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First and second-order Cucker-Smale models with non-universal interaction, time delay and communication failures

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Abstract
In this paper, we deal with first and second-order alignment models with non-universal interaction, time delay and possible lack of connection between the agents. More precisely, we analyze the situation in which the system's agents do not transmit information to all the other agents and also agents that are linked to each other can suspend their interaction at certain times. Moreover, we take into account possible time lags in the interactions. To deal with the considered *non-universal* interaction, a graph topology over the structure of the model has to be considered. Under a so-called *Persistence Excitation Condition*, we establish the exponential convergence to consensus for both models whenever the digraph that describes the interaction between the agents is strongly connected.

Keywords Alignment models · Cucker-Smale model · Non-universal interaction · Communication failures · Time delay

1 Introduction

Multiagent systems have been deeply investigated in these last years, due to their wide application to several scientific disciplines, among others biology [7, 19], economics [1], robotics [6], control theory [5, 36, 42], social sciences [3, 9, 39]. Among them, there is the Hegselmann-Krause opinion formation model [31] and its second-order version, the Cucker-Smale model, introduced in [19] for the description of flocking phenomena, such as the flocking of birds, the swarming of bacteria and the schooling of fish (see also [22, 23]). Also, let us mention the Kuramoto model [34], which describes the behavior of a large set of coupled oscillators. Typically, for the solutions of the aforementioned multiagent

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systems, the convergence to consensus, in the case of the Hegselmann-Krause model, the exhibition of asymptotic flocking, in the case of the Cucker-Smale model, and the asymptotic synchronization, in the case of the case of the Kuramoto model, are investigated.

In multiagent systems, it is important to consider the presence of time delay effects. Indeed, in the applications, one has to take into account certain time lags due to the propagation of information among the agents or to reaction times.

The analysis of the Hegselmann-Krause model and the Cucker-Smale model in the presence of time delays (that can be constant or, more realistically, varying in time), has been carried out by many authors, [10–13, 20, 26, 27, 29, 32, 33, 35, 37, 38]. Most of them require an upper bound on the time delay size in order to obtain converge to consensus results. However, the recent papers [27, 41] prove the asymptotic flocking without requiring any smallness conditions on the time delay, for the Cucker-Smale and Hegselmann-Krause models respectively. Generalizing and extending arguments in [41], the exponential convergence to consensus for both models, in the presence of time-variable time delays, has been proved in [16] and [15], for the first- and second-order model respectively, without assuming the time delay size to be small. We mention also [28, 30] dealing with alignment models with state-dependent delay. Finally, asymptotic synchronization results for Kuramoto oscillators with time delays have been recently achieved in [8] (for the connection between Cucker-Smale and Kuramoto models cf. [21]).

It may happen that the agents involved in an opinion formation or flocking process are not able to exchange information with the whole system's agents, namely each agent can influence or can be influenced only by some agents. In this case, we say we are in the presence of a *non-universal interaction*. To deal with this kind of interaction a network topology over the structure has to be considered (cf. [8, 14]).

A possible scenario that can also occur in the analysis of such models is the one in which the system's particles sometimes suspend the interactions they have with the agents they are linked to. As a consequence, there is a temporary lack of connection between the system's elements that, of course, hinders the convergence to consensus, for the first-order model, or the flocking for the second-order one. Then, it is important to find conditions guaranteeing the system's alignment.

In [4], the convergence to consensus and the asymptotic flocking for a class of Cucker-Smale systems under communication failures, namely with interaction weights possibly degenerating among the system's agents, have been proved under suitable assumptions in the case of symmetric interaction coefficients. The convergence to consensus for a first-order alignment system involving weights depending on the couple of agents that can eventually degenerate has been also proved in [2] under a Persistence Excitation Condition. In the case of nonsymmetric interaction coefficients, the exponential convergence to consensus for the Hegselmann-Krause model with time delay and possible communication failures has been obtained in [17].

In this paper, we analyze the asymptotic behavior of the solutions to first and second-order alignment models, i.e. Hegselmann-Krause and Cucker-Smale models, in the presence of (pair and time-dependent) time delays, non-universal interaction, and possible lack of interaction among connected agents during the evolution. Namely, in these models, time- and pair-dependent weight functions that can degenerate are considered. In this case, the interaction can be missing sometimes not only among agents that are not linked to each other but also among agents that are generally able to exchange information. Under a Persistence Excitation Condition, we establish the exponential consensus and flocking for the Hegselmann-Krause opinion formation model and the Cucker-Smale model whenever the digraph that describes the interaction among the agents is strongly connected.

This is done by dealing with a general influence function (no symmetry or monotonicity assumptions are needed) and without requiring any smallness assumptions on the time delay size. Our result seems very general and greatly improves previous related works [2, 4, 17]. Indeed, in [2] only the first-order model is analyzed, with a less general influence function, in the case of all-to-all interactions. Furthermore, time delay effects are not considered and the convergence to consensus is not achieved exponentially fast. Here, we deal with non-universal interaction, in the presence of time delays, pair and time dependent, and very general influence functions. This generality requires finer and more sophisticated arguments. With respect to [17], where only the first-order model is analyzed, the main novelties are the more general weight and time delay functions (now pair-dependent). Moreover, here, we work in the network topology setting. For the first-order model, we mention also [40] where a consensus result for a linear version of the model is obtained, under a weaker network topology assumption. Finally, the paper [4] deals with the second-order model too. However, the analysis requires symmetry conditions on the weight functions and all-to-all interaction. Moreover, no time delays are included.

The paper is organized as follows. In Section 2, we establish the exponential convergence to consensus for the solutions of the Hegselmann-Krause model, namely the first-order alignment model. In Section 3, the discussion is extended to the second-order version, namely the Cucker-Smale model. The proof of the exponential flocking for the Cucker-Smale model requires more careful analysis with respect to the one carried out in Section 2, since in this case, we have to prove that the agents synchronize their velocities and the distances between the agents' positions are bounded. Finally, in Section 4, some numerical tests illustrate the theoretical results.

2 The first-order alignment model

Consider a finite set of $N \in \mathbb{N}$ agents, with $N \geq 2$. Let $x_i(t) \in \mathbb{R}^d$ be the opinion of the i-th agent at time t. We shall denote with $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the usual norm and scalar product on $\mathbb{R}^d$, respectively. Let us denote with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The interactions between the elements of the system are described by the following Hegselmann-Krause type model:

$$\frac{d}{dt} x_i(t) = \sum_{j:j\neq i} \chi_{ij} b_{ij}(t)(x_j(t-\tau_{ij}(t)) - x_i(t)), \quad t > 0, \ \forall i = 1, \dots, N, \tag{2.1}$$

where the time delay functions $\tau_{ij} : [0, +\infty) \to [0, +\infty)$ are assumed to be continuous and satisfy the following:

$$0 \leq \tau_{ij}(t) \leq \tau, \quad \forall t \geq 0, \ \forall i, j = 1, \dots, N, \tag{2.2}$$

for some positive constant τ.
Here, the terms χ_{ij} are so defined

$$\chi_{ij} = \begin{cases} 1, & \text{if } j \text{ transmits information to } i, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Moreover, the communication rates b_{ij} are of the form

$$b_{ij}(t) := \frac{1}{N-1} \alpha_{ij}(t) \psi(x_i(t), x_j(t-\tau_{ij}(t))), \quad t > 0, \ \forall i, j = 1, \dots, N, \tag{2.4}$$

where the influence function $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is positive, bounded and continuous with

$$K := \|\psi\|_\infty, \tag{2.5}$$

and the weight functions $\alpha_{ij} : [0, +\infty) \to [0, 1]$ are $\mathcal{L}^1$-measurable and satisfy the following Persistence Excitation Condition (cf. [2, 4]):

(PE) there exist two positive constants T and $\tilde{\alpha}$ such that

$$\int_t^{t+T} \alpha_{ij}(s)ds \geq \tilde{\alpha}, \quad \forall t \geq 0, \tag{2.6}$$

for all $i, j = 1, \dots, N$ such that $\chi_{ij} = 1$.

Without loss of generality, we can assume that the positive constant $\tilde{\alpha}$ appearing in (2.6) satisfies $\tilde{\alpha}K < 1$. Let us note that (2.6) becomes relevant when T is large and $\tilde{\alpha}$ is small. In this case, the agents could eventually suspend their interaction for long enough. We also point out that, in the case in which $\alpha_{ij}(t) = 1$, for a.e. $t \geq 0$ and for any $i, j = 1, \dots, N$, i.e. in the case in which the agents do not interrupt their exchange of information, the condition (2.6) is of course satisfied.
Due to the presence of the time delay, the initial conditions are functions defined in the interval $[-\tau, 0]$. The initial conditions

$$x_i(s) = x_i^0(s), \quad \forall s \in [-\tau, 0], \; \forall i = 1, \dots, N, \tag{2.7}$$

are assumed to be continuous functions.
We set

$$C_0 := \max_{i=1,\dots,N} \max_{s\in[-\tau,0]} |x_i(s)|, \tag{2.8}$$

$$\psi_0 := \min_{|y|,|z|\leq C_0} \psi(y, z). \tag{2.9}$$

We will consider a graph topology over the model structure. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph consisting of a finite set $\mathcal{V} = \{1, ..., N\}$ of vertices and a set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ of arcs. We assume that the agents are located at the vertices and interact with each other via the underlying network topology. For each vertex i, we denote by $\mathcal{N}_i$ the set of vertices that directly influence the vertex i, namely

$$\mathcal{N}_i := \{j = 1, \dots, N : \chi_{ij} = 1\}. \tag{2.10}$$

The set $\mathcal{N}_i$ can also be defined in the following way: $j \in \mathcal{N}_i$ if and only if $(i, j) \in \mathcal{E}$. Also, we denote with

$$N_i := |\mathcal{N}_i|. \tag{2.11}$$

Throughout the paper, we will exclude self-loops, i.e. we assume that $i \notin \mathcal{N}_i$ for all $1 \leq i \leq N$. We also denote the network topology via its $(0, 1)$-adjacency matrix $(\chi_{ij})_{ij}$. A *path* in a digraph $\mathcal{G}$ from i_0 to i_p is a finite sequence $i_0, i_1, \dots, i_p$ of distinct vertices such that each successive pair of vertices is an arc of $\mathcal{G}$. The integer p is called *length* of the path. If there exists a path from i to j, then vertex j is said to be *reachable* from vertex i and we define the distance from i to j, in notation $\mathrm{dist}(i, j)$, as the length of the shortest path from i to j. A digraph $\mathcal{G}$ is said to be *strongly connected* if each vertex is reachable from any other vertex. We assume that our digraph $\mathcal{G}$ is strongly connected. We define the *depth* γ of the digraph as follows:

$$\gamma := \max_{i,j=1,\dots,N} \mathrm{dist}(i, j). \tag{2.12}$$

Thus, any particle can be connected to the other individuals of the system via no more than γ intermediate agents. By definition, since $i \notin \mathcal{N}_i$, for all $i = 1, \dots, N$, we have that $\gamma \leq N-1$. Also, since the digraph is strongly connected, $\gamma \geq 1$.

For well-posedness results for alignment models in the presence of time delay effects we refer to classical texts on functional differential equations [24, 25]. Here, we will focus on the asymptotic behavior of the solutions.

Now, we give the rigorous definition of convergence to consensus for solutions of the Hegselmann-Krause model (2.1). We define the diameter $d(\cdot)$ of the solution as

$$d(t) := \max_{i,j=1,\dots,N} |x_i(t) - x_j(t)|, \quad \forall t \geq -\tau.$$

Definition 2.1 We say that a solution $\{x_i\}_{i=1,\dots,N}$ to system (2.1) converges to *consensus* if $d(t) \to 0$, as $t \to \infty$.

We will prove the following exponential convergence to consensus result.

Theorem 2.1 *Assume (2.2) and that the digraph $\mathcal{G}$ is strongly connected. Let $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a positive, bounded, continuous function. Assume that the weight functions $\alpha_{ij} : [0, +\infty) \to [0, 1]$ are $\mathcal{L}^1$-measurable and satisfy* **(PE)**. *Let $x_i^0 : [-\tau, 0] \to \mathbb{R}^d$ be a continuous function, for any $i = 1, \dots, N$. Then, every solution $\{x_i\}_{i=1,\dots,N}$ to (2.1) with the initial conditions (2.7) satisfies the following exponential decay estimate*

$$d(t) \leq C_1 \left(\max_{i,j=1,\dots,N} \max_{r,s \in [-\tau,0]} |x_i(r) - x_j(s)| \right) e^{-C_2 t}, \quad \forall t \geq 0, \tag{2.13}$$

where C_1, C_2 are the positive constants defined as

$$C_1 := \frac{1}{1 - e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)} \left(\frac{\psi_0 \tilde{\alpha}}{N-1} \right)^{\gamma}}, \tag{2.14}$$

$$C_2 := \frac{1}{\gamma(T+\tau)+\tau} \ln \left(\frac{1}{1 - e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)} \left(\frac{\psi_0 \tilde{\alpha}}{N-1} \right)^{\gamma}} \right), \tag{2.15}$$

being $\gamma > 0$ the depth of the digraph, T and $\tilde{\alpha}$ the positive constants in (2.6), and ψ_0 the positive constant in (2.9).

