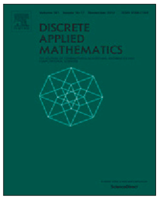
Discrete Applied Mathematics 376 (2025) 359–373

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam



Characterizing and computing in linear time mutual-visibility parameters in distance-hereditary graphs[☆,☆☆]

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ARTICLE INFO

Article history:
Received 25 May 2024
Received in revised form 16 April 2025
Accepted 17 July 2025

Keywords:
Mutual-visibility
Split decomposition
Graph classes
Distance-hereditary graphs
Graph algorithms
Computational complexity

ABSTRACT

The mutual-visibility problem in a graph G asks for the cardinality of a largest set of vertices $X \subseteq V(G)$ so that for any two vertices $x, y \in X$ there is a shortest x, y-path whose internal vertices are all not in X. Variations of this problem are known, based on the extension of the visibility property of vertices that are in and/or outside X. It is known that solving the mutual-visibility problem in all its variations is NP-complete, whereas it has been shown that there are exact formulas for special graph classes like paths, cycles, blocks, cographs, and for the Cartesian product of some simple graphs like paths, cliques and cycles.

In this paper, we study the (variations of) mutual-visibility problem in the context of distance-hereditary graphs. In particular, we introduce the direct canonical decomposition of a graph as a tool for defining useful structural properties of the graphs studied. Then, we show that such properties allow us to devise a linear-time algorithm for solving all the variants of the mutual-visibility problem for distance-hereditary graphs. In turn, this allowed us to show that a recently posed conjecture about the total mutual-visibility number of distance-hereditary graphs holds.

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1. Introduction

Given some points in the Euclidean space, they are mutually visible if and only if, for every possible triple of points, the three points are not collinear. In other words, two points p and q are mutually-visible when no other points belong to the segment pq. A line segment in Euclidean space represents the shortest path between two points, but in more general topologies, this type of path (called geodesic) may not be unique. Then, in general, two points are mutually visible when there exists at least a shortest path between them without further points.

In [27], this concept has been recently extended to mutual-visibility in graphs: mutual-visibility with respect to a set of vertices X is defined in terms of the existence of a shortest path between each pair of vertices of X not containing a third vertex from X. This visibility property is then understood as a kind of non-existence of "obstacles" between the two vertices in the mentioned shortest path, which makes them "visible" to each other. The main motivation comes from applications in the context of communication networks where it may be required that the agents located in a network

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[☆] Preliminary results appeared in the Proceedings of the XII Latin-American Algorithms, Graphs and Optimization Symposium (LAGOS) 2023 (Cicerone and Di Stefano 2023).

[☆☆] The work has been partially supported by the European Union - NextGenerationEU under the Italian Ministry of University and Research (MUR) National Innovation Ecosystem grant ECS00000041 - VITALITY - CUP J97G22000170005, and by the Italian National Group for Scientific Computation (GNCS-INdAM).

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https://doi.org/10.1016/j.dam.2025.07.026

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need to communicate in an *efficient* (i.e., through shortest paths) and *confidential* way (i.e., the messages exchanged do not pass through the location of any other agent).

This graph-based notion of mutual-visibility has remarkably attracted the attention of several investigations, which can be seen in the series of articles [1,7,8,14,15,18–22,28,35,36,36,38,41,43,43]. Some reasons of such interest might come from the following facts.

- There is a close relationship between the mutual-visibility problem and the general position problem [9,37]. The general position problem is also a distance-related topic that has attracted great interest in recent years (e.g., see [32–34,39,44,45]).
- In swarm robotics, one of the typical addressed problem is that of finding a distributed algorithm able to provide coordination abilities to robots so that they can move along the edges of a graph until reaching positions forming some pattern (e.g., see [10,29,42]) or having some required properties (e.g, see [4,40]). In [2,11–13], the required property is that of forming a mutual-visibility set of vertices of the underlying graph.
- Several connections between the mutual-visibility problem and some classical combinatorics topics have been identified. For example, studying the problem of mutual-visibility in the Cartesian product of complete graphs, it was noticed that solving this problem is equivalent to solving an instance of the well-known Zarankiewicz problem (see [20]). Furthermore, in [7] a relation with a classical Bollobás–Wessel theorem was proved, whilst relations with a Turán-type problem on hypergraphs have been observed in [8].
- The standard mutual-visibility problem has been modified in several directions to consider different visibility situations. For example, studying the problem of mutual-visibility in general Cartesian product graphs (see [20]), the notion of independent mutual-visibility was naturally required, so defined, and their first basic properties identified. In [21], a *total* version of the mutual-visibility problem was needed to study the strong product of graphs. This total notion was also a first step in the [19] work, where this total version was further studied, together with two "partially" total versions introduced to close all possible "visibility" situations that could exist between the vertices of a graph.

Formally, given a connected graph G and a set of vertices $X \subseteq V(G)$, two vertices $x, y \in V(G)$ are called to be *X-visible* if there is a shortest x, y-path whose interior vertices do not belong to X. For any given set $X \subseteq V(G)$ of a connected graph G, the following definitions are known from [19].

- *Mutual-visibility set*: if any two vertices of X are X-visible.
- *Outer mutual-visibility set*: if any two vertices $x, y \in X$ and any two vertices $x \in X$ and $y \in \overline{X}$ are X-visible.
- *Dual mutual-visibility set*: if any two vertices $x, y \in X$ and any two vertices $x, y \in \overline{X}$ are X-visible.
- *Total mutual-visibility set*: if any two vertices $x, y \in V(G)$ are X-visible.

Regarding such graph structures, the following parameters are defined as the cardinalities of the largest (respectively) mutual-visibility sets from the above ones. Hence, the *mutual-visibility number* (*dual mutual-visibility number*, *outer mutual-visibility number*, and *total mutual-visibility number*, respectively) of a graph G is denoted as $\mu(G)$ ($\mu_{\rm d}(G)$, $\mu_{\rm o}(G)$, and $\mu_{\rm t}(G)$, respectively). If $\tau \in \{\mu, \mu_{\rm d}, \mu_{\rm o}, \mu_{\rm t}\}$, then we say that $X \subseteq V(G)$ in a *τ-set* if $|X| = \tau(G)$. By definition, we have

$$\mu_{\rm t}(G) \leq \mu_{\rm o}(G) \leq \mu(G) \quad \text{and} \quad \mu_{\rm t}(G) \leq \mu_{\rm d}(G) \leq \mu(G).$$

In [19], it has been shown that computing all the above four mutual-visibility parameters is an NP-complete problem. For such a reason, this problem has been investigated in several graph classes, such as paths, trees, block graphs, cographs, and graphs resulting by applying graph operations like Cartesian product, strong product, and others. In most cases, these studies have revealed that the four mutual-visibility parameters can be expressed by closed formulae for such graph families.

Results. In this work, we study the mutual-visibility in distance-hereditary graphs. These graphs have been introduced by Howorka in [31], and are defined as those graphs in which every connected induced subgraph is isometric, that is the distance between any two vertices in the subgraph is equal to the one in the whole graph. Therefore, any connected induced subgraph of any distance-hereditary graph G "inherits" its distance function from G. This kind of graphs has been rediscovered many times (e.g., see [3]). Since their introduction, dozens of papers have been devoted to them, and different kinds of characterizations have been found: metric, forbidden subgraphs, cycle/chord conditions, level/neighborhood conditions, generative, and more (e.g., see [6]). Note that distance-hereditary graphs include some classes for which the mutual-visibility problem has already been studied: paths, trees, block graphs, and cographs.

To study the (different variations of) mutual-visibility problem in the context of distance-hereditary graphs, we first introduce the *directed canonical decomposition* of a graph, a variant of the well-known split decomposition [5]. Then, we show that the directed canonical decomposition leads to defining useful properties for computing the mutual-visibility number of distance-hereditary graphs. In particular, these properties allow us to devise a linear-time algorithm for computing the mutual-visibility number of a distance-hereditary graph. Finally, we show that the provided algorithm can compute all four mutual-visibility parameters in the studied graph class. In turn, this allows us also to show that a conjecture about the total mutual-visibility number of distance-hereditary graphs posed in [22] holds.

Outline. The paper is organized as follows. Section 2 gives the necessary notation and some preliminary concepts. In Section 3 we introduce the notion of directed canonical decomposition for distance hereditary graphs. Section 4 contains the algorithm that uses the directed canonical decomposition for computing in linear-time the mutual-visibility number of a distance-hereditary graphs. Section 5 gives a characterization for all the mutual-visibility parameters of the class of distance-hereditary graphs. Concluding remarks are provided in Section 6.

