The required existence theory for the systems (4) and (1) will be discussed in later sections.

2.2.1 Smooth Potential Case

We start by introducing the notion of weak solutions to (1). We consider first the case of interaction potentials K_{ij} under assumption (**Pot**).

Definition 2.2 A weak solution to (1) is a N-tuple $\boldsymbol{\rho} = (\rho_i)_{i=1}^N \in \mathcal{C}([0, T), \mathcal{P}(\mathbb{R}^d)^N)$ that satisfies

$$\int_0^T \int_{\mathbb{R}^d} \partial_t \phi_i \rho_i \,\mathrm{d}x\, \mathrm{d}t - \int_0^T \int_{\mathbb{R}^d} \nabla_x \phi_i \cdot \left(\sum_{j=1}^N K_{ij} * \rho_j \right) \rho_i \,\mathrm{d}x\, \mathrm{d}t + \int_{\mathbb{R}^d} \phi_i(0) \rho_{i0} \,\mathrm{d}x = 0,$$

for each $\phi_i \in \mathcal{C}_c^1([0, T); \mathcal{C}_b^1(\mathbb{R}^d))$, as $i = 1, \ldots, N$.

We already mentioned in the previous section that the existence of weak solutions to (1) can be found in Di Francesco and Fagioli (2013). For smooth interaction potentials, the small inertia limit result reads as follows.

Theorem 2.1 *Let $T > 0$. Assume all the potentials satisfying* (**Pot**). *Consider $\mathbf{f}_0 \in \mathcal{P}_c(\mathbb{R}^{2d})^N$. Let $\mathbf{f}^\varepsilon \in \mathcal{C}([0, T); \mathcal{P}_c(\mathbb{R}^{2d})^N)$ be the unique measure solution to* *system* (4). *Let* $\rho_i^\varepsilon(t,x) = \int_{\mathbb{R}^d} f_i^\varepsilon(t,x,v)\,\mathrm{d}v$, *for* $i = 1,\dots,N$. *Then there exists* $\boldsymbol{\rho} \in \mathcal{C}([0,T); \mathcal{P}_1(\mathbb{R}^d)^N)$ *such that for each* $t \in [0,T)$,

$$\boldsymbol{\rho}^\varepsilon(t,\cdot) \xrightarrow{\mathcal{W}_1} \boldsymbol{\rho}(t,\cdot) \quad \text{in } \mathcal{P}_1(\mathbb{R}^d)^N$$

as $\varepsilon \to 0$. *Moreover,* $\boldsymbol{\rho}$ *is a weak solution to system* (1) *in the sense of Definition* 2.2.

Theorem 2.1 is proved in Sect. 3 by extending the results in Fetecau and Sun (2015) where the small inertia limit is proved in the one-species case under regularity assumptions on the interaction kernel.

2.2.2 Singular Potential Case

Now we deal with the case of singular interaction potentials. Precisely, we consider the system (4) with smooth cross-potentials K_{ij}, $i \neq j$ satisfying (**Pot**) and singular self-potentials K_{ii} of the form

$$K_{ii}(x) := \frac{C_i}{|x|^{\alpha_i}}, \tag{5}$$

with $\alpha_i \in (0,d)$, and some positive constants C_i. Note that if $\alpha_i \in ((d-2)\vee 0, d)$, then $K_{ii} * \rho_i = \Lambda^{\alpha_i - d}\rho_i$ with $\Lambda = (-\Delta)^{\frac{1}{2}}$ up to constant. Thus in this case the system (1) becomes the following coupled *fractional porous medium flows* (Caffarelli et al. 2013):

$$\partial_t \rho_i = \nabla \cdot \left(\rho_i \left(\nabla \Lambda^{\alpha_i - d} \rho_i + \sum_{\substack{j=1 \\ j \neq i}}^N \nabla K_{ij} * \rho_j \right) \right),$$

for $i = 1,\dots,N$.

Then our second and main result is stated as follows.

Theorem 2.2 *Let* $T > 0$ *and* $d \geq 1$. *Let* $\mathbf{f}^\varepsilon = (f_i^\varepsilon)_{i=1}^N \in \mathcal{C}([0,T); \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)^N)$ *be a solution to system* (4) *in the sense of distributions, and let* $(\boldsymbol{\rho}, \mathbf{u}) = (\rho_i, u_i)_{i=1}^N$ *be the unique classical solution of the system* (1) *with* $\rho_i > 0$ *on* $\mathbb{R}^d \times [0,T)$, $\partial_t u_i + u_i \cdot \nabla u_i \in L^\infty(\mathbb{R}^d \times (0,T))$, *and if* $\alpha_i < d-2$, $\nabla^{[(d-\alpha_i)/2]+1} u_i \in L^\infty((0,T); L^{\frac{d}{[(d-\alpha_i)/2]}}(\mathbb{R}^d))$ *up to time* $T > 0$ *with the initial data* ρ_{i0}. *If*

$$\sup_{\varepsilon > 0} \sum_{i=1}^N \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v - u_{i0}(x)|^2 f_{i0}^\varepsilon(x,v)\,\mathrm{d}x\,\mathrm{d}v < \infty \tag{6}$$

and

$$\sum_{i=1}^N \int_{\mathbb{R}^d} (\rho_{i0} - \rho_{i0}^\varepsilon) K_{ii} * (\rho_{i0} - \rho_{i0}^\varepsilon)\,\mathrm{d}x + \sum_{i=1}^N W_1(\rho_{i0}, \rho_{i0}^\varepsilon) \to 0, \tag{7}$$

as $\varepsilon \to 0$, *then for each* $i = 1,\dots,N$, *we have*

$$\int_{\mathbb{R}^d} f_i^\varepsilon\,\mathrm{d}v \overset{*}{\rightharpoonup} \rho_i \quad \text{weakly-}* \text{ in } L^2((0,T); \mathcal{M}(\mathbb{R}^d)),$$

$$\int_{\mathbb{R}^d} v\, f_i^\varepsilon \,\mathrm{d}v \overset{*}{\rightharpoonup} \rho_i u_i \quad \textit{weakly-}* \textit{ in } L^2((0,T);\mathcal{M}(\mathbb{R}^d)),$$

and

$$f_i^\varepsilon \overset{*}{\rightharpoonup} \rho_i \delta_{u_i} \quad \textit{weakly-}* \textit{ in } L^2((0,T);\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d)),$$

where we denoted by $\mathcal{M}(\mathbb{R}^n)$ *the space of signed Radon measures on* $\mathbb{R}^n$ *with* $n \in \mathbb{N}$.

Our proof for Theorem 2.2 relies on the modulated energy estimates. For this, we need to establish the existence theory for the kinetic system (4) and the first-order macroscopic system (1) at least locally in time. To be more specific, as stated in Theorem 2.2, it suffices to construct the weak solutions to (4), but for the limit system (1), it is required to show the existence and uniqueness of regular solutions satisfying the regularity conditions of Theorem 2.2.

Remark 2.2 Here we provide some remarks regarding Theorem 2.2.

(i) If we further assume

$$\sum_{i=1}^N \iint_{\mathbb{R}^d\times\mathbb{R}^d} |v-u_{i0}(x)|^2 f_{i0}^\varepsilon(x,v)\,\mathrm{d}x\,\mathrm{d}v \to 0$$

and

$$\frac{1}{\varepsilon}\sum_{i=1}^N \int_{\mathbb{R}^d} (\rho_{i0}-\rho_{i0}^\varepsilon) K_{ii} * (\rho_{i0}-\rho_{i0}^\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon}\sum_{i=1}^N W_1^2(\rho_{i0},\rho_{i0}^\varepsilon) \to 0$$

as $\varepsilon \to 0$, then for each $i = 1,\dots,N$, we have

$$\int_{\mathbb{R}^d} f_i^\varepsilon\,\mathrm{d}v \overset{*}{\rightharpoonup} \rho_i, \qquad \int_{\mathbb{R}^d} v f_i^\varepsilon\,\mathrm{d}v \overset{*}{\rightharpoonup} \rho_i u_i \quad \text{weakly-}* \text{ in } L^\infty((0,T);\mathcal{M}(\mathbb{R}^d)),$$

and

$$f_i^\varepsilon \overset{*}{\rightharpoonup} \rho_i\delta_{u_i} \quad \text{weakly-}* \text{ in } L^\infty((0,T);\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d))$$

as $\varepsilon \to 0$.

(ii) Note that if $\alpha_i > d-1$, then the self-interaction force $\nabla K_{ii} * \rho_i$ is not well defined for $\rho_i \in L^p(\mathbb{R}^d)$ with any p. Thus, we are only able to construct the global-in-time $L^1\cap L^\infty$ solutions to the kinetic system (4) for $\alpha_i \in (0, d-1]$, see Theorem 4.1. We also need to assume additional condition $\nabla K_{ij} \in W^{1,1}(\mathbb{R}^d)$, $i \neq j$ to develop the local-in-time well-posedness of the macroscopic system (1), see Theorem 4.2. In that respect, our results are fully rigorous when the interaction potentials K_{ij}, $i \neq j$, satisfy

$$K_{ij} \in \mathcal{C}^2(\mathbb{R}^d), \quad \nabla K_{ij} \in W^{1,1}\cap W^{1,\infty}(\mathbb{R}^d)$$

and K_{ii} is given as (5) with $\alpha_i \in (0, d-1]$, see also Remark 4.2.

(iii) *Comparison with the one-species results.* The result in Theorem 2.1 is the natural extension of the one in Fetecau and Sun (2015) for the single-species case. Indeed, we adapt most of the techniques from that work and from Cañizo et al. (2009). The comparison with the singular case is more involved. First, existence in both the kinetic and macroscopic cases needs to be established. While in the kinetic case, the existence results turn out to be a natural extension of the results in Choi and Jeong (2023), the macroscopic case seems to require more attention. To the authors' knowledge, the only result addressing existence in the macroscopic case is the one in Di Francesco and Fagioli (2013). Similar to that result, we need to require smoothness for the *cross-interaction* kernels. Specifically, in Di Francesco and Fagioli (2013), the *cross-interaction* kernels are assumed to be globally Lipschitz with a continuous gradient, while the *self-interaction* kernels are required to be $\mathcal{C}^1(\mathbb{R}^d \setminus \{0\})$ and globally Lipschitz. The restriction on the *cross-interaction* is avoidable in the case of symmetric kernels, $K_{ij} = K_{ji}$. In the present work, we are able to relax the assumptions on the *self-interaction* kernels by considering more singular kernels, but we need to assume smoothness for the *cross-interaction*, since the lack of symmetry poses a problem even when extending the single-species results of Choi and Jeong (2021) to our case.

(iv) Unfortunately, we were unable to apply our modulated energy method to investigate the small inertia limit when the cross-interaction potentials are singular. It is important to note that the modulated interaction energy $\int_{\mathbb{R}^d}(\rho_i - \rho_i^\varepsilon)K_{ii} * (\rho_i - \rho_i^\varepsilon)\,\mathrm{d}x$ provides an appropriate measure of distance, for instance, it equals $\|\rho - \rho^\varepsilon\|^2_{\dot{H}^{-1}}$ when $\alpha_i = d - 2$ with $d \geq 3$. However, it remains unclear how to achieve effective control over $\int_{\mathbb{R}^d}(\rho_i - \rho_i^\varepsilon)K_{ij} * (\rho_j - \rho_j^\varepsilon)\,\mathrm{d}x$ for $i \neq j$. Consequently, we restrict our analysis to the case where the cross-interaction potentials are sufficiently regular.

3 Smooth Interaction Potentials

3.1 Well-Posedness for the Kinetic System for $\varepsilon > 0$ Fixed

We start the investigation of the small inertia limit for smooth interaction kernels by studying the well-posedness for system (4) for $\varepsilon > 0$ fixed, in the spirit of Cañizo et al. (2009). We start observing that such existence theory can be studied for a more general class of force fields

$$\mathbf{E}:=(E_i)_{i=1}^N(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{Nd},$$

for $i = 1, \dots, N$, fulfilling the following general set of hypotheses:

$(\mathbf{H_1})$ E_i are continuous on $[0,T] \times \mathbb{R}^d$, for all $i = 1, \dots, N$.

$(\mathbf{H_2})$ There exist some positive constants C_i such that

$$|E_i(t,x)| \leq C_i(1 + |x|),$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$, for all $i = 1, \dots, N$.

(**H$_3$**) E_i are locally Lipschitz with respect to x uniformly in t, for $i = 1, \dots, N$, that is for any compact set $K \subset \mathbb{R}^d$ there exist positive constants L_i such that

$$|E_i(t, x) - E_i(t, y)| \le L_i |x - y|,$$

for all $x, y \in K$, and for all $t \in [0, T]$.

The kinetic system we are going to study takes then the following form

$$\partial_t f_i + v \cdot \nabla_x f_i - \frac{1}{\varepsilon} \nabla_v \cdot (v f_i) + \frac{1}{\varepsilon} E_i \cdot \nabla_v f_i = 0, \tag{8}$$

for $i = 1, \dots, N$. Following the approach in Cañizo et al. (2009), we will construct solutions to (8) by considering the following characteristic system associated with (8)

$$\begin{cases} \frac{\mathrm{d}X}{\mathrm{d}t} = V, \\ \\ \frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{1}{\varepsilon} V + \frac{1}{\varepsilon} E_i(t, X), \end{cases} \tag{9}$$

for $i = 1, \dots, N$. Introducing $P := (X, V) \in \mathbb{R}^d \times \mathbb{R}^d$, and denoting by

$$\Psi_{E_i} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$$

the right-hand side of system (9), we can rewrite system (9) as

$$\frac{\mathrm{d}}{\mathrm{d}t} P = \Psi_{E_i}(t, P), \tag{10}$$

for $i = 1, \dots, N$, subject to the initial condition $P_0 = (X_0, V_0) \in \mathbb{R}^d \times \mathbb{R}^d$. Existence and uniqueness of solutions to system (10) fall into the classical ordinary differential equations theory, see Tikhonov (1952), that provides a vector field $P \in \mathcal{C}^1([0, T]; \mathbb{R}^{2d})$ such that

$$|P| \le |P_0| e^{Ct},$$

for all $t \in [0, T]$ and the constant C depending on T, $|X_0|$, $|V_0|$. We will use the compact notation

$$\Psi_{\mathbf{E}} := (\Psi_{E_i})_{i=1}^N.$$

Let us introduce the time-dependent flow map associated with system (9) by

$$\mathcal{T}_{E_i}^t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d,$$

such that

$$\mathcal{T}_{E_i}^t((X_0, V_0)) = (X(t), V(t)),$$

where $(X(t), V(t))$ is the unique solution to (9) at time $t > 0$ under the initial condition (X_0, V_0).

We are now in the position to introduce the notion of measure solution to system (8). Consider $f_{i0} \in \mathcal{P}_1(\mathbb{R}^{2d})$ the initial datum of the i-th species and let $T > 0$. Then, a measure solution to (8) can be defined as

$$f_i(t) = \mathcal{T}^t_{E_i} \# f_{i0},$$

for $i = 1, \dots, N$. With a slight abuse of notation, for using a compact formulation, we set

$$\mathcal{T}^t_{\mathbf{E}} = \left(\mathcal{T}^t_{E_i}\right)_{i=1}^N,$$

and given the initial datum $\mathbf{f}_0 \in \mathcal{P}(\mathbb{R}^{2d})^N$ and a time $T > 0$, we define the measure solution to (8) as

$$\mathbf{f}(t) = \mathcal{T}^t_{\mathbf{E}} \# \mathbf{f}_0.$$

Referring to system (4), we define the vector field $\mathbf{E}[\mathbf{f}]$ associated with a N-tuple of measures $\mathbf{f}$ as

$$\mathbf{E}[\mathbf{f}] = (E_i[\mathbf{f}])_{i=1}^N = \left(- \sum_{j=1}^N \nabla K_{ij} * \rho_j \right)_{i=1}^N. \tag{11}$$

We can now give the notion of measure solution to system (4) as in Cañizo et al. (2009), Fetecau and Sun (2015).

Definition 3.1 (*Measure solution to* (4)) Fix $T > 0$ and $\varepsilon > 0$. Let $\mathbf{f}_0 \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ be a given initial condition and let $\mathbf{E}[\mathbf{f}]$ be defined as in (11). A N-tuple $\mathbf{f} : [0, T] \to \mathcal{P}_1(\mathbb{R}^{2d})^N$ is a measure solution to system (4) with initial condition $\mathbf{f}_0$ if:

1. the field $\mathbf{E}[\mathbf{f}]$ defined in (11) satisfies the conditions $(\mathbf{H_1})$-$(\mathbf{H_2})$-$(\mathbf{H_3})$;
2. it holds $\mathbf{f}(t) = \mathcal{T}^t_{\mathbf{E}[\mathbf{f}]} \# \mathbf{f}_0$.

3.1.1 A Priori Estimates on the Characteristics System

In this part, we collect some results on the solution to the characteristic system (9). Proofs of the two lemmas below can be obtained directly from system (9) and by definition of $\Psi_{\mathbf{E}}$ in (10), see Cañizo et al. (2009), Fetecau and Sun (2015).

Lemma 3.1 *Fix $T > 0$. Let $\mathbf{E}, \mathbf{D} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{Nd}$ be fields that satisfy $(\mathbf{H_1})$-$(\mathbf{H_2})$-$(\mathbf{H_3})$ and let $\Psi_{\mathbf{E}}, \Psi_{\mathbf{D}}$ as in (10). Consider $R > 0$ and the closed ball $B_R \subset \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$. Then*

1. *$\Psi_{\mathbf{E}}$ is bounded in compact sets, i.e.,*

$$|\Psi_{\mathbf{E}}(t, P)| \le C,$$

for all $P \in B_R$, $t \in [0, T]$ and for some $C > 0$ which depends on R and $\|\mathbf{E}\|_{L^\infty([0,T] \times B^1_R)}$, where B^1_R is the ball in $\mathbb{R}^d$ with radius R.

2. *$\Psi_{\mathbf{E}}$ is locally Lipschitz with respect to X and V, i.e.,*

$$|\Psi_{\mathbf{E}}(t, P_1) - \Psi_{\mathbf{E}}(t, P_2)| \le C(1 + Lip_R(\mathbf{E}))|P_1 - P_2|,$$

for all $P_1, P_2 \in B_R$, $t \in [0, T]$ and $C > 0$.