Remark 2.2 Let us note that the velocity decay C_2 in (2.15) decreases for increasing values of γ, τ and T and decreases for decreasing values of $\tilde{\alpha}$. Indeed, for fixed values of τ, T and $\tilde{\alpha}$, we have

$$\frac{1}{\gamma(T+\tau)+\tau} \to 0, \quad \text{as } \gamma \to +\infty.$$

Moreover, being $N \geq 2$ and $\tilde{\alpha} K < 1$, we have $\frac{\psi_0 \tilde{\alpha}}{N-1} \leq \psi_0 \tilde{\alpha} \leq K \tilde{\alpha} < 1$, from which

$$\left(\frac{\psi_0 \tilde{\alpha}}{N-1} \right)^{\gamma} \to 0, \quad \text{as } \gamma \to +\infty.$$

Also, $e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)} \to 0$, as $\gamma \to +\infty$. Thus, $\ln \left(\frac{1}{1-e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)} \left(\frac{\psi_0 \tilde{\alpha}}{N-1} \right)^{\gamma}} \right) \to 0$, as $\gamma \to +\infty$. Therefore, $C_2 \to 0$, as $\gamma \to +\infty$.

Analogously, $C_2 \to 0$, as $\tau \to +\infty$ or $T \to +\infty$. Finally, for $\tilde{\alpha} \to 0$, it is easy to see that $C_2 \to 0$.
So, C_2 decreases as γ, T, τ grow and as $\tilde{\alpha}$ decays. This is expected since, for large values of γ and T and for small values of $\tilde{\alpha}$, the connection among the agents can be very weak. Furthermore, increasing time lags in the interaction among the agents slow down the convergence to consensus for the Hegselmann-Krause model.

2.1 Preliminary lemmas

Let $\{x_i\}_{i=1,\ldots,N}$ be solution to (2.1) under the initial conditions (2.7). We assume that the hypotheses of Theorem 2.1 are satisfied. We present some auxiliary lemmas.

Definition 2.2 Given a vector $v \in \mathbb{R}^d$, for all $n \in \mathbb{N}_0$ we define

$$\begin{aligned} I_n &:= [n(\gamma(T+\tau)+\tau)-\tau, n(\gamma(T+\tau)+\tau)] \\ m_n^v &:= \quad \min_{j=1,\ldots,N} \min_{s\in I_n} \langle x_j(s), v\rangle, \\ M_n^v &:= \quad \max_{j=1,\ldots,N} \max_{s\in I_n} \langle x_j(s), v\rangle. \end{aligned}$$

Also, we define, for all $n \in \mathbb{N}_0$,

$$\begin{aligned} \tilde{m}_n^v &:= \min_{j=1,\ldots,N} \langle x_j(n(\gamma(T+\tau)+\tau)), v\rangle, \\ \tilde{M}_n^v &:= \max_{j=1,\ldots,N} \langle x_j(n(\gamma(T+\tau)+\tau)), v\rangle. \end{aligned}$$

Lemma 2.3 *For each vector $v \in \mathbb{R}^d$, we have that*

$$m_0^v \le \langle x_i(t), v\rangle \le M_0^v, \tag{2.16}$$

for all $t \ge -\tau$ and for any $i = 1, \ldots, N$.

Proof First of all, we note that the inequalities in (2.16) are satisfied for every $t \in [-\tau, 0]$. Now, let $v \in \mathbb{R}^d$. For all $\epsilon > 0$, we define

$$K^\epsilon := \left\{ t > 0 : \max_{i=1,\ldots,N} \langle x_i(s), v\rangle < M_0^v + \epsilon,\ \forall s \in [0, t) \right\},$$

and

$$S^\epsilon := \sup K^\epsilon.$$

By continuity, we have that $K^\epsilon \neq \emptyset$ and $S^\epsilon > 0$.
We claim that $S^\epsilon = +\infty$. Indeed, suppose by contradiction that $S^\epsilon < +\infty$. By definition of S^ϵ, it turns out that

$$\max_{i=1,\ldots,N} \langle x_i(t), v\rangle < M_0^v + \epsilon, \quad \forall t \in (0, S^\epsilon), \tag{2.17}$$

$$\lim_{t\to S^{\epsilon-}} \max_{i=1,\ldots,N} \langle x_i(t), v\rangle = M_0^v + \epsilon. \tag{2.18}$$

For all $i = 1, \ldots, N$, for $t \in (0, S^\epsilon)$, we have that

$$\frac{d}{dt}\langle x_i(t), v\rangle = \frac{1}{N-1} \sum_{j:j\neq i} \chi_{ij}\alpha_{ij}(t)\psi(x_i(t), x_j(t-\tau_{ij}(t)))\langle x_j(t-\tau_{ij}(t)) - x_i(t), v\rangle.$$

Now, being $t \in (0, S^\epsilon)$, it holds that $t - \tau_{ij}(t) \in (-\tau, S^\epsilon)$. Then, from (2.17)

$$\langle x_j(t-\tau_{ij}(t)), v\rangle < M_0^v + \epsilon, \quad \forall j = 1, \ldots, N, \tag{2.19}$$

where hare we have used the fact that the second inequality in (2.16) is satisfied in $[-\tau, 0]$. Therefore, using (2.5), (2.17), (2.19) and recalling that $\chi_{ij}, \alpha_{ij} \le 1$, for a.e. $t \in (0, S^\epsilon)$ we can write

$$\begin{aligned}\frac{d}{dt}\langle x_i(t), v\rangle &\le \frac{1}{N-1}\sum_{j:j\neq i}\chi_{ij}\alpha_{ij}(t)\psi(x_i(t), x_j(t-\tau_{ij}(t)))(M_0^v + \epsilon - \langle x_i(t), v\rangle)\\ &\le K(M_0^v + \epsilon - \langle x_i(t), v\rangle).\end{aligned}$$

Thus, the Gronwall's inequality yields

$$\begin{aligned}\langle x_i(t), v\rangle &\le e^{-Kt}\langle x_i(0), v\rangle + K(M_0^v + \epsilon)\int_0^t e^{-K(t-s)}ds\\ &= e^{-Kt}\langle x_i(0), v\rangle + (M_0^v + \epsilon)e^{-Kt}(e^{Kt} - 1)\\ &= e^{-Kt}\langle x_i(0), v\rangle + (M_0^v + \epsilon)(1 - e^{-Kt})\\ &\le e^{-Kt}M_0^v + M_0^v + \epsilon - M_0^v e^{-Kt} - \epsilon e^{-Kt}\\ &= M_0^v + \epsilon - \epsilon e^{-Kt}\\ &\le M_0^v + \epsilon - \epsilon e^{-KS^\epsilon},\end{aligned}$$

for all $t \in (0, S^\epsilon)$. We have so proved that, $\forall i = 1, \dots, N$,

$$\langle x_i(t), v\rangle \le M_0^v + \epsilon - \epsilon e^{-KS^\epsilon}, \quad \forall t \in (0, S^\epsilon).$$

Thus, we get

$$\max_{i=1,\dots,N}\langle x_i(t), v\rangle \le M_0^v + \epsilon - \epsilon e^{-KS^\epsilon}, \quad \forall t \in (0, S^\epsilon). \tag{2.20}$$

Letting $t \to S^{\epsilon-}$ in (2.20), from (2.18) we have that

$$M_0^v + \epsilon \le M_0^v + \epsilon - \epsilon e^{-KS^\epsilon} < M_0^v + \epsilon,$$

which is a contradiction. Thus, $S^\epsilon = +\infty$ and

$$\max_{i=1,\dots,N}\langle x_i(t), v\rangle < M_0^v + \epsilon, \quad \forall t > 0.$$

From the arbitrariness of ϵ we can conclude that

$$\max_{i=1,\dots,N}\langle x_i(t), v\rangle \le M_0^v, \quad \forall t > 0,$$

from which

$$\langle x_i(t), v\rangle \le M_0^v, \quad \forall t > 0, \ \forall i = 1, \dots, N.$$

So, the second inequality in (2.16) is proven.
Now, to show that the other inequality holds, fix $v \in \mathbb{R}^d$. Then, for all $i = 1, \dots, N$ and $t > 0$, by applying the second inequality in (2.16) to the vector $-v \in \mathbb{R}^d$ we get

$$\begin{aligned}-\langle x_i(t), v\rangle = \langle x_i(t), -v\rangle &\le \max_{j=1,\dots,N}\max_{s\in[-\tau,0]}\langle x_j(s), -v\rangle\\ &= -\min_{j=1,\dots,N}\min_{s\in[-\tau,0]}\langle x_j(s), v\rangle = -m_0^v,\end{aligned}$$

from which

$$\langle x_i(t), v\rangle \ge m_0^v, \quad \forall t \ge 0, \ \forall i = 1, \dots, N.$$

Thus, also the first inequality in (2.16) is fulfilled. □

Using the same arguments employed in the proof of the previous lemma, one can prove the following more general result.

Lemma 2.4 *For each vector $v \in \mathbb{R}^d$ and for all $n \in \mathbb{N}_0$, we have that*

$$m_n^v \leq \langle x_i(t), v \rangle \leq M_n^v, \tag{2.21}$$

for all $t \geq n(\gamma(T+\tau)+\tau)-\tau$ and for any $i = 1, \dots, N$.

Now, we define the following quantities.

Definition 2.3 For all $n \in \mathbb{N}_0$, we define

$$D_n := \max_{i,j=1,\dots,N} \max_{r,s \in I_n} |x_i(r) - x_j(s)|.$$

Let us note that, for $n = 0$,

$$D_0 := \max_{i,j=1,\dots,N} \max_{r,s \in I_0} |x_i(r) - x_j(s)| = \max_{i,j=1,\dots,N} \max_{r,s \in [-\tau,0]} |x_i(r) - x_j(s)|.$$

So, the exponential decay estimate in (2.13) can be written as

$$d(t) \leq C_1 D_0 e^{-C_2 t}, \quad \forall t \geq 0,$$

where $C_1 > 0$ and $C_2 > 0$ are the constants in (2.14) and (2.15), respectively.

Lemma 2.5 *For each $n \in \mathbb{N}_0$, we have that*

$$|x_i(s) - x_j(t)| \leq D_n, \tag{2.22}$$

for all $s, t \geq n(\gamma(T+\tau)+\tau)-\tau$ and for any $i, j = 1, \dots, N$.

Proof Fix $n \in \mathbb{N}_0$. Let $i, j = 1, \dots, N$ and $s, t \geq n(\gamma(T+\tau)+\tau)-\tau$. Then, if $|x_i(s) - x_j(t)| = 0$, (2.22) is obviously satisfied. So we can assume $|x_i(s) - x_j(t)| > 0$. Let us define the unit vector

$$v = \frac{x_i(s) - x_j(t)}{|x_i(s) - x_j(t)|}.$$

Then, using (2.21) and Cauchy-Schwarz inequality, we have that

$$\begin{aligned} |x_i(s) - x_j(t)| &= \langle x_i(s) - x_j(t), v \rangle = \langle x_i(s), v \rangle - \langle x_j(t), v \rangle \leq M_n^v - m_n^v \\ &\leq \max_{k,l=1,\dots,N} \max_{r,\sigma \in I_n} |x_k(r) - x_l(\sigma)| = D_n. \end{aligned}$$

□

Remark 2.6 Note that (2.22) yields

$$d(t) \leq D_n, \quad \forall t \geq n(\gamma(T+\tau)+\tau)-\tau. \tag{2.23}$$

Moreover, from (2.22) it comes that

$$D_{n+1} \leq D_n, \quad \forall n \in \mathbb{N}_0. \tag{2.24}$$

Next, we show that the agents' opinions are bounded by a constant that depends on the initial data.

Lemma 2.7 *For every $i = 1, \dots, N$, we have that*

$$|x_i(t)| \le C_0, \quad \forall t \ge -\tau, \tag{2.25}$$

where C_0 is the constant defined in (2.8).

