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Affine structures on three-dimensional solvmanifolds and SYZ non-Kähler Mirror Symmetry

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We present a new geometric construction that leads us to new examples of pairs of six-dimensional compact manifolds satisfying a non-Kähler version of Mirror Symmetry as formulated by Lau, Tseng, and Yau using SU(3)-structures. In this new setting, the Calabi-Yau geometry is replaced by the symplectic half-flat geometry on the IIA-side and by the complex-balanced geometry on the IIB-side. The link between the two is provided by the Strominger-Yau-Zaslow construction which relies on the presence of a third space B over which the IIA-side fibers in Lagrangian tori. We will show how to build these examples using the theory of solvmanifolds and how it is linked to the affine geometry of the base of the fibration. Finally, we will describe the action of the Fourier-Mukai transform on semi-flat differential forms and how it realizes the equivalence of the Tseng-Yau cohomology on the IIA-side with the Bott-Chern cohomology on the IIB-side.

Ὁ ἀγεωμὲτρης εἰσίτω, ὡ̔ς μανθάνῃ[1]

Let who is untrained in geometry enter,
so that they can learn it

Entri chi non conosca la geometria,
affinché possa apprenderla

[1] ᾿Αγεωμέτρης μεδείς εισίτω - "Let no one untrained in geometry enter" was the motto over the entrance to Plato's Academy. This slight modification wants to be more inclusive. No gatekeeping in math.

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Chapter 1

Introduction

The interplay between mathematics and theoretical physics has always been fruitful. Very often, new discoveries in physics were anticipated or immediately followed by fundamental advances in mathematics. Think for example to the groundbreaking work of Einstein in general relativity, based on the language of the *absolute differential calculus* just developed by Ricci and Levi-Civita, or to the rising of interest in linear algebra and functional analysis bolstered by the birth of quantum mechanics. However, in the last, say fifty, years the interactions and exchanges were remarkably intensified, both in terms of variety and in terms of the deepness of the topics. The desire to get a better understanding of quantum field theory, and the attempt to build a general framework in which also a quantum formulation of gravity could be encompassed, has produced many new ideas and approaches. Among them, we will deal in particular with the origin of string theory that saw light in the second half of the last century. This is mainly motivated by the fact that the basic ideas of string theory are geometrical in nature. Moreover, the rigorous mathematical formulation has established various challenging (and still open) questions in so many areas of mathematics: algebraic geometry, complex and symplectic differential geometry, geometrical analysis, knot theory, algebras, and category theory just to name a few. In this introductory chapter, we will explain why one, as a mathematician, and in particular as a differential geometer, should keep a mindful eye on these topics. Starting from a brief historical review of string theory, we will explain the framework in which a rigorous mathematical formulation of mirror symmetry can be stated. Then, we will settle down the motivation and the starting point of the present work and how its development would lead to the objective. We conclude the section with an outline of the thesis.

1.1 Historical background

Up to our current knowledge, every physical phenomenon can be described in terms of particles and interactions among them. The Standard Model is our best attempt to get a collective picture, but gravity struggles to be included. While the electromagnetic, weak and strong interactions have a - still not complete, but enough satisfactory - description in terms of an appropriate gauge theory (i.e. in terms of the differential geometry of an associated principal bundle), this is not replicable for the gravitational one. Around the 1920s, a first proposal was already known as the *Kaluza-Klein theory* [55],[58] which was quickly discarded. It supposed a fifth dimension beyond the usual four-dimensional space-time which is compactified on a S^1 with a very small radius. Although unconventional, this assumption made it possible to write, in a unified formalism, both the gravitational and electromagnetic interactions. These two main themes, *unification* and *extra dimensions* will be resumed by string theory half a century later. In the 1960s the physicist Gabriele Veneziano proposed a model for the interaction of hadrons [90] whose mathematical structure, after an observation of Nambu and Susskind, is well understood under the assumption that the fundamental objects of the theory are not point-like particles but one-dimensional strings. Therefore, the idea of strings replacing point-like particles was introduced in the context of strong interactions but it had the drawback of predicting the existence of an unwanted particle of spin 2 and the idea was abandoned. Almost twenty years later, these ideas were revived by Green and Schwarz [43] who reinterpreted the model as a candidate for a quantum theory of gravity, also incorporating supersymmetry. This new kind of symmetry asserts, though not yet observed, that for each fermion there is a supersymmetric partner which is a boson, and vice-versa.

1.1.1 Overview of string theory

While a point-like particle traces a curve (a *world-line*) as it moves, a string propagating in the space-time X would trace a surface (a *world-sheet*) Σ. Studying the mechanics of a string corresponds to studying a map $\sigma : \Sigma \to X$: the action functional is then minimized with respect to a class of surfaces instead of curves. In the process of quantization of the functional, some extra terms, called *anomalies*, emerge but they precisely cancel out when the dimension d of the space-time assumes determined values. In the case of supersymmetric string theory, on a flat space-time X, one gets $d = 10$. So what about these extra six-dimension? Physical arguments imposes the presence, on the internal six-dimensional manifold M, which has to be compact, of a parallel spinor, which in turn, forces M to have holonomy contained in the group SU(3), condition which is fulfilled when M is a three-dimensional Calabi-Yau (we will say more in section 2.2.1). Here it starts the interest of string theorists in studying Kähler geometry and Calabi-Yau manifolds.

1.1.2 Mathematical formulation of mirror symmetry

In particular, they managed to construct a physical model out of the geometric invariants, namely the Dolbeault cohomology groups $H^{p,q}(M)$ of the Calabi-Yau three-fold. Such parameters for the model, called a *superconformal field theory (SCFT)*, were in fact linked to the $h^{2,1}(M)$ and $h^{1,1}(M)$. But here it comes the crucial observation which is at the heart of the entire mirror symmetry: they noticed that another Calabi-Yau three-fold $\check{M}$ with $h^{2,1}(\check{M}) = h^{1,1}(M)$ and $h^{1,1}(\check{M}) = h^{2,1}(M)$ would produce the same observable physics! This has the effect of transferring the geometrical information associated to the symplectic structure of M to the geometrical information associated to the complex structure of $\check{M}$ and vice-versa . Given a Calabi-Yau three-fold M, one can then build two different SCFTs, called A-model, defined in terms of the complexified Kähler class, and B-model, defined in terms of infinitesimal variations of complex structure . Then, at the physical level, mirror symmetry foresees the existence of another Calabi-Yau three-fold $\check{M}$ on which the role of A-model and B-model are swapped. In mathematical terms, this can be formally stated in terms of the isomorphism of cohomology groups

$$H^{p,q}(M) \simeq H^{3-p,q}(\check{M}) \tag{1.1}$$

and can be visualized, in a fancy way, as a symmetry for the Hodge diamond along the oblique - from bottom left to right top - axis:

			1							1			
		0		0					0		0		
	0		1		0			0		101		0	
1		101		101		1	1		1		1		1
	0		1		0			0		101		0	
		0		0					0		0		
			1							1			

Figure 1.1: Mirror diamonds for the quintic

The first explicit computations in this sense were carried out in 1991 by Candelas, de la Ossa, Green, Parkes in their striking paper [20] . There the authors started with a Calabi-Yau three-fold M, represented by a quintic in $\mathbb{P}^4$, and constructed its mirror $\check{M}$ verifying the cohomological relation between them (see the figure). Above all, the big surprise came when they used this correspondence to compute correlation functions for the A-model on M, which were particularly involved, in terms of the correlation functions for the B-model on $\check{M}$ which was an easier task. This had the unexpected consequence of giving a prevision for the number n_d of rational curves in M of given degree d. At that time, the integers n_1 and n_2 were already known but for higher d, the computation by classical algebraic geometry techniques were too complicated. When the prevision for $d = 3$, given by the correspondence in [20], was confirmed some years later, many mathematicians started interesting in mirror symmetry and the topic became increasingly well known also among algebraic geometers. This was the first time that ideas coming from theoretical physics gave a prevision for a conjecture in enumerative geometry. Since the nineties, two major approaches were proposed to explain this mysterious phenomenon: the homological version formulated by Kontsevich [59] and the T-duality one proposed by Strominger, Yau and Zaslow [82]. The first deals with the equivalence of two categories: on one hand one deals with the derived category of the Fukaya category defined in terms of the symplectic geometry, while, on the other, one deals with the derived category of coherent sheaves which relies on the complex structure. Again the mirror conjecture is about a *switch* of geometrical information:

$$\mathcal{D}\mathrm{Fuk}(M) \simeq \mathcal{D}\mathrm{Coh}(\check{M}) \qquad \mathcal{D}\mathrm{Coh}(M) \simeq \mathcal{D}\mathrm{Fuk}(\check{M}) \tag{1.2}$$

The other approach, rather more topological, instead postulates the existence of another space B over which the Calabi-Yau three-fold fibers in special Lagrangian tori. From the physical viewpoint, the SYZ construction refers to the Type II string models which again differentiate in the A-symplectic model and the B-complex model. The duality between the two models is then represented by a duality between the torus fibers. The conjecture of SYZ mirror symmetry can be stated as

Conjecture 1.1.1. (SYZ Mirror Symmetry)
For each Calabi-yau three-fold M there exist another Calabi-Yau three-fold $\check{M}$ and a topological manifold B of dimension three and, possibly singular, fibrations $\pi : M \to B$ and $\check{\pi} : \check{M} \to B$ such that

1. *Let $B^{sing} \subset B$ the locus where $\pi, \check{\pi}$ are singular and let $B_0 = B \backslash B^{sing}$. Then both M and $\check{M}$ fibers in special Lagrangian tori over B_0.*
2. *The torus fiber $T := \pi^{-1}(b)$ is dual to $\check{T} := \check{\pi}^{-1}(b)$ for each $b \in B$.*
3. $H^{p,q}(M) \simeq H^{3-p,q}(\check{M})$

Many examples were constructed but a full comprehension of the picture is still far from being obtained. Partial positive results are also obtained in the semi-flat setting, namely when the fibration is everywhere smooth and so $B = B_0$. The main reference for the topic is the program carried by Gross [5],[44],[45], and together with Siebert [46], see also the survey by Auroux [6] or the paper by Castaño and Matessi [16]. Moreover, also the mechanism which realizes the equivalence between the two sides is not completely understood. A differential-geometric version of the Fourier-Mukai transform, appearing first in the homological approach as an equivalence of derived categories over abelian varieties, is the proposed tool [23],[62], [63]. We will deal with this side of the story, the SYZ program, but in a broader context, the one of non-Kähler geometry.

1.2 Starting point and purpose

Our starting point is the paper by Lau, Tseng, and Yau [61] in which a generalization of the SYZ approach is proposed. If we enlarge the picture by relaxing the Kähler condition one still has a manifestation of mirror symmetry but the single manifold now encompasses just a single string model. The new formulation is based on SU(3)-structures which differentiate into symplectic half-flat manifolds on the IIA side and into complex balanced manifolds on the IIB side. More in details, an SU(3)-structure is represented by a couple of differential forms $(\omega, \Omega) \in \mathcal{A}^2(M, \mathbb{R}) \oplus \mathcal{A}^3(M, \mathbb{C})$ satisfying some properties. If moreover $d\omega = 0$ and $d\mathrm{Re}\Omega = 0$, we are talking about IIA equations while if $d\omega^2 = 0$ and $d\Omega = 0$, they are IIB equations instead. The mechanism of the exchange still requires the existence of an SYZ-fibration, which has to start from the symplectic side. Since the manifolds are no longer Kähler we can not compute their Hodge diamonds by means of the Dolbeault cohomology. Therefore, on the complex side, we will avail the already well-known *Bott-Chern cohomology* while, on the symplectic side, we will make use of the more recent *Tseng-Yau cohomology* [85],[86]. Moreover, the equivalence of the two models is provided by another differential-geometric version of the Fourier-Mukai transform proposed in [61]. We can roughly summarize in the following

Theorem 1.2.1. (Theorem 5.1 and 6.7 in [61]) *Let $M \to B$ a Lagrangian torus bundle associated to a semi-flat supersymmetric $SU(3)$-system of type IIA and $\check{M} \to B$ its SYZ dual. Then the Fourier-Mukai transform gives isomorphisms*

$$H^{3-p,q}_{B,TY}(M) \simeq H^{p,q}_{B,BC}(\check{M})$$

and exchanges the IIA-equations on M with the IIB-equations on $\check{M}$.

Here the "B" subscripts mean that we are restricting to basic, T-invariant forms. We will give a more precise statement in chapter 3. We also remark that correspondence between Tseng-Yau and Bott-Chern cohomologies does not require the half-flat/balanced condition to be present but relies only on the SU(3)-structure plus the Lagrangian fibration showing that mirror symmetry is a phenomenon combining symplectic and complex geometry in a more general way. We will show this with one of our examples. At this point it is worth mentioning that mirror symmetry, and T-duality, can also be described in terms of the *generalized-complex geometry* language introduced by Hitchin [51] and developed by Cavalcanti and Gualtieri [21],[47]. A generalized-complex version of T-duality for nilmanifolds was treated by del Barco, Grama, Soriani [9]. Moreover, also Tseng and Yau reinterpreted their previous analysis in terms of a generalized-complex cohomology theory [87].

The main result of the thesis is the discovery of the SYZ mirror partner, in terms of the above theorem, of almost all the known examples of compact non-Kähler type IIA (symplectic half-flat) six-dimensional manifolds. With few exceptions, all known such examples come from left invariant structures on six-dimensional solvable Lie groups. More precisely, there is a classification ([26],[30]) of solvable Lie groups/Lie algebras admitting left-invariant type IIA structures. All of these admit lattices, i.e. discrete cocompact subgroups. The first result of the thesis is a reinterpretation of this classification in terms of affine structures on three-dimensional solvable Lie groups. While the existing classification for IIA structures is obtained case by case with a purely algebraic classification, we provide a general geometric construction that allows us to build almost all the known examples of compact type IIA manifolds. This has been achieved by blending the theory of action-angle coordinates, coming from a canonical Lagrangian fibration, with the left-invariant affine structure of the base. In [61], the only example, in dimension three, is provided by the (co)tangent bundle of the Heisenberg manifold modulo a lattice. We were able to reinterpret the compact IIA solvmanifolds as the cotangent bundle of a three-dimensional compact solvmanifold B modulo a lattice which is, in turn, related to different affine structures on B. There are several reasons for why the class of manifolds we used was selected among nilmanifolds and the more general solvmanifolds. They are also called by physicists *twisted tori* ([40],[41]) since they can always be seen as a bundle over a torus with another nilmanifold as a fiber (the Mostow bundle [18]). In particular, in the case of 2-step nilmanifolds, the fibration is a principal torus bundle over a torus [71]. This feature makes it quite reasonable to use them to test the effects of T-duality. The complex non-Kähler geometry of nilmanifolds and solvmanifolds is already well and deeply investigated, see [32], [60] and references therein. Instead, for the symplectic side, there are only existence results, cited above, in [26],[30], but no complete classification. Moreover, from the cohomological viewpoint, the availability of Nomizu-like theorems allows us to reduce the computation for the cohomology at the level of Lie algebras which is more tractable. One of the novelties of this work is the first explicit computations for (p, q)-groups in Tseng-Yau cohomology. This leads us to produce therefore the first *Tseng-Yau-Hodge diamonds* and to relate them with the already known *Bott-Chern-Hodge diamonds* for the complex side. Looking at the structure of the algebras we were using for our computation, we noticed that they were sharing a common pattern. In fact, the simply-connected Lie group associated with them had, in each case, the structure of a semidirect product of a three-dimensional solvable Lie group and the abelian $\mathbb{R}^3$. Then, motivated by the symplectic theory of Lagrangian torus fibrations, we investigated the affine integral geometry of such three-dimensional Lie groups and we managed to relate it to the group structure of the six-dimensional Lie groups. This also leads us to produce a common recipe to build a natural symplectic SU(3)-structure on each example. The main results of the present thesis are contained in the article *SYZ mirror symmetry of solvmanifolds* in preparation with my advisor L. Bedulli [14].

1.2.1 Outline

Mirror symmetry lies at the crossroad of symplectic and complex geometry and as the SYZ construction enters the picture, it carries with itself also the affine geometry baggage. Moreover, if one wants to build examples using solvmanifolds, all the geometric structures have to be reconciled with the algebraic structure of the groups. Therefore we start in Chapter 2 with a review of basic facts and fundamental results about all these topics. After that, we explain in detail the SYZ construction and the non-Kähler mirror symmetry formulation as stated by [61] in Chapter 3. We describe our technique to build examples and we show the algebraic structure of the six-dimensional Lie groups that will be used to create the six-dimensional compact solvmanifolds for the mirror pairs. Lastly, in Chapter 4, we carry on the analysis induced by the dual set of action-angle coordinates, we write down the supersymmetric SU(3)-structures, we present their diamonds and show the correspondence of $\breve{\omega}$ and Ω via Fourier-Mukai transform. All the computations for the cohomology are collected in the appendix.

Chapter 2

Preliminaries

This chapter is devoted to recalling the definitions and fixing the notations for each type of geometry we are going to interface with. We also recollect the results of classification needed to build our examples.

2.1 Affine geometry

Definition 2.1.1. *An* ***affine structure*** *on a smooth manifold M is an (equivalence class of an) atlas $\mathcal{U} = \{U_i, \varphi_i\}_i$ whose transition functions are affine, i.e. $\varphi_{ij} \in Aff(\mathbb{R}^n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$. This is equivalent to the existence of a flat, torsionfree connection on the tangent bundle of the manifold. It is called* ***special*** *when the linear part is contained in $SL(n, \mathbb{R})$. If additionally the φ_{ij}'s are in $Aff_{\mathbb{Z}}(\mathbb{R}) = GL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$ the affine structure is said* ***integral****.*

Affine structures can also be described in terms of the more general language of (G, X)-structures *á la Thurston*. A manifold M admits a (G, X)-structure if its transition functions are induced by an element of a Lie group G acting transitively on another manifold X. In our case $X = \mathbb{R}^n$ and G =Aff($\mathbb{R}^n$). Associated to any (G, X)-structure there are two fundamental objects from which we can recover the structure. Let $\tilde{M}$ the universal cover of M and let $\pi_1(M)$ be its fundamental group, then there exists a pair of maps (Dev,hol) with the following properties: Dev: $\tilde{M} \longrightarrow X$ is an immersion and hol: $\pi_1(M) \longrightarrow G$ is a homomorphism of groups such that

$$\begin{array}{ccccc}
\tilde{M} & \xrightarrow{\text{Dev}} & X & \xrightarrow{g} & X \\
\Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle \text{hol}(\gamma)} & & \Big\downarrow{\scriptstyle \text{hol}(\gamma')} \\
\tilde{M} & \xrightarrow{\text{Dev}} & X & \xrightarrow{g} & X
\end{array}$$

commutes for every $g \in G$, $\gamma, \gamma' \in \pi_1(M)$. In particular the square on the right is saying that such a pair is unique up to inner automorphism of the structure. The maps Dev and hol are called the **developing map** and **holonomy representation** respectively. If the developing map is a homeomorphism, then the (G, X)-structure is complete. In general Dev is just injective. The image $\Gamma := \mathrm{hol}(\pi_1(M)) \subset G$ is the *holonomy group* of the (G, X)-structure. Since we are working with $G = \mathrm{Aff}(\mathbb{R}^n)$ we therefore refer to Γ as the **affine holonomy** of the structure and composing hol with the natural homomorphism $\mathrm{Aff}(\mathbb{R}^n) \xrightarrow{\mathrm{Lin}} \mathrm{GL}(n, \mathbb{R})$ we get the **linear holonomy**.

Remark 2.1.1. *The introduction of the developing map serves as a globalization for the (G, X)-structure whose definition is rather in term of local coordinates. Moreover the holonomy representation is the holonomy of a flat connection of a principal G-bundle associated to the structure.*

At this point we can adopt the point of view of [7],[8],[34],[35],[36],[38] and restrict ourselves to left-invariant affine structures on Lie groups. Take a simply-connected Lie group G: an affine structure on it is left-invariant if the left-multiplication map is an automorphism of the structure. Equivalently, left-multiplication map is affine in local charts. In this case, for every $g \in G$ there is a unique affine automorphism $\alpha(g)$ in $\mathrm{Aff}(\mathbb{R}^n)$ such that the diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Dev}} & \mathbb{R}^n \\ \downarrow{\scriptstyle L_g} & & \downarrow{\scriptstyle \alpha(g)} \\ G & \xrightarrow{\mathrm{Dev}} & \mathbb{R}^n \end{array}$$

The map $\alpha : G \longrightarrow \mathrm{Aff}(\mathbb{R}^n)$ is called the **affine representation** of the affine structure: its image $\alpha(G)$ preserves the connected open set $\mathrm{Dev}(G) \subset \mathbb{R}^n$ and acts transitively on it. Thus $\mathrm{Dev}(G)$ is an open orbit of $\alpha(G)$ since Dev is an open map. Moreover, since G is n-dimensional, the isotropy group is discrete and this implies the action is locally simply transitive. The construction can go backward: if we are given an affine representation $\alpha : G \longrightarrow \mathrm{Aff}(\mathbb{R}^n)$ with an open orbit $\mathcal{O} := \alpha(G)x_0$ for some $x_0 \in \mathbb{R}^n$ and dim G=n, then there is a unique left-invariant affine structure on G with developing map $\mathrm{Dev}(g) := \alpha(g) \cdot x_0$. Therefore there is a 1:1 correspondence between left-invariant affine structures on a n-dimensional simply-connected Lie group G and its locally simply-transitive affine actions on $\mathbb{R}^n$. If the open orbit $\mathcal{O}$ is the whole $\mathbb{R}^n$ then the above correspondence can be promoted between simply-transitive affine actions and complete left-invariant affine structures.

Since we are interested in compact affine manifold, assume that the Lie group G, with a left-invariant affine structure, admits a lattice, i.e. a cocompact discrete subgroup. Then the homogeneous space of right cosets $M := \Gamma/G$ inherits an affine structure. Assume, as it will be in all our cases, that $\alpha(\Gamma) \subset \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$, i.e. the restriction $\alpha|_\Gamma$ is an *integral affine representation.* In particular, its linear part is an element of $\mathrm{GL}(n,\mathbb{Z})$. This feature will be fundamental to build a bridge with symplectic geometry.

We now make the following observation that we will use later in section 3.2:

Remark 2.1.2. *Assume $\mathcal{O} = \mathbb{R}^n$. We have seen the affine representation is defined via*

$$\alpha(g) := Dev \circ L_g \circ Dev^{-1} : \mathbb{R}^n \to \mathbb{R}^n \tag{2.1}$$

and $\alpha(g)$ is an affine transformation of $\mathbb{R}^n$, that is is of the form $Av + b$ with $A \in GL(n,\mathbb{R})$ and $b \in \mathbb{R}^n$. Its linear part is then defined as $\lambda := Lin \circ \alpha$ where the map Lin simply sends the affine transformation $(A, b) \mapsto A$ to its linear part. This new linear transformation $\lambda(g)$ of $\mathbb{R}^n$ can be seen as the derivative of the affine transformation $\alpha(g)$:

$$\lambda(g) := d\alpha(g) = dDev \circ (dL_g) \circ dDev^{-1} \tag{2.2}$$

which is just the expression for the (dL_g) in the new coordinates.

This result of Auslander [8] allows us to make a further restriction about our analysis:

Theorem 2.1.1. [8] If G is a simply connected Lie group which has a representation ρ as a simply transitive group of affine motion, then G is solvable.

Therefore we will consider in our work only solvable Lie groups with a focus on the three-dimensional case. Of particular importance is the work of Fried and Goldman [34] in which they classify all possible simply transitive affine action of a solvable unimodular three-dimensional Lie group G, i.e. left-invariant complete affine structure on G. In the following theorem, such transitive actions are presented as subgroups of $\mathrm{Aff}(\mathbb{R}^3)$.

Theorem 2.1.2. *[34] Let G be a simply connected unimodular solvable Lie group acting simply transitively by affine transformations on $\mathbb{R}^3$. Then*

1. *If G is nilpotent then it is conjugate to one of*

   (a)

$$\mathcal{H}_\alpha := \left\{ \begin{pmatrix} 1 & a_{11}t + a_{12}u & a_{21}t + a_{22}u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} s + a_{11}t^2 + a_{22}u^2 + (a_{12} + a_{21})\frac{ut}{2} \\ t \\ u \end{pmatrix} \;\middle|\; s,t,u \in \mathbb{R} \right\}$$

   The conjugacy class of $\mathcal{H}_\alpha$ corresponds to conjugacy class of $\alpha = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ as a bilinear form on $\mathbb{R}^2$;

(b)

$$\mathcal{H}_{b,c} := \left\{ \begin{pmatrix} 1 & cu & bt + \frac{1}{2}cu^2 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} s + (b+c)\ \frac{tu}{2} + \frac{u^3}{6} \\ t + \frac{u^2}{2} \\ u \end{pmatrix} \;\middle|\; s, t, u \in \mathbb{R} \right\}$$

Where $\mathcal{H}_{b,c}$ and $\mathcal{H}_{ab,ac}$ are conjugate for $a > 0$.
In particular $\mathcal{H}_\alpha$ is abelian if and only if α is a symmetric bilinear form. $\mathcal{H}_{b,c}$ is abelian if and only if $b = c$. If the group is not abelian, then it is isomorphic to the Heisenberg group.

2. *If G is not nilpotent, then G is conjugate to one of*

(a)

$$I_\lambda := \left\{ \begin{pmatrix} 1 & \lambda e^s u & \lambda e^{-s} t \\ 0 & e^s & 0 \\ 0 & 0 & e^{-s} \end{pmatrix}, \begin{pmatrix} s + \lambda t u \\ t \\ u \end{pmatrix} \;\middle|\; s, t, u \in \mathbb{R} \right\}$$

(b)

$$D_\lambda := \left\{ \begin{pmatrix} 1 & \lambda(t\cos s - u \sin s)u & \lambda(t \sin s + u \cos s) \\ 0 & \cos s & -\sin s \\ 0 & \sin s & \cos s \end{pmatrix}, \begin{pmatrix} s + \lambda\frac{tu}{2} \\ t \\ u \end{pmatrix} \;\middle|\; s, t, u \in \mathbb{R} \right\}$$

In both cases, the conjugacy class depends only on whether λ is 0 or not.

Remark 2.1.3. In their article Fried and Goldman [34] put aside the groups D_λ since they cannot arise as *crystallographic hulls of affine crystallographic groups*. It will be also excluded by our analysis since it is not completely solvable. We will meet again the groups I_λ andD_λ under the notation of $E(1,1)$ and $E(2)$ respectively.

We end this section by recalling some result about the geometry of affine manifold.

Theorem 2.1.3 ([34])**.** *Let M^3 a closed 3-manifold. The following conditions are equivalent:*

1. *M admits a complete affine structure;*
2. *M is finitely covered by a 2-torus bundle over the circle;*
3. *$\pi_1(M)$ is solvable and M is aspherical*
4. *M has a Riemannian metric locally isometric to a left-invariant metric on a 3-dimensional solvable Lie group.*

Theorem 2.1.4. *[36] Let M be a compact affine manifold whose affine holonomy group is nilpotent. The the following are equivalent*

a. M is complete;

b. the map Dev is surjective;

c. the linear holonomy is unipotent;

d. the linear holonomy preserves volume;

e. the affine holonomy is irreducible;

f. the affine holonomy is indecomposable;

g. M is a complete affine nilmanifold;

h. M has a polynomial volume form;

i. M is orientable and the de Rham cohomology of M is the cohomology of the complex of polynomial exterior forms.

Remark 2.1.4. *Isomorphic Lie groups acting simply transitively by affine trasformations on $\mathbb{R}^n$ are conjugated by a polynomial automorphism of $\mathbb{R}^n$. By this we mean that if α, α' are the affine representation of two isomorphic Lie groups G, G' acting simply transitively on $\mathbb{R}^n$, then there exists a polynomial automorphism F of $\mathbb{R}^n$ such that $\alpha'(g) \cdot v = F(\alpha \cdot F^{-1}(v))$. Equivalently, if Dev,Dev' are the developing maps of α, α' respectively, Dev' $= F \circ$ Dev.*

2.2 Complex geometry

We leave, for the moment, the realm of affine geometry and review the basic notions and the fundamental results in complex geometry. We refer to [54],[69],[91].

Let M be a smooth manifold of even dimension $2n$.

Definition 2.2.1. An ***almost-complex structure*** on M is a vector bundle endomorphism J on TM such that $J^2 = -\mathrm{Id}_{TM}$.

The presence of J allows to decompose the complexified tangent bundle

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$$

where the summands are defined as the $\pm i$-eigenbundle w.r.t. $J_{\mathbb{C}}$, the natural $\mathbb{C}$-linear extension of J. The almost complex structure is said to be **integrable** if the subbundle $T^{0,1}M$ is an integrable distribution, i.e. $[T^{0,1}M, T^{0,1}M] \subseteq T^{0,1}M$.

By tensoriality, we obtain a decomposition on the complexified exterior bundles

$$\Lambda^k_{\mathbb{C}} M = \bigoplus_{k=p+q} \Lambda^{p,q} M = \bigoplus_{k=p+q} \bigwedge^p (T^{1,0}M)^* \otimes \bigwedge^q (T^{0,1}M)^*$$

and on their space of sections

$$\mathcal{A}^k(M, \mathbb{C}) = \bigoplus_{k=p+q} \mathcal{A}^{p,q}(M)$$

This also induces a decomposition for the complexified de Rham operator (we still denote with) d:

$$d = \mu + \partial + \bar{\partial} + \bar{\mu}$$

where μ is a differential operator of bidegree $(2, -1)$ coming from the *Nijenhuis tensor* associated to the almost complex structure J:

$$N_J(X, Y) := [X, Y] + J[X, JY] + J[JX, Y] - [JX, JY] \quad \text{for } X, Y \in TM$$

and

$$\mu + \bar{\mu} = -\frac{1}{4}\big(N_J \otimes \mathrm{Id}_{\mathbb{C}}\big)^*$$

Definition 2.2.2. A ***complex structure*** on M is the datum of an (equivalence class of an) atlas whose transition functions are holomorphic. A complex structure always induces an almost complex structure J while the vice versa holds if and only if the J is integrable (Newlander-Nirenberg theorem).

Proposition 2.2.1. The following are equivalent

1. M has a complex structure
2. $[T^{0,1}M, T^{0,1}M] \subseteq T^{0,1}M$, i.e. $T^{0,1}M$ is integrable distribution
3. $d(\mathcal{A}^{1,0}(M)) \subset \mathcal{A}^{2,0}(M) \oplus \mathcal{A}^{1,1}(M)$, i.e. $d = \partial + \bar{\partial}$
4. $N_J = 0$

The pair (M, J), with J (almost) complex structure, is called an ***(almost) complex manifold***. On a complex manifold, $d^2 = 0$ implies that $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. One can then define these complex analogues of the de Rham cohomology:

$$H^{p,q}_{\partial}(M) := \frac{\text{Ker } \partial : \mathcal{A}^{p,q}(M) \longrightarrow \mathcal{A}^{p+1,q}(M)}{\text{Im } \partial : \mathcal{A}^{p-1,q}(M) \longrightarrow \mathcal{A}^{p,q}(M)} \quad , \quad H^{p,q}_{\bar{\partial}}(M) := \frac{\text{Ker } \bar{\partial} : \mathcal{A}^{p,q}(M) \longrightarrow \mathcal{A}^{p,q+1}(M)}{\text{Im } \bar{\partial} : \mathcal{A}^{p,q-1}(M) \longrightarrow \mathcal{A}^{p,q}(M)}$$

They are naturally isomorphic under complex conjugation but it is more natural to work with the second one which is called the **Dolbeault Cohomology** of M. This choice is related to the fact that "holomorphicity" of functions is defined in terms of the operator $\frac{\partial}{\partial \bar{z}}$.

2.2.1 Kähler geometry and Hodge Theory

Let g be a metric on a complex manifold (M, J), i.e. the assignment, for each point $m \in M$, of a scalar product g_m on $T_m M$. It is said to be **compatible** with J if $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. We can then define an antisymmetric $(0, 2)$-tensor via $\omega(\cdot, \cdot) := g(J\cdot, \cdot)$, called the **fundamental form** of (M, J, g). Posing $h := g + i\omega$ we obtain an **Hermitian structure** and the triple (M, g, J), or simply (M, h), is called an **Hermitian manifold**. If additionally the 2-form ω is closed $d\omega = 0$ then it defines a **Kähler structure** and (M, g, J, ω) is called a **Kähler manifold**.

Remark 2.2.1. *Each of these definitions make sense also for an almost complex structure J. Knowing two of g, J, ω, with the appropriate compatibility relations, determines the third.*

On an almost Hermitian manifold we have some natural linear operators:

- The **Lefschetz operator**

  $$L : \mathcal{A}^k(M, \mathbb{C}) \longrightarrow \mathcal{A}^{k+2}(M, \mathbb{C}) \quad , \quad \alpha \longmapsto \omega \wedge \alpha$$

  In particular

  $$L : \mathcal{A}^{p,q}(M) \longrightarrow \mathcal{A}^{p+1,q+1}(M)$$

- The **Hodge star operator**

  $$* : \mathcal{A}^k(M, \mathbb{C}) \longrightarrow \mathcal{A}^{2n-k}(M, \mathbb{C})$$

  induced by the metric g via

  $$\alpha \wedge *\beta = g(\alpha, \beta) \operatorname{vol}_g$$

  In particular

  $$* : \mathcal{A}^{p,q}(M) \longrightarrow \mathcal{A}^{n-q,n-p}(M)$$

- The **dual Lefschetz operator**

  $$\Lambda := *^{-1} \circ L \circ * : \mathcal{A}^k(M, \mathbb{C}) \longrightarrow \mathcal{A}^{k-2}(M, \mathbb{C})$$

  In particular

  $$\Lambda : \mathcal{A}^{p,q}(M) \longrightarrow \mathcal{A}^{p-1,q-1}(M);$$

- The **$\mathcal{J}$-operator**

$$\mathcal{J} : \mathcal{A}^\bullet(M,\mathbb{C}) \longrightarrow \mathcal{A}^\bullet(M,\mathbb{C}) \quad , \quad \mathcal{J} = \sum_{p,q} i^{p-q}\Pi^{p,q}$$

  where $\Pi^{p,q} : \mathcal{A}^\bullet(M) \longrightarrow \mathcal{A}^{p,q}(M)$. This can be seen as the multiplicative extension of J to the whole exterior algebra $\mathcal{A}^\bullet(M)$

and differential operators:

- The adjoints of ∂ and $\bar\partial$

$$\partial^* := - * \circ \bar\partial \circ * \quad , \quad \bar\partial^* := - * \circ \partial \circ *$$

- The Laplacians

$$\Delta_\partial := \partial^*\partial + \partial\partial^* \quad , \quad \Delta_{\bar\partial} := \bar\partial^*\bar\partial + \bar\partial\bar\partial^*$$

- The d^c-operators

$$d^c := \mathcal{J}^{-1} \circ d \circ \mathcal{J} = -i(\partial - \bar\partial) \quad , \quad d^{c*} := - * \circ d^c \circ *$$

  In particular

$$dd^c = 2i\partial\bar\partial$$

Remark 2.2.2. *The definition of the Lefschetz operator relies only on the 2-form ω and it will have a fundamental role also in the symplectic case.*

The fundamental result in Kähler geometry is

Theorem 2.2.1. *(**Hodge's Theorem**) Let (M, g, J, ω) a compact hermitian manifold. Then*

- *There are orthogonal decompositions*

$$\mathcal{A}^{p,q}(M) = \partial\mathcal{A}^{p-1,q}(M) \oplus \mathcal{H}^{p,q}_\partial(M) \oplus \partial^*\mathcal{A}^{p+1,q}(M)$$
$$\mathcal{A}^{p,q}(M) = \bar\partial\mathcal{A}^{p,q-1}(M) \oplus \mathcal{H}^{p,q}_{\bar\partial}(M) \oplus \bar\partial^*\mathcal{A}^{p,q+1}(M)$$

- *The canonical projection $\mathcal{H}^{p,q}_{\bar\partial}(M) \longrightarrow H^{p,q}_{\bar\partial}(M)$ is an isomorphism.*

If additionally M is Kähler then

- *There is a decomposition*

$$H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(M)$$

Where the space of harmonic forms $\mathcal{H}^{\bullet,\bullet}_{-}(M)$ are defined in terms of the appropriate Laplacians.

Moreover, on a compact Kähler manifold there is a fundamental lemma whose absence characterizes non-Kähler geometries

Lemma 2.2.1 (**$\partial\bar{\partial}$-lemma**). *For a d-closed (p,q)-form on a compact Kähler manifold the following properties are equivalent*

$$d\text{-}exact \Longleftrightarrow \partial\text{-}exact \Longleftrightarrow \bar{\partial}\text{-}exact \Longleftrightarrow \partial\bar{\partial}\text{-}exact$$

and the following theorem by Lefschetz

Theorem 2.2.2 (**Strong Lefschetz Theorem**). *Let (M,g,J,ω) a compact Kähler manifold of (real) dimension $2n$. Then the maps*

$$L^{n-k} : H^k(M,\mathbb{R}) \longrightarrow H^{2n-k}(M,\mathbb{R})$$

are isomorphisms for $k \leq n$

One of the most beautiful consequences of this machinery, available in the Kähler realm, is the possibility to rearrange the information associated to the Dolbeault cohomology in a fancy way:

The numbers $h^{p,q} :=\dim H^{p,q}_{\bar{\partial}}(M)$ are in fact called **Hodge numbers**

$$\begin{array}{ccccccc}
 & & & h^{n,n} & & & \\
 & & h^{n,n-1} & & h^{n-1,n} & & \\
 & \iddots & & \vdots & & \ddots & \\
h^{n,0} & & \cdots & & \cdots & & h^{0,n} \\
 & \ddots & & \vdots & & \iddots & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array} \tag{2.3}$$

while their rearrangement is called the **Hodge diamond** of M. It has natural symmetries given by complex conjugation $h^{p,q} = h^{q,p}$ and by Serre duality $h^{p,q} = h^{n-p,n-q}$.

Calabi-Yau Geometry

Even if we are not going to work with Calabi-Yau manifolds it is worth to spend some words since they furnish the original motivation for the birth of mirror symmetry.

The are several ways to introduce Calabi-Yau structures

Definition 2.2.3. *A (compact) Kähler manifold* (M, g, J, ω) *is called a* ***Calabi-Yau manifold*** *if one of the following equivalent properties holds*

1. *The holonomy of the Kähler metric is contained in* $SU(n)$*.*
2. M *admits a nowhere-vanishing holomorphic* n*-form.*
3. *The canonical bundle* $K_M := \bigwedge^n (T^{1,0}M)^*$ *is holomorphically trivial.*

What makes Calabi-Yau manifolds so special for string theorists? As we mentioned in the introduction, they satisfy the equation imposed by a supersymmetric formulation of gravity. The Ricci-flatness condition is in fact related to the imposition of Einstein equations in vacuum (the internal manifolds M for the theory are also called *vacua*). In classical field theory there are two fundamental objects: a Hilbert space of states and a Hamiltonian function which governs the dynamics. Usually, on curved space-time, the Hilbert space is taken as the L^2-space of differential forms while the Hamiltonian is represented by the Riemannian Laplacian. Then symmetries of the theory are given by linear operators commuting with the Laplacian. When supersymmetry joins the picture one has to enlarge the (Lie) algebra of differential operators to make it closed under commutators. When the manifold is Kähler this is suitably obtained thanks to the *Kähler identities*. The equation coming from supergravity can be written as $\nabla\eta = 0$ for a six-dimensional spinor η. Since $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$, the equation implies that the holomomy reduces to $\mathrm{SU}(3)$. The equations for the spinor can be decoupled into this set of equations

$$d\omega = 0 \quad , \quad d\Omega = 0 \quad , \quad \omega \wedge \Omega = 0 \quad , \quad \frac{1}{8}\Omega \wedge \bar{\Omega} = i^3 \frac{\omega^3}{6} \tag{2.4}$$

for a real two-form ω and a complex three-form Ω. Later, string theorists let *flux compactification* enter the picture which resulted in the internal manifold being no more Kähler and extended the formulation in terms of $\mathrm{SU}(3)$-strucures with torsion [11],[12],[37],[64],[81]. We will see more in section 2.4.

2.2.2 Non-Kähler complex geometry

Relaxing the Kähler condition there is a pletora of different notion of Hermitian metrics and related geometries that are important by their own. We recall some of the most investigated ones:

Definition 2.2.4. *Let M be a hermitian manifold of complex dimension n and let ω its fundamental form. Depending on the equation involving ω we have different definitions:*

- *if $d\omega^{n-1} = 0$ it defines a* ***balanced metric***
- *if $\partial\bar{\partial}\omega = 0$ it defines a* ***strong Kähler with torsion (SKT) metric***
- *if $\partial\bar{\partial}\omega^{n-1} = 0$ it defines a* ***Gauduchon metric***

The study of balanced metrics started with the work of Michelsohn [68] in the context of special Hermitian metrics. They are also called *co-Kähler metrics* since $d\omega^{n-1}$ is equivalent to $d^*\omega = 0$. Instead SKT metrics, also known as *pluriclosed metrics*, were introduced by Bismut at the end of 80's [17]. If a metric is Gauduchon and $\partial\omega^{n-1}$ is $\bar{\partial}$-exact, then it is called *strongly Gauduchon* (sG).

Bott-Chern Cohomology

Let (M, J) be a complex manifold. Without the assumption of a Kähler structure we can not more make use of the Hodge theory. Nevertheless there are other, more general, cohomology theories which encode information about the complex geometry of the manifold.

Definition 2.2.5. *We define the* ***Bott-Chern*** *and* ***Aeppli cohomologies*** *respectively as*

$$H^{p,q}_{BC}(M) := \frac{\text{Ker } d : \mathcal{A}^{p,q}(M) \longrightarrow \mathcal{A}^{p+q+1}(M)}{\text{Im } \partial\bar{\partial} : \mathcal{A}^{p-1,q-1}(M) \longrightarrow \mathcal{A}^{p,q}(M)}$$

$$H^{p,q}_{A}(M) := \frac{\text{Ker } \partial\bar{\partial} : \mathcal{A}^{p,q}(M) \longrightarrow \mathcal{A}^{p+1,q+1}(M)}{\text{Im } \partial : \mathcal{A}^{p-1,q}(M) \longrightarrow \mathcal{A}^{p,q}(M) \oplus \text{Im } \bar{\partial} : \mathcal{A}^{p,q-1}(M) \longrightarrow \mathcal{A}^{p,q}(M)}$$

We recall some properties, [2]:

- Hodge-star operator induces isomorphism $H^{p,q}_{BC}(M) \simeq H^{n-q,n-p}_{A}(M)$
- Complex conjugation induces isomorphisms $H^{p,q}_{BC}(M) \simeq H^{q,p}_{BC}(M)$ and $H^{p,q}_{A}(M) \simeq H^{q,p}_{A}(M)$
- There are natural maps from $H^{p,q}_{BC}(M)$ and $H^{p,q}_{A}(M)$ into $H^{k}_{dR}(M, \mathbb{C})$ which are isomorphism precisely when the $\partial\bar{\partial}$-lemma holds.

- Assume M is also compact, there are natural maps induced by the identity

$$\begin{array}{ccccc} & & H^{p,q}_{BC}(M) & & \\ & \swarrow & \downarrow & \searrow & \\ H^{p,q}_{\partial}(M) & & H^{k}_{dR}(M,\mathbb{C}) & & H^{p,q}_{\bar\partial}(M) \\ & \searrow & \downarrow & \swarrow & \\ & & H^{p,q}_{A}(M) & & \end{array} \tag{2.5}$$

  for $k = p+q$. If the map $H^{p,q}_{BC}(M) \longrightarrow H^{p,q}_{A}(M)$ is injective, then any map in the diagram is an isomorphism and this happens precisely when M satisfies the $\partial\bar\partial$-lemma.

There are natural laplacians associated to both cohomologies:

$$\begin{aligned} \Delta_{BC} &:= (\partial\bar\partial)(\partial\bar\partial)^* + (\partial\bar\partial)^*(\partial\bar\partial) + (\bar\partial^*\partial)(\bar\partial^*\partial)^* + (\bar\partial^*\partial)^*(\bar\partial^*\partial) \\ \Delta_{A} &:= (\partial\bar\partial)^*(\partial\bar\partial) + (\partial\bar\partial)(\partial\bar\partial)^* + (\bar\partial\partial^*)^*(\bar\partial\partial^*) + (\bar\partial\partial^*)(\bar\partial\partial^*)^* \end{aligned} \tag{2.6}$$

but they are not elliptic. Nevertheless,

Theorem 2.2.3. *Let (M, h) be a compact hermitian manifold. Then by defining*

$$\tilde\Delta_{BC} := (\partial\bar\partial)(\partial\bar\partial)^* + (\partial\bar\partial)^*(\partial\bar\partial) + (\bar\partial^*\partial)(\bar\partial^*\partial)^* + (\bar\partial^*\partial)^*(\bar\partial^*\partial) + \bar\partial^*\bar\partial + \partial^*\partial$$

and

$$\tilde\Delta_{A} := \partial\partial^* + \bar\partial\bar\partial^* + (\partial\bar\partial)^*(\partial\bar\partial) + (\partial\bar\partial)(\partial\bar\partial)^* + (\bar\partial\partial^*)^*(\bar\partial\partial^*) + (\bar\partial\partial^*)(\bar\partial\partial^*)^*$$

They have the same principal symbol of Δ_{BC} and Δ_A respectively and by standard elliptic theory one obtains $dim H^{p,q}_{BC}(M) < \infty$ and $dim H^{p,q}_{A}(M) < \infty$.

By setting $h^{p,q}_{BC}$:=dim$H^{p,q}_{BC}(M)$ and $h^{p,q}_{A}$:=dim$H^{p,q}_{A}(M)$ we obtain the **Bott-Chern-Hodge** and **Aeppli-Hodge numbers** so we can have an analogus version of the Hodge diamond for complex non-Kähler manifolds. Clearly, all these notions coincide when the manifold is Kähler.

2.3 Symplectic Geometry

Symplectic geometry is the mathematical formalism underlying the Hamiltonian formulation of classical mechanics. Basically, the phase space of position-momentum configuration of the mechanical system is the prototype of what is called a symplectic manifold, namely a manifold where the change of coordinates are *canonical transformations* for the mechanical system. For the material here we refer to [66],[67],[80].

Linear symplectic spaces

Let V a vector space of dimension $2n$ and let ω be a 2-covector on V. If the linear map $\omega^\flat : V \longrightarrow V^*$ defined by $\omega^\flat(v) = \iota_v\omega$ is invertible, then ω is said **non-degenerate**. A non-degenerate 2-covector is called a **symplectic form**. The pair (V, ω) is then called a *symplectic linear space.*

Unlike in the Riemannian geometry, where the non-degeneracy of the scalar product gives the notion of orthogonal complement, in the symplectic case there are various notions of "orthogonality".

Let (V, ω) a symplectic linear space and let $U \subseteq V$ a subspace. Then the **symplectic complement** U^ω of U is defined as

$$U^\omega = \{v \in V \mid \omega(u, v) = 0 \text{ for all } u \in U\}$$

We can then characterize subspaces in symplectic vector spaces as follows:

- U is **symplectic** if $U \cap U^\omega = \{0\}$;
- U is **isotropic** if $U \subseteq U^\omega$;
- U is **coisotropic** if $U \supseteq U^\omega$;
- U is **Lagrangian** if $U = U^\omega$;

In particular Lagrangian subspaces are the maximal (co)isotropic subspaces since they have $\dim U = \frac{1}{2}\dim V$. By a skew-symmetric version of Gram-Schmidt process, there exists a basis $e_1, \dots, e_n, f_1 \dots, f_n$ of V such that $\omega(e_i, f_j) = \delta_{ij}$, $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ and it is called a **symplectic basis**. Moreover in term of the dual basis

$$\omega = e_1^* \wedge f_1^* + \dots + e_n^* \wedge f_n^*$$

and the matrix associated to ω has expression

$$S_0 = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$$

Compatible linear complex structures

A (linear) complex structure on V is an endomorphism J such that $J^2 = -\mathrm{Id}$. On a symplectic vector space (V, ω), J is said to be ω**-compatible** if $\omega(Ju, Jv) = \omega(u, v)$ and $\omega(u, Ju) > 0$ for all $u \neq 0$. In particular $g_J(u, v) := \omega(u, Jv)$ is a well-defined inner product on V.