3. *For any compact set* $B \subset \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$,

$$\|\Psi_{\mathbf{E}} - \Psi_{\mathbf{D}}\|_{L^\infty(B)} \leq \frac{1}{\varepsilon}\|\mathbf{E} - \mathbf{D}\|_{L^\infty(B^1)}.$$

Now we provide some results that concern the dependence of the characteristics on the field $\mathbf{E}$ and a quantitative bound on the regularity of the flow $\mathcal{T}^t_{\mathbf{E}}$.

Lemma 3.2 *Fix* $T > 0$ *and consider two vector fields* $\mathbf{E}$ *and* $\mathbf{D}$ *satisfying* $(\mathbf{H_1})$-$(\mathbf{H_2})$-$(\mathbf{H_3})$. *Take* $P_0, P_1, P_2 \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$ *and* $R > 0$. *Assume*

$$|\mathcal{T}^t_{\mathbf{E}}(P_0)| \leq R, \quad |\mathcal{T}^t_{\mathbf{D}}(P_0)| \leq R, \quad |\mathcal{T}^t_{\mathbf{E}}(P_1)| \leq R, \quad |\mathcal{T}^t_{\mathbf{E}}(P_2)| \leq R,$$

for $t \in [0, T]$. *Then,*

1. *There is a constant* C *depending on* R *and* $Lip_R(\mathbf{E})$ *such that*

$$|\mathcal{T}^t_{\mathbf{E}}(P_0) - \mathcal{T}^t_{\mathbf{D}}(P_0)| \leq \frac{e^{Ct} - 1}{C\varepsilon} \sup_{s\in[0,T)} \|\mathbf{E}(s) - \mathbf{D}(s)\|_{L^\infty(B^1_R)},$$

for $t \in [0, T]$.

2. *There is a constant* C *depending on* R *such that*

$$|\mathcal{T}^t_{\mathbf{E}}(P_1) - \mathcal{T}^t_{\mathbf{E}}(P_2)| \leq |P_1 - P_2| e^{C\int_0^t (Lip_R(\mathbf{E}(s))+1)\,\mathrm{d}s},$$

for $t \in [0, T]$.

3. *There is a constant* C *depending on* R *and* $\|\mathbf{E}\|_{L^\infty([0,T]\times B^1_R)}$ *such that*

$$|\mathcal{T}^t_{\mathbf{E}}(P_0) - \mathcal{T}^s_{\mathbf{E}}(P_0)| \leq C|t - s|,$$

for $s, t \in [0, T]$.

Remark 3.1 Note that the sub-linearity assumption on the vector field $\mathbf{E}$ ensures global existence for solution for $t \in \mathbb{R}$. The boundedness assumption in Lemma 3.2 on the initial flow $\mathcal{T}^t_{\mathbf{E}}(P_0)$ is only needed to prove a quantitative estimate on the flow map for every time $t \in [0, T]$. Moreover, Lemma 3.2 ensures that the flow $\mathcal{T}^t_{\mathbf{E}}$ is Lipschitz on $B_R \subset \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$, with constant

$$Lip_R(\mathcal{T}^t_{\mathbf{E}}) \leq e^{C\int_0^t (Lip_R(\mathbf{E}(s))+1)\,\mathrm{d}s},$$

for $t \in [0, T]$.

In the following lemmas, we collect some contraction results in the Wasserstein distance $\mathcal{W}_1$ that are crucial in proving the existence of measure solutions for (8). What we reproduce is the extension to multiple species of the results in Canizo et al. (2009, Lemmas 3.11, 3.12, and 3.13), see also Fetecau and Sun (2015).

Lemma 3.3 *Let* $\mathbf{E}, \mathbf{D} : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ *be two Borel measurable maps and let* $\mathbf{f} \in \mathcal{P}_1(\mathbb{R}^d)^N$. *Then*

$$\mathcal{W}_1\big(\mathbf{E}\#\mathbf{f}, \mathbf{D}\#\mathbf{f}\big) \leq \|\mathbf{E} - \mathbf{D}\|_{L^\infty(\operatorname{supp}\mathbf{f})}.$$

Lemma 3.4 *Let* $T > 0$. *Let* $\mathbf{E} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{Nd}$ *be a field that satisfies* $(\mathbf{H_1})$-$(\mathbf{H_2})$-$(\mathbf{H_3})$ *and let* $\mathbf{f}$ *be a* N*-tuple of measures on* $\mathbb{R}^d$ *with compact support contained in a ball* $B_R \subset \mathbb{R}^d$. *Then, there exists a positive constant* C *depending on* N, R *and* $\|\mathbf{E}\|_{L^\infty([0,T]\times B_R^1)}$ *such that*

$$\mathcal{W}_1(\mathcal{T}_{\mathbf{E}}^t\#\mathbf{f}, \mathcal{T}_{\mathbf{E}}^s\#\mathbf{f}) \leq C|t - s|,$$

for any $s, t \in [0, T]$.

Lemma 3.5 *Let* $\mathcal{T} : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ *be a Lipschitz map and let* $\mathbf{f}, \mathbf{g} \in \mathcal{P}_1(\mathbb{R}^d)^N$ *both have compact support contained in a ball* B_R. *Then*

$$\mathcal{W}_1\big(\mathcal{T}\#\mathbf{f}, \mathcal{T}\#\mathbf{g}\big) \leq L\mathcal{W}_1(\mathbf{f}, \mathbf{g}),$$

where L *is the Lipschitz constant of* $\mathcal{T}$ *on the ball* B_R.

3.1.2 Existence and Uniqueness for Smooth Potentials

We turn now to the existence and uniqueness of measure solutions to system (4). We first provide the following preliminary lemmas, whose proof is straightforward and we omit.

Lemma 3.6 *Assume the potentials* K_{ij} *under assumption* (**Pot**). *Let* $\mathbf{f} \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ *be with compact support contained in a ball* $B_R \subset \mathbb{R}^{2d}$. *Set* $B_R^1 := \{x \ : \ (x, v) \in B_R\}$, *for all* $v \in \mathbb{R}^d$. *Consider the vector field defined in* (11). *Then,*

$$\|\mathbf{E}[\mathbf{f}]\|_{L^\infty(B_R^1)} \leq \Xi, \quad and \quad Lip_R(\mathbf{E}[\mathbf{f}]) \leq \Upsilon,$$

where the constants Ξ *and* Υ *are defined by*

$$\Xi := \sum_{i,j=1}^N \|\nabla K_{ij}\|_{L^\infty(B_{2R})},$$

and

$$\Upsilon := \sum_{i,j=1}^N Lip_{2R}(\nabla K_{ij}).$$

Lemma 3.7 *Assume the potentials* K_{ij} *as in* (**Pot**). *Let* $\mathbf{f}, \mathbf{g} \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ *and* $R > 0$. *Then,*

$$\|\mathbf{E}[\mathbf{f}] - \mathbf{E}[\mathbf{g}]\|_{L^\infty(B_R^1)} \leq \Upsilon\mathcal{W}_1(\mathbf{f}, \mathbf{g}).$$

Existence and uniqueness of measure solutions to the kinetic system (4) are stated and proved in the following theorem.

Theorem 3.1 *Assume the potentials K_{ij} under assumption (**Pot**), and let $\mathbf{f}_0 \in \mathcal{P}_c(\mathbb{R}^{2d})^N$. Then there exists a unique measure solution $\mathbf{f} \in \mathcal{P}_c(\mathbb{R}^{2d})^N$ to system (4) with initial condition $\mathbf{f}_0$ in the sense of Definition 3.1. In particular,*

$$\mathbf{f} \in \mathcal{C}([0, +\infty); \mathcal{P}_c(\mathbb{R}^{2d})^N), \tag{12}$$

and there exists an increasing function $R = R(T)$ such that for all $T > 0$,

$$\operatorname{supp}(\mathbf{f}) \subset B_{R(T)} \subset \mathbb{R}^d \times \mathbb{R}^d, \tag{13}$$

for all $t \in [0, T]$.

Proof Let $\mathbf{f}_0$ be such that

$$\operatorname{supp}(\mathbf{f}_0) \subset B_{R_0} \subset \mathbb{R}^d \times \mathbb{R}^d,$$

for some $R_0 > 0$. In order to prove the existence and uniqueness of the solution, we are going to use a contraction argument. In particular, we introduce the metric space

$$\mathcal{F} = \left\{ \mathbf{f} \in \mathcal{C}((0, T], \mathcal{P}_c(\mathbb{R}^{2d})^N) \,:\, \operatorname{supp}(\mathbf{f}) \subset B_R \text{ for all } t \in [0, T] \right\},$$

where $R := 2R_0$ and $T > 0$ is a fixed time we will choose later. This metric space is equipped with the distance $\mathcal{W}_1$, see Definition 2.1. In this space, we define a map as follows. For $\mathbf{f} \in \mathcal{F}$, consider $\mathbf{E}[\mathbf{f}]$ defined as in (11). Then, by Lemmas 3.6 and 3.7 and by assumption (**Pot**), we obtain that $\mathbf{E}[\mathbf{f}]$ satisfies ($\mathbf{H_1}$)-($\mathbf{H_2}$)-($\mathbf{H_3}$) and thus we can define

$$\Gamma[\mathbf{f}](t) := \mathcal{T}^t_{\mathbf{E}[\mathbf{f}]} \# \mathbf{f}_0.$$

The aim is to prove that this map is a contraction and its unique fixed point in $\mathcal{F}$ is the solution to (4). We start proving that the operator $\Gamma[\mathbf{f}]$ is well posed in the space $\mathcal{F}$. From Lemma 3.6, we have that

$$\|\mathbf{E}[\mathbf{f}]\|_{L^\infty([0,T]\times B^1_R)} \leq \Xi,$$

and from Lemma 3.2,

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{T}^t_{\mathbf{E}[\mathbf{f}]}(P) \right| \leq C_1,$$

for all $P \in B_{R_0} \subset \mathbb{R}^d \times \mathbb{R}^d$, with C_1 depending on R_0 and Ξ. For $T < R_0/C_1$, we have that $\mathcal{T}^t_{\mathbf{E}[\mathbf{f}]} \# \mathbf{f}_0$ has support contained in B_R for all $t \in [0, T]$. Then, for each $t \in [0, T]$, $\Gamma[\mathbf{f}](t) \in \mathcal{P}_c(\mathbb{R}^{2d})^N$ and the map $t \mapsto \Gamma[\mathbf{f}](t)$ is continuous by Lemma 3.4. Thus, the map $\Gamma : \mathcal{F} \to \mathcal{F}$ is well defined.

We show now that the map is a contraction, i.e., considering two functions $\mathbf{f}, \mathbf{g} \in \mathcal{F}$ and taking $\Gamma[\mathbf{f}]$ and $\Gamma[\mathbf{g}]$, we want to prove that

$$\mathcal{W}_1(\Gamma[\mathbf{f}], \Gamma[\mathbf{g}]) \leq C\mathcal{W}_1(\mathbf{f}, \mathbf{g})$$

for $0 < C < 1$ which does not depend on the functions $\mathbf{f}$ and $\mathbf{g}$. By definition of Γ, we have that

$$\mathcal{W}_1(\Gamma[\mathbf{f}], \Gamma[\mathbf{g}]) = \mathcal{W}_1(\mathcal{T}^t_{\mathbf{E}[\mathbf{f}]}\#\mathbf{f}_0, \mathcal{T}^t_{\mathbf{E}[\mathbf{g}]}\#\mathbf{f}_0).$$

Using Lemmas 3.3, 3.2 and 3.7, the above distance can be estimated as follows

$$\begin{aligned} \mathcal{W}_1(\mathcal{T}^t_{\mathbf{E}[\mathbf{f}]}\#\mathbf{f}_0, \mathcal{T}^t_{\mathbf{E}[\mathbf{g}]}\#\mathbf{f}_0) &\leq \|\mathcal{T}^t_{\mathbf{E}[\mathbf{f}]} - \mathcal{T}^t_{\mathbf{E}[\mathbf{g}]}\|_{L^\infty(\operatorname{supp}\mathbf{f}_0)} \\ &\leq C(t) \sup_{s\in[0,T]} \|\mathbf{E}[\mathbf{f}](s) - \mathbf{E}[\mathbf{g}](s)\|_{L^\infty(B^1_R)} \\ &\leq C(t)\Upsilon\mathcal{W}_1(\mathbf{f}, \mathbf{g}), \end{aligned}$$

where $C(t) = (e^{C_2 t} - 1)/\varepsilon C_2$ is the function in the statement of Lemma 3.2, with C_2 a constant depending on R and Υ. Therefore, we obtain that

$$\mathcal{W}_1(\Gamma[\mathbf{f}], \Gamma[\mathbf{g}]) \leq C(t)\Upsilon\mathcal{W}_1(\mathbf{f}, \mathbf{g}).$$

Since it holds that

$$\lim_{t\to 0} C(t) = 0,$$

we get

$$\mathcal{W}_1(\Gamma[\mathbf{f}], \Gamma[\mathbf{g}]) \leq C(T)\Upsilon\mathcal{W}_1(\mathbf{f}, \mathbf{g}).$$

We can choose T small enough so that $C(T)\Upsilon < 1$. In this way, the functional Γ is contractive and then there is a unique fixed point of Γ in $\mathcal{F}$. By construction, it is easy to see that this fixed point of Γ is a solution to (4) on $[0, T]$. Finally, since the growth of characteristic is bounded, we can construct a unique global solution satisfying (12) and (13). □

We now state $\mathcal{W}_1-$stability for solutions to system (4) that can be proved by using a triangulation argument that mixes the estimates obtained in Lemmas 3.3, 3.2 and 3.7 and Grönwall's inequality.

Proposition 3.1 *Assume that the potentials K_{ij} are under assumption (**Pot**). Let $\mathbf{f}_0, \mathbf{g}_0 \in \mathcal{P}_c(\mathbb{R}^{2d})^N$, and consider the solutions $\mathbf{f}, \mathbf{g}$ to (4) with initial conditions $\mathbf{f}_0$ and $\mathbf{g}_0$, respectively. Then, there exists an increasing function $r(t) : [0, \infty) \to \mathbb{R}^+$ with $r(0) = 1$ that depends only on the supports of $\mathbf{f}_0$ and $\mathbf{g}_0$ such that*

$$\mathcal{W}_1(\mathbf{f}(t), \mathbf{g}(t)) \leq r(t)\mathcal{W}_1(\mathbf{f}_0, \mathbf{g}_0), \tag{14}$$

for $t \geq 0$.

Theorem 3.2 *Let $T > 0$ be a positive time. Assume K_{ij} as in (**Pot**). Let $\mathbf{f}_0 \in \mathcal{C}^1(\mathbb{R}^{2d})^N \cap L^1(\mathbb{R}^{2d})^N$ with compact support. Then system (4) has a solution $\mathbf{f}^\varepsilon \in \mathcal{C}([0,T];\mathcal{C}^1(\mathbb{R}^{2d})^N)$ with initial datum $\mathbf{f}_0$.*

Sketch of Proof We provide a sketch of the proof. The details can be found in Glassey (1996) for the Vlasov–Poisson system and the Vlasov–Maxwell system. The proof is divided into three steps. One first constructs an approximating sequence $\mathbf{f}^{\varepsilon,n} \in \mathcal{C}([0,T];\mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d)^N)$ by iterations, defining $\mathbf{f}^{\varepsilon,n+1}$ to be the solution of

$$\begin{aligned}
&\partial_t f_i^{\varepsilon,n+1} + v\cdot\nabla_x f_i^{\varepsilon,n+1} - \frac{1}{\varepsilon}\nabla_v\cdot(v f_i^{\varepsilon,n+1}) + \frac{1}{\varepsilon}E_i^n\cdot\nabla_v f_i^{\varepsilon,n+1} = 0,\\
&f_i^{\varepsilon,n+1}(0,x,v) = f_{i0}(x,v),
\end{aligned}$$

as $i = 1,\dots,N$. Once we observe that the characteristics associated with the system above depend on n, but still satisfy the features in lemmas above, then it holds that $\mathbf{f}^{\varepsilon,n} \in \mathcal{C}^1([0,T];\mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d)^N)$ is uniformly bounded with respect to n, since all the constants in the estimates above depend on the support of the initial datum and the Lipschitz constant of the kernels. Next, showing that $\mathbf{f}^{\varepsilon,n}$ is a Cauchy sequence in $\mathcal{C}([0,T];\mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d)^N)$ converging to the solution to (4), we complete the proof. □

3.2 Uniform Estimates in ε

In this part, we gather some uniform in ε estimates we will use to prove the convergence of solutions to (4) toward the solution to (1) as $\varepsilon \to 0$. For this reason, we make the ε-dependence explicit, i.e., we deal with the system

$$\partial_t f_i^\varepsilon + v\cdot\nabla_x f_i^\varepsilon = \frac{1}{\varepsilon}\nabla_v\cdot\left(\left(v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon\right) f_i^\varepsilon\right), \tag{15}$$

for $i = 1,\dots,N$, equipped with initial data

$$f_i^\varepsilon(t,x,v)|_{t=0} = f_{i0}^\varepsilon(x,v) \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d), \tag{16}$$

and where

$$\rho_i^\varepsilon(t,x) = \int_{\mathbb{R}^d} f_i^\varepsilon(t,x,v)\,\mathrm{d}v.$$

Throughout this section, we assume the initial data with compact support.

Proposition 3.2 *Assume all the potentials under assumption (**Pot**). Let $\mathbf{f}^\varepsilon$ be a solution to the system (15)–(16) as proved in Theorem 3.1. Then, there exists an increasing function $R(T)$ independent on ε such that for all $T > 0$,*

$$\operatorname{supp}(\mathbf{f}^\varepsilon)(t) \subset B_{R(T)},$$

for all $t \in [0,T]$ and $\varepsilon > 0$. The function $R(T)$ depends only on the support of $\mathbf{f}_0^\varepsilon$ and Ξ.

The proof easily follows by noticing that the support of $\mathbf{f}^\varepsilon$ evolves according to the flow associated with the characteristic system (9) and by the bounds

$$\left|\sum_{j=1}^{N}\nabla K_{ij}*\rho_j^\varepsilon\right|\le\sum_{j=1}^{N}\|\nabla K_{ij}\|_{L^\infty}=:C_{i,1},$$

for $i=1,\dots,N$, as in Fetecau and Sun (2015).