Proof Given $i = 1, \dots, N$ and $t \ge -\tau$, if $|x_i(t)| = 0$, then trivially $C_0 \ge |x_i(t)|$. On the contrary, if $|x_i(t)| > 0$, we define

$$v = \frac{x_i(t)}{|x_i(t)|},$$

which is a unit vector. Then, by applying (2.16) and by using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |x_i(t)| = \langle x_i(t), v\rangle \le M_0^v &= \max_{j=1,\dots,N} \max_{s\in[-\tau,0]} \langle x_j(s), v\rangle \\ &\le \max_{j=1,\dots,N} \max_{s\in[-\tau,0]} |x_j(s)||v| = \max_{j=1,\dots,N} \max_{s\in[-\tau,0]} |x_j(s)| = C_0, \end{aligned}$$

and (2.25) is satisfied. □

Remark 2.8 From the estimate (2.25), since the influence function ψ is continuous, we deduce that

$$\psi(x_i(t), x_j(t-\tau_{ij}(t))) \ge \psi_0, \tag{2.26}$$

for all $t \ge 0$, for all $i, j = 1, \dots, N$, where ψ_0 is the positive constant in (2.9).

2.2 Consensus estimate

In order to prove the consensus result, we need the following crucial proposition, inspired by a previous argument in [27].

Proposition 2.9 *For all $v \in \mathbb{R}^d$, it holds*

$$m_0^v + \Gamma(\tilde{M}_0^v - m_0^v) \le \langle x_i(t), v\rangle \le M_0^v - \Gamma(M_0^v - \tilde{m}_0^v), \tag{2.27}$$

for all $t \in I_1$ and for all $i = 1, \dots, N$, where Γ is the positive constant defined as follows

$$\Gamma := e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)} \left(\frac{\psi_0\tilde{\alpha}}{N-1}\right)^{\gamma}. \tag{2.28}$$

Remark 2.10 Let us note that $\Gamma \in (0, 1)$ since we have assumed $\tilde{\alpha}K < 1$. Moreover, by definition of Γ, the positive constants C_1, C_2 in (2.13) can be rewritten in the following way (see (2.14) and (2.15)):

$$\begin{aligned} C_1 &= \frac{1}{1-\Gamma}, \\ C_2 &= \frac{1}{\gamma(T+\tau)+\tau} \ln\left(\frac{1}{1-\Gamma}\right). \end{aligned}$$

Proof of Proposition 2.9 Fix $v \in \mathbb{R}^d$. Let $L = 1, \dots, N$ be such that $\langle x_L(0), v\rangle = \tilde{m}_0^v$. Note that from (2.16), $M_0^v \geq \tilde{m}_0^v$. Then, for a.e. $t \in [0, \gamma(T+\tau)+\tau]$, using (2.16) we have

$$\begin{aligned}
\frac{d}{dt}\langle x_L(t), v\rangle &= \sum_{j:j\neq L} \chi_{Lj}(t) b_{Lj}(t)(\langle x_j(t-\tau_{Lj}(t)), v\rangle - \langle x_L(t), v\rangle) \\
&\leq \sum_{j:j\neq L} \chi_{Lj} b_{Lj}(t)(M_0^v - \langle x_L(t), v\rangle) \\
&\leq \frac{K}{N-1}\sum_{j:j\neq L}(M_0^v - \langle x_L(t), v\rangle) = K(M_0^v - \langle x_L(t), v\rangle).
\end{aligned}$$

Thus, the Gronwall's inequality yields

$$\begin{aligned}
\langle x_L(t), v\rangle &\leq e^{-Kt}\langle x_L(0), v\rangle + M_0^v(1-e^{-Kt}) \\
&= e^{-Kt}\tilde{m}_0^v + M_0^v(1-e^{-Kt}) \\
&= M_0^v - e^{-Kt}(M_0^v - \tilde{m}_0^v) \\
&\leq M_0^v - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v).
\end{aligned}$$

Hence,

$$\langle x_L(t), v\rangle \leq M_0^v - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v), \quad \forall t \in [0, \gamma(T+\tau)+\tau]. \tag{2.29}$$

Now, let $i_1 = 1, \dots, N \setminus \{L\}$ be such that $\chi_{i_1 L} = 1$. Such an index i_1 exists since the digraph is strongly connected. Then, for a.e. $t \in [\tau, \gamma(T+\tau)+\tau]$, from (2.29) we get

$$\begin{aligned}
\frac{d}{dt}\langle x_{i_1}(t), v\rangle &= \sum_{j\neq i_1, L} \chi_{i_1 j} b_{i_1 j}(t)(\langle x_j(t-\tau_{i_1 j}(t)), v\rangle - \langle x_{i_1}(t), v\rangle) \\
&\quad + b_{i_1 L}(t)(\langle x_L(t-\tau_{i_1 L}(t)), v\rangle - \langle x_{i_1}(t), v\rangle) \\
&\leq \sum_{j\neq i_1, L} \chi_{i_1 j} b_{i_1 j}(t)(M_0^v - \langle x_{i_1}(t), v\rangle) \\
&\quad + b_{i_1 L}(t)\left(M_0^v - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v) - \langle x_{i_1}(t), v\rangle\right) \\
&= (M_0^v - \langle x_{i_1}(t), v\rangle)\sum_{j\neq i_1, L} \chi_{i_1 j} b_{i_1 j}(t) \\
&\quad + b_{i_1 L}(t)\left(M_0^v - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v) - \langle x_{i_1}(t), v\rangle\right).
\end{aligned}$$

Note that

$$\sum_{j\neq i_1, L} \chi_{i_1 j} b_{i_1 j}(t) = \sum_{j\neq i_1} \chi_{i_1 j} b_{i_1 j}(t) - b_{i_1 L}(t) \leq \frac{K}{N-1}\sum_{j\neq i_1}\chi_{i_1 j} - b_{i_1 L}(t) = \frac{KN_{i_1}}{N-1} - b_{i_1 L}(t).$$

Thus, it comes that

$$\begin{aligned}
\frac{d}{dt}\langle x_{i_1}(t), v\rangle \leq\ & \frac{KN_{i_1}}{N-1}(M_0^v - \langle x_{i_1}(t), v\rangle) - b_{i_1L}(t)(M_0^v - \langle x_{i_1}(t), v\rangle) \\
& + b_{i_1L}(t)\left(M_0^v - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v) - \langle x_{i_1}(t), v\rangle\right) \\
=\ & \frac{KN_{i_1}}{N-1}(M_0^v - \langle x_{i_1}(t), v\rangle) - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)b_{i_1L}(t) \\
\leq\ & \frac{KN_{i_1}}{N-1}(M_0^v - \langle x_{i_1}(t), v\rangle) - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\alpha_{i_1L}(t)\frac{\psi_0}{N-1} \\
=\ & \frac{KN_{i_1}}{N-1}M_0^v - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\alpha_{i_1L}(t)\frac{\psi_0}{N-1} - \frac{KN_{i_1}}{N-1}\langle x_{i_1}(t), v\rangle.
\end{aligned}$$

Hence, the Gronwall's estimate yields

$$\begin{aligned}
\langle x_{i_1}(t), v\rangle \leq\ & e^{-\frac{KN_{i_1}}{N-1}(t-\tau)}\langle x_{i_1}(\tau), v\rangle) + M_0^v(1 - e^{-\frac{KN_{i_1}}{N-1}(t-\tau)}) \\
& - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\frac{\psi_0}{N-1}\int_\tau^t \alpha_{i_1L}(s)e^{-\frac{KN_{i_1}}{N-1}(t-s)}ds \\
\leq\ & e^{-\frac{KN_{i_1}}{N-1}(t-\tau)}M_0^v + M_0^v(1 - e^{-\frac{KN_{i_1}}{N-1}(t-\tau)}) \\
& - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)e^{-K\gamma(T+\tau)}\frac{\psi_0}{N-1}\int_\tau^t \alpha_{i_1L}(s)ds \\
=\ & M_0^v - e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\frac{\psi_0}{N-1}\int_\tau^t \alpha_{i_1L}(s)ds,
\end{aligned}$$

for all $t \in [\tau, \gamma(T+\tau)+\tau]$. In particular, for $t \in [T+\tau, \gamma(T+\tau)+\tau]$, we find

$$\langle x_{i_1}(t), v\rangle \leq M_0^v - e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\frac{\psi_0}{N-1}\tilde{\alpha}, \tag{2.30}$$

where here we have used the fact that, from (2.6),

$$\int_\tau^t \alpha_{i_1L}(s)ds \geq \int_\tau^{T+\tau} \alpha_{i_1L}(s)ds \geq \tilde{\alpha}.$$

Let us note that, if $\gamma = 1$, estimate (2.30) holds for each agent. If $\gamma > 1$, let us consider an index $i_2 \in \{1, \ldots, N\} \setminus \{i_1\}$ such that $\chi_{i_2i_1} = 1$. Then, for a.e. $t \in [T+2\tau, \gamma(T+\tau)+\tau]$, from (2.30) it comes that

$$\begin{aligned}
\frac{d}{dt}\langle x_{i_2}(t), v\rangle =\ & \sum_{j\neq i_1,i_2} \chi_{i_2j}b_{i_2j}(t)(\langle x_j(t-\tau_{i_2j}(t)), v\rangle - \langle x_{i_2}(t), v\rangle) \\
& + b_{i_2i_1}(t)(\langle x_{i_1}(t-\tau_{i_2i_1}(t)), v\rangle - \langle x_{i_2}(t), v\rangle) \\
\leq\ & (M_0^v - \langle x_{i_2}(t), v\rangle)\sum_{j\neq i_1,i_1} \chi_{i_2j}b_{i_2j}(t) \\
& + b_{i_2i_1}(t)\left(M_0^v - e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\frac{\psi_0}{N-1}\tilde{\alpha} - \langle x_{i_2}(t), v\rangle\right).
\end{aligned}$$

Thus, arguing as above,

$$\begin{aligned}
\frac{d}{dt}\langle x_{i_2}(t), v\rangle \leq & \frac{KN_{i_2}}{N-1}(M_0^v - \langle x_{i_2}(t), v\rangle) - b_{i_2 i_1}(t)(M_0^v - \langle x_{i_2}(t), v\rangle) \\
& + b_{i_2 i_1}(t)\left(M_0^v - e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\frac{\psi_0}{N-1}\tilde{\alpha} - \langle x_{i_2}(t), v\rangle\right) \\
= & \frac{KN_{i_2}}{N-1}(M_0^v - \langle x_{i_2}(t), v\rangle) - b_{i_2 i_1}(t)e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\frac{\psi_0}{N-1}\tilde{\alpha} \\
\leq & \frac{KN_{i_2}}{N-1}(M_0^v - \langle x_{i_2}(t), v\rangle) - \alpha_{i_2 i_1}(t)e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\left(\frac{\psi_0}{N-1}\right)^2\tilde{\alpha} \\
= & \frac{KN_{i_2}}{N-1}M_0^v - \alpha_{i_2 i_1}(t)e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\left(\frac{\psi_0}{N-1}\right)^2\tilde{\alpha} \\
& - \frac{KN_{i_2}}{N-1}\langle x_{i_2}(t), v\rangle.
\end{aligned}$$

Again, using Gronwall's estimate it comes that

$$\begin{aligned}
\langle x_{i_2}(t), v\rangle \leq & e^{-\frac{KN_{i_2}}{N-1}(t-T-2\tau)}\langle x_{i_2}(T+2\tau), v\rangle + M_0^v(1 - e^{-\frac{KN_{i_2}}{N-1}(t-T-2\tau)}) \\
& - e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\left(\frac{\psi_0}{N-1}\right)^2\tilde{\alpha}\int_{T+2\tau}^t \alpha_{i_2 i_1}(s)e^{-\frac{KN_{i_2}}{N-1}(t-s)}ds \\
\leq & M_0^v - e^{-K(3\gamma(T+\tau)-T)}(M_0^v - \tilde{m}_0^v)\left(\frac{\psi_0}{N-1}\right)^2\tilde{\alpha}\int_{T+2\tau}^t \alpha_{i_2 i_1}(s)ds,
\end{aligned}$$

for all $t \in [T+2\tau, \gamma(T+\tau)+\tau]$. In particular, for $t \in [2T+2\tau, \gamma(T+\tau)+\tau]$, the condition (2.6) yields

$$\langle x_{i_2}(t), v\rangle \leq M_0^v - e^{-K(3\gamma(T+\tau)-T)}(M_0^v - \tilde{m}_0^v)\left(\frac{\psi_0}{N-1}\right)^2\tilde{\alpha}^2. \tag{2.31}$$

Finally, iterating the above procedure along the path $i_0, i_1, \ldots, i_r$, $r \leq \gamma$, that starts from $i_0 = L$ we find the following upper bound

$$\langle x_{i_k}(t), v\rangle \leq M_0^v - e^{-K((k+1)\gamma(T+\tau)-\left(\sum_{l=0}^{k-1} l\right)(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\left(\frac{\psi_0\tilde{\alpha}}{N-1}\right)^k, \tag{2.32}$$

for all $1 \leq k \leq r$ and for all $t \in [k(T+\tau), \gamma(T+\tau)+\tau]$. In particular, if the path has length γ, for $k = \gamma$, since $\sum_{l=0}^{\gamma-1} l = \frac{\gamma(\gamma-1)}{2}$, inequality (2.32) reads as

$$\langle x_{i_\gamma}(t), v\rangle \leq M_0^v - e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\left(\frac{\psi_0\tilde{\alpha}}{N-1}\right)^\gamma, \tag{2.33}$$

for all $t \in [\gamma(T+\tau), \gamma(T+\tau)+\tau]$.
Let us note that (2.33) holds for every agent in the path starting from $i_0 = L$ for $t \in [\gamma(T+\tau), \gamma(T+\tau)+\tau]$. Then, from the arbitrariness of the path and since the digraph is strongly connected, (2.33) holds for all the agents.