2. Notation and preliminaries

In this work, we consider undirected graphs and, unless otherwise stated, all graphs in the paper are connected. Given a graph G, $V(G)$ and $E(G)$ are used to denote its vertex set and its edge set, respectively. We use standard terminologies from [6,30], some of which are briefly reviewed here.

If $X \subseteq V(G)$, then $G[X]$ denotes the subgraph of G induced by X, that is the maximal subgraph of G with vertex set X. $N_G(X)$ is the open neighborhood of X in G, that is $N_G(X) = \{v \in V(G) \setminus X : \exists u \in X, uv \in E(G)\}$. If $X = \{u\}$ we simply write $N_G(u)$ instead of $N_G(\{u\})$. If $|N_G(u)| = 1$, u is called *pendant* vertex.

We call two graphs G and H *isomorphic*, and write $G \sim H$, if there exists a bijection $\varphi : V(G) \to V(H)$ such that $uv \in E(G) \Leftrightarrow \varphi(u)\varphi(v) \in E(H)$ for all $u, v \in V(G)$. Such a bijection φ is called *isomorphism*.

Concerning graph operations, given two distinct graphs G and H, the *union* $G \cup H$ is the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$, while the *join* $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

An *edge cut-set* of a connected graph G is any subset $E' \subseteq E(G)$ of edges that, if removed, forms a graph with more than one connected component. An edge cut-set of size 1 is simply called *cut-edge*. Similarly, a *vertex cut-set* is any subset $V' \subseteq V(G)$ such that the induced subgraph $G[V(G) \setminus V']$ has more than one connected component. A vertex cut-set of size 1 is called *cut-vertex*.

A graph G is *biconnected* if it has no cut-vertices. The *complete graph* (or *clique*) K_n, $n \geq 1$, is the graph with n vertices where each pair of distinct vertices is adjacent. The *path graph* P_n, $n \geq 2$, is the graph with $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ such that v_i is adjacent to v_j if $|i - j| = 1$. A *complete bipartite graph* $K_{m,n}$ is a graph whose vertices can be partitioned into two subsets V_1 and V_2, with $|V_1| = m$ and $|V_2| = n$, such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. A *star graph* corresponds to any $K_{1,n}$. G is a *block graph* if each biconnected component (i.e., a "block") of G is a clique. A *cograph* is a graph which contains no induced path on four vertices. Cographs can be characterized in many different ways, see [23]. For instance, cographs are precisely the graphs that can be obtained from K_1 using a sequence of disjoint unions and joins of graphs. Block graphs and cographs form subclasses of distance-hereditary graphs [6].

2.1. Mutual-visibility

Concerning the mutual-visibility, it is easy to observe that $\mu(G) \geq 1$ for each graph G (indeed, any vertex $u \in V(G)$ is a mutual-visibility set of G). From [27] we also know that computing the mutual-visibility number of a graph is NP-complete. Concerning small values of μ, we know that:

- $\mu(G) = 1$ if and only if $G \sim K_1$;
- $\mu(G) = 2$ if and only if $G \sim P_n$, $n \geq 2$;

Moreover, a partial characterization for $\mu(G) = 3$ is provided in [20]. The following lemma asserts that there always exists a maximum mutual-visibility set for a graph G without cut-vertices.

Lemma 1 (*[27, Lemma 2.5]*). *Let C be the set of the cut-vertices in a graph G. There exists a μ-set X for G such that $X \cap C = \emptyset$.*

2.2. Split decomposition and distance-hereditary graphs

We will follow the definition of split decomposition in [5]. Let G be a graph. A *split* in G is a vertex partition (X, Y) of G such that $|X|, |Y| \geq 2$ and every vertex of $N_G(X)$ is adjacent to every vertex of $N_G(Y)$. Notice that not all connected graphs have a split, and those that do not have a split are called *prime graphs*. A *marked graph* D is a connected graph with a subset of edges $M(D)$, called *marked edges*, that form a matching such that every edge in $M(D)$ is a cut-edge. The ends of the marked edges are called *marked vertices*, and the components of $(V(D), E(D) \setminus M(D))$ are called *bags* of D. The edges in $E(D) \setminus M(D)$ are called *unmarked edges*, and the vertices that are not marked are called *unmarked vertices*. If (X, Y) is a split in G, then we construct a marked graph D that consists of the vertex set $V(G) \cup \{x', y'\}$ for two distinct new vertices $x', y' \notin V(G)$ and the edge set $E(G[X]) \cup E(G[Y]) \cup \{x'y'\} \cup E'$, where we define $x'y'$ as marked and

$$E' := \{xx' | x \in X \wedge \exists y \in Y : xy \in E(G)\} \cup \{y'y | y \in Y \wedge \exists x \in X : xy \in E(G)\}.$$

The marked graph D is called a *simple decomposition* of G (e.g., see Fig. 1). The following proposition can be easily observed from the definition of split.

Proposition 2. *Let (X, Y) be a split of a graph G. If $X \setminus N_G(Y)$ ($Y \setminus N_G(X)$, resp.) is not empty, then $N_G(Y)$ ($N_G(X)$, resp.) is a vertex cut-set of G.*

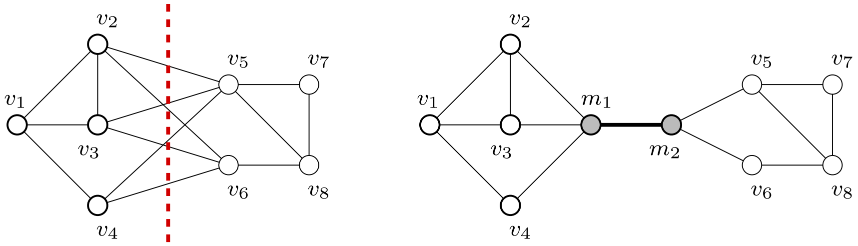

Fig. 1. A graph G along with the marked graph forming a simple decomposition of G. Gray vertices and bold lines are used to visualize marked vertices and marked edges, respectively. Notice that each obtained bag is isomorphic to an induced subgraph of G.

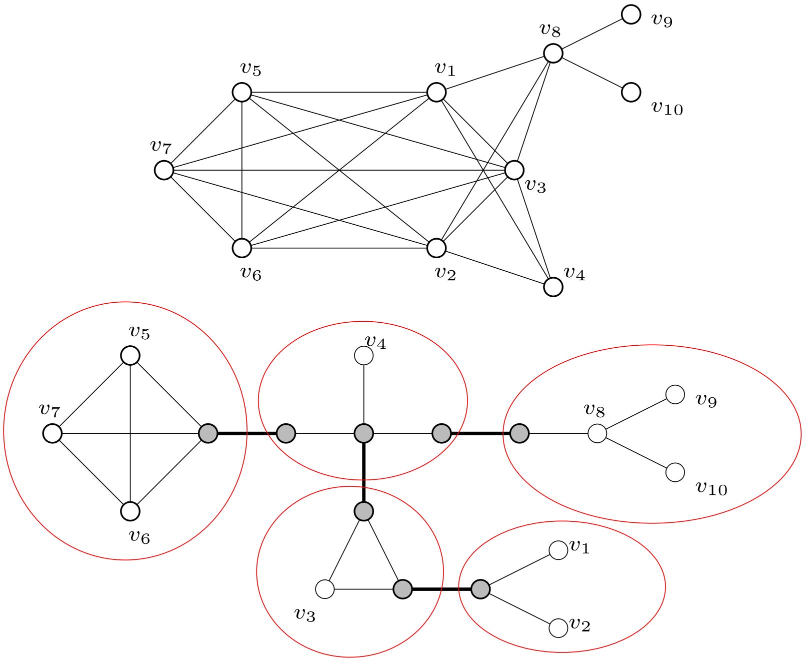

Fig. 2. A distance-hereditary graph G along with D_G. Red ovals, gray vertices, and bold lines are used to visualize the bags, the marked vertices, and the marked edges of D_G.

Fig. 1 shows a split (X, Y), where $X = \{v_1, v_2, v_3, v_4\}$, $Y = \{v_5, v_6, v_7, v_8\}$, and $N_G(Y) = \{v_2, v_3, v_4\}$. According to the above proposition, $N_G(Y)$ is a vertex cut-set (indeed, removing all its vertices would disconnect v_1 from each vertex in Y).

A *split decomposition* of a graph G is a marked graph D defined inductively to be either G (if G is prime) or a marked graph defined from a split decomposition D' of G by replacing a connected component H of $(V(D'), E(D') \setminus M(D'))$ with a simple decomposition of H. Fig. 2 shows a split decomposition of a graph (as explained later, the inductive process is stopped when a clique is produced).