3.2.1 Estimate for Smooth Solutions

We first produce uniform in ε estimates in the case of smooth solutions, namely solutions given by Theorem 3.2. In the next subsection, we will deal with uniform in ε estimates for measure solutions.

Proposition 3.3 *Assume all the potentials under assumption* (**Pot**). *Suppose that the initial datum* $\mathbf{f}_0$ *in* (16) *has a finite first moment in* v*, i.e.,* $|v|f_{i0}\in L^1(\mathbb{R}^d\times\mathbb{R}^d)$ *for all* $i=1,\dots,N$*. Let* $\mathbf{f}^\varepsilon$ *be the classical solution to* (15)*, as in Theorem* 3.2*. Then there exist some positive constants* C_i*, as* $i=1,\dots,N$*, and a function* $M(\varepsilon)$ *depending on* ε*, such that*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|v+\sum_{j=1}^{N}\nabla K_{ij}*\rho_j^\varepsilon\right|f_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}v\le C_iM(\varepsilon),\tag{17}$$

for all $t\in[0,T]$*, where* C_i *depends on* $\|(1+|v|)f_{i0}\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}$ *and* Ξ*. Moreover,*

$$\lim_{\varepsilon\downarrow 0}M(\varepsilon)=0.$$

Proof We set

$$I_i(t)=\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|v+\sum_{j=1}^{N}\nabla K_{ij}*\rho_j^\varepsilon\right|f_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}v,$$

for $i=1,\dots,N$. We want to prove that there exist C_i and $M(\varepsilon)$ as in the statement such that

$$\sup_{t\in[0,T]}I_i(t)\le C_iM(\varepsilon),$$

for a small ε. Straightforward computation shows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_i(t)=&\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\partial_t\left|v+\sum_{j=1}^{N}\nabla K_{ij}*\rho_j^\varepsilon\right|\right)f_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}v\\&+\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|v+\sum_{j=1}^{N}\nabla K_{ij}*\rho_j^\varepsilon\right|\partial_tf_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

By using system (15) and integration by parts, we get

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|v+\sum_{j=1}^N \nabla K_{ij}*\rho_j^\varepsilon\right|\partial_t f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v$$
$$=\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(v\cdot\nabla_x\left|v+\sum_{j=1}^N \nabla K_{ij}*\rho_j^\varepsilon\right|\right) f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v$$
$$-\frac{1}{\varepsilon}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|v+\sum_{j=1}^N \nabla K_{ij}*\rho_j^\varepsilon\right| f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v.$$

Therefore, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}I_i(t)=-\frac{1}{\varepsilon}I_i(t)+I_i^1(t)+I_i^2(t),$$

with

$$I_i^1(t)=\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\partial_t\left|v+\sum_{j=1}^N \nabla K_{ij}*\rho_j^\varepsilon\right|\right) f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v,$$
$$I_i^2(t)=\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(v\cdot\nabla_x\left|v+\sum_{j=1}^N \nabla K_{ij}*\rho_j^\varepsilon\right|\right) f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v,$$

for $i=1,\ldots,N$. In order to obtain our claim, we want to show that $I_i^1(t)$ and $I_i^2(t)$ are bounded linearly by $I_i(t)$ and then derive a differential inequality to bound $I_i(t)$. By setting

$$m_i:=\int_{\mathbb{R}^d} v f_i \,\mathrm{d}v,$$

integrating (15) in v, we have that

$$\partial_t\rho_i^\varepsilon+\nabla_x\cdot m_i^\varepsilon=0, \tag{18}$$

and the conservation of masses

$$\|\rho_i^\varepsilon(t)\|_{L^1(\mathbb{R}^d)}=\|f_{i0}\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)},$$

for all $t>0$ and for all $i=1,\ldots,N$. Thus, using the equation for ρ_i^ε in (18), we can preliminary estimate

$$\left|\partial_t\left|v+\sum_{j=1}^N \nabla K_{ij}*\rho_j^\varepsilon\right|\right|\le\left|\sum_{j=1}^N \nabla K_{ij}*\partial_t\rho_j^\varepsilon\right|\le\sum_{j=1}^N\left|\Delta K_{ij}*m_j^\varepsilon\right|.$$

By adding and subtracting $\left(\sum_{h=1}^N \nabla K_{ih} * \rho_h^\varepsilon\right) f_j^\varepsilon$ in the absolute value in the right-hand side of the inequality above, and using assumption (**Pot**) we get

$$
\begin{aligned}
&\sum_{j=1}^N \left| \Delta K_{ij} * \int_{\mathbb{R}^d} w f_j^\varepsilon \, dw \right| \\
&\quad \le \sum_{j=1}^N \left| \Delta K_{ij} * \left[\int_{\mathbb{R}^d} \left(w f_j^\varepsilon + \left(\sum_{h=1}^N \nabla K_{ih} * \rho_j^\varepsilon \right) f_j^\varepsilon \right) dw \right.\right. \\
&\qquad \left.\left. - \int_{\mathbb{R}^d} \left(\sum_{h=1}^N \nabla K_{ih} * \rho_h^\varepsilon \right) f_j^\varepsilon \, dw \right]\right| \\
&\quad \le \sum_{j=1}^N \left| \Delta K_{ij} * \int_{\mathbb{R}^d} \left(w + \sum_{h=1}^N \nabla K_{ih} * \rho_h^\varepsilon \right) f_j^\varepsilon \, dw \right| \\
&\qquad + \sum_{j=1}^N \left| \Delta K_{ij} * \int_{\mathbb{R}^d} \left(\sum_{h=1}^N \nabla K_{ih} * \rho_h^\varepsilon \right) f_j^\varepsilon \, dw \right| \\
&\quad \le \sum_{j=1}^N \| \Delta K_{ij} \|_{L^\infty} \int_{\mathbb{R}^d} \left| w + \sum_{h=1}^N \nabla K_{ih} * \rho_h^\varepsilon \right| f_j^\varepsilon \, dw \\
&\qquad + \sum_{j=1}^N \| \Delta K_{ij} * \rho_j^\varepsilon \|_{L^\infty} \| \sum_{h=1}^N \nabla K_{ih} * \rho_h^\varepsilon \|_{L^\infty} \\
&\quad \le \sum_{j=1}^N \| \Delta K_{ij} \|_{L^\infty} \int_{\mathbb{R}^d} \left| w + \sum_{h=1}^N \nabla K_{ih} * \rho_h^\varepsilon \right| f_j^\varepsilon \, dw \\
&\qquad + \sum_{j=1}^N \| \Delta K_{ij} \|_{L^\infty} \| \nabla K_{ij} \|_{L^\infty} \| \rho_h^\varepsilon \|_{L^1}^2
\end{aligned}
$$

Thus, integrating the above inequality in x and v we obtain

$$
\begin{aligned}
|I_i^1(t)| &\le \sum_{j=1}^N \| \Delta K_{ij} \|_{L^\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| v + \sum_{h=1}^N \nabla K_{ih} * \rho_h^\varepsilon \right| f_i^\varepsilon(x, v, t) f_j^\varepsilon(x, w, t) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}w \\
&\quad + \sum_{j=1}^N \| \Delta K_{ij} \|_{L^\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|v| + |w|) f_j^\varepsilon(x, w, t) f_i^\varepsilon(x, v, t) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}w \\
&\quad + \sum_{j=1}^N \| \Delta K_{ij} \|_{L^\infty} \| \nabla K_{ij} \|_{L^\infty} \| \rho_j^\varepsilon \|_{L^1}^2 \| \rho_i^\varepsilon \|_{L^1} \\
&\le \sum_{j=1}^N \| \Delta K_{ij} \|_{L^\infty} I_i(t) \| \rho_j^\varepsilon \|_{L^1}
\end{aligned}
$$

$$+\sum_{j=1}^{N}\|\Delta K_{ij}\|_{L^\infty}\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|v|f_i^\varepsilon(x,v,t)\,\mathrm{d}x\,\mathrm{d}v+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|w|f_j^\varepsilon(x,w,t)\,\mathrm{d}x\,\mathrm{d}w\right)$$
$$+\sum_{j=1}^{N}\|\Delta K_{ij}\|_{L^\infty}\|\nabla K_{ij}\|_{L^\infty}\|\rho_j^\varepsilon\|_{L^1}^2\|\rho_i^\varepsilon\|_{L^1}.$$

Since $f_i^\varepsilon \in \mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d)$ for all $i=1,\ldots,N$, we obtain that for each i there exist two positive constants A_i^1 and A_i^2 depending on Ξ and all $\|(1+|v|)f_{i0}\|_{L^1}$ such that

$$|I_i^1(t)|\le A_i^1 I_i(t)+A_i^2.$$

Concerning the terms I_i^2, we can estimate

$$\begin{aligned}|I_i^2(t)| &\le \sum_{j=1}^{N}\int_{\mathbb{R}^d\times\mathbb{R}^d}|v||\Delta K_{ij}*\rho_j^\varepsilon|f_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}v\\
&\le \sum_{j=1}^{N}\|\Delta K_{ij}\|_{L^\infty}\|\rho_j^\varepsilon\|_{L^1}\left[\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|v+\sum_{h=1}^{N}\nabla K_{ih}\rho_h^\varepsilon\right|f_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}v\right.\\
&\qquad\left.+\sum_{h=1}^{N}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|\nabla K_{ih}*\rho_h^\varepsilon\right|f_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}v\right]\\
&\le \sum_{j=1}^{N}\|\Delta K_{ij}\|_{L^\infty}\|\rho_j^\varepsilon\|_{L^1}\left[I_i(t)+\sum_{h=1}^{N}\|\nabla K_{ih}\|_{L^\infty}\|\rho_h^\varepsilon\|_{L^1}\|\rho_i^\varepsilon\|_{L^1}\right].\end{aligned}$$

Thus, we derive that for each i there exist two positive constants B_i^1 and B_i^2 depending on Ξ and all $\|f_{i0}\|_{L^1}$ such that

$$|I_i^2(t)|\le B_i^1 I_i(t)+B_i^2.$$

Hence, considering the estimates above, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}I_i(t)\le -\frac{1}{\varepsilon}I_i(t)+C_i^1 I_i(t)+C_i^2, \tag{19}$$

where $C_i^k=A_i^k+B_i^k$ for $i=1,\ldots,N$ and $k=1,2$. Furthermore, at time $t=0$ we get

$$I_i(0)\le \||v|f_{i0}\|_{L^1}+D_i, \tag{20}$$

as $i=1,\ldots,N$, where the positive constants D_i depend on Ξ and all $\|f_{i0}\|_{L^1}$. Combining (19) and (20) and using Grönwall's lemma, we obtain that

$$\sup_{t\in[0,T]} I_i(t)\le C_i M_i(\varepsilon),$$

where the constants C_i depend on Ξ and all $\|(1+|v|)f_{i0}\|_{L^1}$. Finally, is it enough to note that $M(\varepsilon) := \max_i\{M_i(\varepsilon)\}$ decays to 0 as $\varepsilon \to 0$. □

3.2.2 Estimate for Measure Solutions

In this section, our aim is to find an estimate as in (17) for a measure solution $\mathbf{f}$ to system (15). In order to proceed, we introduce the mollifier

$$\gamma^{(n)}(x, v) = n^{2d}\gamma^{(1)}(nx, nv) \in \mathcal{C}_c^\infty(\mathbb{R}^{2d}),$$

where

$$\mathrm{supp}(\gamma^{(1)}) \subset \overline{B_1(0)} \subset \mathbb{R}^{2d}, \qquad \gamma^{(1)} \geq 0, \qquad \iint_{\mathbb{R}^{2d}} \gamma^{(1)}(x, v)\,\mathrm{d}x\,\mathrm{d}v = 1,$$
$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|\gamma^{(1)}(x, v)\,\mathrm{d}x\,\mathrm{d}v \leq 1.$$

Now, let $\mathbf{f}_0 \in \mathcal{P}_c(\mathbb{R}^{2d})^N$ and $\varepsilon > 0$ fixed. Define

$$\mathbf{f}_0^{(n)} = \mathbf{f}_0 * \gamma^{(n)}, \tag{21}$$

i.e.,

$$f_{i0}^{(n)} = f_{i0} * \gamma^{(n)} \in \mathcal{C}^2(\mathbb{R}^{2d}),$$

for all $i = 1, \ldots, N$. The following is a classical result concerning the mollifier $\gamma^{(1)}$, see Ambrosio (2003).

Lemma 3.8 *Let* $\mathbf{f} \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ *be with* $\mathrm{supp}(\mathbf{f}) \subset B_{R_0} \subset \mathbb{R}^{2d}$. *Then*

(i) $\mathrm{supp}(\mathbf{f}^{(n)}) \subset B_{R_0+1}$ *for all* $n \geq 1$.
(ii) $\mathbf{f}^{(n)} \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ *and*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v| f_i^{(n)}(x, v)\,\mathrm{d}x\,\mathrm{d}v$$

are uniformly bounded, for all $i = 1, \ldots, N$.
(iii) $\{\mathbf{f}^{(n)}\}_{n\geq 1}$ *is a Cauchy sequence in* $\mathcal{P}_1(\mathbb{R}^{2d})^N$ *equipped with the Wasserstein distance* $\mathcal{W}_1$ *and* $\|\mathbf{f}^{(n)} - \mathbf{f}\|_{\mathcal{W}_1} \to 0$ *as* $n \to +\infty$.

Consider that the approximating sequence $\mathbf{f}^{\varepsilon,(n)}$ satisfying the system

$$\partial_t f_i^{\varepsilon,(n)} + v \cdot \nabla_x f_i^{\varepsilon,(n)} = \frac{1}{\varepsilon}\nabla_v \cdot \left(\left(v + \sum_{j=1}^N \nabla K_{ij} * \rho_i^{\varepsilon,(n)}\right) f_i^{\varepsilon,(n)}\right), \tag{22}$$

for $i = 1, \ldots, N$, equipped with initial data

$$f_i^{\varepsilon,(n)}\,|_{t=0} = f_{i0}^{(n)}(x, v),$$

with

$$\rho_i^{\varepsilon,(n)} = \int_{\mathbb{R}^d} f_i^{\varepsilon,(n)} \, \mathrm{d}v.$$

The results in Theorem 3.2, Proposition 3.2 and Proposition 3.1, together with Lemma 3.8, ensure the following approximation result.

Lemma 3.9 *Assume all the potentials under assumption (**Pot**). Let $\mathbf{f}_0 \in \mathcal{P}_c(\mathbb{R}^{2d})^N$ and $\mathbf{f}_0^{(n)}$ as in (21). Then for each $T > 0$ there exists a solution $\mathbf{f}^{\varepsilon,(n)} \in \mathcal{C}([0,T); \mathcal{C}^1(\mathbb{R}^{2d})^N)$ to (22) whose support depends only on T and Ξ and is uniformly bounded both in ε and n. Furthermore, if $\mathbf{f}^{\varepsilon} \in \mathcal{C}([0,T), \mathcal{P}_c(\mathbb{R}^{2d})^N)$ is the unique measure solution to (15) as provided in Theorem 3.1, then*

$$\mathbf{f}^{\varepsilon,(n)}(t,\cdot,\cdot) \xrightarrow{\mathcal{W}_1} \mathbf{f}^{\varepsilon}(t,\cdot,\cdot) \quad \text{in } \mathcal{P}_1(\mathbb{R}^{2d})^N,$$

uniformly in t as $n \to \infty$.

Proof Since $\mathbf{f}_0^{(n)} \in \mathcal{C}_c^2(\mathbb{R}^{2d})^N$, we can apply Theorem 3.2 and we find that there exists a smooth solution $\mathbf{f}^{\varepsilon,(n)} \in \mathcal{C}([0,t), \mathcal{C}^1(\mathbb{R}^{2d})^N)$ to (15) for every $\varepsilon > 0$ and every $n \geq 1$, with compact support. By Proposition 3.2, we have that $\operatorname{supp}(\mathbf{f}^{\varepsilon,(n)})$ is independent of ε and depends on T, Ξ and the support of $\mathbf{f}_0^{(n)}$. Since by Lemma 3.8 $\operatorname{supp}(\mathbf{f}_0^{(n)})$ is contained in a ball for all $n \geq 1$, we deduce that $\operatorname{supp}(\mathbf{f}^{\varepsilon,(n)})$ is uniformly bounded both in ε and in n for all $t \in [0,T]$. Now, let $\mathbf{f}^{\varepsilon} \in \mathcal{C}([0,t), \mathcal{P}_1(\mathbb{R}^{2d})^N)$ be the unique solution to (15) as in Theorem 3.1. By Proposition 3.1, for all $t \geq 0$,

$$\|\mathbf{f}^{\varepsilon,(n)} - \mathbf{f}^{\varepsilon}\|_{\mathcal{W}_1} \leq r(T)\|\mathbf{f}_0^{(n)} - \mathbf{f}_0\|_{\mathcal{W}_1}.$$

By Lemma 3.8, we have the assertion. □

From this result, it follows that

$$\boldsymbol{\rho}^{\varepsilon,(n)}(t,\cdot) \to \boldsymbol{\rho}^{\varepsilon}(t,\cdot) \quad \text{weakly as measures} \tag{23}$$

for each $t \in [0,T)$ as $n \to \infty$, where $\boldsymbol{\rho} = (\rho_i)_{i=1}^N$.

Lemma 3.10 *Let $\mathbf{f}^{\varepsilon}$ be the solution to (15) obtained as the limit of approximating sequences $\mathbf{f}^{\varepsilon,(n)}$ as in Lemma 3.9. Then, for all $t \geq 0$, $\nabla K_{ij} * \rho_j^{\varepsilon}$ are continuous functions in $\mathbb{R}^d$ for all $i, j = 1 \ldots, N$ and*

$$\nabla K_{ij} * \rho_j^{\varepsilon,(n)}(t,\cdot) \to \nabla K_{ij} * \rho_j^{\varepsilon}(t,\cdot)$$

strongly in $L^{\infty}_{loc}(\mathbb{R}^d)$, as $n \to \infty$.