Remark 2.3.1. *Once we are given an inner product g on a linear symplectic space (V, ω) we can produce a canonical compatible complex structure J. In general the metric g_J will be different from g. In fact we can define a skew-symmetric endomorphism $A : V \longrightarrow V$ via the identity*

$$\omega(u,v) = g(Au,v)$$

*In particular AA^t is symmetric and positive, i.e. $g(AA^tu,u) = g(A^tu,A^tu) > 0$ for all $u \neq 0$. Therefore, by the spectral theorem, the operator $\sqrt{AA^t}$ is well-defined and commutes with A. By setting $J := (\sqrt{AA^t})^{-1}A$ we get the desired compatible linear structure. The factorization $A = \sqrt{AA^t}J$ is called **polar decomposition** of A.*

Let J be a compatible complex structure on (V,ω). If L is a Lagrangian subspace of (V,ω), then also JL is Lagrangian and $JL = L^\perp$ with respect to g_J. Another consequence of compatibility is that one can take the f_i in the symplectic basis as $f_i = Je_i$.

2.3.1 Symplectic manifolds

Let M be a smooth manifold of dimension $2n$.

Definition 2.3.1. *A **symplectic structure** on M is a non-degenerate differential 2-form ω which is closed. Therefore $\omega^n \neq 0$ and $d\omega = 0$. The pair (M,ω) is called a **symplectic manifold**.*

Given a symplectic manifold (M,ω), a submanifold $N \subseteq M$ is said to be

- **symplectic** if T_mN is a symplectic subspace of T_mM $m \in N$.
- **isotropic** if T_mN is a isotropic subspace of T_mM $m \in N$.
- **coisotropic** if T_mN is a coisotropic subspace of T_mM $m \in N$.
- **Lagrangian** if T_mN is a Lagrangian subspace of T_mM for all $m \in N$.

Another fundamental difference between symplectic structures and Riemannian metrics is that there is no local obstruction to a symplectic structure being locally equivalent to the standard linear model

Theorem 2.3.1 (Darboux)**.** *Let (M,ω) be a $2n$-dimensional symplectic manifold. For any $m \in M$, there are smooth coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_n)$, centered at m, in which ω has coordinate representation*

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i.$$

*These coordinates are called **Darboux coordinates**.*

Cotangent bundles

The prototypical examples of symplectic manifolds are provided by cotangent bundles of smooth manifolds. Define on the total space of $\pi : T^*M \longrightarrow M$ the **tautological 1-form** as

$$\tau_{(m,\alpha)} = d\pi^*_{(m,\alpha)}\alpha$$

where we denoted with (m, α) a point in T^*M. If $x_1, \ldots, x_n$ are local coordinates around m and $y_1, \ldots, y_n$ are the coordinate expression for $\alpha = y_i dx^i$, then

$$\tau = y_i dx^i$$

and clearly $\omega_{\text{can}} := -d\tau = dx^i \wedge dy^i$ defines a symplectic structure on T^*M. In general if σ is a section for π, that is σ is a smooth 1-form, σ is closed as differential forms if and only if $\sigma(M)$ is a Lagrangian submanifold of $(T^*M, \omega_{\text{can}})$.

2.3.2 Lagrangian Fibrations

A coisotropic submanifold $N \subseteq (M, \omega)$ is such that $TN^\omega \subseteq TN$ and can be characterized in the following manner:

$$T_nN = \{v \in T_nM \; | v(F) = 0 \text{ for all } F \in C^\infty(M)_N\}$$
$$\text{Ann}(T_nN) = \{\alpha \in T_n^*M \; | \alpha = dF|_n \text{ for some } F \in C^\infty(M)_N\}$$

where $C^\infty(M)_N := \{F \in C^\infty(M) \mid F|_N = 0\}$. In particular the map $\omega^\flat : T_nM \longrightarrow T_n^*M$ allow us to identify $\text{Ann}(TN)$ with $TN^\omega \subseteq TM|_N$ and dF with X_F. We have

Lemma 2.3.1. *The following are equivalent:*

1. *For all $F \in C^\infty(M)_N$, X_F is tangent to N;*

2. *$C^\infty(M)_N$ is a Poisson subalgebra of $C^\infty(M)$.*

3. *N is a coisotropic submanifold of M.*

Lemma 2.3.2. *Suppose $F : (M, \omega) \longrightarrow \mathbb{R}^k$ is a submersion and that the components of $F = (F_1, \ldots, F_k)$ Poisson commute, that is $\{F_i, F_j\} = 0$. Then the fibers of F are coisotropic submanifolds of M of codimension k.*

Let us now specialize in the case maximally coisotropic case $k = n$.

Definition 2.3.2. *Let (M, ω) be a symplectic manifold. A **Lagrangian fibration** is a fibration $\pi : (M, \omega) \longrightarrow B$ such that every fiber is a Lagrangian submanifold of M, in particular $dimB = n$.*

We will see that when M and B are compact the fibers must be tori. In general they are of the form $\mathbb{T}^k \times \mathbb{R}^{n-k}$.

Affine torus bundles

Let G be a Lie group and X be a *principal homogeneous G-space*, that is X is equipped with a free, transitive action of G. In particular dim$G$=dim$X$ and we can identify $X = G.x \simeq G$ for any choice of a $x \in X$, up to a translation in G. If G is a torus we say X is an *affine torus*, if G is a vector space instead we say X is an *affine vector space*. Let V be a vector space acting transitively on X and that dim V = dim $X = n$. The stabilizer V_x is a discrete subgroup of V, not depending on x. Therefore we can see X as a principal homogeneous $H := V/V_x$-space. Since H is compact, connected and abelian it is of the form of a vector space times a torus. In particular the space X has a product structure of an affine torus times an affine vector space.
Extend now this construction to fiber bundles: we want to define an action of a *group bundle* to a given fiber bundle $E \to B$. By a *group bundle* we mean a fiber bundle $\mathcal{G} \to B$ with fibers carrying a group structure and bundle charts being fiberwise group isomorphism with a given group G. Then we can define smooth maps

$$\mathcal{G} \times_B E \longrightarrow E$$

that are fiberwise group actions. We are interested in the case when the model group is a torus. Then

Definition 2.3.3. *We say a fibration $\pi : M \longrightarrow B$ is an **affine torus bundle** if it is equipped with a fiberwise, free, transitive action of a torus bundle $\mathcal{T} \longrightarrow B$.*

When the fibration $\pi : M \to B$ has compact fibers we can mimic the construction above for vector spaces and obtain easily such a torus bundle action. In fact, suppose we are given a vector bundle $E \to B$, dim$E$ =dim$M$, with a transitive, fiberwise action of E on M. Then we can construct the stabilizer bundle $\Lambda \to B$ for the fiber bundle action and define the torus bundle as the quotient bundle $\mathcal{T} := E/\Lambda \to B$. In the context of symplectic geometry the vector bundle E will be represented by the cotangent bundle of the base T^*B while the torus-action is related to theory of *action-angle coordinates* (Arnol'd-Liouville Theorem).

Remark 2.3.2. *The presence of any global section $\sigma : B \to M$ would identify the two fiber bundles M and $\mathcal{T}$. We will construct our examples in a way that a global section always exists. Moreover, if $\mathcal{T} \to B$ is trivial, then $M \to B$ is a principal torus bundle. The bundle $\Lambda \to B$ is therefore related to the monodromy of the fibrations. In our treatment it will play a prominent role and the absence of triviality will give rise to a rich geometric interpretation.*

In the following we will resume the construction by Duistermaat [28] following [67]

Theorem 2.3.2. *Let $(M, \omega) \stackrel{\pi}{\to} B$ be a Lagrangian fibration with compact, connected fibers. Then there is a canonical, fiberwise transitive vector bundle action $T^*B \times$*

$M \to M$. Thus every Lagrangian fibration has canonically the structure of an affine torus bundle.

Proof. For each 1-form on the base $\alpha \in \mathcal{A}^1(B)$ we can define a vertical vector field X_α in the following way

$$\iota_{X_\alpha}\omega = -\pi^*\alpha$$

Let $VM := \ker d\pi$ the vertical bundle of M relative to π and let $\mathfrak{X}_V(M) := \Gamma(VM)$. For any vector field $Y \in \mathfrak{X}_V(M)$ we have

$$\omega(X_\alpha, Y) = \iota_Y \iota_{X_\alpha}\omega = -\iota_Y \pi^*\alpha = 0$$

since α is a basic 1-form. Since VM is a Lagrangian subbundle, this implies that $X_\alpha \in \mathfrak{X}_V(M)$. Such construction extends to an isomorphism of vector bundles

$$VM \simeq \pi^* T^* B$$

We now exploit this map to define an action of T^*B on M as follows: let $\Phi_\alpha : M \to M$ the time-one flow associated to X_α. Since X_α is vertical, so is the flow i.e. it preserves the fibers of π. Define the fiber bundle map

$$T^*B \times_B M \longrightarrow M \quad , \quad (\alpha_b, m) \longmapsto \Phi_\alpha(m)$$

.

To check this is indeed a vector bundle action we just need the flows commuting for each α. Take then $\alpha_1, \alpha_2 \in \mathcal{A}^1(B)$. Let $X_{\alpha_1}, X_{\alpha_2}$ the associated vertical vector fields.

$$\begin{aligned}
\iota_{[X_{\alpha_1}, X_{\alpha_2}]}\omega &= (\mathcal{L}_{X_{\alpha_1}}\iota_{X_{\alpha_2}} - \iota_{X_{\alpha_2}}\mathcal{L}_{X_{\alpha_1}})\omega \\
&= -\mathcal{L}_{X_{\alpha_1}}\pi^*\alpha_2 - \iota_{X_{\alpha_2}} d\iota_{X_{\alpha_1}}\omega \\
&= -\mathcal{L}_{X_{\alpha_1}}\pi^*\alpha_2 + \iota_{X_{\alpha_2}}\pi^* d\alpha_1 \\
&= 0
\end{aligned} \tag{2.7}$$

Non-degeneracy of ω implies $[X_{\alpha_1}, X_{\alpha_2}] = 0$. Moreover since each map $T^*_{\pi(m)}B \to V_m$ is an isomorphism, the action is fiberwise transitive.

□

Let Λ be the bundle of stabilizers for this T^*B-action

$$\Lambda_b = \{\alpha \in T^*_b B \mid \Phi_\alpha(m) = m\}$$

and since the group (which is pointwise just the vector space $T^*_b B$) has the same dimension of the orbit, Λ_b must be a discrete subgroup. In particular it is a lattice by compactness assumption. Therefore Λ is also called *period lattice (bundle)*. We set

$$\mathcal{T} := T^*B/\Lambda \longrightarrow B$$

Moreover the canonical symplectic form on T^*B is preserved by Λ, which becomes a Lagrangian submanifold, and descends to $\mathcal{T}$ turning the $\mathcal{T} \to B$ into a Lagrangian fibration.

The covering group of the covering bundle $\Lambda \to B$ is a homomorphism

$$\rho_b : \pi_1(B, b) \longrightarrow \mathrm{Aut}(\Lambda_b) \simeq \mathrm{Aut}(H_1(F_b), \mathbb{Z}) \simeq GL(n, \mathbb{Z})$$

called the **monodromy** of the bundle $\pi : M \longrightarrow B$. Here F_b denotes the fiber over b.

Remark 2.3.3. *Up to taking the inverse transpose of ρ we note that the monodromy is nothing else that the holonomy of the flat connection associated to the affine structure on the base.*

Action-angles coordinates

Here we review the theory of action-angle coordinates as described in [28],[48],[67]. Take a point $b \in B$ and a basis $\beta_1(b), \ldots, \beta_n(b)$ of $\Lambda_b = \Lambda \cap T^*_b B$. Then there are unique differential forms, say β_j, defined in some neighborhood of b such that form a local basis for Λ. Since by the preceding observation Λ is a Lagrangian submanifold, the β_j define a Lagrangian section of T^*B. In particular $d\beta_j = 0$ so they are locally exact:

$$\beta_j = 2\pi dr_j$$

where $r_1, \ldots, r_n$ are functions on B. The fact the β_i are linear independent implies the r_j are local coordinates around b. Let $\theta_1, \ldots, \theta_n$ be the corresponding dual variables on T^*B near $\pi^{-1}(B)$. The lattice subbundle is described by $\theta_j \in 2\pi\mathbb{Z}$, $j = 1, \ldots, n$.

Thus $(r_1, \ldots, r_n, \theta_1, \ldots, \theta_n)$ form a system of local coordinates on M known as **action-angle coordinates** and the symplectic form on M is locally given as

$$\omega = d\tau$$

and $\tau = \sum_i^n r_i d\theta_i$ is a well-defined 1-form on M. Note that

$$r_i(c) = \frac{1}{2\pi} \int_{\gamma_i(c)} \tau$$

where $\gamma_i(c)$ is the curve in the fiber above c given by $0 \leq \theta_i \leq 2\pi$, $\theta_j = 0, j \neq i$

Theorem 2.3.3. *[48] Under the hypothesis of Theorem 2.3.2, local action angle coordinates exist. If τ is a 1-form on M such that $\omega = d\tau$ and if $\gamma_i(c)$ are smoothly varying curves in the fiber above c whose homotopy classes $[\gamma_i(c)], i = 1, \ldots, n$ form* *a basis for the fundamental group of the fiber over c for each c, then the functions r_i give action variables, whose dual variables give the angle variables.*

Assume for simplicity the symplectic form $\omega = d\tau$ is exact. Consider the local system $\xi \to B$ whose fiber over b is the abelian group $H_1(\pi^{-1}(b), \mathbb{Z}) \simeq \mathbb{Z}^n$. Let $q : \tilde{B} \to B$ be the universal cover and let $\tilde{\xi} = q^*\xi$. Since $\tilde{B}$ is simply-connected, $\tilde{\xi}$ is trivial. Let $c_1, \dots, c_n$ be a $\mathbb{Z}$-basis of continuous sections of $\tilde{\xi} \to B$.

Definition 2.3.4. *The **flux map** is defined to be the map $\mathcal{R} : \tilde{B} \to \mathbb{R}^n$ given by*

$$\mathcal{R}(\tilde{b}) = (r_1(\tilde{b}), \dots, r_n(\tilde{b})) := \left(\frac{1}{2\pi} \int_{c_1(\tilde{b})} \tau, \dots, \frac{1}{2\pi} \int_{c_n(\tilde{b})} \tau \right). \tag{2.8}$$

Lemma 2.3.3. *Suppose $\tilde{U} \subseteq \tilde{B}$ and $U \subseteq B$ are open subsets such that $q|_{\tilde{U}} : \tilde{U} \to U$ is a diffeomorphism. Then $\mathcal{R} \circ (q|_{\tilde{U}}) : U \to \mathbb{R}^n$ gives action coordinates on U.*

For a generic, not necessary exact, symplectic form ω the definition of action coordinates has to be modified. Denote with $\tilde{\pi} : q^*M \to \tilde{B}$ the pullback of the universal cover. Fix a basepoint $\tilde{b}_0 \in \tilde{B}$. Given a point $\tilde{b} \in \tilde{B}$, pick a path $\gamma : [0,1] \to \tilde{B}$ from $\tilde{b}_0$ to $\tilde{b}$. A *family of loops* over γ is a homotopy $C : S^1 \times [0,1] \to q^*M$ satisfying $\tilde{\pi}(C(s,t)) = \gamma(t)$, i.e. if t is fixed, $C(s,t)$ is a loop in $\tilde{\pi}^{-1}(\gamma(t))$. For $k = 1, \dots, n$ pick a family of loops C_k over γ with $C_k(\cdot, t) \in c_k(\gamma(t))$ for all $t \in [0,1]$. Define

$$\mathcal{R}(\tilde{b}) = (r_1(\tilde{b}), \dots, r_n(\tilde{b})) \quad , \quad r_k(\tilde{b}) = \int_{C_k} \omega \tag{2.9}$$

This definition does not depend on :

- the basis $c_1, \dots, c_n$ of $q^*\xi$
- the basepoint $\tilde{b}_0$
- the path γ
- the family of loops C_k over γ.

Instead, if we change the basepoint, the resulting flux map differs just by a translation. In particular, if we change the basis of sections $c_1, \dots, c_n$ by an element of $GL(n, \mathbb{Z})$ then the result is to apply a $\mathbb{Z}$-linear transformation to the flux map. Moreover the integral affine structure on $\tilde{B}$ descends to B. This observation brings us back to the realm of (integral) affine geometry and we can note that this flux map, or just the action coordinates, are nothing else that the developing map for the affine structure induced on the base. This establish the fundamental connection between affine geometry and symplectic geometry.

This story can also be interpreted in terms of the cohomology of a sheaf as we can see from Duistermaat [28]. The choice of angle coordinates is determined by the choice of a Lagrangian section $\lambda_i : U_i \to M$ and on the intersections $U_i \cap U_j$ one has

$$\lambda_i(b) = \mu_{ij}(b)(\lambda_i(b))$$

for a uniquely determined Lagrangian section $\mu_{ij} : U_i \cap U_j \to \mathcal{T}$. The μ_{ij} constitute a cocycle which modulo coboundaries determines the bundle $\pi : M \to B$ with symplectic structure and Lagrangian fibers.

Therefore, let $\mathcal{L}(\mathcal{T})$ be the sheaf of germs of Lagrangian sections $B \to \mathcal{T}$, then the structure of M is determined by the base B, the Lagrangian covering $\Lambda \subset T^*B$ and the cohomology class $[\mu] \in H^1(B, \mathcal{L}(\mathcal{T}))$. We have a short exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \mathcal{L}(T^*B) \longrightarrow \mathcal{L}(\mathcal{T}) \longrightarrow 0$$

which induces a long exact sequence

$$0 \to H^0(B,\Lambda) \to H^0(B,\mathcal{L}(T^*B)) \to H^0(B,\mathcal{L}(\mathcal{T})) \xrightarrow{\delta} H^1(B,\Lambda) \to$$
$$\to H^1(B,\mathcal{L}(T^*B)) \to H^1(B,\mathcal{L}(\mathcal{T})) \xrightarrow{\delta} H^2(B,\Lambda) \to \cdots$$

.

Then the class

$$\nu = \delta[\mu] \in H^2(B,\Lambda)$$

is called the **Chern class** of the fibration $\pi : M \to B$. If the covering bundle $\Lambda \to B$ is trivial, then $\Lambda \simeq \mathbb{Z}^n$ and $H^2(B,\Lambda) \simeq (H^2(B,\mathbb{Z}))^n$.

We note moreover that

$$0 \to H^i(B, C^\infty(\mathcal{T})) \xrightarrow{\delta} H^{i+1}(B,\Lambda) \to 0 \quad \text{for} \quad i \geq 1$$

In particular $H^1(B, C^\infty(\mathcal{T})) \xrightarrow{\delta} H^2(B,\Lambda)$ is an isomorphism, which means that the structure of $\pi : M \to B$ as smooth bundle is governed by the Chern class. We end the section with the following result by Duistermaat concerning the topological relationship between the fibrations:

Theorem 2.3.4. *The following are equivalent*

1. *$M \simeq \mathcal{T}$ as smooth bundles*
2. *There exists a global section $\sigma : B \to M$ for $\pi : M \to B$*
3. *The Chern class $\delta[\mu] \in H^2(B,\Lambda)$ is trivial*

In particular if (2) *hold also the following are equivalent*

- *$M \simeq \mathcal{T}$ as a symplectic manifold fibered over B with Lagrangian fibers;*

- *$M \to B$ admits a global Lagrangian section $\sigma : B \to M$;*
- *The Chern class vansishes and for any section $\sigma : B \to M$ the 2-form $\sigma^*\omega$ is exact on B.*

Theorem 2.3.5 (Global action-angle coordinates)**.** *The fibration $\pi : M \to B$ is topologically trivial if and only if the monodromy and the Chern class are trivial. Moreover the following are equivalent*

1. *There is a smooth map $(\mathcal{R}, \Theta) : M \to \mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n$ such that*
   - $\omega = \sum_{i=1}^n d\theta_i \wedge dr_i$
   - *The r_i are constant along the fibers of π*
   - *θ is injective on each fiber of π*
2. *The fibration $\pi : M \longrightarrow B$ is topologically trivial and ω is exact*

2.3.3 Tseng-Yau cohomology

In a series of paper [84],[85],[86],[87] ,Tseng and Yau introduced a new cohomology theory suited for symplectic manifolds. Pursuing an idea already present in the work of Brylinski [19] they developed in full generality a symplectic analogue of Hodge theory. In this section we will recall the basic facts about it. We have already met the **Lefschetz operator** $L = \omega \wedge \cdot$, its dual Λ and the Hodge star operator $*$. In the same spirit of Kähler identities, it is reasonable to consider the commutator $d^\Lambda := [d, \Lambda] = d\Lambda - \Lambda d$. It is a differential operator of degree -1:

$$d^\Lambda : \mathcal{A}^k(M) \longrightarrow \mathcal{A}^{k-1}(M)$$

In the same way as in the Riemannian setting, we can define a **symplectic Hodge-star operator** $*_s$ using the *symplectic volume form* $\frac{\omega^n}{n!}$ (also called *Liouville volume form*) via the identity

$$\alpha \wedge *_s\beta = \omega^{-1}(\alpha, \beta)\frac{\omega^n}{n!}$$

In particular we can rewrite the operator d^Λ as the *symplectic adjoint* of the de Rham differential d:

$$d^\Lambda = (-1)^k *_s d *_s$$

and also

$$\Lambda = *_s L *_s$$

One has $(d^\Lambda)^2 = 0$ and $dd^\Lambda + d^\Lambda d = 0$. Also the operator dd^Λ has an important role and it is of degree 0. We recall the definition of the *counting operator* H as

$$H := \sum_k (n-k)\Pi^k$$

where Π^k is the projector onto forms of degree k. Then, there are the following commutators relations for d, d^Λ and dd^Λ with $\mathfrak{sl}_2$-operators L, Λ, H.

Lemma 2.3.4.

$$\begin{array}{ccc} [d,L]=0 \quad , & [d,\Lambda]=d^\Lambda \quad , & [d,H]=d \\ [d^\Lambda,L]=d \quad , & [d^\Lambda,\Lambda]=0 \quad , & [d^\Lambda,H]=-d^\Lambda \\ [dd^\Lambda,L]=0 \quad , & [dd^\Lambda,\Lambda]=0 \quad , & [dd^\Lambda,H]=0 \end{array}$$

.

Then Tseng and Yau found that the symplectic analogue of Bott-Chern and Aeppli cohomologies are

$$H^k_{d+d^\Lambda}(M) := \frac{\operatorname{Ker}\{d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M,\} \cap \operatorname{Ker}\{d^\Lambda : \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)\}}{\operatorname{Im}\{dd^\Lambda : \mathcal{A}^k(M) \longrightarrow \mathcal{A}^k(M)\}}$$

$$H^k_{dd^\Lambda}(M) := \frac{\operatorname{Ker}\{dd^\Lambda : \mathcal{A}^k(M) \longrightarrow \mathcal{A}^k(M)\}}{\operatorname{Im}\{d : \mathcal{A}^{k-1}(M) \to \mathcal{A}^k(M)\} \oplus \operatorname{Im}\{d^\Lambda : \mathcal{A}^{k+1}(M) \to \mathcal{A}^k(M)\}}$$

For each operator $d^\Lambda, d + d^\Lambda, dd^\Lambda$, they developed the Hodge theory associated to each Laplacian. In particular the cohomology groups are finite-dimensional and there is a pairing, as for Bott-Chern and Aeppli, such that $H^k_{d+d^\Lambda}(M) \simeq H^{2n-k}_{dd^\Lambda}(M)$. Moreover there is a symplectic version of the $\partial\bar{\partial}$-lemma:

Lemma 2.3.5 (**dd^Λ-lemma/Definition**)**.** *Let α be a d-closed and d^Λ-closed differential form. We say that the dd^Λ-lemma holds if the following properties are equivalent:*

(i) α is d-exact;

(ii) α is d^Λ-exact;

(iii) α is dd^Λ-exact.

so that

Proposition 2.3.1 ([85])**.** *On a compact symplectic manifold (M, ω), the dd^Λ-lemma holds, or equivalently the strong Lefschetz property is satisfied, if and only if the canonical homomorphism $H^k_{d+d^\Lambda}(M) \to H^k_{dR}(M)$ is an isomorphism for all k.*

From now on we will refer to $H^\bullet_{d+d^\Lambda}(M)$ as the **Tseng-Yau cohomology** of M and it will represent the symplectic cohomology involved in the formulation of non-Kähler mirror symmetry.

2.4 SU(n)-geometry

We take as definition of SU(n)-structure the one used in [61].

Definition 2.4.1. *Let M a real $2n$-dimensional smooth manifold. A **SU(n)-structure** on M is the datum of a couple of differential forms (ω, Ω) satisfying the following properties*

- *Ω is a nowhere-vanishing decomposable complex n-form such that by defining*

$$T^{0,1}M := \{v \in TM \otimes \mathbb{C} | \iota_v \Omega = 0\} \tag{2.10}$$

  and letting $T^{1,0}M$ its complex conjugate, one has a splitting

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M \tag{2.11}$$

  which induces an almost-complex structure J on M. Then Ω is a type $(n, 0)$ w.r.t. this J.

- *ω is a non-degenerate real $(1, 1)$-form w.r.t J and it is such that $\omega(\cdot, J\cdot)$ is an Hermitian metric.*

From both properties one deduces that

$$\begin{aligned} &\omega \wedge \Omega = 0 \\ &\Omega \wedge \bar{\Omega} = i^n \cdot F \cdot \frac{\omega^n}{n!} \end{aligned} \tag{2.12}$$

for some nowhere-vanishing function F on M which is called the *conformal factor of the SU(n)-structure.*

Remark 2.4.1. *If both forms are closed, $d\omega = d\Omega = 0$, then the J is integrable and the ω is symplectic turning M into a Calabi-Yau manifold*

This is equivalent to the common one used in [15],[24],[33]:

Definition 2.4.2. *Let M be a $2n$-dimensional smooth manifold and let $\mathcal{L}(M)$ its $GL(2n, \mathbb{R})$-principal bundle of linear frames. A **SU(n)-structure** on M is a SU(n)-reduction of $\mathcal{L}(M)$.*

Therefore a SU(n)-structure on M is determined by the choice of the following data

- an almost complex structure J;
- a J-Hermitian metric g;

- a complex $(n,0)$-form Ω of constant norm.

It is important to cite here also the characterization for SU(3)-structures done by Hitchin [52],[53]

We recall that a *stable* 3-form, in the sense of Hitchin, is a three-form such that $\lambda(\psi_+(p)) < 0$ for each $p \in M$. The map λ is pointwise defined as follow [53]: set $V := T_pM$ and fix a volume form η. Consider the canonical isomorphism $A : \Lambda^5(V^*) \to V \otimes \Lambda^6(V^*)$ defined via $A(\xi) = v \otimes \eta$, where $\iota_v \eta = \xi$. Then define for a fixed $\rho \in \Lambda^3(V^*)$ the maps

$$K_\rho : V \to V \otimes \Lambda^6(V^*) \quad , \quad K_\rho(v) = A(\iota_v \rho \wedge \rho)$$

and

$$\lambda : \Lambda^3(V^*) \to (\Lambda^6(V^*))^{\otimes 2} \quad , \quad \lambda(\rho) := \frac{1}{6}\mathrm{tr} K_\rho^2$$

Then, if $\lambda(\rho) \neq 0$, the form $\sqrt{|\lambda(\rho)|} \in \Lambda^6(V^*)$ defines a volume form by choosing the orientation of V for which ω^3 is positively oriented. Moreover if $\lambda(\rho) < 0$, ρ defines an almost complex structure $J = J_\rho$ via $J_\rho := -\frac{1}{\sqrt{-\lambda(\rho)}} K_\rho$. In our case $\rho = \psi_+$ and by setting $\psi_- = J\psi_+$ one can define a complex-volume form $\Omega = \psi_+ + i\psi_-$.

So an $SU(3)$-structure is the datum of

- an almost symplectic structure ω
- a *stable* three-form ψ_+

such that $\omega \wedge \psi^+ = 0$ and $\omega(\cdot, J_{\psi^+} \cdot)$ defines a positive definite Hermitian form. Moreover $\Omega \wedge \bar{\Omega} = c\frac{\omega^3}{3!}$ for a constant c.

2.4.1 Supersymmetric systems of type IIA/IIB in dimension three

We now focus on the three-dimensional case and differentiate the structure into two models: let M be a smooth six-dimensional real manifold admitting an SU(3)-structure defined by a couple (ω, Ω) in the sense of definition 2.4.1. Then the system

$$\mathrm{IIA} : \begin{cases} \mathrm{d}\omega = 0 \\ \mathrm{d}\ \mathrm{Re}\ \Omega = 0 \\ \Omega \wedge \bar{\Omega} = -i \cdot F \cdot \frac{\omega^3}{6} \\ \mathrm{dd}^\Lambda (F \cdot \mathrm{Im}\ \Omega) = \rho_A \end{cases} \tag{2.13}$$

defines a **symplectic half-flat geometry** on M and we refer to the triple (M, ω, Ω) as **supersymmetric SU(3)-structure of type IIA**

Remark 2.4.2. *We recall that an SU(3)-structure (ω, Ω) such that $d(\omega \wedge \omega) = 0$ and $dRe\Omega = 0$ is called* ***half-flat****.*

We will deal only with examples of symplectic half-flat structures coming from solvmanifolds. For other interesting examples, also non compact, one can look in [73],[74],[93].

Instead, the system

$$\text{IIB}: \begin{cases} \mathrm{d}\omega^2 = 0 \\ \mathrm{d}\Omega = 0 \\ \Omega \wedge \bar{\Omega} = -i \cdot F \cdot \frac{\omega^3}{6} \\ 2i\partial\bar{\partial}(F^{-1} \cdot \omega) = \rho_B \end{cases} \tag{2.14}$$

defines a **complex balanced geometry** on M and we refer to the triple (M, ω, Ω) as **supersymmetric SU(3)-structure of type IIB**. From now on we will use "check" superscripts to denote the components of a IIB-system.

Remark 2.4.3. The last equations in both systems has to be taken as definitions for the flux forms ρ_A and ρ_B. Their presence is related to the presence of torsion. We will not deal with them in particular but we remark their importance in the context of geometric PDEs associated to them (see for example the recent survey by D. H. Phong [72]) .

Remark 2.4.4. The definition of IIA/IIB systems can be extended to an arbitrary dimension. The definition for a supersymmetric SU(n)-structure of type IIB is straightforward. Instead, for the type IIA system one has to take

$$\text{IIA}: \begin{cases} \mathrm{d}\omega = 0 \\ \mathrm{d}(\pi_\Delta^{n,0} \cdot \Omega) = 0 \\ \mathrm{d}(\pi_\Delta^{1,n-1} \cdot \Omega) = 0 \\ \Omega \wedge \bar{\Omega} = -i \cdot F \cdot \frac{\omega^3}{6} \\ \mathrm{d}\mathrm{d}^\Lambda\big(F \cdot (\pi_\Delta^{n-1,1} \cdot \Omega + \pi_\Delta^{0,n} \cdot \Omega)\big) = \rho_A \end{cases} \tag{2.15}$$

where Δ is a Lagrangian distribution with respect the (p, q)-decomposition of forms is taken (see section 2.3.3). Clearly, for $n = 3$ one recovers the above definition.

Torsion of a SU(3)-structure

We have already mentioned that the formulation of string theory in the non-Kähler setting is related to the presence of torsion in the structures defining the geometry. The obstruction of a SU(3)-structure to be Calabi-Yau is encoded in the so called *torsion forms* which are described in [15],[24]:

$$\begin{cases} d\omega = \nu_0\ \psi^+ + \alpha_0\ \psi^- + \nu_1 \wedge \omega + \nu_3 \\ d\psi^+ = \pi_0\ \omega^2 + \pi_1 \wedge \psi^+ - \pi_2 \wedge \omega \\ d\psi^- = \sigma_0\ \omega^2 + \sigma_1 \wedge \psi^+ - \sigma_2 \wedge \omega \end{cases} \tag{2.16}$$

where we set $\Omega = \psi^+ + i\psi^-$ and $\alpha_0, \nu_0, \pi_0, \sigma_0 \in C^\infty(M)$, $\nu_1, \pi_1, \sigma_1 \in \Lambda^1 M$, $\nu_3 \in \Lambda^2_{12}M, \pi_2, \sigma_2 \in \Lambda^2_8 M$.

We also recall the decomposition of $\Lambda^\bullet M$ as $\mathfrak{su}(3)$-module:

$$\begin{array}{l} \Lambda^2 M = \Lambda^2_1 M \oplus \Lambda^2_6 M \oplus \Lambda^2_8 M \\ \Lambda^3 M = \Lambda^3_{Re} M \oplus \Lambda^3_{Im} M \oplus \Lambda^3_6 M \oplus \Lambda^3_{12} M \\ \Lambda^4 M = \Lambda^4_1 M \oplus \Lambda^4_6 M \oplus \Lambda^4_8 M \end{array} \tag{2.17}$$

$$\begin{array}{l} \Lambda^2_1 = \mathbb{R}\omega \\ \Lambda^2_6 M = \Big\{ *_s (\alpha \wedge \psi^+) \big| \alpha \in \Lambda^1 M \Big\} = \Big\{ \varphi \in \Lambda^2 M \big|\ J\varphi = -\varphi \Big\} \\ \Lambda^2_8 M = \Big\{ \varphi \in \Lambda^2 M \big| \varphi \wedge \psi^+ = 0 \text{ and } *_s \varphi = -\varphi \wedge \omega\ \Big\} = \\ = \Big\{ \varphi \in \Lambda^2 M \big| J\varphi = \varphi\ ,\ \varphi \wedge \omega^2 = 0 \Big\} \end{array} \tag{2.18}$$

and

$$\begin{array}{l} \Lambda^3_{Re} = \mathbb{R}\psi^+ \\ \Lambda^3_{Im} = \mathbb{R}\psi^- \\ \Lambda^3_6 M = \Big\{ \alpha \wedge \omega \big| \alpha \in \Lambda^1 M \Big\} = \Big\{ \gamma \in \Lambda^3 M \big|\ *_s \gamma = \gamma \Big\} \\ \Lambda^3_{12} M = \Big\{ \gamma \in \Lambda^3 M \big| \gamma \wedge \omega = 0\ ,\ \gamma \wedge \psi^+ = 0 \text{ and} \gamma \wedge \psi^- = 0\ \Big\} \end{array} \tag{2.19}$$

In the case of a *symplectic half-flat structure* the equations (2.16) reduce to

$$\begin{cases} d\omega = 0 \\ d\psi^+ = 0 \\ d\psi^- = -\sigma_2 \wedge \omega \end{cases} \tag{2.20}$$

while for *complex balanced* one to

$$\begin{cases} d\omega = \nu_3 \\ d\psi^+ = 0 \\ d\psi^- = 0 \end{cases} \tag{2.21}$$

Refined Tseng-Yau Cohomology

Let (ω, Ω) be an SU(3)-structure on M, here we are not specifying the type of the structure. We recall that a **real polarization** w.r.t a non-degenerate 2-form ω is an integrable distribution $\Delta \subseteq T_{\mathbb{C}}M$ such that $\omega|_{\Delta} = 0$ and $\bar{\Delta} = \Delta$ (see [48],[92] for the example). Set $\Delta^{\perp}$ for the g-orthogonal complement of Δ where the metric g is the one associated to ω and Ω in the defining SU(3)-structure. The orthogonal decomposition $TM = \Delta \oplus \Delta^{\perp}$ extends to the space of differential forms

$$\mathcal{A}^{\bullet}(M) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_{\Delta}(M)$$

where $\mathcal{A}^{p,q}_{\Delta}(M)$ ranges over the p Δ-directions and q $\Delta^{\perp}$-directions.

We are now in position to define the cohomology we will use to compute the symplectic invariants of a IIA structure.

Definition 2.4.3. *Let (ω, Ω) be an $SU(3)$-structure on M such that ω is symplectic and let Δ be a real polarization with respect to ω. The* ***refined Tseng-Yau cohomology*** *of M is defined as*

$$H^{p,q}_{TY,\Delta}(M) := \frac{\text{Ker } (d + d^{\Lambda}) \cap \mathcal{A}^{p,q}_{\Delta}(M)}{\text{Im } (dd^{\Lambda}) \cap \mathcal{A}^{p,q}_{\Delta}(M)}$$

When a supersymmetric SU(3)-structure (M, ω, Ω) is the total space of a Lagrangian fibration $\pi : M \to B$, the vertical Lagrangian distribution induced by π will be chosen as the real polarization.

2.5 Lie groups and Lie algebras

We are interested in studying the properties of compact solvmanifolds. We therefore recall some basic facts about solvable Lie groups and solvable Lie algebras. The main reference in this sense is the paper by Bock [18]. There the author studied and classified in a comprehensive way the algebra, and geometry, of solvmanifolds up to dimension six.

Definition 2.5.1. A **solvable** (respectively **nilpotent**) **Lie algebra** $\mathfrak{g}$ is a Lie algebra such that its derived series $\mathfrak{g} \geq [\mathfrak{g}, \mathfrak{g}] \geq [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \geq \cdots$ (lower center series $\mathfrak{g} \geq [\mathfrak{g}, \mathfrak{g}] \geq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \geq \cdots$) terminates in a finite number of steps. A Lie algebra is said **completely solvable** if it admits a chain of ideals L_i such that

$$0 = L_0 \subset L_1 \subset \cdots \subset L_n = \mathfrak{g}$$

with $\dim L_i = i$. Equivalently, if $\mathfrak{g}$ is defined over a field $\mathbb{K}$, it is completely solvable if and only if the eigenvalues of ad_X are in $\mathbb{K}$ for all $X \in \mathfrak{g}$. A Lie group G is said solvable (nilpotent, completely solvable respectively) if it is its Lie algebra.

Remark 2.5.1. We will be interested in left-invariant structures defined on the Lie groups. In the following, when taking the quotient by a lattice, the action is always meant by left translation and we will adopt the expression G/Γ for the space of right cosets.

Definition 2.5.2. *A (compact)* ***solvmanifold*** *is a quotient of a solvable Lie group modulo a lattice.*

In dimension three the only unimodular, solvable Lie groups (not compact) are

1. The abelian $(\mathbb{R}^3, +)$
2. The **Heisenberg group**

$$H_3(\mathbb{R}) := \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \Big| x_1, x_2, x_3 \in \mathbb{R} \right\}$$

3. The universal cover of the group of rigid motion of the Minkowski plane

$$E(1,1) := \left\{ \begin{pmatrix} e^{x_1} & 0 & 0 & x_2 \\ 0 & e^{-x_1} & 0 & x_3 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big| x_1, x_2, x_3 \in \mathbb{R} \right\}$$

4. The universal cover of the group of rigid motion of Euclidean plane

$$E(2) := \left\{ \begin{pmatrix} \cos t & -\sin t & 0 & x \\ \sin t & \cos t & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big| x, y, t \in \mathbb{R} \right\}$$

with corresponding Lie algebras

1. $$\mathfrak{a}_3 = (0, 0, 0)$$

2. $$\mathfrak{h}_3 = (0, 0, 12)$$

3. $$\mathfrak{e}(1,1) = (13, -23, 0)$$

4. $$\mathfrak{e}(2) = (23, -13, 0)$$

We are adopting the convention that $(0,0,12)$ stands for a basis $\{E_1, E_2, E_3\}$ ($\{e^1, e^2, e^3\}$) for the (co)algebra such that $[E_1, E_2] = E_3$ and the other brackets vanish ($de^1 = de^2 = 0$ and $de^3 = -e^{12}$).

Remark 2.5.2. *The Heisenberg group is nilpotent while $E(1,1)$ is completely solvable. This will have consequences for our forthcoming constructions.*

Remark 2.5.3. *Each of these solvable Lie groups has a structure of semidirect product $\mathbb{R} \ltimes_{\mu_i} \mathbb{R}^2$ where the action is one of*

$$\mu_1 = 0 \quad , \quad \mu_2(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad , \quad \mu_3(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix} \quad , \quad \mu_4(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Each of these solvable Lie groups admits a lattice Γ. For simplicity we will present just one possible lattice and consider the solvmanifold obtained by quotienting by it.

1. $$\Gamma = \mathbb{Z}^3$$
   so that $\mathbb{R}^3/\mathbb{Z}^3 \simeq \mathbb{T}^3$

2. $$\Gamma = H_3(\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| n_1, n_2, n_3 \in \mathbb{Z} \right\}$$
   so that $H_3(\mathbb{R})/H_3(\mathbb{Z})$ is the **Heisenberg manifold**

3. $$\Gamma = \Gamma_t := t\mathbb{Z} \ltimes_{\mu_3} \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix} \rangle_{\mathbb{Z}} \quad \text{, for } t = \log \frac{3+\sqrt{5}}{2}$$
   so that $E(1,1)/\Gamma_t$ correspond to the (compact) Sol geometry in Thurston's classification. An element $\gamma \in \Gamma_t$ can be written as
   $$\begin{pmatrix} e^{tn_1} & 0 & 0 & n_2 + e^t n_3 \\ 0 & e^{-tn_1} & 0 & n_2 + e^{-t} n_3 \\ 0 & 0 & 1 & tn_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
   see Bock [18] or Auslander [29].

4.

$$\Gamma = \Gamma_\pi := \pi\mathbb{Z} \ltimes_{\mu_4} \mathbb{Z}^2$$

so that $E(2)/\Gamma_\pi$ is a compact solvmanifold with $b_1 = 1$

if we take 2π instead of π we get a quotient diffeomorphic to a three-torus $E(2)/\Gamma_{2\pi} \simeq \mathbb{T}$ which has $b_1 = 3$ ([18]).

In fact, the diffeomorphism type of a solvmanifold is governed by the fundamental group which is isomorphic to the lattice Γ.

Theorem 2.5.1 (Mostow [75])**.** *Let (G_1/Γ_1) (G_2/Γ_2) two solvmanifold. Then any isomorphsim $\varphi : \Gamma_1 \longrightarrow \Gamma_2$ extends to an equivariant diffeomorphism $\Phi : G_1 \longrightarrow G_2$.*

The hypothesis of nilpotence or complete solvability is crucial for cohomological computations:

Theorem 2.5.2 (Nomizu [70])**.** *Let G be a simply connected nilpotent Lie group with a discrete subgroup Γ. Assume that $X := \Gamma\backslash G$ is compact. Then, the de Rham cohomology of X can be represented by G-invariant forms*

This has been then extended by Hattori to the completely solvable case

Theorem 2.5.3 (Hattori [50])**.** *Let G/Γ be a solvmanifold. Then*

1. *The natural inclusion of the Chevalley-Eilenberg complex into the de Rham complex $(\bigwedge^\bullet \mathfrak{g}^*, \delta) \to (\mathcal{A}^\bullet(G/\Gamma), d)$ induces an injection in cohomology.*

2. *If G is completely-solvable, then the inclusion is a quasi-isomorphism.*

3. *If $Ad(\Gamma)$ and $Ad(G)$ have the same Zariski closure, then the inclusion is a quasi-isomorphism.*

and more recently Kasuya gave a useful tool also for the generic solvable case

Theorem 2.5.4 (Kasuya [56])**.** *Let G be a simply connected solvable real Lie group and let $\mathfrak{g}$ be its Lie algebra. Assume it contains a lattice Γ. Let $\alpha_1, \ldots, \alpha_n$ complex characters for the semi-simple representation $\Psi : G \to Aut(\mathfrak{g})$ induced by a semi-simple complement in $\mathfrak{g}$ and associated to a basis $X_1, \ldots, X_n$ of $\mathfrak{g}_{\mathbb{C}}$. Then by defining*

$$\mathcal{A}^p_\Gamma = span\langle \alpha_I x_I \mid I \subset \{1, \ldots, n\}, |I| = n, \alpha_I|_\Gamma = 1\rangle \quad x_1, \ldots, x_n \, dual\ basis\ of\ \mathfrak{g}^*_{\mathbb{C}}$$

the inclusion of the sub-complex $\mathcal{A}^\bullet_\Gamma$ in the complex valued de Rham complex $\mathcal{A}^\bullet_{\mathbb{C}}(G/\Gamma)$ is a quasi-isomorphism.

The Nomizu-Hattori type theorem has been extended to Bott-Chern and Tseng-Yau cohomologies by Angella and Kasuya in [3],[4], also for the non completely-solvable case under suitable assumptions. The result for the symplectic cohomologies was already done by Macrì [65].

2.5.1 Classification of structures on six-dimensional solvable Lie algebras

In this section we collect the classification results for the structures of our interest on Lie algebras.

Nilpotent Lie algebras

A **complex structure** on a nilpotent Lie algebra $\mathfrak{g}$ is an endomorphism $J : \mathfrak{g} \to \mathfrak{g}$ such that $J^2 = -\mathrm{Id}$ for which exists a basis $\{\psi^i\}_{i=1}^n$ of the i-eigenspace $\mathfrak{g}^{1,0}$ relative to the extension $J^{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$ such that

$$d\psi^i \in \mathrm{span}\langle \psi^1, \ldots, \psi^{i-1} \rangle$$

When the subalgebra $\mathfrak{g}^{1,0}$ is abelian, the complex structure J is said *abelian* and consequently $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{1,1} \mathfrak{g}^*$. Instead, it is said *complex-parallelizable* if $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0} \mathfrak{g}^*$. Moreover if the basis $\psi^1, \ldots, \psi^n$ satisfies

$$d\psi^i \in \bigwedge^2 \langle \psi^1, \ldots, \psi^{i-1}, \psi^{\bar{1}}, \ldots, \psi^{\overline{i-1}} \rangle$$

it is called *nilpotent*.

A first list of nilpotent Lie algebras (NLA from now on) admitting complex and/or symplectic structures was given by Salamon [76]. There are 34 classes of isomorphism of NLA: 18 of them admit a complex structure, 26 of them admit a symplectic structure while 15 of them admit both.

Among the complex ones, the NLA admitting a balanced structure were classified by Latorre,Ugarte,Villacampa [60] where also computations for their Bott-Chern cohomology were provided. This was achieved exploiting the computations in [22],[88],[89].

Theorem 2.5.5 ([22],[60],[88])**.** *Let $\mathfrak{g}$ be an NLA of dimension 6. Then, $\mathfrak{g}$ has a complex structure if and only if it is isomorphic to one of the following Lie algebras:*

$\mathfrak{h}_1 = (0, 0, 0, 0, 0, 0)$
$\mathfrak{h}_2 = (0, 0, 0, 0, 12, 34)$
$\mathfrak{h}_3 = (0, 0, 0, 0, 0, 12 + 34)$
$\mathfrak{h}_4 = (0, 0, 0, 0, 12, 14 + 23)$
$\mathfrak{h}_5 = (0, 0, 0, 0, 13 + 42, 14 + 23)$
$\mathfrak{h}_6 = (0, 0, 0, 0, 12, 13)$
$\mathfrak{h}_7 = (0, 0, 0, 12, 13, 23)$
$\mathfrak{h}_8 = (0, 0, 0, 0, 0, 12)$
$\mathfrak{h}_9 = (0, 0, 0, 0, 12, 14 + 25)$
$\mathfrak{h}_{10} = (0, 0, 0, 12, 13, 14)$
$\mathfrak{h}_{11} = (0, 0, 0, 12, 13, 14 + 23)$
$\mathfrak{h}_{12} = (0, 0, 0, 12, 13, 24)$
$\mathfrak{h}_{13} = (0, 0, 0, 12, 13 + 14, 24)$
$\mathfrak{h}_{14} = (0, 0, 0, 12, 14, 13 + 42)$
$\mathfrak{h}_{15} = (0, 0, 0, 12, 13 + 42, 14 + 23)$
$\mathfrak{h}_{16} = (0, 0, 0, 12, 14, 24)$
$\mathfrak{h}_{19}^- = (0, 0, 0, 12, 13, 14 - 35)$
$\mathfrak{h}_{26}^+ = (0, 0, 12, 13, 23, 14 + 25)$

Moreover:

a. *Any complex structure on* $\mathfrak{h}_{19}^-$ *and* $\mathfrak{h}_{26}^+$ *is non-nilpotent;*

b. *For* $1 \leq k \leq 16$*, any complex structure on* $\mathfrak{h}_k$ *is nilpotent;*

c. *Any complex structure on* $\mathfrak{h}_1$*,*$\mathfrak{h}_3$*,*$\mathfrak{h}_8$*,*$\mathfrak{h}_9$ *is abelian;*

d. *There exist both abelian and non-abelian nilpotent complex structures on* $\mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5$ *and* $\mathfrak{h}_{15}$*;*

e. *Any complex structure on* $\mathfrak{h}_6, \mathfrak{h}_7, \mathfrak{h}_{10}, \mathfrak{h}_{11}, \mathfrak{h}_{12}, \mathfrak{h}_{13}, \mathfrak{h}_{14}$ *and* $\mathfrak{h}_{16}$ *is not abelian.*

f. *Any complex structure on* $\mathfrak{h}_6$ *and* $\mathfrak{h}_{19}^-$ *has compatible metrics which are balanced;*

g. *On the Lie algebras* $\mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5$ *there exist complex structure having balanced compatible metrics but also not admitting such metrics. On* $\mathfrak{h}_3$ *there is only a complex structure admitting compatible balanced metrics;*

h. *There exists an SKT metric on* $\mathfrak{g}$ *if and only if is isomorphic to* $\mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5$ *or* $\mathfrak{h}_8$*.*

i. *There exists an sG metric on* $\mathfrak{g}$ *if and only if it is isomorphic to* $\mathfrak{h}_k$*, for* $k = 1, \ldots, 6$ *or* $\mathfrak{h}_{19}^-$*.*

From points f. and g. we get that the only NLAs we are possibly interested in are

$$\begin{array}{ll} \mathfrak{h}_2 = (0,0,0,0,12,34) & \mathfrak{h}_5 = (0,0,0,0,13+42,14+23) \\ \mathfrak{h}_3 = (0,0,0,0,0,12+34) & \mathfrak{h}_6 = (0,0,0,0,12,13) \\ \mathfrak{h}_4 = (0,0,0,0,12,14+23) & \mathfrak{h}_{19}^- = (0,0,0,12,23,14-35) \end{array}$$

On the symplectic side instead, a result of classification is represented by the work of Conti, Tomassini [26]. General half-flat NLA were classified by the first author in [25]. Also in [10] appeared nilpotent examples which correspond to some we have constructed.