For a marked edge xy connecting two bags X and Y in a split decomposition D, the *recomposition* of D along xy is the split decomposition D' obtained by making each vertex in $N_X(x')$ adjacent to each vertex in $N_Y(y')$ and by removing x', y', and their adjacent edges. For a split decomposition D, let $G[D]$ denote the graph obtained from D by recomposing all marked edges. By definition, if D is a split decomposition of G, then $G[D] = G$. Since each marked edge of a split decomposition D is a cut-edge and all marked edges form a matching, if we contract all unmarked edges in D, then we obtain a tree. We call it the *decomposition tree* of G associated with D and denote it by $T(D)$. To distinguish the vertices of $T(D)$ from the vertices of G or D, the vertices of $T(D)$ will be called *nodes*. Obviously, the nodes of $T(D)$ are in bijection with the bags of D. Two bags of D are called *neighbor bags* if their corresponding nodes in $T(D)$ are adjacent.

A split decomposition D of G is called a *canonical split decomposition* (or canonical decomposition for short) if each bag of D is either a prime graph, a star, or a complete graph, and D is not the refinement of a decomposition with the same property. Fig. 2 shows a graph G along with its canonical decomposition.

Theorem 3 (*Cunningham and Edmonds [25], Dahlhaus [26]*)**.** *Every connected graph G has a unique canonical decomposition, up to isomorphism, and it can be computed in time $O(|V(G)| + |E(G)|)$.*

From Theorem 3, we can talk about only one canonical decomposition of a graph G because all canonical decompositions of G are isomorphic. Let D be a split decomposition of a graph G with bags that are either prime graphs, complete graphs or stars (it is not necessarily a canonical decomposition). The type of a bag of D is either P, K, or S, depending on whether it is a prime graph, a complete graph, or a star. The type of a marked edge uv is AB where A and B are the types of bags containing u and v, respectively. If $A = \mathsf{S}$ or $B = \mathsf{S}$, then we can replace S with S_p or S_c depending on whether the end of the marked edge is a pendant vertex (i.e., a leaf) or the center of the star.

Theorem 4 (*Bouchet [5]*). *Let D be a split decomposition of a graph with bags that are either complete graphs or stars. Then D is a canonical decomposition if and only if it has no marked edge of type KK or $\mathsf{S}_p\mathsf{S}_c$.*

From now on, for the sake of simplicity, we denote by D_G the canonical decomposition of G, by M_G the marked edges of D_G, and by T_G the decomposition tree associated with D_G. We remark on the well-known property for which each bag of D_G is isomorphic to an induced subgraph of G. We use the following characterization of distance-hereditary graphs.

Theorem 5 (*Bouchet [5]*). *A graph G is distance-hereditary if and only if each bag of D_G is of type K or S.*

According to this theorem, if G is distance-hereditary we denote any bag of D_G as K-bag or S-bag.

An *alternating path* is any path connecting two vertices in D_G such that an unmarked edge and a marked edge alternatively appear in the path (e.g., see the unique shortest path from v_1 to v_7 in the canonical decomposition shown in Fig. 2).

Lemma 6 (*Courcelle [24]*). *Let G be a graph. Then, $xy \in E(G)$ if and only if there exists an alternating path between x and y in D_G. This alternating path is moreover unique.*

We conclude this section by providing a useful statement.

Lemma 7. *Let G be a graph. If $v \in V(G)$ is the center of a S-bag of D_G, then it is a cut-vertex of G.*

Proof. If D_G has only one bag, according to the hypothesis G is a star with v as the central vertex. In this case, the statement trivially follows. Assume now that D_G has at least two bags and let X be the set of vertices of D_G forming the S-bag of which v is the center. Let $u_1, u_2 \in X \setminus \{v\}$. Since D_G has at least two bags, we can assume that u_1 is marked. Consider any alternating path $v, u_1, w_1, w_2, \ldots, w_k$, with $k \geq 2$ and $w_k \in V(G)$. Since u_1w_1 is a marked edge, by Proposition 2 we easily get that removing v in G would disconnect w_k from either u_2 (if u_2 is unmarked) or u_2' (if u_2 is marked and $u_2, \ldots, u_2'$ is any alternating path from u_2 to an unmarked vertex u_2'). □

Since this lemma proves that the center v of each S-bag of D_G is a cut-vertex of G, for the sake of clarity, we simply call *σ-vertex* each of such vertices. We remark that, by definition, σ-vertices are always unmarked.

3. The directed canonical decomposition

In this section, we introduce the notion of *directed canonical decomposition* of a distance hereditary graph G. It constitutes the main ingredient for the construction of the algorithm capable of computing a μ-set of G. Such an algorithm will be described in the next section.

Definition 1. Let G be a distance-hereditary graph.

- The *directed canonical decomposition* of G is denoted as $\overrightarrow{D}_G$, and corresponds to the mixed graph obtained from D_G according to the following edge-replacing operation: for each marked edge uv of type $\mathsf{S}_c\mathsf{X}$, with $\mathsf{X} \in \{\mathsf{S}_c, \mathsf{K}\}$, remove uv from M_G and insert an edge directed from u to v into M_G.
- A directed decomposition tree $\overrightarrow{T}_G$ can be associated with $\overrightarrow{D}_G$ in the same way T_G has been associated with the canonical decomposition D_G.

Note that, according to this definition, when forming $\overrightarrow{D}_G$, a marked edge uv is replaced by one oriented edge (when uv is of type $\mathsf{S}_c\mathsf{K}$) or by two opposite oriented edges (when uv is of type $\mathsf{S}_c\mathsf{S}_c$). For the sake of simplicity, we call *arrow* an oriented marked edge $a = (u, v)$, and we call *opposite* two distinct arrows joining the same pair of vertices (see Fig. 3). Given an arrow $a = (u, v)$, we call u (v, resp.) *tail* (*head*, resp.) of a.

Definition 2. Let $a = (u, v)$ be an arrow of $\overrightarrow{D}_G$. We denote by

- $\overrightarrow{D}^t_G(a)$ ($\overrightarrow{D}^h_G(a)$, resp.) the subgraph of $\overrightarrow{D}_G$ obtained by removing a (and the arrow (v, u) if present) and containing the tail u (the head v, resp.);
- $V^h_G(a)$ ($V^t_G(a)$, resp.) the set containing all the unmarked vertices in $\overrightarrow{D}^h_G(a)$ ($\overrightarrow{D}^t_G(a)$, resp.) reachable from v (u, resp.) via an alternating path.

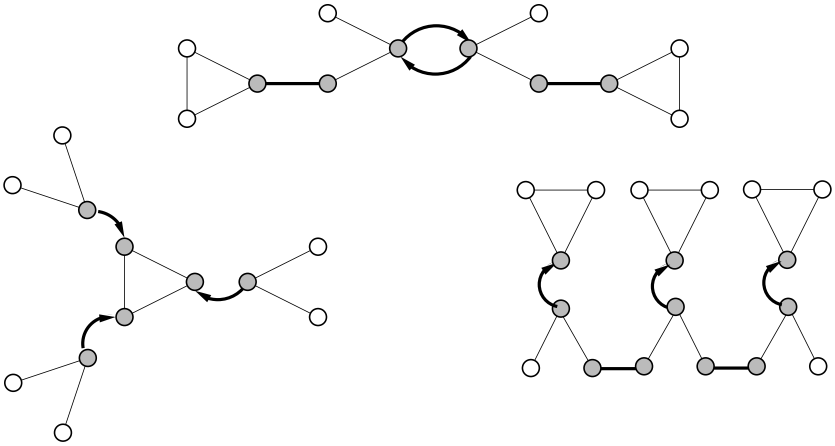

Fig. 3. Three examples of directed canonical decomposition. In particular: *(top)* two opposite t-arrows, *(bottom-left)* head-connected t-arrows, and *(bottom-right)* tail-connected t-arrows.

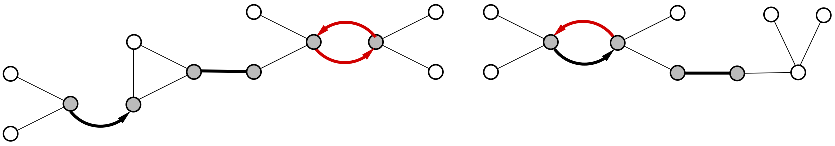

Fig. 4. Two directed canonical decompositions (t-arrows are represented in red).

Definition 3. Given a distance-hereditary graph G, an arrow $a = (u, v)$ in $\overrightarrow{D}_G$ is called *terminal* (t-arrow, for short) if both the following conditions hold:

1. $V_G^h(a)$ does not contain a σ-vertex;
2. $\overrightarrow{D}_G^h(a)$ does not contain an alternating v, u'-path in D_G followed by an arrow (u', v').

Fig. 4 shows examples of *t-arrows*.