Proof Given the regularity of $\mathbf{f}^{\varepsilon,(n)}$, i.e., $\mathbf{f}^{\varepsilon,(n)} \in \mathcal{C}([0,T); \mathcal{C}^1(\mathbb{R}^{2d})^2)$ and the assumption (**Pot**), we get the continuity of the convolutions. Moreover, we can easily estimate

$$|\nabla K_{ij} * \rho_j^{\varepsilon,(n)}| \leq \|\nabla K_{ij}\|_{L^{\infty}},$$

and for all $x_1, x_2 \in \mathbb{R}^d$, we have

$$|\nabla K_{ij} * \rho_j^{\varepsilon,(n)}(x_1) - \nabla K_{ij} * \rho_j^{\varepsilon,(n)}(x_2)| \leq \|\nabla K_{ij}\|_{L^\infty}|x_1 - x_2|,$$

for $i, j = 1, \ldots, N$. Thus, the sequences

$$\{\nabla K_{ij} * \rho_j^{\varepsilon,(n)}\}_{n\geq 1}$$

are equicontinuous and uniformly bounded. Hence, by Ascoli–Arzelà theorem, they strongly converge on a subsequence on compact sets in $\mathbb{R}^d$. Furthermore, by (23) we have that the limit functions are

$$\nabla K_{ij} * \rho_j^\varepsilon,$$

respectively. These limit functions are also continuous on $\mathbb{R}^d$ by inequalities above (using ρ_j^ε in place of $\rho_j^{\varepsilon,(n)}$). Then the assertion follows. □

Since the approximating sequence $\mathbf{f}^{\varepsilon,(n)}$ is smooth, we can apply to it Proposition 3.3 with fixed ε. In particular, with $n \geq 1$ fixed, we can say that there exist N positive constants C_i depending on Ξ and all $\|(1+|v| f_{i0}^{(n)})\|_{L^1}$ and a function $M(\varepsilon)$ depending on ε such that for ε small enough

$$\left| \iint_{\mathbb{R}^{2d}} \left(v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^{\varepsilon,(n)} \right) f_i^{\varepsilon,(n)} \, \mathrm{d}x \, \mathrm{d}v \right| \leq C_i M(\varepsilon).$$

By part (ii) in Lemma 3.8, we have that $\|(1 + |v| f_{i0}^{(n)})\|_{L^1}$ are uniformly bounded in n for all $i = 1, \ldots, N$, thus the function $M(\varepsilon)$ and the constants C_i can be chosen independent on n. Therefore, the estimates

$$\left| \iint_{\mathbb{R}^{2d}} \left(v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^{\varepsilon,(n)} \right) f_i^{\varepsilon,(n)} \, \mathrm{d}x \, \mathrm{d}v \right| \leq C_i M(\varepsilon) \tag{24}$$

hold for all $n \geq 1$ and $t \in [0, T]$, as $i = 1, \ldots, N$.

Proposition 3.4 *Assume $\varepsilon > 0$ fixed such that (24) holds and assume that assumptions in Lemma 3.9 are satisfied. Then for any $(\phi_i)_{i=1}^N \in \mathcal{C}_b(\mathbb{R}^{2d})^N$ there exist N constants $\overline{C}_i$ such that*

$$\left| \iint_{\mathbb{R}^{2d}} \phi_i(x, v) \left(v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon(x) \right) f_i^\varepsilon(x, v) \, \mathrm{d}x \, \mathrm{d}v \right| \leq \overline{C}_i M(\varepsilon)$$

hold for all $t \in [0, T]$, as $i = 1, \ldots, N$. In particular, the constants $\overline{C}_i$ are independent of ε and t, and $\overline{C}_i = \|\phi_i\|_{L^\infty} C_i$, where C_i are constants depending on all $\iint_{\mathbb{R}^{2d}} (1 + |v|) f_{i0} \, \mathrm{d}x \, \mathrm{d}v$ and Ξ.

Proof Multiplying (24) by ϕ_i, we have

$$\left| \iint_{\mathbb{R}^{2d}} \phi(x,v)\left(v + \sum_{j=1}^{N} \nabla K_{ij} * \rho_j^{\varepsilon,(n)}\right) f_i^{\varepsilon,(n)}\, \mathrm{d}x\, \mathrm{d}v \right| \le C_i \|\phi_i\|_{L^\infty} M(\varepsilon), \tag{25}$$

where C_i are constants depending on Ξ and the first moment of f_{i0} in v. Let $\Omega(T)$ be the common support of $\mathbf{f}^{\varepsilon,(n)}(t)$ for all $\varepsilon > 0$, $n \ge 1$ and $t \in [0, T]$. Then, by Lemma 3.10 and Proposition 2.1, we obtain that for each $t \in [0, T]$,

$$\begin{aligned}
&\iint_{\mathbb{R}^{2d}} \phi_i(x,v)\left(v + \sum_{j=1}^{N} \nabla K_{ij} * \rho_j^{\varepsilon,(n)}\right) f_i^{\varepsilon,(n)}\, \mathrm{d}x\mathrm{d}v \\
&= \iint_{\Omega(T)} \phi_i(x,v)\left(v + \sum_{j=1}^{N} \nabla K_{ij} * \rho_j^{\varepsilon,(n)}\right) f_i^{\varepsilon,(n)}\, \mathrm{d}x\mathrm{d}v
\end{aligned}$$

converges to

$$\iint_{\mathbb{R}^{2d}} \phi_i(x,v)\left(v + \sum_{j=1}^{N} \nabla K_{ij} * \rho_j^{\varepsilon}\right) f_i^{\varepsilon}\, \mathrm{d}x\mathrm{d}v$$

as $n \to \infty$, for all $i = 1, \dots, N$. Therefore, considering the limit as $n \to \infty$ in (25), we find the assertion. □

3.3 Small Inertia Limit

This subsection is finally devoted to the proof of Theorem 2.1. More precisely, we consider $\mathbf{f}^\varepsilon$ solution to (4), satisfying the uniform bounds as stated in Proposition 3.4 and we show that the marginals

$$\rho_i^\varepsilon(t,x) = \int_{\mathbb{R}^d} f_i^\varepsilon(t,x,v)\, \mathrm{d}v$$

converge to a solution $\boldsymbol{\rho} = (\rho_i)_{i=1}^N$ to the first-order system

$$\partial_t \rho_i - \nabla \cdot \left(\left(\sum_{j=1}^{N} \nabla K_{ij} * \rho_j \right) \rho_i \right) = 0, \tag{26}$$

for $i = 1, \dots, N$, equipped with initial data

$$\rho_i(t,x)\mid_{t=0} = \rho_{i0}(x).$$

Proof of Theorem 2.1 We start noting that, for each $\phi_i \in \mathcal{C}_c^1([0,T);\mathcal{C}_b^1(\mathbb{R}^{2d}))$, the measure solution $\mathbf{f}^\varepsilon$ satisfies

$$\int_0^T \iint_{\mathbb{R}^{2d}} \partial_t \phi_i f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t + \iint_{\mathbb{R}^{2d}} \phi_i(0) f_{i0}\,\mathrm{d}x\,\mathrm{d}v + \int_0^T \iint_{\mathbb{R}^{2d}} \nabla_x \phi_i \cdot v f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t$$
$$- \frac{1}{\varepsilon}\int_0^T \iint_{\mathbb{R}^{2d}} \nabla_v \phi_i \cdot \left(v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right) f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t = 0, \tag{27}$$

for all $i = 1,\dots,N$. Consider $\psi_i \in \mathcal{C}_c^1(0,T)$, and $\chi_i \in \mathcal{C}_b^1(\mathbb{R}^d)$, as $i = 1,\dots,N$, and define

$$\phi_i(t,x,v) = \psi_i(t)\chi_i(x). \tag{28}$$

Using the test functions defined in (28) in system (27), we have

$$\int_0^T \psi_i'(t) \int_{\mathbb{R}^d} \chi_i(x)\rho_i^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \psi_i(t) \iint_{\mathbb{R}^{2d}} \nabla_x \chi_i(x) \cdot v f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t.$$

Set

$$\xi_i(t) := \int_{\mathbb{R}^d} \chi_i(x)\rho_i^\varepsilon(t,x)\,\mathrm{d}x.$$

Thus, it follows

$$\int_0^T \psi_i'(t)\xi_i(t)\,\mathrm{d}t = -\int_0^T \psi_i(t) \iint_{\mathbb{R}^{2d}} \nabla_x \chi_i(x) \cdot v f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t,$$

for any $\psi_i \in \mathcal{C}_c^1(0,T)$. Therefore, we deduce that the weak derivative of ξ_i is

$$\xi_i'(t) = \iint_{\mathbb{R}^{2d}} \nabla_x \chi_i \cdot v f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v \in L^\infty(0,T).$$

Let $\Omega(T)$ be the common support of $\mathbf{f}^\varepsilon$ for every $\varepsilon > 0$ and for all $t \in [0,T]$. By Theorem 3.1, $\mathbf{f}^\varepsilon$ is uniformly supported on $\Omega(T)$, thus

$$\|\xi_i\|_{W^{1,\infty}(0,T)} \le C_i(T)\|\chi_i\|_{\mathcal{C}_b^1(\mathbb{R}^d)}, \tag{29}$$

where C_i depend on T and are independent of ε. Since $\xi_i(t)$ are uniformly bounded in $W^{1,\infty}(0,T)$, as $i = 1,\dots,N$, by Ascoli–Arzelà theorem there exist a subsequence ε_k and a function $\mu_i(t) \in \mathcal{C}([0,T))$ such that

$$\int_{\mathbb{R}^d} \chi_i(x)\rho_i^{\varepsilon_k}(x,t)\,\mathrm{d}x \to \mu_i(t) \tag{30}$$

uniformly on $[0,T)$ as $\varepsilon_k \to 0$. Furthermore, Proposition 3.2 ensures that the support of $\mathbf{f}^\varepsilon$ is uniformly bounded in ε; then, the sequence $\boldsymbol{\rho}^\varepsilon(t,\cdot)$ is tight. By Prokhorov's Theorem, for each $t \in [0,T)$, $\boldsymbol{\rho}^\varepsilon(t,\cdot)$ converges weakly-*, up to a subsequence, to $\boldsymbol{\rho}(t,\cdot) \in \mathcal{P}(\mathbb{R}^d)^N$. By Proposition 2.1, we have that this implies convergence in $\mathcal{P}_1(\mathbb{R}^d)^N$ with respect to $\mathcal{W}_1$-distance. Hence, for each $t > 0$, there exists a subsequence of $\boldsymbol{\rho}^{\varepsilon_k}$ denoted by $\boldsymbol{\rho}^{\varepsilon_{k_n}}$, where k_n may depend on time, such that

$$\boldsymbol{\rho}^{\varepsilon_{k_n}}(t,\cdot) \xrightarrow{\mathcal{W}_1} \boldsymbol{\rho}(t,\cdot) \quad \text{in } \mathcal{P}_1(\mathbb{R}^d)^N$$

as $\varepsilon_{k_n} \to 0$. It follows that for each $t \in [0,T)$ and all $\chi_i \in \mathcal{C}_b^1(\mathbb{R}^d)$ we get

$$\int_{\mathbb{R}^d} \rho_i^{\varepsilon_{k_n}}(t,x)\chi_i(x)\,\mathrm{d}x \to \int_{\mathbb{R}^d} \rho_i(t,x)\chi_i(x)\,\mathrm{d}x \tag{31}$$

as $\varepsilon_{k_n} \to 0$. The limit $\mu_i(t)$ in (30) is unique at each $t \in [0,T)$. Combining this with (31), we deduce that the sequence $\boldsymbol{\rho}_{\varepsilon_k}(t,\cdot)$, with ε_k independent of time, and $\boldsymbol{\rho}(t,\cdot) \in \mathcal{P}_1(\mathbb{R}^d)^N$ satisfy

$$\int_{\mathbb{R}^d} \chi_i(x)\rho_i^{\varepsilon_k}(t,x)\,\mathrm{d}x \to \int_{\mathbb{R}^d} \chi_i(x)\rho_i(t,x)\,\mathrm{d}x \tag{32}$$

uniformly on $[0,T)$ as $\varepsilon_k \to 0$, for any $\chi_i \in \mathcal{C}_b^1(\mathbb{R}^d)$. Moreover,

$$\boldsymbol{\rho}^{\varepsilon_k}(t,\cdot) \xrightarrow{\mathcal{W}_1} \boldsymbol{\rho}(t,\cdot) \quad \text{in } \mathcal{P}_1(\mathbb{R}^d)^N \tag{33}$$

as $\varepsilon_k \to 0$. Now we want to prove that in (32) we can consider test functions χ_i depending also on t. In particular, taking $\zeta_i(t,x) \in \mathcal{C}_c([0,T);\mathcal{C}_b^1(\mathbb{R}^d))$ we have that

$$\int_{\mathbb{R}^d} \zeta_i(t,x)\rho_i^{\varepsilon_k}(t,x)\,\mathrm{d}x$$

are equicontinuous on $[0,T)$. Indeed, considering $s,t \in [0,T)$,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \zeta_i(t,x)\rho_i^{\varepsilon_k}(t,x)\,\mathrm{d}x - \int_{\mathbb{R}^d} \zeta_i(s,x)\rho_i^{\varepsilon_k}(s,x)\,\mathrm{d}x \right| \\
&\quad \le \int_{\mathbb{R}^d} |\zeta_i(t,x) - \zeta_i(s,x)|\rho_i^{\varepsilon_k}(t,x)\,\mathrm{d}x + \left| \int_{\mathbb{R}^d} \zeta_i(s,x)[\rho_i^{\varepsilon_k}(t,x) - \rho_i^{\varepsilon_k}(s,x)]\,\mathrm{d}x \right| \\
&\quad \le \sup_{x\in\mathbb{R}^d} |\zeta_i(t,x) - \zeta_i(s,x)| + C_i(T) \sup_{t\in(0,T)} \|\zeta_i\|_{\mathcal{C}_b^1(\mathbb{R}^d)} |t-s|.
\end{aligned}$$

Since ζ_i is uniformly continuous on $[0,T) \times \mathbb{R}^d$, then

$$\sup_{x\in\mathbb{R}^d} |\zeta_i(t,x) - \zeta_i(s,x)| \to 0 \quad \text{as } |t-s| \to 0,$$

and we get equicontinuity. Thus, up to a subsequence,

$$\int_{\mathbb{R}^d} \zeta_i(t,x)\rho_i^{\varepsilon_k}(t,x)\,\mathrm{d}x \to \int_{\mathbb{R}^d} \zeta_i(t,x)\rho_i(t,x)\,\mathrm{d}x \tag{34}$$

uniformly on $[0, T)$ as $\varepsilon_k \to 0$, for any test functions $\zeta_i \in \mathcal{C}_c([0, T); \mathcal{C}_b^1(\mathbb{R}^d))$. Now, set

$$\Omega_1(T) := \{x \ : \ (x, v) \in \Omega(T)\},$$

that is a ball in $\mathbb{R}^d$, uniform in v. We can deduce that Ω_1 is bounded and both $\mathrm{supp}(\boldsymbol{\rho})$ and $\mathrm{supp}(\boldsymbol{\rho}^{\varepsilon_k})$ are in $\Omega_1(T)$ for all $t \in [0, T]$. Consider $\Psi_i \in \mathcal{C}_c^1([0, T); \mathcal{C}_b^1(\mathbb{R}^d))$ and let $\phi_i(x, v, t) = \Psi_i(x, t)$ in (27), as $i = 1, \dots, N$. Hence,

$$\int_0^T \int_{\mathbb{R}^d} \partial_t \Psi_i \rho_i^{\varepsilon_k} \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \iint_{\mathbb{R}^{2d}} \nabla_x \Psi_i \cdot v f_i^{\varepsilon_k} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int_{\mathbb{R}^d} \Psi_i(0) \rho_{i0}(x) \, \mathrm{d}x = 0. \tag{35}$$

Regarding the first integral in (35), by (34) we have that

$$\int_0^T \int_{\mathbb{R}^d} \partial_t \Psi_i \rho_i^{\varepsilon_k} \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_{\mathbb{R}^d} \partial_t \Psi_i \rho_i \, \mathrm{d}x \, \mathrm{d}t,$$

as $\varepsilon_k \to 0$. Concerning the integrand of the second term in (35), it can be rewritten as

$$\begin{aligned} \iint_{\mathbb{R}^{2d}} \nabla_x \Psi_i \cdot v f_i^{\varepsilon_k} \, \mathrm{d}x \, \mathrm{d}v &= \iint_{\mathbb{R}^{2d}} \nabla_x \Psi_i \cdot \left(v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^{\varepsilon_k} \right) f_i^{\varepsilon_k} \, \mathrm{d}x \, \mathrm{d}v \\ &\quad - \iint_{\mathbb{R}^{2d}} \nabla_x \Psi_i \cdot \left(\sum_{j=1}^N \nabla K_{ij} * \rho_j^{\varepsilon_k} \right) f_i^{\varepsilon_k} \, \mathrm{d}x \, \mathrm{d}v. \end{aligned}$$

By Proposition 3.4, we have that

$$\iint_{\mathbb{R}^{2d}} \nabla_x \Psi_i \cdot \left(v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^{\varepsilon_k} \right) f_i^{\varepsilon_k} \, \mathrm{d}x \, \mathrm{d}v \to 0, \tag{36}$$

as $\varepsilon_k \to 0$, uniformly in t. The families $\{\nabla K_{ij} * \rho_j^{\varepsilon_k}(t, \cdot)\}$ are bounded in $W^{1,\infty}(\mathbb{R}^d)$ for all $t \in [0, T)$. In particular,

$$\|\nabla K_{ij} * \rho_j^{\varepsilon_k}(t, \cdot)\|_{W^{1,\infty}(\mathbb{R}^d)} \le \|\nabla K_{ij}\|_{W^{1,\infty}(\mathbb{R}^d)}.$$

Now, we want to prove that $\{\nabla K_{ij} * \rho_j^{\varepsilon_k}\}$ are equicontinuous in t. In order to use inequalities in (29) with the kernels in places of χ_i, we should mollify K_{ij}. Let

$$K_{ij}^{(n)} = K_{ij} * \gamma^{(n)},$$

where $\gamma^{(n)}$ is the mollifier defined in Sect. 3.2.2. It follows that

$$\nabla K_{ij}^{(n)} = \nabla K_{ij} * \gamma^{(n)},$$

; thus, we have

$$\|\nabla K_{ij}^{(n)}\|_{\mathcal{C}_b^1} \le \|\nabla K_{ij}\|_{W^{1,\infty}},$$

for all $n \ge 1$. Now, considering the mollified interaction kernels acting on the i-th species in estimates (29) in places of χ_i, we get

$$\sup_x \|\sum_{j=1}^N \nabla K_{ij}^{(n)} * \rho_j^{\varepsilon_k}\|_{W^{1,\infty}(0,T)} \le C_i(T) \sum_{j=1}^N \|\nabla K_{ij}^{(n)}\|_{\mathcal{C}_b^1} \le C_i(T) \sum_{j=1}^N \|\nabla K_{ij}\|_{W^{1,\infty}}.$$