The only symplectic half-flat NLA's are

$$\mathfrak{h}_6 = (0,0,0,0,12,13) \quad \text{and} \quad \mathfrak{h}_7 = (0,0,0,12,13,23)$$

We remark there is no in literature a corresponding treatment, as done for the complex balanced condition, on the symplectic side. Below we will present a useful lemma which helps understanding when a Lie algebra can not admit a symplectic half-flat structure. That is a slightly improvement of the criterion used in [26] and [30]: it is still a computational method but it has the advantage to be defined in terms of the symplectic cohomology of the algebra.

Solvable Lie algebras

The list of six-dimensional solvable Lie algebras (SLA from now on) is notably more numerous. Among all SLA we will consider only the unimodular one. The SLA (non-nilpotent) admitting a complex balanced structure are presented in [32].

Theorem 2.5.6 ([32])**.** *Let $\mathfrak{g}$ be a unimodular (non nilpotent) solvable Lie algebra of dimension 6. Then, $\mathfrak{g}$ admits a complex structure with a non-zero closed (3, 0)-form if and only if it is isomorphic to one in the following list:*

$$
\begin{aligned}
&\mathfrak{g}_1 = A_{5,7}^{-1,-1,1} \oplus \mathbb{R} = (15, -25, -35, 45, 0, 0)\\
&\mathfrak{g}_2^\alpha = A_{5,17}^{-\alpha,\alpha,1} \oplus \mathbb{R} = (\alpha\ 15 + 25, -15 + \alpha\ 25, -\alpha\ 35 + 45, -35 - \alpha\ 45, 0, 0) \quad \alpha \geq 0\\
&\mathfrak{g}_3 = \mathfrak{e}(2) \oplus \mathfrak{e}(1,1) = (0, -13, 12, 0, -46, -45)\\
&\mathfrak{g}_4 = A_{6,37}^{0,0,1} = (23, -36, 26, -56, 46, 0)\\
&\mathfrak{g}_5 = A_{6,82}^{0,1,1} = (24 + 35, 26, 36, -46, -56, 0)\\
&\mathfrak{g}_6 = A_{6,88}^{0,0,1} = (24 + 35, -36, 26, -56, 46, 0)\\
&\mathfrak{g}_7 = B_{6,6}^{1} = (24 + 35, 46, 56, -26, -36, 0)\\
&\mathfrak{g}_8 = N_{6,118}^{0,-1,-1} = (-16 + 25, -15 - 26, 36 - 45, 35 + 46, 0, 0)\\
&\mathfrak{g}_9 = B_{6,4}^{1} = (45, 15 + 36, 14 - 26 + 56, -56, 46, 0)
\end{aligned}
$$

While for special metrics

Theorem 2.5.7 ([32])**.** *Let $(M = G/\Gamma, J)$ be a 6-dimensional solvmanifold endowed with an invariant complex structure J with holomorphically trivial canonical bundle, and denote by $\mathfrak{g}$ the Lie algebra of G.*

- *Then, (M, J) has an SKT metric if and only if $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_2^0$ or $\mathfrak{g}_4$*
- *Then, (M, J) has a balanced metric if and only if $\mathfrak{g}$ is isomorphic to one of $\mathfrak{g}_1, \mathfrak{g}_2^\alpha, \mathfrak{g}_3, \mathfrak{g}_5, \mathfrak{g}_7, \mathfrak{g}_8$. Moreover, in such cases, any J admits a balanced metric expect for the first two complex structures on $\mathfrak{g}_8$.*
- *Then (M, J) has an sG metric if and only if $\mathfrak{g}$ is isomorphic to one of $\mathfrak{g}_1, \mathfrak{g}_2^\alpha, \mathfrak{g}_3, \mathfrak{g}_5, \mathfrak{g}_7, \mathfrak{g}_8$. Moreover any invariant Hermitian metric is sG.*

On the other side, the list of SLA admitting a symplectic half-flat structure are listed in [30].

Theorem 2.5.8 ([30])**.** *A unimodular (non-Abelian) solvable Lie algebra $\mathfrak{g}$ has a symplectic half-flat structure if and only if it is isomorphic to one in the following list:*

$\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1) = (0, -13, -23, 0, -46, -45)$
$\mathfrak{g}_{5,1} \oplus \mathbb{R} = (0, 0, 0, 0, 12, 13)$
$A_{5,7}^{-1,-1,1} \oplus \mathbb{R} = (15, -25, -35, 45, 0, 0)$
$\mathfrak{g}_{6,N3} = (0, 0, 0, 12, 13, 23)$
$A_{5,17}^{-\alpha,\alpha,1} \oplus \mathbb{R} = (\alpha\ 15 + 25, -15 + \alpha\ 25, -\alpha\ 35 + 45, -35 - \alpha\ 45, 0, 0) \quad \alpha \geq 0$
$\mathfrak{g}_{6,38}^{0} = (23, -36, 26, 26 - 56, 36 + 46, 0)$
$\mathfrak{g}_{6,54}^{0,-1} = (16 + 35, -26 + 45, 36, -46, 0, 0)$
$\mathfrak{g}_{6,118}^{0,-1,-1} = (-16 + 25, -15 - 26, 36 - 45, 35 + 46, 0, 0)$

Some of these already appeared as examples in [83] ($\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$ and $\mathfrak{g}_{6,54}^{0,-1}$) while the nilpotent $\mathfrak{g}_{5,1} \oplus \mathbb{R}$ in [13].

Remark 2.5.4. *Solvmanifolds have been proposed as compactification space in string theory by various authors [1],[31],[40],[41],[42]. One of the results of the thesis is the discovery that some of these NLAs and SLAs can be paired by a mirror symmetry relation.*

A useful lemma

In the case of a *symplectic half-flat structure* the equations (2.16) reduce to

$$\begin{cases} d\omega = 0 \\ d\psi^{+} = 0 \\ d\psi^{-} = -\sigma_2 \wedge \omega \end{cases} \tag{2.22}$$

and there is an additional identity involving the operator d^{Λ}:

$$\begin{aligned} d^{\Lambda}\psi^{-} &= (d\Lambda - \Lambda d)\psi^{-} = -\Lambda d\psi^{-} = \\ &= \Lambda(\sigma_2 \wedge \omega) = \Lambda L \sigma_2 = \sigma_2 \end{aligned} \tag{2.23}$$

since

$$\Lambda\psi^{-} = - * L * \psi^{-} = - * L\psi^{+} = - * (\omega \wedge \psi^{+}) = 0 \tag{2.24}$$

Lemma 2.5.1. *Let M be a compact six-dimensional manifold with a syumplectic half-flat $SU(3)$-structure represented by $(\omega, \Omega = \psi^{+} + i\psi^{-})$. Then $d\sigma_2 = 0$ implies $\sigma_2 = 0$.*

Proof. Let $\langle \alpha, \beta \rangle_s = \int_M \alpha \wedge *_s \beta$ the scalar product induced on $\Lambda^{\bullet} M$ by the symplectic hodge star operator $*_s = J* = *J$. Then

$$
\begin{aligned}
||\sigma_2||_s^2 &= \langle \sigma_2, \sigma_2 \rangle_s = \\
&= \int_M \sigma_2 \wedge *_s \sigma_2 = (\sigma_2 \in \Lambda^2_8 M) \\
&= -\int_M \sigma_2 \wedge \sigma_2 \wedge \omega = (\text{by } (2.16)) \\
&= \int_M \sigma_2 \wedge d\psi^- = (\text{Leibniz}) \\
&= -\int_M d(\sigma_2 \wedge \psi^-) + \int_M d\sigma_2 \wedge \psi^- = (\text{ Stokes Theorem}) \\
&= -\int_M d\sigma_2 \wedge \psi^- = 0 \; (\text{ by hypothesis})
\end{aligned}
\tag{2.25}
$$

Therefore, since $\langle , \rangle$ is non degenerate we get $\sigma_2 = 0$. □

Remark 2.5.5. *[49] A solvmanifold G/Γ is Kähler if and only if it is a finite quotient of a complex torus which has the structure of a complex torus bundle over a complex torus. If G is completely solvable, then G/Γ is Kähler if and only if it is a complex torus.*

Corollary 2.5.1. *On a compact, six-dimensional, symplectic solvmanifold $M = G/\Gamma$, if the operator dd^Λ is zero when restricted to $\Lambda^3\mathfrak{g}^*$, then M cannot admit a symplectic half-flat $SU(3)$-structure, unless it is a torus.*

Proof. By the identity 2.23, $dd^\Lambda\gamma = 0$ for every 3-form γ would imply $dd^\Lambda\psi^- = d\sigma_2 = 0$. Then for the previous lemma there would not be torsion anymore and the manifold should be Calabi-Yau. This can happen only if M is a torus. □

Therefore, in view of this lemma, at least for non-toric solvmanifolds, the condition $\mathrm{dd}^\Lambda\mathrm{Im}\Omega = 0$ is complementary to the condition $\mathrm{dRe}\Omega = 0$. This is the same that happens for SKT and balanced structure in the complex non-Kähler case. It is then reasonable to call a manifold (M, ω, Ω) such that $\mathrm{dd}^\Lambda\mathrm{Im}\Omega = 0$ a **symplectic SKT manifold**. This is in accordance with the fact that, in presence of torsion, also flux-forms make their appearance and they are related to the dd^Λ and $\partial\bar{\partial}$ operators.

Chapter 3

Constructing SYZ pairs

In this chapter we first recall briefly the setting as described by Lau,Tseng,Yau [61] and then we explain the method by which we intend to produce examples that satisfy all the required properties.

3.1 Strominger-Yau-Zaslow picture

Let $\pi : M \to B$ be a Lagrangian fibration with compact connected fibers. As we have already seen, the fibers are necessarily tori and there is an induced integral affine structure on B. By the Arnol'd-Liouville theorem, every $b \in B$ has an open neighborhood $U \subseteq B$ such that $(\pi^{-1}(U), \omega)$ is symplectomorphic to $(T^*U/\Lambda^*, \omega_{can})$ where $\Lambda^* \subset T^*U$ is the lattice induced by the integral affine structure. Around $\pi^{-1}(b)$ there exist local coordinates $\{r_1, \dots, r_n, \theta_1, \dots, \theta_n\}$ such that the lattice bundle Λ^* is generated by $dr_1, \dots, dr_n$ and the symplectic form ω is in Darboux coordinates $\omega = \sum_{i=1}^n d\theta_i \wedge dr_i$. The dual torus bundle $\check{\pi} : \check{M} \to B$ is locally obtained by $\check{\pi}^{-1}(U) \simeq TU/\Lambda$ where Λ is the dual lattice generated by taking $\{\frac{\partial}{\partial r_1}, \dots, \frac{\partial}{\partial r_n}\}$. That is, we are just dualizing, fiberwise, the torus fibration.

Remark 3.1.1. *$\check{M}$ can be also interpreted as*

$$\check{M} := \{(b, \nabla) \mid r \in B\ ,\ \nabla \textit{ is a flat } U(1)\textit{-connection on } \pi^{-1}(b)\}$$

The dual bundle map $\check{\pi} : \check{M} \to B$ is then given by forgetting the fiberwise connection.

The dual total space $\check{M}$ is endowed with a canonical complex structure: for each $b \in B$ there exists an open subset $\check{U} \subseteq B$ containing b and a biholomorphism $\check{\pi}^{-1}(\check{U}) \simeq TU/\Lambda$, where Λ is the dual lattice bundle of Λ^* generated by $\{\frac{\partial}{\partial r_1}, \dots, \frac{\partial}{\partial r_n}\}$. Then dual coordinates on the fiber of TB are denoted as $\check{\theta}_1, \dots, \check{\theta}_n$ and one can take $z_i = \check{\theta}_i + ir_i$ as complex coordinates on $\check{M}$. If the affine structure on the base can be taken with special linear part, then one can define a holomorphic volume form on $\check{M}$ which is locally given by

$$\check{\Omega}_{\text{can}} = \bigwedge_{i=1}^{n} dz_i = \bigwedge_{i=1}^{n} \left(d\check{\theta}_i + i dr_i\right)$$

Definition 3.1.1. We will refer to M and $\check{M}$ in this construction as a **semi-flat mirror pair** or just an **SYZ (mirror) pair**.

Semi-flat setting and SU(3)-structures

We now return to the three-dimensional case and, using the notation coming from the SYZ construction, we will restrict our attention to a subset of the algebra of differential forms.

Definition 3.1.2. *We denote with $\mathcal{A}^k_B(M, \mathbb{C})$ the space of complex-valued k-forms on M which depend only on the base, also called **semi-flat (differential) forms**. An element $\phi \in \mathcal{A}^k_B(M, \mathbb{C})$ is locally written as*

$$\phi = \sum_{I,J} a_{IJ}(r) d\theta_I \wedge dr_J$$

where $I = (i_1, \ldots, i_p)$, $J = (j_1, \ldots, j_q)$ are multi-indices and $p + q = k$, (r_i, θ_i) are action-angle coordinates and $a_{IJ}(r)$ are complex-valued functions on B with $r = (r_1, \ldots, r_n)$.

If the total space of the fibration $\pi : M \to B$ admits a SU(3)- structure (not just a symplectic form), we denote with $\mathcal{A}^{p,q}_{B,\Delta}(M) \subset \mathcal{A}^{p,q}_{\Delta}(M)$ the space of **semi-flat** (p, q)**-forms**. Since in our construction the choice of the Lagrangian distribution will be indeed induced by the fibration itself we will omit the subscript for the distribution and we will write just $\mathcal{A}^{p,q}_B(M)$ (see section 2.4.1). Clearly the p-directions in Δ correspond to $d\theta_1, d\theta_2, d\theta_3$ while the q-directions for the orthogonal $\Delta^\perp$ to dr_1, dr_2, dr_3.

Similarly, we will denote with $\mathcal{A}^{p,q}_B(\check{M})$ the semi-flat (p,q)-forms on the SYZ-dual $\check{M}$ which are locally written as.

$$\check{\phi} = \sum_{I,J} a_{IJ}(r) dz_I \wedge d\bar{z}_J$$

Definition 3.1.3. *Let (M, ω, Ω) a supersymmetric SU(3) system of type IIA. Assume that $M \to B$ has the structure of a Lagrangian-torus bundle. If the defining forms ω, Ω are taken in $\mathcal{A}^\bullet_B(M, \mathbb{C})$ then (M, ω, Ω) is said to be a **supersymmetric semi-flat SU(3)-structure of type IIA**. Analogously, if $(\check{M}, \check{\omega}, \check{\Omega})$ is a supersymmetric SU(3) system of type IIB such that $\check{\omega}$ and $\check{\Omega}$ are in $\mathcal{A}^{\bullet,\bullet}_B(M)$, we say that $(\check{M}, \check{\omega}, \check{\Omega})$ is a **supersymmetric semi-flat SU(3)-structure of type IIB**.*

Also the (refined) Tseng-Yau and the Bott-Chern cohomology can be restricted to the semi-flat forms only:

$$
\begin{aligned}
H^{p,q}_{TY,B}(M) &:= \frac{\text{Ker }(d + d^{\Lambda}) \cap \mathcal{A}^{p,q}_B(M)}{\text{Im }(dd^{\Lambda}) \cap \mathcal{A}^{p,q}_B(M)} \\
H^{p,q}_{BC,B}(\check{M}) &:= \frac{\text{Ker }(d) \cap \mathcal{A}^{p,q}_B(\check{M})}{\text{Im }(\partial\bar{\partial}) \cap \mathcal{A}^{p,q}_B(\check{M})}
\end{aligned}
\tag{3.1}
$$

On the complex SYZ-dual, the (p, q)-decomposition on forms is taken with respect to the complex polarization induced by the aforementioned complex structure $\check{\Omega}$. However, on $\check{M}$, we can take another polarization induced by the dual action-angle coordinates $\{r_i, \check{\theta}_i\}$.

This extra structure in the complex side allow us to define a new operator:

Definition 3.1.4. *The* ***polarization switch operator*** $\mathcal{P}$ *on* $\mathcal{A}^{\bullet,\bullet}_B(\check{M})$ *is defined as the operator which acts as a switch on the basic wedges as*

$$dz_I \longleftrightarrow d\check{\theta}_I \quad , \quad d\bar{z}_J \longleftrightarrow dr_J$$

Therefore if

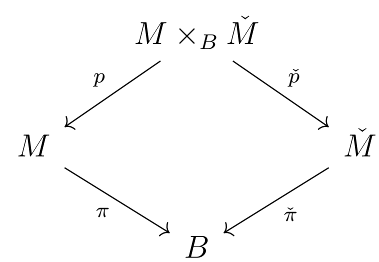
$$\check{\phi} = \sum_{I,J} a_{IJ}(r) dz_I \wedge d\bar{z}_J$$

one has

$$\mathcal{P} \cdot \check{\phi} = \sum_{I,J} a_{IJ}(r) d\check{\theta}_I \wedge dr_J$$

3.1.1 Fourier-Mukai Transform and Mirror Symmetry

We are now in place to define the main tool of the construction which realizes the *mirror transform*. Let $\pi : (M, \omega, \Omega) \to B$ be a supersymmetric semi-flat SU(3)-system of type IIA. Let $\check{\pi} : \check{M} \to B$ its SYZ-dual and consider their fiber product over B:



On the Poincaré bundle line over $M \times_B \check{M}$ there is a universal connection which locally is written as $\mathrm{d} + i\check{\theta}_i d\theta_i + i\theta_i d\check{\theta}_i$. Its curvature form is

$$F = 2i \sum_i^3 d\check{\theta}_i \wedge d\theta_i \tag{3.2}$$

We can finally define

Definition 3.1.5. *Let $\phi \in \mathcal{A}_B^\bullet(M)$ and $\check{\phi} \in \mathcal{A}_B^\bullet(\check{M})$. Their **Fourier-Mukai transforms** are defined as*

$$\begin{aligned} FT \cdot \check{\phi} &:= p_*\Big((\check{p}^*(\mathcal{P} \cdot \check{\phi})) \wedge \exp \frac{F}{2i}\Big) \\ FT \cdot \phi &:= \mathcal{P}^{-1} \cdot \Big(\check{p}_*\big((p^*\phi) \wedge \exp \frac{-F}{2i}\big)\Big) \end{aligned} \tag{3.3}$$

where the pushforward maps $p_*, \check{p}_*$ are just the integration along the fibers.

We recall here the main properties of the Fourier-Mukai transform:

Proposition 3.1.1 ([61])**.** *We have*

- $FT^2 = (-1)^{\frac{n(n-1)}{2}} Id$
- $FT \circ \bar{\partial} \cdot \check{\phi} = \frac{(-1)^n i}{2} \cdot d \circ FT \cdot \phi$
- $FT \circ \partial \cdot \check{\phi} = \frac{(-1)^n i}{2} \cdot d^\Lambda \circ FT \cdot \phi$

We note also that

Lemma 3.1.1. *The Fourier-Mukai transform intertwines, up to a sign, complex conjugation with the symplectic Hodge star operator, that is*

$$FT \cdot \bar{\check{\phi}} = *_s \, FT \cdot \check{\phi}$$

Proof. Denote with $c : \mathcal{A}_B^{\bullet,\bullet}(\check{M}, \mathbb{C}) \to \mathcal{A}_B^{\bullet,\bullet}(\check{M}, \mathbb{C})$ the complex conjugation.

On the basic element $dz_I \wedge d\bar{z}_J$ the Fourier-Mukai transform acts as

$$FT(dz_I \wedge d\bar{z}_J) = d\theta_{I^c} \wedge dr_J$$

Then, by straightforward computation:

$$FT\Big(c(dz_I \wedge d\bar{z}_J)\Big) = FT\Big(dz_J \wedge d\bar{z}_I\Big) = d\theta_{J^c} \wedge dr_I$$

while

$$*_s\Big(d\theta_K \wedge dr_L\Big) = *\Big(J_\Omega\big(d\theta_K \wedge dr_L\big)\Big) = *\Big(d\theta_L \wedge dr_K\Big) = d\theta_{L^c} \wedge dr_{K^c}$$

Then by setting $K = I^c$ and $L = J$ we get the claim.

□

Using Proposition 3.1.1 the authors in [61] prove the following

Theorem 3.1.1 ([61]). *Fourier-Mukai transform induces an isomorphism of double complexes*

$$\Big(\mathcal{A}_B^{\bullet,\bullet}(M,\mathbb{C}), \frac{(-1)^n i}{2}d, \frac{(-1)^n i}{2}d^\Lambda\Big) \simeq \Big(\mathcal{A}_B^{\bullet,\bullet}(\check{M},\mathbb{C}), \bar{\partial}, \partial\Big)$$

and at level of cohomologies

$$H^{n-p,q}_{B,TY}(M,\mathbb{C}) \simeq H^{p,q}_{B,BC}(\check{M}) \tag{3.4}$$

The last isomorphism is precisely the mirror symmetric relation between the diamonds associated to the two different cohomology theories. This is the non-Kähler version of mirror symmetry.

Finally we can state the main result of [61] for which we want to produce concrete examples:

Theorem 3.1.2 ([61]). *Let (M,ω) and $(\check{M},\check{\Omega})$ a semif-flat SYZ-pair. Let $\check{\omega}$ be a real $(1,1)$-form in $\mathcal{A}_B^{1,1}(\check{M})$ and set $\Omega = FT(e^{2\check{\omega}})$. Then*

1. *The triple $(\check{M},\check{\omega},\check{\Omega})$ forms a SU(n)-structure if and only if (M,ω,Ω) forms a SU(n)-structure. Moreover the conformal factors are related by the relation $F\check{F} = 2^{2n}$;*

2. *(M,ω,Ω) is supersymmetric of type IIA if and only if $(\check{M},\check{\omega},\check{\Omega})$ is supersymmetric of type IIB;*

3. *Under Fourier-Mukai transform the fluxes ρ_A and ρ_B correspond to each other up to a constant multiple.*

3.2 Strategy for constructing semi-flat SYZ mirror pairs from affine structures on Lie groups

In this section we will show how to produce pairs of compact six-dimensional solvmanifolds which admit semi-flat supersymmetric SU(3)-structure and satisfy the relation (3.4) and the Theorem 3.1.2.

We started from the computation of the Tseng-Yau cohomology for the NLA's we knew were admitting a symplectic half-flat structure and we noticed that, under the right (p,q)-decomposition, the numbers could be related to the Bott-Chern numbers in [60]. This would have meant that the nilmanifolds underlying these NLAs were possible candidates as mirror pairs. As mentioned in the preliminaries, a (2-step) nilmanifold fits in a natural torus fibration over a torus which comes from the study of the commutator subgroup of the nilpotent Lie group (see Palais and Stewart [71]). Nevertheless, this fibration is not, in general, Lagrangian. Moreover, since we started working with homogeneous manifolds, it was reasonable to look among nilmanifolds also for the base B. This had limited the possibility to only two options: a three-torus or a Heisenberg manifold. At this point, we noticed that all the nilpotent Lie groups involved in our analysis could be obtained as a semidirect product $G \ltimes_\rho \mathbb{R}^3$, where G was the three-dimensional Heisenberg group or the additive $\mathbb{R}^3$. Different choices of the acting homomorphism ρ would lead to different six-dimensional nilpotent Lie groups. We then reversed the point of view and started with the group G and considered its cotangent bundle T^*G which is globally a trivial vector bundle $G \times \mathbb{R}^3$ since the group is parallelizable. Then, the construction of the six-dimensional Lie group $G \ltimes_\rho \mathbb{R}^3$ was just endow the cotangent bundle T^*G with a group structure. The six-dimensional manifold is obtained by quotienting by the lattice $\Gamma \ltimes_\rho \mathbb{Z}^3$ so that

$$M = G \ltimes_\rho \mathbb{R}^3 / \Gamma \ltimes_\rho \mathbb{Z}^3 \xrightarrow{\pi} B := G/\Gamma \tag{3.5}$$

is an honest submersion between compact, smooth manifolds. The natural projection on the first factor $G \ltimes_\rho \mathbb{R}^3 \xrightarrow{\tilde{\pi}} G$, being also obviously a group homomorphism, descends to a well-defined map π between the quotients. However, we still had to let the symplectic geometry enter the picture. In Lau, Tseng, and Yau [61] the example given is represented by the Lagrangian torus bundle $T^*B/\Lambda^* \to B$ where B is the Heisenberg manifold and the symplectic form is the canonical one. We then reinterpreted this example in our language and realized the total space T^*B/Λ^* of the fibration as a nilmanifold, namely we realized it as a homogeneous space for the nilpotent Lie group $G \ltimes_\rho \mathbb{R}^3$ (see the lemmas below). A posteriori, looking under the lens of section 2.3.2, this construction corresponds to having a Lagrangian fibration $\pi : M \to B$ with a global section that makes the (symplectic) identification of M with the torus bundle T^*B/Λ^*. Nevertheless, we remark that the presence of monodromy, related to the homomorphism ρ, makes the topology of the total space highly non-trivial. The fundamental observation we made at this point was the possibility to relate all this construction with the integral affine geometry of the base B. In fact, it was already well-known that the monodromy of the fibration $M \xrightarrow{\pi} B$, once a basis for the $H_1(\pi^{-1}(b), \mathbb{Z})$ is fixed, is just the inverse transpose of the linear holonomy of the affine structure of the base. What we have done was just conciliate this feature with the group structure of our spaces. This has been fundamental to exploit all the advantages of working with homogeneous spaces. We then extended this construction to the (completely) solvable case with $G = E(1,1)$ producing more new examples. This should work also in the non completely solvable case with the appropriate minor modification but we have not treated this in our discussion. We will say more in the conclusions. In the following, we will explain in detail the construction in full generality.

Let G be a simply connected, unimodular, solvable n-dimensional Lie group and let $\Gamma \subset G$ be a lattice. Recalling the notation in section 2.1, let $\alpha : G \to \mathrm{Aff}(\mathbb{R}^n)$ an affine representation and assume that its restriction to Γ, say $\mathfrak{a} := \alpha|_\Gamma$, restricts to an integral affine representation of the lattice $\mathfrak{a} : \Gamma \to \mathrm{Aff}_\mathbb{Z}(\mathbb{R}^n)$. That is, we have chosen a left-invariant affine structure on G that descends on an integral affine structure on the quotient $B := G/\Gamma$. Denote with λ and $\mathfrak{l}$ the linear part of α and $\mathfrak{a}$ respectively, that is $\lambda : G \to \mathrm{GL}(n,\mathbb{R})$ and $\mathfrak{l} : \Gamma \to \mathrm{GL}(n,\mathbb{Z})$.

Consider now the natural projection $\tilde{\pi} : T^*G \to G$. Since G is parallelizable, one has $T^*G \simeq G \times \mathbb{R}^n$. We want to endow T^*G with a group structure induced by G and its affine representation via semidirect product:

$$T^*G = G \ltimes_\varphi \mathbb{R}^n$$

where $\varphi := \lambda^{-T}$ (here $^{-T}$ denotes the dual representation). For $g, g' \in G$ and $v, v' \in \mathbb{R}^n$, the group law is therefore given by

$$(g, v) \cdot (g', v') = (gg', v + \varphi(g)v').$$

Clearly this construction endows also the tangent bundle TG of a group structure via $G \ltimes_\lambda \mathbb{R}^n$ in the same way. Set, just as a matter of notation, $T^*\Gamma := \Gamma \ltimes_\mathfrak{f} \mathbb{Z}^n$ and consider the quotient $M := T^*G/T^*\Gamma$ (analogously set $T\Gamma := \Gamma \ltimes_\mathfrak{l} \mathbb{Z}^n$ so that $\check{M} := TG/T\Gamma$ will be in the sequel the dual fibration). The projection $\tilde{\pi}$ is equivariant with the actions (indeed it is a group homomorphism) and therefore it descends to a well-defined map on the quotients:

$$\pi : M \to B$$

.

Moreover $T^*G/T^*\Gamma$ can be identified with T^*B/Λ^* where Λ^* is the lattice in T^*B generated by $\{dr_1, \ldots, dr_n\}$ and $\{r_1, \ldots, r_n\}$ are the affine coordinates of the structure induced by α. This identification can be shown as follows.

First of all, also B is parallelizable and therefore T^*B is globally a product $B \times \mathbb{R}^n$ as well. Moreover $B \times \mathbb{R}^n$ is the quotient of $T^*G = G \ltimes_\varphi \mathbb{R}^n$ by its subgroup $\Gamma \times \{0\} \simeq \Gamma$. Thus we can define a natural action of T^*G on T^*B via

$$(g, y) \cdot (\Gamma h, \nu) = (\Gamma h g^{-1}, y + \varphi(g)\nu) \tag{3.6}$$

for $g, h \in G$ and $y, \nu \in \mathbb{R}^n$. We are tacitly identifying the $\mathbb{R}^n$ factors.

Lemma 3.2.1. *The map defined in (3.6) is indeed an action. Moreover it is transitive.*

Proof. Clearly $(e, 0) \cdot (\Gamma h, \nu) = (\Gamma h, \nu)$. We check it is an action

$$
\begin{aligned}
(g, y) \cdot \Big((g', y') \cdot (\Gamma h, \nu)\Big) &= (g, y) \cdot (\Gamma h g'^{-1}, y' + \varphi(g')\nu) \\
&= (\Gamma h g'^{-1} g^{-1}, y + \varphi(g) y' + \varphi(g)\varphi(g')\nu) \\
&= (\Gamma h (gg')^{-1}, y + \varphi(g) y' + \varphi(gg')\nu)
\end{aligned} \tag{3.7}
$$

while

$$
(gg', y + \varphi(g)y') \cdot (\Gamma h, \nu) = (\Gamma h(gg')^{-1}, y + \varphi(g)y + \varphi(gg')\nu) \tag{3.8}
$$

In order to show that this is transitive let $(\Gamma h, \nu)$ and $(\Gamma k, \mu)$ two different points in T^*B and just take the element in T^*G of the form $(g, y) := (k^{-1}h, \mu - \varphi(k^{-1}h)\nu)$. By plugging this in the action we get $(g, y) \cdot (\Gamma h, \nu) = (\Gamma k, \mu)$

□

Also the lattice Λ^* acts on T^*B by translations:

$$
(\Gamma h, l) \cdot (\Gamma h, v) = (\Gamma h, l + v) \tag{3.9}
$$

where $l = \sum_i^n m_i dr_i$ and the m_i's are in $\mathbb{Z}$. We claim that these two actions are indeed compatible, namely

Lemma 3.2.2. *The action 3.6 and 3.9 commute*

Proof.

$$
\begin{aligned}
(g, y) \cdot \Big((\Gamma h, l) \cdot (\Gamma h, \nu)\Big) &= (g, y) \cdot \big(\Gamma h, \nu + l\big) \\
&= (\Gamma h g^{-1}, y + \varphi(g)(\nu + l)) \\
&= (\Gamma h g^{-1}, y + \varphi(g)\nu + \varphi(g) l)
\end{aligned} \tag{3.10}
$$

while

$$
\begin{aligned}
(\Gamma h g^{-1}, l') \cdot \Big((g, y) \cdot (\Gamma h, \nu)\Big) &= (\Gamma h g^{-1}, l') \cdot \big(\Gamma h g^{-1}, y + \varphi(g)\nu\big) \\
&= (\Gamma h g^{-1}, y + \varphi(g)\nu + l') \\
&= (\Gamma h g^{-1}, y + \varphi(g)\nu + \varphi(g) l)
\end{aligned} \tag{3.11}
$$

Where in the last equality the fact that $l' = \varphi(g) l$ follows from the change of basis for covectors from the point Γh to $\Gamma h g^{-1}$. We indeed note that φ is just the $(dL_g)^*$ in the basis $dr_1, \ldots, dr_n$ (see the remark 2.1.2) .

□

Therefore the T^*G-action on TB descends on T^*B/Λ^*. Clearly we can also see from the construction, and the last observation, that the stabilizer of this action is indeed $\Gamma \ltimes_\varphi \mathbb{Z}^n$. In fact, under our initial hypothesis, for $\gamma \in \Gamma$, $\varphi(\gamma)$ takes values in $\mathrm{GL}(n, \mathbb{Z})$ and therefore $\varphi(\gamma)l$ is trivial mod Λ^*. This exhibits T^*B/Λ^* as a homogeneous space for the action of $T^*G \simeq G \ltimes_\varphi \mathbb{R}^n$ with stabilizer $T^*\Gamma = \Gamma \ltimes_{\mathfrak{f}} \mathbb{Z}^n$. Therefore $T^*G/T^*\Gamma \simeq T^*B/\Lambda^*$. We will refer to them commonly with M. Clearly, by taking the dual correspondent for each ingredient, one can obtain the same for $TG/T\Gamma$ and TB/Λ and we will refer to them commonly with $\check{M}$. A posteriori we can do also the following observation

Remark 3.2.1. *The fibration $\tilde{\pi} : T^*G \to G$ has a natural global section $\tilde{\sigma} : G \to T^*G = G \times \mathbb{R}^n$, namely its zero-section as a vector bundle. We note that $\tilde{\sigma}$ commutes with the action of $T^*\Gamma$ and therefore descends to a global section $\sigma : B \to M$. From what we have seen in the section 2.3.2, the presence of the global section allows us to identify the bundle $\pi : M \to B$ with the torus bundle $\tau : T^*B/\Lambda^* \to B$ so that $\pi : M \to B$ is a honest Lagrangian torus bundle.*

The SU(n)-structure

Starting from a solvable n-dimensional Lie group, a left-invariant affine structure on it, and the choice of a lattice, we produced a recipe to construct a compact $2n$-dimensional smooth manifolds M admitting a Lagrangian torus fibration over a compact n-dimensional smooth manifold B. How can we endow M with an SU(n)-structure? Having already a symplectic structure ω, it remains for us to define a complex n-form with the properties satisfying definition 2.4.1. We will achieve this by blending the theory of action-angle coordinates with the group structure underlying our construction. Recall by section 2.3.2, that action coordinates on B correspond to the local coordinates given by the developing map. In particular, they are global on $\tilde{B} = G$. In the same way, action-angle coordinates on M lift to global coordinates on $\tilde{M} = T^*G$. In these coordinates $\omega = \sum_{i=1}^n d\theta_i \wedge dr_i$. In other words, action-angle coordinates allow us to symplectically identify (T^*G, ω) with $(\mathbb{R}^{2n}, \omega_{\text{can}})$. We claim that ω is indeed left-invariant with respect to the group structure on T^*G.

Lemma 3.2.3. *Left multiplication map of T^*G acts by symplectomorphisms of (T^*G, ω).*

Proof. Let $h = (g, v) \in T^*G = G \times \mathbb{R}^n$. Under the identification $(T^*G, \omega) \simeq (\mathbb{R}^{2n}, \omega_{\text{can}})$ described above, the differential of the left-multiplication map $L_h : T^*G \to T^*G$

$$(g', v') \longmapsto (g, v) \cdot (g', v') = (gg', v + \varphi(g)v)$$

is represented, pointwise, by matrices of the form $\begin{pmatrix} \lambda(g) & 0 \\ 0 & \lambda^{-T}(g) \end{pmatrix}$ which are indeed symplectic. Therefore, one gets $L_h^*\omega = \omega$. □

Naively, one would be tempted to define the complex n-form as $\Omega := \bigwedge_{i=1}^{n}(d\theta_i + idr_i)$. Unfortunately, this is not left-invariant. Nevertheless, we can argue as follows. Action-angle coordinates give a global coframe $dr_1, \dots, dr_n, d\theta_1, \dots, d\theta_n$ on T^*G. We can obtain a left-invariant coframe by simply applying the differential of left-multiplication as above. Define the following 1-forms:

$$e^i = \begin{cases} L_h^* d\theta_i & \text{for} \quad i = 1, \dots, n \\ L_h^* dr_i & \text{for} \quad i = n+1, \dots, 2n \end{cases} \tag{3.12}$$

Since the symplectic form ω is left-invariant, it can be written as

$$\omega = \sum_{i=1}^{n} d\theta_i \wedge dr_i = \sum_{i=1}^{n} L_h^* d\theta_i \wedge L_h^* dr_i = \sum_{i=1}^{n} e^i \wedge e^{i+n} \tag{3.13}$$

Consequently, by defining

$$\Omega := \bigwedge_{i=1}^{n} L_h^*(d\theta_i + idr_i) = \bigwedge_{i=1}^{n}(L_h^* d\theta_i + iL_h^* dr_i) = \bigwedge_{i=1}^{n}(e^i + ie^{n+i}) \tag{3.14}$$

we obtain a left-invariant complex n-form with the desired properties.

Therefore the triple (M, ω, Ω) define a symplectic SU(n)-structure together with the structure of a Lagrangian torus fibration over the n-dimensional manifold B. In particular, the SU(n)-structure is *semi-flat*, in the sense that the defining form (ω, Ω) are semi-flat by construction.

This procedure can be dualized by taking TG and the coordinates $r_1, \dots, r_n, \check{\theta}_1, \dots, \check{\theta}_n$ introduced at the beginning of the chapter. We have already a complex structure defined by the complex n-form $\check{\Omega} = \bigwedge_{i=1}^{n}(d\check{\theta}_i + idr_i)$. Namely $\check{J}(dr_i) = d\check{\theta}_i$ and $\check{J}(d\check{\theta}_i) = -dr_i$. The Lemma 3.2.3 has its dual version: for $\check{h} = (g, \check{v})$, the left multiplication map $\check{L}_{\check{h}} : TG \to TG$

$$(g', \check{v}') \longmapsto (g, \check{v}) \cdot (g', \check{v}') = (gg', \check{v} + \lambda(g)\check{v}')$$

is represented, pointwise, by matrices of the form $\begin{pmatrix} \lambda(g) & 0 \\ 0 & \lambda(g) \end{pmatrix}$

which are complex with respect to the complex structure induced by $\check{\Omega} = \bigwedge_{i=1}^{n}(d\check{\theta}_i + idr_i)$. So we get

Lemma 3.2.4. *Left multiplication map of TG acts by biholomorphisms (with respect to the complex structure $\check{J}$ induced by $\check{\Omega}$).*

Nevertheless, in order to obtain again the left-invariance $\check{L}_{\check{h}}^* \check{\Omega} = \check{\Omega}$ the hypothesis of a special affine structure on the group G is needed since it implies that the determinant of the matrix $\operatorname{diag}(\lambda, \lambda)$ is indeed one.

We need to build the two-form $\check{\omega}$. Again the naive definition via $\sum_{i=1}^n d\check{\theta}_i \wedge dr_i$ is not left-invariant. Dually, we have a global coframe on TG given by $dr_1, \ldots, dr_n, d\check{\theta}_1, \ldots, d\check{\theta}_n$. So define the following 1-forms:

$$\check{e}^i = \begin{cases} \check{L}^*_{\check{h}} d\check{\theta}_i & \text{for} \quad i = 1, \ldots, n \\ \check{L}^*_{\check{h}} dr_i & \text{for} \quad i = n+1, \ldots, 2n \end{cases} \tag{3.15}$$

Analogously, the complex n-form is left-invariant

$$\check{\Omega} = \bigwedge_{i=1}^{n} (d\check{\theta}_i + i dr_i) = \bigwedge_{i=1}^{n} (\check{L}^*_{\check{h}} d\check{\theta}_i + i \check{L}^*_{\check{h}} dr_i) = \bigwedge_{i=1}^{n} (\check{e}^i + i e^{n+i}) \tag{3.16}$$

and, by defining

$$\check{\omega} := \sum_{i=1}^{n} \check{L}^*_{\check{h}} d\check{\theta}_i \wedge L^*_{\check{h}} dr_i = \sum_{i=1}^{n} \check{e}^i \wedge \check{e}^{n+i} \tag{3.17}$$

we obtain a left-invariant $(1,1)$-form with the desired properties. The triple $(\check{M}, \check{\omega}, \check{\Omega})$ defines a complex SU(n)-structure with a torus fibration over B dual to $(M, \omega, \Omega) \longrightarrow B$. Again, the forms $(\check{\omega}, \check{\Omega})$ are *semi-flat* by construction.

We therefore obtained a pair (M, ω, Ω) and $(\check{M}, \check{\omega}, \check{\Omega})$ of manifolds admitting SU(n)-structures which form also a (semi-flat) SYZ pair.

Remark 3.2.2. *At the linear algebraic level, the prototypical examples of symplectic and complex linear spaces are given by the direct sums $V \oplus V^*$ and $V \oplus V$ respectively which are none else that the linear approximation, at each point of our spaces. In this sense then, at the infinitesimal level, this explains the switch between symplectic and complex geometry performed by T-duality.*

We summarize all the results of this section in the following statement.

Theorem 3.2.1. *Let G a simply connected, unimodular, solvable, n-dimensional Lie group and let $\Gamma \subset G$ be a lattice and set $B := G/\Gamma$. Let α be a special affine representation of G induced by a developing map Dev and assume that the affine holonomy $\mathfrak{a} := \alpha|_\Gamma$ is integral. Denote with $\lambda, \mathfrak{l}$ their linear parts respectively and with $\varphi, \mathfrak{f}$ their inverse transpose. Set $M := G \ltimes_\varphi \mathbb{R}^n / \Gamma \ltimes_\mathfrak{f} \mathbb{Z}^n$ and $\check{M} := G \ltimes_\lambda \mathbb{R}^n / \Gamma \ltimes_\mathfrak{l} \mathbb{Z}^n$*

1. *There is a transitive action of $T^*G = G \ltimes_\varphi \mathbb{R}^n$ on T^*B/Λ with stabilizer $\Gamma \ltimes_\mathfrak{f} \mathbb{Z}^n$ which realizes T^*B/Λ^* as a homogeneous space for the solvable Lie group T^*G. Analogously for TG and TB/Λ;*

2. *$M \simeq T^*B/\Lambda^*$ and $\check{M} \simeq TB/\Lambda$ as torus-fibration. In particular M admits a symplectic structure ω such that (M, ω) is symplectomorphic to $(T^*B/\Lambda^*, \omega_{can})$ and $\check{M}$ admits a complex structure $\check{\Omega}$ such that $(\check{M}, \check{\Omega})$ is biholomorphic to $(TB/\Lambda, \check{\Omega}_{can})$;*

3. *Moreover, M admits a complex three-form Ω and $\check{M}$ admits a real $(1,1)$-form $\check{\omega}$ such that (M, ω, Ω) and $(\check{M}, \check{\omega}, \check{\Omega})$ are semi-flat, SYZ dual, supersymmetric $SU(n)$-systems. In particular $FT(e^{2\check{\omega}}) = \Omega$.*

Therefore (M, ω, Ω) and $(\check{M}, \check{\omega}, \check{\Omega})$ satisfy the theorem 3.1.2 ([61]) and the relation 3.4.

The rest of the thesis is devoted to showing concrete examples for this construction when $n = 3$.

3.3 Affine structures and representations

For what we have seen, recall in section 2.5 we have excluded $E(2)$, the only possibility for G in dimension three is one of $H_3(\mathbb{R}), E(1,1)$ or the abelian $(\mathbb{R}^3, +)$. We now describe some affine representation of G and the correspondent affine coordinates. Different choice for the developing map will lead to different affine representations.

3.3.1 Heisenberg group $H_3(\mathbb{R})$

Choose as developing map

$$
\begin{aligned}
&\mathrm{Dev} : H_3(\mathbb{R}) \longrightarrow \mathbb{R}^3 \\
&\begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\end{aligned}
\tag{3.18}
$$

For $g = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$ we compute $\alpha = \mathrm{Dev} \circ L_g \circ \mathrm{Dev}^{-1}$:

$$\begin{aligned}
\alpha(g)(v) &= \mathrm{Dev} \circ L_g \circ \mathrm{Dev}^{-1}(v) \\
&= \mathrm{Dev} \circ L_g \left(\begin{pmatrix} 1 & v_1 & v_3 \\ 0 & 0 & v_2 \\ 0 & 0 & 1 \end{pmatrix} \right) \\
&= \mathrm{Dev} \left(\begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 0 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & v_1 & v_3 \\ 0 & 1 & v_2 \\ 0 & 0 & 1 \end{pmatrix} \right) \\
&= \mathrm{Dev} \left(\begin{pmatrix} 1 & x_1 + v_1 & x_3 + v_3 + x_1 v_2 \\ 0 & 1 & x_2 + v_2 \\ 0 & 0 & 1 \end{pmatrix} \right) \\
&= \begin{pmatrix} x_1 + v_1 \\ x_2 + v_2 \\ x_3 + v_3 + x_1 v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x_1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\end{aligned} \tag{3.19}$$

Thus we obtained a homomorphism from $H_3(\mathbb{R})$ to $\mathrm{Aff}(\mathbb{R}^3) = \mathrm{GL}(3,\mathbb{R}) \ltimes \mathbb{R}^3$

$$\begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x_1 & 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) \tag{3.20}$$

Affine coordinates on G are then defined by setting

$$\begin{cases} r_1 = x_1 \\ r_2 = x_2 \\ r_3 = x_3 \end{cases} \tag{3.21}$$

If we perform the same computation in (3.19) for $\gamma \in H_3(\mathbb{Z})$ we get

$$\alpha(\gamma)(v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & n_1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \tag{3.22}$$

so that the assignment

$$\begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & n_1 & 1 \end{pmatrix}, \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \right) \tag{3.23}$$

is a well-defined homomorphism from $H_3(\mathbb{Z})$ to $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^3) = \mathrm{GL}(3,\mathbb{Z}) \ltimes \mathbb{R}^3$ (the translation part is $\mathbb{Z}^3$ indeed, but we are interested just in the linear part).

Set now for future reference

$$\begin{aligned} \lambda_{N,1} &:= \mathrm{Lin} \circ \alpha : g \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x_1 & 1 \end{pmatrix} \\ \mathfrak{l}_{N,1} &:= \mathrm{Lin} \circ \mathfrak{a} : \gamma \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & n_1 & 1 \end{pmatrix} \end{aligned} \tag{3.24}$$

Twisted developing map for $H_3(\mathbb{R})$

Take now as developing map

$$\begin{aligned} &\mathrm{Dev} : H_3(\mathbb{R}) \longrightarrow \mathbb{R}^3 \\ &\begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ \lambda x_2 \\ (\lambda - 1)x_3 + x_1 x_2 \end{pmatrix} \end{aligned} \tag{3.25}$$

with inverse is $\mathrm{Dev}^{-1} : \mathbb{R}^3 \to H_3(\mathbb{R})$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & v_1 & \frac{1}{\lambda-1}(v_3 - \frac{v_1 v_2}{\lambda}) \\ 0 & 1 & \frac{v_2}{\lambda} \\ 0 & 0 & 1 \end{pmatrix}$$

where $\lambda \in \mathbb{R} \backslash \{0, 1\}$.