3.1. Structural properties of t-arrows

The following statements provide some structural properties of t-arrows.

Lemma 8. *Given a distance-hereditary graph G, if $a = (x, y)$ and $b = (y, x)$ are opposite t-arrows in $\overrightarrow{D}_G$, then they are the only t-arrows in $\overrightarrow{D}_G$.*

Proof. Let us assume that there is an arrow $e = (u, v)$ in $\overrightarrow{D}_G^h(a) = \overrightarrow{D}_G^t(b)$. On one hand, since a is a t-arrow, there is no alternating path connecting the marked vertex y to the marked vertex u, tail of e. Since b is a t-arrow, there is no alternating path connecting the marked vertex y to the marked vertex v, head of e. On the other hand, any shortest path P connecting y to u or to v, that is not alternating, must contain two consecutive unmarked edges (two consecutive marked edges are not possible). Consider the first two consecutive unmarked edges encountered on P starting from y. These two edges cannot belong to a K-bag otherwise P would not be a shortest path. Hence, the two edges must belong to a S-bag. If the center of this S-bag is a marked vertex, then the corresponding marked edge is an arrow, but it is not possible since a is a t-arrow. If the center of the S-bag is unmarked, then it is a σ-vertex and a would not be a t-arrow, by definition. So $\overrightarrow{D}_G^h(a)$ does not contain any t-arrow.

By symmetry, also $\overrightarrow{D}_G^h(b) = \overrightarrow{D}_G^t(a)$ does not contain t-arrows. □

The following three results show how the t-arrows appear in a directed canonical decomposition of a graph.

Lemma 9. *If G is a distance-hereditary graph, then in $\overrightarrow{D}_G$ there is no path $x, y, u_1, u_2, \ldots, u_k, x', y'$ in which both (x, y) and (x', y') are t-arrows.*

Proof. Assume that there exists in $\overrightarrow{D}_G$ a path $x, y, u_1, u_2, \ldots, u_k, x', y'$ in which both (x, y) and (x', y') are t-arrows. By the definition of t-arrow, such a path cannot be an alternating path. Then, consider any shortest path P defined as

$x, y, u'_1, u'_2, \ldots, u'_{k'}, x', y'$. Notice that $k' \geq 1$ otherwise P reduces to x, y, x', y', an alternating path. As shown in the proof of Lemma 8, P contains two consecutive unmarked edges that belong to a S-bag. In the same proof, it is shown that the presence of this S-bag implies that (x, y) cannot be t-arrow, a contradiction. □

Assuming that $\overrightarrow{D}_G$ has more than one t-arrow, then either Lemma 8 holds and there are only two opposite t-arrows, or, by Lemma 9, a shortest path connecting the vertices of two t-arrows (x, y) and (x', y') is either in the form $x, y = u_0, u_1, u_2, \ldots, u_k = y', x'$ or in the form $y, x = u_0, u_1, u_2, \ldots, u_k = x', y'$, $k \geq 1$. If the path is in the first form, we say that the arrows are *head-connected*, otherwise *tail-connected*. Fig. 3 shows examples of head-connected and tail-connected t-arrows.

Lemma 10. *Let G be a distance-hereditary graph. If $\overrightarrow{D}_G$ has more than one t-arrow, they are (cf. Fig. 3):*

- *two and opposite, or*
- *pairwise head-connected, or*
- *pairwise tail-connected.*

Proof. The first case follows directly from Lemma 8. Suppose now that in $\overrightarrow{D}_G$ there are not two opposite t-arrows. By contradiction assume that all the t-arrows in $\overrightarrow{D}_G$ are neither pairwise head-connected nor pairwise tail-connected. Then there must exist three t-arrows $a = (x, y)$, $b = (x', y')$, and $c = (x'', y'')$ such that two of them, say a and b, are head-connected and the third t-arrow c that is tail-connected with a or b. Without loss of generality, assume that c is tail-connected with a. Then, there must exist a path $y, x, u_1, \ldots, u_k, x'', y''$. Hence, the path $x', y', v_1, \ldots, v_{k'}, y, x, u_1, \ldots, u_k, x'', y''$ contradicts with Lemma 9. □

Interestingly, for a distance-hereditary G, if the t-arrows of $\overrightarrow{D}_G$ are pairwise head-connected they are in a particular configuration.

Lemma 11. *Let G be a distance-hereditary graph such that all the pairs of t-arrows in $\overrightarrow{D}_G$ are head-connected. Then, the heads of all the t-arrows belong to a unique* K*-bag.*

Proof. Let $a = (x, y)$ and $b = (x', y')$ be two t-arrows. Assume that the shortest path P connecting the two heads contains an edge e from a S-bag. If P is an alternating path, one of the two vertices of e is the center of the S-bag and is marked. The corresponding marked edge is an arrow: impossible because one between a and b would not be a t-arrow. If P is not an alternating path, there are two edges, say e and e', of a S-bag in P that are both incident to the center of the S-bag. Whether or not the center is a σ-vertex, both a and b would not be t-arrows. Then, the path between a and b is an alternating path and contains only edges from K-bags. But, by Theorem 4, there cannot be two adjacent K-bags and then the K-bag is unique and P consists only of one edge from this clique. □

The next statement concerns the case in which G has no arrows.

Lemma 12. *Given a distance-hereditary graph G, $\overrightarrow{D}_G$ has no arrows if and only if G is a block graph.*

Proof. ($\Leftarrow$) Assume that a is an arrow of $\overrightarrow{D}_G$. Recomposing the marked edge in D_G corresponding to a would lead to a biconnected subgraph of G that is not a clique, contradicting the definition of block graph.

($\Rightarrow$) In this case, all the marked edges in D_G are of type KS_p or $\mathsf{S}_p\mathsf{S}_p$ whose recompositions lead to biconnected subgraphs of G that are cliques: K_2 in the latter case, and K_n, $n \geq 3$, in the former. □

The last result of this section restricts Lemma 10 to the case in which G is not a cograph.

Lemma 13. *Let G be a distance-hereditary graph but not a cograph. If $\overrightarrow{D}_G$ has more than one t-arrow, they are pairwise tail-connected.*

Proof. Assume that G has more than one t-arrow. Lemma 10 states that the t-arrows of $\overrightarrow{D}_G$ are (1) two and opposite, or (2) pairwise head-connected, or (3) pairwise tail-connected. If there exists a σ-vertex v in $\overrightarrow{D}_G$, according to the definition of t-arrow (cf. Definition 3), the existence of v implies that the above first two cases cannot occur. Then, in the remainder of this proof, we assume there are no σ-vertices in $\overrightarrow{D}_G$.

In D_G, there must exist a pair of bags B' and B'', joined by a marked edge $e = (v', v'')$ such that v' is in B' and v'' is in B'', with the following properties:

(*i*) removing e from D_G divides D_G into two sub-decompositions denoted as D'_G and D''_G, where D'_G contains v' and D''_G contains v'';
(*ii*) recomposing all the bags in D'_G gets a cograph G';
(*iii*) recomposing the cograph G' from (ii) with B'' gets a distance-hereditary graph G''.

These properties hold otherwise G would be a cograph, against the hypothesis. Note that G' can be assumed to have a sort of "maximality" property because it cannot be further extended toward a cograph using the bag B''.

We recall that, according to Theorems 4 and 5, in D_G the bags are of type K or S only, and there are no marked edges of type KK or $\mathsf{S}_p\mathsf{S}_c$. It is easy to observe that if B'' is a K-bag or a S-bag where v'' is the central vertex, then also G'' is a cograph, against the maximality of G'. Then, B'' is S-bag where v'' is a pendant vertex.

Consider now the directed decomposition $\overrightarrow{D}_G$. We can assume that removing e from D_G divides $\overrightarrow{D}_G$ into two directed sub-decompositions $\overrightarrow{D}'_G$ and $\overrightarrow{D}''_G$, where $\overrightarrow{D}'_G$ contains v' and $\overrightarrow{D}''_G$ contains v''. Since B'' is S-bag where v'' is a pendant vertex, then there must exist a t-arrow a'' in $\overrightarrow{D}''_G$ not oriented toward $\overrightarrow{D}'_G$.

By recalling that v' is a marked vertex in B', we now analyze three cases according to B' and v'. We will show that one of them cannot occur, while, in the other two cases, a t-arrow a' in $\overrightarrow{D}'_G$ not oriented toward $\overrightarrow{D}''_G$ must exist.