Furthermore,

$$\left\| \sum_{j=1}^N \nabla K_{ij}^{(n)} * \rho_j^{\varepsilon_k} \right\|_{W^{1,\infty}(\mathbb{R}^d)} \le \sum_{j=1}^N \|\nabla K_{ij}^{(n)}\|_{W^{1,\infty}} \le \sum_{j=1}^N \|\nabla K_{ij}\|_{W^{1,\infty}};$$

thus, we find that

$$\left\| \sum_{j=1}^N \nabla K_{ij}^{(n)} * \rho_j^{\varepsilon_k} \right\|_{W^{1,\infty}(\mathbb{R}^d \times (0,T))} \le (1 + C_i(T)) \sum_{j=1}^N \|\nabla K_{ij}\|_{W^{1,\infty}}.$$

Therefore, for all $x, y \in \mathbb{R}^d$ and $s, t \in [0, T)$, we get

$$\begin{aligned} &\left| \sum_{j=1}^N \left[\nabla K_{ij}^{(n)} * \rho_j^{\varepsilon_k}(t, x) - \nabla K_{ij}^{(n)} * \rho_j^{\varepsilon_k}(s, y)\right] \right| \\ &\qquad \le (C_i(T) + 1)\left(\sum_{j=1}^N \|\nabla K_{ij}\|_{W^{1,\infty}} \right)\big(|t - s| + |x - y|\big). \end{aligned} \tag{37}$$

Since ∇K_{ij} are continuous, by Lemma 3.10 we get

$$\nabla K_{ij}^{(n)} \to \nabla K_{ij},$$

uniformly on compact sets in $\mathbb{R}^d$. Since

$$\left| \sum_{j=1}^N \left[\nabla K_{ij}^{(n)} * \rho_j^{\varepsilon_k}(t, x) - \nabla K_{ij} * \rho_j^{\varepsilon_k}(t, x)\right] \right| \le \sup_x \left[\left| \sum_{j=1}^N \nabla K_{ij}^{(n)}(x) - \nabla K_{ij}(x) \right| \right],$$

we have that for any compact set $A \subset \mathbb{R}^d$,

$$\sum_{j=1}^N \nabla K_{ij}^{(n)} * \rho_j^{\varepsilon_k}(t, x) \xrightarrow{n\to\infty} \sum_{j=1}^N \nabla K_{ij} * \rho_j^{\varepsilon_k}(t, x),$$

uniformly for $t \in [0, T)$, for $x \in A$, $k \in \mathbb{N}$. Therefore, considering the limit as $n \to \infty$ in (37) on a compact set $A \subset \mathbb{R}^d$, we get

$$\left| \sum_{j=1}^{N} \left[\nabla K_{ij} * \rho_j^{\varepsilon_k}(t, x) - \nabla K_{ij} * \rho_j^{\varepsilon_k}(s, y) \right] \right| \\ \leq (C(T) + 1) \left(\sum_{j=1}^{N} \| \nabla K_{ij} \|_{W^{1,\infty}} \right) \big(|t - s| + |x - y| \big).$$

Thus, by Ascoli–Arzelà theorem, there exist N subsequences still denoted by $\rho_i^{\varepsilon_k}$, as $i = 1, \dots, N$, such that

$$\sum_{j=1}^{N} \nabla K_{ij} * \rho_j^{\varepsilon_k} \to \sum_{j=1}^{N} \nabla K_{ij} * \rho_j,$$

as $\varepsilon_k \to 0$, strongly in $L^\infty([0, T) \times A)$, with $A \subset \mathbb{R}^d$ compact set. Hence, for every $t \in [0, T)$,

$$\left| \int_{\mathbb{R}^d} \nabla \Psi_i \cdot \left[\sum_{j=1}^{N} (\nabla K_{ij} * \rho_j^{\varepsilon_k}) \rho_i^{\varepsilon_k} - \sum_{j=1}^{N} (\nabla K_{ij} * \rho_j) \rho_i \right] \mathrm{d}x \right| \\ \leq \sum_{j=1}^{N} \| \nabla K_{ij} * \rho_j^{\varepsilon_k} - \nabla K_{ij} * \rho_j \|_{L^\infty(\Omega_1(T))} \| \nabla \Psi_i \|_{L^\infty} \\ + \sum_{j=1}^{N} \left| \int_{\Omega_1(T)} \nabla \Psi_i \cdot (\nabla K_{ij} * \rho_j)(\rho_i^{\varepsilon_k} - \rho_i) \, \mathrm{d}x \right|,$$

and the first term goes to zero as $\varepsilon_k \to 0$ uniformly on $[0, T)$ and the second integral vanishes as $\varepsilon_k \to 0$ by (33). Combining this with (36), we obtain that, for each $t \in (0, T)$,

$$\iint_{\mathbb{R}^{2d}} \nabla \Psi_i \cdot v f_i^{\varepsilon_k} \, \mathrm{d}x \, \mathrm{d}v \to - \int_{\mathbb{R}^d} \nabla \Psi_i \cdot \left[\sum_{j=1}^{N} (\nabla K_{ij} * \rho_j) \rho_i \right] \mathrm{d}x$$

as $\varepsilon_k \to 0$. Finally, define

$$\Omega_2(T) = \{ v \in \mathbb{R}^d \, : \, (x, v) \in \Omega(T) \}.$$

We have that $\Omega_2(T)$ is bounded for all $t \in (0, T)$ and the following uniform estimate holds:

$$\left| \iint_{\Omega_1(T) \times \mathbb{R}^d} \nabla \Psi_i \cdot v f_i^{\varepsilon_k} \, \mathrm{d}x \, \mathrm{d}v \right| \leq D_i \| \nabla \Psi_i \|_{L^\infty(\mathbb{R}^d)},$$

where the constant D_i depends only on $\Omega_2(T)$. This implies, by Lebesgue's dominated convergence theorem, that

$$\int_0^T \iint_{\mathbb{R}^{2d}} \nabla\Psi_i \cdot v f_i^{\varepsilon_k} \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \to -\int_0^T \int_{\mathbb{R}^d} \nabla\Psi_i \cdot \left[\sum_{j=1}^N (\nabla K_{ij} * \rho_j)\rho_i\right] \mathrm{d}x\,\mathrm{d}t$$

as $\varepsilon_k \to 0$. Thus, the limiting N-tuple of measures $\boldsymbol{\rho} \in \mathcal{C}([0,T); \mathcal{P}(\mathbb{R})^N)$ is a solution to system (26) in the weak sense. □

We now give two corollaries concerning the uniqueness of solutions to system (1).

Corollary 3.1 *Assume that the assumptions in Theorem 2.1 and Proposition 3.1 hold. Then, the N-tuple $\boldsymbol{\rho} \in \mathcal{C}([0,T); \mathcal{P}_1(\mathbb{R}^d)^N)$ obtained in Theorem 2.1 is the unique solution to system (1).*

Proof The proof follows by Proposition 3.1. Indeed, if we assume that there are two solutions starting from the same initial datum, by (14) we have the statement. □

Corollary 3.2 *Assume that assumptions in Theorem 2.1 hold. Moreover, assume that the cross-interaction kernels are equal, i.e., $H{:=}K_{ij}$, for all $i \neq j$. Then the solution to system (1) obtained in Theorem 2.1 is unique.*

Proof Since $\boldsymbol{\rho} \in \mathcal{C}([0,T), \mathcal{P}_1(\mathbb{R}^d)^N)$ is a weak solution to (26), by Fetecau and Sun (2015, Theorem 5.1) and the references therein, we can say that $\boldsymbol{\rho}$ is the push-forward of $\boldsymbol{\rho}_0$ via the flow $\mathcal{T}^t_{\mathbf{E}[\mathbf{f}]}$ where $\mathbf{E}[\mathbf{f}] = (E_i[\mathbf{f}])_{i=1}^N$ with

$$E_i[\mathbf{f}] = -\sum_{j=1}^N \nabla K_{ij} * \rho_j \in L^\infty([0,T) \times \mathbb{R}^d),$$

that is

$$\boldsymbol{\rho} = \mathcal{T}^t_{\mathbf{E}[\mathbf{f}]} \# \boldsymbol{\rho}_0.$$

Furthermore, $\boldsymbol{\rho}(t,\cdot)$ has compact support and it is narrowly continuous in time, since we get that $\boldsymbol{\rho}(t,\cdot) \in \mathcal{C}([0,T); \mathcal{P}_1(\mathbb{R}^d)^N)$ where the continuity is in the $\mathcal{W}_1$ metric (see Proposition 2.1). Then $\boldsymbol{\rho}$ is the unique solution to (26) in the mass transportation sense. □

4 Singular Interaction Potentials

In this section, we investigate the case of singular self-interaction potentials and smooth cross-potentials and provide the details of the proof of Theorem 2.2. For this, we first discuss the existence of solutions to the coupled kinetic and first-order macroscopic equations. We recall the kinetic and macroscopic order systems:

$$\partial_t f_i + v \cdot \nabla_x f_i = \frac{1}{\varepsilon}\nabla_v \cdot (v f_i) + \frac{1}{\varepsilon}\left(\sum_{j=1}^N \nabla K_{ij} * \rho_j\right) \cdot \nabla_v f_i, \qquad (38)$$

for $i = 1, \ldots, N$, where $\rho_i(t, x)$ is the macroscopic population density of the i-th species, i.e.,

$$\rho_i(t, x) = \int_{\mathbb{R}^d} f_i(t, x, v)\,\mathrm{d}v$$

and

$$\begin{cases} \partial_t \rho_i = \nabla \cdot (\rho_i u_i), \\ u_i = \sum_{j=1}^N \nabla K_{ij} * \rho_j, \end{cases} \tag{39}$$

for $i = 1, \ldots, N$. Here the cross-potentials K_{ij}, $i \neq j$, are given as in (**Pot**) and singular self-potentials K_{ii} are of the form

$$K_{ii}(x) = \frac{C_i}{|x|^{\alpha_i}}, \tag{40}$$

with $\alpha_i \in (0, d)$ and some positive constants C_i.

In the following two subsections, we establish the existence theory for the systems (38) and (39) satisfying required regularity conditions stated in Theorem 2.2, respectively. As mentioned in Remark 2.2, due to some technical difficulties, we construct the global/local-in-time solutions to the systems (38) and (39) in a little more restrictive setting. Precisely, the global-in-time existence of weak solutions to the kinetic system (38) is obtained for $\alpha_i \in (0, d-1]$ under the assumption that

$$K_{ij} \in \mathcal{C}^2(\mathbb{R}^d), \quad \nabla K_{ij} \in W^{1,\infty}(\mathbb{R}^d)$$

for $i \neq j$ and $\alpha_i \in (0, d-1]$. The local-in-time existence and uniqueness of classical solutions to (39) are constructed under the assumption that

$$K_{ij} \in \mathcal{C}^2(\mathbb{R}^d), \quad \nabla K_{ij} \in W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d)$$

for $i \neq j$ and $\alpha_i \in (0, d)$.

4.1 Existence for Solution to the Kinetic System

In this subsection, motivated from Choi and Jeong (2023), we investigate the global-in-time existence of weak solutions to the kinetic system (38) when $\alpha_i \in (0, d-1]$.

We start by considering a regularized version of the system (38). For this purpose, we perturb the self-potentials and consider the following system

$$\partial_t f_i^\delta + v \cdot \nabla_x f_i^\delta = \frac{1}{\varepsilon}\nabla_v \cdot (v f_i^\delta) + \frac{1}{\varepsilon}\nabla_v \cdot \left(\left(\sum_{j=1}^N \nabla K_{ij}^\delta * \rho_j^\delta\right) f_i^\delta\right), \tag{41}$$

for $i = 1, \ldots, N$, with

$$K_{ii}^\delta(x) := \frac{C_i}{|x|^{\alpha_i} + \delta},$$

and

$$\rho_i^\delta(t,x):=\int_{\mathbb{R}^d} f_i^\delta(t,x,v)\,\mathrm{d}v.$$

In system (41), we set $K_{ij}^\delta:=K_{ij}$, for $i\neq j$, in order to keep the notation to a minimum. Notice that the global-in-time existence and uniqueness of a weak solution to the regularized system (41) follow by the results developed in Sect. 3, since the force fields $\nabla K_{ij}^\delta * \rho_j^\delta$ are bounded and Lipschitz continuous. Throughout this subsection, we assume $\alpha_i \in (0, d-1]$.

4.1.1 Uniform in δ Estimates

In this part, we gather some uniform in δ estimates that we will apply for proving the existence of solutions to system (38). Let us begin with L^∞ bound estimates.

Lemma 4.1 *Let $T>0$ and $\mathbf{f}^\delta:=(f_1^\delta,\dots,f_N^\delta)$ be the weak solution to* (41) *on the interval $[0,T]$ in the sense of Definition 3.1. Then we have*

$$\sup_{0\le t\le T}\|f_i^\delta(\cdot,\cdot,t)\|_{L^p}\le \|f_{i0}^\delta\|_{L^p}e^{\frac{d}{\varepsilon}(1-\frac{1}{p})T},$$

for $p\in[1,+\infty)$, and

$$\sup_{0\le t\le T}\|f_i^\delta(\cdot,\cdot,t)\|_{L^\infty}\le \|f_{i0}^\delta\|_{L^\infty}e^{\frac{d}{\varepsilon}T}.$$

Proof By integrating by parts with respect to x and v, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\mathbb{R}^d}(f_i^\delta)^p\,\mathrm{d}x\,\mathrm{d}v &= -\frac{1}{\varepsilon}p(p-1)\iint_{\mathbb{R}^d\times\mathbb{R}^d}(f_i^\delta)^{p-2}\nabla_x f_i^\delta\cdot v f_i^\delta\,\mathrm{d}x\,\mathrm{d}v\\ &-\frac{1}{\varepsilon}p(p-1)\iint_{\mathbb{R}^d\times\mathbb{R}^d}(f_i^\delta)^{p-2}\nabla_v f_i^\delta\cdot\left(\sum_{j=1}^N K_{ij}^\delta*\rho_j^\delta\right)f_i^\delta\,\mathrm{d}x\,\mathrm{d}v\\ &+p(p-1)\iint_{\mathbb{R}^d\times\mathbb{R}^d}(f_i^\delta)^{p-2}\nabla_x f_i^\delta\cdot v f_i^\delta\,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\mathbb{R}^d}(f_i^\delta)^p\,\mathrm{d}x\,\mathrm{d}v = d\frac{1}{\varepsilon}(p-1)\iint_{\mathbb{R}^d\times\mathbb{R}^d}(f_i^\delta)^p\,\mathrm{d}x\,\mathrm{d}v,$$

for $p\in[1,+\infty)$. Therefore, by Grönwall's lemma we have

$$\|f_i^\delta(\cdot,\cdot,t)\|_{L^p}^p=\|f_{i0}^\delta\|_{L^p}^p e^{\frac{d}{\varepsilon}(p-1)t}.$$

Then, it follows that

$$\sup_{0\le t\le T}\|f_i^\delta(\cdot,\cdot,t)\|_{L^p}\le \|f_{i0}^\delta\|_{L^p}e^{\frac{d}{\varepsilon}(1-\frac{1}{p})T},$$

for $p \in [1, +\infty)$. Sending $p \to +\infty$ in the previous line, we obtain that

$$\sup_{0\le t\le T} \|f_i^\delta(\cdot,\cdot,t)\|_{L^\infty} \le \|f_{i0}^\delta\|_{L^\infty} e^{\frac{d}{\varepsilon}T},$$

that concludes the proof. □

Now we state a lemma that points out the relationship between the local density and the kinetic energy (cf. Golse and Saint-Raymond 1999, Lemma 3.1) that we will use to estimate the interaction energy. Notice that in the next result we consider generic functions and we do not work along the solutions to system (41).

The proofs of the following two lemmas are similar to the ones in Choi and Jeong (2023, Lemma 2.2) and Choi and Jeong (2023, Lemma 2.3), thus we omit the details.

Lemma 4.2 *Assume that $f_i \in L^1_+ \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $|v|^2 f_i \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$, for $i = 1, \dots, N$. Then, there exists a positive constant C such that*

$$\|\rho_i\|_{L^{\frac{d+2}{d}}} \le C\|f_i\|_{L^\infty}^{\frac{2}{d+2}} \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 f_i \,\mathrm{d}x\,\mathrm{d}v\right)^{\frac{d}{d+2}}.$$

In particular, we find that

$$\|\rho_i\|_{L^p} \le C\|f_i\|_{L^\infty}^{\frac{2}{d+2}\beta} \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 f_i \,\mathrm{d}x\,\mathrm{d}v\right)^{\frac{d}{d+2}\beta} \|\rho_i\|_{L^1}^{1-\beta},$$

for all $p \in [1, \frac{d+2}{d}]$, with $\rho_i = \int_{\mathbb{R}^d} f_i \,\mathrm{d}v$ and $\beta = \frac{d+2}{2}(1 - \frac{1}{p})$.

Let us now provide a bound estimate on the interaction energy.

Lemma 4.3 *Let $T > 0$ and $\mathbf{f}^\delta$ be the weak solution to (41) on the interval $[0, T]$. Then*

$$\left|\iint_{\mathbb{R}^d\times\mathbb{R}^d} K_{ii}^\delta(x-y)\rho_i^\delta(x)\rho_i^\delta(y)\,\mathrm{d}x\,\mathrm{d}y\right| \le C_i\|\rho_{i0}\|_{L^1}^{2-\frac{5}{2d}\alpha_i} \|\rho_i^\delta\|_{L^{\frac{d+2}{d}}}^{\frac{5}{2d}\alpha_i},$$

where $C_i > 0$ is independent of δ.