Now, let $R = 1, \ldots, N$ be such that $\tilde{M}_0^v = \langle x_R(0), v\rangle$. Then, arguing as before, we get

$$\langle x_R(t), v\rangle \geq m_0^v(1 + e^{-K(\gamma(T+\tau)+\tau)}(\tilde{M}_0^v - m_0^v)), \quad \forall t \in [0, \gamma(T+\tau)+\tau]. \tag{2.34}$$

Employing the same arguments used above, we can conclude that

$$\langle x_i(t), v\rangle \geq m_0^v + e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)}(\tilde{M}_0^v - m_0^v)\left(\frac{\psi_0\tilde{\alpha}}{N-1}\right)^\gamma,$$

for all $t \in [\gamma(T+\tau), \gamma(T+\tau)+\tau]$ and for all $i = 1, \dots, N$. Finally, we can deduce that estimate (2.27) holds. □

The following proposition generalizes the previous one in successive time intervals. Its proof is analogous to the previous one, so we omit it.

Proposition 2.11 *Let $v \in \mathbb{R}^d$. For any $n \in \mathbb{N}_0$, it holds*

$$m_n^v + \Gamma(\tilde{M}_n^v - m_n^v) \le \langle x_i(t), v\rangle \le M_n^v - \Gamma(M_n^v - \tilde{m}_n^v), \tag{2.35}$$

for all $t \in I_{n+1}$ and for all $i = 1, \dots, N$, where Γ is the positive constant in (2.28).

Now, we are able the consensus Theorem 2.1.

Proof of Theorem 2.1 Fix $v \in \mathbb{R}^d$. Let us define the quantities

$$\mathcal{D}_n^v := M_n^v - m_n^v, \quad \forall n \in \mathbb{N}_0,$$

where M_n^v, m_n^v are the constants introduced in Definition 2.2. Note that, for all $n \in \mathbb{N}_0$, we have $\mathcal{D}_n^v \ge 0$, being $M_n^v \ge m_n^v$.
Let $\Gamma \in (0, 1)$ be the constant in (2.28). We claim that

$$\mathcal{D}_{n+1}^v \le (1-\Gamma)\mathcal{D}_n^v, \quad \forall n \in \mathbb{N}_0. \tag{2.36}$$

Indeed, fix $n \in \mathbb{N}_0$. Let $i, j = 1, \dots, N$ and $s, t \in I_{n+1}$ be such that $\langle x_i(s), v\rangle = M_{n+1}^v$ and $\langle x_j(t), v\rangle = m_{n+1}^v$. Then, applying Lemma 2.11, we can write

$$\begin{aligned}\mathcal{D}_{n+1}^v &= M_{n+1}^v - m_{n+1}^v = \langle x_i(s), v\rangle - \langle x_j(t), v\rangle \\ &\le M_n^v - m_n^v - \Gamma(M_n^v - \tilde{m}_n^v) - \Gamma(\tilde{M}_n^v - m_n^v).\end{aligned} \tag{2.37}$$

Now, we distinguish four cases.
Case I) Assume that $M_n^v - \tilde{m}_n^v = 0$ and $\tilde{M}_n^v - m_n^v = 0$. Then, since from (2.21)

$$m_n^v \le \tilde{m}_n^v \le \tilde{M}_n^v = m_n^v,$$

we get

$$m_n^v = \tilde{m}_n^v = M_n^v.$$

As a consequence, (2.37) becomes

$$\mathcal{D}_{n+1}^v \le 0 = (1-\Gamma)\mathcal{D}_n^v.$$

Case II) Assume that $M_n^v - \tilde{m}_n^v = 0$ and $\tilde{M}_n^v - m_n^v > 0$. Then, since from (2.21)

$$\tilde{m}_n^v \le \tilde{M}_n^v \le M_n^v = \tilde{m}_n^v,$$

we can write

$$\tilde{M}_n^v = M_n^v.$$

As a consequence, (2.37) becomes

$$\mathcal{D}_{n+1}^v \le M_n^v - m_n^v - \Gamma\tilde{M}_n^v + \Gamma m_n^v = (1-\Gamma)(M_n^v - m_n^v) = (1-\Gamma)\mathcal{D}_n^v.$$

Case III) Assume that $M_n^v - \tilde{m}_n^v > 0$ and $\tilde{M}_n^v - m_n^v = 0$. Then, from (2.21) we have

$$m_n^v \le \tilde{m}_n^v \le \tilde{M}_n^v = m_n^v,$$

from which

$$\tilde{m}_n^v = m_n^v.$$

As a consequence, (2.37) becomes

$$\mathcal{D}^v_{n+1} \leq M^v_n - m^v_n - \Gamma M^v_n + \Gamma \tilde{m}^v_n = (1-\Gamma)(M^v_n - m^v_n) = (1-\Gamma)\mathcal{D}^v_n.$$

Case IV) Assume that $M^v_n - \tilde{m}^v_n > 0$ and $\tilde{M}^v_n - m^v_n > 0$. In this case, using the fact that $\tilde{M}^v_n \geq \tilde{m}^v_n$, from (2.37) we get

$$\mathcal{D}^v_{n+1} \leq (1-\Gamma)(M^v_n - m^v_n) - \Gamma \tilde{M}^v_n + \Gamma \tilde{m}^v_n \leq (1-\Gamma)(M^v_n - m^v_n) = (1-\Gamma)\mathcal{D}^v_n.$$

Hence, (2.36) is fulfilled.
As a consequence, since the positive constant Γ in (2.36) does not depend of the choice of the vector v, we find the following estimate :

$$D_{n+1} \leq (1-\Gamma) D_n, \quad \forall n \in \mathbb{N}_0. \tag{2.38}$$

To see this, fix $n \in \mathbb{N}$. Let $i, j = 1, \dots, N$ and $s, t \in I_{n+1}$ be such that

$$D_{n+1} = |x_i(s) - x_j(t)|.$$

Let us define the unit vector

$$v = \frac{x_i(s) - x_j(t)}{|x_i(s) - x_j(t)|}.$$

Then, using (2.21) and (2.36),

$$\begin{aligned} D_{n+1} &= \langle x_i(s) - x_j(t), v\rangle = \langle x_i(s), v\rangle - \langle x_j(t), v\rangle \\ &\leq M^v_{n+1} - m^v_{n+1} = \mathcal{D}^v_{n+1} \\ &\leq (1-\Gamma)\mathcal{D}^v_n = (1-\Gamma)(M^v_n - m^v_n) \\ &\leq (1-\Gamma) \max_{k,l=1,\dots,N} \max_{r,w \in I_n} |x_k(r) - x_l(w)| = (1-\Gamma) D_n. \end{aligned}$$

Thus, (2.38) holds true.
Now, from (2.38) it comes that

$$D_n \leq (1-\Gamma)^n D_0, \quad \forall n \in \mathbb{N}_0. \tag{2.39}$$

Let us note that (2.39) can be rewritten as

$$D_n \leq e^{-nC_2(\gamma(T+\tau)+\tau)} D_0, \quad \forall n \in \mathbb{N}_0, \tag{2.40}$$

where C_2 is the positive constant in (2.15).
Now, let $t \geq 0$. Thus, $t \in [n(\gamma(T+\tau)+\tau), (n+1)(\gamma(T+\tau)+\tau)]$, for some $n \in \mathbb{N}_0$. Then, using (2.23) and (2.40), it comes that

$$d(t) \leq D_n \leq e^{-nC_2(\gamma(T+\tau)+\tau)} D_0 \leq e^{-C_2(t-\gamma(T+\tau)-\tau)} D_0 = e^{-C_2 t} C_1 D_0,$$

where C_1 is the positive constant in (2.14). This concludes our proof. □

Remark 2.12 Assume that $\chi_{ij} = 1$, for all $i, j = 1, \dots, N$, $i \neq j$, i.e. assume that the interaction is all-to-all. Then, if for all $i, j = 1, \dots, N$ we have $\alpha_{ij}(t) = \alpha(t)$, for a.e. $t \geq 0$, and $\tau_{ij}(t) = \tau(t)$, for all $t \geq 0$, where $\alpha(\cdot)$ and $\tau(\cdot)$ are suitable functions satisfying (2.6) and (2.2) respectively, the constants C_1, C_2 in Theorem 2.1 can be chosen independently of the number of agents (see [17]).

3 The second-order alignment model

Consider a finite set of $N \in \mathbb{N}$ particles, with $N \geq 2$. Let $x_i(t) \in \mathbb{R}^d$ and $v_i(t) \in \mathbb{R}^d$ denote the position and the velocity of the i-th particle at time t, respectively. We shall denote with $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the usual norm and scalar product in $\mathbb{R}^d$, respectively. The interactions between the elements of the system are described by the following Cucker-Smale type model with pair and time variable time delays,

$$\begin{cases} \frac{d}{dt} x_i(t) = v_i(t), & t > 0, \ \forall i = 1, \ldots, N, \\ \frac{d}{dt} v_i(t) = \sum_{j: j \neq i} \chi_{ij} c_{ij}(t)(v_j(t - \tau_{ij}(t)) - v_i(t)), & t > 0, \ \forall i = 1, \ldots, N, \end{cases} \tag{3.41}$$

where the time delay functions $\tau_{ij} : [0, +\infty) \to [0, +\infty)$ are as in (2.2) and the terms χ_{ij} are defined as in (2.3).
Here, the communication rates c_{ij} are of the form

$$c_{ij}(t) := \frac{1}{N-1} \alpha_{ij}(t) \tilde{\psi}(|x_i(t) - x_j(t - \tau_{ij}(t))|), \quad t > 0, \ \forall i, j = 1, \ldots, N, \tag{3.42}$$

where $\tilde{\psi} : [0, +\infty) \to \mathbb{R}$ is a positive, bounded, and continuous function and

$$\tilde{K} := \|\tilde{\psi}\|_\infty,$$

and the weight functions $\alpha_{ij} : [0, +\infty) \to [0, 1]$ are $\mathcal{L}^1$-measurable and satisfy the Persistence Excitation Condition **(PE)**. Without loss of generality, we can assume that the positive constant $\tilde{\alpha}$ appearing in (2.6) satisfies $\tilde{\alpha} \tilde{K} < 1$.
The initial conditions

$$x_i(s) = x_i^0(s), \quad v_i(s) = v_i^0(s), \quad \forall s \in [-\tau, 0], \ \forall i = 1, \ldots, N, \tag{3.43}$$

are assumed to be continuous functions.
We set

$$C_0^V := \max_{i=1,\ldots,N} \ \max_{s \in [-\tau, 0]} |v_i(s)|, \tag{3.44}$$

$$M_0^X := \max_{i=1,\ldots,N} \ \max_{s,t \in [-\tau, 0]} |x_i(s) - x_i(t)|. \tag{3.45}$$

We define the space and velocity diameters as follows

$$d_X(t) := \max_{i,j=1,\ldots,N} |x_i(t) - x_j(t)|, \quad \forall t \geq -\tau,$$
$$d_V(t) := \max_{i,j=1,\ldots,N} |v_i(t) - v_j(t)|, \quad \forall t \geq -\tau.$$

Definition 3.1 We say that a solution $\{(x_i, v_i)\}_{i=1,\ldots,N}$ to system (3.41) exhibits *asymptotic flocking* if it satisfies the two following conditions:

1. there exists a positive constant d^* such that
$$\sup_{t \geq -\tau} d_X(t) \leq d^*;$$
2. $\lim_{t \to +\infty} d_V(t) = 0.$

Our main result is the following.