- Since there are no marked edges of type $\mathsf{S}_p\mathsf{S}_c$ in $\overrightarrow{D}_G$ and v'' is a pendant vertex of B'', B' cannot be a S-bag with v' in its center.
- If B' is a S-bag with v' pendant vertex, then the analysis above can be symmetrically applied to B' thus providing a t-arrow a' in $\overrightarrow{D}'_G$ not oriented toward $\overrightarrow{D}''_G$.
- Assume that B' is a K-bag. This implies there must exist one or more S-bags in $\overrightarrow{D}'_G$, otherwise extending G' with B'' still gets a cograph, against the maximality of G'.
  Since there are no σ-vertices in $\overrightarrow{D}_G$, all the centers of these S-bags of $\overrightarrow{D}'_G$ are marked vertices that generate arrows in $\overrightarrow{D}_G$. If all such arrows are oriented toward $\overrightarrow{D}''_G$, then it can be easily observed that the marked vertex v' in the K-bag is a universal vertex in G'. In such a case, extending the cograph G' with B'' when v' is a universal vertex still gets a cograph, against the maximality of G'. Hence, there exists in $\overrightarrow{D}'_G$ an arrow not oriented toward $\overrightarrow{D}''_G$. Also in this case, this arrow implies that there must exist a t-arrow a' in $\overrightarrow{D}'_G$ not oriented toward $\overrightarrow{D}''_G$.

According to the definition of t-arrow, the presence of a' in $\overrightarrow{D}'_G$ and a'' in $\overrightarrow{D}''_G$ shows that the t-arrows of $\overrightarrow{D}_G$ are pairwise tail-connected. □

3.2. T-arrows and mutual-visibility

The following statements motivate the introduction of t-arrows with respect to the need of computing a μ-set of a distance-hereditary graph G.

Lemma 14. *Given a distance-hereditary graph G and a set $X \subseteq V(G)$ and an arrow a in $\overrightarrow{D}_G$, then two vertices $x, y \in V^t_G(a)$ are X-visible if there exists $w \in V^h_G(a) \setminus X$.*

Proof. Since there are two alternating paths from x to w and from w to y, $w \in N(x) \cap N(y)$. This implies that x and y are at distance at most two. If x and y are adjacent, then they are obviously X-visible. If they are at distance two, path x, w, y is a shortest path, and, since w is not in X, vertices x and y are X-visible. □

Lemma 15. *Given a distance-hereditary graph G, a set $X \subseteq V(G)$, and two arrows $a = (u, v)$ and $b = (u', v')$ in $\overrightarrow{D}_G$ connected by an alternating path $u, v = v_0, v_1, \ldots, v_k = u', v'$, two vertices $x, y \in V^t_G(a)$ are X-visible if there exists $w \in V^h_G(b) \setminus X$.*

Proof. The proof reduces to Lemma 14 by simply observing that $V^t_G(a) \subseteq V^t_G(b)$ and $V^h_G(a) \supseteq V^h_G(b)$. □

4. Computing a μ-set

We have already remarked that the directed canonical decomposition of a distance hereditary graph G has been formulated to be a valid tool for computing a μ-set of G. The algorithm we propose for this purpose is simply denoted as $\mathcal{A}$ and reported in Fig. 6. Basically, it first computes $\overrightarrow{D}_G$ and then performs specific actions according to the number and the structure of the t-arrows of $\overrightarrow{D}_G$. In particular, the strategy underlying $\mathcal{A}$ is based on the following steps, in order:

- it is initially assumed that the entire $V(G)$ can play the role of a μ-set denoted as X;
- then, if there are σ-vertices in $\overrightarrow{D}_G$, they are removed from X (indeed, Lemma 7 states that each σ-vertex is a cut-vertex in G, and hence, by Lemma 1, it can be correctly eliminated from X);
- finally, t-arrows of $\overrightarrow{D}_G$ are analyzed:
  - if there are no arrows at all (in this case, Lemma 12 states that G is a block graph), or no t-arrows, then it is possible to prove that the computed set X is indeed a μ-set of G;
  - if there are t-arrows, Lemmata 14 and 15 state that some vertices must be further removed from X. For such cases, $\mathcal{A}$ follows three different approaches according to the structural characterization given by Lemma 10.

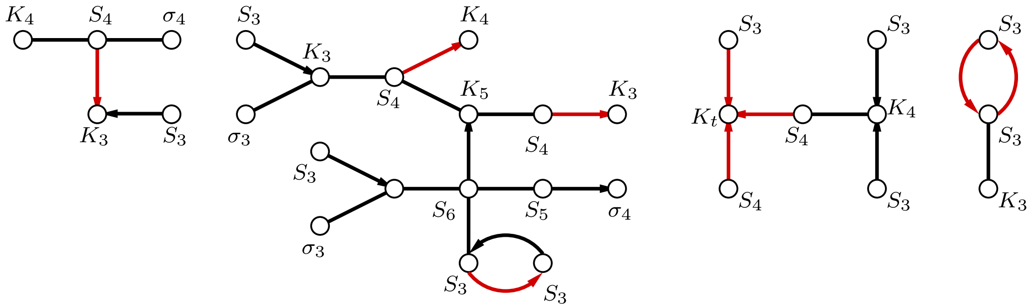

Fig. 5. Four directed decomposition trees named, in order, $\overrightarrow{T}_G$, $\overrightarrow{T}_{G_1}$, $\overrightarrow{T}_{G_2}$, and $\overrightarrow{T}_{G_3}$ (t-arrows are represented in red). Each vertex is labeled with K_n, S_n, or σ_n to recall the corresponding bag in $\overrightarrow{D}_G$ (a clique, a star without σ vertex, a start with σ vertex). The subscript n denotes the size of that bag. For instance, $\overrightarrow{T}_G$ corresponds to the directed canonical decomposition shown in Fig. 2.

To describe the algorithm in terms of these three approaches, in what follows we use the examples illustrated by the directed decomposition trees named $\overrightarrow{T}_{G_1}$, $\overrightarrow{T}_{G_2}$, and $\overrightarrow{T}_{G_3}$ and shown in Fig. 5.

Tree $\overrightarrow{T}_{G_1}$ is obtained from a graph G_1 in which its directed canonical decomposition has many t-arrows that are pairwise tail-connected. In this case, $\mathcal{A}$ starts by removing three σ vertices from $X = V(G)$, and then proceeds to execute the block of Lines 6–10. In such a block, for each t-arrow a, an arbitrary vertex $w \in V^h_{G_1}(a)$ is removed from the current set X. After all the t-arrows are processed, X is returned as a μ-set of G.

In the second example, $\overrightarrow{T}_{G_2}$ is obtained from a graph G_2 in which its directed canonical decomposition has pairwise head-connected t-arrows. In this case, $\mathcal{A}$ executes the block of Lines 11–23. According to Lemma 11, the heads of all the t-arrows belong to a unique K-bag (see K_t in the example). If K_t has an unmarked vertex w (in the example, K_t should have $t \geq 4$ vertices), then $X \setminus \{w\}$ is returned. This is because for each arrow a in $\overrightarrow{D}_{G_2}$ each pair of elements $u, v \in V^t_{G_2}(a)$ are connected to w, and the removal of w from X makes them X-visible. However, if w does not exist because all the vertices of K_t are marked, still there are cases in which the removal of a single vertex is sufficient. These cases occur when a generic directed canonical decomposition $\overrightarrow{D}_G$ has a "special subgraph" as in the following definition.

Definition 4. Let $a = (u, v)$ be a t-arrow of $\overrightarrow{D}_G$. We say that $\overrightarrow{D}^h_G(a)$ is *special* if one of the following conditions holds:

- $\overrightarrow{D}^h_G(a)$ is a S-bag with three vertices: v is the center and it is marked, x and y are the pendant vertices and are both unmarked,
- $\overrightarrow{D}^h_G(a)$ is composed by a S-bag and a K-bag. The S-bag has three vertices: v is the center and it is marked, x is an unmarked vertex, and y is a marked pendant vertex connected, via a marked edge, to the K-bag.

In both cases, the vertex x is called *special vertex* of $\overrightarrow{D}^h_G(a)$.

It follows that, if in the K-bag K_t in $\overrightarrow{T}_{G_2}$ has $t = 3$ marked vertices, then Lines 16–18 are executed. There, just one special vertex w is removed from X before returning this set as μ-set of G_2. Note that, in case there are no special subgraphs in $\overrightarrow{D}_{G_2}$, two vertices w and w' are removed from X at Line 21. In the next section, we will prove that this is the minimum number of vertices to remove from X for X to be a mutually visible set.

Concerning the last example, $\overrightarrow{T}_{G_3}$ is obtained from a graph G_3 in which its directed canonical decomposition contains just two opposite t-arrows a and b. In the example, $\mathcal{A}$ executes the block of Lines 24–33. This is because there are special subgraphs, and hence it is enough to remove just one special vertex from X (Line 26 or 29). In general, if there are no special subgraphs, the algorithm eliminates two unmarked vertices from X (one from $V^t_G(a)$ and the other from $V^t_G(b)$, cf. Lines 31–32).