Proof We recall the classical Hardy–Littlewood–Sobolev inequality, that is,

$$\left|\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mu(x)|x-y|^{-\lambda}\nu(y)\,\mathrm{d}x\,\mathrm{d}y\right| \le C_{p,\lambda,d}\|\mu\|_{L^p}\|\nu\|_{L^q},$$

for $\mu \in L^p(\mathbb{R}^d)$, $\nu \in L^q(\mathbb{R}^d)$, $1 < p, q < \infty$, $1/p + 1/q + \lambda/d = 2$, and $0 < \lambda < d$. By L^p-interpolation we know that for $1 \le p, q \le \gamma$,

$$\|\rho_i\|_{L^p} \le \|\rho_i\|_{L^1}^{1-a}\|\rho_i\|_{L^\gamma}^a, \qquad \frac{1}{p} = 1 - a + \frac{a}{\gamma},$$

and

$$\|\rho_i\|_{L^q} \le \|\rho_i\|_{L^1}^{1-b}\|\rho_i\|_{L^\gamma}^{b}, \qquad \frac{1}{q} = 1 - b + \frac{b}{\gamma}.$$

Thus

$$\|\rho_i\|_{L^p}\|\rho_i\|_{L^q} \le \|\rho_i\|_{L^1}^{2-(a+b)}\|\rho_i\|_{L^\gamma}^{a+b}.$$

If $1/p + 1/q + \lambda/d = 2$, then

$$a + b = \frac{\gamma}{\gamma - 1}\frac{\lambda}{d}.$$

If we take $\gamma = \frac{d+2}{d}$ and $\lambda = \alpha_i$, we obtain

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d} K_{ii}^\delta(x-y)\rho_i^\delta(x)\rho_i^\delta(y)\,\mathrm{d}x\,\mathrm{d}y &\le \iint_{\mathbb{R}^d\times\mathbb{R}^d} K_{ii}(x-y)\rho_i^\delta(x)\rho_i^\delta(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\le C_i\|\rho_i^\delta\|_{L^p}\|\rho_i^\delta\|_{L^q} \\ &\le C_i\|\rho_i^\delta\|_{L^1}^{2-\frac{5}{2d}\alpha_i}\|\rho_i^\delta\|_{L^{\frac{d+2}{d}}}^{\frac{5}{2d}\alpha_i} \\ &\le C_i\|\rho_{i0}\|_{L^1}^{2-\frac{5}{2d}\alpha_i}\|\rho_i^\delta\|_{L^{\frac{d+2}{d}}}^{\frac{5}{2d}\alpha_i},\end{aligned}$$

with $C_i > 0$ independent of δ. □

Next we prove a uniform in δ estimate on the second moments of the weak solution $\mathbf{f}^\delta$ to system (41).

Proposition 4.1 *Let $T > 0$ and $\mathbf{f}^\delta$ be the weak solution to system (41) on the interval $[0, T]$. Assume that*

$$\int_{\mathbb{R}^d} \rho_{i0} K_{ii} * \rho_{i0}\,\mathrm{d}x < \infty.$$

Then the following estimate on the second moment holds:

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{|x|^2}{2} + \frac{|v|^2}{2}\right) f_i^\delta\,\mathrm{d}x\,\mathrm{d}v + \frac{1}{\varepsilon}\int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 f_i^\delta\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s \le C,$$

for all $t \in [0, T]$ and for some $C > 0$ independent of δ.

Proof A direct computation gives that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 f_i^\delta\,\mathrm{d}x\,\mathrm{d}v\right) \\ &\quad = -\frac{1}{\varepsilon}\sum_{\substack{j=1\\ j\ne i}}^N \iint_{\mathbb{R}^d\times\mathbb{R}^d}(\nabla K_{ij}^\delta * \rho_j^\delta)\cdot v f_i^\delta\,\mathrm{d}x\,\mathrm{d}v - \frac{1}{\varepsilon}\iint_{\mathbb{R}^d\times\mathbb{R}^d}(\nabla K_{ii}^\delta * \rho_i^\delta)\cdot v f_i^\delta\,\mathrm{d}x\,\mathrm{d}v \\ &\qquad - \frac{1}{\varepsilon}\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 f_i^\delta\,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

For $i \neq j$, we have that $|\nabla K_{ij} * \rho_j| \leq \|\nabla K_{ij}\|_{L^\infty}$, thus

$$\left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\nabla K_{ij}^\delta * \rho_j^\delta) \cdot v f_i^\delta \, \mathrm{d}x \, \mathrm{d}v \right| \leq \|\nabla K_{ij}^\delta\|_{L^\infty} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v| f_i^\delta \, \mathrm{d}x \, \mathrm{d}v.$$

If, instead, $i = j$, by using (18) in Proposition 3.3, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} K_{ii}^\delta (x-y) \rho_i^\delta(x) \rho_i^\delta(y) \, \mathrm{d}x \, \mathrm{d}y \right) &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} K_{ii}^\delta (x-y) \partial_t \rho_i^\delta(x) \rho_i^\delta(y) \, \mathrm{d}x \, \mathrm{d}y \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\nabla K_{ii}^\delta * \rho_i^\delta) \cdot v f_i^\delta \, \mathrm{d}x \, \mathrm{d}v. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 f_i^\delta \, \mathrm{d}x \, \mathrm{d}v \right) + \frac{1}{2\varepsilon} \frac{\mathrm{d}}{\mathrm{d}t} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} K_{ii}^\delta (x-y) \rho_i^\delta(x) \rho_i^\delta(y) \, \mathrm{d}x \, \mathrm{d}y \right) \\ &\quad = -\frac{1}{\varepsilon} \sum_{\substack{j=1 \\ j \neq i}}^N \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\nabla K_{ij} * \rho_j^\delta) \cdot v f_i^\delta \, \mathrm{d}x \, \mathrm{d}v - \frac{1}{\varepsilon} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 f_i^\delta \, \mathrm{d}x \, \mathrm{d}v. \end{aligned}$$

In the spatial variable, we have the following estimate for the second-order moment

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x|^2}{2} f_i^\delta \, \mathrm{d}x \, \mathrm{d}v \right) &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x|^2}{2} \partial_t f_i^\delta \, \mathrm{d}x \, \mathrm{d}v \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} x \cdot v f_i^\delta \, \mathrm{d}x \, \mathrm{d}v \leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{|x|^2}{2} + \frac{|v|^2}{2} \right) f_i^\delta \, \mathrm{d}x \, \mathrm{d}v. \end{aligned}$$

Now, considering the estimates above, we obtain

$$\begin{aligned} &\iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{|x|^2}{2} + \frac{|v|^2}{2} \right) f_i^\delta \, \mathrm{d}x \, \mathrm{d}v + \frac{1}{2\varepsilon} \iint_{\mathbb{R}^d \times \mathbb{R}^d} K_{ii}^\delta (x-y) \rho_i^\delta(x) \rho_i^\delta(y) \, \mathrm{d}x \, \mathrm{d}y \\ &\quad + \frac{1}{\varepsilon} \int_0^t \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{f_i^\delta} |v f_i^\delta|^2 \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}s \\ &\leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{|x|^2}{2} + \frac{|v|^2}{2} \right) f_{i0}^\delta \, \mathrm{d}x \, \mathrm{d}v + \frac{1}{2\varepsilon} \iint_{\mathbb{R}^d \times \mathbb{R}^d} K_{ii}^\delta (x-y) \rho_{i0}^\delta(x) \rho_{i0}^\delta(y) \, \mathrm{d}x \, \mathrm{d}y \\ &\quad + \frac{1}{\varepsilon} \sum_{\substack{j=1 \\ j \neq i}}^N \int_0^t \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\nabla K_{ij}^\delta * \rho_j^\delta) \cdot v f_i^\delta \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}s \\ &\quad + \int_0^t \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{|x|^2}{2} + \frac{|v|^2}{2} \right) f_i^\delta \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}s. \end{aligned}$$

By Lemmas 4.2 and 4.3, we know that

$$\left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} K_{ii}^\delta(x-y)\rho_i^\delta(x)\rho_i^\delta(y)\,\mathrm{d}x\,\mathrm{d}y \right| \le C,$$

where C is independent of δ. We get

$$\begin{aligned}
&\iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{|v|^2}{2} + \frac{|x|^2}{2}\right) f_i^\delta \,\mathrm{d}x\,\mathrm{d}v + \frac{1}{\varepsilon}\int_0^t \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{f_i^\delta}|v f_i^\delta|^2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s \\
&\quad \le \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{|x|^2}{2} + \frac{|v|^2}{2}\right) f_{i0}^\delta \,\mathrm{d}x\,\mathrm{d}v + C\int_0^t \iint_{\mathbb{R}^d \times \mathbb{R}^d} (|v|^2 + |x|^2) f_i^\delta \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s + C
\end{aligned}$$

for some $C > 0$ independent of δ. Then, by Grönwall's lemma we obtain the result. □

Remark 4.1 From Proposition 4.1, we deduce the following estimates on total energy for f_i^δ, as $i = 1, \ldots, N$, and for all $t \in [0, T]$:

$$\begin{aligned}
&\sum_{i=1}^N \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|v|^2}{2} f_i^\delta \,\mathrm{d}x\,\mathrm{d}v + \frac{1}{2\varepsilon}\sum_{i=1}^N \int_{\mathbb{R}^d} \rho_i^\delta K_{ii} * \rho_i^\delta \,\mathrm{d}x \\
&\quad + \frac{1}{\varepsilon}\int_0^t \sum_{i=1}^N \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 f_i^\delta \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s \\
&\quad \le \sum_{i=1}^N \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|v|^2}{2} f_{i0}^\delta \,\mathrm{d}x\,\mathrm{d}v + \frac{1}{2\varepsilon}\sum_{i=1}^N \int_{\mathbb{R}^d} \rho_{i0}^\delta K_{ii} * \rho_{i0}^\delta \,\mathrm{d}x + \frac{1}{\varepsilon}\sum_{i\neq j}\|\nabla K_{ij}\|_{L^\infty} t.
\end{aligned}$$

4.1.2 Existence of Weak Solution to the Kinetic System

Now, we prove the existence of weak solutions to system (38). For this purpose, we need the following lemma, cf. Glasse (1996, Section 7), Karper et al. (2013, Lemma 2.6).

Lemma 4.4 *Let $\{f^n\}_n$ be bounded in $L^p_{loc}([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ with $1 < p < \infty$, and $\{G^n\}_n$ be bounded in $L^p_{loc}([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$. Assume that f^n and G^n satisfy*

$$\partial_t f^n + v \cdot \nabla_x f^n = \nabla_v \cdot G^n, \qquad f^n\,|_{t=0} = f_0 \in L^p(\mathbb{R}^d \times \mathbb{R}^d),$$

and

$$f^n \text{ is bounded in } L^\infty(\mathbb{R}^d \times \mathbb{R}^d),$$
$$(|v|^2 + |x|^2) f^n \text{ is bounded in } L^\infty((0,T); L^1(\mathbb{R}^d \times \mathbb{R}^d)).$$

Then, for any $q < \frac{d+2}{d+1}$, the sequence

$$\left\{ \int_{\mathbb{R}^d} f^n \,\mathrm{d}v \right\}_n$$

is relatively compact in $L^q((0,T)\times\mathbb{R}^d)$.

The existence result of weak solutions to system (38) is contained in the following theorem.

Theorem 4.1 *Assume that the initial datum* $\mathbf{f}_0$ *satisfies*

$$f_{i0}\in L^1_+\cap L^\infty(\mathbb{R}^d\times\mathbb{R}^d),\qquad (|x|^2+|v|^2)f_{i0}\in L^1(\mathbb{R}^d\times\mathbb{R}^d),$$

and

$$(K_{ii}*\rho_{i0})f_{i0}\in L^1(\mathbb{R}^d\times\mathbb{R}^d)\quad \textit{with}\ \alpha_i\in(0,d-1].$$

Then there exists a weak solution $\mathbf{f}$ *to* (38) *such that*

$$\mathbf{f}\in\mathcal{C}([0,T];\mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d)^N).$$

Proof By the uniform in δ bound estimates obtained above we know

$$\|f_i^\delta\|_{L^\infty((0,T);L^p(\mathbb{R}^d\times\mathbb{R}^d))}+\|\rho_i^\delta\|_{L^\infty((0,T);L^q(\mathbb{R}^d))}\le C,$$

with $p\in[1+\infty]$, $q\in[1,\frac{d+2}{d}]$, $C>0$ independent of δ. Therefore, by compactness theory, we have that as $\delta\to 0$, up to a subsequence,

$$\begin{aligned} f_i^\delta &\overset{*}{\rightharpoonup} f_i \quad \text{in}\quad L^\infty((0,T);L^p(\mathbb{R}^d\times\mathbb{R}^d)),\quad p\in[1,+\infty],\\ \rho_i^\delta &\overset{*}{\rightharpoonup} \rho_i \quad \text{in}\quad L^\infty((0,T);L^p(\mathbb{R}^d)),\quad p\in[1,\tfrac{d+2}{d}]. \end{aligned}$$

Set

$$G_i^\delta:=\frac{1}{\varepsilon}vf_i^\delta+\frac{1}{\varepsilon}\sum_{j=1}^N(\nabla K_{ij}^\delta*\rho_j^\delta)f_i^\delta.$$

We want to prove that $G_i^\delta\in L^p_{loc}([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$ for some $p\in(1,\infty)$, in order to apply Lemma 4.4. We need to check the self-interaction terms. Let $q<2$. Then

$$\begin{aligned} \iint_{\mathbb{R}^d\times\mathbb{R}^d}|vf_i^\delta|^q\,\mathrm{d}x\,\mathrm{d}v &= \iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(|v|^2f_i^\delta\right)^{\frac{q}{2}}(f_i^\delta)^{\frac{q}{2}}\,\mathrm{d}x\,\mathrm{d}v\\ &\le\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2f_i^\delta\,\mathrm{d}x\,\mathrm{d}v\right)^{\frac{q}{2}}\|f_i^\delta\|^{\frac{q}{2}}_{L^{\frac{q}{2-q}}}. \end{aligned}$$

For the second term, by using the Calderón–Zygmund lemma, we obtain that for $\alpha_i=d-1$

$$\|(\nabla K_{ii}^\delta*\rho_i^\delta)f_i^\delta\|_{L^p}\le C\|f_i^\delta\|_{L^\infty}\|\nabla K_{ii}*\rho_i^\delta\|_{L^p}\le C\|f_i^\delta\|_{L^\infty}\|\rho_i^\delta\|_{L^p},$$

for $p < \frac{d+2}{d}$. On the other hand, when $\alpha_i < d-1$, for any $q_i < \frac{d+2}{d}$, we choose $p_i > 0$ such that

$$\frac{1}{p_i} = 1 + \frac{1}{q_i} - \frac{\alpha_i + 1}{d}.$$

Then, it is clear that $p_i < \frac{d+2}{d}$ and by the Hardy–Littlewood–Sobolev inequality, we deduce

$$\|(\nabla K_{ii}^\delta * \rho_i^\delta) f_i^\delta\|_{L^{q_i}} \leq C\|f_i^\delta\|_{L^\infty}\|\nabla K_{ii} * \rho_i^\delta\|_{L^{q_i}} \leq C\|f_i^\delta\|_{L^\infty}\|\rho_i^\delta\|_{L^{p_i}}.$$

Thus, by Lemma 4.4, we get

$$\rho_i^\delta \to \rho_i \text{ in } L^q((0,T)\times\mathbb{R}^d) \text{ and a.e.,}$$

up to a subsequence, as $\delta \to 0$, for $q < \frac{d+2}{d+1}$. Now we want to prove that

$$(\nabla K_{ii}^\delta * \rho_i^\delta) f_i^\delta \to (\nabla K_{ii} * \rho_i) f_i,$$

in the sense of distributions. Let $\Psi_i \in \mathcal{C}_c^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$.

$$\begin{aligned}
&\int_0^T \iint_{\mathbb{R}^d\times\mathbb{R}^d} [(\nabla K_{ii}^\delta * \rho_i^\delta) f_i^\delta - (\nabla K_{ii} * \rho_i) f_i]\Psi_i \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s \\
&= \int_0^T \int_{\mathbb{R}^d} (\nabla(K_{ii}^\delta - K_{ii}) * \rho_i)\rho_{i,\Psi}\,\mathrm{d}x\,\mathrm{d}s + \int_0^T \int_{\mathbb{R}^d} \nabla K_{ii}^\delta * (\rho_i^\delta - \rho_i)\rho_{i,\Psi}^\delta\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^T \int_{\mathbb{R}^d} (\nabla K_{ii}^\delta * \rho_i)(\rho_{i,\Psi}^\delta - \rho_{i,\Psi})\,\mathrm{d}x\,\mathrm{d}s \\
&=: I + II + III,
\end{aligned}$$

with $\rho_{i,\Psi} := \int_{\mathbb{R}^d} f_i \Psi \,\mathrm{d}v$ and $\rho_{i,\Psi}^\delta := \int_{\mathbb{R}^d} f_i^\delta \Psi\,\mathrm{d}v$. Thanks to the uniform in δ estimate for f_i^δ in $L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ and the compact support of Ψ_i, we find

$$\rho_{i,\Psi},\ \rho_{i,\Psi}^\delta \in L^p((0,T); L^q(\mathbb{R}^d)),$$

for any $p, q \in [1,\infty]$, uniformly in δ.

Estimate of I. We have that $|(\nabla K_{ii}^\delta * \rho_i)\rho_{i,\Psi}| \leq |\nabla K_{ii} * \rho_i||\rho_{i,\Psi}|$ and $(\nabla K_{ii}^\delta * \rho_i)\rho_{i,\Psi}$ converges pointwise to $(\nabla K_{ii} * \rho_i)\rho_{i,\Psi}$ as $\delta \to 0$. When $\alpha_i = d-1$, we use the Calderón–Zygmund lemma to deduce

$$\int_0^T \int_{\mathbb{R}^d} (|\nabla K_{ii}| * \rho_i)|\rho_{i,\Psi}|\,\mathrm{d}x\,\mathrm{d}s \leq C\|\rho_i\|_{L^p(\mathbb{R}^d\times(0,T))}\|\rho_{i,\Psi}\|_{L^{p'}(\mathbb{R}^d\times(0,T))}$$

for any $p \in (1, \frac{d+2}{d})$, where p' is the Hölder conjugate of p. Moreover, when $\alpha_i < d-1$, by Hardy–Littlewood–Sobolev inequality, we get

$$\int_0^T \int_{\mathbb{R}^d} (|\nabla K_{ii}| * \rho_i)|\rho_{i,\Psi}| \,\mathrm{d}x \,\mathrm{d}s \leq \int_0^T \iint_{\mathbb{R}^d \times \mathbb{R}^d} \rho_i(x)|x-y|^{-(\alpha_i+1)}|\rho_{i,\Psi}|(y) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}s$$
$$\leq C\|\rho_i\|_{L^{p_i}(\mathbb{R}^d \times (0,T))} \|\rho_{i,\Psi}\|_{L^{p_i'}((0,T);L^{q_i}(\mathbb{R}^d))},$$

where

$$p_i \in \left(1, \frac{d+2}{d}\right), \quad \frac{\alpha_i+1}{d} = 1 - \frac{1}{q_i} + \frac{1}{p_i}, \tag{42}$$

and p_i' is the Hölder conjugate of p_i. Therefore, by Lebesgue's dominated convergence theorem, we obtain that I vanishes as $\delta \to 0$.