By doing the same computation for (3.19):

$$
\begin{aligned}
\alpha(g)(v) &= \mathrm{Dev}\circ L_g \circ \mathrm{Dev}^{-1}(v)\\
&= \mathrm{Dev}\circ L_g\left(\begin{pmatrix}1 & v_1 & \frac{1}{\lambda-1}(v_3-\frac{v_1v_2}{\lambda})\\ 0 & 1 & \frac{v_2}{\lambda}\\ 0 & 0 & 1\end{pmatrix}\right)\\
&= \mathrm{Dev}\left(\begin{pmatrix}1 & x_1 & x_3\\ 0 & 0 & x_2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & v_1 & \frac{1}{\lambda-1}(v_3-\frac{v_1v_2}{\lambda})\\ 0 & 1 & \frac{v_2}{\lambda}\\ 0 & 0 & 1\end{pmatrix}\right)\\
&= \mathrm{Dev}\left(\begin{pmatrix}1 & x_1+v_1 & \frac{1}{\lambda-1}(v_3-\frac{v_1v_2}{\lambda})+x_3+\frac{x_1v_2}{\lambda}\\ 0 & 1 & x_2+\frac{v_2}{\lambda}\\ 0 & 0 & 1\end{pmatrix}\right)\\
&= \begin{pmatrix}x_1+v_1\\ \lambda x_2+v_2\\ (\lambda-1)x_3+v_3-\frac{v_1v_2}{\lambda}+\frac{\lambda-1}{\lambda}x_1v_2+(x_1+v_1)(x_2+\frac{v_2}{\lambda})\end{pmatrix}\\
&= \begin{pmatrix}x_1+v_1\\ \lambda x_2+v_2\\ (\lambda-1)x_3+v_3+x_1v_2+x_2v_1+x_1x_2\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ x_2 & x_1 & 1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}+\begin{pmatrix}x_1\\ \lambda x_2\\ (\lambda-1)x_3+x_1x_2\end{pmatrix}
\end{aligned} \tag{3.26}
$$

we obtain the following linear representations for $H_3(\mathbb{R})$ and $H_3(\mathbb{Z})$ respectively

$$
\lambda_{N,2} : g \longmapsto \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ x_2 & x_1 & 1\end{pmatrix} \quad \text{and} \quad \iota_{N,2} : \gamma \longmapsto \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ n_2 & n_1 & 1\end{pmatrix} \tag{3.27}
$$

and affine coordinates

$$
\begin{cases} r_1 = x_1\\ r_2 = \lambda x_2\\ r_3 = (\lambda-1)x_3 + x_1x_2 \end{cases} \tag{3.28}
$$

3.3.2 $E(1,1)$

Choose as developing map

$$
\begin{aligned}
&\mathrm{Dev} : E(1,1) \longrightarrow \mathbb{R}^3\\
&\begin{pmatrix}e^{x_1} & 0 & 0 & x_2\\ 0 & e^{-x_1} & 0 & x_3\\ 0 & 0 & 1 & x_1\\ 0 & 0 & 0 & 1\end{pmatrix} \longmapsto \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}
\end{aligned} \tag{3.29}
$$

with inverse $\mathrm{Dev}^{-1} : \mathbb{R}^3 \to E(1,1)$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \longmapsto \begin{pmatrix} e^{v_1} & 0 & 0 & v_2 \\ 0 & e^{-v_1} & 0 & v_3 \\ 0 & 0 & 1 & v_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For $g = \begin{pmatrix} e^{x_1} & 0 & 0 & x_2 \\ 0 & e^{-x_1} & 0 & x_3 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$ we compute α:

$$\begin{aligned}
\alpha(g)(v) &= \mathrm{Dev} \circ L_g \circ \mathrm{Dev}^{-1}(v) \\
&= \mathrm{Dev} \circ L_g \left(\begin{pmatrix} e^{v_1} & 0 & 0 & v_2 \\ 0 & e^{-v_1} & 0 & v_3 \\ 0 & 0 & 1 & v_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \\
&= \mathrm{Dev} \left(\begin{pmatrix} e^{x_1} & 0 & 0 & x_2 \\ 0 & e^{-x_1} & 0 & x_3 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{v_1} & 0 & 0 & v_2 \\ 0 & e^{-v_1} & 0 & v_3 \\ 0 & 0 & 1 & v_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \\
&= \mathrm{Dev} \left(\begin{pmatrix} e^{x_1+v_1} & 0 & 0 & x_2 + e^{x_1} v_2 \\ 0 & e^{-x_1-v_1} & 0 & x_3 + e^{-x_1} v_3 \\ 0 & 0 & 1 & x_1 + v_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \\
&= \begin{pmatrix} x_1 + v_1 \\ x_2 + e^{x_1} v_2 \\ x_3 + e^{-x_1} v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{x_1} & 0 \\ 0 & 0 & e^{-x_1} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\end{aligned} \tag{3.30}$$

Thus we obtained a homomorphism from $E(1,1)$ to $\mathrm{Aff}(\mathbb{R}^3) = \mathrm{GL}(3,\mathbb{R}) \ltimes \mathbb{R}^3$

$$\begin{pmatrix} e^{x_1} & 0 & 0 & x_2 \\ 0 & e^{-x_1} & 0 & x_3 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longmapsto \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{x_1} & 0 \\ 0 & 0 & e^{-x_1} \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) \tag{3.31}$$

Take now an element γ in Γ_t. It is of the form

$$\gamma = \begin{pmatrix} e^{tn_1} & 0 & 0 & n_2 + e^t n_3 \\ 0 & e^{-tn_1} & 0 & n_2 + e^{-t} n_3 \\ 0 & 0 & 1 & tn_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $n_1, n_2, n_3 \in \mathbb{Z}$ and for a fixed $t = \log \frac{3+\sqrt{5}}{2}$ (see section 2.5). If we compute again the integral affine representation $\alpha(\gamma)(v)$ we get

$$\begin{pmatrix}1 & 0 & 0\\ 0 & e^{tn_1} & 0\\ 0 & 0 & e^{-tn_1}\end{pmatrix}$$

as linear part which does not lie in $\mathrm{GL}(3,\mathbb{Z})$. Nevertheless it is conjugated to an element of $\mathrm{GL}(3,\mathbb{Z})$ as the following identity shows

$$\begin{pmatrix}1 & 0 & 0\\ 0 & e^{t} & 0\\ 0 & 0 & e^{-t}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & e^{t}\\ 0 & 1 & e^{-t}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & e^{t}\\ 0 & 1 & e^{-t}\end{pmatrix}^{-1} \tag{3.32}$$

So that

$$\begin{pmatrix}1 & 0 & 0\\ 0 & e^{n_1 t} & 0\\ 0 & 0 & e^{-n_1 t}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & e^{t}\\ 0 & 1 & e^{-t}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 3\end{pmatrix}^{n_1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & e^{t}\\ 0 & 1 & e^{-t}\end{pmatrix}^{-1} \tag{3.33}$$

Therefore, though the linear part has not integer entries, it represents an automorphism of the lattice generated by $\left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\e^{t}\\e^{-t}\end{pmatrix} \right\rangle_{\mathbb{Z}}$ inside $\mathbb{R}^3$. We will denote this lattice with $\mathbb{Z}^3_t$.

Finally set

$$\begin{aligned}\lambda_{S,1} &:= \mathrm{Lin}\circ\alpha : g \longmapsto \begin{pmatrix}1 & 0 & 0\\ 0 & e^{x_1} & 0\\ 0 & 0 & e^{-x_1}\end{pmatrix}\\ \mathfrak{l}_{S,1} &:= \mathrm{Lin}\circ\mathfrak{a} : \gamma \longmapsto \begin{pmatrix}1 & 0 & 0\\ 0 & e^{tn_1} & 0\\ 0 & 0 & e^{-tn_1}\end{pmatrix}\end{aligned} \tag{3.34}$$

while affine coordinates are then defined as:

$$\begin{cases} r_1 = x_1\\ r_2 = x_2\\ r_3 = x_3\end{cases} \tag{3.35}$$

Twisted developing map for $E(1,1)$

Take now as developing map

$$\begin{aligned} \mathrm{Dev} : E(1,1) &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} e^{x_1} & 0 & 0 & x_2 \\ 0 & e^{-x_1} & 0 & x_3 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} &\longmapsto \begin{pmatrix} x_1 + x_2x_3 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned} \tag{3.36}$$

with inverse $\mathrm{Dev}^{-1} : \mathbb{R}^3 \to E(1,1)$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \longmapsto \begin{pmatrix} e^{v_1 - v_2v_3} & 0 & 0 & v_2 \\ 0 & e^{-v_1+v_2v_3} & 0 & v_3 \\ 0 & 0 & 1 & v_1 - v_2v_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Again we compute

$$\begin{aligned} \alpha(g)(v) &= \mathrm{Dev} \circ L_g \circ \mathrm{Dev}^{-1}(v) \\ &= \mathrm{Dev} \circ L_g \left(\begin{pmatrix} e^{v_1 - v_2v_3} & 0 & 0 & v_2 \\ 0 & e^{-v_1+v_2v_3} & 0 & v_3 \\ 0 & 0 & 1 & v_1 - v_2v_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \\ &= \mathrm{Dev} \left(\begin{pmatrix} e^{x_1} & 0 & 0 & x_2 \\ 0 & e^{-x_1} & 0 & x_3 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{v_1 - v_2v_3} & 0 & 0 & v_2 \\ 0 & e^{-v_1+v_2v_3} & 0 & v_3 \\ 0 & 0 & 1 & v_1 - v_2v_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \\ &= \mathrm{Dev} \left(\begin{pmatrix} e^{x_1+v_1 - v_2v_3} & 0 & 0 & x_2 + e^{x_1}v_2 \\ 0 & e^{-x_1-v_1+v_2v_3} & 0 & x_3 + e^{-x_1}v_3 \\ 0 & 0 & 1 & x_1 + v_1 - v_2v_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} x_1 + v_1 - v_2v_3 + (x_2 + e^{x_1}v_2)(x_3 + e^{-x_1}v_3) \\ x_2 + e^{x_1}v_2 \\ x_3 + e^{-x_1}v_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + v_1 + x_2x_3 + x_2e^{-x_1}v_3 + x_3e^{x_1}v_2 \\ x_2 + e^{x_1}v_2 \\ x_3 + e^{-x_1}v_3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & x_3e^{x_1} & x_2e^{-x_1} \\ 0 & e^{x_1} & 0 \\ 0 & 0 & e^{-x_1} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} x_1 + x_2x_3 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned} \tag{3.37}$$

and we obtain a different affine representation for $E(1,1)$:

$$\begin{pmatrix} e^{x_1} & 0 & 0 & x_2 \\ 0 & e^{-x_1} & 0 & x_3 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longmapsto \left(\begin{pmatrix} 1 & x_3 e^{x_1} & x_2 e^{-x_1} \\ 0 & e^{x_1} & 0 \\ 0 & 0 & e^{-x_1} \end{pmatrix}, \begin{pmatrix} x_1 + x_2 x_3 \\ x_2 \\ x_3 \end{pmatrix} \right) \tag{3.38}$$

Take $\gamma \in \Gamma_t$. If we compute $\alpha(\gamma)(v)$ we obtain as linear part

$$\begin{pmatrix} 1 & e^{tn_1}(n_2 + e^{-t} n_3) & e^{-tn_1}(n_2 + e^{t} n_3) \\ 0 & e^{tn_1} & 0 \\ 0 & 0 & e^{-tn_1} \end{pmatrix} \tag{3.39}$$

which, again, does not lie in $\mathrm{GL}(3,\mathbb{Z})$. If we want to show this is still conjugate to an integral matrix as in (3.39), the computation is rather more cumbersome. Nevertheless, take as generators for Γ_t:

$$\gamma_1 := \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad \gamma_2 := \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad \gamma_3 := \begin{pmatrix} 1 & 0 & 0 & e^t \\ 0 & 1 & 0 & e^{-t} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.40}$$

so that

$$\lambda(\gamma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \quad , \quad \lambda(\gamma_2) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad \lambda(\gamma_3) = \begin{pmatrix} 1 & e^t & e^{-t} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.41}$$

We observe:

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & 1 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & 1 & e^{-t} \end{pmatrix}^{-1} \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & 1 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & 1 & e^{-t} \end{pmatrix}^{-1} \\ \begin{pmatrix} 1 & e^t & e^{-t} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & 1 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 3 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & 1 & e^{-t} \end{pmatrix} \end{aligned} \tag{3.42}$$

and again we can interpret a matrix of the form (3.39) as an automorphism of the lattice $\mathbb{Z}_t^3$ as in previous example.

Therefore the linear representations are:

$$\begin{aligned} \lambda_{S,2} &:= \mathrm{Lin} \circ \alpha : g \longmapsto \begin{pmatrix} 1 & x_3 e^{x_1} & x_2 e^{-x_1} \\ 0 & e^{x_1} & 0 \\ 0 & 0 & e^{-x_1} \end{pmatrix} \\ \mathfrak{l}_{S,2} &:= \mathrm{Lin} \circ \mathfrak{a} : \gamma \longmapsto \begin{pmatrix} 1 & e^{tn_1}(n_2 + e^{-t} n_3) & e^{-tn_1}(n_2 + e^{t} n_3) \\ 0 & e^{tn_1} & 0 \\ 0 & 0 & e^{-tn_1} \end{pmatrix} \end{aligned} \tag{3.43}$$

Finally, affine coordinates are defined as:

$$\begin{cases} r_1 = x_1 + x_2 x_3 \\ r_2 = x_2 \\ r_3 = x_3 \end{cases} \tag{3.44}$$

Remark 3.3.1. *It is nice to observe that the integral properties of the previous matrices are linked to the algebraic properties of the roots of the polynomial $x^2 - kx + 1$ for $x = e^t$. Indeed, the identities obtained in this section can be seen using repeatedly the identity $e^t + e^{-t} = k = 3$.*

3.3.3 $(\mathbb{R}^3, +)$

If we take as developing map the "identity map", as above in the first choice for both groups, we will obtain a trivial affine representation for the group $G = \mathbb{R}^3$. This would imply trivial linear representation and therefore trivial monodromy. Consequently, this would lead to a six-dimensional example isomorphic to a six-torus and the fibration being trivial. We will then exclude this from our analysis.

Twisted developing map for $(\mathbb{R}^3, +)$

Take $\mathbb{R}^3$ with coordinates (x_1, x_3, x_5) and choose as developing map

$$\mathrm{Dev} : \begin{pmatrix} x_1 \\ x_3 \\ x_5 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_3 \\ x_5 + x_1 x_3 \end{pmatrix} \tag{3.45}$$

with inverse

$$\mathrm{Dev}^{-1} : \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \longmapsto \begin{pmatrix} v_1 \\ v_2 \\ v_3 - v_1 v_2 \end{pmatrix} \tag{3.46}$$

We compute the representation:

$$\begin{aligned}\alpha(g)(v) =&\mathrm{Dev}\circ L_g \circ \mathrm{Dev}^{-1}(v) = \\ &\mathrm{Dev}\circ L_g\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 - v_1 v_2\end{pmatrix}\right) = \\ &\mathrm{Dev}\left(\begin{pmatrix} x_1 \\ x_3 \\ x_5\end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 - v_1 v_2\end{pmatrix}\right) \\ &\mathrm{Dev}\left(\begin{pmatrix} x_1 + v_1 \\ x_3 + v_2 \\ x_5 + v_3 - v_1 v_2\end{pmatrix}\right) = \\ &\begin{pmatrix} x_1 + v_1 \\ x_3 + v_2 \\ x_5 + v_3 + x_1 x_3 + x_1 v_2 + x_3 v_1\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_3 & x_1 & 1\end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3\end{pmatrix} + \begin{pmatrix} x_1 \\ x_3 \\ x_5 + x_1 x_3\end{pmatrix}\end{aligned} \tag{3.47}$$

And analogously for $\gamma \in \mathbb{Z}^3$. Therefore the linear representation are

$$\begin{aligned}\lambda_{\mathbb{T}} := \mathrm{Lin}\circ\alpha : g \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_3 & x_1 & 1\end{pmatrix} \\ \mathfrak{l}_{\mathbb{T}} := \mathrm{Lin}\circ\mathfrak{a} : \gamma \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ n_3 & n_1 & 1\end{pmatrix}\end{aligned} \tag{3.48}$$

and affine coordinates defined as:

$$\begin{cases} r_1 = x_1 \\ r_2 = x_2 \\ r_3 = x_3 + x_1 x_2 \end{cases} \tag{3.49}$$

3.3.4 Bonus map for $H_3(\mathbb{R})$

In this example we are going to take a particularly structure that leads to a six-dimensional pair of examples of SU(3)-manifolds which don't admit IIA/IIB structures but still satisfy the cohomological aspect of non-Kähler mirror symmetry.

Choose as developing map

$$\begin{aligned}\mathrm{Dev} : H_3(\mathbb{R}) \longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1\end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 + \frac{x_1^2}{2} \\ x_3 + \frac{x_1^3}{6}\end{pmatrix}\end{aligned} \tag{3.50}$$

with inverse $\mathrm{Dev}^{-1} : \mathbb{R}^3 \to H_3(\mathbb{R})$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & v_1 & v_3 - \frac{v_1^3}{6} \\ 0 & 1 & v_2 - \frac{v_1^2}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

We compute

$$\begin{aligned}
\alpha(g)(v) &= \mathrm{Dev} \circ L_g \circ \mathrm{Dev}^{-1}(v) \\
&= \mathrm{Dev} \circ L_g \left(\begin{pmatrix} 1 & v_1 & v_3 - \frac{v_1^3}{6} \\ 0 & 1 & v_2 - \frac{v_1^2}{2} \\ 0 & 0 & 1 \end{pmatrix} \right) \\
&= \mathrm{Dev} \left(\begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & v_1 & v_3 - \frac{v_1^3}{6} \\ 0 & 1 & v_2 - \frac{v_1^2}{2} \\ 0 & 0 & 1 \end{pmatrix} \right) \\
&= \mathrm{Dev} \left(\begin{pmatrix} 1 & x_1 + v_1 & x_3 + v_3 - \frac{v_1^3}{6} + x_1 v_2 - \frac{x_1 v_1^2}{2} \\ 0 & 1 & x_2 + v_2 - \frac{v_1^2}{2} \\ 0 & 0 & 1 \end{pmatrix} \right) \\
&= \begin{pmatrix} x_1 + v_1 \\ x_2 + v_2 - \frac{v_1^2}{2} + \frac{(x_1+v_1)^2}{2} \\ x_3 + v_3 - \frac{v_1^3}{6} + x_1 v_2 - \frac{x_1 v_1^2}{2} + \frac{(x_1+v_1)^3}{6} \end{pmatrix} \\
&= \begin{pmatrix} x_1 + v_1 \\ x_2 + v_2 + x_1 v_1 + \frac{x_1^2}{2} \\ x_3 + v_3 + x_1 v_2 + \frac{x_1^3}{6} + \frac{x_1^2 v_1}{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ \frac{x_1^2}{2} & x_1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 + \frac{x_1^2}{2} \\ x_3 + \frac{x_1^3}{6} \end{pmatrix}
\end{aligned} \tag{3.51}$$

Thus we obtained a homomorphism from $H_3(\mathbb{R})$ to $\mathrm{Aff}(\mathbb{R}^3) = \mathrm{GL}(3, \mathbb{R}) \ltimes \mathbb{R}^3$

$$\begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \left(\begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ \frac{x_1^2}{2} & x_1 & 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) \tag{3.52}$$

Doing the same also for $\gamma \in H_3(\mathbb{Z})$ we get linear representations

$$\begin{aligned} \lambda_Y &:= \mathrm{Lin} \circ \alpha : g \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ \frac{x_1^2}{2} & x_1 & 1 \end{pmatrix} \\ \mathfrak{l}_Y &:= \mathrm{Lin} \circ \mathfrak{a} : \gamma \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ n_1 & 1 & 0 \\ \frac{n_1^2}{2} & n_1 & 1 \end{pmatrix} \end{aligned} \tag{3.53}$$

and affine coordinates:

$$\begin{cases} r_1 = x_1 \\ r_2 = x_2 + \frac{x_1^2}{2} \\ r_3 = x_3 + \frac{x_1^3}{6} \end{cases} \tag{3.54}$$

3.3.5 Recap for linear representations

We recollect here the linear representations of the groups $H_3(\mathbb{R}), E(1,1), (\mathbb{R}^3,+)$ that will be used to construct the six-dimensional Lie groups.

	$\mathrm{Dev}(g)$	λ	φ
N_1	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -x_1 \\ 0 & 0 & 1 \end{pmatrix}$
N_2	$\begin{pmatrix} x_1 \\ \lambda x_2 \\ (\lambda-1)x_3 + x_1x_2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & x_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -x_2 \\ 0 & 1 & -x_1 \\ 0 & 0 & 1 \end{pmatrix}$
S_1	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{x_1} & 0 \\ 0 & 0 & e^{-x_1} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-x_1} & 0 \\ 0 & 0 & e^{x_1} \end{pmatrix}$
S_2	$\begin{pmatrix} x_1 + x_2x_3 \\ x_2 \\ x_3 \end{pmatrix}$	$\begin{pmatrix} 1 & e^{x_1}x_3 & e^{-x_1}x_2 \\ 0 & e^{x_1} & 0 \\ 0 & 0 & e^{-x_1} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ -x_3 & e^{-x_1} & 0 \\ -x_2 & 0 & e^{x_1} \end{pmatrix}$
$\mathbb{T}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 + x_1x_2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & x_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -x_2 \\ 0 & 1 & -x_1 \\ 0 & 0 & 1 \end{pmatrix}$
Y	$\begin{pmatrix} x_1 \\ x_2 + \frac{x_1^2}{2} \\ x_3 + \frac{x_1^3}{6} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ \frac{x_1^2}{2} & x_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -x_1 & \frac{x_1^2}{2} \\ 0 & 1 & -x_1 \\ 0 & 0 & 1 \end{pmatrix}$

Table 3.1: Linear representation associated to affine structures

3.4 From 3d to 6d

In this section we build the six-dimensional Lie groups $G \ltimes_\rho \mathbb{R}^3$ where the action ρ is given by one of the preceding linear representations. We describe the group law and its Lie (co)algebra. Also we relate the algebras obtained with the ones from the various classifications.

3.4.1 $G_{N,1} := H_3(\mathbb{R}) \ltimes_{\varphi^{N,1}} \mathbb{R}^3$

Group law reads

$$\left(\begin{pmatrix}1 & x_1 & x_3\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}\right)\left(\begin{pmatrix}1 & x_1' & x_3'\\ 0 & 1 & x_2'\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}v_1'\\ v_2'\\ v_3'\end{pmatrix}\right) = \left(\begin{pmatrix}1 & x_1+x_1' & x_3+x_3'+x_1x_2'\\ 0 & 1 & x_2+x_2'\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}v_1+v_1'\\ v_2+v_2'-x_1v_3'\\ v_3+v_3'\end{pmatrix}\right) \tag{3.55}$$

Rewrite this as

$$(x_1+x_1', x_2+x_2', x_3+x_3', x_4+x_4', x_5+x_5'+x_1x_2', x_6+x_6'-x_1x_3') \tag{3.56}$$

We compute the derivative of the new left multiplication

$$\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & x_1 & 0 & 0 & 1 & 0\\ 0 & 0 & -x_1 & 0 & 0 & 1\end{pmatrix}. \tag{3.57}$$

from which we get the following basis of left-invariant vector fields

$$\begin{aligned} E_1 &= \frac{\partial}{\partial x_1} \quad , \quad E_2 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_5} \quad , \quad E_3 = \frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial x_6},\\ E_4 &= \frac{\partial}{\partial x_4} \quad , \quad E_5 = \frac{\partial}{\partial x_5} \quad , \quad E_6 = \frac{\partial}{\partial x_6} \end{aligned} \tag{3.58}$$

The only non trivial brackets are

$$[E_1, E_2] = E_5 \quad \text{and} \quad [E_1, E_3] = -E_6 \tag{3.59}$$

and dually

$$\begin{aligned} e^1 = dx_1 \quad &, \quad e^2 = dx_2 \quad , \quad e^3 = dx_3, \\ e^4 = dx_4 \quad &, \quad e^5 = dx_5 - x_1 dx_2 \quad , \quad de^6 = dx_6 + x_1 dx_3. \end{aligned} \tag{3.60}$$

$$de^5 = -dx_1 \wedge dx_2 = -e^{12} \quad \text{and} \quad de^6 = dx_1 \wedge dx_3 = e^{13} \tag{3.61}$$

After the linear change $x_3 \mapsto -x_3$, this Lie algebra corresponds to $\mathfrak{g}_{5,1} \oplus \mathbb{R} = (0,0,0,0,12,13)$ in [30]. We will denote it with $\mathfrak{g}_{N,1}$.

3.4.2 $\check{G}_{N,1} := H_3(\mathbb{R}) \ltimes_{\lambda_{N,1}} \mathbb{R}^3$

Group law reads

$$\left(\begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & x_1' & x_3' \\ 0 & 1 & x_2' \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} \right) = \left(\begin{pmatrix} 1 & x_1 + x_1' & x_3 + x_3' + x_1 x_2' \\ 0 & 1 & x_2 + x_2' \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 + v_1' \\ v_2 + v_2' \\ v_3 + v_3' + x_1 v_2' \end{pmatrix} \right) \tag{3.62}$$

Rewrite this as

$$(x_1 + x_1', x_2 + x_2', x_3 + x_3', x_4 + x_4', x_5 + x_5' + x_1 x_2', x_6 + x_6' + x_1 x_3') \tag{3.63}$$

We compute the derivative of the new left multiplication

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & x_1 & 0 & 0 & 1 \end{pmatrix}. \tag{3.64}$$

from which we get the following basis of left-invariant vector fields

$$\begin{aligned} E_1 = \frac{\partial}{\partial x_1} \quad &, \quad E_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_5} \quad , \quad E_3 = \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_6}, \\ E_4 = \frac{\partial}{\partial x_4} \quad &, \quad E_5 = \frac{\partial}{\partial x_5} \quad , \quad E_6 = \frac{\partial}{\partial x_6} \end{aligned} \tag{3.65}$$

The only non trivial brackets are

$$[E_1, E_2] = E_5 \quad \text{and} \quad [E_1, E_3] = E_6 \tag{3.66}$$

and dually

$$\begin{aligned} &e^1 = dx_1 \quad , \quad e^2 = dx_2 \quad , \quad e^3 = dx_3, \\ &e^4 = dx_4 \quad , \quad e^5 = dx_5 - x_1 dx_2 \quad , \quad de^6 = dx_6 - x_1 dx_3. \end{aligned} \tag{3.67}$$

$$de^5 = -dx_1 \wedge dx_2 = -e^{12} \quad \text{and} \quad de^6 = dx_1 \wedge dx_3 = -e^{13} \tag{3.68}$$

which again corresponds to $\mathfrak{g}_{5,1} \oplus \mathbb{R} = (0,0,0,0,12,13)$ in [30]. We will denote it with $\check{\mathfrak{g}}_{N,1}$.

Clearly, $G_{N,1}$ and $\check{G}_{N,1}$ ($\mathfrak{g}_{N,1}$ and $\check{\mathfrak{g}}_{N,1}$) are isomorphic as Lie groups (algebras).

3.4.3 $G_{N,2} := H_3(\mathbb{R}) \ltimes_{\varphi^{N,2}} \mathbb{R}^3$

Group law reads

$$\left(\begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & x_1' & x_3' \\ 0 & 1 & x_2' \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} \right) = \\ \left(\begin{pmatrix} 1 & x_1 + x_1' & x_3 + x_3' + x_1 x_2' \\ 0 & 1 & x_2 + x_2' \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 + v_1' - x_2 v_3' \\ v_2 + v_2' - x_1 v_3' \\ v_3 + v_3' \end{pmatrix} \right) \tag{3.69}$$

Rewrite this as

$$(x_1 + x_1', x_2 + x_2', x_3 + x_3', x_4 + x_4' + x_1 x_2', x_5 + x_5' - x_1 x_3', x_6 + x_6' - x_2 x_3') \tag{3.70}$$

We compute the derivative of the new left multiplication

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -x_1 & 0 & 1 & 0 \\ 0 & 0 & -x_2 & 0 & 0 & 1 \end{pmatrix}. \tag{3.71}$$

from which we get the following basis of left-invariant vector fields

$$\begin{aligned} &E_1 = \frac{\partial}{\partial x_1} \quad , \quad E_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_4} \quad , \quad E_3 = \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_5} - x_2 \frac{\partial}{\partial x_6}, \\ &E_4 = \frac{\partial}{\partial x_4} \quad , \quad E_5 = \frac{\partial}{\partial x_5} \quad , \quad E_6 = \frac{\partial}{\partial x_6} \end{aligned} \tag{3.72}$$

The only non trivial brackets are

$$[E_1, E_2] = E_4 \quad , \quad [E_1, E_3] = -E_5 \quad , \quad [E_2, E_3] = -E_6 \tag{3.73}$$

and dually

$$\begin{aligned} &e^1 = dx_1 \quad , \quad e^2 = dx_2 \quad , \quad e^3 = dx_3, \\ &e^4 = dx_4 - x_1 dx_2 \quad , \quad e^5 = dx_5 + x_1 dx_3 \quad , \quad de^6 = dx_6 + x_2 dx_3. \end{aligned} \tag{3.74}$$

$$de^4 = -dx_1 \wedge dx_2 = -e^{12} \quad , \quad de^5 = dx_1 \wedge dx_3 = e^{13} \quad , \quad de^6 = dx_2 \wedge dx_3 = e^{23} \tag{3.75}$$

After the same linear change $x_3 \mapsto -x_3$, this Lie algebra corresponds to $\mathfrak{g}_{6,N3} = (0,0,0,12,13,23)$ in [30]. We will denote it with $\mathfrak{g}_{N,2}$.

3.4.4 $\check{G}_{N,2} := H_3(\mathbb{R}) \ltimes_{\lambda_{N,2}} \mathbb{R}^3$

Group law reads

$$\left(\begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & x'_1 & x'_3 \\ 0 & 1 & x'_2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v'_1 \\ v'_2 \\ v'_3 \end{pmatrix} \right) = \\ \left(\begin{pmatrix} 1 & x_1 + x'_1 & x_3 + x'_3 + x_1 x'_2 \\ 0 & 1 & x_2 + x'_2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 + v'_1 \\ v_2 + v'_2 \\ v_3 + v'_3 + x_2 v'_1 + x_1 v'_2 \end{pmatrix} \right) \tag{3.76}$$

Rewrite this as

$$(x_1 + x'_1, x_2 + x'_2, x_3 + x'_3, x_4 + x'_4, x_5 + x'_5 + x_1 x'_2, x_6 + x'_6 + x_1 x'_4 + x_2 x'_3) \tag{3.77}$$

We compute the derivative of the new left multiplication

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & x_2 & x_1 & 0 & 1 \end{pmatrix}. \tag{3.78}$$

from which we get the following basis of left-invariant vector fields

$$\begin{aligned} &E_1 = \frac{\partial}{\partial x_1} \quad , \quad E_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_5} \quad , \quad E_3 = \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_6}, \\ &E_4 = \frac{\partial}{\partial x_4} + x_1 \frac{\partial}{\partial x_6} \quad , \quad E_5 = \frac{\partial}{\partial x_5} \quad , \quad E_6 = \frac{\partial}{\partial x_6} \end{aligned} \tag{3.79}$$

The only non trivial brackets are

$$[E_1, E_2] = E_5 \quad , \quad [E_1, E_4] = E_6 \quad , \quad [E_2, E_3] = E_6 \tag{3.80}$$

and dually

$$\begin{array}{lll} e^1 = dx_1 \quad , & e^2 = dx_2 \quad , & e^3 = dx_3, \\ e^4 = dx_4 \quad , & e^5 = dx_5 - x_1 dx_2 \quad , & de^6 = dx_6 - x_1 dx_4 - x_2 dx_3. \end{array} \tag{3.81}$$

$$de^5 = -dx_1 \wedge dx_2 = -e^{12} \quad \text{and} \quad de^6 = -dx_1 \wedge dx_4 - dx_2 \wedge dx_3 = -e^{14} - e^{23} \tag{3.82}$$

which corresponds to $\mathfrak{h}_4 = (0,0,0,0,12,14+23)$ in [60]. We will denote it with $\breve{\mathfrak{g}}_{N,2}$.

3.4.5 $G_{S,1} := E(1,1) \ltimes_{\varphi^{S,1}} \mathbb{R}^3$

Group law reads

$$\left(\begin{pmatrix} e^{x_1} & 0 & 0 & x_2 \\ 0 & e^{-x_1} & 0 & x_3 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right) \left(\begin{pmatrix} e^{x_1'} & 0 & 0 & x_2' \\ 0 & e^{-x_1'} & 0 & x_3' \\ 0 & 0 & 1 & x_1' \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} \right) =$$
$$\left(\begin{pmatrix} e^{x_1+x_1'} & 0 & 0 & x_2 + e^{x_1} x_2' \\ 0 & e^{-(x_1+x_1')} & 0 & x_3 + e^{-x_1} x_3' \\ 0 & 0 & 1 & x_1 + x_1' \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 + v_1' \\ v_2 + e^{-x_1} v_2' \\ v_3 + e^{x_1} v_3' \end{pmatrix} \right) \tag{3.83}$$

Rewrite this as

$$(x_1 + e^{-x_5} x_1', x_2 + e^{x_5} x_2', x_3 + e^{x_5} x_3', x_4 + e^{-x_5} x_4', x_5 + x_5', x_6 + x_6') \tag{3.84}$$

The derivative of the new left multiplication is

$$\begin{pmatrix} e^{-x_5} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{x_5} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{x_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-x_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.85}$$

which gives the following basis of left-invariant vector fields

$$\begin{aligned} E_1 = e^{-x_5}\frac{\partial}{\partial x_1} \quad &, \quad E_2 = e^{x_5}\frac{\partial}{\partial x_2} \quad , \quad E_3 = e^{x_5}\frac{\partial}{\partial x_3}, \\ E_4 = e^{-x_5}\frac{\partial}{\partial x_4} \quad &, \quad E_5 = \frac{\partial}{\partial x_5} \quad , \quad E_6 = \frac{\partial}{\partial x_6} \end{aligned} \tag{3.86}$$

The only non-trivial brackets are

$$\begin{aligned} [E_1, E_5] = e^{-x_5}\frac{\partial}{\partial x_1} = E_1 \quad &, \quad [E_2, E_5] = -e^{x_5}\frac{\partial}{\partial x_2} = -E_2 \\ [E_3, E_5] = -e^{x_5}\frac{\partial}{\partial x_3} = -E_3 \quad &, \quad [E_4, E_5] = e^{-x_5}\frac{\partial}{\partial x_4} = E_4 \end{aligned} \tag{3.87}$$

dually we obtain a basis of left-invariant 1-forms

$$\begin{aligned} e^1 = e^{x_5}dx_1 \quad &, \quad e^2 = e^{-x_5}dx_2 \quad , \quad e^3 = e^{-x_5}dx_3 \\ e^4 = e^{x_5}dx_4 \quad &, \quad e^5 = dx_5 \quad , \quad e^6 = dx_6 \end{aligned} \tag{3.88}$$

with

$$\begin{aligned} de^1 = -e^{15} \quad &, \quad de^2 = e^{25} \quad , \quad de^3 = e^{35} \\ de^4 = -e^{45} \quad &, \quad de^5 = 0 \quad , \quad de^6 = 0 \end{aligned} \tag{3.89}$$

Such a Lie algebra corresponds to $A_{5,7}^{-1,-1,1} \oplus \mathbb{R} = (15, -25, -35, 45, 0, 0)$ in [30]. We denote it with $\mathfrak{g}_{S,1}$

3.4.6 $\check{G}_{S,1} := E(1,1) \ltimes_{\lambda_{S,1}} \mathbb{R}^3$

Group law reads

$$\left(\begin{pmatrix} e^{x_1} & 0 & 0 & x_2 \\ 0 & e^{-x_1} & 0 & x_3 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right) \left(\begin{pmatrix} e^{x_1'} & 0 & 0 & x_2' \\ 0 & e^{-x_1'} & 0 & x_3' \\ 0 & 0 & 1 & x_1' \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} \right) = \left(\begin{pmatrix} e^{x_1+x_1'} & 0 & 0 & x_2 + e^{x_1}x_2' \\ 0 & e^{-(x_1+x_1')} & 0 & x_3 + e^{-x_1}x_3' \\ 0 & 0 & 1 & x_1 + x_1' \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 + v_1' \\ v_2 + e^{x_1}v_2' \\ v_3 + e^{-x_1}v_3' \end{pmatrix} \right) \tag{3.90}$$

which has the same group law of $G_{S,1}$

$$(x_1 + e^{x_5}x_1', x_2 + e^{x_5}x_2', x_3 + e^{-x_5}x_3', x_4 + e^{x_5}x_4', x_5 + x_5', x_6 + x_6') \tag{3.91}$$

and therefore gives the same Lie algebra $A_{5,7}^{-1,-1,1} \oplus \mathbb{R} = (15, -25, -35, 45, 0, 0)$. We refer to it with $\check{\mathfrak{g}}_B^{S,1}$: even if it is the same, we do this in order to distinguish the two sides. Clearly $G_{S,1}$ and $\check{G}_{S,1}$ ($\mathfrak{g}_{S,1}$ and $\check{\mathfrak{g}}_{S,1}$) are isomorphic as Lie groups (algebras).

3.4.7 $G_{S,2} := E(1,1) \ltimes_{\varphi^{S,2}} \mathbb{R}^3$

Group law reads

$$\left(\begin{pmatrix} e^{x_1} & 0 & 0 & x_2 \\ 0 & e^{-x_1} & 0 & x_3 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right)\left(\begin{pmatrix} e^{x_1'} & 0 & 0 & x_2' \\ 0 & e^{-x_1'} & 0 & x_3' \\ 0 & 0 & 1 & x_1' \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix}\right) =$$
$$\left(\begin{pmatrix} e^{x_1+x_1'} & 0 & 0 & x_2 + e^{x_1}x_2' \\ 0 & e^{-(x_1+x_1')} & 0 & x_3 + e^{-x_1}x_3' \\ 0 & 0 & 1 & x_1 + x_1' \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 + v_1' \\ v_2 + e^{x_1}v_2' - x_3v_1' \\ v_3 + e^{-x_1}v_3' - x_2v_1' \end{pmatrix}\right) \tag{3.92}$$

Rewrite this as

$$(x_1 + e^{-x_6}x_1' - x_3x_5', x_2 + e^{x_6}x_2' - x_4x_5', x_3 + e^{-x_6}x_3', x_4 + e^{x_6}x_4', x_5 + x_5', x_6 + x_6') \tag{3.93}$$

The derivative of the new left multiplication is

$$\begin{pmatrix} e^{-x_6} & 0 & 0 & 0 & -x_3 & 0 \\ 0 & e^{x_6} & 0 & 0 & -x_4 & 0 \\ 0 & 0 & e^{-x_6} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{x_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.94}$$

which gives the following basis of left-invariant vector fields

$$\begin{aligned} &E_1 = e^{-x_6}\frac{\partial}{\partial x_1} \quad , \quad E_2 = e^{x_6}\frac{\partial}{\partial x_2} \quad , \quad E_3 = e^{-x_6}\frac{\partial}{\partial x_3}, \\ &E_4 = e^{x_6}\frac{\partial}{\partial x_4} \quad , \quad E_5 = \frac{\partial}{\partial x_5} - x_3\frac{\partial}{\partial x_1} - x_4\frac{\partial}{\partial x_2} \quad , \quad E_6 = \frac{\partial}{\partial x_6} \end{aligned} \tag{3.95}$$

The only non-trivial brackets are

$$\begin{aligned}
&[E_1,E_6]=e^{-x_6}\frac{\partial}{\partial x_1}=E_1 \quad , \quad [E_3,E_5]=e^{-x_6}\frac{\partial}{\partial x_1}=E_1\\
&[E_2,E_6]=-e^{x_6}\frac{\partial}{\partial x_2}=-E_2 \quad , \quad [E_4,E_5]=e^{x_6}\frac{\partial}{\partial x_2}=E_2\\
&[E_3,E_6]=e^{-x_6}\frac{\partial}{\partial x_3}=E_3 \quad , \quad [E_4,E_6]=-e^{x_6}\frac{\partial}{\partial x_4}=-E_4
\end{aligned} \tag{3.96}$$

dually we obtain a basis of left-invariant 1-forms

$$\begin{aligned}
&e^1=e^{x_6}dx_1+x_3e^{x_6}dx_5 \quad , \quad e^2=e^{-x_6}dx_2+x_4e^{-x_6}dx_5 \quad , \quad e^3=e^{x_6}dx_3\\
&e^4=e^{-x_6}dx_4 \quad , \quad e^5=dx_5 \quad , \quad e^6=dx_6
\end{aligned} \tag{3.97}$$

with

$$\begin{aligned}
&de^1=-e^{16}-e^{35} \quad , \quad de^2=e^{26}-e^{45} \quad , \quad de^3=-e^{36}\\
&de^4=e^{46} \quad , \quad de^5=0 \quad , \quad de^6=0
\end{aligned} \tag{3.98}$$

After the linear change $x_5 \mapsto -x_5$, this Lie algebra corresponds to $\mathfrak{g}_{6,54}^{0,-1} = (16+35,-26+45,36,-46,0,0)$ in [30]. We will denote it with $\mathfrak{g}_{S,2}$.

3.4.8 $\check{G}_{S,2} := E(1,1) \ltimes_{\lambda_{S,2}} \mathbb{R}^3$

Group law reads

$$\begin{aligned}
&\left(\begin{pmatrix} e^{x_1} & 0 & 0 & x_2\\ 0 & e^{-x_1} & 0 & x_3\\ 0 & 0 & 1 & x_1\\ 0 & 0 & 0 & 1\end{pmatrix}, \begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix}\right)\left(\begin{pmatrix} e^{x_1'} & 0 & 0 & x_2'\\ 0 & e^{-x_1'} & 0 & x_3'\\ 0 & 0 & 1 & x_1'\\ 0 & 0 & 0 & 1\end{pmatrix}, \begin{pmatrix} v_1'\\ v_2'\\ v_3'\end{pmatrix}\right)=\\
&\left(\begin{pmatrix} e^{x_1+x_1'} & 0 & 0 & x_2+e^{x_1}x_2'\\ 0 & e^{-(x_1+x_1')} & 0 & x_3+e^{-x_1}x_3'\\ 0 & 0 & 1 & x_1+x_1'\\ 0 & 0 & 0 & 1\end{pmatrix}, \begin{pmatrix} v_1+v_1'+x_3e^{x_1}v_2'+x_2e^{-x_1}v_3'\\ v_2+e^{x_1}v_2'\\ v_3+e^{-x_1}v_3'\end{pmatrix}\right)
\end{aligned} \tag{3.99}$$

$$\begin{aligned}
(x_1,\dots,x_6)(x_1',\dots,x_6') = (&x_1+x_1'+x_4e^{-x_6}x_2'+x_3e^{x_6}x_5', x_2+e^{-x_6}x_2',\\
&x_3+e^{-x_6}x_3', x_4+e^{x_6}x_4', x_5+e^{x_6}x_5', x_6+x_6')
\end{aligned} \tag{3.100}$$

The derivative of the new left multiplication is

$$\begin{pmatrix} 1 & x_4e^{-x_6} & 0 & 0 & x_3e^{x_6} & 0 \\ 0 & e^{-x_6} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-x_6} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{x_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{x_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.101}$$

which gives the following basis of left-invariant vector fields

$$\begin{aligned} &E_1 = \frac{\partial}{\partial x_1} \quad , \quad E_2 = x_4e^{-x_6}\frac{\partial}{\partial x_1} + e^{-x_6}\frac{\partial}{\partial x_2} \quad , \quad E_3 = e^{-x_6}\frac{\partial}{\partial x_3} \\ &E_4 = e^{x_6}\frac{\partial}{\partial x_4} \quad , \quad E_5 = x_3e^{x_6}\frac{\partial}{\partial x_1} + e^{x_6}\frac{\partial}{\partial x_5} \quad , \quad E_6 = \frac{\partial}{\partial x_6} \end{aligned} \tag{3.102}$$

The only non-trivial brackets are

$$\begin{aligned} &[E_2, E_6] = x_4e^{-x_6}\frac{\partial}{\partial x_1} + e^{-x_6}\frac{\partial}{\partial x_2} = E_2 \quad , \quad [E_2, E_4] = -\frac{\partial}{\partial x_1} = -E_1 \\ &[E_3, E_6] = e^{-x_6}\frac{\partial}{\partial x_3} = E_3 \quad , \quad [E_3, E_5] = \frac{\partial}{\partial x_1} = E_1 \\ &[E_4, E_6] = -e^{x_6}\frac{\partial}{\partial x_4} = -E_4 \quad , \quad [E_5, E_6] = -x_3e^{x_6}\frac{\partial}{\partial x_1} - e^{x_6}\frac{\partial}{\partial x_5} = -E_5 \end{aligned} \tag{3.103}$$

dually we obtain a basis of left-invariant 1-forms

$$\begin{aligned} &e^1 = dx_1 - x_4dx_2 - x_3dx_5 \quad , \quad e^2 = e^{x_6}dx_2 \quad , \quad e^3 = e^{x_6}dx_3 \\ &e^4 = e^{-x_6}dx_4 \quad , \quad e^5 = e^{-x_6}dx_5 \quad , \quad e^6 = dx_6 \end{aligned} \tag{3.104}$$

with

$$\begin{aligned} &de^1 = e^{24} - e^{35} \quad , \quad de^2 = -e^{26} \quad , \quad de^3 = -e^{36} \\ &de^4 = e^{46} \quad , \quad de^5 = e^{56} \quad , \quad de^6 = 0 \end{aligned} \tag{3.105}$$

This Lie algebra corresponds to $\mathfrak{g}_5 = (24 + 35, 26, 36, -46, -56, 0)$ in [32]. We will denote it with $\check{\mathfrak{g}}_{S,2}$.

3.4.9 $G_{\mathbb{T}} := \mathbb{R}^3 \ltimes_{\varphi^{\mathbb{T}}} \mathbb{R}^3$

The resulting six-dimensional Lie group has group law

$$(x_1 + x_1', x_2 + x_2' - x_1x_6', x_3 + x_3', x_4 + x_4' - x_3x_6', x_5 + x_5', x_6 + x_6') \tag{3.106}$$

The differential of left multiplication is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -x_3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.107}$$

which gives the following basis of left-invariant vector fields

$$\begin{array}{lll} E_1 = \dfrac{\partial}{\partial x_1} \quad , & , E_2 = \dfrac{\partial}{\partial x_2} \quad , & E_3 = \dfrac{\partial}{\partial x_3} \\ E_4 = \dfrac{\partial}{\partial x_4} & E_5 = \dfrac{\partial}{\partial x_5} & E_6 = \dfrac{\partial}{\partial x_6} - x_1 \dfrac{\partial}{\partial x_2} - x_3 \dfrac{\partial}{\partial x_4} \end{array} \tag{3.108}$$

The only non-trivial brackets are

$$[E_1, E_6] = -E_2 \qquad [E_3, E_6] = -E_4 \tag{3.109}$$

and dually

$$\begin{array}{lll} e^1 = dx_1 \quad , & e^2 = dx_2 + x_1 dx_6 \quad , & e^3 = dx_3 \\ e^4 = dx_4 + x_3 dx_6 \quad , & e^5 = dx_5 \quad , & e^6 = dx_6 \end{array} \tag{3.110}$$

with

$$\begin{array}{lll} de^1 = 0 \quad , & de^2 = e^{16} \quad , & de^3 = 0 \\ de^4 = e^{36} \quad , & de^5 = 0 \quad , & de^6 = 0 \end{array} \tag{3.111}$$

The algebra obtained, up to reordering, is isomorphic to $\mathfrak{g}_{N,1}$ but we denote with $\mathfrak{g}_{\mathbb{T}} = (0, -16, 0, -36, 0, 0)$.

3.4.10 $\check{G}_{\mathbb{T}} := \mathbb{R}^3 \ltimes_{\lambda_{\mathbb{T}}} \mathbb{R}^3$

The resulting six-dimensional Lie group has group law

$$(x_1 + x_1', x_2 + x_2', x_3 + x_3', x_4 + x_4', x_5 + x_5', x_6 + x_6' + x_1 x_2' + x_3 x_4') \tag{3.112}$$

The differential of left multiplication is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & x_1 & 0 & x_3 & 0 & 1 \end{pmatrix} \tag{3.113}$$

which gives the following basis of left-invariant vector fields

$$\begin{array}{lll} E_1 = \dfrac{\partial}{\partial x_1} \quad , & E_2 = \dfrac{\partial}{\partial x_2} + x_1 \dfrac{\partial}{\partial x_6} \quad , & E_3 = \dfrac{\partial}{\partial x_3} \\ E_4 = \dfrac{\partial}{\partial x_4} + x_3 \dfrac{\partial}{\partial x_6} & E_5 = \dfrac{\partial}{\partial x_5} & E_6 = \dfrac{\partial}{\partial x_6} \end{array} \tag{3.114}$$

The only non-trivial brackets are

$$[E_1, E_2] = E_6 \qquad [E_3, E_4] = E_6 \tag{3.115}$$

and dually

$$\begin{array}{lll} e^1 = dx_1 \quad , & e^2 = dx_2 \quad , & e^3 = dx_3 \\ e^4 = dx_4 \quad , & e^5 = dx_5 \quad , & e^6 = dx_6 - x_1 dx_2 - x_3 dx_4 \end{array} \tag{3.116}$$

with

$$\begin{array}{lll} de^1 = 0 \quad , & de^2 = 0 \quad , & de^3 = 0 \\ de^4 = 0 \quad , & de^5 = 0 \quad , & de^6 = -e^{12} - e^{34} \end{array} \tag{3.117}$$

The algebra obtained is isomorphic to $\mathfrak{h}_3 = (0, 0, 0, 0, 0, 12 + 34)$ in [60] but we denote with $\breve{\mathfrak{g}}_{\mathbb{T}} = (0, 0, 0, 0, 0, 12 + 34)$.