4.1. Correctness and complexity

We provide in this section the correctness and complexity of $\mathcal{A}$. To this end, we first give some useful lemmas.

Lemma 16. *Given a distance-hereditary graph G, if a is a t-arrow of G, then $V^h_G(a)$ is the set of all unmarked vertices of $\overrightarrow{D}^h_G(a)$.*

Proof. We recall that $V^h_G(a)$ is the set that contains all the unmarked vertices in $\overrightarrow{D}^h_G(a)$ accessible from v via an alternating path.

Let $a = (u, v)$ and assume that there is a vertex $x \in V(G)$ in $\overrightarrow{D}^h_G(a)$ which is not in $V^h_G(a)$, that is x is not reachable from v via an alternating path. Then the shortest path from v to x must contain at least a vertex that is the center of

```
Algorithm $\mathcal{A}$:
Input: a distance-hereditary graph $G$.
Output: a $\mu$-set $X$ of G.

Compute $\overrightarrow{D}_G$;
Let $S$ be the set of the $\sigma$-vertices of $\overrightarrow{D}_G$;
Let $X = V(G) \setminus S$;
Let $A$ be a set containing all the t-arrows of $\overrightarrow{D}_G$;
if $A = \emptyset$ then return $X$;
if $A$ contains either a single t-arrow or pairwise tail-connected t-arrows then
    forall t-arrow $a$ in $A$ do
        Let $w \in V^h_G(a)$;
        $X := X \setminus \{w\}$
    return $X$
if $A$ contains pairwise head-connected t-arrows then
    Let $K$ be the K-bag containing all the heads of the t-arrows in $A$;
    if $K$ has an unmarked vertex $w$ then
        $X := X \setminus \{w\}$;
    else
        Let $C := \{\overrightarrow{D}^t_G(a) \mid a \in A\}$;
        if $C$ contains a special element $D$ then
            $X := X \setminus \{w\}$, with $w$ special vertex of $D$;
        else
            Let $D$ and $D'$ be two distinct elements of $C$;
            Let $w \in V(D)$ and $w' \in V(D')$ be unmarked vertices;
            $X := X \setminus \{w, w'\}$ ;
    return $X$
if $A$ contains two opposite t-arrows $a$ and $b$ then
    if $\overrightarrow{D}^h_G(a)$ is special then
        $X := X \setminus \{w\}$, with $w \in V^h_G(a)$ special vertex of $\overrightarrow{D}^h_G(a)$
    else
        if $\overrightarrow{D}^h_G(b)$ is special then
            $X := X \setminus \{w'\}$, with $w' \in V^h_G(b)$ special vertex of $\overrightarrow{D}^h_G(b)$
        else
            Let $w \in V^t_G(a)$ and $w' \in V^t_G(b)$ be unmarked vertices;
            $X := X \setminus \{w, w'\}$
    return $X$
```

Fig. 6. Algorithm for computing a μ-set X of any distance-hereditary graph G.

a S-bag. Consider the first of these vertices starting from v. If it is marked, then the corresponding marked edge is an arrow. Otherwise, it is a σ-vertex. In both cases, a cannot be a t-arrow. □

Lemma 17. *Let G be a distance-hereditary graph and $X \subseteq V(G)$. Let x and y be distinct vertices in $V(G)$, and let P be the shortest x, y-path in $\overrightarrow{D}_G$. Let Σ be the set of the internal σ-vertices of P and let A be the set of the arrows $a = (u, v)$ in $\overrightarrow{D}_G$ such that only u is in P. Vertices x and y are X-visible if and only if $\Sigma \cap X = \emptyset$ and for each $a \in A$ there exists a vertex $w \in V^h_G(a)$ such that $w \notin X$.*

Proof. ($\Longleftarrow$) If Σ and A are both empty, P is an alternating x, y-path. By Lemma 6, then x and y are adjacent in G and hence mutually visible. Otherwise, when $\Sigma \neq \emptyset$ or $A \neq \emptyset$, P is not an alternating path (i.e., in P there are pairs of consecutive unmarked edges that are both incident to a center of a S-bag). These centers belong to Σ (if they are σ-vertices) or are tails of the arrows in A. Let $u_1, u_2, \ldots, u_k$ be the sequence of these centers in P, ordered from x to y. Denote as v_i the vertex u_i if u_i is a σ-vertex or a vertex $w \in V^h_G(a) \setminus X$ if u_i is the tail of an arrow $a \in A$. Notice that, according to the assumptions, $v_i \notin X$ for each $1 \leq i \leq k$. Then, $x = v_0, v_1, v_2, \ldots, v_k, v_{k+1} = y$ is a path in G as there is an alternating path between v_i and v_{i+1}, for each $0 \leq i \leq k$, in D_G and then an edge between v_i and v_{i+1} in G by Lemma 6. Call this path Q. Path Q is an induced path in G since there are no alternating paths connecting non-consecutive vertices v_i in D_G and then, since G is a distance-hereditary graph, the path is a shortest path in G. So Q is a shortest path in G and no internal vertex of Q is in X, then x and y are X-visible.

($\Longrightarrow$) Let us assume that $\Sigma \cap X \neq \emptyset$. Let u be a vertex in $\Sigma \cap X$. Since P passes through u and by Lemma 1, vertex u is a cut-vertex in G, then each shortest path in G between x and y passes through u. Then x and y are not X-visible. Assume now that there exists $a = (u, v) \in A$ such that all the vertices in $V^h_G(a)$ are in X. Consider any shortest path Q between x and y in G. By Lemma 6 each edge in Q corresponds to an alternating path in D_G. Then, sequences of alternating paths connect x and y in $\overrightarrow{D}_G$, and since P is unique in $\overrightarrow{D}_G$, one of these alternating paths must pass through the tail u. Then, a vertex of $V^h_G(a)$ is in Q. Since $V^h_G(a) \subseteq X$, x and y are not X-visible. □

Lemma 18. *Given a distance-hereditary graph G, Algorithm $\mathcal{A}$ computes a μ-set of G.*

Proof. Algorithm $\mathcal{A}$ first computes $\overrightarrow{D}_G$ and then a set X defined as $V(G)$ minus the σ-vertices of $\overrightarrow{D}_G$ (i.e., minus the cut-vertices of G). By Lemma 1, X is a candidate to be a mutual-visibility set. In the following, we first prove that X is indeed a mutual-visibility set, and then we prove that it is also a μ-set of G.

Concerning the mutual-visibility property, four cases must be analyzed according to the number of t-arrows in $\overrightarrow{D}_G$.

(*i*) *$\overrightarrow{D}_G$ has no t-arrows.* In this case, Algorithm $\mathcal{A}$ returns X. Let us show that X is a mutual-visibility set. If $\overrightarrow{D}_G$ has no arrows at all, by Lemma 12, G is a block graph. A μ-set for a block graph is given by all its vertices but its cut-vertices [27]. Assume now that $\overrightarrow{D}_G$ has arrows that are not t-arrows. Then, by definition of t-arrow, $V^h_G(a)$ contains a σ-vertex for each arrow in $\overrightarrow{D}_G$. Then, since all the σ-vertices are not in X, each pair of vertices x and y are X-visible by Lemma 17.

(*ii*) *$\overrightarrow{D}_G$ has one t-arrow or the t-arrows in $\overrightarrow{D}_G$ are pairwise tail-connected.* In this case Algorithm $\mathcal{A}$ returns X minus a vertex in $V^h_G(a)$ for each t-arrow a in $\overrightarrow{D}^h_G(a)$. Lemma 15 ensures that for each arrow a in $\overrightarrow{D}_G$, $V^h_G(a)$ has a vertex not in X, and then, by Lemma 17, any pair of vertices $x, y \in X$ are in mutual-visibility.

(*iii*) *The t-arrows in $\overrightarrow{D}_G$ are pairwise head-connected.* By Lemma 11 all the heads of the t-arrows are all in a K-bag. Algorithm $\mathcal{A}$ returns X minus an unmarked vertex w of the K-bag, if it exists. Then, since $w \in V^h_G(a)$ for each t-arrow a, by Lemma 15 also any arrow b has an unmarked vertex in $V^h_G(b)$. By considering that the cut-vertices of G are not part of X, by Lemma 17 any pair of vertices $x, y \in X$ are in mutual-visibility.
If all the vertices of the K-bag are marked, Algorithm $\mathcal{A}$ returns X minus one or two vertices chosen among the vertices in $\overrightarrow{D}^t_G(a)$ for some t-arrows a. Let C be the set defined as in Line 16. If C does not contain a special subgraph, then Algorithm $\mathcal{A}$ chooses two vertices from two subgraphs in C and then for each arrow a in $\overrightarrow{D}_G$, $V^h_G(a)$ has a vertex not in X. By Lemmas 15 and 17, any pair of vertices $x, y \in X$ are in mutual-visibility. If C contains a special subgraph, Algorithm $\mathcal{A}$ removes from X its special vertex and again for each arrow a in $\overrightarrow{D}_G$, $V^h_G(a)$ has a vertex not in X.