Estimate of II. As in the previous estimate, we have that for $\alpha_i < d-1$

$$\left| \int_0^T \int_{\mathbb{R}^d} \nabla K_{ii} * (\rho_i^\delta - \rho_i)\rho_{i,\Psi}^\delta \,\mathrm{d}x \,\mathrm{d}s \right|$$
$$\leq \int_0^T \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\rho_i^\delta - \rho_i|(x)|x-y|^{-(\alpha_i+1)}|\rho_{i,\Psi}^\delta|(y) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}s$$
$$\leq C\|\rho_i^\delta - \rho_i\|_{L^{p_i}(\mathbb{R}^d \times (0,T))} \|\rho_i \Psi\|_{L^{p_i'}((0,T);L^{q_i}(\mathbb{R}^d))},$$

with $p_i \in (1, \frac{d+2}{d+1})$ and q_i is chosen as in (42). The case $\alpha_i = d-1$ can be handled similarly to the estimate of I. Thus, $II \to 0$ as $\delta \to 0$.

Estimate of III. As said, we know that

$$(\nabla K_{ii}^\delta * \rho_i)\Psi \in L^1((0,T); L^q(\mathbb{R}^d)),$$

with $q < 2$ uniformly in δ. Then, since $f_i^\delta \overset{*}{\rightharpoonup} f_i$, we obtain that $III \to 0$ as $\delta \to 0$. We conclude that $\mathbf{f}$ is a weak solution to system (38). □

4.2 Existence and Uniqueness of Regular Solutions to the Macroscopic System

Our aim here is to prove that, under suitable assumptions on the parameter, there exists a unique regular solution to system (39). We only focus on the case $-2 < \alpha_i - d < 0$ for all $i = 1, \ldots, N$. Note that the case $\alpha_i \leq d-2$ for all $i = 1, \ldots, N$ can be handled by the classical well-posedness theory. Moreover, the case $\alpha_i \leq d-2$ for some i can be taken into account by a simple modification of our arguments. In this subsection, we consider that the cross-interaction potentials K_{ij}, $i \neq j$, satisfy

$$K_{ij} \in \mathcal{C}^2(\mathbb{R}^d), \quad \nabla K_{ij} \in W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d)$$

for $i \neq j$ and the self-interaction potentials are given by

$$K_{ii}(x) = \frac{C_i}{|x|^{\alpha_i}} \quad \text{with } \alpha_i \in (d-2, d).$$

Let us first recall Moser-type inequalities.

Lemma 4.5 *(i) Let $s > 0$, $r \in (1, \infty)$, and $p_1, p_2, q_1, q_2 \in (1, \infty]$. Then we have*

$$\|\Lambda^s(fg)\|_{L^r} \lesssim \|\Lambda^s f\|_{L^{p_1}}\|g\|_{L^{q_1}} + \|f\|_{L^{p_2}}\|\Lambda^s g\|_{L^{q_2}}$$

for $f \in \dot{W}^{s,p_1} \cap L^{p_2}$ and $g \in \dot{W}^{s,q_2} \cap L^{q_1}$, where $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$.
(ii) Let $s > 0$. If $f \in \dot{W}^{1,\infty} \cap \dot{H}^s$ and $g \in L^\infty \cap \dot{H}^{s-1}$, then we obtain

$$\|[\Lambda^s, f]g\|_{L^2} \lesssim \|f\|_{\dot{H}^s}\|g\|_{L^\infty} + \|\nabla f\|_{L^\infty}\|g\|_{\dot{H}^{s-1}}.$$

Here $[\cdot,\cdot]$ denotes the commutator operator, i.e., $[A, B] = AB - BA$.

We then present a priori estimate of solutions of (39) in the lemma below.

Lemma 4.6 *Let $d \geq 1$, $s > \frac{d}{2} + 3$, $-2 < \alpha_i - d < 0$ for $i = 1, \dots, N$, and $T > 0$. Let $\boldsymbol{\rho} \in \mathcal{C}([0, T]; H^s(\mathbb{R}^d)^N)$ be a smooth solution to (39) decaying fast at infinity on a time interval $[0, T]$. Then by choosing a sufficiently small $T^* > 0$ depending on $\boldsymbol{\rho}_0$, we have*

$$\|\boldsymbol{\rho}(t)\|_{H^s} \leq 4\|\boldsymbol{\rho}_0\|_{H^s}, \quad \textit{for all } 0 \leq t \leq T^*.$$

Proof We first notice that our main system can be rewritten as

$$\begin{cases} \partial_t \rho_i + \nabla \cdot (\rho_i u_i) = 0, \\ u_i = -\Lambda^{\alpha_i - d}\nabla\rho_i - \nabla V_i[\boldsymbol{\rho}], \end{cases}$$

for $i = 1, \dots, N$, with

$$V_i[\boldsymbol{\rho}] = \sum_{j \neq i} K_{ij} * \rho_j .$$

Then for $s > \frac{d}{2} + 3$, we find

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s \rho_i\|_{L^2}^2 &= \int_{\mathbb{R}^d} \Lambda^s \rho_i \cdot \Lambda^s(\nabla \cdot (\rho_i \Lambda^{\alpha_i - d}\nabla\rho_i))\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^d} \Lambda^s \rho_i \cdot \Lambda^s(\nabla \cdot (\rho_i \nabla V_i[\boldsymbol{\rho}])\,\mathrm{d}x \\ &=: I + II, \end{aligned}$$

where I can be estimated as

$$|I| \leq C\|\rho_i\|_{H^s}^3 \leq C\|\boldsymbol{\rho}\|_{H^s}^3.$$

due to a direct consequence of Choi and Jeong (2021, Section 2.2). Next, note that

$$\|\nabla^2 V_i[\boldsymbol{\rho}]\|_{L^\infty} \leq \sum_{j \neq i}\|\nabla^2 K_{ij} * \rho_j\|_{L^\infty} \leq \sum_{j \neq i}\|\nabla^2 K_{ij}\|_{L^1}\|\rho_j\|_{L^\infty}$$

$$\leq C \sum_{j \neq i} \|\rho_j\|_{H^s} \leq C\|\boldsymbol{\rho}\|_{H^s} \tag{43}$$

and

$$\|\nabla^2 V_i[\boldsymbol{\rho}]\|_{H^s} \leq \sum_{j \neq i} \|\nabla^2 K_{ij} * \rho_j\|_{H^s} \leq \sum_{j \neq i} \|\nabla^2 K_{ij}\|_{L^1} \|\rho_j\|_{H^s} \leq C\|\boldsymbol{\rho}\|_{H^s}.$$

Then we estimate

$$\begin{aligned}
II &= \int_{\mathbb{R}^d} (\Lambda^s \rho_i) \Lambda^s (\nabla \rho_i \cdot \nabla V_i[\boldsymbol{\rho}] + \rho_i \Delta V_i[\boldsymbol{\rho}])\, \mathrm{d}x \\
&= \int_{\mathbb{R}^d} (\Lambda^s \rho_i) \nabla(\Lambda^s \rho_i) \cdot \nabla V_i[\boldsymbol{\rho}]\, \mathrm{d}x + \int_{\mathbb{R}^d} (\Lambda^s \rho_i)[\Lambda^s, \nabla V_i[\boldsymbol{\rho}]] \nabla \rho_i\, \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d} (\Lambda^s \rho_i) \Lambda^s (\rho_i \Delta V_i[\boldsymbol{\rho}])\, \mathrm{d}x \\
&\leq \|\Delta V_i[\boldsymbol{\rho}]\|_{L^\infty} \|\Lambda^s \rho_i\|_{L^2}^2 + \|\Lambda^s \rho_i\|_{L^2} (\|[\Lambda^s, \nabla V_i[\boldsymbol{\rho}]] \nabla \rho_i\|_{L^2} \\
&\quad + \|\Lambda^s (\rho_i \Delta V_i[\boldsymbol{\rho}])\|_{L^2}).
\end{aligned}$$

We now use Lemma 4.5 to deduce

$$\|[\Lambda^s, \nabla V_i[\boldsymbol{\rho}]] \nabla \rho_i\|_{L^2} \lesssim \|\nabla V_i[\boldsymbol{\rho}]\|_{H^s} \|\nabla \rho_i\|_{L^\infty} + \|\nabla^2 V_i[\boldsymbol{\rho}]\|_{L^\infty} \|\nabla \rho_i\|_{H^{s-1}} \leq C\|\boldsymbol{\rho}\|_{H^s}^2$$

and

$$\|\Lambda^s (\rho_i \Delta V_i[\boldsymbol{\rho}])\|_{L^2} \lesssim \|\Lambda^s \rho_i\|_{L^2} \|\Delta V_i[\boldsymbol{\rho}]\|_{L^\infty} + \|\rho_i\|_{L^2} \|\Lambda^s \Delta V_i[\boldsymbol{\rho}]\|_{L^\infty} \leq C\|\boldsymbol{\rho}\|_{H^s}^2.$$

Hence, by combining all of the above estimates, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{\rho}\|_{H^s} \leq C\|\boldsymbol{\rho}\|_{H^s}^2,$$

and subsequently, by choosing a $T^* > 0$ small enough, we conclude the desired result. □

Remark 4.2 By using similar estimates as in the proof of Lemma 4.6, we readily obtain

$$\|u_i\|_{H^{s-1}} \leq \|\Lambda^{\alpha_i - d} \nabla \rho_i\|_{H^{s-1}} + \|\nabla V_i[\boldsymbol{\rho}]\|_{H^{s-1}} \leq C\|\boldsymbol{\rho}\|_{H^s}$$

and thus $\|u_i\|_{W^{1,\infty}} \leq C\|u_i\|_{H^{s-1}} \leq C\|\boldsymbol{\rho}\|_{H^s}$ due to $s > \frac{d}{2} + 3$. Moreover, we find

$$\begin{aligned}
\|\partial_t u_i\|_{L^\infty} &\leq \|\Lambda^{\alpha_i - d} \nabla(\nabla \cdot (\rho_i u_i))\|_{L^\infty} + \sum_{j \neq i} \|\nabla K_{ij} * \nabla \cdot (\rho_j u_j)\|_{L^\infty} \\
&\leq C\|\rho_i u_i\|_{H^{s-1}} + \sum_{j \neq i} \|\rho_j u_j\|_{H^{s-1}}
\end{aligned}$$

$$\leq C\|\boldsymbol{\rho}\|_{H^s}.$$

We now present the result on the existence and uniqueness of regular solutions to system (39).

Theorem 4.2 *Let $d \geq 1$, $s > \frac{d}{2}+3$, and $-2 \leq \alpha_i - d \leq 0$ for $i = 1, \ldots, N$. Then, the system (39) admits a local-in-time unique solution $\rho_i \in L^1 \cap H^s(\mathbb{R}^d)$, i.e., for any non-negative initial datum $\boldsymbol{\rho}_0 \in (L^1 \cap H^s(\mathbb{R}^d))^N$, there exists $T = T(\boldsymbol{\rho}_0) > 0$ and a unique non-negative solution $\boldsymbol{\rho} \in \mathcal{C}([0,T); L^1 \cap H^s(\mathbb{R}^d))^N$ to (39) with $\boldsymbol{\rho}(t=0) = \boldsymbol{\rho}_0$.*

Proof *(existence)*: The existence of solutions $\rho_i \in L^\infty(0, T^*; L^1 \cap H^s(\mathbb{R}^d))$ can be obtained by combining the a priori estimate in Lemma 4.6 and the classical approximation arguments, see Choi and Jeong (2021) for instance. Here we sketch the proof for the existence and uniqueness of regular solutions to (39). We first approximate the system (39) by

$$\begin{cases} \partial_t \rho_i^{n+1} + \nabla \cdot (\rho_i^{n+1} u_i^n) = 0, \\ u_i^n = -\Lambda^{\alpha_i - d} \nabla \rho_i^n - \nabla V_i[\boldsymbol{\rho}^n], \end{cases}$$

for $i = 1, \ldots, N$, with

$$V_i[\boldsymbol{\rho}^n] = \sum_{j \neq i} K_{ij} * \rho_j^n .$$

Here the initial datum and first iteration step are given by

$$\rho_i^n(0,x) = \rho_i(0,x) \quad \text{for all } i = 1, \ldots, N \text{ and for all } n \geq 1$$

and

$$\rho_i^0(t,x) = \rho_i(0,x) \quad \text{for all } i = 1, \ldots, N.$$

We then use the a priori estimate in Lemma 4.6 to derive

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq t \leq T^*} \|\boldsymbol{\rho}^n(t)\|_{H^s} \leq 4\|\boldsymbol{\rho}_0\|_{H^s}$$

for some $T^* > 0$. Next, we show that the approximate sequence $\{\rho_i^n\}_{n \in \mathbb{N}}$ is Cauchy in $L^\infty(0, T^*; H^1(\mathbb{R}^d))$. Let us write $\rho_i^{n,n-1} = \rho_i^n - \rho_i^{n-1}$ and $u_i^{n,n-1} = u_i^n - u_i^{n-1}$. Then $\rho_i^{n,n-1}$ satisfies

$$\partial_t \rho_i^{n,n-1} + \nabla \cdot (\rho_i^{n,n-1} u_i^n + \rho_i^{n-1} u_i^{n,n-1}) = 0.$$

We now estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_i^{n,n-1}\|_{L^2}^2 \leq C(\|\nabla u_i^n\|_{L^\infty}\|\rho_i^{n,n-1}\|_{L^2}^2 + \|\nabla \rho_i^{n-1}\|_{L^\infty}\|\rho_i^{n,n-1}\|_{L^2}\|\rho_i^{n,n-1}\|_{H^1}$$
$$+\|\rho_i^{n-1}\|_{L^\infty}\|\rho_i^{n,n-1}\|_{H^1}^2),$$

thanks to (43) and $s > \frac{d}{2} + 3$. Since

$$\|\nabla u_i^n\|_{L^\infty} \lesssim \|\Lambda^{\alpha_i - d}\nabla^2 \rho_i^n\|_{L^\infty} + \|\nabla^2 V_i[\boldsymbol{\rho}^n]\|_{L^\infty} \lesssim \|\boldsymbol{\rho}^n\|_{H^s},$$

we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_i^{n,n-1}\|_{L^2}^2 \le C\|\rho_i^{n,n-1}\|_{H^1}^2. \tag{44}$$

We next estimate $\|\nabla \rho_i^{n,n-1}\|_{H^1}$. To take the notation to a minimum we set with ∂ the partial derivative with respect to a generic space variable, i.e., $\partial := \partial_{x_k}$. Note that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial\rho_i^{n,n-1}\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^d} \nabla\cdot(\partial\rho_i^{n,n-1}u_i^n)\partial\rho_i^{n,n-1}\,\mathrm{d}x - \int_{\mathbb{R}^d} \nabla\cdot(\rho_i^{n,n-1}\partial u_i^n)\partial\rho_i^{n,n-1}\,\mathrm{d}x \\
&\qquad - \int_{\mathbb{R}^d} \nabla\cdot(\partial\rho_i^{n-1}u_i^{n,n-1})\partial\rho_i^{n,n-1}\,\mathrm{d}x - \int_{\mathbb{R}^d} \nabla\cdot(\rho_i^{n-1}\partial u_i^{n,n-1})\partial\rho_i^{n,n-1}\,\mathrm{d}x.
\end{aligned}$$

We deduce that the right-hand side of the above can be bounded by

$$\begin{aligned}
&C(\|\boldsymbol{\rho}^n\|_{H^s} + \|\boldsymbol{\rho}^{n-1}\|_{H^s})\|\rho_i^{n,n-1}\|_{H^1}^2 + (\|\nabla\cdot(\partial\rho_i^{n-1}\nabla V_i[\boldsymbol{\rho}^{n,n-1}])\|_{L^2} \\
&\quad + \|\nabla\cdot(\rho_i^{n-1}\partial\nabla V_i[\boldsymbol{\rho}^{n,n-1}])\|_{L^2})\|\rho_i^{n,n-1}\|_{H^1}.
\end{aligned}$$

Here we further estimate

$$\begin{aligned}
&\|\nabla\cdot(\partial\rho_i^{n-1}\nabla V_i[\boldsymbol{\rho}^{n,n-1}])\|_{L^2} + \|\nabla\cdot(\rho_i^{n-1}\partial\nabla V_i[\boldsymbol{\rho}^{n,n-1}])\|_{L^2} \\
&\quad \le C\|\nabla^2\rho_i^{n-1}\|_{L^\infty}\|\nabla V_i[\boldsymbol{\rho}^{n,n-1}]\|_{L^2} + \|\partial\rho_i^{n-1}\|_{L^\infty}\|\nabla^2 V_i[\boldsymbol{\rho}^{n,n-1}]\|_{L^2} \\
&\qquad + \|\rho_i^{n-1}\|_{L^\infty}\|\nabla^3 V_i[\boldsymbol{\rho}^{n,n-1}]\|_{L^2} \\
&\quad \le C\sum_{j\neq i}\|\rho_j^{n,n-1}\|_{H^1}\|\boldsymbol{\rho}^{n-1}\|_{H^s}
\end{aligned}$$

to yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^N\|\partial\rho_i^{n,n-1}\|_{L^2}^2 \le C\sum_{i=1}^N\|\rho_i^{n,n-1}\|_{H^1}^2$$

from which we obtain that $\{\rho_i^n\}_{n\in\mathbb{N}}$ is Cauchy in $L^\infty(0,T^*;H^1(\mathbb{R}^d))$, and thus, we have the existence of limit function ρ_i such that $\rho_i^n \to \rho_i$ in $L^\infty(0,T^*;H^1(\mathbb{R}^d))$ for all $i = 1,\ldots,N$. Moreover, due to the uniform-in-n bound estimate of $\|\boldsymbol{\rho}^n(t)\|_{L^\infty(0,T^*;H^s)}$ and Gagliardo–Nirenberg interpolation inequality, we have $\rho_i^n \to \rho_i$ in $L^\infty(0,T^*;H^{s-\varepsilon}(\mathbb{R}^d))$ for all $i = 1,\ldots,N$ for any small $\varepsilon > 0$. Then we can readily show the limit function ρ_i is the solution to the system (39).