3.4.11 $G_Y := H_3(\mathbb{R}) \ltimes_{\varphi^Y} \mathbb{R}^3$

Group law reads

$$\left(\begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & x_1' & x_3' \\ 0 & 1 & x_2' \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} \right) = \\ \left(\begin{pmatrix} 1 & x_1 + x_1' & x_3 + x_3' + x_1 x_2' \\ 0 & 1 & x_2 + x_2' \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 + v_1' - x_1 v_2' + \frac{x_1^2}{2} v_3' \\ v_2 + v_2' - x_1 v_3' \\ v_3 + v_3' \end{pmatrix} \right) \tag{3.118}$$

Rewrite this as

$$(x_1 + x_1', x_2 + x_2', x_3 + x_3', x_4 + x_4' + x_1 x_2', x_5 + x_5' - x_1 x_3', x_6 + x_6' - x_1 x_5' + \frac{x_1^2}{2} x_3') \tag{3.119}$$

We compute the derivative of the new left multiplication

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -x_1 & 0 & 1 & 0 \\ 0 & 0 & \frac{x_1^2}{2} & 0 & -x_1 & 1 \end{pmatrix}. \tag{3.120}$$

from which we get the following basis of left-invariant vector fields

$$\begin{aligned} &E_1 = \frac{\partial}{\partial x_1} \quad , \quad E_2 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_4} \quad , \quad E_3 = \frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial x_5} + \frac{x_1^2}{2}\frac{\partial}{\partial x_6}, \\ &E_4 = \frac{\partial}{\partial x_4} \quad , \quad E_5 = \frac{\partial}{\partial x_5} - x_1\frac{\partial}{\partial x_6} \quad , \quad E_6 = \frac{\partial}{\partial x_6} \end{aligned} \tag{3.121}$$

The only non trivial brackets are

$$[E_1, E_2] = E_4 \quad , \quad [E_1, E_3] = -E_5 \quad , \quad [E_1, E_5] = -E_6 \tag{3.122}$$

and dually

$$\begin{aligned} &e^1 = dx_1 \quad , \quad e^2 = dx_2 \quad , \quad e^3 = dx_3, \\ &e^4 = dx_4 - x_1 dx_2 \quad , \quad e^5 = dx_5 + x_1 dx_3 \quad , \quad e^6 = dx_6 + x_1 dx_5 + \frac{x_1^2}{2} dx_3. \end{aligned} \tag{3.123}$$

$$de^4 = -dx_1 \wedge dx_2 = -e^{12} \quad , \quad de^5 = dx_1 \wedge dx_3 = e^{13} \quad , \quad de^6 = dx_1 \wedge (dx_5 + x_1 dx_3) = e^{15} \tag{3.124}$$

Up to linear changes, this Lie algebra corresponds to $\mathfrak{h}_{10} = (0, 0, 0, 12, 13, 14)$ in [60]. We will denote it with $\mathfrak{g}_Y$.

3.4.12 $\check{G}_Y := H_3(\mathbb{R}) \ltimes_{\lambda_Y} \mathbb{R}^3$

Group law reads

$$\begin{aligned} &\left(\begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & x_1' & x_3' \\ 0 & 1 & x_2' \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} \right) = \\ &\qquad \left(\begin{pmatrix} 1 & x_1 + x_1' & x_3 + x_3' + x_1 x_2' \\ 0 & 1 & x_2 + x_2' \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v_1 + v_1' \\ v_2 + v_2' + x_1 v_1' \\ v_3 + v_3' + x_1 v_2' + \frac{x_1^2}{2} v_1' \end{pmatrix} \right) \end{aligned} \tag{3.125}$$

Rewrite this as

$$(x_1+x_1', x_2+x_2', x_3+x_3', x_4+x_4'+x_1x_2', x_5+x_5'+x_1x_3', x_6+x_6'+x_1x_5'+\frac{x_1^2}{2}x_3') \quad (3.126)$$

We compute the derivative of the new left multiplication

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 1 & 0 \\ 0 & 0 & \frac{x_1^2}{2} & 0 & x_1 & 1 \end{pmatrix}. \quad (3.127)$$

from which we get the following basis of left-invariant vector fields

$$\begin{aligned} &E_1 = \frac{\partial}{\partial x_1} \quad , \quad E_2 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_4} \quad , \quad E_3 = \frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_5} + \frac{x_1^2}{2}\frac{\partial}{\partial x_6}, \\ &E_4 = \frac{\partial}{\partial x_4} \quad , \quad E_5 = \frac{\partial}{\partial x_5} + x_1\frac{\partial}{\partial x_6} \quad , \quad E_6 = \frac{\partial}{\partial x_6} \end{aligned} \quad (3.128)$$

The only non trivial brackets are

$$[E_1, E_2] = E_4 \quad , \quad [E_1, E_3] = E_5 \quad , \quad [E_1, E_5] = E_6 \quad (3.129)$$

and dually

$$\begin{aligned} &e^1 = dx_1 \quad , \quad e^2 = dx_2 \quad , \quad e^3 = dx_3, \\ &e^4 = dx_4 - x_1dx_2 \quad , \quad e^5 = dx_5 - x_1dx_3 \quad , \quad e^6 = dx_6 - x_1dx_5 + \frac{x_1^2}{2}dx_3. \end{aligned} \quad (3.130)$$

$$de^4 = -dx_1\wedge dx_2 = -e^{12} \quad , \quad de^5 = dx_1\wedge dx_3 = -e^{13} \quad , \quad de^6 = -dx_1\wedge(dx_5 - x_1dx_3) = -e^{15} \quad (3.131)$$

which, again, corresponds to $\mathfrak{h}_{10} = (0, 0, 0, 12, 13, 14)$ in [60]. We will denote it with $\check{\mathfrak{g}}_Y$.

Chapter 4

Supersymmetric Structures and Fourier-Mukai Transforms

In this chapter we present all the explicit computation involving the semi-flat supersymmetric SU(3)-structures of type IIA/IIB. We write the expression for the forms (ω, Ω), as defined in the previous chapter, we check their properties and show the cohomology diamond associated. Then, for each mirror pair, we describe explicitly the Fourier-Mukai transform realizing $FT(e^{2\check{\omega}}) = \Omega$. From example to example the forms

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56} \\ \Omega &= \left(e^1 + ie^2\right) \wedge \left(e^3 + ie^4\right) \wedge \left(e^5 + ie^6\right) \end{aligned} \tag{4.1}$$

will change their expression according to a reordering of the basis. This has been done to relate the structures with the ones already known in the literature and in order to maintain an accordance with the classifying results cited in section 2.5. For each pair we are not writing the explicit correspondence among the flux forms ρ_A and ρ_B since they follow directly from the computation in the appendix.

4.1 $(M_{N,1}, \check{M}_{N,1})$

IIA Equations on $\mathfrak{g}_{N,1} = (0, 0, 0, 0, 12, -13)$

Take

$$\begin{aligned} \omega &= e^{41} + e^{62} + e^{35} \\ \Omega &= (e^4 + ie^1) \wedge (e^6 + ie^2) \wedge (e^3 + ie^5) \end{aligned} \tag{4.2}$$

We have

$$\begin{aligned}\text{Re }\Omega &= e^{463} - e^{425} - e^{165} - e^{123} \\ \text{Im }\Omega &= e^{465} + e^{423} + e^{163} - e^{125}\end{aligned} \tag{4.3}$$

$$\begin{aligned}&d\omega = e^{132} + e^{123} = 0 \\ &\text{d Re }\Omega = de^{463} - de^{425} - de^{165} - de^{123} = 0 \\ &\text{d Im }\Omega = de^{465} + de^{423} + de^{163} - de^{125} = -e^{4135} - e^{4612} \neq 0\end{aligned} \tag{4.4}$$

and

$$\frac{1}{8}\Omega \wedge \bar{\Omega} = -\mathrm{i}e^{123456} = -\mathrm{i}\frac{\omega^3}{6}. \tag{4.5}$$

which implies $F \equiv 8$.

TY Diamond for $\mathfrak{g}_{N,1}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & & 2 & & \\ & 2 & & 6 & & 3 & \\ 1 & & 5 & & 6 & & 1 \\ & 2 & & 6 & & 3 & \\ & & 2 & & 2 & & \\ & & & 1 & & & \end{array} \tag{4.6}$$

IIB Equations on $\check{\mathfrak{g}}_{N,1} = (0, 0, 0, 0, 12, 13)$

Take

$$\begin{aligned}\check{\omega} &= \check{e}^{41} + \check{e}^{32} + \check{e}^{65} \\ \check{\Omega} &= (\check{e}^4 + i\check{e}^1) \wedge (\check{e}^3 + i\check{e}^2) \wedge (\check{e}^6 + i\check{e}^5)\end{aligned} \tag{4.7}$$

We have

$$\begin{aligned} \text{Re } \check{\Omega} &= \check{e}^{436} - \check{e}^{425} - \check{e}^{135} - \check{e}^{126} \\ \text{Im } \check{\Omega} &= \check{e}^{435} + \check{e}^{426} + \check{e}^{136} - \check{e}^{125} \end{aligned} \tag{4.8}$$

$$\begin{aligned} &d\check{\omega} = -\check{e}^{135} + \check{e}^{612} \neq 0 \\ &d\check{\omega}^2 = 2d\check{\omega} \wedge \check{\omega} = 2(-\check{e}^{135} + \check{e}^{126}) \wedge (\check{e}^{41} + \check{e}^{32} + \check{e}^{65}) = 0 \\ &\text{d Re } \check{\Omega} = d\check{e}^{436} - d\check{e}^{425} - d\check{e}^{135} - d\check{e}^{126} = 0 \\ &\text{d Im } \check{\Omega} = d\check{e}^{435} + d\check{e}^{426} + d\check{e}^{136} - d\check{e}^{125} = -\check{e}^{4312} - \check{e}^{4213} = 0 \end{aligned} \tag{4.9}$$

and

$$\frac{1}{8}\check{\Omega} \wedge \bar{\check{\Omega}} = -\mathrm{i}\check{e}^{123456} = -\mathrm{i}\frac{\check{\omega}^3}{6}. \tag{4.10}$$

that implies $\check{F} \equiv 8$.

BC Diamond for $\check{\mathfrak{g}}_{N,1}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 3 & & \\ & 2 & & 6 & & 2 & \\ 1 & & 6 & & 6 & & 1 \\ & 2 & & 5 & & 2 & \\ & & 2 & & 2 & & \\ & & & 1 & & & \end{array} \tag{4.11}$$

Fourier-Mukai Transform and Mirror Duality for $M_{N,1}$ and $\check{M}_{N,1}$

The SYZ-dual fibrations are

$$
\begin{array}{ccc}
M_{N,1} = \frac{H_3(\mathbb{R})\ltimes_{\varphi^{N,1}}\mathbb{R}^3}{H_3(\mathbb{Z})\ltimes_{\mathfrak{j}^{N,1}}\mathbb{Z}^3} & & \check{M}_{N,1} = \frac{H_3(\mathbb{R})\ltimes_{\lambda_{N,1}}\mathbb{R}^3}{H_3(\mathbb{Z})\ltimes_{\iota_{N,1}}\mathbb{Z}^3} \\
 & \overset{\pi}{\searrow} \qquad \overset{\check{\pi}}{\swarrow} & \\
 & B = H_3(\mathbb{R})/H_3(\mathbb{Z}) &
\end{array}
\tag{4.12}
$$

We start from the symplectic side where the symplectic form ω

$$\omega = e^{41} + e^{62} + e^{35} = dx_4 \wedge dx_1 + dx_6 \wedge dx_2 + dx_3 \wedge dx_5 \tag{4.13}$$

gives as action coordinates on the base and angle coordinates on the fibers respectively:

$$\begin{cases} r_1 = x_1 \\ r_2 = x_2 \\ r_3 = x_5 \end{cases} , \quad \begin{cases} \theta_1 = x_4 \\ \theta_2 = x_6 \\ \theta_3 = x_3 \end{cases} \tag{4.14}$$

and $\omega = \sum_{i=1}^{3} d\theta_i \wedge dr_i$.

Rewrite the coframe of differential 1-forms in action-angle coordinates:

$$
\begin{aligned}
e^1 &= dx_1 = dr_1 \\
e^2 &= dx_2 = dr_2 \\
e^3 &= dx_3 = d\theta_3 \\
e^4 &= dx_4 = d\theta_1 \\
e^5 &= dx_5 - x_1 dx_2 = dr_3 - r_1 dr_2 \\
e^6 &= dx_6 + x_1 dx_3 = d\theta_2 + r_1 d\theta_3
\end{aligned}
\tag{4.15}
$$

The expression of the three-form Ω in these coordinates is

$$
\begin{aligned}
\Omega &= (e^4 + ie^1) \wedge (e^6 + ie^2) \wedge (e^3 + ie^5) = \\
&= \big(d\theta_1 + idr_1\big) \wedge \big((d\theta_2 + r_1 d\theta_3) + idr_2\big) \wedge \big(d\theta_3 + i(dr_3 - r_1 dr_2)\big)
\end{aligned}
\tag{4.16}
$$

On the complex side, the dual action-angle coordinates are

$$\begin{cases} r_1 = x_1 \\ r_2 = x_2 \\ r_3 = x_5 \end{cases} , \quad \begin{cases} \check{\theta}_1 = x_4 \\ \check{\theta}_2 = x_3 \\ \check{\theta}_3 = x_6 \end{cases} \tag{4.17}$$

and

$$\begin{aligned}
\check{e}^1 &= dx_1 = dr_1 \\
\check{e}^2 &= dx_2 = dr_2 \\
\check{e}^3 &= dx_3 = d\check{\theta}_2 \\
\check{e}^4 &= dx_4 = d\check{\theta}_1 \\
\check{e}^5 &= dx_5 - x_1 dx_2 = dr_3 - r_1 dr_2 \\
\check{e}^6 &= dx_6 - x_1 dx_3 = d\check{\theta}_3 - r_1 d\check{\theta}_2
\end{aligned} \tag{4.18}$$

Define complex $(1,0)$-forms:

$$\psi^1 = \check{e}^4 + i\check{e}^1 \quad , \quad \psi^2 = \check{e}^3 + i\check{e}^2 \quad , \quad \psi^3 = \check{e}^6 + i\check{e}^5 \tag{4.19}$$

and complex coordinates

$$\begin{cases} z_1 = \check{\theta}_1 + ir_1 = x_4 + ix_1 \\ z_2 = \check{\theta}_2 + ir_2 = x_3 + ix_2 \\ z_3 = \check{\theta}_3 + ir_3 = x_6 + ix_5 \end{cases} \tag{4.20}$$

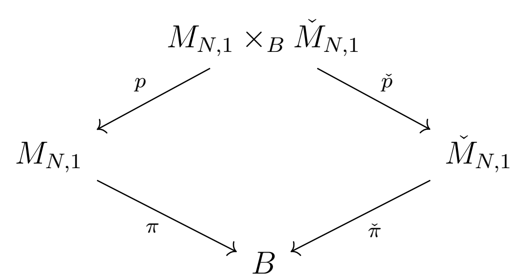
so that

$$\begin{aligned}
\psi^1 &= d\check{\theta}_1 + idr_1 = dz_1 \\
\psi^2 &= d\check{\theta}_2 + idr_2 = dz_2 \\
\psi^3 &= (d\check{\theta}_3 - r_1 d\check{\theta}_2) + i(dr_3 - r_1 dr_2) = dz_3 - r_1 dz_2
\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}
\check{\omega} &= \check{e}^{41} + \check{e}^{32} + \check{e}^{65} \\
&= \frac{i}{2}(\psi^{1\bar{1}} + \psi^{2\bar{2}} + \psi^{3\bar{3}}) \\
&= \frac{i}{2}\Big(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + (dz_3 - r_1 dz_2) \wedge (d\bar{z}_3 - r_1 d\bar{z}_2)\Big) \\
&= \frac{i}{2}\Big(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge \big((1 + r_1^2)\, d\bar{z}_2 - r_1 d\bar{z}_3\big) + dz_3 \wedge (d\bar{z}_3 - r_1 d\bar{z}_2)\Big)
\end{aligned} \tag{4.22}$$

Now, consider the fiber product



and recall the definition of the Fourier-Mukai tranform

$$FT \cdot \check{\phi} := p_*\Big(\big(\check{p}^*(\mathcal{P} \cdot \check{\phi})\big) \wedge \exp \frac{F}{2i}\Big) \tag{4.23}$$

where

$$F = 2\mathrm{i} \sum_{i=1}^{3} d\check{\theta}_i \wedge d\theta_i \tag{4.24}$$

so that

$$\begin{aligned} \mathcal{P} \cdot (2\check{\omega}) = i\Big(& d\check{\theta}_1 \wedge dr_1 + d\check{\theta}_2 \wedge \big((1 + r_1^2)dr_2 - r_1 dr_3\big) + \\ & + d\check{\theta}_3 \wedge \big(dr_3 - r_1 dr_2\big)\Big) \end{aligned} \tag{4.25}$$

Set

$$\begin{aligned} \eta_1 &:= dr_1 \\ \eta_2 &:= (1 + r_1^2)dr_2 - r_1 dr_3 \\ \eta_3 &:= dr_3 - r_1 dr_2 \end{aligned} \tag{4.26}$$

and take the product

$$\begin{aligned} e^{\mathcal{P}\cdot 2\check{\omega}} \wedge e^{\frac{F}{2i}} &= e^{i\sum_{i=1}^{3} d\check{\theta}_i \wedge \eta_i} \wedge e^{\sum_{i=1}^{3} d\check{\theta}_i \wedge d\theta_i} \\ &= e^{\sum_{i=1}^{3} d\check{\theta}_i \wedge (d\theta_i + i\eta_i)} \end{aligned} \tag{4.27}$$

Integrating this along the $\check{\theta}_i$'s we get

$$\begin{aligned} & (d\theta_1 + i\eta_1) \wedge (d\theta_2 + i\eta_2) \wedge (d\theta_3 + i\eta_3) \\ & = \Big(d\theta_1 + idr_1\Big) \wedge \Big(d\theta_2 + i\big((1 + r_1^2)dr_2 - r_1 dr_3\big)\Big) \wedge \Big(d\theta_3 + i\big(dr_3 - r_1 dr_2\big)\Big) \\ & = \big(d\theta_1 + idr_1\big) \wedge \big((d\theta_2 + r_1 d\theta_3) + idr_2\big) \wedge \big(d\theta_3 + i(dr_3 - r_1 dr_2)\big) \end{aligned} \tag{4.28}$$

that coincide with (4.16), showing that $FT(e^{2\check{\omega}}) = \Omega$.

4.2 $(M_{N,2}, \check{M}_{N,2})$

IIA Equations on $\mathfrak{g}_{N,2} = (0, 0, 0, 12, -13, -23)$

Take, for $\lambda \in \mathbb{R} \backslash \{0, 1\}$

$$\begin{aligned} \omega &= e^{61} + \lambda e^{52} + (\lambda - 1)e^{34} \\ \Omega &= (e^6 + ie^1) \wedge (e^5 + i\lambda e^2) \wedge (e^3 + i(\lambda - 1)e^4) \end{aligned} \tag{4.29}$$

We have

$$\begin{aligned} \text{Re } \Omega &= e^{653} - \lambda(\lambda-1)e^{642} - (\lambda-1)e^{154} - \lambda e^{123} \\ \text{Im } \Omega &= (\lambda-1)e^{654} + \lambda e^{623} + e^{153} - \lambda(\lambda-1)e^{124} \end{aligned} \tag{4.30}$$

$$\begin{aligned} d\omega &= e^{231} + \lambda e^{132} + (\lambda-1)e^{312} = (1-\lambda+\lambda-1)e^{123} = 0 \\ \text{d Re } \Omega &= de^{653} - \lambda(\lambda-1)de^{642} - (\lambda-1)de^{154} - \lambda de^{123} = 0 \\ \text{d Im } \Omega &= (\lambda-1)de^{654} + \lambda de^{623} + de^{153} - \lambda(\lambda-1)de^{124} \\ &= (\lambda-1)(-e^{2345} + e^{1346} + e^{1256}) \neq 0 \end{aligned} \tag{4.31}$$

and

$$\frac{1}{8}\Omega \wedge \bar{\Omega} = -\mathrm{i}e^{123456} = -\mathrm{i}\frac{\omega^3}{6}. \tag{4.32}$$

which implies $F \equiv 8$.

TY Diamonds for $\mathfrak{g}_{N,2}$

Depending on the value of the parameter $\lambda \in \mathbb{R}\backslash\{0,1\}$ we have three possible diamonds

For $\lambda = -1$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 2 & & \\ & 2 & & 6 & & 3 & \\ 1 & & 4 & & 7 & & 1 \\ & 2 & & 6 & & 3 & \\ & & 1 & & 2 & & \\ & & & 1 & & & \end{array} \tag{4.33}$$

For $\lambda = 2$ or $\lambda = \frac{1}{2}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 2 & & \\ & 2 & & 6 & & 3 & \\ 1 & & 5 & & 6 & & 1 \\ & 2 & & 6 & & 3 & \\ & & 1 & & 2 & & \\ & & & 1 & & & \end{array} \tag{4.34}$$

Otherwise

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 2 & & \\ & 2 & & 6 & & 3 & \\ 1 & & 4 & & 6 & & 1 \\ & 2 & & 6 & & 3 & \\ & & 1 & & 2 & & \\ & & & 1 & & & \end{array} \tag{4.35}$$

IIB Equations on $\check{\mathfrak{g}}_{N,2} = (0,0,0,0,12,14+23)$

Take, for $\lambda \in \mathbb{R}\backslash\{0,1\}$,

$$\begin{aligned} \check{\omega} &= \check{e}^{31} + \lambda \check{e}^{42} + (\lambda - 1)\check{e}^{65} \\ \check{\Omega} &= (\check{e}^3 + i\check{e}^1) \wedge (\check{e}^4 + i\lambda\check{e}^2) \wedge (\check{e}^6 + i(\lambda-1)\check{e}^5) \end{aligned} \tag{4.36}$$

We have

$$\begin{aligned} \text{Re } \check{\Omega} &= \check{e}^{346} - \lambda(\lambda-1)\check{e}^{325} - (\lambda-1)\check{e}^{145} - \lambda\check{e}^{126} \\ \text{Im } \check{\Omega} &= (\lambda-1)\check{e}^{345} + \lambda\check{e}^{326} + \check{e}^{146} - \lambda(\lambda-1)\check{e}^{125} \end{aligned} \tag{4.37}$$

$$\begin{aligned} d\check{\omega} &= (\lambda-1)(-e^{145} - e^{235} + e^{612}) \neq 0 \\ d\check{\omega}^2 &= 2d\check{\omega} \wedge \check{\omega} = 2(\lambda-1)(-e^{145} - e^{235} + e^{126}) \wedge (\check{e}^{31} + \lambda\check{e}^{42} + (\lambda-1)\check{e}^{65}) = 0 \\ \text{d Re } \check{\Omega} &= d\check{e}^{346} - \lambda(\lambda-1)d\check{e}^{325} - d\check{e}^{145} - \lambda d\check{e}^{126} = 0 \\ \text{d Im } \check{\Omega} &= (\lambda-1)d\check{e}^{345} + \lambda d\check{e}^{326} + d\check{e}^{146} - \lambda(\lambda-1)d\check{e}^{125} \\ &= (1-\lambda)\check{e}^{1234} + \lambda\check{e}^{1234} - e^{1234} = 0 \end{aligned} \tag{4.38}$$

and

$$\frac{1}{8}\check{\Omega} \wedge \bar{\check{\Omega}} = -\mathrm{i}\check{e}^{123456} = -\mathrm{i}\frac{\check{\omega}^3}{6}. \tag{4.39}$$

that implies $\check{F} \equiv 8$.

BC Diamond for $\mathfrak{g}_B^{N,2}$

Depending on the value of the parameter $\lambda \in \mathbb{R}\backslash\{0,1\}$ we have three possibly diamonds

For $\lambda = -1$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 3 & & \\ & 2 & & 7 & & 2 & \\ 1 & & 6 & & 6 & & 1 \\ & 1 & & 4 & & 1 & \\ & & 2 & & 2 & & \\ & & & 1 & & & \end{array} \tag{4.40}$$

For $\lambda = 2$ or $\lambda = \frac{1}{2}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 3 & & \\ & 2 & & 6 & & 2 & \\ 1 & & 6 & & 6 & & 1 \\ & 1 & & 5 & & 1 & \\ & & 2 & & 2 & & \\ & & & 1 & & & \end{array} \tag{4.41}$$

Otherwise

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 3 & & \\ & 2 & & 6 & & 2 & \\ 1 & & 6 & & 6 & & 1 \\ & 1 & & 4 & & 1 & \\ & & 2 & & 2 & & \\ & & & 1 & & & \end{array} \tag{4.42}$$

Fourier-Mukai Transform and Mirror Duality for $M_{N,2}$ and $\check{M}_{N,2}$

The SYZ-dual fibrations are

$$\begin{array}{ccc} M_{N,2} = \frac{H_3(\mathbb{R}) \ltimes_{\varphi^{N,2}} \mathbb{R}^3}{H_3(\mathbb{Z}) \ltimes_{\mathfrak{f}^{N,2}} \mathbb{Z}^3} & & \check{M}_{N,2} = \frac{H_3(\mathbb{R}) \ltimes_{\lambda_{N,2}} \mathbb{R}^3}{H_3(\mathbb{Z}) \ltimes_{\iota_{N,2}} \mathbb{Z}^3} \\ & \searrow^{\pi} \quad {}^{\check{\pi}}\swarrow & \\ & B = H_3(\mathbb{R})/H_3(\mathbb{Z}) & \end{array} \tag{4.43}$$

We start from the symplectic side where the symplectic form ω

$$\begin{aligned} \omega &= e^{61} + \lambda e^{52} + (\lambda - 1)e^{34} \\ &= (dx_6 + x_2 dx_3) \wedge dx_1 + \lambda(dx_5 + x_1 dx_3) \wedge dx_2 + (\lambda - 1)dx_3 \wedge (dx_4 - x_1 dx_2) \\ &= dx_6 \wedge dx_1 + x_2 dx_3 \wedge dx_1 + \lambda dx_5 \wedge dx_2 + \lambda x_1 dx_3 \wedge dx_2 + \\ &+ (\lambda - 1)dx_3 \wedge dx_4 - (\lambda - 1)x_1 dx_3 \wedge dx_2 \\ &= dx_6 \wedge dx_1 + \lambda dx_5 \wedge dx_2 + (\lambda - 1)dx_3 \wedge dx_4 + dx_3 \wedge d(x_1 x_2) \\ &= dx_6 \wedge dx_1 + dx_5 \wedge \lambda dx_2 + dx_3 \wedge d\big((\lambda - 1)x_4 + x_1 x_2\big) \end{aligned} \tag{4.44}$$

gives as action coordinates on the base and angle coordinates on the fibers respectively:

$$\begin{cases} r_1 = x_1 \\ r_2 = \lambda x_2 \\ r_3 = (\lambda - 1)x_4 + x_1 x_2 \end{cases} \quad , \quad \begin{cases} \theta_1 = x_6 \\ \theta_2 = x_5 \\ \theta_3 = x_3 \end{cases} \tag{4.45}$$

and $\omega = \sum_{i=1}^{3} d\theta_i \wedge dr_i$

Rewrite the coframe of differential 1-forms in action-angle coordinates:

$$\begin{aligned} e^1 &= dx_1 = dr_1 \\ e^2 &= dx_2 = \frac{dr_2}{\lambda} \\ e^3 &= dx_3 = d\theta_3 \\ e^4 &= dx_4 - x_1 dx_2 = \frac{1}{\lambda - 1} dr_3 - \frac{r_2}{\lambda(\lambda - 1)} dr_1 - \frac{r_1}{\lambda - 1} dr_2, \\ e^5 &= dx_5 + x_1 dx_3 = d\theta_2 + r_1 d\theta_3 \\ e^6 &= dx_6 + x_2 dx_3 = d\theta_1 + \frac{r_2}{\lambda} d\theta_3 \end{aligned} \tag{4.46}$$

The expression of the three-form Ω in these coordinate is

$$\begin{aligned}\Omega &= (e^6 + ie^1)\wedge(e^5 + i\lambda e^2)\wedge(e^3 + i(\lambda-1)e^4) = \\ &= \Big((d\theta_1 + \frac{r_2}{\lambda}d\theta_3) + idr_1\Big)\wedge\Big((d\theta_2 + r_1 d\theta_3) + idr_2\Big)\wedge\Big(d\theta_3 + i(dr_3 - \frac{r_2}{\lambda}dr_1 - r_1 dr_2)\Big)\end{aligned} \tag{4.47}$$

On the complex side, the dual action-angle coordinates are

$$\begin{cases} r_1 = x_1 \\ r_2 = \lambda x_2 \\ r_3 = (\lambda-1)x_5 + x_1x_2 \end{cases} \quad , \quad \begin{cases} \check{\theta}_1 = x_3 \\ \check{\theta}_2 = x_4 \\ \check{\theta}_3 = x_6 \end{cases} \tag{4.48}$$

and

$$\begin{aligned} \check{e}^1 &= dx_1 = dr_1 \\ \check{e}^2 &= dx_2 = \frac{dr_2}{\lambda} \\ \check{e}^3 &= dx_3 = d\check{\theta}_1 \\ \check{e}^4 &= dx_4 = d\check{\theta}_2 \\ \check{e}^5 &= dx_5 - x_1dx_2 = \frac{1}{\lambda-1}dr_3 - \frac{r_2}{\lambda(\lambda-1)}dr_1 - \frac{r_1}{\lambda-1}dr_2 \\ \check{e}^6 &= dx_6 - x_1dx_4 - x_2dx_3 = d\check{\theta}_3 - r_1 d\check{\theta}_2 - \frac{r_2}{\lambda}d\check{\theta}_1 \end{aligned} \tag{4.49}$$

Define complex $(1,0)$-forms:

$$\psi^1 = \check{e}^3 + i\check{e}^1 \quad , \quad \psi^2 = \check{e}^4 + i\lambda\check{e}^2 \quad , \quad \psi^3 = \check{e}^6 + i(\lambda-1)\check{e}^5 \tag{4.50}$$

and complex coordinates

$$\begin{cases} z_1 = \check{\theta}_1 + ir_1 = x_3 + ix_1 \\ z_2 = \check{\theta}_2 + ir_2 = x_4 + i\lambda x_2 \\ z_3 = \check{\theta}_3 + ir_3 = x_6 + i\big((\lambda-1)x_5 + x_1x_2\big) \end{cases} \tag{4.51}$$

so that

$$\begin{aligned} \psi^1 &= d\check{\theta}_1 + idr_1 = dz_1 \\ \psi^2 &= d\check{\theta}_2 + idr_2 = dz_2 \\ \psi^3 &= (d\check{\theta}_3 - r_1d\check{\theta}_2 - \frac{r_2}{\lambda}d\check{\theta}_1) + i(dr_3 - r_1dr_2 - \frac{r_2}{\lambda}dr_1) = dz_3 - r_1dz_2 - \frac{r_2}{\lambda}dz_1 \end{aligned} \tag{4.52}$$

and

$$\begin{aligned}
\check{\omega} &= \check{e}^{31} + \lambda \check{e}^{42} + (\lambda - 1)\check{e}^{65} \\
&= \frac{i}{2}(\psi^{1\bar{1}} + \psi^{2\bar{2}} + \psi^{3\bar{3}}) \\
&= \frac{i}{2}\Big(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + (dz_3 - r_1 dz_2 - \frac{r_2}{\lambda}dz_1) \wedge (d\bar{z}_3 - r_1 d\bar{z}_2 - \frac{r_2}{\lambda}d\bar{z}_1)\Big) \\
&= \frac{i}{2}\Big(dz_1 \wedge \big((1 + \frac{r_2^2}{\lambda^2})d\bar{z}_1 + \frac{r_1 r_2}{\lambda}d\bar{z}_2 - \frac{r_2}{\lambda}d\bar{z}_3\big) + \\
&+ dz_2 \wedge \big(\frac{r_1 r_2}{\lambda}d\bar{z}_1 + (1 + r_1^2)\ d\bar{z}_2 - r_1 d\bar{z}_3\big) + dz_3 \wedge \big(-\frac{r_2}{\lambda}d\bar{z}_1 - r_1 d\bar{z}_2 + d\bar{z}_3\big)\Big)
\end{aligned} \tag{4.53}$$

Now

$$\begin{aligned}
\mathcal{P} \cdot (2\check{\omega}) = i\Big(& d\check{\theta}_1 \wedge \big((1 + \frac{r_2^2}{\lambda^2})dr_1 + \frac{r_1 r_2}{\lambda}dr_2 - \frac{r_2}{\lambda}dr_3\big) + \\
&+ d\check{\theta}_2 \wedge \big(\frac{r_1 r_2}{\lambda}dr_1 + (1 + r_1^2)\ dr_2 - r_1 dr_3\big) + \\
&+ d\check{\theta}_3 \wedge \big(-\frac{r_2}{\lambda}dr_1 - r_1 dr_2 + dr_3\big)\Big)
\end{aligned} \tag{4.54}$$

Set

$$\begin{aligned}
\eta_1 &:= (1 + \frac{r_2^2}{\lambda^2})dr_1 + \frac{r_1 r_2}{\lambda}dr_2 - \frac{r_2}{\lambda}dr_3 \\
\eta_2 &:= \frac{r_1 r_2}{\lambda}dr_1 + (1 + r_1^2)dr_2 - r_1 dr_3 \\
\eta_3 &:= -\frac{r_2}{\lambda}dr_1 - r_1 dr_2 + dr_3
\end{aligned} \tag{4.55}$$

and take the product

$$\begin{aligned}
e^{\mathcal{P}\cdot 2\check{\omega}} \wedge e^{\frac{F}{2i}} &= e^{i\sum_{i=1}^{3} d\check{\theta}_i \wedge \eta_i} \wedge e^{\sum_{i=1}^{3} d\check{\theta}_i \wedge d\theta_i} \\
&= e^{\sum_{i=1}^{3} d\check{\theta}_i \wedge (d\theta_i + i\eta_i)}
\end{aligned} \tag{4.56}$$

Integrating this along the $\check{\theta}_i$'s we get

$$\begin{aligned}
&(d\theta_1 + i\eta_1) \wedge (d\theta_2 + i\eta_2) \wedge (d\theta_3 + i\eta_3) = \\
&= \Big(d\theta_1 + i\big((1 + \frac{r_2^2}{\lambda^2})dr_1 + \frac{r_1 r_2}{\lambda}dr_2 - \frac{r_2}{\lambda}dr_3\big)\Big) \\
&\wedge \Big(d\theta_2 + i\big(\frac{r_1 r_2}{\lambda}dr_1 + (1 + r_1^2)dr_2 - r_1 dr_3\big)\Big) \wedge \\
&\wedge \Big(d\theta_3 + i\big(-\frac{r_2}{\lambda}dr_1 - r_1 dr_2 + dr_3\big)\Big) \\
&= \Big((d\theta_1 + \frac{r_2}{\lambda}d\theta_3) + idr_1\Big) \wedge \Big((d\theta_2 + r_1 d\theta_3) + idr_2\Big) \wedge \Big(d\theta_3 + i(dr_3 - \frac{r_2}{\lambda}dr_1 - r_1 dr_2)\Big)
\end{aligned} \tag{4.57}$$

that coincide with (4.47), showing that $FT(e^{2\check{\omega}}) = \Omega$.

4.3 $(M_{S,1}, \check{M}_{S,1})$

IIA Equations on $\mathfrak{g}_{S,1} = (15, -25, -35, 45, 0, 0)$

Take

$$\begin{aligned} \omega &= e^{31} + e^{42} + e^{65} \\ \Omega &= (e^3 + ie^1) \wedge (e^4 + ie^2) \wedge (e^6 + ie^5) \end{aligned} \tag{4.58}$$

We have

$$\begin{aligned} \text{Re } \Omega &= e^{346} - e^{325} - e^{145} - e^{126} \\ \text{Im } \Omega &= e^{345} + e^{326} + e^{146} - e^{125} \end{aligned} \tag{4.59}$$

$$\begin{aligned} &d\omega = 0 \\ &d\text{Re } \Omega = 0 \\ &d\text{Im } \Omega = -2(e^{2356} + e^{1456}) \end{aligned} \tag{4.60}$$

and

$$\frac{1}{8}\Omega \wedge \bar{\Omega} = -i \; e^{123456} = -i\frac{\omega^3}{6} \tag{4.61}$$

with $F \equiv 8$

TY Diamond for $\mathfrak{g}_{S,1}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 3 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ & 1 & & 3 & & 1 & \\ & & 1 & & 1 & & \\ & & & 1 & & & \end{array} \tag{4.62}$$

IIB Equations on $\check{\mathfrak{g}}_{S,1}$ $= (15, -25, -35, 45, 0, 0)$

Take

$$\begin{aligned} \check{\omega} &= \check{e}^{41} + \check{e}^{32} + \check{e}^{65} \\ \check{\Omega} &= (\check{e}^4 + i\check{e}^1) \wedge (\check{e}^3 + i\check{e}^2) \wedge (\check{e}^6 + i\check{e}^5) \end{aligned} \tag{4.63}$$

We have

$$\begin{aligned} \text{Re } \check{\Omega} &= \check{e}^{436} - \check{e}^{425} - \check{e}^{135} - \check{e}^{126} \\ \text{Im } \check{\Omega} &= \check{e}^{435} + \check{e}^{426} + \check{e}^{136} - \check{e}^{125} \end{aligned} \tag{4.64}$$

$$\begin{aligned} &d\check{\omega} = -\check{e}^{451} + \check{e}^{415} + \check{e}^{352} - \check{e}^{325} = 2(-\check{e}^{145} + \check{e}^{235}) \neq 0 \\ &d\check{\omega}^2 = 2d\check{\omega} \wedge \check{\omega} = 2(-\check{e}^{14532} + \check{e}^{23541}) = 0 \\ &d\text{Re } \check{\Omega} = 0 \\ &d\text{Im } \check{\Omega} = 0 \end{aligned} \tag{4.65}$$

and

$$\frac{1}{8}\check{\Omega} \wedge \bar{\check{\Omega}} = -i\ \check{e}^{123456} = -i\frac{\check{\omega}^3}{6} \tag{4.66}$$

that implies $\check{F} \equiv 8$.

BC Diamond for $\check{\mathfrak{g}}_{S,1}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 3 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ & 1 & & 3 & & 1 & \\ & & 1 & & 1 & & \\ & & & 1 & & & \end{array} \tag{4.67}$$

Fourier-Mukai Transform and Mirror Duality for $M_{S,1}$ and $\check{M}_{S,1}$

The SYZ-dual fibrations are

$$\begin{array}{ccc} M_{S,1} = \frac{E(1,1)\ltimes_{\varphi_{S,1}}\mathbb{R}^3}{\Gamma_t\ltimes_{\mathfrak{f}_{S,1}}\mathbb{Z}_t^3} & & \check{M}_{S,1} = \frac{E(1,1)\ltimes_{\lambda_{S,1}}\mathbb{R}^3}{\Gamma_t\ltimes_{\mathfrak{l}_{S,1}}\mathbb{Z}_t^3} \\ & \searrow^{\pi} \quad {}^{\check{\pi}}\swarrow & \\ & B = E(1,1)/\Gamma_t & \end{array} \tag{4.68}$$

We start from the symplectic side where the symplectic form ω

$$\begin{aligned} \omega &= e^{31} + e^{42} + e^{65} = e^{-x_5}dx_3 \wedge e^{x_5}dx_1 + e^{x_5}dx_4 \wedge e^{-x_5}dx_2 + dx_6 \wedge dx_5 \\ &= dx_4 \wedge dx_1 + dx_3 \wedge dx_2 + dx_6 \wedge dx_5 \end{aligned} \tag{4.69}$$

gives as action coordinates on the base and angle coordinates on the fibers respesctively

$$\begin{cases} r_1 = x_1 \\ r_2 = x_2 \\ r_3 = x_5 \end{cases} , \quad \begin{cases} \theta_1 = x_3 \\ \theta_2 = x_4 \\ \theta_3 = x_6 \end{cases} \tag{4.70}$$

and $\omega = \sum_{i=1}^{3} d\theta_i \wedge dr_i$.
Rewrite the coframe of differential 1-forms in action-angle coordinates

$$\begin{aligned} e^1 &= e^{x_5} dx_1 = e^{r_3} dr_1 \\ e^2 &= e^{-x_5} dx_2 = e^{-r_3} dr_2 \\ e^3 &= e^{-x_5} dx_3 = e^{-r_3} d\theta_1 \\ e^4 &= e^{x_5} dx_4 = e^{r_3} d\theta_2 \\ e^5 &= dx_5 = dr_3 \\ e^6 &= dx_6 = d\theta_3 \end{aligned} \tag{4.71}$$

The expression of the three-form Ω in these coordinates is

$$\begin{aligned} \Omega &= (e^3 + ie^1) \wedge (e^4 + ie^2) \wedge (e^6 + ie^5) \\ &= (e^{-r_3} d\theta_1 + ie^{r_3} dr_1) \wedge (e^{r_3} d\theta_2 + ie^{-r_3} dr_2) \wedge (d\theta_3 + idr_3) \\ &= e^{-r_3}(d\theta_1 + ie^{2r_3} dr_1) \wedge e^{r_3}(d\theta_2 + ie^{-2r_3} dr_2) \wedge (d\theta_3 + idr_3) \\ &= (d\theta_1 + ie^{2r_3} dr_1) \wedge (d\theta_2 + ie^{-2r_3} dr_2) \wedge (d\theta_3 + idr_3) \end{aligned} \tag{4.72}$$

On the complex side instead, the dual action-angle coordinates are

$$\begin{cases} r_1 = x_1 \\ r_2 = x_2 \\ r_3 = x_5 \end{cases} , \quad \begin{cases} \check{\theta}_1 = x_4 \\ \check{\theta}_2 = x_3 \\ \check{\theta}_3 = x_6 \end{cases} \tag{4.73}$$

and

$$\begin{aligned} \check{e}^1 &= e^{x_5} dx_1 = e^{r_3} dr_1 \\ \check{e}^2 &= e^{-x_5} dx_2 = e^{-r_3} dr_2 \\ \check{e}^3 &= e^{-x_5} dx_3 = e^{-r_3} d\check{\theta}_2 \\ \check{e}^4 &= e^{x_5} dx_4 = e^{r_3} d\check{\theta}_1 \\ \check{e}^5 &= dx_5 = dr_3 \\ \check{e}^6 &= dx_6 = \check{d}\theta_3 \end{aligned} \tag{4.74}$$

Define complex $(1,0)$-forms:

$$\psi^1 = \check{e}^4 + i\check{e}^1 \quad , \quad \psi^2 = \check{e}^3 + i\check{e}^2 \quad , \quad \psi^3 = \check{e}^6 + i\check{e}^5 \tag{4.75}$$

and complex coordinates

$$\begin{cases} z_1 = \check{\theta}_1 + ir_1 = x_4 + ix_1 \\ z_2 = \check{\theta}_2 + ir_2 = x_3 + ix_2 \\ z_3 = \check{\theta}_3 + ir_3 = x_6 + ix_5 \end{cases} \tag{4.76}$$

so that

$$\begin{aligned}\psi^1 &= e^4 + ie^1 = e^{x_5} d\check{\theta}_1 + ie^{x_5} dr_1 = e^{x_5} dz_1 \\ \psi^2 &= e^3 + ie^2 = e^{-x_5} d\check{\theta}_2 + ie^{-x_5} dr_2 = e^{-x_5} dz_2 \\ \psi^3 &= e^6 + ie^5 = d\check{\theta}_3 + idr_3 = dz_3\end{aligned} \tag{4.77}$$

and

$$\begin{aligned}\check{\omega} &= \check{e}^{41} + \check{e}^{32} + \check{e}^{65} \\ &= \frac{i}{2}(\psi^{1\bar{1}} + \psi^{2\bar{2}} + \psi^{3\bar{3}}) \\ &= \frac{i}{2}\Big(e^{2r_3} dz_1 \wedge d\bar{z}_1 + e^{-2r_3} dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3\Big)\end{aligned} \tag{4.78}$$

Now

$$\mathcal{P} \cdot (2\check{\omega}) = i\big(e^{2r_3} d\check{\theta}_1 \wedge dr_1 + e^{-2r_3} d\check{\theta}_2 \wedge dr_2 + d\check{\theta}_3 \wedge dr_3\big) \tag{4.79}$$

Set

$$\begin{aligned}\eta_1 &:= e^{2r_3} dr_1 \\ \eta_2 &:= e^{-2r_3} dr_2 \\ \eta_3 &:= dr_3\end{aligned} \tag{4.80}$$

Take the product

$$\begin{aligned}e^{\mathcal{P}\cdot 2\check{\omega}} \wedge e^{\frac{F}{2i}} &= e^{i\sum_{i=1}^3 d\check{\theta}_i \wedge \eta_i} \wedge e^{\sum_{i=1}^3 d\check{\theta}_i \wedge d\theta_i} \\ &= e^{\sum_{i=1}^3 d\check{\theta}_i \wedge (d\theta_i + i\eta_i)}\end{aligned} \tag{4.81}$$

Integrating this along the $\check{\theta}_i$'s we get

$$\begin{aligned}&(d\theta_1 + i\eta_1) \wedge (d\theta_2 + i\eta_2) \wedge (d\theta_3 + i\eta_3) \\ &= (d\theta_1 + ie^{2r_3} dr_1) \wedge (d\theta_2 + ie^{-2r_3} dr_2) \wedge (d\theta_3 + idr_3)\end{aligned} \tag{4.82}$$

which agrees with (4.72), showing that $FT(e^{2\check{\omega}}) = \Omega$.

4.4 $(M_{S,2}, \check{M}_{S,2})$

IIA Equations on $\mathfrak{g}_{S,2} = (16 + 35, -26 + 45, 36, -46, 0, 0)$

Take

$$\begin{aligned} \omega &= e^{14} + e^{23} + e^{56} \\ \Omega &= (e^1 + ie^4) \wedge (e^2 + ie^3) \wedge (e^5 + ie^6) \end{aligned} \tag{4.83}$$

We have

$$\begin{aligned} \text{Re } \Omega &= e^{125} - e^{136} - e^{426} - e^{435} \\ \text{Im } \Omega &= e^{126} + e^{135} + e^{425} - e^{436} \end{aligned} \tag{4.84}$$

$$\begin{aligned} d\omega &= -e^{164} - e^{354} - e^{146} + e^{263} - e^{453} + e^{236} = 0 \\ d\text{Re } \Omega &= 0 \\ d\text{Im } \Omega &= -e^{2356} + e^{1456} - 2e^{1356} - 2e^{2456} \end{aligned} \tag{4.85}$$

and

$$\frac{1}{8}\Omega \wedge \bar{\Omega} = -i \ e^{123456} = -i\frac{\omega^3}{6} \tag{4.86}$$

with $F \equiv 8$.