Theorem 3.1 *Assume (2.2) and that the digraph $\mathcal{G}$ is strongly connected. Let $\tilde{\psi} : [0, +\infty) \to \mathbb{R}$ be a positive, bounded, continuous function that satisfies*

$$\int_0^{+\infty} \left(\min_{r \in [0,t]} \tilde{\psi}(r) \right)^{\gamma} dt = +\infty, \tag{3.46}$$

where γ is the depth of the digraph. Assume that the weight functions $\alpha_{ij} : [0, +\infty) \to [0, 1]$ are $\mathcal{L}^1$-measurable and satisfy **(PE)***. Moreover, let $x_i^0, v_i^0 : [-\tau, 0] \to \mathbb{R}^d$ be continuous functions, for any $i = 1, \dots, N$. Then, for every solution $\{(x_i, v_i)\}_{i=1,\dots,N}$ to (3.41) with the initial conditions (3.43), there exists a positive constant d^* such that*

$$\sup_{t \geq -\tau} d_X(t) \leq d^*, \tag{3.47}$$

and the following exponential decay estimate holds

$$d_V(t) \leq C_3 \left(\max_{i,j=1,\dots,N} \max_{r,s \in [-\tau,0]} |v_i(r) - v_j(s)| \right) e^{-C_4 t}, \quad \forall t \geq 0, \tag{3.48}$$

where C_3, C_4 are the positive constants defined as

$$C_3 := \frac{1}{1 - e^{-K\left(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau\right)} \left(\frac{\tilde{\alpha}}{N-1}\right)^{\gamma} \left(\min_{r \in [0,d^*]} \psi(r)\right)^{\gamma}}, \tag{3.49}$$

$$C_4 := \frac{1}{\gamma(T+\tau)+\tau} \ln \left(\frac{1}{1 - e^{-K\left(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau\right)} \left(\frac{\tilde{\alpha}}{N-1}\right)^{\gamma} \left(\min_{r \in [0,d^*]} \psi(r)\right)^{\gamma}} \right), \tag{3.50}$$

being $\gamma > 0$ the depth of the digraph, T and $\tilde{\alpha}$ the positive constants in (2.6).

Remark 3.2 Let us note that, if the function $\tilde{\psi}$ is nonincreasing and the interaction is universal, i.e. $\gamma = 1$, then the condition (3.46) reduces to

$$\int_0^{+\infty} \tilde{\psi}(t) dt = +\infty,$$

which is the classical assumption to obtain unconditional flocking (see e.g. [41]). Since here we deal with an influence function not necessarily monotonic and the interaction is not universal, we require the stronger assumption (3.46) (cf. [15] for the case of universal interaction).

Remark 3.3 Let us note that, analogously to the first-order model (see Remark 2.2), the decay velocity C_4 tends to 0 as γ, T or τ goes to $+\infty$ and as $\tilde{\alpha} \to 0$. Therefore, C_4 decreases for increasing values of γ, T and τ and for decreasing values of $\tilde{\alpha}$.

3.1 Preliminary lemmas

Let $\{x_i, v_i\}_{i=1,\dots,N}$ be solution to (3.41) under the initial conditions (3.43). We assume that the hypotheses of Theorem 3.1 are satisfied. The following lemmas hold. We omit their proofs since they can be proved using the same arguments employed in Section 2.

Definition 3.2 Given a vector $v \in \mathbb{R}^d$, for all $n \in \mathbb{N}_0$ we define

$$r_n^v := \min_{j=1,\dots,N} \min_{s \in I_n} \langle v_j(s), v \rangle,$$
$$R_n^v := \max_{j=1,\dots,N} \max_{s \in I_n} \langle v_j(s), v \rangle,$$

where, as in the previous section,

$$I_n = [n(\gamma(T+\tau)+\tau)-\tau, n(\gamma(T+\tau)+\tau)].$$

Also, we define, for all $n \in \mathbb{N}_0$,

$$\tilde{r}_n^v := \min_{j=1,\dots,N} \langle v_j(n(\gamma(T+\tau)+\tau)), v \rangle,$$
$$\tilde{R}_n^v := \max_{j=1,\dots,N} \langle v_j(n(\gamma(T+\tau)+\tau)), v \rangle.$$

Lemma 3.4 *For each vector $v \in \mathbb{R}^d$ and for any $n \in \mathbb{N}_0$, we have that*

$$r_n^v \leq \langle v_i(t), v \rangle \leq R_n^v, \tag{3.51}$$

for all $t \geq n(\gamma(T+\tau)+\tau)-\tau$ and for any $i = 1, \dots, N$.

Definition 3.3 For all $n \in \mathbb{N}_0$, we define

$$F_n := \max_{i,j=1,\dots,N} \max_{r,s \in I_n} |v_i(r) - v_j(s)|.$$

Remark 3.5 Let us note that

$$F_0 := \max_{i,j=1,\dots,N} \max_{r,s \in I_0} |v_i(r) - v_j(s)| = \max_{i,j=1,\dots,N} \max_{r,s \in [-\tau,0]} |v_i(r) - v_j(s)|.$$

Then, the exponential decay estimate in (3.48) can be written as

$$d_V(t) \leq C_3 F_0 e^{-C_4 t}, \quad \forall t \geq 0,$$

where $C_3 > 0$ and $C_4 > 0$ are the constants defined in (3.49) and (3.50), respectively.

Lemma 3.6 *For each $n \in \mathbb{N}_0$, we have that*

$$|v_i(s) - v_j(t)| \leq F_n, \tag{3.52}$$

for all $s, t \geq n(\gamma(T+\tau)+\tau)-\tau$ and for any $i, j = 1, \dots, N$.

Remark 3.7 Let us note that (3.52) yields

$$d_V(t) \leq F_n, \quad \forall t \geq n(\gamma(T+\tau)+\tau)-\tau. \tag{3.53}$$

Furthermore, from (3.52) it follows that

$$F_{n+1} \leq F_n, \quad \forall n \in \mathbb{N}_0. \tag{3.54}$$

Also, arguing as in Section 2, we can find a bound on the velocities $|v_i(t)|$, which is uniform with respect to t and $i = 1, \dots, N$.

Lemma 3.8 *For every $i = 1, \dots, N$, we have that*

$$|v_i(t)| \leq C_0^V, \quad \forall t \geq -\tau, \tag{3.55}$$

where C_0^V is the constant defined in (3.44).

We now provide the following result, in which an estimate of the position diameters is established.

Lemma 3.9 *For every* $i, j = 1, \ldots, N$*, we get*

$$|x_i(t) - x_j(t - \tau_{ij}(t))| \le \tau C_0^V + M_0^X + d_X(t), \quad \forall t \ge 0, \tag{3.56}$$

where C_0^V *and* M_0^X *are the positive constants in* (3.44) *and* (3.45)*, respectively.*

Proof Given $i, j = 1, \ldots, N$ and $t \ge 0$, we have

$$\begin{aligned} |x_i(t) - x_j(t - \tau_{ij}(t))| &\le |x_i(t) - x_j(t)| + |x_j(t) - x_j(t - \tau_{ij}(t))| \\ &\le d_X(t) + |x_j(t) - x_j(t - \tau_{ij}(t))|. \end{aligned} \tag{3.57}$$

Now, we estimate

$$|x_j(t) - x_j(t - \tau_{ij}(t))|.$$

If $t - \tau_{ij}(t) > 0$, from (2.2) and (3.55) we get

$$\begin{aligned} |x_j(t) - x_j(t - \tau_{ij}(t))| &\le \int_{t-\tau_{ij}(t)}^t |v_j(s)| ds \\ &\le C_0^V \tau_{ij}(t) \le \tau C_0^V. \end{aligned}$$

On the other hand, if $t - \tau_{ij}(t) \le 0$, then $t \le \tau$ and

$$\begin{aligned} |x_j(t) - x_j(t - \tau_{ij}(t))| &\le |x_j(0) - x_j(t - \tau_{ij}(t))| + \int_0^t |v_j(s)| ds \\ &\le M_0^X + tC_0^V \le M_0^X + \tau C_0^V. \end{aligned}$$

Therefore, in both cases,

$$|x_j(t) - x_j(t - \tau_{ij}(t))| \le M_0^X + \tau C_0^V,$$

from which (3.57) becomes

$$|x_i(t) - x_j(t - \tau_{ij}(t))| \le M_0^X + \tau C_0^V + d_X(t).$$

□

3.2 The flocking estimate

To prove the flocking result we need, as before, a crucial proposition. First of all, we give the following definition.

Definition 3.4 We define

$$\tilde{\phi}(t) := \min \left\{ \tilde{\psi}(r) : r \in \left[0, \tau C_0^V + M_X^0 + \max_{s \in [-\tau, t]} d_X(s) \right] \right\},$$

for all $t \ge -\tau$.

Remark 3.10 Let us note that from (3.56)

$$\tilde{\psi}(|x_i(t) - x_j(t - \tau_{ij}(t))|) \ge \tilde{\phi}(t), \quad \forall t \ge 0, \forall i, j = 1, \ldots, N.$$

from which

$$c_{ij}(t) \ge \frac{1}{N-1} \alpha_{ij}(t) \tilde{\phi}(t), \quad \forall t \ge 0, \forall i, j = 1, \ldots, N. \tag{3.58}$$

Proposition 3.11 *For all* $v \in \mathbb{R}^d$, *it holds*

$$r_0^v + \Gamma_1(\tilde{R}_0^v - r_0^v) \le \langle v_i(t), v\rangle \le R_0^v - \Gamma_1(R_0^v - \tilde{r}_0^v), \tag{3.59}$$

for all $t \in I_1$ *and for all* $i = 1, \dots, N$, *where* Γ_1 *is the positive constant defined as follows*

$$\Gamma_1 := e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)} \left(\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)\tilde{\alpha}}{N-1}\right)^{\gamma}. \tag{3.60}$$

Remark 3.12 Let us note that $\Gamma_1 \in (0,1)$ since we have assumed $\tilde{\alpha}\tilde{K} < 1$.

Proof of Proposition 3.11 Fix $v \in \mathbb{R}^d$. Let $L = 1, \dots, N$ be such that $\langle v_L(0), v\rangle = \tilde{r}_0^v$. Note that from (3.51), $R_0^v \ge \tilde{r}_0^v$. Then, for a.e. $t \in [0, \gamma(T+\tau)+\tau]$, from (3.51)

$$\begin{aligned}
\frac{d}{dt}\langle v_L(t), v\rangle &= \sum_{j:j\neq L} \chi_{Lj} c_{Lj}(t)(\langle v_j(t-\tau_{Lj}(t)), v\rangle - \langle v_L(t), v\rangle) \\
&\le \sum_{j:j\neq L} \chi_{Lj} c_{Lj}(t)(R_0^v - \langle v_L(t), v\rangle) \\
&\le \frac{\tilde{K}}{N-1} \sum_{j:j\neq L} (R_0^v - \langle v_L(t), v\rangle) = \tilde{K}(R_0^v - \langle v_L(t), v\rangle).
\end{aligned}$$

Thus, the Gronwall's inequality yields

$$\begin{aligned}
\langle v_L(t), v\rangle &\le e^{-\tilde{K}t}\langle v_L(0), v\rangle + R_0^v(1 - e^{-\tilde{K}t}) \\
&= R_0^v - e^{-\tilde{K}t}(R_0^v - \tilde{r}_0^v) \\
&\le R_0^v - e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v).
\end{aligned}$$

Therefore, we have

$$\langle v_L(t), v\rangle \le R_0^v - e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v), \quad \forall t \in [0, \gamma(T+\tau)+\tau]. \tag{3.61}$$

Now, let $i_1 = 1, \dots, N \setminus \{L\}$ be such that $\chi_{i_1 L} = 1$. Such an index i_1 exists since the digraph is strongly connected. Then, for a.e. $t \in [\tau, \gamma(T+\tau)+\tau]$, from (3.61) we get

$$\begin{aligned}
\frac{d}{dt}\langle v_{i_1}(t), v\rangle &= \sum_{j\neq i_1, L} \chi_{i_1 j} c_{i_1 j}(t)(\langle v_j(t-\tau_{i_1 j}(t)), v\rangle - \langle v_{i_1}(t), v\rangle) \\
&\quad + c_{i_1 L}(t)(\langle v_L(t-\tau_{i_1 L}(t)), v\rangle - \langle v_{i_1}(t), v\rangle) \\
&\le \sum_{j\neq i_1, L} \chi_{i_1 j} c_{i_1 j}(t)(R_0^v - \langle v_{i_1}(t), v\rangle) \\
&\quad + c_{i_1 L}(t)\left(R_0^v - e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v) - \langle v_{i_1}(t), v\rangle\right) \\
&= (R_0^v - \langle v_{i_1}(t), v\rangle) \sum_{j\neq i_1, L} \chi_{i_1 j} c_{i_1 j}(t) \\
&\quad + c_{i_1 L}(t)\left(R_0^v - e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v) - \langle v_{i_1}(t), v\rangle\right).
\end{aligned}$$