(uniqueness): Let ρ_i and $\tilde{\rho}_i$ be two solutions to (39) obtained in the above. We also let u_i and $\tilde{u}_i$ be corresponding vector fields. Then defining $g_i := \rho_i - \tilde{\rho}_i$ and

$w_i = u_i - \tilde{u}_i$, we find that g_i satisfies

$$\partial_t g_i + \nabla \cdot (g_i u_i + \tilde{\rho}_i w_i) = 0.$$

Then by using almost the same argument as above, we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^N \|g_i\|_{H^1}^2 \le C \sum_{i=1}^N \|g_i\|_{H^1}^2$$

and this concludes that

$$\rho_i \equiv \tilde{\rho}_i$$

provided that $\rho_i(0) = \tilde{\rho}_i(0)$ for all $i = 1, \dots, N$. This completes the proof. □

4.3 Small Inertia Limit

In this subsection, we show the rigorous small inertia limit of the system (38) providing Theorem 2.2. Since we want to study the behavior of solutions to kinetic system (38) with respect to the inertia parameter $\varepsilon > 0$, we explicit the ε-dependence; namely, we define $\mathbf{f}^\varepsilon = (f_i^\varepsilon)_{i=1}^N$ to be a weak solution to the system:

$$\partial_t f_i^\varepsilon + v \cdot \nabla_x f_i^\varepsilon = \frac{1}{\varepsilon}\nabla_v \cdot (v f_i^\varepsilon) + \frac{1}{\varepsilon}\left(\sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon\right) \cdot \nabla_v f_i^\varepsilon, \tag{45}$$

for $i = 1, \dots, N$, with smooth cross-potentials as in (**Pot**) and singular self-potentials of the form (40).

We quantitatively show the convergence of solutions $\mathbf{f}^\varepsilon$ toward the system

$$\begin{cases} \partial_t \rho_i = \nabla \cdot (\rho_i u_i), \\ u_i = \sum_{j=1}^N \nabla K_{ij} * \rho_j, \end{cases} \tag{46}$$

for $i = 1, \dots, N$, as $\varepsilon \to 0$.

In order to measure the error between solutions to the systems (45) and (46), we employ the modulated energy method. For this, we first recall from Choi and Jung (2024) (see also Nguyen et al. (2022)) the following modulated interaction energy estimates.

Theorem 4.3 *Let $T > 0$ and K be given by*

$$K(x) = \frac{1}{|x|^\alpha} \quad \textit{with } \alpha \in (0, d).$$

Suppose that the pairs $(\bar{\rho}, \bar{u})$ and (ρ, u) satisfy the followings:

(i) $(\bar\rho,\bar u)$ *and* (ρ,u) *satisfy the continuity equations in the sense of distribution:*

$$\partial_t\bar\rho+\nabla\cdot(\bar\rho\bar u)=0 \quad and \quad \partial_t\rho+\nabla\cdot(\rho u)=0,$$

(ii) $(\bar\rho,\bar u)$ *and* (ρ,u) *satisfy the energy inequality:*

$$\sup_{0\le t\le T}\left(\int_{\mathbb{R}^d}\bar\rho|\bar u|^2\,\mathrm{d}x+\int_{\mathbb{R}^d}\bar\rho K*\bar\rho\,\mathrm{d}x\right)<\infty,$$

and

$$\sup_{0\le t\le T}\left(\int_{\mathbb{R}^d}\rho|u|^2\,\mathrm{d}x+\int_{\mathbb{R}^d}\rho K*\rho\,\mathrm{d}x\right)<\infty,$$

(iii) $\bar\rho,\rho\in\mathcal{C}((0,T);L^1(\mathbb{R}^d))$, $\nabla u\in L^\infty(\mathbb{R}^d\times(0,T))$ *and if* $\alpha<d-2$,

$$\begin{cases}\nabla^{[(d-\alpha)/2]+1}u\in L^\infty(0,T;L^{\frac{d}{[(d-\alpha)/2]}}(\mathbb{R}^d)) & \text{if } \alpha\in(0,d-2)\setminus(d-2\mathbb{N}),\\ \nabla^{\frac{d-\alpha}{2}}u\in L^\infty(0,T;L^{\frac{2d}{d-\alpha-2}}(\mathbb{R}^d)) & \text{if } \alpha\equiv d \mod 2,\end{cases}$$

where $d-2\mathbb{N}:=\{d-2n \;\; n\in\mathbb{N}\}$ *and* $[\,\cdot\,]$ *denotes the floor function.*

Then we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}(\rho-\bar\rho)K*(\rho-\bar\rho)\,\mathrm{d}x\le\int_{\mathbb{R}^d}\bar\rho(u-\bar u)\cdot\nabla K*(\rho-\bar\rho)\,\mathrm{d}x+C\int_{\mathbb{R}^d}(\rho-\bar\rho)K*(\rho-\bar\rho)\,\mathrm{d}x$$

for $t\in[0,T)$ *and some* $C>0$ *which depends only on* α, d *and* $\|\nabla u\|_{L^\infty(\mathbb{R}^d\times(0,T))}$, *and if* $d<\alpha-2$, *additionally*

$$\begin{cases}\|\nabla^{[(d-\alpha)/2]+1}u\|_{L^\infty((0,T);L^{\frac{d}{[(d-\alpha)/2]-1}})}, & \text{if } \alpha\in(0,d-2)\setminus(d-2\mathbb{N}),\\ \|\nabla^{(d-\alpha)/2}u\|_{L^\infty((0,T);L^{\frac{2d}{d-\alpha-2}})}, & \text{if } \alpha\equiv d \bmod 2.\end{cases}$$

Remark 4.3 If we define the macroscopic velocity as

$$u(t,x)=\frac{\int_{\mathbb{R}^d}vf(t,x,v)\,\mathrm{d}v}{\int_{\mathbb{R}^d}f(t,x,v)\,\mathrm{d}v},$$

then, by denoting u_i^ε the macroscopic velocity corresponding to ρ_i^ε, we have

$$\rho_i^\varepsilon|u_i^\varepsilon|^2\le\int_{\mathbb{R}^d}|v|^2f_i^\varepsilon\,\mathrm{d}v,$$

and thus by Proposition 4.1, we obtain that for $\varepsilon>0$

$$\sum_{i=1}^N\int_{\mathbb{R}^d}\rho_i^\varepsilon|u_i^\varepsilon|^2\,\mathrm{d}x<\infty$$

on some time interval $[0, T]$.

We also recall from Carrillo et al. (2021, Lemma 4.1) (see also Choi 2021, Proposition 3.1), Villani (2009, Theorem 23.9), Ambrosio et al. (2008), Carrillo and Choi (2021), Figalli and Kang (2019)) the following lemma which gives a relation between the 1-Wasserstein distance and modulated kinetic energy.

Lemma 4.7 *Let $T > 0$ and $\bar{\rho} : [0, T] \to \mathcal{P}(\mathbb{R}^d)$ be a narrowly continuous solution of*

$$\partial_t \bar{\rho} + \nabla \cdot (\bar{\rho}\bar{u}) = 0,$$

that is, $\bar{\rho}$ is continuous in the duality with continuous bounded functions, for a Borel vector field $\bar{u}$ satisfying

$$\int_0^T \int_{\mathbb{R}^d} |\bar{u}(x,t)|^p \bar{\rho}(x,t) \, \mathrm{d}x \, \mathrm{d}t < \infty$$

for some $p > 1$. Let $\rho \in \mathcal{C}([0, T]; \mathcal{P}_p(\mathbb{R}^d))$ be a solution of the following continuity equation

$$\partial_t \rho + \nabla \cdot (\rho u) = 0$$

with the velocity fields $u \in L^\infty((0, T); \dot{W}^{1,\infty}(\mathbb{R}^d))$. Then there exists a $C_{u,T} > 0$ depending only on T and $\|\nabla u\|_{L^\infty}$ such that for all $t \in [0, T]$

$$W_1^2(\rho, \bar{\rho}) \le C_{u,T} \left(W_1^2(\rho_0, \bar{\rho}_0) + \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon - u|^2 \, \mathrm{d}x \, \mathrm{d}s \right),$$

where ρ^ε and u^ε are defined in Remark 4.3.

Remark 4.4 Since

$$\rho_i^\varepsilon |u_i^\varepsilon - u_i|^2 \le \int_{\mathbb{R}^d} f_i^\varepsilon |v - u_i|^2 \, \mathrm{d}v,$$

Lemma 4.7 particularly implies

$$W_1^2(\rho_i, \bar{\rho}_i) \le C_{u,T} \left(W_1^2(\rho_{i0}, \bar{\rho}_{i0}) + \int_0^t \iint_{\mathbb{R}^d \times \mathbb{R}^d} f_i^\varepsilon |v - u_i|^2 \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}s \right),$$

for $i = 1, \ldots, N$.

We are now in a position to provide the details of proof for Theorem 2.2.

Proof of Theorem 2.2 We first rewrite the system (46) as

$$\begin{aligned} &\partial_t \rho_i + \nabla \cdot (\rho_i u_i) = 0, \\ &\varepsilon \partial_t u_i + \varepsilon u_i \cdot \nabla u_i = -u_i - \sum_{j=1}^N \nabla K_{ij} * \rho_j + \varepsilon \mathrm{e}_i, \end{aligned}$$

where $\mathrm{e}_i:=\partial_t u_i + u_i \cdot \nabla u_i$, for $i = 1, \dots, N$. For the error estimates, we consider the modulated kinetic and interaction energies:

$$\mathcal{E}_K(f_i^\varepsilon|\rho_i, u_i):=\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i - v|^2 f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v + \frac{1}{2\varepsilon}\int_{\mathbb{R}^d}(\rho_i - \rho_i^\varepsilon)K_{ii} * (\rho_i - \rho_i^\varepsilon)\,\mathrm{d}x.$$

Straightforward computation yields that for each $i = 1, \dots, N$

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i - v|^2 f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v + \frac{1}{\varepsilon}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i - v|^2 f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v\\
&\quad = \iint_{\mathbb{R}^d\times\mathbb{R}^d}(u_i - v)\otimes(v - u_i) : \nabla_x u_i f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v - \iint_{\mathbb{R}^d\times\mathbb{R}^d}(v - u_i)\cdot \mathrm{e}_i f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v\\
&\qquad + \frac{1}{\varepsilon}\iint_{\mathbb{R}^d\times\mathbb{R}^d}(v - u_i)\cdot\left(\sum_{j=1}^N \nabla K_{ij} * (\rho_j - \rho_j^\varepsilon)\right) f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v\\
&\quad =: I + II + III,
\end{aligned}$$

where

$$I \le \|\nabla u_i\|_{L^\infty}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i - v|^2 f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v,$$

and

$$II \le \varepsilon\|\mathrm{e}_i\|_{L^\infty}^2 + \frac{1}{4\varepsilon}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i - v|^2 f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v.$$

For III, we use $\nabla K_{ij} \in W^{1,\infty}$ for $i, j = 1, \dots, N$ with $i \neq j$ to obtain

$$\begin{aligned}
III \le& \frac{1}{\varepsilon}\int_{\mathbb{R}^d}\rho_i^\varepsilon(u_i^\varepsilon - u_i)\cdot\nabla K_{ii} * (\rho_i - \rho_i^\varepsilon)\,\mathrm{d}x\\
&+ \frac{1}{\varepsilon}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i - v|^2 f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v\right)^{1/2}\sum_{j\neq i}\|\nabla K_{ij}\|_{W^{1,\infty}} W_1(\rho_j, \rho_j^\varepsilon)\\
\le& \frac{1}{\varepsilon}\int_{\mathbb{R}^d}\rho_i^\varepsilon(u_i^\varepsilon - u_i)\cdot\nabla K_{ii} * (\rho_i - \rho_i^\varepsilon)\,\mathrm{d}x + \frac{1}{4\varepsilon}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i - v|^2 f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v\\
&+ \frac{c_K^2}{\varepsilon}\sum_{j\neq i} W_1^2(\rho_j, \rho_j^\varepsilon),
\end{aligned}$$

where

$$c_K := \max_{i=1,\dots,N}\sum_{j\neq i}\|\nabla K_{ij}\|_{W^{1,\infty}}$$

and we used

$$\begin{aligned}
&\left|\iint_{\mathbb{R}^d\times\mathbb{R}^d}(v - u_i)\cdot\left(\sum_{j\neq i}\nabla K_{ij} * (\rho_j - \rho_j^\varepsilon)\right) f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v\right|\\
&\quad \le \sum_{j\neq i}\|\nabla K_{ij} * (\rho_j - \rho_j^\varepsilon)\|_{L^\infty}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i - v|^2 f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v\right)^{1/2}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} f_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}v\right)^{1/2}
\end{aligned}$$

$$\leq \sum_{j\neq i}\|\nabla K_{ij}\|_{W^{1,\infty}} W_1(\rho_j,\rho_j^\varepsilon)\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i-v|^2 f_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}v\right)^{1/2}$$

due to $\|f_i^\varepsilon\|_{L^1}=1$ for all $\varepsilon>0$ and $i=1,\ldots,N$. This implies

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i-v|^2 f_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}v+\frac{1}{2\varepsilon}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i-v|^2 f_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}v\\
&\quad\leq C\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i-v|^2 f_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}v+\frac{1}{\varepsilon}\int_{\mathbb{R}^d}\rho_i^\varepsilon(u_i^\varepsilon-u_i)\cdot\nabla K_{ii}*(\rho_i-\rho_i^\varepsilon)\,\mathrm{d}x\\
&\qquad+\frac{c_K^2}{\varepsilon}\sum_{j\neq i}W_1^2(\rho_j,\rho_j^\varepsilon)+C\varepsilon
\end{aligned}$$

for some $C>0$ depends only on $\|\nabla u_i\|_{L^\infty}$ and $\|\mathrm{e}_i\|_{L^\infty}$. We then apply Theorem 4.3 and Lemma 4.7 to deduce

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_K(f_i^\varepsilon|\rho_i,u_i)+\frac{1}{2\varepsilon}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i-v|^2 f_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}v\\
&\quad\leq C\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i-v|^2 f_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}v+\frac{1}{\varepsilon}\sum_{i=1}^N\int_{\mathbb{R}^d}(\rho_i-\rho_i^\varepsilon)K_{ii}*(\rho_i-\rho_i^\varepsilon)\,\mathrm{d}x\right)+C\varepsilon\\
&\qquad+\frac{Cc_K^2}{\varepsilon}\sum_{j\neq i}W_1^2(\rho_{j0},\rho_{j0}^\varepsilon)+\frac{Cc_K^2}{\varepsilon}\sum_{j\neq i}\int_0^t\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_j-v|^2 f_j^\varepsilon\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s\\
&\quad\leq C\mathcal{E}_K(f_i^\varepsilon|\rho_i,u_i)+C\varepsilon+\frac{Cc_K^2}{\varepsilon}\sum_{j\neq i}W_1^2(\rho_{j0},\rho_{j0}^\varepsilon)\\
&\qquad+\frac{Cc_K^2}{\varepsilon}\sum_{j\neq i}\int_0^t\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_j-v|^2 f_j^\varepsilon\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s.
\end{aligned}$$

We now sum over $i=1,\ldots,N$ to derive

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^N\mathcal{E}_K(f_i^\varepsilon|\rho_i,u_i)+\frac{1}{2\varepsilon}\sum_{i=1}^N\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i-v|^2 f_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}v\\
&\quad\leq c_1\sum_{i=1}^N\mathcal{E}_K(f_i^\varepsilon|\rho_i,u_i)+c_2\varepsilon+\frac{c_2}{\varepsilon}\sum_{i=1}^N W_1^2(\rho_{i0},\rho_{i0}^\varepsilon)\\
&\qquad+\frac{c_2}{\varepsilon}\sum_{i=1}^N\int_0^t\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u_i-v|^2 f_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s,
\end{aligned}$$

where c_1 and c_2 are positive constant independent of $\varepsilon > 0$. In particular, we take $c_1 > 4c_2$. Then applying Grönwall's lemma to the above yields

$$
\begin{aligned}
&\sum_{i=1}^N \mathcal{E}_K(f_i^\varepsilon|\rho_i, u_i) + \frac{1}{\varepsilon}\left(\frac{1}{2} - \frac{c_2}{c_1}\right) \sum_{i=1}^N \int_0^t e^{c_1(t-s)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_i - v|^2 f_i^\varepsilon \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}s \\
&\quad \le \sum_{i=1}^N \mathcal{E}_K(f_{i0}^\varepsilon|\rho_{i0}, u_{i0}) e^{c_1 t} + \frac{c_2}{c_1}(e^{c_1 t} - 1)\left(\varepsilon + \frac{1}{\varepsilon}\sum_{i=1}^N W_1^2(\rho_{i0}, \rho_{i0}^\varepsilon)\right) \\
&\qquad - \frac{c_2}{c_1 \varepsilon} \sum_{i=1}^N \int_0^t \iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_i - v|^2 f_i^\varepsilon \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}s,
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\sum_{i=1}^N \mathcal{E}_K(f_i^\varepsilon|\rho_i, u_i) + \frac{1}{\varepsilon} \sum_{i=1}^N \int_0^t \iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_i - v|^2 f_i^\varepsilon \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}s \\
&\quad \le c_0 \sum_{i=1}^N \mathcal{E}_K(f_{i0}^\varepsilon|\rho_{i0}, u_{i0}) + \frac{c_0}{\varepsilon} \sum_{i=1}^N W_1^2(\rho_{i0}, \rho_{i0}^\varepsilon) + c_0 \varepsilon,
\end{aligned}
$$

where $c_0 > 0$ is independent of $\varepsilon > 0$. We finally use (6), (7), and Choi and Jung (2024, Lemma 4.2) to conclude our desired result. □

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