TY Diamond for $\mathfrak{g}_{S,2}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 0 & & 2 & & 1 & \\ 1 & & 2 & & 1 & & 1 \\ & 0 & & 2 & & 1 & \\ & & 1 & & 1 & & \\ & & & 1 & & & \end{array} \tag{4.87}$$

IIB Equations on $\check{\mathfrak{g}}_{S,2} = (24+35, 26, 36, -46, -56, 0)$

Take

$$\begin{aligned}\check{\omega} &= \check{e}^{54} + \check{e}^{23} + \check{e}^{16}\\ \check{\Omega} &= (\check{e}^5 + i\check{e}^4) \wedge (\check{e}^2 + i\check{e}^3) \wedge (\check{e}^1 + i\check{e}^6)\end{aligned} \tag{4.88}$$

We have

$$\begin{aligned}\text{Re } \check{\Omega} &= \check{e}^{521} - \check{e}^{536} - \check{e}^{426} - \check{e}^{431}\\ \text{Im } \check{\Omega} &= \check{e}^{562} + \check{e}^{531} + \check{e}^{421} - \check{e}^{436}\end{aligned} \tag{4.89}$$

$$\begin{aligned}d\check{\omega} &= 2\check{e}^{456} + 2\check{e}^{236} + \check{e}^{246} - \check{e}^{356} \neq 0\\ d\check{\omega}^2 &= 2d\check{\omega} \wedge \check{\omega} = 4(\check{e}^{45623} + \check{e}^{23654}) = 0\\ d\text{Re } \check{\Omega} &= \check{e}^{5621} + \check{e}^{5261} - \check{e}^{4631} - \check{e}^{4361} = 0\\ d\text{Im } \check{\Omega} &= \check{e}^{5631} + \check{e}^{5361} + \check{e}^{5324} + \check{e}^{4621} + \check{e}^{4261} - \check{e}^{4235} = 0\end{aligned} \tag{4.90}$$

and

$$\frac{1}{8}\check{\Omega} \wedge \bar{\check{\Omega}} = -i\ e^{123456} = -i\frac{\check{\omega}^3}{6} \tag{4.91}$$

and $\check{F} \equiv 8$

BC Diamond for $\check{\mathfrak{g}}_{S,2}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 1 & & 1 & \\ 1 & & 2 & & 2 & & 1 \\ & 1 & & 2 & & 1 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{4.92}$$

Fourier-Mukai Transform and Mirror Duality for $M_{S,2}$ and $\check{M}_{S,2}$

The SYZ-dual fibrations are

$$\begin{array}{ccc} M_{S,2} = \frac{E(1,1)\ltimes_{\varphi^{S,2}}\mathbb{R}^3}{\Gamma_t\ltimes_{\mathfrak{f}^{S,2}}\mathbb{Z}_t^3} & & \check{M}_{S,2} = \frac{E(1,1)\ltimes_{\lambda_{S,2}}\mathbb{R}^3}{\Gamma_t\ltimes_{\iota_{S,2}}\mathbb{Z}_t^3} \\ & \searrow^{\pi} \qquad {}^{\check{\pi}}\swarrow & \\ & B = E(1,1)/\Gamma_t & \end{array} \tag{4.93}$$

We start from the symplectic side where the symplectic form ω

$$\begin{aligned} \omega = e^{14} + e^{23} + e^{56} = \\ &= (e^{x_6}dx_1 + x_3e^{x_6}dx_5)\wedge e^{-x_6}dx_4 + (e^{-x_6}dx_2 + x_4e^{-x_6}dx_5)\wedge e^{x_6}dx_3 + dx_5\wedge dx_6 \\ &= dx_1\wedge dx_4 + x_3dx_5\wedge dx_4 + dx_2\wedge dx_3 + x_4dx_5\wedge dx_3 + dx_5\wedge dx_6 \\ &= dx_1\wedge dx_4 + dx_2\wedge dx_3 + dx_5\wedge d(x_6 + x_4x_3) \end{aligned} \tag{4.94}$$

gives as action coordinates on the base and angle coordinates on the fibers respectively

$$\begin{cases} r_1 = x_4 \\ r_2 = x_3 \\ r_3 = x_6 + x_4x_3 \end{cases} , \qquad \begin{cases} \theta_1 = x_1 \\ \theta_2 = x_2 \\ \theta_3 = x_5 \end{cases} \tag{4.95}$$

and $\omega = \sum_{i=1}^{3} d\theta_i\wedge dr_i$.

Rewrite the coframe of differential 1-forms in action-angle coordinates

$$\begin{aligned} e^1 &= e^{x_6}dx_1 + x_3e^{x_6}dx_5 = e^{r_3-r_1r_2}d\theta_1 + r_2e^{r_3-r_1r_2}d\theta_3 \\ e^2 &= e^{-x_6}dx_2 + x_4e^{-x_6}dx_5 = e^{-r_3+r_1r_2}d\theta_2 + r_1e^{-r_3+r_1r_2}d\theta_3 \\ e^3 &= e^{x_6}dx_3 = e^{r_3-r_1r_2}dr_2 \\ e^4 &= e^{-x_6}dx_4 = e^{-r_3+r_1r_2}dr_1 \\ e^5 &= dx_5 = d\theta_3 \\ e^6 &= dx_6 = dr_3 - r_2dr_1 - r_1dr_2 \end{aligned} \tag{4.96}$$

The expression of the three-form Ω in these coordinates is

$$
\begin{aligned}
\Omega &= (e^1 + ie^4) \wedge (e^2 + ie^3) \wedge (e^5 + ie^6) \\
&= \Big((e^{r_3 - r_1 r_2} d\theta_1 + r_2 e^{r_3 - r_1 r_2} d\theta_3) + i e^{-r_3 + r_1 r_2} dr_1 \Big) \wedge \Big((e^{-r_3 + r_1 r_2} d\theta_2 + \\
&+ r_1 e^{-r_3 + r_1 r_2} d\theta_3) + i e^{r_3 - r_1 r_2} dr_2 \Big) \wedge \Big(d\theta_3 + i(dr_3 - r_2 dr_1 - r_1 dr_2 \Big) \\
&= e^{r_3 - r_1 r_2} \Big((d\theta_1 + r_2 d\theta_3) + i e^{-2(r_3 - r_1 r_2)} dr_1 \Big) \wedge e^{-r_3 + r_1 r_2} \Big((d\theta_2 + r_1 d\theta_3) + i e^{2(r_3 - r_1 r_2)} dr_2 \Big) \\
&\wedge \Big(d\theta_3 + i(dr_3 - r_2 dr_1 - r_1 dr_2) \Big) \\
&= \Big((d\theta_1 + r_2 d\theta_3) + i e^{-2(r_3 - r_1 r_2)} dr_1 \Big) \wedge \Big((d\theta_2 + r_1 d\theta_3) + i e^{2(r_3 - r_1 r_2)} dr_2 \Big) \wedge \\
&\wedge \Big(d\theta_3 + i(dr_3 - r_2 dr_1 - r_1 dr_2) \Big)
\end{aligned}
\tag{4.97}
$$

On the complex side instead, the dual action-angle coordinates are

$$
\begin{cases} r_1 = x_4 \\ r_2 = x_3 \\ r_3 = x_6 + x_4 x_3 \end{cases} \quad , \quad \begin{cases} \check{\theta}_1 = x_5 \\ \check{\theta}_2 = x_2 \\ \check{\theta}_3 = x_1 \end{cases}
\tag{4.98}
$$

and

$$
\begin{aligned}
\check{e}^1 &= dx_1 - x_4 dx_2 - x_3 dx_5 = d\check{\theta}_3 - r_1 d\check{\theta}_2 - r_2 d\check{\theta}_1 \\
\check{e}^2 &= e^{x_6} dx_2 = e^{r_3 - r_1 r_2} d\check{\theta}_2 \\
\check{e}^3 &= e^{x_6} dx_3 = e^{r_3 - r_1 r_2} dr_2 \\
\check{e}^4 &= e^{-x_6} dx_4 = e^{-r_3 + r_1 r_2} dr_1 \\
\check{e}^5 &= e^{-x_6} dx_5 = e^{-r_3 + r_1 r_2} d\check{\theta}_1 \\
\check{e}^6 &= dx_6 = dr_3 - r_2 dr_1 - r_1 dr_2
\end{aligned}
\tag{4.99}
$$

Define complex $(1,0)$-forms:

$$
\psi^1 = \check{e}^5 + i\check{e}^4 \quad , \quad \psi^2 = \check{e}^2 + i\check{e}^3 \quad , \quad \psi^3 = \check{e}^1 + i\check{e}^6
\tag{4.100}
$$

and complex coordinates

$$
\begin{cases} z_1 = \check{\theta}_1 + ir_1 = x_5 + ix_4 \\ z_2 = \check{\theta}_2 + ir_2 = x_2 + ix_3 \\ z_3 = \check{\theta}_3 + ir_3 = x_1 + i(x_6 + x_4 x_3) \end{cases}
\tag{4.101}
$$

so that

$$\begin{aligned}
\psi^1 &= e^{-r_3+r_1r_2} d\check{\theta}_1 + i e^{-r_3+r_1r_2} dr_1 = e^{-r_3+r_1r_2} dz_1 \\
\psi^2 &= e^{r_3-r_1r_2} d\check{\theta}_2 + i e^{r_3-r_1r_2} dr_2 = e^{r_3-r_1r_2} dz_2 \\
\psi^3 &= (d\check{\theta}_3 - r_2 d\check{\theta}_1 - r_1 d\check{\theta}_2) + i(dr_3 - r_2 dr_1 - r_1 dr_2) = dz_3 - r_2 dz_1 - r_1 dz_2
\end{aligned} \tag{4.102}$$

and

$$\begin{aligned}
\check{\omega} &= \check{e}^{54} + \check{e}^{23} + \check{e}^{16} = \\
&= \frac{i}{2}(\psi^{1\bar{1}} + \psi^{2\bar{2}} + \psi^{3\bar{3}}) = \\
&= \frac{i}{2}\Big(e^{-2(r_3-r_1r_2)} dz_1 \wedge d\bar{z}_1 + e^{2(r_3-r_1r_2)} dz_2 \wedge d\bar{z}_2 + \\
&+ (dz_3 - r_2 dz_1 - r_1 dz_2) \wedge (d\bar{z}_3 - r_2 d\bar{z}_1 - r_1 d\bar{z}_2)\Big)
\end{aligned} \tag{4.103}$$

Now

$$\begin{aligned}
\mathcal{P} \cdot (2\check{\omega}) = i\Big(& e^{-2(r_3-r_1r_2)} d\check{\theta}_1 \wedge dr_1 + e^{2(r_3-r_1r_2)} d\check{\theta}_2 \wedge dr_2 + \\
&+ (d\check{\theta}_3 - r_2 d\check{\theta}_1 - r_1 d\check{\theta}_2) \wedge (dr_3 - r_2 dr_1 - r_1 dr_2)\Big) = \\
i\Big(& d\check{\theta}_1 \wedge \big((e^{-2(r_3-r_1r_2)} + r_2^2) dr_1 + r_1 r_2 dr_2 - r_2 dr_3\big) + \\
&+ d\check{\theta}_2 \wedge \big(r_1 r_2 dr_1 + (e^{2(r_3-r_1r_2)} + r_1^2) dr_2 - r_1 dr_3\big) + \\
&+ d\check{\theta}_3 \wedge \big(-r_2 dr_1 - r_1 dr_2 + dr_3\big)\Big)
\end{aligned} \tag{4.104}$$

Set

$$\begin{aligned}
\eta_1 &:= e^{-2(r_3-r_1r_2)} dr_1 \\
\eta_2 &:= e^{2(r_3-r_1r_2)} dr_2 \\
\eta_3 &:= dr_3 - r_2 dr_1 - r_1 dr_2
\end{aligned} \tag{4.105}$$

and

$$\begin{aligned}
\tilde{\eta}_1 &:= \eta_1 - r_2 \eta_3 \\
\tilde{\eta}_2 &:= \eta_2 - r_1 \eta_3 \\
\tilde{\eta}_3 &:= \eta_3
\end{aligned} \tag{4.106}$$

so that we can rewrite the two-form as

$$\mathcal{P}\cdot(2\check{\omega}) = i\sum_{i=1}^{3} d\check{\theta}_i\wedge\tilde{\eta}_i \tag{4.107}$$

Take the product

$$\begin{aligned} e^{\mathcal{P}\cdot 2\check{\omega}}\wedge e^{\frac{F}{2i}} &= e^{i\sum_{i=1}^{3} d\check{\theta}_i\wedge\tilde{\eta}_i}\wedge e^{\sum_{i=1}^{3} d\check{\theta}_i\wedge d\theta_i} \\ &= e^{\sum_{i=1}^{3} d\check{\theta}_i\wedge(d\theta_i+i\tilde{\eta}_i)} \end{aligned} \tag{4.108}$$

Integrating this along the $\check{\theta}_i$'s we get

$$\begin{aligned} &(d\theta_1+i\tilde{\eta}_1)\wedge(d\theta_2+i\tilde{\eta}_2)\wedge(d\theta_3+i\tilde{\eta}_3) = \\ &\big(d\theta_1+i(\eta_1-r_2\eta_3)\big)\wedge\big(d\theta_2+i(\eta_2-r_1\eta_3)\big)\wedge\big(d\theta_3+i\eta_3\big) = \\ &\big(d\theta_1+i\eta_1\big)\wedge\big(d\theta_2+i\eta_2\big)\wedge\big(d\theta_3+i\eta_3\big)+ \\ &+\big(r_2\eta_3\wedge\eta_2\wedge d\theta_3+r_1\eta_1\wedge\eta_3\wedge d\theta_3\big)+i\big(-r_1 d\theta_1\wedge\eta_3\wedge d\theta_3-r_2\eta_3\wedge d\theta_2\wedge d\theta_3\big) \end{aligned} \tag{4.109}$$

while the expression for Ω in (4.97)

$$\begin{aligned} &\Big((d\theta_1+r_2 d\theta_3)+ie^{-2(r_3-r_1r_2)}dr_1\Big)\wedge\Big((d\theta_2+r_1 d\theta_3)+ie^{2(r_3-r_1r_2)}dr_2\Big)\wedge \\ &\wedge\Big(d\theta_3+i(dr_3-r_2dr_1-r_1dr_2)\Big) \\ &=\big((d\theta_1+r_2d\theta_3)+i\eta_1\big)\wedge\big((d\theta_2+r_1d\theta_3)+i\eta_2\big)\wedge\big(d\theta_3+i\eta_3\big) \\ &=\big((d\theta_1+i\eta_1\big)\wedge\big((d\theta_2+i\eta_2\big)\wedge\big(d\theta_3+i\eta_3\big)+ \\ &+\big(-r_2d\theta_3\wedge\eta_2\wedge\eta_3-r_1\eta_1\wedge d\theta_3\wedge\eta_3\big)+i\big(r_2d\theta_3\wedge d\theta_2\wedge\eta_3+r_1d\theta_1\wedge d\theta_3\wedge\eta_3\big) \end{aligned} \tag{4.110}$$

and they indeed coincide, showing that $FT(e^{2\check{\omega}})=\Omega$.

4.5 $(M_{\mathbb{T}},\check{M}_{\mathbb{T}})$

IIA Equations on $\mathfrak{g}_{\mathbb{T}}=(0,-16,0,-36,0,0)\simeq(0,0,0,0,12,13)$

$$\begin{aligned} \omega &= e^{41}+e^{23}+e^{65} \\ \Omega &= (e^4+ie^1)\wedge(e^2+ie^3)\wedge(e^6+ie^5) \end{aligned} \tag{4.111}$$

Take

$$\begin{aligned} \mathrm{Re}\,\Omega &= e^{426}-e^{435}-e^{125}-e^{136} \\ \mathrm{Im}\,\Omega &= e^{425}+e^{436}+e^{126}-e^{135} \end{aligned} \tag{4.112}$$

$$\begin{aligned} d\omega &= e^{136} - e^{136} = 0 \\ d\text{Re }\Omega &= 0 \\ d\text{Im }\Omega &= -e^{2536} - e^{1456} \neq 0 \end{aligned} \tag{4.113}$$

and

$$\frac{1}{8}\Omega \wedge \bar{\Omega} = -i\ e^{123456} = -i\frac{\omega^3}{6} \tag{4.114}$$

with $F \equiv 8$.

TY Diamond for $\mathfrak{g}_{\mathbb{T}}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 3 & & \\ & 2 & & 6 & & 3 & \\ 1 & & 4 & & 7 & & 1 \\ & 2 & & 6 & & 3 & \\ & & 1 & & 3 & & \\ & & & 1 & & & \end{array} \tag{4.115}$$

IIB Equations on $\check{\mathfrak{g}}_{\mathbb{T}}$ $= (0, 0, 0, 0, 0, 12 + 34)$

$$\begin{aligned} \check{\omega} &= \check{e}^{41} + \check{e}^{23} + \check{e}^{65} \\ \check{\Omega} &= (\check{e}^4 + i\check{e}^1) \wedge (\check{e}^2 + i\check{e}^3) \wedge (\check{e}^6 + i\check{e}^5) \end{aligned} \tag{4.116}$$

We have

$$\begin{aligned} \text{Re }\check{\Omega} &= \check{e}^{426} - \check{e}^{435} - \check{e}^{125} - \check{e}^{136} \\ \text{Im }\check{\Omega} &= \check{e}^{425} + \check{e}^{126} + \check{e}^{126} - \check{e}^{135} \end{aligned} \tag{4.117}$$

$$\begin{aligned} d\check{\omega} &= -\check{e}^{125} - \check{e}^{345} \neq 0 \\ d\check{\omega}^2 &= 2d\check{\omega} \wedge \check{\omega} = 0 \\ d\mathrm{Re}\ \check{\Omega} &= 0 \\ d\mathrm{Im}\ \check{\Omega} &= -\check{e}^{4312} - \check{e}^{1234} = 0 \end{aligned} \tag{4.118}$$

and

$$\frac{1}{8}\check{\Omega} \wedge \bar{\check{\Omega}} = -i\ \check{e}^{123456} = -i\frac{\check{\omega}^3}{6} \tag{4.119}$$

and $\check{F} \equiv 8$

BC Diamond for $\check{\mathfrak{g}}_{\mathbb{T}}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 3 & & \\ & 3 & & 7 & & 3 & \\ 1 & & 6 & & 6 & & 1 \\ & 1 & & 4 & & 1 & \\ & & 2 & & 2 & & \\ & & & 1 & & & \end{array} \tag{4.120}$$

Fourier-Mukai Transform and Mirror Duality for $M_{\mathbb{T}}$ and $\check{M}_{\mathbb{T}}$

The SYZ-dual fibrations are

$$\begin{array}{ccc} M_{\mathbb{T}} = \frac{\mathbb{R}^3 \ltimes_{\varphi^{\mathbb{T}}} \mathbb{R}^3}{\mathbb{Z}^3 \ltimes_{\mathfrak{f}^{\mathbb{T}}} \mathbb{Z}^3} & & \check{M}_{\mathbb{T}} = \frac{\mathbb{R}^3 \ltimes_{\lambda_{\mathbb{T}}} \mathbb{R}^3}{\mathbb{Z}^3 \ltimes_{\mathfrak{l}_{\mathbb{T}}} \mathbb{Z}^3} \\ & \overset{\pi}{\searrow} \quad \overset{\check{\pi}}{\swarrow} & \\ & B = \mathbb{R}^3/\mathbb{Z}^3 & \end{array} \tag{4.121}$$

We start from the symplectic side where the symplectic form ω

$$\begin{aligned}\omega &= e^{41} + e^{23} + e^{65} = \\ &= (dx_4 + x_3 dx_6) \wedge dx_1 + (dx_2 + x_1 dx_6) \wedge dx_3 + dx_6 \wedge dx_5 \\ &= dx_4 \wedge dx_1 + x_3 dx_6 \wedge dx_1 + dx_2 \wedge dx_3 + x_1 dx_6 \wedge dx_3 + dx_6 \wedge dx_5 \\ &= dx_4 \wedge dx_1 + dx_2 \wedge dx_3 + dx_6 \wedge d(x_5 + x_1 x_3)\end{aligned} \tag{4.122}$$

gives as action coordinates on the base and angle coordinates on the fibers respectively

$$\begin{cases} r_1 = x_1 \\ r_2 = x_3 \\ r_3 = x_5 + x_1 x_3 \end{cases} , \quad \begin{cases} \theta_1 = x_4 \\ \theta_2 = x_2 \\ \theta_3 = x_6 \end{cases} \tag{4.123}$$

and $\omega = \sum_{i=1}^{3} d\theta_i \wedge dr_i$.
Rewrite the coframe of differential 1-forms in action-angle coordinates

$$\begin{aligned} e^1 &= dx_1 = dr_1 \\ e^2 &= dx_2 + x_1 dx_6 = d\theta_2 + r_1 d\theta_3 \\ e^3 &= dx_3 = dr_2 \\ e^4 &= dx_4 + x_3 dx_6 = d\theta_1 + r_2 d\theta_3 \\ e^5 &= dx_5 = dr_3 - r_1 dr_2 - r_2 dr_1 \\ e^6 &= dx_6 = d\theta_3 \end{aligned} \tag{4.124}$$

The expression of the three-form Ω in these coordinates is

$$\begin{aligned}\Omega &= (e^4 + ie^1) \wedge (e^2 + ie^3) \wedge (e^6 + ie^5) \\ &= \Big((d\theta_1 + r_2 d\theta_3) + i dr_1\Big) \wedge \Big((d\theta_2 + r_1 d\theta_3) + i dr_2\Big) \wedge \Big(d\theta_3 + i(dr_3 - r_2 dr_1 - r_1 dr_2)\Big) \\ &= \Big(d\theta_1 + i\big((1 + r_2^2) dr_1 + r_1 r_2 dr_2 - r_2 dr_3\big)\Big) \wedge \Big(d\theta_2 + i\big(r_1 r_2 dr_1 + (1 + r_1^2) dr_2 - r_1 dr_3\big)\Big) \wedge \\ &\wedge \Big(d\theta_3 + i\big(-r_2 dr_1 - r_1 dr_2 + dr_3\big)\Big)\end{aligned} \tag{4.125}$$

On the complex side instead, the dual action-angle coordinates are

$$\begin{cases} r_1 = x_1 \\ r_2 = x_3 \\ r_3 = x_5 + x_1 x_3 \end{cases} , \quad \begin{cases} \check{\theta}_1 = x_4 \\ \check{\theta}_2 = x_2 \\ \check{\theta}_3 = x_6 \end{cases} \tag{4.126}$$

and

$$\begin{aligned}
\check{e}^1 &= dx_1 = dr_1 \\
\check{e}^2 &= dx_2 = d\check{\theta}_2 \\
\check{e}^3 &= dx_3 = dr_2 \\
\check{e}^4 &= dx_4 = d\check{\theta}_1 \\
\check{e}^5 &= dx_5 = dr_3 - r_2 dr_1 - r_1 dr_2 \\
\check{e}^6 &= dx_6 - x_1 dx_2 - x_3 dx_4 = d\check{\theta}_3 - r_1 d\check{\theta}_2 - r_2 d\check{\theta}_1
\end{aligned} \tag{4.127}$$

Define complex $(1,0)$-forms:

$$\psi^1 = \check{e}^4 + i\check{e}^1 \quad , \quad \psi^2 = \check{e}^2 + i\check{e}^3 \quad , \quad \psi^3 = \check{e}^6 + i\check{e}^5 \tag{4.128}$$

and complex coordinates

$$\begin{cases}
z_1 = \check{\theta}_1 + ir_1 = x_4 + ix_1 \\
z_2 = \check{\theta}_2 + ir_2 = x_2 + ix_3 \\
z_3 = \check{\theta}_3 + ir_3 = x_6 + i(x_5 + x_1 x_3)
\end{cases} \tag{4.129}$$

so that

$$\begin{aligned}
\psi^1 &= dz_1 \\
\psi^2 &= dz_2 \\
\psi^3 &= dz_3 - r_2 dz_1 - r_1 dz_2
\end{aligned} \tag{4.130}$$

and

$$\begin{aligned}
\check{\omega} &= \check{e}^{41} + \check{e}^{23} + \check{e}^{65} \\
&= \frac{i}{2}(\psi^{1\bar{1}} + \psi^{2\bar{2}} + \psi^{3\bar{3}}) \\
&= \frac{i}{2}\Big(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + (dz_3 - r_2 dz_1 - r_1 dz_2) \wedge (d\bar{z}_3 - r_2 d\bar{z}_1 - r_1 d\bar{z}_2)\Big) \\
&= \frac{i}{2}\Big(dz_1 \wedge \big((1 + r_2^2) d\bar{z}_1 + r_1 r_2 d\bar{z}_2 - r_2 d\bar{z}_3\big) + \\
&+ dz_2 \wedge \big(r_1 r_2 d\bar{z}_1 + (1 + r_1^2) d\bar{z}_2 - r_1 d\bar{z}_3\big) + \\
&+ dz_3 \wedge \big(d\bar{z}_3 - r_2 d\bar{z}_1 - r_1 d\bar{z}_2\big)\Big)
\end{aligned} \tag{4.131}$$

$$\begin{aligned}
\mathcal{P} \cdot (2\check{\omega}) = i\Big(& d\check{\theta}_1 \wedge \big((1 + r_2^2) dr_1 + r_1 r_2 dr_2 - r_2 dr_3\big) + \\
&+ d\check{\theta}_2 \wedge \big(r_1 r_2 dr_1 + (1 + r_1^2) dr_2 - r_1 dr_3\big) + \\
&+ d\check{\theta}_3 \wedge \big(- r_2 dr_1 - r_1 dr_2 + dr_3\big)\Big)
\end{aligned} \tag{4.132}$$

Set

$$\begin{aligned}\eta_1 &:= (1+r_2^2)dr_1 + r_1r_2dr_2 - r_2dr_3 \\ \eta_2 &:= r_1r_2dr_1 + (1+r_1^2)dr_2 - r_1dr_3 \\ \eta_3 &:= dr_3 - r_2dr_1 - r_1dr_2\end{aligned} \tag{4.133}$$

so that we can rewrite the two-form as

$$\mathcal{P}\cdot(2\breve{\omega}) = i\sum_{i=1}^{3} d\check{\theta}_i \wedge \eta_i \tag{4.134}$$

Take the product

$$\begin{aligned}e^{\mathcal{P}\cdot 2\breve{\omega}} \wedge e^{\frac{F}{2i}} &= e^{i\sum_{i=1}^{3} d\check{\theta}_i\wedge\eta_i} \wedge e^{\sum_{i=1}^{3} d\check{\theta}_i\wedge d\theta_i} \\ &= e^{\sum_{i=1}^{3} d\check{\theta}_i\wedge(d\theta_i+i\eta_i)}\end{aligned} \tag{4.135}$$

integrating this along the $\check{\theta}_i$'s we get

$$\begin{aligned}&(d\theta_1+i\eta_1)\wedge(d\theta_2+i\eta_2)\wedge(d\theta_3+i\eta_3) = \\ &\Big(d\theta_1 + i\big((1+r_2^2)dr_1 + r_1r_2dr_2 - r_2dr_3\big)\Big)\wedge\Big(d\theta_2 + i\big(r_1r_2dr_1 + (1+r_1^2)dr_2 - r_1dr_3\big)\Big)\wedge \\ &\wedge\Big(d\theta_3 + i\big(-r_2dr_1 - r_1dr_2 + dr_3\big)\Big) = \\ &\Big((d\theta_1 + r_2d\theta_3) + idr_1\Big)\wedge\Big((d\theta_2 + r_1d\theta_3) + idr_2\Big)\wedge\Big(d\theta_3 + i(dr_3 - r_2dr_1 - r_1dr_2)\Big)\end{aligned} \tag{4.136}$$

which agrees with (4.125), showing that $FT(e^{2\breve{\omega}}) = \Omega$.

4.6 $(M_Y, \check{M}_Y)$

The NLA $\mathfrak{g}_Y = \check{\mathfrak{g}}_Y = (0,0,0,12,13,15)$ has both a symplectic and a complex structure (see [76]), but it does not admit any half-flat nor balanced structure. Nevertheless, it fits in our SYZ construction and leads to a new mirror pair. In this section we will show only the fibration and the diamonds. Cohomology computations are, as above, in the appendix.

Mirror Duality for M_Y and $\check{M}_Y$

The SYZ-dual fibrations are

$$
\begin{array}{ccc}
M_Y = \frac{H_3(\mathbb{R}) \ltimes_{\varphi^Y} \mathbb{R}^3}{H_3(\mathbb{Z}) \ltimes_{\mathfrak{f}^Y} \mathbb{Z}^3} & & \check{M}_Y = \frac{H_3(\mathbb{R}) \ltimes_{\lambda_Y} \mathbb{R}^3}{H_3(\mathbb{Z}) \ltimes_{\mathfrak{l}_Y} \mathbb{Z}^3} \\
& \searrow^{\pi} \qquad {}^{\check{\pi}}\swarrow & \\
& B = H_3(\mathbb{R})/H_3(\mathbb{Z}) &
\end{array}
\tag{4.137}
$$

The symplectic SU(3)-system on M_Y is

$$
\begin{aligned}
\omega &= e^{61} + e^{52} + e^{34} \\
\Omega &= \left(e^6 + ie^1\right) \wedge \left(e^5 + ie^2\right) \wedge \left(e^3 + ie^4\right)
\end{aligned}
\tag{4.138}
$$

Indeed

$$d\omega = 0$$

but

$$d\mathrm{Re}\Omega = d(e^{653} - e^{624} - e^{154} - e^{123}) = -e^{1245}$$

and

$$d\mathrm{Im}\Omega = d(e^{654} + e^{623} + e^{153} - e^{124}) = -e^{1346} + e^{1235}$$

While on $\check{M}_Y$, the complex SU(3)-structure is given by

$$
\begin{aligned}
\check{\omega} &= \check{e}^{31} + \check{e}^{52} + \check{e}^{64} \\
\check{\Omega} &= \left(\check{e}^3 + i\check{e}^1\right) \wedge \left(\check{e}^5 + i\check{e}^2\right) \wedge \left(\check{e}^6 + i\check{e}^4\right)
\end{aligned}
\tag{4.139}
$$

with

$$
\begin{gathered}
d\check{\omega} = -\check{e}^{123} + \check{e}^{124} + \check{e}^{145}, \\
d\check{\omega}^2 = d\check{\omega} \wedge \check{\omega} = \check{e}^{12346}
\end{gathered}
$$

and

$$d\mathrm{Re}\check{\Omega} = d(\check{e}^{356} - \check{e}^{324} - \check{e}^{154} - \check{e}^{126}) = 0$$

$$d\mathrm{Im}\check{\Omega} = d(\check{e}^{354} + \check{e}^{326} + \check{e}^{156} - \check{e}^{124}) = 0$$

TY Diamond for $\mathfrak{g}_Y$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 2 & & \\ & 1 & & 5 & & 3 & \\ 1 & & 4 & & 5 & & 1 \\ & 1 & & 5 & & 3 & \\ & & 1 & & 2 & & \\ & & & 1 & & & \end{array} \tag{4.140}$$

BC Diamond for $\check{\mathfrak{g}}_Y$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 3 & & \\ & 2 & & 5 & & 2 & \\ 1 & & 5 & & 5 & & 1 \\ & 1 & & 4 & & 1 & \\ & & 1 & & 1 & & \\ & & & 1 & & & \end{array} \tag{4.141}$$

Fourier-Mukai Transform and Mirror Duality for M_Y and $\check{M}_Y$

The symplectic form on M_Y

$$
\begin{aligned}
\omega &= e^{61} + e^{52} + e^{34} \\
&= (dx_6 + x_1 dx_5 + \frac{x_1^2}{2} dx_3) \wedge dx_1 + (dx_5 + x_1 dx_3) \wedge dx_2 + dx_3 \wedge (dx_4 - x_1 dx_2) \\
&= dx_6 \wedge dx_1 + x_1 dx_5 \wedge dx_1 + \frac{x_1^2}{2} dx_3 \wedge dx_1 + dx_5 \wedge dx_2 + x_1 dx_3 \wedge dx_2 + \\
&+ dx_3 \wedge dx_4 - x_1 dx_3 \wedge dx_2 \\
&= dx_6 \wedge dx_1 + dx_5 \wedge (dx_2 + x_1 dx_1) + dx_3 \wedge (dx_4 + \frac{x_1^2}{2} dx_1)
\end{aligned}
\tag{4.142}
$$

gives action-angle coordinates

$$
\begin{cases} r_1 = x_1 \\ r_2 = x_2 + \frac{x_1^2}{2} \\ r_3 = x_4 + \frac{x_1^3}{6} \end{cases} \quad , \quad \begin{cases} \theta_1 = x_6 \\ \theta_2 = x_5 \\ \theta_3 = x_3 \end{cases} \tag{4.143}
$$

and ω can be written as $\omega = \sum_{i=1}^{3} d\theta_i \wedge dr_i$.

Rewrite the coframe of differential 1-forms in action-angle coordinates

$$
\begin{aligned}
e^1 &= dx_1 = dr_1 \\
e^2 &= dx_2 = dr_2 - r_1 dr_1 \\
e^3 &= dx_3 = d\theta_3 \\
e^4 &= dx_4 - x_1 dx_2 = dr_3 + \frac{r_1^2}{2} dr_1 - r_1 dr_2 \\
e^5 &= dx_5 + x_1 dx_3 = d\theta_2 + r_1 d\theta_3 \\
e^6 &= dx_6 + x_1 dx_5 + \frac{x_1^2}{2} dx_3 = d\theta_1 + r_1 d\theta_2 + \frac{r_1^2}{2} d\theta_3
\end{aligned}
\tag{4.144}
$$

The expression of the three-form Ω in these coordinates is

$$
\begin{aligned}
\Omega &= (e^6 + ie^1) \wedge (e^5 + ie^2) \wedge (e^3 + ie^4) \\
&= \Big((d\theta_1 + r_1 d\theta_2 + \frac{r_1^2}{2} d\theta_3) + i dr_1 \Big) \wedge \Big((d\theta_2 + r_1 d\theta_3) + i(dr_2 - r_1 dr_1) \Big) \wedge \\
&\wedge \Big(d\theta_3 + i(dr_3 + \frac{r_1^2}{2} dr_1 - r_1 dr_2) \Big) \\
&= \Big(d\theta_1 + i\big((1 + r_1^2 + \frac{r_1^4}{4}) dr_1 + (-r_1 - \frac{r_1^3}{2}) dr_2 + \frac{r_1^2}{2} dr_3 \Big) \wedge \\
&\Big(d\theta_2 + i\big((-r_1 - \frac{r_1^3}{2}) dr_1 + (1 + r_1^2) dr_2 - r_1 dr_3 \big) \Big) \wedge \\
&\Big(d\theta_3 + i\big(dr_3 + \frac{r_1^2}{2} dr_1 - r_1 dr_2 \big) \Big)
\end{aligned}
\tag{4.145}
$$

On the complex side instead, the dual action-angle coordinates are

$$
\begin{cases} r_1 = x_1 \\ r_2 = x_2 + \frac{x_1^2}{2} \\ r_3 = x_4 + \frac{x_1^3}{6} \end{cases} \quad , \quad \begin{cases} \check{\theta}_1 = x_3 \\ \check{\theta}_2 = x_5 \\ \check{\theta}_3 = x_6 \end{cases}
\tag{4.146}
$$

and

$$
\begin{aligned}
\check{e}^1 &= dx_1 = dr_1 \\
\check{e}^2 &= dx_2 = dr_2 - r_1 dr_1 \\
\check{e}^3 &= dx_3 = d\check{\theta}_1 \\
\check{e}^4 &= dx_4 - x_1 dx_2 = dr_3 + \frac{r_1^2}{2} dr_1 - r_1 dr_2 \\
\check{e}^5 &= dx_5 - x_1 dx_3 = d\check{\theta}_2 - r_1 d\theta_1 \\
\check{e}^6 &= dx_6 - x_1 dx_5 + \frac{x_1^2}{2} dx_3 = d\check{\theta}_3 - r_1 d\check{\theta}_2 + \frac{r_1^2}{2} d\check{\theta}_1
\end{aligned}
\tag{4.147}
$$

Define complex $(1,0)$-forms:

$$
\psi^1 = \check{e}^3 + i\check{e}^1 \quad , \quad \psi^2 = \check{e}^5 + i\check{e}^2 \quad , \quad \psi^3 = \check{e}^6 + i\check{e}^4
\tag{4.148}
$$

and complex coordinates

$$
\begin{cases} z_1 = \check{\theta}_1 + ir_1 = x_3 + ix_1 \\ z_2 = \check{\theta}_2 + ir_2 = x_5 + i(x_2 + \frac{x_1^2}{2}) \\ z_3 = \check{\theta}_3 + ir_3 = x_6 + i(x_4 + \frac{x_1^3}{6}) \end{cases}
\tag{4.149}
$$

so that

$$\begin{aligned}\psi^1 &= dz_1 \\ \psi^2 &= dz_2 - r_1 dz_1 \\ \psi^3 &= dz_3 - r_1 dz_2 + \frac{r_1^2}{2} dz_1\end{aligned} \tag{4.150}$$

and

$$\begin{aligned}\breve{\omega} &= \breve{e}^{31} + \breve{e}^{52} + \breve{e}^{64} \\ &= \frac{i}{2}(\psi^{1\bar{1}} + \psi^{2\bar{2}} + \psi^{3\bar{3}}) \\ &= \frac{i}{2}\Big(dz_1 \wedge d\bar{z}_1 + (dz_2 - r_1 dz_1) \wedge (d\bar{z}_2 - r_1 d\bar{z}_2) + (dz_3 + \frac{r_1^2}{2} dz_1 - r_1 dz_2) \wedge (d\bar{z}_3 + \frac{r_1^2}{2} d\bar{z}_1 - r_1 d\bar{z}_2)\Big) \\ &= \frac{i}{2}\Big(dz_1 \wedge \big((1 + r_1^2 + \frac{r_1^4}{4}) d\bar{z}_1 + (-r_1 - \frac{r_1^3}{2}) d\bar{z}_2 + \frac{r_1^2}{2} d\bar{z}_3\big) + \\ &+ dz_2 \wedge \big((-r_1 - \frac{r_1^3}{2}) d\bar{z}_1 + (1 + r_1^2) d\bar{z}_2 - r_1 d\bar{z}_3\big) + \\ &+ dz_3 \wedge (d\bar{z}_3 + \frac{r_1^2}{2} d\bar{z}_1 - r_1 d\bar{z}_2)\Big)\end{aligned} \tag{4.151}$$

$$\begin{aligned}\mathcal{P} \cdot (2\breve{\omega}) = i\Big(&d\breve{\theta}_1 \wedge \big((1 + r_1^2 + \frac{r_1^4}{4}) dr_1 + (-r_1 - \frac{r_1^3}{2}) dr_2 + \frac{r_1^2}{2} dr_3\big) + \\ &+ d\breve{\theta}_2 \wedge \big((-r_1 - \frac{r_1^3}{2}) dr_1 + (1 + r_1^2) dr_2 - r_1 dr_3\big) + \\ &+ d\breve{\theta}_3 \wedge \big(\frac{r_1^2}{2} dr_1 - r_1 dr_2 + dr_3\big)\Big)\end{aligned} \tag{4.152}$$

Set

$$\begin{aligned}\eta_1 &:= (1 + r_1^2 + \frac{r_1^4}{4}) dr_1 + (-r_1 - \frac{r_1^3}{2}) dr_2 + \frac{r_1^2}{2} dr_3 \\ \eta_2 &:= (-r_1 - \frac{r_1^3}{2}) dr_1 + (1 + r_1^2) dr_2 - r_1 dr_3 \\ \eta_3 &:= dr_3 + \frac{r_1^2}{2} dr_1 - r_1 dr_2\end{aligned} \tag{4.153}$$

so that we can rewrite the two-form as

$$\mathcal{P} \cdot (2\breve{\omega}) = i \sum_{i=1}^{3} d\breve{\theta}_i \wedge \eta_i \tag{4.154}$$

Take the product

$$e^{\mathcal{P}\cdot 2\check{\omega}} \wedge e^{\frac{F}{2i}} = e^{i\sum_{i=1}^{3} d\check{\theta}_i\wedge\eta_i} \wedge e^{\sum_{i=1}^{3} d\check{\theta}_i\wedge d\theta_i} \\ = e^{\sum_{i=1}^{3} d\check{\theta}_i\wedge(d\theta_i+i\eta_i)} \tag{4.155}$$

integrating this along the $\check{\theta}_i$'s we get

$$\begin{aligned}&(d\theta_1 + i\eta_1)\wedge(d\theta_2 + i\eta_2)\wedge(d\theta_3 + i\eta_3) = \\ &\Big(d\theta_1 + i\big((1 + r_1^2 + \frac{r_1^4}{4})dr_1 + (-r_1 - \frac{r_1^3}{2})dr_2 + \frac{r_1^2}{2}dr_3\Big)\wedge \\ &\Big(d\theta_2 + i\big((-r_1 - \frac{r_1^3}{2})dr_1 + (1 + r_1^2)dr_2 - r_1 dr_3\big)\Big)\wedge \\ &\Big(d\theta_3 + i\big(dr_3 + \frac{r_1^2}{2}dr_1 - r_1 dr_2\big)\Big)\end{aligned} \tag{4.156}$$

which agrees with (4.145), showing that $FT(e^{2\check{\omega}}) = \Omega$.

Chapter 5

Conclusions

In this last chapter, we sum up the focal points of the thesis and we do some remarks that may inspire further developments. The main achievement of the thesis has been the production of concrete examples of pairs of compact manifolds satisfying the demanding and intricate properties of semi-flat SYZ dual fibrations. We also presented the first examples of mirror non-Kähler diamonds. This was interlaced with the classification results for IIA/IIB structures on solvmanifolds: we have found an appropriate setting in which the construction turned out to be realizable. But the picture is not fully complete. We remark that:

- As initially stated, we presented the mirror partner for each of the known sympletic half-flat (completely solvable) solvmanifold with correspondent Lie algebra:

Symplectic half-flat SLA	Complex-balanced SLA
$(0,0,0,0,12,-13)$	$(0,0,0,0,12,13)$
$(0,0,0,12,-13,-23)$	$(0,0,0,0,12,14+23)$
$(15,-25,-35,45,0,0)$	$(15,-25,-35,45,0,0)$
$(16+35,-26+45,36,-46,0,0)$	$(24+35,26,36,-46,-56,0)$
$(0,-16,0,-36,0,0)$	$(0,0,0,0,0,12+34)$
Symplectic SLA	Complex SLA
$(0,0,0,-12,13,15)$	$(0,0,0,12,13,15)$

Table 5.1: Mirror symmetric SLA's

- As one can see from the classification in [30] we excluded the algebras:
  - $A_{5,17}^{-\alpha,\alpha,1} \oplus \mathbb{R}$ and $\mathfrak{g}_{6,118}^{0,-1,-1}$ are not completely solvable, nevertheless their simply connected Lie groups fit in our construction choosing as starting

3-dimensional Lie group $E(2)$ (and choosing $\alpha = 0$ in the first algebra). We indeed computed the TY cohomologies and checked they correspond to the BC cohomologies of the complex balanced $\mathfrak{g}_2^0$ and $\mathfrak{g}_8$. The latter represents the SLA underlying the Nakamura manifold. To make the construction work completely there is left only to incorporate the analysis of theorem 2.5.4([56]).

  - $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$ is completely solvable but the SU(3)-structure given in [30] cannot be associated to a real polarization induced by any torus-fibration.
  - $\mathfrak{g}_{6,38}^0$ is not completely solvable and it seems not possible to interpret its Lie group as a semi-direct product of the form $G \ltimes_\rho \mathbb{R}^3$ but we do not have a proof.

  We guess that our construction is applicable also on these two last algebras if one allows the fibration to have singularities;

- From the point of view of classification of structures on solvmanifolds we just hit the tip of the iceberg. In fact, there is no symplectic version of the analysis carried out in [60] for complex balanced nilmanifolds. This is related to the fact that the condition on the three-form Ω of having only the real part closed is rather more difficult to be translated into algebraic terms. It would be a nice result to classify all the possible symplectic half-flat structures on a fixed symplectic solvmanifold. Moreover having at hand eventually the Fourier-Mukai transform, when the SYZ fibration exists, it is, in theory, possible to translate every feature of complex non-Kähler geometry into symplectic terms and vice versa (see for the example the corollary 2.5.1 at the end of section 2.5). For example, using the classification of symplectic structures on six-dimensional NLA's given by Goze, Khakimdjanov and Medina [39], we have applied this corollary to exclude the symplectic structure ω_2 on the algebra corresponding to $(0,0,0,0,12,13)$ ((23^{rd}) in [39]) and the symplectic structure ω_3 on the algebra corresponding to $(0,0,0,12,13,23)$ ((18^{th}) in [39]) from admitting a symplectic half-flat structure;

- Using the result of classification for affine structure on three-solvmanifolds by Fried and Goldman, it is possible to extend the list of mirror symmetric SU(3) structures, not necessarily half-flat/balanced, among the six-dimensional solvmanifolds. In fact, it would be a challenging work to classify all possible six-dimensional Lie groups that can be obtained by changing the developing map of the affine structures on the three-dimensional Lie group. The SYZ construction would then produce a mirror pair for each possible choice;

- *The role of* Δ. We want to remark that the choice of the base of the fibration B is indeed crucial in the analysis. Though the total spaces $M_{N,1}$ and $M_{\mathbb{T}}$ are the same symplectic manifold, the choice of a different base, which in turn induces a different Lagrangian distribution, originates two different mirrors, complex balanced partners;

- In the description of the pair $(M_{N,2}, \check{M}_{N,2})$, the SU(3)-structure of type IIA on $M_{N,2}$ was given in [83] (the symplectic structure also in [39]). Applying the Fourier-Mukai transform to Ω_λ we noticed that for $\lambda = -1$ the corresponding SU(3)-structure of type IIB on $\mathfrak{h}_4 = (0, 0, 0, 12, 14 + 23)$ in [60] is not figuring in the complex balanced classification of NLA's;

- In section 2.3.2 we have seen how, following Duistermaat [28], the invariants of a Lagrangian fibration are the Chern class and the monodromy. In particular, a prominent role is played by the cohomology group $H^2(B, \Lambda)$ which parametrizes the classes of isomorphism of Lagrangian fibrations up to symplectomorphism as described by the work of Sepe [77],[78],[79] which, in turn, is based on an idea of Dazord and Delzant [27]. Therefore, is in theory possible to apply the technique presented in [77] to all our examples and obtain a full classification, at least at the topological level.

Both territories of mirror symmetry and of non-Kähler geometry are still pretty unexplored. They are promising and inspiring topics and it is quite likely that the interaction between the two would produce numerous discoveries and a better understanding for both sides.

Appendix A

Cohomology Computations

Table A.1: Tseng-Yau cohomology for SLA's

SLA	$H^{1,0}_{TY}$	$H^{0,1}_{TY}$	$H^{2,0}_{TY}$	$H^{1,1}_{TY}$	$H^{0,2}_{TY}$	$H^{2,1}_{TY}$	$H^{1,2}_{TY}$
$\mathfrak{g}_{N,1}$	2	2	2	6	3	5	6
$\mathfrak{g}_{N,2}^{-1}$	1	2	2	6	3	4	7
$\mathfrak{g}_{N,2}^{2,\frac{1}{2}}$	1	2	2	6	3	5	6
$\mathfrak{g}_{N,2}^{\lambda}$	1	2	2	6	3	4	6
$\mathfrak{g}_{S,1}$	1	1	1	3	1	3	3
$\mathfrak{g}_{S,2}$	1	1	0	2	1	2	1
$\mathfrak{g}_{\mathbb{T}}$	1	3	2	6	3	4	7
$\mathfrak{g}_{Y}$	1	2	1	5	3	4	5

Table A.2: Bott-Chern cohomology for SLA's

SLA	$H^{1,0}_{BC}$	$H^{2,0}_{BC}$	$H^{1,1}_{BC}$	$H^{2,1}_{BC}$	$H^{3,1}_{BC}$	$H^{2,2}_{BC}$	$H^{3,2}_{BC}$
$\check{\mathfrak{g}}_{N,1}$	2	2	5	6	2	6	3
$\check{\mathfrak{g}}_{N,2}^{-1}$	2	1	4	6	2	7	3
$\check{\mathfrak{g}}_{N,2}^{2,\frac{1}{2}}$	2	1	5	6	2	6	3
$\check{\mathfrak{g}}_{N,2}^{\lambda}$	2	1	4	6	2	6	3
$\check{\mathfrak{g}}_{S,1}$	1	1	3	3	1	3	1
$\check{\mathfrak{g}}_{S,2}$	0	1	2	2	1	1	1
$\check{\mathfrak{g}}_{\mathbb{T}}$	2	1	4	6	3	7	3
$\check{\mathfrak{g}}_{Y}$	1	1	4	5	2	5	3

Here are reported the dimensions of the TY/BC cohomology groups for the solvable Lie algebras involved. For the Lie algebra $\mathfrak{g}^{\lambda}_{N,2}$ we distinguish three cases depending on the real parameter $\lambda \in \mathbb{R}\backslash\{0,1\}$ already appeared in section 3.3.1.

We have said that the choice of working with solvable Lie groups/algebras was also motivated by the possibility of computing the cohomology of the solvmanifolds in terms of their associated Lie algebras. The main tool that allows us to exploit this feature is the result of Angella and Kasuya ([3] Theorem 1.3). The required fact is that, with respect to both the operators d, d^{Λ} on one side, and $\partial, \bar{\partial}$ on the other, the inclusion of invariant forms is a quasi-isomorphism. In the symplectic case, this is guaranteed thanks to the result of Macrì ([65]). Instead, for the Bott-Chern cohomology, we have a general result only in the nilpotent case (see Corollary 2.7 in [3]). In the solvable case the situation is more problematic: there is only a result for the solvmanifolds which are of splitting type or are complex-parallelizable (see [3]). Therefore for the Lie algebras $\mathfrak{g}_{N,-}, \mathfrak{g}_{\mathbb{T}}, \mathfrak{g}_{Y}$ and their mirrors those numbers represent also the TY/BC cohomology of the solvmanifolds. This is valid also for $\mathfrak{g}_{S,1}$ and $\check{\mathfrak{g}}_{S,1}$, since it is of splitting type (this corresponds to the example 3.1 in [3] case (iii)). For the remaining $\mathfrak{g}_{S,2}$ the result still holds in view of [65], but for its mirror $\check{\mathfrak{g}}_{S,2}$ we can not obtain the same conclusion in a direct way. Nevertheless, we can a posteriori get the BC-cohomology for the solvmanifolds using Theorem 6.7 in [61] which gives the correspondence with the TY-cohomology. There is a procedure by Kasuya [57] for a generic solvmanifold that relies on the semisimple splitting of the Lie algebra and the structure of the fixed lattice. This could be used for this remaining case and also for the non-completely solvable ones.