Note that

$$\sum_{j\neq i_1, L} \chi_{i_1 j} c_{i_1 j}(t) = \sum_{j\neq i_1} \chi_{i_1 j} c_{i_1 j}(t) - c_{i_1 L}(t) \le \frac{\tilde{K}}{N-1} \sum_{j\neq i_1, L} \chi_{i_1 j} - c_{i_1 L}(t) = \frac{\tilde{K} N_{i_1}}{N-1} - c_{i_1 L}(t).$$

Thus, from (3.58) it comes that

$$\begin{aligned}\frac{d}{dt}\langle v_{i_1}(t), v\rangle &\leq \frac{\tilde{K}N_{i_1}}{N-1}(R_0^v - \langle v_{i_1}(t), v\rangle) - c_{i_1L}(t)(R_0^v - \langle v_{i_1}(t), v\rangle) \\ &\quad + c_{i_1L}(t)\left(R_0^v - e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v) - \langle v_{i_1}(t), v\rangle\right) \\ &\leq \frac{\tilde{K}N_{i_1}}{N-1}(R_0^v - \langle v_{i_1}(t), v\rangle) - \alpha_{i_1L}(t)\frac{\tilde{\phi}(t)}{N-1}e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v) \\ &= \frac{\tilde{K}N_{i_1}}{N-1}R_0^v - \alpha_{i_1L}(t)\frac{\tilde{\phi}(t)}{N-1}e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v) - \frac{\tilde{K}N_{i_1}}{N-1}\langle v_{i_1}(t), v\rangle.\end{aligned}$$

Hence, the Gronwall's estimate yields

$$\begin{aligned}\langle v_{i_1}(t), v\rangle &\leq e^{-\frac{\tilde{K}N_{i_1}}{N-1}(t-\tau)}\langle v_{i_1}(\tau), v\rangle) + R_0^v(1 - e^{-\frac{\tilde{K}N_{i_1}}{N-1}(t-\tau)}) \\ &\quad - e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\frac{1}{N-1}\int_\tau^t \tilde{\phi}(s)\alpha_{i_1L}(s)e^{-\frac{\tilde{K}N_{i_1}}{N-1}(t-s)}ds \\ &\leq e^{-\frac{\tilde{K}N_{i_1}}{N-1}(t-\tau)}R_0^v + R_0^v(1 - e^{-\frac{\tilde{K}N_{i_1}}{N-1}(t-\tau)}) \\ &\quad - e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)e^{-\tilde{K}\gamma(T+\tau)}\frac{1}{N-1}\int_\tau^t \tilde{\phi}(s)\alpha_{i_1L}(s)ds \\ &= R_0^v - e^{-\tilde{K}(2\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\frac{1}{N-1}\int_\tau^t \tilde{\phi}(s)\alpha_{i_1L}(s)ds,\end{aligned}$$

for all $t \in [\tau, \gamma(T+\tau)+\tau]$. Note that, since $\tilde{\phi}$ is a nonincreasing function,

$$\tilde{\phi}(t) \geq \tilde{\phi}(\gamma(T+\tau)+\tau), \quad \forall t \in [0, \gamma(T+\tau)+\tau]. \tag{3.62}$$

Then, we can write

$$\langle v_{i_1}(t), v\rangle \leq R_0^v - e^{-\tilde{K}(2\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{N-1}\int_\tau^t \alpha_{i_1L}(s)ds,$$

for all $t \in [\tau, \gamma(T+\tau)+\tau]$. In particular, for $t \in [T+\tau, \gamma(T+\tau)+\tau]$, we find

$$\langle v_{i_1}(t), v\rangle \leq R_0^v - e^{-\tilde{K}(2\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{N-1}\tilde{\alpha}, \tag{3.63}$$

where here we have used the fact that (2.6) implies the following inequality

$$\int_\tau^t \alpha_{i_1L}(s)ds \geq \int_\tau^{T+\tau} \alpha_{i_1L}(s)ds \geq \tilde{\alpha}.$$

Now, if $\gamma = 1$, (3.63) holds for each agent. On the other hand, if $\gamma > 1$, let us consider an index $i_2 \in \{1, \ldots, N\} \setminus \{i_1\}$ such that $\chi_{i_2i_1} = 1$. Then, for a.e. $t \in [T+2\tau, \gamma(T+\tau)+\tau]$, from (3.63) it comes that

$$
\begin{aligned}
\frac{d}{dt}\langle v_{i_2}(t), v\rangle = & \sum_{j\neq i_1,i_2} \chi_{i_2 j} c_{i_2 j}(t)(\langle v_j(t-\tau_{i_2 j}(t)), v\rangle - \langle v_{i_2}(t), v\rangle) \\
& + c_{i_2 i_1}(t)(\langle v_{i_1}(t-\tau_{i_2 i_1}(t)), v\rangle - \langle v_{i_2}(t), v\rangle) \\
\leq & (R_0^v - \langle v_{i_2}(t), v\rangle) \sum_{j\neq i_1,i_2} \chi_{i_2 j} c_{i_2 j}(t) \\
& + c_{i_2 i_1}(t)\left(R_0^v - e^{-\tilde{K}(2\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{N-1}\tilde{\alpha} - \langle v_{i_2}(t), v\rangle\right).
\end{aligned}
$$

Hence, arguing as above we obtain

$$
\begin{aligned}
\frac{d}{dt}\langle v_{i_2}(t), v\rangle \leq & \frac{\tilde{K}N_{i_2}}{N-1}(R_0^v - \langle v_{i_2}(t), v\rangle) - c_{i_2 i_1}(t)(R_0^v - \langle v_{i_2}(t), v\rangle) \\
& + c_{i_2 i_1}(t)\left(R_0^v - e^{-\tilde{K}(2\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{N-1}\tilde{\alpha} - \langle v_{i_2}(t), v\rangle\right) \\
\leq & \frac{\tilde{K}N_{i_2}}{N-1}(R_0^v - \langle v_{i_2}(t), v\rangle) - \alpha_{i_2 i_1}(t) e^{-\tilde{K}(2\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v) \\
& \times \frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{(N-1)^2}\tilde{\phi}(t)\tilde{\alpha}.
\end{aligned}
$$

Again, using Gronwall's estimate it comes that

$$
\begin{aligned}
\langle v_{i_2}(t), v\rangle \leq & e^{-\frac{\tilde{K}N_{i_2}}{N-1}(t-T-2\tau)}\langle v_{i_2}(T+2\tau), v\rangle) + R_0^v(1 - e^{-\frac{\tilde{K}N_{i_2}}{N-1}(t-T-2\tau)}) \\
& - e^{-\tilde{K}(2\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{(N-1)^2}\tilde{\alpha}\int_{T+2\tau}^t \tilde{\phi}(s)\alpha_{i_2 i_1}(s) e^{-\frac{\tilde{K}N_{i_2}}{N-1}(t-s)} ds \\
\leq & R_0^v - e^{-\tilde{K}(3\gamma(T+\tau)-T)}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{(N-1)^2}\tilde{\alpha}\int_{T+2\tau}^t \tilde{\phi}(s)\alpha_{i_2 i_1}(s) ds,
\end{aligned}
$$

for all $t \in [T+2\tau, \gamma(T+\tau)+\tau]$. In particular, for $t \in [2T+2\tau, \gamma(T+\tau)+\tau]$, the condition (2.6) and the inequality (3.62) imply that

$$
\langle v_{i_2}(t), v\rangle \leq R_0^v - e^{-\tilde{K}(3\gamma(T+\tau)-T)}(R_0^v - \tilde{r}_0^v)\left(\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{N-1}\right)^2 \tilde{\alpha}^2. \tag{3.64}
$$

Finally, iterating the above procedure along the path $i_0, i_1, \ldots, i_r$, with $r \leq \gamma$, starting from $i_0 = L$ we find the following upper bound

$$
\langle v_{i_k}(t), v\rangle \leq R_0^v - e^{-K((k+1)\gamma(T+\tau)-(T+\tau)\left(\sum_{l=0}^{k-1} l\right)+\tau)}(R_0^v - \tilde{r}_0^v)\left(\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)\tilde{\alpha}}{N-1}\right)^k, \tag{3.65}
$$

for all $1 \leq k \leq r$ and for all $t \in [k(T+\tau), \gamma(T+\tau)+\tau]$. In particular, if the path has length γ, for $k = \gamma$, since $\sum_{l=0}^{\gamma-1} l = \frac{\gamma(\gamma-1)}{2}$, inequality (3.65) reads as

$$
\langle v_{i_\gamma}(t), v\rangle \leq R_0^v - e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\left(\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)\tilde{\alpha}}{N-1}\right)^\gamma, \tag{3.66}
$$

for all $t \in [\gamma(T+\tau), \gamma(T+\tau)+\tau]$. Arguing as in Proposition 2.9, we can say that (3.66) holds for every $i = 1, \ldots, N$.

Now, let $R = 1, \dots, N$ be such that $\tilde{R}_0^v = \langle v_R(0), v\rangle$. Then, arguing as before, we get

$$\langle v_R(t), v\rangle \geq r_0^v + e^{-K(\gamma(T+\tau)+\tau)}(\tilde{R}_0^v - r_0^v), \quad \forall t \in [0, \gamma(T+\tau)+\tau]. \tag{3.67}$$

Employing the same arguments used above, we can conclude that

$$\langle v_i(t), v\rangle \geq r_0^v + e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)}(\tilde{R}_0^v - r_0^v)\left(\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)\tilde{\alpha}}{N-1}\right)^{\gamma},$$

for all $t \in [\gamma(T+\tau), \gamma(T+\tau)+\tau]$ and for all $i = 1, \dots, N$. Finally, we can deduce that estimate (3.59) holds. □

The following proposition extends the previous one in successive time intervals. We omit its proof since it is analogous to the previous one.

Proposition 3.13 *Let $v \in \mathbb{R}^d$. For any $n \in \mathbb{N}$, it holds*

$$r_n^v + \Gamma_{n+1}(\tilde{R}_n^v - r_n^v) \leq \langle v_i(t), v\rangle \leq R_n^v - \Gamma_{n+1}(R_n^v - \tilde{r}_n^v), \tag{3.68}$$

for all $t \in I_{n+1}$ and for all $i = 1, \dots, N$, where Γ_{n+1} is the positive constant defined as

$$\Gamma_{n+1} := e^{-K\left(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau\right)}\left(\frac{\tilde{\phi}((n+1)(\gamma(T+\tau)+\tau))\tilde{\alpha}}{N-1}\right)^{\gamma}. \tag{3.69}$$

Remark 3.14 Let us note that from (3.68) it comes that

$$R_{n+1}^v - r_{n+1}^v \leq (1-\Gamma_{n+1})(R_n^v - r_n^v), \quad \forall n \in \mathbb{N}_0. \tag{3.70}$$

where $\Gamma_{n+1} \in (0, 1)$ is the constant in (3.69).
Indeed, given $n \in \mathbb{N}_0$, let $i, j = 1, \dots, N$ and $s, t \in I_{n+1}$ be such that $\langle v_i(s), v\rangle = R_{n+1}^v$ and $\langle v_j(t), v\rangle = r_{n+1}^v$. Then, applying Lemma 3.13, we can write

$$\begin{aligned} R_{n+1}^v - r_{n+1}^v &= \langle v_i(s), v\rangle - \langle v_j(t), v\rangle \\ &\leq R_n^v - r_n^v - \Gamma_{n+1}(R_n^v - \tilde{r}_n^v) - \Gamma_{n+1}(\tilde{R}_n^v - r_n^v). \end{aligned} \tag{3.71}$$

Then, arguing as in the proof of Theorem 2.1, we get that estimate (3.70) holds.
Also, setting $C^* := e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)}\left(\frac{\tilde{\alpha}}{N-1}\right)^{\gamma}$, it holds that

$$\Gamma_{n+1} = C^*(\tilde{\phi}((n+1)(\gamma(T+\tau)+\tau)))^{\gamma}, \quad \forall n \in \mathbb{N}_0. \tag{3.72}$$

As a consequence, (3.70) can be written as

$$R_{n+1}^v - r_{n+1}^v \leq (1 - C^*(\tilde{\phi}((n+1)(\gamma(T+\tau)+\tau)))^{\gamma})(R_n^v - r_n^v), \quad \forall n \in \mathbb{N}_0. \tag{3.73}$$

In particular, from (3.70) and (3.73), arguing as in Theorem 2.1, it comes that

$$F_{n+1} \leq (1-\Gamma_{n+1})F_n, \quad \forall n \in \mathbb{N}_0, \tag{3.74}$$

or, equivalently,

$$F_{n+1} \leq (1 - C^*(\tilde{\phi}((n+1)(\gamma(T+\tau)+\tau)))^{\gamma})F_n, \quad \forall n \in \mathbb{N}_0, \tag{3.75}$$

Now, we are able to prove Theorem 3.1.