(*iv*) *$\overrightarrow{D}_G$ has exactly two opposite t-arrows.* Call $a = (u, v)$ and $b = (v, u)$ the t-arrows. In case $\overrightarrow{D}^h_G(a)$ and $\overrightarrow{D}^h_G(b)$ are not special, Algorithm $\mathcal{A}$ returns X minus two vertices: a vertex w_a in $V^h_G(a)$ and a vertex w_b in $V^h_G(b)$. Note that $V^h_G(a) = V^t_G(b)$ and $V^h_G(b) = V^t_G(a)$ and by Lemma 16, these sets are formed by all the unmarked vertices of $\overrightarrow{D}^h_G(a)$ and $\overrightarrow{D}^h_G(b)$ respectively. Let x, y be two vertices in X and let P be a shortest path connecting them in D_G. If P passes on u and v, then there exists an alternating path connecting x and y in $\overrightarrow{D}_G$, hence they are adjacent in G and hence in mutual-visibility. If P passes on u only (v only, resp.) x and y are in mutual-visibility as shown in Case (*ii*), whereas if P passes neither on u nor v, x and y are in mutual-visibility as shown in Case (*i*).
If at least one of $\overrightarrow{D}^h_G(a)$ and $\overrightarrow{D}^h_G(b)$ is special, say $\overrightarrow{D}^h_G(a)$, then it is sufficient to remove only one vertex in $\overrightarrow{D}^h_G(a)$ from X to make all the vertices of X in mutual-visibility. In this case, Algorithm $\mathcal{A}$ correctly returns X minus a vertex w_a in $V^h_G(a)$.

According to the above analysis, X is a mutual-visibility set of G.

We now show that X is indeed a μ-set of G. By Lemma 1 we assume that X does not contain any vertex in the set C of the cut vertices of G. If $\overrightarrow{D}_G$ has no t-arrows, the set X returned by algorithm $\mathcal{A}$ is given by all the vertices in $V(G) \setminus C$. Then X is a μ-set. If $\overrightarrow{D}_G$ has only one t-arrow or tail-connected t-arrows, X is given by all the vertices in $V(G) \setminus C$ but one vertex for each $V^h_G(a)$, where a is a t-arrow. Since $V^h_G(a) \cap V^h_G(b) = \emptyset$ for each pair a, b of t-arrows, by Lemmata 1 and 17 X is a μ-set. Indeed, if $V^h_G(a) \subseteq X$ for a certain t-arrow a, then there would be a pair of vertices in $D^t_G(a)$ not in mutual-visibility. For the same reason, when a and b are opposite t-arrows, X cannot contain one vertex in $\overrightarrow{D}^h_G(a)$ and one vertex in $\overrightarrow{D}^h_G(b)$ (unless $\overrightarrow{D}^h_G(a)$ and $\overrightarrow{D}^h_G(b)$ are special subgraphs of $\overrightarrow{D}_G$). In this case, X is again a μ-set. As discussed, if at least one of $\overrightarrow{D}^h_G(a)$ and $\overrightarrow{D}^h_G(b)$ is a special component, it is sufficient to remove only one vertex in $\overrightarrow{D}^h_G(a)$ from X in order to make it a μ-set.

As for the last case where the t-arrows are head-connected, $X = V(G) \setminus (C \cup \{w\})$, where w is an unmarked vertex in the K-bag containing all the heads of the t-arrows, if it exists. The removal of a vertex w is needed, otherwise a pair of unmarked vertices in $\overrightarrow{D}^t_G(a)$, for each t-arrow a, would not be in visibility. Then X is a μ-set. If such a vertex does not

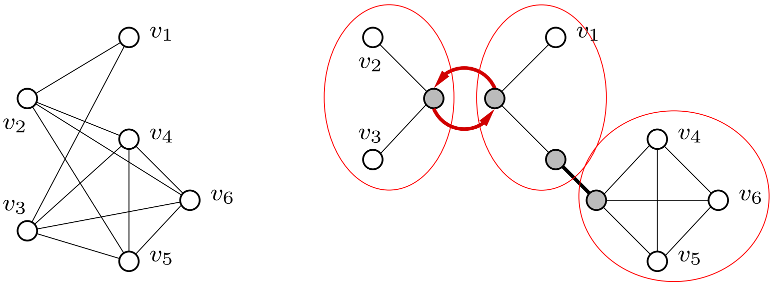

Fig. 7. A big-μ cograph $G = (K_1 \cup K_3) + H$ without universal vertices where $V(K_1) = \{v_1\}$, $V(K_3) = \{v_4, v_5, v_6\}$, and $V(H) = \{v_2, v_3\}$. In the directed decomposition $\overrightarrow{D}_G$ of G, t-arrows are represented in red.

exist, consider a t-arrow a. To make all the pairs of vertices in $\overrightarrow{D}^t_G(a)$ in mutual-visibility, at least one vertex w in $\overrightarrow{D}^t_G(b)$, for some arrow $b \neq a$, must be not in X. Then all the pairs of vertices in $\overrightarrow{D}^t_G(a')$, are in mutual-visibility, for each $a' \neq b$. To make in mutual-visibility all the pairs of vertices in $\overrightarrow{D}^t_G(b)$, when $\overrightarrow{D}^t_G(b)$ is not special, we need to remove a further vertex w' from X. Then $|X| = |V(G) \setminus C| - 2$, and X is a μ-set. Finally, if $\overrightarrow{D}^t_G(b)$ is special, Algorithm $\mathcal{A}$ chooses the special vertex w of $\overrightarrow{D}^t_G(b)$ and then no further vertex should be removed from X. Hence $|X| = |V(G) \setminus C| - 1$ and X is a μ-set. □

Theorem 19. *If G is a distance-hereditary graph, then $\mu(G)$ can be computed in linear time.*

Proof. Lemma 18 shows that Algorithm $\mathcal{A}$ computes a μ-set of G. Concerning the computational complexity of $\mathcal{A}$, observe that D_G can be computed in linear time (cf. Theorem 3). Notice that the number of bags in D_G is linear in the number of vertices of G. It follows that the additional structures $\overrightarrow{D}_G$ and $\overrightarrow{T}_G$ can be computed from D_G in the same time bound. The same holds for the set S of all the σ vertices of $\overrightarrow{D}_G$. Concerning the set A of all the t-arrows of $\overrightarrow{D}_G$, it can be computed by a visit in post-order of $\overrightarrow{T}_G$ rooted at any vertex. The cycle after Line 6 can be computed in linear time since the computation of all the $V^h_G(a)$ for each $a \in A$ is bounded by the size of $\overrightarrow{D}_G$. If A contains two opposite t-arrows, the algorithm checks if $\overrightarrow{D}^h_G(a)$ is special, an operation that can be done in linear time. The time to perform Lines 11 and 13 is bounded by the size of the K-bag K, whereas the computation of $\overrightarrow{D}^t_G(a)$ for each $a \in A$ at Line 16 is bounded by the size of the whole $\overrightarrow{D}_G$, which is linear. □

5. On the other mutual-visibility parameters

In [22], a complete characterization of the four mutual-visibility parameters for the class of cographs has been provided. In this section, we extend this characterization to the class of distance-hereditary graphs. To this end, we first recall some definitions and results from [22].

Definition 5. [22, Definition 4.3] A *big-μ* graph is any graph G defined as $G = (K_1 \cup K_t) + H$, where K_1, K_t, and H are three distinct graphs and $t \geq 0$ (i.e., K_t can be an empty graph).

From this definition, it follows that each non-trivial clique is a big-μ graph (it is sufficient to take K_t empty and H as a clique). Consequently, observe that if G is a big-μ graph, then $\mu(G) = n(G)$ when K_t is empty and H isomorphic to a clique, and $\mu(G) = n(G) - 1$ otherwise. This observation explains the term big-μ.

Corollary 20 (*[22, Corollary 4.6]*). *Let G be a cograph. Then $\mu(G) > \mu_{\mathrm{t}}(G)$ if and only if G is a big-μ graph $G = (K_1 \cup K_t) + H$ with no universal vertices.*

In [19], any graph G such that $\mu(G) = \mu_{\mathrm{t}}(G)$ is called a *(μ, μ_{t})-graph*. Notice that when a graph G is a (μ, μ_{t})-graph all four mutual-visibility parameters of that graph get the same value. Corollary 20 implies that the smallest cograph G which is not a (μ, μ_{t})-graph corresponds to the cycle $C_4 = (K_1 \cup K_t) + H$, with $t = 1$ and $H = K_1 \cup K_1$.