A.0.1 Tseng-Yau Cohomology

For each SLA we recall the notation, the symplectic form ω and its dual Lefschetz operator, the complementary Lagrangian distribution $\Delta, \Delta^{\perp}$ induced by the fibration. Then we compute the TY cohomology exhibiting also the generators for the groups.

$\mathfrak{g}_{N,1} = (0,0,0,0,12,-13)$

$$\omega = e^{41} + e^{62} + e^{35} \quad , \quad \Lambda = \iota_1\iota_4 + \iota_2\iota_6 + \iota_5\iota_3$$

$$\Delta = \langle e^4, e^6, e^3 \rangle \quad , \quad \Delta^{\perp} = \langle e^1, e^2, e^5 \rangle$$

	d	Λ	dΛ	Λd	d^Λ	$d+d^\Lambda$	dd^Λ
e^1	0	0	0	0	0	0	0
e^2	0	0	0	0	0	0	0
e^3	0	0	0	0	0	0	0
e^4	0	0	0	0	0	0	0
e^5	$-e^{12}$	0	0	0	0	$-e^{12}$	0
e^6	e^{13}	0	0	0	0	e^{13}	0

	d	Λ	dΛ	Λd	d^Λ	$d+d^\Lambda$	dd^Λ
e^{12}	0	0	0	0	0	0	0
e^{13}	0	0	0	0	0	0	0
e^{14}	0	-1	0	0	0	0	0
e^{15}	0	0	0	0	0	0	0
e^{16}	0	0	0	0	0	0	0
e^{23}	0	0	0	0	0	0	0
e^{24}	0	0	0	0	0	0	0
e^{25}	0	0	0	0	0	0	0
e^{26}	e^{123}	-1	0	0	0	e^{123}	0
e^{34}	0	0	0	0	0	0	0
e^{35}	e^{123}	1	0	0	0	e^{123}	0
e^{36}	0	0	0	0	0	0	0
e^{45}	e^{124}	0	0	e^2	e^2	$e^{124}+e^2$	0
e^{46}	$-e^{134}$	0	0	e^3	$-e^3$	$-e^{134}-e^3$	0
e^{56}	$-e^{126}-e^{135}$	0	0	0	0	$-e^{126}-e^{135}$	0

	d	Λ	dΛ	Λd	d^Λ	d+d^Λ	dd^Λ
e^{123}	0	0	0	0	0	0	0
e^{124}	0	$-e^2$	0	0	0	0	0
e^{125}	0	0	0	0	0	0	0
e^{126}	0	$-e^1$	0	0	0	0	0
e^{134}	0	$-e^3$	0	0	0	0	0
e^{135}	0	e^1	0	0	0	0	0
e^{136}	0	0	0	0	0	0	0
e^{145}	0	$-e^5$	e^{12}	0	e^{12}	e^{12}	0
e^{146}	0	$-e^6$	$-e^{13}$	0	$-e^{13}$	$-e^{13}$	0
e^{156}	0	0	0	0	0	0	0
e^{234}	0	0	0	0	0	0	0
e^{235}	0	e^2	0	0	0	0	0
e^{236}	0	e^3	0	0	0	0	0
e^{245}	0	0	0	0	0	0	0
e^{246}	$-e^{1234}$	e^4	0	e^{23}	$-e^{23}$	$-e^{1234}-e^{23}$	0
e^{256}	$-e^{1235}$	e^5	$-e^{12}$	$-e^{12}$	0	$-e^{1235}$	0
e^{345}	$-e^{1234}$	$-e^4$	0	e^{23}	$-e^{23}$	$-e^{1234}-e^{23}$	0
e^{346}	0	0	0	0	0	0	0
e^{356}	e^{1236}	e^6	e^{13}	e^{13}	0	e^{1236}	0
e^{456}	$e^{1246}+e^{1345}$	0	0	$e^{26}+e^{35}$	$-e^{26}-e^{35}$	$e^{1246}+e^{1345}-e^{26}-e^{35}$	$-2e^{123}$

	d	Λ	dΛ	Λd	d^Λ	d+d^Λ	dd^Λ
e^{1234}	0	$-e^{23}$	0	0	0	0	0
e^{1235}	0	e^{12}	0	0	0	0	0
e^{1236}	0	e^{13}	0	0	0	0	0
e^{1245}	0	e^{25}	0	0	0	0	0
e^{1246}	0	$e^{26}+e^{14}$	e^{123}	0	e^{123}	e^{123}	0
e^{1256}	0	e^{15}	0	0	0	0	0
e^{1345}	0	$e^{35}-e^{14}$	e^{123}	0	e^{123}	e^{123}	0
e^{1346}	0	e^{36}	0	0	0	0	0
e^{1356}	0	e^{16}	0	0	0	0	0
e^{1456}	0	$-e^{56}$	$e^{126}+e^{135}$	0	$e^{126}+e^{135}$	$e^{126}+e^{135}$	0
e^{2345}	0	$-e^{24}$	0	0	0	0	0
e^{2346}	0	$-e^{34}$	0	0	0	0	0
e^{2356}	0	$e^{26}-e^{35}$	0	0	0	0	0
e^{2456}	e^{12345}	$-e^{45}$	$-e^{124}$	$-e^{235}-e^{124}$	e^{235}	$e^{12345}+e^{235}$	0
e^{3456}	$-e^{12346}$	$-e^{46}$	e^{134}	$e^{236}+e^{134}$	$-e^{236}$	$-e^{12346}-e^{236}$	0

	d	Λ	dΛ	Λd	d^Λ	d+d^Λ	dd^Λ
e^{12345}	0	$-e^{235}-e^{124}$	0	0	0	0	0
e^{12346}	0	$-e^{236}-e^{134}$	0	0	0	0	0
e^{12356}	0	$-e^{135}-e^{126}$	0	0	0	0	0
e^{12456}	0	$-e^{256}-e^{145}$	$-e^{1235}$	0	$-e^{1235}$	$-e^{1235}$	0
e^{13456}	0	$-e^{356}-e^{146}$	e^{1236}	0	e^{1236}	e^{1236}	0
e^{23456}	0	$-e^{246}-e^{345}$	0	0	0	0	0

$$
\begin{aligned}
H^1_{TY} &= \langle e^1, e^2, e^3, e^4 \rangle \\
H^2_{TY} &= \langle e^{12}, e^{13}, e^{14}, e^{15}, e^{16}, e^{23}, e^{24}, e^{25}, e^{26} - e^{35}, e^{34}, e^{36} \rangle \\
H^3_{TY} &= \langle e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{156}, e^{234}, e^{235}, e^{236}, e^{245}, e^{246} - e^{345}, e^{346} \rangle \\
H^4_{TY} &= \langle e^{1234}, e^{1235}, e^{1236}, e^{1245}, e^{1246} - e^{1345}, e^{1256}, e^{1346}, e^{1356}, e^{2345}, e^{2346}, e^{2356} \rangle \\
H^5_{TY} &= \langle e^{12345}, e^{12346}, e^{12356}, e^{23456} \rangle
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
H^{(0,0)^\Delta}_{TY} &= \langle 1 \rangle \\
H^{(1,0)^\Delta}_{TY} &= \langle e^3, e^4 \rangle \\
H^{(0,1)^\Delta}_{TY} &= \langle e^1, e^2 \rangle \\
H^{(2,0)^\Delta}_{TY} &= \langle e^{34}, e^{36} \rangle \\
H^{(0,2)^\Delta}_{TY} &= \langle e^{12}, e^{15}, e^{25} \rangle \\
H^{(1,1)^\Delta}_{TY} &= \langle e^{13}, e^{14}, e^{16}, e^{23}, e^{24}, e^{26} - e^{35} \rangle \\
H^{(3,0)^\Delta}_{TY} &= \langle e^{436} \rangle \\
H^{(0,3)^\Delta}_{TY} &= \langle e^{125} \rangle \\
H^{(2,1)^\Delta}_{TY} &= \langle e^{134}, e^{136}, e^{234}, e^{236}, e^{246} - e^{345} \rangle \\
H^{(1,2)^\Delta}_{TY} &= \langle e^{124}, e^{126}, e^{135}, e^{156}, e^{235}, e^{245} \rangle \\
H^{(3,1)^\Delta}_{TY} &= \langle e^{1346}, e^{2346} \rangle \\
H^{(2,2)^\Delta}_{TY} &= \langle e^{1234}, e^{1236}, e^{1246} - e^{1345}, e^{1356}, e^{2345}, e^{2356} \rangle \\
H^{(1,3)^\Delta}_{TY} &= \langle e^{1235}, e^{1245}, e^{1256} \rangle \\
H^{(3,2)^\Delta}_{TY} &= \langle e^{12346}, e^{23456} \rangle \\
H^{(2,3)^\Delta}_{TY} &= \langle e^{12345}, e^{12356} \rangle \\
H^{(3,3)^\Delta}_{TY} &= \langle e^{123456} \rangle
\end{aligned}
\tag{A.2}
$$

$\mathfrak{g}_{N,2} = (0, 0, 0, 12, -13, -23)$

$$
\omega_\lambda = e^{61} + \lambda e^{52} + (\lambda - 1) e^{34} \quad , \quad \Lambda_\lambda = \iota_1 \iota_6 + \frac{1}{\lambda} \iota_2 \iota_5 + \frac{1}{\lambda - 1} \iota_4 \iota_3
$$

$$
\Delta = \langle e^6, e^5, e^3 \rangle \quad , \quad \Delta^\perp = \langle e^1, e^2, e^4 \rangle
$$

	d	Λ	dΛ	Λd	d$^\Lambda$	d+d$^\Lambda$	dd$^\Lambda$
e^1	0	0	0	0	0	0	0
e^2	0	0	0	0	0	0	0
e^3	0	0	0	0	0	0	0
e^4	$-e^{12}$	0	0	0	0	$-e^{12}$	0
e^5	e^{13}	0	0	0	0	e^{13}	0
e^6	e^{23}	0	0	0	0	e^{23}	0

	d	Λ	dΛ	Λd	d$^\Lambda$	d+d$^\Lambda$	dd$^\Lambda$
e^{12}	0	0	0	0	0	0	0
e^{13}	0	0	0	0	0	0	0
e^{14}	0	0	0	0	0	0	0
e^{15}	0	0	0	0	0	0	0
e^{16}	$-e^{123}$	-1	0	0	0	$-e^{123}$	0
e^{23}	0	0	0	0	0	0	0
e^{24}	0	0	0	0	0	0	0
e^{25}	e^{123}	$-\frac{1}{\lambda}$	0	0	0	e^{123}	0
e^{26}	0	0	0	0	0	0	0
e^{34}	e^{123}	$\frac{1}{\lambda-1}$	0	0	0	e^{123}	0
e^{35}	0	0	0	0	0	0	0
e^{36}	0	0	0	0	0	0	0
e^{45}	$-e^{125}-e^{134}$	0	0	$-\frac{1}{\lambda(\lambda-1)}e^1$	$\frac{1}{\lambda(\lambda-1)}e^1$	$-e^{125}-e^{134}+\frac{1}{\lambda(\lambda-1)}e^1$	0
e^{46}	$-e^{126}-e^{234}$	0	0	$-\frac{\lambda}{\lambda-1}e^2$	$\frac{\lambda}{\lambda-1}e^2$	$-e^{126}-e^{234}+\frac{\lambda}{\lambda-1}e^2$	0
e^{56}	$e^{136}-e^{235}$	0	0	$\frac{\lambda-1}{\lambda}e^3$	$-\frac{\lambda-1}{\lambda}e^3$	$e^{136}-e^{235}+\frac{\lambda-1}{\lambda}e^3$	0

	d	Λ	dΛ	Λd	d^Λ	$d+d^\Lambda$	dd^Λ
e^{123}	0	0	0	0	0	0	0
e^{124}	0	0	0	0	0	0	0
e^{125}	0	$\frac{1}{\lambda}e^1$	0	0	0	0	0
e^{126}	0	e^2	0	0	0	0	0
e^{134}	0	$\frac{1}{\lambda-1}e^1$	0	0	0	0	0
e^{135}	0	0	0	0	0	0	0
e^{136}	0	e^3	0	0	0	0	0
e^{145}	0	0	0	0	0	0	0
e^{146}	e^{1234}	e^4	$-e^{12}$	$\frac{1}{\lambda-1}e^{12}$	$-\frac{\lambda}{\lambda-1}e^{12}$	$e^{1234}-\frac{\lambda}{\lambda-1}e^{12}$	0
e^{156}	e^{1235}	e^5	e^{13}	$\frac{1}{\lambda}e^{13}$	$\frac{\lambda-1}{\lambda}e^{13}$	$e^{1235}+\frac{\lambda-1}{\lambda}e^{13}$	0
e^{234}	0	$\frac{1}{\lambda-1}e^2$	0	0	0	0	0
e^{235}	0	$\frac{1}{\lambda}e^3$	0	0	0	0	0
e^{236}	0	0	0	0	0	0	0
e^{245}	$-e^{1234}$	$\frac{1}{\lambda}e^4$	$-\frac{1}{\lambda}e^{12}$	$-\frac{1}{\lambda-1}e^{12}$	$\frac{1}{\lambda(\lambda-1)}e^{12}$	$-e^{1234}+\frac{1}{\lambda(\lambda-1)}e^{12}$	0
e^{246}	0	0	0	0	0	0	0
e^{256}	e^{1236}	$-\frac{1}{\lambda}e^6$	$-\frac{1}{\lambda}e^{23}$	$-e^{23}$	$\frac{\lambda-1}{\lambda}e^{23}$	$e^{1236}+\frac{\lambda-1}{\lambda}e^{23}$	0
e^{345}	e^{1235}	$\frac{1}{\lambda-1}e^5$	$-\frac{1}{\lambda-1}e^{13}$	$\frac{1}{\lambda}e^{13}$	$\frac{1}{\lambda(\lambda-1)}e^{13}$	$e^{1235}+\frac{1}{\lambda(\lambda-1)}e^{13}$	0
e^{346}	e^{1236}	$\frac{1}{\lambda-1}e^6$	$\frac{1}{\lambda-1}e^{23}$	$-e^{23}$	$\frac{\lambda}{\lambda-1}e^{23}$	$e^{1236}+\frac{\lambda}{\lambda-1}e^{23}$	0
e^{356}	0	0	0	0	0	0	0
e^{456}	ϵ	0	0	$-\kappa$	κ	$\epsilon+\kappa$	$-2\frac{\lambda^2-\lambda+1}{\lambda(\lambda-1)}e^{123}$

where $\epsilon = -e^{1256} - e^{1346} + e^{2345}$ and $\kappa = \frac{1}{\lambda(\lambda-1)}e^{16} - \frac{\lambda}{\lambda-1}e^{25} - \frac{\lambda-1}{\lambda}e^{34}$

	d	Λ	dΛ	Λd	d^Λ	$d+d^\Lambda$	dd^Λ
e^{1234}	0	$\frac{1}{\lambda-1}e^{12}$	0	0	0	0	0
e^{1235}	0	$\frac{1}{\lambda}e^{13}$	0	0	0	0	0
e^{1236}	0	$-e^{23}$	0	0	0	0	0
e^{1245}	0	$\frac{1}{\lambda}e^{14}$	0	0	0	0	0
e^{1246}	0	$-e^{24}$	0	0	0	0	0
e^{1256}	0	$-e^{25}-\frac{1}{\lambda}e^{16}$	$\frac{1-\lambda}{\lambda}e^{123}$	0	$\frac{1-\lambda}{\lambda}e^{123}$	$\frac{1-\lambda}{\lambda}e^{123}$	0
e^{1345}	0	$\frac{1}{\lambda-1}e^{15}$	0	0	0	0	0
e^{1346}	0	$-e^{34}+\frac{1}{\lambda-1}e^{16}$	$-\frac{\lambda}{\lambda-1}e^{123}$	0	$-\frac{\lambda}{\lambda-1}e^{123}$	$-\frac{\lambda}{\lambda-1}e^{123}$	0
e^{1356}	0	$-e^{35}$	0	0	0	0	0
e^{1456}	$-e^{12345}$	$-e^{45}$	$e^{125}+e^{134}$	$-\frac{1}{\lambda-1}e^{125}+\frac{1}{\lambda}e^{134}$	$\frac{\lambda}{\lambda-1}e^{125}+\frac{1-\lambda}{\lambda}e^{134}$	$-e^{12345}+\frac{\lambda}{\lambda-1}e^{125}+\frac{1-\lambda}{\lambda}e^{134}$	0
e^{2345}	0	$-\frac{1}{\lambda}e^{34}+\frac{1}{\lambda-1}e^{25}$	$\frac{1}{\lambda(\lambda-1)}e^{123}$	0	$\frac{1}{\lambda(\lambda-1)}e^{123}$	$\frac{1}{\lambda(\lambda-1)}e^{123}$	0
e^{2346}	0	$\frac{1}{\lambda-1}e^{26}$	0	0	0	0	0
e^{2356}	0	$\frac{1}{\lambda}e^{36}$	0	0	0	0	0
e^{2456}	$-e^{12346}$	$\frac{1}{\lambda}e^{46}$	$-\frac{1}{\lambda}e^{126}-\frac{1}{\lambda}e^{234}$	$-e^{234}-\frac{1}{\lambda-1}e^{126}$	$\frac{\lambda-1}{\lambda}e^{234}+\frac{1}{\lambda(\lambda-1)}e^{126}$	$-e^{12346}+\frac{\lambda-1}{\lambda}e^{234}+\frac{1}{\lambda(\lambda-1)}e^{126}$	0
e^{3456}	e^{12356}	$\frac{1}{\lambda-1}e^{56}$	$\frac{1}{\lambda-1}e^{136}-\frac{1}{\lambda}e^{235}$	$e^{235}+\frac{1}{\lambda}e^{136}$	$\frac{1}{\lambda(\lambda-1)}e^{136}-\frac{\lambda}{\lambda-1}e^{235}$	$e^{12356}+\frac{1}{\lambda(\lambda-1)}e^{136}-\frac{\lambda}{\lambda-1}e^{235}$	0

	d	Λ	dΛ	Λd	d^Λ	$d+d^\Lambda$	dd^Λ
e^{12345}	0	$-\frac{1}{\lambda}e^{134}+\frac{1}{\lambda-1}e^{125}$	0	0	0	0	0
e^{12346}	0	$\frac{1}{\lambda-1}e^{126}+e^{234}$	0	0	0	0	0
e^{12356}	0	$e^{235}+\frac{1}{\lambda}e^{136}$	0	0	0	0	0
e^{12456}	0	$e^{245}+\frac{1}{\lambda}e^{146}$	$\frac{1-\lambda}{\lambda}e^{1234}$	0	$\frac{1-\lambda}{\lambda}e^{1234}$	$\frac{1-\lambda}{\lambda}e^{1234}$	0
e^{13456}	0	$\frac{\lambda}{\lambda-1}e^{1235}$	$\frac{\lambda}{\lambda-1}e^{1235}$	0	$\frac{\lambda}{\lambda-1}e^{1235}$	$\frac{\lambda}{\lambda-1}e^{1235}$	0
e^{23456}	0	$\frac{1}{\lambda-1}e^{256}-\frac{1}{\lambda}e^{346}$	$\frac{1}{\lambda(\lambda-1)}e^{1236}$	0	$\frac{1}{\lambda(\lambda-1)}e^{1236}$	$\frac{1}{\lambda(\lambda-1)}e^{1236}$	0

$$
\begin{aligned}
H^1_{TY} &= \langle e^1, e^2, e^3\rangle \\
H^2_{TY} &= \langle e^{12}, e^{13}, e^{14}, e^{15}, e^{23}, e^{24}, e^{26}, e^{35}, e^{36}, e^{16}+e^{25}, e^{16}+e^{34}\rangle \\
H^3_{TY} &= \langle e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{145}, e^{234}, e^{235}, e^{236}, e^{246}, e^{356}, \gamma_\lambda\rangle \\
H^4_{TY} &= \langle e^{1234}, e^{1235}, e^{1236}, e^{1245}, e^{1246}, e^{1345}, e^{1356}, e^{2346}, e^{2356}, \lambda^2 e^{2345}+e^{1346}, \\
&\qquad (\lambda-1)^2 e^{2345}+e^{1256}\rangle \\
H^5_{TY} &= \langle e^{12345}, e^{12346}, e^{12356}\rangle
\end{aligned}
\tag{A.3}
$$

where γ_λ is one of

$$
\gamma_\lambda = \begin{cases} e^{146}+e^{245} & \text{if } \lambda=-1 \\ e^{156}-e^{345} & \text{if } \lambda=2 \\ e^{256}-e^{346} & \text{if } \lambda=\frac{1}{2} \\ 0 \quad \text{otherwise} \end{cases}
\tag{A.4}
$$

$$
\begin{aligned}
H^{(0,0)^\Delta}_{TY} &= \langle 1\rangle \\
H^{(1,0)^\Delta}_{TY} &= \langle e^3\rangle \\
H^{(0,1)^\Delta}_{TY} &= \langle e^1, e^2\rangle \\
H^{(2,0)^\Delta}_{TY} &= \langle e^{35}, e^{36}\rangle \\
H^{(1,1)^\Delta}_{TY} &= \langle e^{13}, e^{15}, e^{23}, e^{26}, e^{16}+e^{25}, e^{16}+e^{34}\rangle \\
H^{(0,2)^\Delta}_{TY} &= \langle e^{12}, e^{14}, e^{24}\rangle \\
H^{(3,0)^\Delta}_{TY} &= \langle e^{356}\rangle \\
H^{(2,1)^\Delta}_{TY} &= \langle e^{135}, e^{136}, e^{235}, e^{236}, \gamma'_\lambda\rangle \\
H^{(1,2)^\Delta}_{TY} &= \langle e^{125}, e^{126}, e^{134}, e^{145}, e^{234}, e^{246}, \gamma''_\lambda\rangle \\
H^{(0,3)^\Delta}_{TY} &= \langle e^{124}\rangle \\
H^{(3,1)^\Delta}_{TY} &= \langle e^{1356}, e^{2356}\rangle \\
H^{(2,2)^\Delta}_{TY} &= \langle e^{1235}, e^{1236}, e^{1345}, e^{2346}, \lambda^2 e^{2345}+e^{1346}, (\lambda-1)^2 e^{2345}+e^{1256}\rangle \\
H^{(1,3)^\Delta}_{TY} &= \langle e^{1234}, e^{1245}, e^{1246}\rangle \\
H^{(3,2)^\Delta}_{TY} &= \langle e^{12356}\rangle \\
H^{(2,3)^\Delta}_{TY} &= \langle e^{12345}, e^{12346}\rangle \\
H^{(2,3)^\Delta}_{TY} &= \langle e^{123456}\rangle \\
H^{(3,3)^\Delta}_{TY} &= \langle e^{123456}\rangle
\end{aligned}
\tag{A.5}
$$

where

$$\gamma'_\lambda = \begin{cases} e^{146}+e^{245} & \text{if } \lambda=-1 \\ 0 & \text{otherwise} \end{cases} \quad , \quad \gamma''_\lambda = \begin{cases} e^{156}-e^{345} & \text{if } \lambda=2 \\ e^{256}-e^{346} & \text{if } \lambda=\frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad , \tag{A.6}$$

$\mathfrak{g}_{S,1} = (15, -25, -35, 45, 0, 0)$

$$\omega = e^{31}+e^{42}+e^{65} \quad , \quad \Lambda = \iota_1\iota_3+\iota_2\iota_4+\iota_5\iota_6$$

$$\Delta = \langle e^3, e^4, e^6 \rangle \quad , \quad \Delta^\perp = \langle e^1, e^2, e^5 \rangle$$

	d	Λ	dΛ	Λd	d^Λ	d+d^Λ	dd^Λ
e^1	$-e^{15}$	0	0	0	0	$-e^{15}$	0
e^2	e^{25}	0	0	0	0	e^{25}	0
e^3	e^{35}	0	0	0	0	e^{35}	0
e^4	$-e^{45}$	0	0	0	0	$-e^{45}$	0
e^5	0	0	0	0	0	0	0
e^6	0	0	0	0	0	0	0

	d	Λ	dΛ	Λd	d^Λ	d+d^Λ	dd^Λ
e^{12}	0	0	0	0	0	0	0
e^{13}	0	-1	0	0	0	0	0
e^{14}	$2e^{145}$	0	0	0	0	$2e^{145}$	0
e^{15}	0	0	0	0	0	0	0
e^{16}	$-e^{156}$	0	0	e^1	$-e^1$	$-e^{123}-e^1$	e^{15}
e^{23}	$-2e^{235}$	0	0	0	0	$-2e^{235}$	0
e^{24}	0	-1	0	0	0	0	0
e^{25}	0	0	0	0	0	0	0
e^{26}	e^{256}	0	0	$-e^2$	e^2	$e^{256}+e^2$	e^{25}
e^{34}	0	0	0	0	0	0	0
e^{35}	0	0	0	0	0	0	0
e^{36}	e^{356}	0	0	$-e^3$	e^3	$e^{356}+e^3$	e^{35}
e^{45}	0	0	0	0	0	0	0
e^{46}	$-e^{456}$	0	0	e^4	$-e^4$	$-e^{456}-e^4$	e^{45}
e^{56}	0	-1	0	0	0	0	0

	d	Λ	dΛ	Λd	d$^\Lambda$	d+d$^\Lambda$	dd$^\Lambda$
e^{123}	e^{1235}	e^{2}	e^{25}	$-e^{25}$	0	e^{1235}	0
e^{124}	$-e^{1245}$	$-e^{1}$	e^{15}	e^{15}	0	$-e^{1245}$	0
e^{125}	0	0	0	0	0	0	0
e^{126}	0	0	0	0	0	0	0
e^{134}	$-e^{1345}$	$-e^{4}$	e^{45}	e^{45}	0	$-e^{1345}$	0
e^{135}	0	$-e^{5}$	0	0	0	0	0
e^{136}	0	$-e^{6}$	0	0	0	0	0
e^{145}	0	0	0	0	0	0	0
e^{146}	$2e^{1456}$	0	0	$-2e^{14}$	$2e^{14}$	$2e^{1456}+2e^{14}$	$4e^{145}$
e^{156}	0	$-e^{1}$	e^{15}	0	e^{15}	e^{15}	0
e^{234}	e^{2345}	e^{3}	e^{35}	e^{35}	0	e^{2345}	0
e^{235}	0	0	0	0	0	0	0
e^{236}	$-2e^{2356}$	0	0	$2e^{23}$	$-2e^{23}$	$-2e^{2356}-2e^{23}$	$4e^{235}$
e^{245}	0	$-e^{5}$	0	0	0	0	0
e^{246}	0	$-e^{6}$	0	0	0	0	0
e^{256}	0	$-e^{2}$	$-e^{25}$	0	$-e^{25}$	$-e^{25}$	0
e^{345}	0	0	0	0	0	0	0
e^{346}	0	0	0	0	0	0	0
e^{356}	0	$-e^{3}$	$-e^{35}$	0	$-e^{35}$	$-e^{35}$	0
e^{456}	0	$-e^{4}$	e^{45}	0	e^{45}	e^{45}	0

	d	Λ	dΛ	Λd	d$^\Lambda$	d+d$^\Lambda$	dd$^\Lambda$
e^{1234}	0	$e^{24}+e^{13}$	0	0	0	0	0
e^{1235}	0	e^{25}	0	0	0	0	0
e^{1236}	e^{12356}	e^{26}	e^{256}	$e^{256}-e^{123}$	e^{123}	$e^{12356}+e^{123}$	e^{1235}
e^{1245}	0	$-e^{15}$	0	0	0	0	0
e^{1246}	$-e^{12456}$	$-e^{16}$	e^{156}	$e^{156}+e^{124}$	$-e^{124}$	$-e^{12456}-e^{124}$	e^{1245}
e^{1256}	0	$-e^{12}$	0	0	0	0	0
e^{1345}	0	$-e^{45}$	0	0	0	0	0
e^{1346}	$-e^{13456}$	$-e^{46}$	e^{456}	$e^{456}+e^{134}$	$-e^{134}$	$-e^{13456}-e^{134}$	e^{1345}
e^{1356}	0	$-e^{56}-e^{13}$	0	0	0	0	0
e^{1456}	0	$-e^{14}$	$-2e^{145}$	0	$-2e^{145}$	$-2e^{145}$	0
e^{2345}	0	e^{35}	0	0	0	0	0
e^{2346}	e^{23456}	e^{36}	e^{356}	$-e^{234}+e^{356}$	e^{234}	$e^{23456}+e^{234}$	e^{2345}
e^{2356}	0	$-e^{23}$	$2e^{235}$	0	$2e^{235}$	$2e^{235}$	0
e^{2456}	0	$-e^{56}-e^{24}$	0	0	0	0	0
e^{3456}	0	$-e^{34}$	0	0	0	0	0

	d	Λ	dΛ	Λd	d$^\Lambda$	d+d$^\Lambda$	dd$^\Lambda$
e^{12345}	0	$e^{135}+e^{245}$	0	0	0	0	0
e^{12346}	0	$e^{246}+e^{136}$	0	0	0	0	0
e^{12356}	0	$e^{256}-e^{123}$	$-e^{1235}$	0	$-e^{1235}$	$-e^{1235}$	0
e^{12456}	0	$-e^{156}-e^{124}$	e^{1245}	0	e^{1245}	e^{1245}	0
e^{13456}	0	$-e^{456}-e^{134}$	e^{1345}	0	e^{1345}	e^{1345}	0
e^{23456}	0	$e^{356}-e^{234}$	$-e^{2345}$	0	$-e^{2345}$	$-e^{2345}$	0

$$\begin{aligned}
H^1_{TY} &= \langle e^5, e^6\rangle \\
H^2_{TY} &= \langle e^{12}, e^{13}, e^{24}, e^{34}, e^{56}\rangle \\
H^3_{TY} &= \langle e^{125}, e^{126}, e^{135}, e^{136}, e^{245}, e^{246}, e^{345}, e^{346}\rangle \\
H^4_{TY} &= \langle e^{1234}, e^{1256}, e^{1356}, e^{2456}, e^{3456}\rangle \\
H^5_{TY} &= \langle e^{12345}, e^{12346}\rangle
\end{aligned} \tag{A.7}$$

$$\begin{aligned}
H^{(0,0)^\Delta}_{TY} &= \langle 1\rangle \\
H^{(1,0)^\Delta}_{TY} &= \langle e^6\rangle \\
H^{(0,1)^\Delta}_{TY} &= \langle e^5\rangle \\
H^{(2,0)^\Delta}_{TY} &= \langle e^{34}\rangle \\
H^{(1,1)^\Delta}_{TY} &= \langle e^{13}, e^{24}, e^{56}\rangle \\
H^{(0,2)^\Delta}_{TY} &= \langle e^{12}\rangle \\
H^{(3,0)^\Delta}_{TY} &= \langle e^{346}\rangle \\
H^{(2,1)^\Delta}_{TY} &= \langle e^{136}, e^{246}, e^{345}\rangle \\
H^{(1,2)^\Delta}_{TY} &= \langle e^{126}, e^{135}, e^{245}\rangle \\
H^{(0,3)^\Delta}_{TY} &= \langle e^{125}\rangle \\
H^{(3,1)^\Delta}_{TY} &= \langle e^{3456}\rangle \\
H^{(2,2)^\Delta}_{TY} &= \langle e^{1234}, e^{1356}, e^{2456}\rangle \\
H^{(1,3)^\Delta}_{TY} &= \langle e^{1256}\rangle \\
H^{(3,2)^\Delta}_{TY} &= \langle e^{12346}\rangle \\
H^{(2,3)^\Delta}_{TY} &= \langle e^{12345}\rangle \\
H^{(3,3)^\Delta}_{TY} &= \langle e^{123456}\rangle
\end{aligned} \tag{A.8}$$

$\mathfrak{g}_{S,2} = (16 + 35, -26 + 45, 36, -46, 0, 0)$

$$\omega = e^{14} + e^{23} + e^{56} \quad , \quad \Lambda = \iota_4\iota_1 + \iota_3\iota_2 + \iota_6\iota_5$$

$$\Delta = \langle e^1, e^2, e^5\rangle \quad , \quad \Delta^\perp = \langle e^4, e^3, e^6\rangle$$

	d	Λ	dΛ	Λd	d$^\Lambda$	d+d$^\Lambda$	dd$^\Lambda$
e^1	$-e^{16}-e^{35}$	0	0	0	0	$-e^{16}-e^{35}$	0
e^2	$e^{26}-e^{45}$	0	0	0	0	$e^{26}-e^{45}$	0
e^3	$-e^{36}$	0	0	0	0	$-e^{36}$	0
e^4	e^{46}	0	0	0	0	e^{46}	0
e^5	0	0	0	0	0	0	0
e^6	0	0	0	0	0	0	0

	d	Λ	dΛ	Λd	d$^\Lambda$	d+d$^\Lambda$	dd$^\Lambda$
e^{12}	$e^{145}-e^{235}$	0	0	0	0	$e^{145}-e^{235}$	0
e^{13}	$2e^{136}$	0	0	0	0	$2e^{136}$	0
e^{14}	e^{345}	1	0	0	0	e^{345}	0
e^{15}	e^{156}	0	0	e^1	$-e^1$	$e^{156}-e^1$	$e^{16}+e^{35}$
e^{16}	$-e^{356}$	0	0	$-e^3$	e^3	$-e^{356}+e^3$	$-e^{36}$
e^{23}	$-e^{345}$	1	0	0	0	$-e^{345}$	0
e^{24}	$-2e^{246}$	0	0	0	0	$-2e^{246}$	0
e^{25}	$-e^{256}$	0	0	$-e^2$	e^2	$-e^{256}+e^2$	$e^{26}-e^{45}$
e^{26}	$-e^{456}$	0	0	$-e^4$	e^4	$-e^{456}+e^4$	e^{46}
e^{34}	0	0	0	0	0	0	0
e^{35}	e^{356}	0	0	e^3	$-e^3$	$e^{356}-e^3$	e^{36}
e^{36}	0	0	0	0	0	0	0
e^{45}	$-e^{456}$	0	0	$-e^4$	e^4	$-e^{456}+e^4$	e^{46}
e^{46}	0	0	0	0	0	0	0
e^{56}	0	1	0	0	0	0	0

	d	Λ	dΛ	Λd	d$^\Lambda$	d+d$^\Lambda$	dd$^\Lambda$
e^{123}	$e^{1345} - e^{1236}$	e^{1}	$-e^{16} - e^{35}$	$-e^{16} - e^{35}$	0	$e^{1345} - e^{1236}$	0
e^{124}	$e^{2345} + e^{1246}$	$-e^{2}$	$-e^{26} + e^{45}$	$-e^{26} + e^{45}$	0	$e^{2345} + e^{1246}$	0
e^{125}	0	0	0	0	0	0	0
e^{126}	$e^{1456} - e^{2356}$	0	0	$e^{14} - e^{23}$	$e^{23} - e^{14}$	$e^{1456} - e^{2356} + e^{23} - e^{14}$	$-2e^{345}$
e^{134}	$-e^{1346}$	$-e^{3}$	e^{36}	e^{36}	0	$-e^{1346}$	0
e^{135}	$-2e^{1356}$	0	0	$-2e^{13}$	$2e^{13}$	$-2e^{1356} + 2e^{13}$	$4e^{136}$
e^{136}	0	0	0	0	0	0	0
e^{145}	0	e^{5}	0	0	0	0	0
e^{146}	e^{3456}	e^{6}	0	e^{34}	$-e^{34}$	$e^{3456} - e^{34}$	0
e^{156}	0	e^{1}	$-e^{16} - e^{35}$	0	$-e^{16} - e^{35}$	$-e^{16} - e^{35}$	0
e^{234}	e^{2346}	e^{4}	e^{46}	e^{46}	0	e^{2346}	0
e^{235}	0	e^{5}	0	0	0	0	0
e^{236}	$-e^{3456}$	e^{6}	0	$-e^{34}$	e^{34}	$-e^{3456} + e^{34}$	0
e^{245}	$2e^{2456}$	0	0	$2e^{24}$	$-2e^{24}$	$2e^{2456} - 2e^{24}$	e^{246}
e^{246}	0	0	0	0	0	0	0
e^{256}	0	e^{2}	$e^{26} - e^{45}$	0	$e^{26} - e^{45}$	$e^{26} - e^{45}$	0
e^{345}	0	0	0	0	0	0	0
e^{346}	0	0	0	0	0	0	0
e^{356}	0	e^{3}	$-e^{36}$	0	$-e^{36}$	$-e^{36}$	0
e^{456}	0	e^{4}	e^{46}	0	e^{46}	e^{46}	0

	d	Λ	dΛ	Λd	d$^\Lambda$	d+d$^\Lambda$	dd$^\Lambda$
e^{1234}	0	$e^{23} + e^{14}$	0	0	0	0	0
e^{1235}	e^{12356}	e^{15}	e^{156}	$e^{156} + e^{123}$	$-e^{123}$	$e^{12356} - e^{123}$	$-e^{1345} + e^{1236}$
e^{1236}	e^{13456}	e^{16}	$-e^{356}$	$-e^{356} + e^{134}$	$-e^{134}$	$e^{12456} - e^{134}$	$-e^{1346}$
e^{1245}	$-e^{12456}$	$-e^{25}$	e^{256}	$e^{256} - e^{124}$	e^{124}	$-e^{12456} + e^{124}$	$e^{2345} + e^{1246}$
e^{1246}	e^{23456}	$-e^{26}$	e^{456}	$e^{456} + e^{234}$	$-e^{234}$	$-^{23456} - e^{234}$	$-e^{2346}$
e^{1256}	0	e^{12}	$e^{145} - e^{235}$	0	$e^{145} - e^{235}$	$e^{145} - e^{235}$	0
e^{1345}	e^{13456}	$-e^{35}$	$-e^{356}$	$e^{134} - e^{356}$	$-e^{134}$	$e^{13456} - e^{134}$	e^{1346}
e^{1346}	0	$-e^{36}$	0	0	0	0	0
e^{1356}	0	$-e^{13}$	$2e^{136}$	0	$2e^{136}$	$2e^{136}$	0
e^{1456}	0	$e^{56} + e^{14}$	e^{345}	0	e^{345}	e^{345}	0
e^{2345}	$-e^{23456}$	e^{45}	$-e^{456}$	$-e^{456} - e^{234}$	e^{234}	$-e^{23456} + e^{234}$	e^{2346}
e^{2346}	0	e^{46}	0	0	0	0	0
e^{2356}	0	$e^{56} + e^{23}$	$-e^{345}$	0	$-e^{345}$	$-e^{345}$	0
e^{2456}	0	e^{24}	$-2e^{246}$	0	$-2e^{246}$	$-2e^{246}$	0
e^{3456}	0	e^{34}	0	0	0	0	0

	d	Λ	dΛ	Λd	d$^\Lambda$	d+d$^\Lambda$	dd$^\Lambda$
e^{12345}	0	$e^{145} + e^{235}$	0	0	0	0	0
e^{12346}	0	$e^{236} + e^{146}$	0	0	0	0	0
e^{12356}	0	$e^{156} + e^{123}$	$e^{1345} - e^{1236}$	0	$e^{1345} - e^{1236}$	$e^{1345} - e^{1236}$	0
e^{12456}	0	$-e^{256} + e^{124}$	$e^{2345} + e^{1246}$	0	$e^{2345} + e^{1246}$	$e^{2345} + e^{1246}$	0
e^{13456}	0	$-e^{356} + e^{134}$	$-e^{1346}$	0	$-e^{1346}$	$-e^{1346}$	0
e^{23456}	0	$e^{456} + e^{234}$	e^{2346}	0	e^{2346}	e^{2346}	0

$$
\begin{aligned}
H^1_{TY} &= \langle e^5, e^6 \rangle \\
H^2_{TY} &= \langle e^{14} + e^{23}, e^{34}, e^{56} \rangle \\
H^3_{TY} &= \langle e^{125}, e^{145}, e^{235}, e^{346}, e^{236} + e^{146} \rangle \\
H^4_{TY} &= \langle e^{1234}, e^{3456}, e^{1456} + e^{2356} \rangle \\
H^5_{TY} &= \langle e^{12345}, e^{12346} \rangle
\end{aligned} \tag{A.9}
$$

$$
\begin{aligned}
H^{(0,0)^\Delta}_{TY} &= \langle 1 \rangle \\
H^{(1,0)^\Delta}_{TY} &= \langle e^5 \rangle \\
H^{(0,1)^\Delta}_{TY} &= \langle e^6 \rangle \\
H^{(2,0)^\Delta}_{TY} &= 0 \\
H^{(1,1)^\Delta}_{TY} &= \langle e^{14} + e^{23}, e^{56} \rangle \\
H^{(0,2)^\Delta}_{TY} &= \langle e^{34} \rangle \\
H^{(3,0)^\Delta}_{TY} &= \langle e^{125} \rangle \\
H^{(2,1)^\Delta}_{TY} &= \langle e^{145}, e^{235}, \rangle \\
H^{(1,2)^\Delta}_{TY} &= \langle e^{236} + e^{146}, \rangle \\
H^{(0,3)^\Delta}_{TY} &= \langle e^{346} \rangle \\
H^{(3,1)^\Delta}_{TY} &= 0 \\
H^{(2,2)^\Delta}_{TY} &= \langle e^{1234}, e^{1456} + e^{2356} \rangle \\
H^{(3,1)^\Delta}_{TY} &= \langle e^{3456} \rangle \\
H^{(3,2)^\Delta}_{TY} &= \langle e^{12345} \rangle \\
H^{(2,3)^\Delta}_{TY} &= \langle e^{12346} \rangle \\
H^{(3,3)^\Delta}_{TY} &= \langle e^{123456} \rangle
\end{aligned} \tag{A.10}
$$

$\mathfrak{g}_{\mathbb{T}} = (0, -16, 0, -36, 0, 0) \simeq (0, 0, 0, 0, 12, 13)$

$$\omega = e^{41} + e^{23} + e^{65} \quad , \quad \Lambda = \iota_1\iota_4 + \iota_3\iota_2 + \iota_5\iota_6$$

$$\Delta = \langle e^4, e^2, e^6 \rangle \quad , \quad \Delta^\perp = \langle e^1, e^3, e^5 \rangle$$

	d	Λ	dΛ	Λd	d$^{\Lambda}$	d+d$^{\Lambda}$	dd$^{\Lambda}$
e^1	0	0	0	0	0	0	0
e^2	e^{16}	0	0	0	0	e^{16}	0
e^3	0	0	0	0	0	0	0
e^4	e^{36}	0	0	0	0	e^{36}	0
e^5	0	0	0	0	0	0	0
e^6	0	0	0	0	0	0	0

	d	Λ	dΛ	Λd	d$^{\Lambda}$	d+d$^{\Lambda}$	dd$^{\Lambda}$
e^{12}	0	0	0	0	0	0	0
e^{13}	0	0	0	0	0	0	0
e^{14}	$-e^{136}$	-1	0	0	0	$-e^{136}$	0
e^{15}	0	0	0	0	0	0	0
e^{16}	0	0	0	0	0	0	0
e^{23}	$-e^{136}$	1	0	0	0	$-e^{136}$	0
e^{24}	$-e^{146} - e^{236}$	0	0	0	0	$-e^{146} - e^{236}$	0
e^{25}	$-e^{156}$	0	0	e^1	$-e^1$	$-e^{156} - e^1$	0
e^{26}	0	0	0	0	0	0	0
e^{34}	0	0	0	0	0	0	0
e^{35}	0	1	0	0	0	0	0
e^{36}	0	0	0	0	0	0	0
e^{45}	$-e^{356}$	0	0	e^3	$-e^3$	$-e^{356} - e^3$	0
e^{46}	0	0	0	0	0	0	0
e^{56}	0	-1	0	0	0	0	0

	d	Λ	dΛ	Λd	d^Λ	d+d^Λ	dd^Λ
e^{123}	0	e^1	0	0	0	0	0
e^{124}	e^{1236}	e^2	e^{16}	e^{16}	0	e^{1236}	0
e^{125}	0	0	0	0	0	0	0
e^{126}	0	0	0	0	0	0	0
e^{134}	0	e^3	0	0	0	0	0
e^{135}	0	0	0	0	0	0	0
e^{136}	0	0	0	0	0	0	0
e^{145}	e^{1356}	$-e^5$	0	$-e^{13}$	e^{13}	$e^{1356}+e^{13}$	0
e^{146}	0	$-e^6$	0	0	0	0	0
e^{156}	0	$-e^1$	0	0	0	0	0
e^{234}	e^{1346}	e^4	e^{36}	e^{36}	0	e^{1346}	0
e^{235}	e^{1356}	e^5	0	$-e^{13}$	e^{13}	$e^{1356}+e^{13}$	0
e^{236}	0	e^6	0	0	0	0	0
e^{245}	$e^{1456}+e^{2356}$	0	0	$-e^{14}-e^{23}$	$e^{14}+e^{23}$	$e^{1456}+e^{2356}+e^{14}+e^{23}$	$-2e^{136}$
e^{246}	0	0	0	0	0	0	0
e^{256}	0	$-e^2$	$-e^{16}$	0	$-e^{16}$	$-e^{16}$	0
e^{345}	0	0	0	0	0	0	0
e^{346}	0	0	0	0	0	0	0
e^{356}	0	$-e^3$	0	0	0	0	0
e^{456}	0	$-e^4$	$-e^{36}$	0	$-e^{36}$	$-e^{36}$	0

	d	Λ	dΛ	Λd	d^Λ	d+d^Λ	dd^Λ
e^{1234}	0	$-e^{23}+e^{14}$	0	0	0	0	0
e^{1235}	0	e^{15}	0	0	0	0	0
e^{1236}	0	e^{16}	0	0	0	0	0
e^{1245}	$-e^{1235}$	e^{25}	$-e^{156}$	$e^{123}-e^{156}$	$-e^{123}$	$-e^{1235}-e^{123}$	0
e^{1246}	0	e^{26}	0	0	0	0	0
e^{1256}	0	$-e^{12}$	0	0	0	0	0
e^{1345}	0	e^{35}	0	0	0	0	0
e^{1346}	0	e^{36}	0	0	0	0	0
e^{1356}	0	$-e^{13}$	0	0	0	0	0
e^{1456}	0	$-e^{14}-e^{56}$	e^{136}	0	e^{136}	e^{136}	0
e^{2345}	$-e^{13456}$	e^{45}	$-e^{356}$	$-e^{356}+e^{134}$	$-e^{134}$	$-e^{13456}-e^{134}$	0
e^{2346}	0	e^{46}	0	0	0	0	0
e^{2356}	0	$e^{56}-e^{23}$	e^{136}	0	e^{136}	e^{136}	0
e^{2456}	0	$-e^{24}$	$e^{146}+e^{236}$	0	$e^{146}+e^{236}$	$e^{146}+e^{236}$	0
e^{3456}	0	$-e^{34}$	0	0	0	0	0

	d	Λ	dΛ	Λd	d^Λ	d+d^Λ	dd^Λ
e^{12345}	0	$e^{145}-e^{235}$	0	0	0	0	0
e^{12346}	0	$e^{146}-e^{236}$	0	0	0	0	0
e^{12356}	0	$e^{156}-e^{123}$	0	0	0	0	0
e^{12456}	0	$e^{256}-e^{124}$	$-e^{1236}$	0	$-e^{1236}$	$-e^{1236}$	0
e^{13456}	0	$e^{356}-e^{134}$	0	0	0	0	0
e^{23456}	0	$e^{456}-e^{234}$	$-e^{1346}$	0	$-e^{1346}$	$-e^{1346}$	0

$$
\begin{aligned}
H^1_{TY} &= \langle e^1, e^3, e^5, e^6 \rangle \\
H^2_{TY} &= \langle e^{12}, e^{13}, e^{14} - e^{23}, e^{15}, e^{16}, e^{26}, e^{34}, e^{35}, e^{36}, e^{46}, e^{56} \rangle \\
H^3_{TY} &= \langle e^{123}, e^{125}, e^{126}, e^{134}, e^{135}, e^{145} - e^{235}, e^{146}, e^{156}, e^{236}, e^{246}, e^{345}, e^{346}, e^{356} \rangle \\
H^4_{TY} &= \langle e^{1234}, e^{1235}, e^{1236}, e^{1246}, e^{1256}, e^{1345}, e^{1346}, e^{1356}, e^{2346}, e^{3456}, e^{1456} - e^{2356} \rangle \\
H^5_{TT} &= \langle e^{12345}, e^{12346}, e^{12356}, e^{13456} \rangle
\end{aligned}
\tag{A.11}
$$

$$
\begin{aligned}
H^{(0,0)^\Delta}_{TY} &= \langle 1 \rangle \\
H^{(1,0)^\Delta}_{TY} &= \langle e^6 \rangle \\
H^{(0,1)^\Delta}_{TY} &= \langle e^1, e^3, e^5 \rangle \\
H^{(2,0)^\Delta}_{TY} &= \langle e^{26}, e^{46} \rangle \\
H^{(0,2)^\Delta}_{TY} &= \langle e^{13}, e^{15}, e^{35} \rangle \\
H^{(1,1)^\Delta}_{TY} &= \langle e^{12}, e^{14} - e^{23}, e^{16}, e^{34}, e^{36}, e^{56} \rangle \\
H^{(3,0)^\Delta}_{TY} &= \langle e^{246} \rangle \\
H^{(0,3)^\Delta}_{TY} &= \langle e^{135} \rangle \\
H^{(2,1)^\Delta}_{TY} &= \langle e^{126}, e^{146}, e^{236}, e^{346} \rangle \\
H^{(1,2)^\Delta}_{TY} &= \langle e^{123}, e^{125}, e^{134}, e^{145} - e^{235}, e^{156}, e^{345}, e^{356} \rangle \\
H^{(3,1)^\Delta}_{TY} &= \langle e^{1246}, e^{2346} \rangle \\
H^{(2,2)^\Delta}_{TY} &= \langle e^{1234}, e^{1236}, e^{1256}, e^{1346}, e^{3456}, e^{1456} - e^{2356} \rangle \\
H^{(1,3)^\Delta}_{TY} &= \langle e^{1235}, e^{1345}, e^{1356} \rangle \\
H^{(3,2)^\Delta}_{TY} &= \langle e^{12346} \rangle \\
H^{(2,3)^\Delta}_{TY} &= \langle e^{12345}, e^{12356}, e^{13456} \rangle \\
H^{(3,3)^\Delta}_{TY} &= \langle e^{123456} \rangle
\end{aligned}
\tag{A.12}
$$