Proof of Theorem 3.1 Let $\{(x_i, v_i)\}_{i=1,\dots,N}$ be solution to (3.41) under the initial conditions (3.43). Let us define

$$\tilde{\Gamma}_{n+1} = \frac{\Gamma_{n+1}}{\gamma(T+\tau)+\tau}, \quad \forall n \in \mathbb{N}_0.$$

Let us introduce the function $\mathcal{E} : [-\tau, +\infty) \to [0, +\infty)$,

$$\mathcal{E}(t) := \begin{cases} F_0, \quad t \in [-\tau, \gamma(T+\tau)+\tau], \\ \mathcal{E}(n(\gamma(T+\tau)+\tau)) \left(1 - \tilde{\Gamma}_{n+1}(t - n(\gamma(T+\tau)+\tau))\right), \\ \qquad t \in (n(\gamma(T+\tau)+\tau), (n+1)(\gamma(T+\tau)+\tau)], \ n \geq 1. \end{cases}$$

By definition, $\mathcal{E}$ is continuous, nonnegative and nonincreasing. Moreover, we claim that

$$F_n \leq \mathcal{E}(t), \quad \forall t \in [-\tau, n(\gamma(T+\tau)+\tau)], \ \forall n \in \mathbb{N}_0. \tag{3.76}$$

We prove this by induction. For $n = 1$, from (3.54) we can immediately say that

$$F_1 \leq F_0 = \mathcal{E}(t), \quad \forall t \in [-\tau, \gamma(T+\tau)+\tau].$$

Now, assume that (3.76) holds for some $n \geq 1$. We have to show that (3.76) is true also for $n+1$. From the induction hypothesis and by using again (3.54), we have that

$$F_{n+1} \leq F_n \leq \mathcal{E}(t),$$

for all $t \in [-\tau, n(\gamma(T+\tau)+\tau)]$. It lasts to prove that $F_{n+1} \leq \mathcal{E}(t)$, for all $t \in (n(\gamma(T+\tau)+\tau), (n+1)(\gamma(T+\tau)+\tau)]$. From (3.74), it comes that

$$\begin{aligned} \mathcal{E}(t) &\geq \mathcal{E}((n+1)(\gamma(T+\tau)+\tau)) = \mathcal{E}(n(\gamma(T+\tau)+\tau))(1 - \tilde{\Gamma}_{n+1}(\gamma(T+\tau)+\tau)) \\ &= (1 - \Gamma_{n+1})F_n \geq F_{n+1}, \end{aligned}$$

for all $t \in (n(\gamma(T+\tau)+\tau), (n+1)(\gamma(T+\tau)+\tau)]$, where in the above inequalities we have used the fact that $\mathcal{E}$ is nonincreasing. Hence, (3.76) is proven.
Now, for almost all time (see [15] for further details)

$$\frac{d}{dt} \max_{s\in[-\tau,t]} d_X(s) \leq \left| \frac{d}{dt} d_X(t) \right| \leq d_V(t). \tag{3.77}$$

Next, let us define the function $\mathcal{W} : [-\tau, +\infty) \to [0, +\infty)$,

$$\mathcal{W}(t) := (\gamma(T+\tau)+\tau)\mathcal{E}(t) + C^* \int_0^{\tau C_0^V + M_0^X + \max_{s\in[-\tau, t+\gamma(T+\tau)+\tau]} d_X(s)} \left(\min_{\sigma\in[0,r]} \tilde{\psi}(\sigma) \right)^\gamma dr,$$

for all $t \geq -\tau$. By construction, $\mathcal{W}$ is continuous. Also, for each $n \geq 1$ and for a.e. $t \in (n(\gamma(T+\tau)+\tau), (n+1)(\gamma(T+\tau)+\tau))$, from (3.53), (3.76) and (3.77) it follows that

$$\begin{aligned} \frac{d}{dt}\mathcal{W}(t) &= (\gamma(T+\tau)+\tau)\frac{d}{dt}\mathcal{E}(t) + C^*(\tilde{\phi}(t+\gamma(T+\tau)+\tau))^\gamma \frac{d}{dt} \max_{s\in[-\tau, t+(\gamma(T+\tau)+\tau)]} d_X(s) \\ &\leq -\mathcal{E}(n\gamma(T+\tau)+\tau) C^*(\tilde{\phi}((n+1)(\gamma(T+\tau)+\tau)))^\gamma \\ &\quad + C^*(\tilde{\phi}(t+\gamma(T+\tau)+\tau))^\gamma d_V(t+(\gamma(T+\tau)+\tau)) \\ &\leq C^* F_n(-(\tilde{\phi}((n+1)(\gamma(T+\tau)+\tau)))^\gamma + (\tilde{\phi}((n+1)(\gamma(T+\tau)+\tau))^\gamma)) = 0. \end{aligned}$$

Then,

$$\frac{d}{dt}\mathcal{W}(t) \leq 0, \quad \text{a.e. } t > \gamma(T+\tau)+\tau, \tag{3.78}$$

which implies

$$\mathcal{W}(t) \le \mathcal{W}(\gamma(T+\tau)+\tau), \quad \forall t \ge \gamma(T+\tau)+\tau. \tag{3.79}$$

Now, by definition of $\mathcal{W}$, being $\mathcal{E}$ a nonnegative function, we have

$$C^* \int_0^{\tau C_0^V + M_0^X + \max_{s\in[-\tau, t+\gamma(T+\tau)+\tau]} d_X(s)} \left(\min_{\sigma\in[0,r]} \tilde{\psi}(\sigma) \right)^\gamma dr \le \mathcal{W}(\gamma(T+\tau)+\tau),$$

for all $t \ge \gamma(T+\tau)+\tau$. Letting $t \to \infty$ in the above inequality, we can conclude that

$$C^* \int_0^{\tau C_0^V + M_0^X + \sup_{s\in[-\tau, +\infty)} d_X(s)} \left(\min_{\sigma\in[0,r]} \tilde{\psi}(\sigma) \right)^\gamma dr \le \mathcal{W}(\gamma(T+\tau)+\tau). \tag{3.80}$$

Finally, since the function $\tilde{\psi}$ satisfies property (3.46), from (3.80), we can conclude that there exists a positive constant d^* such that

$$\tau C_0^V + M_0^X + \sup_{s\in[-\tau,+\infty)} d_X(s) \le d^*. \tag{3.81}$$

Now, let us define

$$\hat{\phi} := \min_{r\in[0,d^*]} \tilde{\psi}(r).$$

Note that $\hat{\phi} > 0$. Also, (3.81) yields

$$\hat{\phi} \le \tilde{\phi}(t), \quad \forall t \ge -\tau. \tag{3.82}$$

Then, from (3.75) and (3.82) we have

$$F_{n+1} \le (1 - C^*\hat{\phi}^\gamma) F_n, \quad \forall n \in \mathbb{N}_0. \tag{3.83}$$

Thus, thanks to an induction argument, we can write

$$F_n \le (1 - C^*\hat{\phi}^\gamma)^n F_0, \quad \forall n \in \mathbb{N}_0. \tag{3.84}$$

Note that $C^*\hat{\phi}^\gamma = e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)} \left(\frac{\tilde{\alpha}}{N-1}\right)^\gamma \left(\min_{r\in[0,d^*]} \tilde{\psi}(r)\right)^\gamma$. Thus, the positive constants C_3 and C_4 in (3.48) can be rewritten in the following way (see (3.49) and (3.50)):

$$C_3 = \frac{1}{1 - C^*\hat{\phi}^\gamma},$$

$$C_4 = \frac{1}{(\gamma(T+\tau)+\tau)} \ln\left(\frac{1}{1 - C^*\hat{\phi}^\gamma}\right).$$

As a consequence, inequality (3.84) can be rewritten as

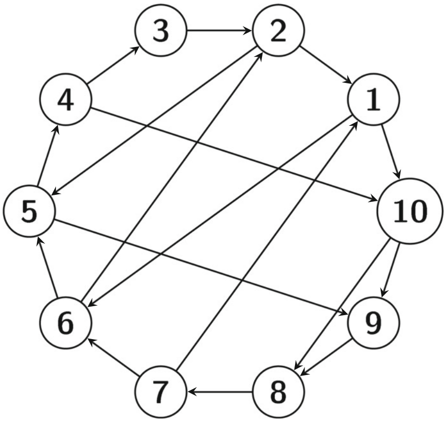
$$F_n \le e^{-nC_4(\gamma(T+\tau)+\tau)} F_0, \quad \forall n \in \mathbb{N}_0, \tag{3.85}$$

where C_4 is the positive constant in (3.50).

Finally, let $t \ge 0$. Then, $t \in [n(\gamma(T+\tau)+\tau), (n+1)(\gamma(T+\tau)+\tau)]$, for some $n \in \mathbb{N}_0$. Then, using (3.53) and (3.85)

$$d_V(t) \le F_n \le e^{-nC_4(\gamma(T+\tau)+\tau)} F_0 \le e^{-C_4(t-\gamma(T+\tau)-\tau)} F_0 = e^{-C_4 t} C_3 F_0,$$

where C_3 is the positive constant in (3.49). This concludes our proof. □



$$\chi_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Fig. 1 Strongly connected digraph and its adjacency matrix, $\gamma = 6$

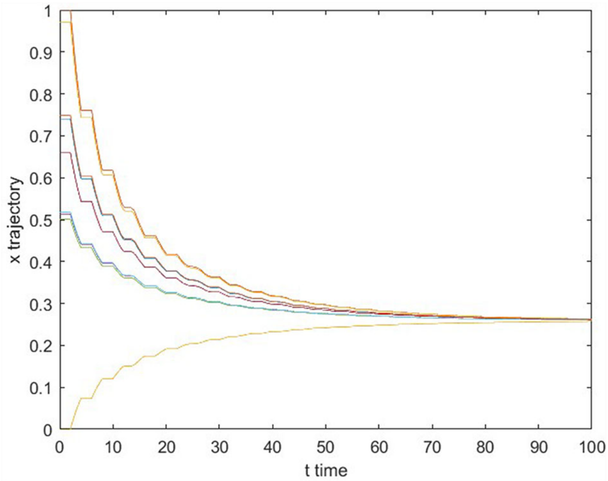


Fig. 2 Numerical solution of the first-order model in the case $\gamma = 6$.

4 Numerical simulations

In this section, we present some numerical simulations for the first-order model (2.1) and the second-order model (3.41) in the one-dimensional case, i.e. $d = 1$, to give evidence to the theoretical results. We consider the influence functions in the definitions (2.4) and (3.42) defined by

$$\psi(r, r') = \tilde{\psi}(r, r') = \psi^*(|r - r'|), \quad r, r' \in [0, +\infty).$$

In particular, we assume that the function $\psi^*(\cdot)$ takes the form

$$\psi^*(r) := e^{-(r-1)^2}, \quad r \in [0, +\infty). \tag{4.86}$$

Meanwhile, for simplicity, the weight functions $\alpha_{ij}(\cdot)$ for all $i, j = 1, \ldots, N$ coincide with a piecewise functions $\alpha(\cdot)$ equal to 1 or 0 alternately in time intervals of length 2. The initial conditions were set to be constant and drawn from a random distribution in the interval

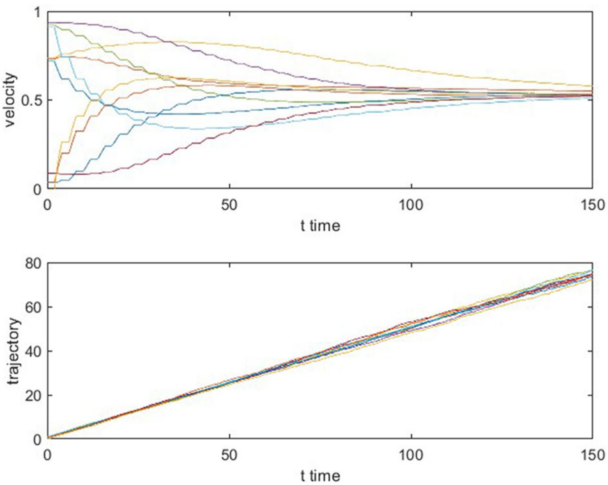
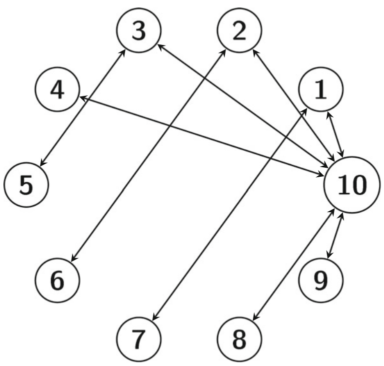


Fig. 3 Numerical solution of the second-order model in the case $\gamma = 6$.



$$\chi_{ij} = \begin{pmatrix} 0&0&0&0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&1&1&0 \end{pmatrix}$$

Fig. 4 Strongly connected digraph and its adjacency matrix, $\gamma = 4$

[0, 1]. To produce the tests, we used MATLAB environment. The solutions of the systems are computed using the MATLAB functions *dde23*, which computes the solution of a given delay differential equations (DDEs) with a constant time delay vector, and *ode45*, which performs the solution of an ordinary differential equation (ODE). In all our simulations, we consider a discretization of the time with a time step of 0.1 and the size of the population $N = 10$. For any agent $i \in \{1, \ldots, N\}$, we assume the time delays τ_{ij} given by a constant vector in which each entrance takes value in the interval [0, 1] as follows:

$$(\tau_{i1}, \ldots, \tau_{iN}) = (0.3804,\ 0.5678,\ 0.0759,\ 0.0540,\\ 0.5308,\ 0.7792,\ 0.9340,\ 0.1299,\ 0.5688,\ 0.4649).$$

Another strongly connected digraph is considered in Figure 4, with depth $\gamma = 4$. Figure 5 and Figure 6 provide numerical tests for such a network structure for the first and second-order model respectively.