For example, the graph G in Fig. 7 is a big-μ cograph without universal vertices, and in $\overrightarrow{D}_G$ there are two opposite t-arrows a and b. Since both $\overrightarrow{D}^h_G(a)$ and $\overrightarrow{D}^h_G(b)$ are special, Algorithm $\mathcal{A}$ computes a μ-set of G at Line 26 by eliminating a special vertex from $V^h_G(a)$. It is worth to remark that, regardless of the eliminated vertex, the final result is not a total mutual-visibility (thus confirming Corollary 20).

The following statement extends the analysis to all the visibility parameters.

Theorem 21 (*[22, Theorem 4.7]*). *If G is a cograph, then $\mu(G) = \mu_{\mathrm{t}}(G)$, or both $\mu(G) = \mu_{\mathrm{d}}(G) = n(G) - 1$ and $\mu_{\mathrm{t}}(G) = \mu_{\mathrm{o}}(G) = n(G) - 2$ hold.*

Concerning distance-hereditary graphs, the following property has been conjectured in [22].

Conjecture 22 (*[22, Conjecture 4.8]*). *If G is a distance-hereditary graph but not a big-μ cograph without universal vertices, then $\mu(G) = \mu_t(G)$.*

We now address the problem of characterizing the value of the four mutual-visibility parameters in the context of distance-hereditary graphs. The subsequent Theorem 24 provides the required characterization. In turn, this will imply showing that the statement in Conjecture 22 holds.

Lemma 23. *Let G be a distance-hereditary graph. If G is not a cograph, then Algorithm $\mathcal{A}$ compute a μ-set which is also a μ_t-set of G.*

Proof. Assume now that G is not a cograph. By Lemma 13, $\overrightarrow{D}_G$ has either no t-arrows or pairwise tail-connected t-arrows. This implies that Algorithm $\mathcal{A}$, when processing G, executes only the code block in Lines 1–10. As a consequence, there are two possible cases to analyze:

- *$\mathcal{A}$ returns X at Line 5*. This implies that there are no t-arrows in $\overrightarrow{D}_G$. As remarked in the proof of Lemma 18 (cf. case (*i*)), each pair of distinct vertices x and y are X-visible by Lemma 17.
- *$\mathcal{A}$ returns X at Line 6*. This implies that has one t-arrow or pairwise tail-connected t-arrows. As remarked in the proof of Lemma 18 (cf. case (*ii*)), each pair of distinct vertices x and y are X-visible by Lemma 17.

Since each two distinct vertices of G are X-visible in both cases, X is a total mutual-visibility set of G. As Lemma 18 shows that X is a μ-set of G, we finally get that $\mu(G) = \mu_t(G)$ holds. □

Fig. 8 shows three versions of a graph G' obtained by extending the big-μ cograph G of Fig. 7 by adding a pendant vertex v_7 in different ways. In each extension, G' is a distance-hereditary graph, not a cograph. According to Lemma 23, Algorithm $\mathcal{A}$ always computes a mutual-visibility set X which is both a μ-set and a μ_t-set of G'. In fact, in the first case $X = V(G') \setminus \{v_2, v\}$ with $v \in \{v_1, v_4, v_5, v_6\}$, in the second case $X = V(G') \setminus \{v_6, v\}$ with $v \in \{v_2, v_3\}$, and in the last case $X = V(G') \setminus \{v_1, v\}$ with $v \in \{v_2, v_3\}$. It can be easily checked that X is also a μ_t-set of G' in each of the three cases.

Theorem 24. *Let G be a distance-hereditary graph. If G is a big-μ cograph without a universal vertex, then $\mu(G) = \mu_d(G) = n(G) - 1$ and $\mu_t(G) = \mu_o(G) = n(G) - 2$, otherwise $\mu(G) = \mu_o(G) = \mu_d(G) = \mu_t(G)$.*

Proof. If G is a cograph (regardless of whether it is a large-μ graph with no universal vertex or not), then the statement holds according to Corollary 20 and Theorem 21. If G is a distance-hereditary but not a cograph, then Lemma 23 guarantees that the four mutual-visibility parameters have the same value for G. □

6. Conclusions

In this work, we have introduced the directed canonical decomposition, along with a couple of associated structural properties, as a tool for efficiently computing the mutual-visibility number (and its variants) of distance-hereditary graphs. This naturally suggests some further research directions. The first idea is applying the same tool for computing the general position number gp(G) for a distance-hereditary graph G. Another possibility is to use the same approach to calculate the mutual visibility number for other well-known graph classes, such as circle graphs and parity graphs. In fact, the graphs belonging to these classes have been characterized according to the canonical decomposition [6]. It is also possible to find a class in [16] and a whole hierarchy of classes in [17], all defined according to the split composition, which could be investigated along the same line of research.

Acknowledgments

S. Cicerone and G. Di Stefano were partially supported by the European Union - NextGenerationEU under the Italian Ministry of University and Research (MUR) National Innovation Ecosystem grant ECS00000041 - VITALITY - CUP J97G22000170005, and by the Italian National Group for Scientific Computation (GNCS-INdAM).

Data availability

No data was used for the research described in the article.

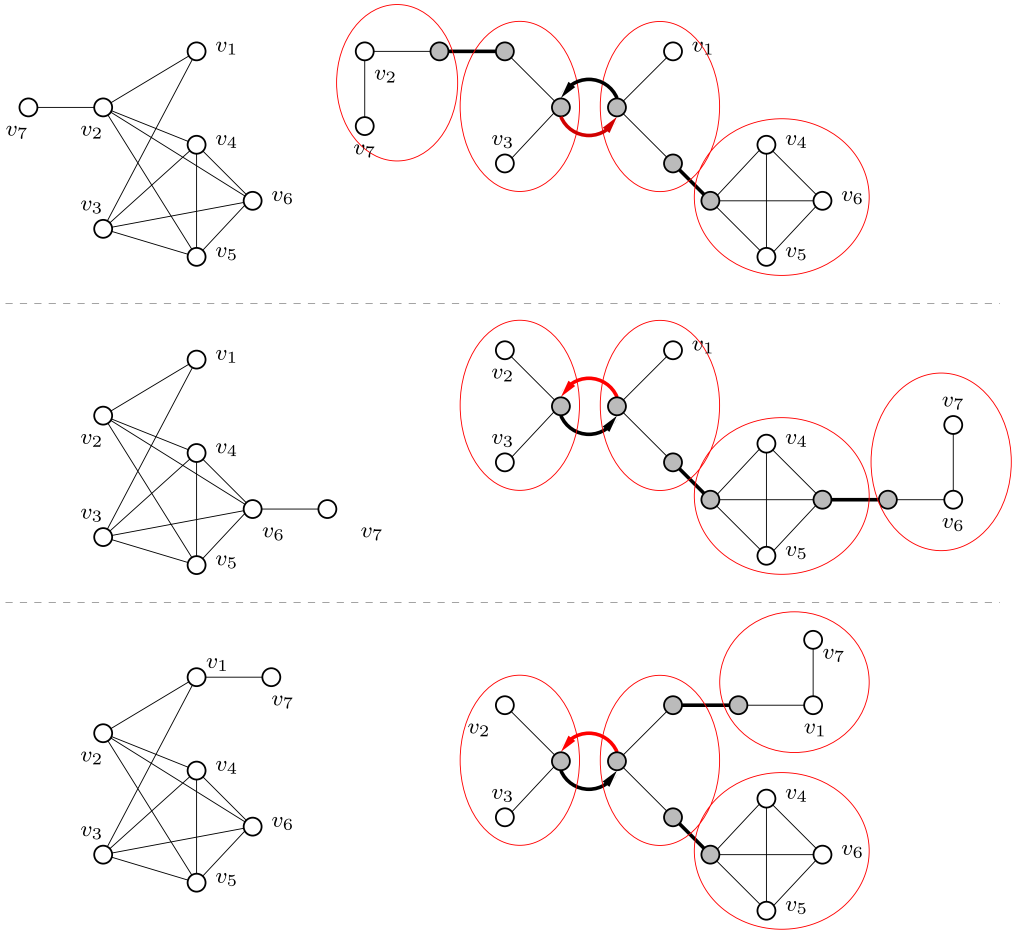

Fig. 8. The big-μ cograph G of Fig. 7 extended in three ways by adding a pendant vertex v_7. Each time, a distance-hereditary graph which is not a cograph is obtained.

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