In Figure 1, we draw the network structure which describes the interaction between the agents and the adjacency matrix associated with that structure. For this particular case, the

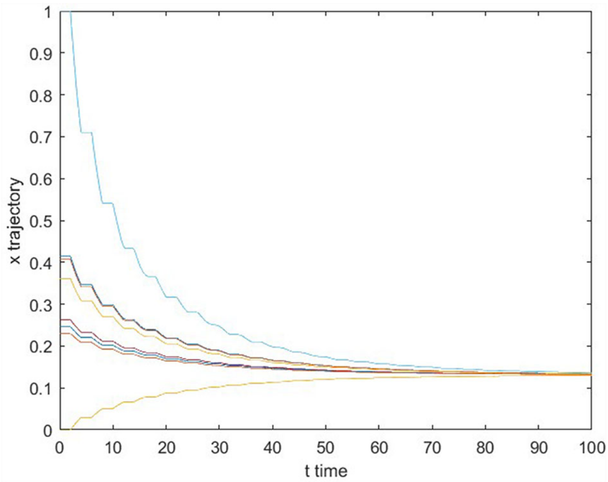


Fig. 5 Numerical solution of the first-order model in the case $\gamma = 4$.

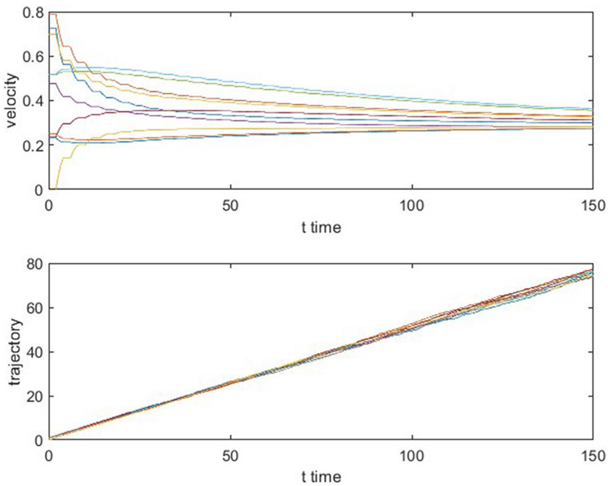


Fig. 6 Numerical solution of the second-order model in the case $\gamma = 4$.

depth of the graph is $\gamma = 6$. In this framework, in Figure 2 we perform the dynamic of the solution for the first-order Cucker-Smale and in Figure 3 we compute the solutions for the second-order model. One can see that the opinion and velocity trajectories of all the agents converge to the same limit, ensuring consensus in the first case and flocking for the second case. Notice that we can see the effect of the presence of the function $\alpha(\cdot)$, that causes a lack of connection in some time intervals.

5 Conclusions

In this paper, we have analyzed the asymptotic behavior of solutions to first and second-order Cucker-Smale models with pair and time-dependent time delays, possible lack of connection among the system's agents during the system's evolution and non-universal interaction.

Under a so-called Persistence Excitation Condition, we were able to prove the exponential consensus for both the first and second-order model, provided that the digraph describing the interaction among the agents is strongly connected (see Theorem 2.1 and Theorem 3.1). This was done without assuming any smallness conditions on the time delay size and any monotonicity assumptions on the influence function. Finally, some numerical examples of our theoretical results were illustrated.

As future research directions, different network topologies could be analyzed. Moreover, some ideas here developed can be useful for the analysis, still widely open, of Hegselmann-Krause and Cucker-Smale models with changing-sign interactions (cf. [18]).

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

References

1. Marsan, G.A., Bellomo, N., Egidi, M.: Towards a mathematical theory of complex socio-economical systems by functional subsystems representation. Kinet. Relat. Models **1**, 249–278 (2008)
2. Ancona, F., Bentaibi, M., Rossi, F.: Consensus under Persistence Excitation, Preprint (2024) arXiv:2403.07549
3. Bellomo, N., Herrero, M.A., Tosin, A.: On the dynamics of social conflict: Looking for the Black Swan. Kinet. Relat. Models **6**, 459–479 (2013)
4. Bonnet, B., Flayac, E.: Consensus and flocking under communication failures for a class of Cucker-Smale systems. Systems Control Lett. **152**, 10pp (2021)

5. Borzì, A., Wongkaew, S.: Modeling and control through leadership of a refined flocking system. Math. Models Methods Appl. Sci. **25**, 255–282 (2015)
6. Bullo, F., Cortés, J., Martínez, S.: Distributed control of robotic networks: a mathematical approach to motion coordination algorithms. Princeton series in applied mathematics. Princeton University Press, Princeton (2009)
7. Camazine, S., Deneubourg, J.L., Franks, N.R., Sneyd, J., Theraulaz, G., Bonabeau, E.: Self-Organization in Biological Systems. Princeton University Press, Princeton, NJ (2001)
8. Carty, C., Choi, Y.-P., Cicolani, C., Pignotti, C.: Asymptotic synchronization of Kuramoto oscillators with time delay and non-universal interaction. Z. Angew. Math. Phys. **76**, Paper No. 34, 30 pp (2025)
9. Ceragioli, F., Frasca, P.: Continuous and discontinuous opinion dynamics with bounded confidence. Nonlinear Anal. Real World Appl. **13**, 1239–1251 (2012)
10. Choi, Y.-P., Haskovec, J.: Cucker-Smale model with normalized communication weights and time delay. Kinet. Relat. Models **10**, 1011–1033 (2017)
11. Choi, Y.-P., Li, Z.: Emergent behavior of Cucker-Smale flocking particles with heterogeneous time delays. Appl. Math. Lett. **86**, 49–56 (2018)
12. Choi, Y.-P., Paolucci, A., Pignotti, C.: Consensus of the Hegselmann-Krause opinion formation model with time delay. Math. Methods Appl. Sci. **44**, 4560–4579 (2021)
13. Choi, Y.-P., Pignotti, C.: Emergent behavior of Cucker-Smale model with normalized weights and distributed time delays. Netw. Heterog. Media **14**, 789–804 (2019)
14. Cicolani, C., Pignotti, C.: Opinion dynamics of two populations with time-delayed coupling. Math. Methods Appl. Sci. **48**, 5731–5744 (2025)
15. Continelli, E.: Asymptotic flocking for the Cucker-Smale model with time-variable time delays. Acta Appl. Math. **188**, Paper No. 15, 23 pp (2023)
16. Continelli, E., Pignotti, C.: Consensus for Hegselmann-Krause type models with time variable time delays. Math. Methods Appl. Sci. **46**, 18916–18934 (2023)
17. Continelli, E., Pignotti, C.: Convergence to consensus results for Hegselmann-Krause type models with attractive-lacking interaction. Math. Control Relat. Fields **14**, 1408–1427 (2024)
18. Continelli, E., Pignotti, C.: Hegselmann-Krause and Cucker-Smale type models with attractive-repulsive interaction, Commun. Math. Sci. **23**, 1547–1582 (2025)
19. Cucker, F., Smale, S.: Emergent behaviour in flocks. IEEE Trans. Autom. Control **52**, 852–862 (2007)
20. Dong, J.-G., Ha, S.-Y., Doheon, K., Jeongho, K.: Time-delay effect on the flocking in an ensemble of thermomechanical Cucker-Smale particles. J. Differential Equations **266**, 2373–2407 (2019)
21. Ha, S.-Y., Lattanzio, C., Rubino, B., Slemrod, M.: Flocking and synchronization of particle models. Quart. Appl. Math. **69**, 91–103 (2011)
22. Ha, S.-Y., Liu, J.-G.: A simple proof of the Cucker-Smale flocking dynamics and mean-field limit. Commun. Math. Sci. **7**, 297–325 (2009)
23. Ha, S.-Y., Tadmor, E.: From particle to kinetic and hydrodynamic descriptions of flocking. Kinet. Relat. Models **1**, 415–435 (2008)
24. Halanay, A.: Differential Equations: Stability. Time Lags, Academic Press, New York and London, Oscillations (1966)
25. Hale, J.K., Lunel, S.M.V.: Introduction to functional differential equations, Applied Mathematical Sciences, vol. 99. Springer, New York (1993)
26. Haskovec, J.: A simple proof of asymptotic consensus in the Hegselmann-Krause and Cucker-Smale models with normalization and delay. SIAM J. Appl. Dyn. Syst. **20**, 130–148 (2021)
27. Haskovec, J.: Direct proof of unconditional asymptotic consensus in the Hegselmann-Krause model with transmission-type delay. Bull. Lond. Math. Soc. **53**, 1312–1323 (2021)
28. Haskovec, J.: Cucker-Smale model with finite speed of information propagation: well-posedness, flocking and mean-field limit. Kinet. Relat. Models **16**, 394–422 (2023)
29. Haskovec, J., Markou, I.: Asymptotic flocking in the Cucker-Smale model with reaction-type delays in the non-oscillatory regime. Kinet. Relat. Models **13**, 795–813 (2020)
30. Haskovec, J., Cartabia, M.R.: Optimal condition for asymptotic consensus in the Hegselmann-Krause model with finite speed of information propagation. Proc. Amer. Math. Soc. **151**, 3871–3878 (2023)
31. Hegselmann, R., Krause, U.: Opinion dynamics and bounded confidence models, analysis, and simulation. J. Artif. Soc. Soc. Simul. **5**, 1–24 (2002)
32. Liu, Y., Wu, J.: Flocking and asymptotic velocity of the Cucker-Smale model with processing delay. J. Math. Anal. Appl. **415**, 53–61 (2014)
33. Lu, J., Ho, D.W.C., Kurths, J.: Consensus over directed static networks with arbitrary finite communications delays. Phys. Rev. E **80**, 066121, 7 pp (2009)
34. Kuramoto, Y.: Chemical oscillations, waves, and turbulence, Springer Ser. Synergetics, vol. 19. Springer-Verlag (1984)

35. Paolucci, A.: Convergence to consensus for a Hegselmann-Krause-type model with distributed time delay. Minimax Theory Appl. **6**, 379–394 (2021)
36. Piccoli, B., Rossi, F., Trélat, E.: Control to flocking of the kinetic Cucker-Smale model. SIAM J. Math. Anal. **47**, 4685–4719 (2015)
37. Pignotti, C., Reche Vallejo, I.: Flocking estimates for the Cucker-Smale model with time lag and hierarchical leadership. J. Math. Anal. Appl. **464**, 1313–1332 (2018)
38. Pignotti, C., Trélat, E.: Convergence to consensus of the general finite-dimensional Cucker-Smale model with time-varying delays. Commun. Math. Sci. **16**, 2053–2076 (2018)
39. Pilyugin, S.Y., Campi, M.C.: Opinion formation in voting processes under bounded confidence. Netw. Heterog. Media **14**, 617–632 (2019)
40. Proskurnikov, A.V., Calafiore, G.C.: Delay robustness of consensus algorithms: continuous-time theory. IEEE Trans. Automat. Control **68**, 5301–5316 (2023)
41. Cartabia, M.R.: Cucker-Smale model with time delay. Discrete Contin. Dynam. Systems **42**, 2409–2432 (2022)
42. Wongkaew, S., Caponigro, M., Borzì, A.: On the control through leadership of the Hegselmann-Krause opinion formation model. Math. Models Methods Appl. Sci. **25**, 565–585 (2015)