$\mathfrak{g}_Y = (0,0,0,12,-13,-15) \simeq (0,0,0,12,13,14)$

$$\omega = e^{61} + e^{52} + e^{34} \quad , \quad \Lambda = \iota_1\iota_6 + \iota_2\iota_5 + \iota_4\iota_3$$

$$\Delta = \langle e^6, e^5, e^3 \rangle \quad , \quad \Delta^\perp = \langle e^1, e^2, e^4 \rangle$$

	d	Λ	dΛ	Λd	d$^\Lambda$	d+d$^\Lambda$	dd$^\Lambda$
e^1	0	0	0	0	0	0	0
e^2	0	0	0	0	0	0	0
e^3	0	0	0	0	0	0	0
e^4	$-e^{12}$	0	0	0	0	$-e^{12}$	0
e^5	e^{13}	0	0	0	0	e^{13}	0
e^6	e^{15}	0	0	0	0	e^{15}	0

	d	Λ	dΛ	Λd	d$^\Lambda$	d+d$^\Lambda$	dd$^\Lambda$
e^{12}	0	0	0	0	0	0	0
e^{13}	0	0	0	0	0	0	0
e^{14}	0	0	0	0	0	0	0
e^{15}	0	0	0	0	0	0	0
e^{16}	0	-1	0	0	0	0	0
e^{23}	0	0	0	0	0	0	0
e^{24}	0	0	0	0	0	0	0
e^{25}	e^{123}	-1	0	0	0	e^{123}	0
e^{26}	e^{125}	0	0	$-e^1$	e^1	$e^{125}+e^1$	0
e^{34}	e^{123}	1	0	0	0	e^{123}	0
e^{35}	0	0	0	0	0	0	0
e^{36}	e^{135}	0	0	0	0	e^{135}	0
e^{45}	$-e^{125}-e^{134}$	0	0	0	0	$-e^{125}-e^{134}$	0
e^{46}	$-e^{126}+e^{145}$	0	0	$-e^2$	e^2	$-e^{126}+e^{145}+e^2$	0
e^{56}	$-e^{136}$	0	0	e^3	$-e^3$	$-e^{136}-e^3$	0

	d	Λ	dΛ	Λd	d$^\Lambda$	d+d$^\Lambda$	dd$^\Lambda$
e^{123}	0	0	0	0	0	0	0
e^{124}	0	0	0	0	0	0	0
e^{125}	0	$-e^1$	0	0	0	0	0
e^{126}	0	e^2	0	0	0	0	0
e^{134}	0	e^1	0	0	0	0	0
e^{135}	0	0	0	0	0	0	0
e^{136}	0	e^3	0	0	0	0	0
e^{145}	0	0	0	0	0	0	0
e^{146}	0	e^4	$-e^{12}$	0	$-e^{12}$	$-e^{12}$	0
e^{156}	0	e^5	e^{13}	0	e^{13}	e^{13}	0
e^{234}	0	e^2	0	0	0	0	0
e^{235}	0	e^3	0	0	0	0	0
e^{236}	e^{1235}	0	0	e^{13}	$-e^{13}$	$e^{1235}-e^{13}$	0
e^{245}	$-e^{1245}$	e^4	$-e^{12}$	$-e^{12}$	0	$-e^{1234}$	0
e^{246}	e^{1245}	0	0	e^{14}	$-e^{14}$	$e^{1245}-e^{14}$	0
e^{256}	e^{1236}	$-e^6$	$-e^{15}$	$-e^{23}$	$-e^{15}+e^{23}$	$e^{1235}-e^{15}+e^{23}$	0
e^{345}	e^{1235}	e^5	e^{13}	e^{13}	0	e^{1235}	0
e^{346}	e^{1236}	e^6	e^{15}	$-e^{23}$	$e^{15}+e^{23}$	$e^{1236}+e^{15}+e^{23}$	0
e^{356}	0	0	0	0	0	0	0
e^{456}	$-e^{1256}-e^{1346}$	0	0	$e^{25}+e^{34}$	$-e^{25}-e^{34}$	$-e^{1256}-e^{1346}-e^{25}-e^{34}$	$-2e^{123}$

	d	Λ	dΛ	Λd	d$^\Lambda$	d+d$^\Lambda$	dd$^\Lambda$
e^{1234}	0	e^{12}	0	0	0	0	0
e^{1235}	0	e^{13}	0	0	0	0	0
e^{1236}	0	$-e^{23}$	0	0	0	0	0
e^{1245}	0	e^{14}	0	0	0	0	0
e^{1246}	0	$-e^{24}$	0	0	0	0	0
e^{1256}	0	$-e^{25}-e^{16}$	$-e^{123}$	0	$-e^{123}$	$-e^{123}$	0
e^{1345}	0	e^{15}	0	0	0	0	0
e^{1346}	0	$-e^{34}+e^{16}$	$-e^{123}$	0	$-e^{123}$	$-e^{123}$	0
e^{1356}	0	$-e^{35}$	0	0	0	0	0
e^{1456}	0	$-e^{45}$	$e^{125}+e^{134}$	0	$e^{125}+e^{134}$	$e^{125}+e^{134}$	0
e^{2345}	0	$e^{25}-e^{34}$	0	0	0	0	0
e^{2346}	e^{12345}	e^{26}	e^{125}	$-e^{134}+e^{125}$	e^{134}	$e^{12345}+e^{134}$	0
e^{2356}	0	e^{36}	e^{135}	0	e^{135}	e^{135}	0
e^{2456}	$-e^{12346}$	e^{46}	$-e^{126}+e^{145}$	$-e^{234}-e^{126}$	$e^{234}+e^{145}$	$-e^{12346}+e^{234}+e^{145}$	0
e^{3456}	e^{12356}	e^{56}	e^{136}	$e^{235}+e^{136}$	$-e^{235}$	$e^{12356}-e^{235}$	0

	d	Λ	dΛ	Λd	d$^\Lambda$	d+d$^\Lambda$	dd$^\Lambda$
e^{12345}	0	$-e^{134}+e^{123}$	0	0	0	0	0
e^{12346}	0	$e^{234}+e^{126}$	0	0	0	0	0
e^{12356}	0	$e^{235}+e^{136}$	0	0	0	0	0
e^{12456}	0	$e^{245}+e^{146}$	$-e^{1234}$	0	$-e^{1234}$	$-e^{1234}$	0
e^{13456}	0	$e^{345}+e^{156}$	e^{1235}	0	e^{1235}	e^{1235}	0
e^{23456}	0	$-e^{346}+e^{256}$	$-e^{1345}$	0	$-e^{1345}$	$-e^{1345}$	0

$$
\begin{aligned}
H^1_{TY} &= \langle e^1, e^2, e^3 \rangle\\
H^2_{TY} &= \langle e^{12}, e^{13}, e^{14}, e^{15}, e^{16}, e^{23}, e^{24}, e^{25} - e^{34}, e^{35} \rangle\\
H^3_{TY} &= \langle e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{145}, e^{234}, e^{235}, e^{236} - e^{345} + e^{156}, e^{356} \rangle\\
H^4_{TY} &= \langle e^{1234}, e^{1235}, e^{1236}, e^{1245}, e^{1246}, e^{1256} - e^{1346}, e^{1345}, e^{1356}, e^{2345} \rangle\\
H^5_{TY} &= \langle e^{12345}, e^{12346}, e^{12356} \rangle
\end{aligned} \tag{A.13}
$$

$$
\begin{aligned}
H^{(0,0)^\Delta}_{TY} &= \langle 1 \rangle\\
H^{(1,0)^\Delta}_{TY} &= \langle e^3 \rangle\\
H^{(0,1)^\Delta}_{TY} &= \langle e^1, e^2 \rangle\\
H^{(2,0)^\Delta}_{TY} &= \langle e^{35} \rangle\\
H^{(0,2)^\Delta}_{TY} &= \langle e^{12}, e^{14}, e^{24} \rangle\\
H^{(1,1)^\Delta}_{TY} &= \langle e^{13}, e^{15}, e^{16}, e^{23}, e^{25} - e^{34} \rangle\\
H^{(3,0)^\Delta}_{TY} &= \langle e^{356} \rangle\\
H^{(0,3)^\Delta}_{TY} &= \langle e^{124} \rangle\\
H^{(2,1)^\Delta}_{TY} &= \langle e^{135}, e^{136}, e^{235}, e^{236} - e^{345} + e^{156} \rangle\\
H^{(1,2)^\Delta}_{TY} &= \langle e^{125}, e^{126}, e^{134}, e^{145}, e^{234} \rangle\\
H^{(3,1)^\Delta}_{TY} &= \langle e^{1356} \rangle\\
H^{(2,2)^\Delta}_{TY} &= \langle e^{1235}, e^{1236}, e^{1256} - e^{1346}, e^{1345}, e^{2345} \rangle\\
H^{(1,3)^\Delta}_{TY} &= \langle e^{1234}, e^{1245}, e^{1246} \rangle\\
H^{(3,2)^\Delta}_{TY} &= \langle e^{12356} \rangle\\
H^{(2,3)^\Delta}_{TY} &= \langle e^{12345}, e^{12346} \rangle\\
H^{(3,3)^\Delta}_{TY} &= \langle e^{123456} \rangle
\end{aligned} \tag{A.14}
$$

A.0.2 Bott-Chern Cohomology

For each SLA we recall the notation, the complex three-form $\check{\Omega}$ and the basis of $(1,0)$-forms with their differentials. Then we compute the BC cohomology exhibiting also the generators for the groups. This is in according with the computation in ([60]).

$\check{\mathfrak{g}}_{N,1} = (0,0,0,0,12,13)$

$$\check{\Omega} = (e^4 + ie^1) \wedge (e^3 + ie^2) \wedge (e^6 + ie^5)$$

$$
\begin{aligned}
\psi^1 &= e^4 + ie^1 \quad , \quad d\psi^1 = 0 \\
\psi^2 &= e^3 + ie^2 \quad , \quad d\psi^2 = 0 \\
\psi^3 &= e^6 + ie^5 \quad , \quad d\psi^3 = -e^{13} - ie^{12} = \frac{i}{2}(\psi^{12} - \psi^{1\bar{2}})
\end{aligned}
\tag{A.15}
$$

	∂	$\bar\partial$	d	$\partial\bar\partial$
ψ^1	0	0	0	0
ψ^2	0	0	0	0
ψ^3	$\frac{i}{2}\psi^{12}$	$-\frac{i}{2}\psi^{1\bar2}$	$\frac{i}{2}(\psi^{12} - \psi^{1\bar2})$	0
$\psi^{\bar1}$	0	0	0	0
$\psi^{\bar2}$	0	0	0	0
$\psi^{\bar3}$	$-\frac{i}{2}\psi^{2\bar1}$	$-\frac{i}{2}\psi^{\bar1\bar2}$	$-\frac{i}{2}(\psi^{2\bar1} + \psi^{\bar1\bar2})$	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
ψ^{12}	0	0	0	0
ψ^{13}	0	0	0	0
ψ^{23}	0	$-\frac{i}{2}\psi^{12\bar2}$	$-\frac{i}{2}\psi^{12\bar2}$	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{123\bar1}$	0	0	0	0
$\psi^{123\bar2}$	0	0	0	0
$\psi^{123\bar3}$	0	$\frac{i}{2}\psi^{123\bar1\bar2}$	$\frac{i}{2}\psi^{123\bar1\bar2}$	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{123\bar1\bar2}$	0	0	0	0
$\psi^{123\bar1\bar3}$	0	0	0	0
$\psi^{123\bar2\bar3}$	0	0	0	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
ψ^{11}	0	0	0	0
$\psi^{1\bar2}$	0	0	0	0
$\psi^{1\bar3}$	$\frac{i}{2}\psi^{12\bar1}$	$\frac{i}{2}(\psi^{1\bar1\bar2})$	$\frac{i}{2}(\psi^{12\bar1} + \psi^{1\bar1\bar2})$	0
$\psi^{2\bar1}$	0	0	0	0
$\psi^{2\bar2}$	0	0	0	0
$\psi^{2\bar3}$	0	$\frac{i}{2}\psi^{2\bar1\bar2}$	$\frac{i}{2}\psi^{2\bar1\bar2}$	0
$\psi^{3\bar1}$	$\frac{i}{2}\psi^{12\bar1}$	$\frac{i}{2}(\psi^{1\bar1\bar2})$	$\frac{i}{2}(\psi^{12\bar1} + \psi^{1\bar1\bar2})$	0
$\psi^{3\bar2}$	$\frac{i}{2}\psi^{12\bar2}$	0	$\frac{i}{2}\psi^{12\bar2}$	0
$\psi^{3\bar3}$	$\frac{i}{2}(\psi^{1\bar2\bar3} - \psi^{23\bar1})$	$\frac{i}{2}(\psi^{1\bar2\bar3} + \psi^{3\bar1\bar2})$	$\frac{i}{2}(\psi^{12\bar3} - \psi^{2\bar3\bar1} + \psi^{1\bar2\bar3} + \psi^{3\bar1\bar2})$	$-\frac{1}{2}\psi^{12\bar1\bar2}$

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{12\bar1}$	0	0	0	0
$\psi^{12\bar2}$	0	0	0	0
$\psi^{12\bar3}$	0	$-\frac{i}{2}\psi^{12\bar1\bar2}$	$-\frac{i}{2}\psi^{12\bar1\bar2}$	0
$\psi^{13\bar1}$	0	0	0	0
$\psi^{13\bar2}$	0	0	0	0
$\psi^{13\bar3}$	$\frac{i}{2}\psi^{123\bar1}$	$-\frac{i}{2}\psi^{13\bar1\bar2}$	$\frac{i}{2}\psi^{123\bar1}-\psi^{13\bar1\bar2}$	0
$\psi^{23\bar1}$	0	$-\frac{i}{2}\psi^{12\bar1\bar2}$	$-\frac{i}{2}\psi^{12\bar1\bar2}$	0
$\psi^{23\bar2}$	0	0	0	0
$\psi^{23\bar3}$	0	$-\frac{i}{2}(\psi^{12\bar2\bar3}+\psi^{23\bar1\bar2})$	$-\frac{i}{2}(\psi^{12\bar2\bar3}+\psi^{23\bar1\bar2})$	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{12\bar1\bar2}$	0	0	0	0
$\psi^{12\bar1\bar3}$	0	0	0	0
$\psi^{12\bar2\bar3}$	0	0	0	0
$\psi^{13\bar1\bar2}$	0	0	0	0
$\psi^{13\bar1\bar3}$	0	0	0	0
$\psi^{13\bar2\bar3}$	$-\frac{i}{2}\psi^{123\bar1\bar2}$	0	$-\frac{i}{2}\psi^{123\bar1\bar2}$	0
$\psi^{23\bar1\bar2}$	0	0	0	0
$\psi^{23\bar1\bar3}$	0	$\frac{i}{2}\psi^{12\bar1\bar2\bar3}$	$\frac{i}{2}\psi^{12\bar1\bar2\bar3}$	0
$\psi^{23\bar2\bar3}$	0	0	0	0

$$\begin{aligned}
&H^{1,0}_{BC}=\langle\psi^1,\psi^2\rangle \quad , \quad H^{2,0}_{BC}=\langle\psi^{12},\psi^{13}\rangle\\
&H^{1,1}_{BC}=\langle\psi^{1\bar1},\psi^{1\bar2},\psi^{2\bar1},\psi^{2\bar2},\psi^{1\bar3}-\psi^{3\bar1}\rangle\\
&H^{2,1}_{BC}=\langle\psi^{12\bar1},\psi^{12\bar2},\psi^{13\bar1},\psi^{13\bar2},\psi^{23\bar2},\psi^{12\bar3}-\psi^{23\bar1}\rangle\\
&H^{2,2}_{BC}=\langle\psi^{12\bar1\bar3},\psi^{12\bar2\bar3},\psi^{13\bar1\bar2},\psi^{13\bar1\bar3},\psi^{23\bar1\bar2},\psi^{23\bar2\bar3}\rangle\\
&H^{3,1}_{BC}=\langle\psi^{123\bar1},\psi^{123\bar2}\rangle \quad , \quad H^{3,2}_{BC}=\langle\psi^{123\bar1\bar2},\psi^{123\bar1\bar3},\psi^{123\bar2\bar3}\rangle
\end{aligned} \tag{A.16}$$

$\check{\mathfrak{g}}_{N,2}=(0,0,0,0,12,14+23)$

$$\check\Omega_\lambda=(e^3+ie^1)\wedge(e^4+i\lambda e^2)\wedge(e^6+i(\lambda-1)e^5)$$

$$\begin{aligned}
&\psi^1=e^3+ie^1 \quad , \quad d\psi^1=0\\
&\psi^2=e^4+i\lambda e^2 \quad , \quad d\psi^2=0\\
&\psi^3=e^6+i(\lambda-1)e^5 \quad , \quad d\psi^3=-e^{14}-e^{23}-i(\lambda-1)e^{12}=\frac{i}{2}\left(\Big(\frac{\lambda-1}{\lambda}\Big)\psi^{12}+\frac{1}{\lambda}\psi^{1\bar2}+\psi^{2\bar1}\right)
\end{aligned} \tag{A.17}$$

	∂	$\bar\partial$	d	$\partial\bar\partial$
ψ^{1}	0	0	0	0
ψ^{2}	0	0	0	0
ψ^{3}	$\frac{i}{2}\frac{\lambda-1}{\lambda}\psi^{12}$	$\frac{i}{2}(\frac{1}{\lambda}\psi^{1\bar2}+\psi^{2\bar1})$	$\frac{i}{2}\left(\left(\frac{\lambda-1}{\lambda}\right)\psi^{12}+\frac{1}{\lambda}\psi^{1\bar2}+\psi^{2\bar1}\right)$	0
$\psi^{\bar1}$	0	0	0	0
$\psi^{\bar2}$	0	0	0	0
$\psi^{\bar3}$	$\frac{i}{2}(\frac{1}{\lambda}\psi^{2\bar1}+\psi^{1\bar2})$	$-\frac{i}{2}\frac{\lambda-1}{\lambda}\psi^{\bar1\bar2}$	$\frac{i}{2}\left(\left(\frac{\lambda-1}{\lambda}\right)\psi^{\bar1\bar2}+\frac{1}{\lambda}\psi^{1\bar2}+\psi^{2\bar1}\right)$	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
ψ^{12}	0	0	0	0
ψ^{13}	0	$-\frac{i}{2}\psi^{12\bar1}$	$-\frac{i}{2}\psi^{12\bar1}$	0
ψ^{23}	0	$\frac{i}{2}\frac{1}{\lambda}\psi^{12\bar2}$	$\frac{i}{2}\frac{1}{\lambda}\psi^{12\bar2}$	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{123\bar1}$	0	0	0	0
$\psi^{123\bar2}$	0	0	0	0
$\psi^{123\bar3}$	0	$\frac{i}{2}\frac{\lambda-1}{\lambda}\psi^{123\bar1\bar2}$	$\frac{i}{2}\frac{\lambda-1}{\lambda}\psi^{123\bar1\bar2}$	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{123\bar1\bar2}$	0	0	0	0
$\psi^{123\bar1\bar3}$	0	0	0	0
$\psi^{123\bar2\bar3}$	0	0	0	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{1\bar1}$	0	0	0	0
$\psi^{1\bar2}$	0	0	0	0
$\psi^{1\bar3}$	$-\frac{i}{2}\frac{1}{\lambda}\psi^{12\bar1}$	$\frac{i}{2}\frac{\lambda-1}{\lambda}\psi^{1\bar1\bar2}$	$\frac{i}{2}(-\frac{1}{\lambda}\psi^{12\bar1}+\frac{\lambda-1}{\lambda}\psi^{1\bar1\bar2})$	0
$\psi^{2\bar1}$	0	0	0	0
$\psi^{2\bar2}$	0	0	0	0
$\psi^{2\bar3}$	$\frac{i}{2}\psi^{12\bar2}$	$\frac{i}{2}\frac{\lambda-1}{\lambda}\psi^{2\bar1\bar2}$	$\frac{i}{2}(\psi^{12\bar2}+\frac{\lambda-1}{\lambda}\psi^{2\bar1\bar2})$	0
$\psi^{3\bar1}$	$\frac{i}{2}\frac{\lambda-1}{\lambda}\psi^{12\bar1}$	$-\frac{i}{2}\frac{1}{\lambda}\psi^{1\bar1\bar2}$	$\frac{i}{2}(\frac{\lambda-1}{\lambda}\psi^{12\bar1})-\frac{1}{\lambda}\psi^{1\bar1\bar2}$	0
$\psi^{3\bar2}$	$\frac{i}{2}\frac{\lambda-1}{\lambda}\psi^{12\bar2}$	$\frac{i}{2}\psi^{2\bar1\bar2}$	$\frac{i}{2}(\frac{\lambda-1}{\lambda}\psi^{12\bar2}+\psi^{2\bar1\bar2})$	0
$\psi^{3\bar3}$	$\frac{i}{2}(\frac{\lambda-1}{\lambda}\psi^{12\bar3}+\frac{1}{\lambda}\psi^{23\bar1}+\psi^{13\bar2})$	$\frac{i}{2}(\frac{1}{\lambda}\psi^{1\bar2\bar3}+\psi^{2\bar1\bar3}+\frac{\lambda-1}{\lambda}\psi^{3\bar1\bar2})$	$\partial\psi^{3\bar3}+\bar\partial\psi^{3\bar3}$	$-\frac{1}{2}\frac{\lambda^2-\lambda+1}{\lambda^2}\psi^{12\bar1\bar2}$

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{12\bar1}$	0	0	0	0
$\psi^{12\bar2}$	0	0	0	0
$\psi^{12\bar3}$	0	$-\frac{i}{2}\frac{\lambda-1}{\lambda}\psi^{12\bar1\bar2}$	$-\frac{i}{2}\frac{\lambda-1}{\lambda}\psi^{12\bar1\bar2}$	0
$\psi^{13\bar1}$	0	0	0	0
$\psi^{13\bar2}$	0	$-\frac{i}{2}\psi^{12\bar1\bar2}$	$-\frac{i}{2}\psi^{12\bar1\bar2}$	0
$\psi^{13\bar3}$	0	$-\frac{i}{2}(\psi^{12\bar1\bar3}+\frac{\lambda-1}{\lambda}\psi^{13\bar1\bar2})$	$-\frac{i}{2}(\psi^{12\bar1\bar3}+\frac{\lambda-1}{\lambda}\psi^{13\bar1\bar2})$	0
$\psi^{23\bar1}$	0	$-\frac{i}{2}\frac{1}{\lambda}\psi^{12\bar1\bar2}$	$-\frac{i}{2}\frac{1}{\lambda}\psi^{12\bar1\bar2}$	0
$\psi^{23\bar2}$	0	0	0	0
$\psi^{23\bar3}$	0	$\frac{i}{2}(\frac{1}{\lambda}\psi^{12\bar2\bar3}-\frac{\lambda-1}{\lambda}\psi^{23\bar1\bar2})$	$\frac{i}{2}(\frac{1}{\lambda}\psi^{12\bar2\bar3}-\frac{\lambda-1}{\lambda}\psi^{23\bar1\bar2})$	0

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
$\psi^{12\bar{1}\bar{2}}$	0	0	0	0
$\psi^{12\bar{1}\bar{3}}$	0	0	0	0
$\psi^{12\bar{2}\bar{3}}$	0	0	0	0
$\psi^{13\bar{1}\bar{2}}$	0	0	0	0
$\psi^{13\bar{1}\bar{3}}$	0	0	0	0
$\psi^{13\bar{2}\bar{3}}$	$\frac{i}{2}\frac{1}{\lambda}\psi^{123\bar{1}\bar{2}}$	$-\frac{i}{2}\psi^{12\bar{1}\bar{2}\bar{3}}$	$\frac{i}{2}(\frac{1}{\lambda}\psi^{123\bar{1}\bar{2}} - \psi^{12\bar{1}\bar{2}\bar{3}})$	0
$\psi^{23\bar{1}\bar{2}}$	0	0	0	0
$\psi^{23\bar{1}\bar{3}}$	$\frac{i}{2}\psi^{123\bar{1}\bar{2}}$	$-\frac{i}{2}\frac{1}{\lambda}\psi^{12\bar{1}\bar{2}\bar{3}}$	$\frac{i}{2}(\psi^{123\bar{1}\bar{2}} - \frac{1}{\lambda}\psi^{12\bar{1}\bar{2}\bar{3}})$	0
$\psi^{23\bar{2}\bar{3}}$	0	0	0	0

$$\begin{aligned} &H^{1,0}_{BC} = \langle\psi^1,\psi^2\rangle \quad , \quad H^{2,0}_{BC} = \langle\psi^{12}\rangle \\ &H^{1,1}_{BC} = \langle\psi^{1\bar{1}},\psi^{1\bar{2}},\psi^{2\bar{1}},\psi^{2\bar{2}},\mu_\lambda\rangle \\ &H^{2,1}_{BC} = \langle\psi^{12\bar{1}},\psi^{12\bar{2}},\psi^{13\bar{1}},\psi^{23\bar{2}}\rangle \\ &H^{2,2}_{BC} = \langle\psi^{12\bar{1}\bar{3}},\psi^{12\bar{2}\bar{3}},\psi^{13\bar{1}\bar{2}},\psi^{13\bar{1}\bar{3}},\psi^{23\bar{1}\bar{2}},\psi^{23\bar{2}\bar{3}},\eta_\lambda\rangle \\ &H^{3,1}_{BC} = \langle\psi^{123\bar{1}},\psi^{123\bar{2}}\rangle \quad , \quad H^{3,2}_{BC} = \langle\psi^{123\bar{1}\bar{2}},\psi^{123\bar{1}\bar{3}},\psi^{123\bar{2}\bar{3}}\rangle \end{aligned} \tag{A.18}$$

where

$$\mu_\lambda = \begin{cases} \psi^{1\bar{3}} + \psi^{3\bar{1}} & \text{if} \quad \lambda = 2 \\ \psi^{2\bar{3}} + \psi^{3\bar{2}} & \text{if} \quad \lambda = \frac{1}{2} \\ 0 \quad \text{otherwise} \end{cases} \quad , \quad \eta_\lambda = \begin{cases} \psi^{13\bar{2}\bar{3}} + \psi^{23\bar{1}\bar{3}} & \text{if} \quad \lambda = -1 \\ 0 \quad \text{otherwise} \end{cases} \quad , \tag{A.19}$$

$\check{\mathfrak{g}}_{S,1} = (15, -25, -35, 45, 0, 0)$

$$\check{\Omega} = (e^4 + ie^1) \wedge (e^3 + ie^2) \wedge (e^6 + ie^5)$$

$$\begin{aligned} &\psi^1 = e^4 + ie^1 \quad , \quad d\psi^1 = -e^{45} - ie^{15} = \frac{i}{2}(\psi^{13} - \psi^{1\bar{3}}) \\ &\psi^2 = e^3 + ie^2 \quad , \quad d\psi^2 = e^{35} + ie^{25} = \frac{i}{2}(-\psi^{23} + \psi^{2\bar{3}}) \\ &\psi^3 = e^6 + ie^5 \quad , \quad d\psi^3 = 0 \end{aligned} \tag{A.20}$$

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
ψ^1	$\frac{i}{2}\psi^{13}$	$-\frac{i}{2}\psi^{13}$	$\frac{i}{2}(\psi^{13} - \psi^{13})$	$-\frac{1}{4}\psi^{133}$
ψ^2	$-\frac{i}{2}\psi^{23}$	$\frac{i}{2}\psi^{2\bar{3}}$	$\frac{i}{2}(-\psi^{23} + \psi^{2\bar{3}})$	$\frac{1}{4}\psi^{23\bar{3}}$
ψ^3	0	0	0	0
$\psi^{\bar{1}}$	$-\frac{i}{2}\psi^{3\bar{1}}$	$-\frac{i}{2}\psi^{\bar{1}\bar{3}}$	$-\frac{i}{2}(\psi^{3\bar{1}} + \psi^{\bar{1}\bar{3}})$	$-\frac{1}{4}\psi^{3\bar{1}\bar{3}}$
$\psi^{\bar{2}}$	$\frac{i}{2}\psi^{3\bar{2}}$	$\frac{i}{2}\psi^{\bar{2}\bar{3}}$	$\frac{i}{2}(\psi^{3\bar{2}} + \psi^{\bar{2}\bar{3}})$	$\frac{1}{4}\psi^{3\bar{2}\bar{3}}$
$\psi^{\bar{3}}$	0	0	0	0

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
ψ^{12}	0	0	0	0
ψ^{13}	0	$\frac{i}{2}\psi^{13\bar{3}}$	$\frac{i}{2}\psi^{13\bar{3}}$	0
ψ^{23}	0	$\frac{i}{2}\psi^{23\bar{3}}$	$\frac{i}{2}\psi^{23\bar{3}}$	0

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
$\psi^{123\bar{1}}$	0	$\frac{i}{2}\psi^{123\bar{1}\bar{3}}$	$\frac{i}{2}\psi^{123\bar{1}\bar{3}}$	0
$\psi^{123\bar{2}}$	0	$-\frac{i}{2}\psi^{123\bar{2}\bar{3}}$	$-\frac{i}{2}\psi^{123\bar{2}\bar{3}}$	0
$\psi^{123\bar{3}}$	0	0	0	0

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
$\psi^{123\bar{1}\bar{2}}$	0	0	0	0
$\psi^{123\bar{1}\bar{3}}$	0	0	0	0
$\psi^{123\bar{2}\bar{3}}$	0	0	0	0

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
$\psi^{1\bar{1}}$	$i\psi^{13\bar{1}}$	$i\psi^{1\bar{1}\bar{3}}$	$i(\psi^{13\bar{1}} + \psi^{1\bar{1}\bar{3}})$	$-\psi^{13\bar{1}\bar{3}}$
$\psi^{1\bar{2}}$	0	0	0	0
$\psi^{1\bar{3}}$	$\frac{i}{2}\psi^{13\bar{3}}$	0	$\frac{i}{2}(\psi^{13\bar{3}}$	0
$\psi^{2\bar{1}}$	0	0	0	0
$\psi^{2\bar{2}}$	$i\psi^{23\bar{2}}$	$-i\psi^{2\bar{2}\bar{3}}$	$i(\psi^{23\bar{2}} - \psi^{2\bar{2}\bar{3}})$	$\psi^{23\bar{2}\bar{3}}$
$\psi^{2\bar{3}}$	$-\frac{i}{2}\psi^{23\bar{3}}$	0	$-\frac{i}{2}\psi^{2\bar{3}\bar{3}}$	0
$\psi^{3\bar{1}}$	0	$\frac{i}{2}\psi^{3\bar{1}\bar{3}}$	$\frac{i}{2}\psi^{3\bar{1}\bar{3}}$	0
$\psi^{3\bar{2}}$	0	$-\frac{i}{2}\psi^{3\bar{2}\bar{3}}$	$-\frac{i}{2}\psi^{32\bar{3}}$	0
$\psi^{3\bar{3}}$	0	0	0	0

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
$\psi^{12\bar{1}}$	$-\frac{i}{2}\psi^{123\bar{1}}$	$-\frac{i}{2}\psi^{12\bar{1}\bar{3}}$	$-\frac{i}{2}(\psi^{123\bar{1}} + \psi^{12\bar{1}\bar{3}})$	$\frac{1}{4}\psi^{123\bar{1}\bar{3}}$
$\psi^{12\bar{2}}$	$\frac{i}{2}\psi^{123\bar{2}}$	$\frac{i}{2}\psi^{12\bar{2}\bar{3}}$	$\frac{i}{2}(\psi^{123\bar{2}} + \psi^{12\bar{2}\bar{3}})$	$-\frac{1}{4}\psi^{123\bar{2}\bar{3}}$
$\psi^{12\bar{3}}$	0	0	0	0
$\psi^{13\bar{1}}$	0	$-i\psi^{13\bar{1}\bar{3}}$	$-i\psi^{13\bar{1}\bar{3}}$	0
$\psi^{13\bar{2}}$	0	0	0	0
$\psi^{13\bar{3}}$	0	0	0	0
$\psi^{23\bar{1}}$	0	0	0	0
$\psi^{23\bar{2}}$	0	$i\psi^{23\bar{2}\bar{3}}$	$i\psi^{23\bar{2}\bar{3}}$	0
$\psi^{23\bar{3}}$	0	0	0	0

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
$\psi^{12\bar{1}\bar{2}}$	0	0	0	0
$\psi^{12\bar{1}\bar{3}}$	$-\frac{i}{2}\psi^{123\bar{1}\bar{3}}$	0	$-\frac{i}{2}\psi^{123\bar{1}\bar{3}}$	0
$\psi^{12\bar{2}\bar{3}}$	$\frac{i}{2}\psi^{123\bar{2}\bar{3}}$	0	$\frac{i}{2}\psi^{123\bar{2}\bar{3}}$	0
$\psi^{13\bar{1}\bar{2}}$	0	$\frac{i}{2}\psi^{13\bar{1}\bar{2}\bar{3}}$	$\frac{i}{2}\psi^{13\bar{1}\bar{2}\bar{3}}$	0
$\psi^{13\bar{1}\bar{3}}$	0	0	0	0
$\psi^{13\bar{2}\bar{3}}$	0	0	0	0
$\psi^{23\bar{1}\bar{2}}$	0	$-\frac{i}{2}\psi^{23\bar{1}\bar{2}\bar{3}}$	$-\frac{i}{2}\psi^{23\bar{1}\bar{2}\bar{3}}$	0
$\psi^{23\bar{1}\bar{3}}$	0	0	0	0
$\psi^{23\bar{2}\bar{3}}$	0	0	0	0

$$
\begin{aligned}
H^{1,0}_{BC} &= \langle \psi^{3} \rangle \quad , \quad H^{2,0}_{BC} = \langle \psi^{12} \rangle \\
H^{1,1}_{BC} &= \langle \psi^{1\bar{2}}, \psi^{2\bar{1}}, \psi^{3\bar{3}} \rangle \\
H^{2,1}_{BC} &= \langle \psi^{12\bar{3}}, \psi^{13\bar{2}}, \psi^{23\bar{1}} \rangle \\
H^{2,2}_{BC} &= \langle \psi^{12\bar{1}\bar{2}}, \psi^{13\bar{2}\bar{3}}, \psi^{23\bar{1}\bar{3}} \rangle \\
H^{3,1}_{BC} &= \langle \psi^{123\bar{3}} \rangle \quad , \quad H^{3,2}_{BC} = \langle \psi^{123\bar{1}\bar{2}} \rangle
\end{aligned} \tag{A.21}
$$

$\check{\mathfrak{g}}_{S,2} = (-24+35, 26, 36, -46, -56, 0)$

$$\check{\Omega} = (e^5 + ie^4) \wedge (e^2 + ie^3) \wedge (e^1 + ie^6)$$

$$
\begin{aligned}
\psi^1 &= e^5 + ie^4 \quad , \quad d\psi^1 = e^{56} + ie^{46} = \frac{i}{2}(-\psi^{13} + \psi^{1\bar{3}}) \\
\psi^2 &= e^2 + ie^3 \quad , \quad d\psi^2 = -e^{26} - ie^{36} = \frac{i}{2}(\psi^{23} - \psi^{2\bar{3}}) \\
\psi^3 &= e^1 + ie^6 \quad , \quad d\psi^3 = e^{24} - e^{35} = \frac{i}{2}(\psi^{1\bar{2}} + \psi^{2\bar{1}})
\end{aligned} \tag{A.22}
$$

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
ψ^1	$-\frac{i}{2}\psi^{13}$	$\frac{i}{2}\psi^{1\bar{3}}$	$\frac{i}{2}(-\psi^{13} + \psi^{1\bar{3}})$	$\frac{1}{4}(\psi^{13\bar{3}} + \psi^{12\bar{1}})$
ψ^2	$\frac{i}{2}\psi^{23}$	$-\frac{i}{2}\psi^{2\bar{3}}$	$\frac{i}{2}(\psi^{23} - \psi^{2\bar{3}})$	$\frac{1}{4}(\psi^{23\bar{3}} + \psi^{12\bar{2}})$
ψ^3	0	$\frac{i}{2}(\psi^{1\bar{2}} + \psi^{2\bar{1}})$	$\frac{i}{2}(\psi^{1\bar{2}} + \psi^{2\bar{1}})$	0
$\psi^{\bar{1}}$	$\frac{i}{2}\psi^{3\bar{1}}$	$\frac{i}{2}\psi^{\bar{1}\bar{3}}$	$\frac{i}{2}(\psi^{3\bar{1}} + \psi^{\bar{1}\bar{3}})$	$\frac{1}{4}(-\psi^{3\bar{1}\bar{3}} + \psi^{1\bar{1}\bar{2}})$
$\psi^{\bar{2}}$	$-\frac{i}{2}\psi^{3\bar{2}}$	$-\frac{i}{2}\psi^{\bar{2}\bar{3}}$	$-\frac{i}{2}(\psi^{3\bar{2}} + \psi^{\bar{2}\bar{3}})$	$\frac{1}{4}(-\psi^{3\bar{2}\bar{3}} + \psi^{2\bar{1}\bar{2}})$
$\psi^{\bar{3}}$	$\frac{i}{2}(\psi^{2\bar{1}+\psi^{1\bar{2}}})$	0	$\frac{i}{2}(\psi^{2\bar{1}+\psi^{1\bar{2}}})$	0

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
ψ^{12}	0	0	0	0
ψ^{13}	0	$-\frac{i}{2}(\psi^{13\bar{3}} + \psi^{12\bar{1}})$	$-\frac{i}{2}(\psi^{13\bar{3}} + \psi^{12\bar{1}})$	0
ψ^{23}	0	$\frac{i}{2}(\psi^{23\bar{3}} + \psi^{12\bar{2}})$	$\frac{i}{2}(\psi^{23\bar{3}} + \psi^{12\bar{2}})$	0

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
$\psi^{1\bar{1}}$	$-i\psi^{13\bar{1}}$	$-i\psi^{1\bar{1}\bar{3}}$	$-i(\psi^{13\bar{1}} + \psi^{1\bar{1}\bar{3}})$	$-\psi^{13\bar{1}\bar{3}}$
$\psi^{1\bar{2}}$	0	0	0	0
$\psi^{1\bar{3}}$	$-\frac{i}{2}(\psi^{13\bar{3}} + \psi^{12\bar{1}})$	0	$-\frac{i}{2}(\psi^{13\bar{3}} + \psi^{12\bar{1}})$	0
$\psi^{2\bar{1}}$	0	0	0	0
$\psi^{2\bar{2}}$	$i\psi^{23\bar{2}}$	$i\psi^{2\bar{2}\bar{3}}$	$i(\psi^{23\bar{2}} + \psi^{2\bar{2}\bar{3}})$	$-\psi^{23\bar{2}\bar{3}}$
$\psi^{2\bar{3}}$	$\frac{i}{2}(\psi^{23\bar{3}} + \psi^{12\bar{2}})$	0	$\frac{i}{2}(\psi^{23\bar{3}} + \psi^{12\bar{2}})$	0
$\psi^{3\bar{1}}$	0	$-\frac{i}{2}(\psi^{1\bar{1}\bar{2}} + \psi^{3\bar{1}\bar{3}})$	$-\frac{i}{2}(\psi^{1\bar{1}\bar{2}} + \psi^{3\bar{1}\bar{3}})$	0
$\psi^{3\bar{2}}$	0	$\frac{i}{2}(\psi^{2\bar{1}\bar{2}} + \psi^{3\bar{2}\bar{3}})$	$\frac{i}{2}(\psi^{2\bar{1}\bar{2}} + \psi^{3\bar{2}\bar{3}})$	0
$\psi^{3\bar{3}}$	$-\frac{i}{2}(\psi^{23\bar{1}} + \psi^{13\bar{2}})$	$\frac{i}{2}(\psi^{1\bar{2}\bar{3}} + \psi^{2\bar{1}\bar{3}})$	$\frac{i}{2}(-\psi^{23\bar{1}} - \psi^{13\bar{2}} + \psi^{1\bar{2}\bar{3}} + \psi^{2\bar{1}\bar{3}})$	$-\frac{1}{2}\psi^{12\bar{1}\bar{2}}$

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{123\bar1}$	0	$-\frac{i}{2}\psi^{123\bar1\bar3}$	$-\frac{i}{2}\psi^{123\bar1\bar3}$	0
$\psi^{123\bar2}$	0	$\frac{i}{2}\psi^{123\bar2\bar3}$	$\frac{i}{2}\psi^{123\bar2\bar3}$	0
$\psi^{123\bar3}$	0	0	0	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{123\bar1\bar2}$	0	0	0	0
$\psi^{123\bar1\bar3}$	0	0	0	0
$\psi^{123\bar2\bar3}$	0	0	0	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{12\bar1}$	$\frac{i}{2}\psi^{123\bar1}$	$\frac{i}{2}\psi^{12\bar1\bar3}$	$\frac{i}{2}(\psi^{123\bar1}+\psi^{12\bar1\bar3})$	$-\frac{1}{4}\psi^{123\bar1\bar3}$
$\psi^{12\bar2}$	$-\frac{i}{2}\psi^{123\bar2}$	$-\frac{i}{2}\psi^{12\bar2\bar3}$	$-\frac{i}{2}(\psi^{123\bar2}+\psi^{12\bar2\bar3})$	$-\frac{1}{4}\psi^{123\bar2\bar3}$
$\psi^{12\bar3}$	0	0	0	0
$\psi^{13\bar1}$	0	$-i\psi^{13\bar1\bar3}$	$-i\psi^{13\bar1\bar3}$	0
$\psi^{13\bar2}$	0	$-i\psi^{12\bar1\bar2}$	$-i\psi^{12\bar1\bar2}$	0
$\psi^{13\bar3}$	$-\frac{i}{2}\psi^{123\bar1}$	$-\frac{i}{2}\psi^{12\bar1\bar3}$	$-\frac{i}{2}(\psi^{123\bar1}+\psi^{12\bar1\bar3})$	$\frac{1}{4}\psi^{123\bar1\bar3}$
$\psi^{23\bar1}$	0	$-i\psi^{12\bar1\bar2}$	$-i\psi^{12\bar1\bar2}$	0
$\psi^{23\bar2}$	0	$-i\psi^{23\bar2\bar3}$	$-i\psi^{23\bar2\bar3}$	0
$\psi^{23\bar3}$	$\frac{i}{2}\psi^{123\bar2}$	$\frac{i}{2}\psi^{12\bar2\bar3}$	$\frac{i}{2}(\psi^{123\bar2}+\psi^{12\bar2\bar3})$	$\frac{1}{4}\psi^{123\bar2\bar3}$

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{12\bar1\bar2}$	0	0	0	0
$\psi^{12\bar1\bar3}$	$\frac{i}{2}\psi^{123\bar1\bar3}$	0	$\frac{i}{2}\psi^{123\bar1\bar3}$	0
$\psi^{12\bar2\bar3}$	$-\frac{i}{2}\psi^{123\bar2\bar3}$	0	$-\frac{i}{2}\psi^{123\bar2\bar3}$	0
$\psi^{13\bar1\bar2}$	0	$-\frac{i}{2}\psi^{13\bar1\bar2\bar3}$	$-\frac{i}{2}\psi^{13\bar1\bar2\bar3}$	0
$\psi^{13\bar1\bar3}$	0	0	0	0
$\psi^{13\bar2\bar3}$	$\frac{i}{2}\psi^{123\bar1\bar2}$	$-\frac{i}{2}\psi^{12\bar1\bar2\bar3}$	$\frac{i}{2}(\psi^{123\bar1\bar2}-\psi^{12\bar1\bar2\bar3})$	0
$\psi^{23\bar1\bar2}$	0	$\frac{i}{2}\psi^{23\bar1\bar2\bar3}$	$\frac{i}{2}\psi^{23\bar1\bar2\bar3}$	0
$\psi^{23\bar1\bar3}$	$\frac{i}{2}\psi^{123\bar1\bar2}$	$-\frac{i}{2}\psi^{12\bar1\bar2\bar3}$	$\frac{i}{2}(\psi^{123\bar1\bar2}-\psi^{12\bar1\bar2\bar3})$	0
$\psi^{23\bar2\bar3}$	0	0	0	0

$$
\begin{aligned}
&H^{1,0}_{BC} = 0 \quad , \quad H^{2,0}_{BC} = \langle\psi^{12}\rangle \\
&H^{1,1}_{BC} = \langle\psi^{1\bar2}, \psi^{2\bar1}\rangle \\
&H^{2,1}_{BC} = \langle\psi^{12\bar3}, \psi^{13\bar2} - \psi^{23\bar1}\rangle \\
&H^{2,2}_{BC} = \langle\psi^{13\bar2\bar3} - \psi^{23\bar1\bar3}\rangle \\
&H^{3,1}_{BC} = \langle\psi^{123\bar3}\rangle \quad , \quad H^{3,2}_{BC} = \langle\psi^{123\bar1\bar2}\rangle
\end{aligned} \tag{A.23}
$$

$\check{\mathfrak{g}}_{\mathbb{T}} = (0,0,0,0,0,12+34)$

$$\check\Omega = (e^4 + ie^1)\wedge(e^2 + ie^3)\wedge(e^6 + ie^5)$$

$$
\begin{aligned}
&\psi^1 = e^4 + ie^1 \quad , \quad d\psi^1 = 0 \\
&\psi^2 = e^2 + ie^3 \quad , \quad d\psi^2 = 0 \\
&\psi^3 = e^6 + ie^5 \quad , \quad d\psi^3 = -i(e^{12} + e^{34}) = \frac{i}{2}(\psi^{1\bar2} + \psi^{2\bar1})
\end{aligned} \tag{A.24}
$$

	∂	$\bar\partial$	d	$\partial\bar\partial$
ψ^{1}	0	0	0	0
ψ^{2}	0	0	0	0
ψ^{3}	0	$\frac{i}{2}(\psi^{1\bar2}+\psi^{2\bar1})$	$\frac{i}{2}(\psi^{1\bar2}+\psi^{2\bar1})$	0
$\psi^{\bar1}$	0	0	0	0
$\psi^{\bar2}$	0	0	0	0
$\psi^{\bar3}$	$\frac{i}{2}(\psi^{1\bar2}+\psi^{2\bar1})$	0	$\frac{i}{2}(\psi^{1\bar2}+\psi^{2\bar1})$	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
ψ^{12}	0	0	0	0
ψ^{13}	0	$-\frac{i}{2}\psi^{12\bar1}$	$-\frac{i}{2}\psi^{12\bar1}$	0
ψ^{23}	0	$\frac{i}{2}\psi^{12\bar2}$	$\frac{i}{2}\psi^{12\bar2}$	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{123\bar1}$	0	0	0	0
$\psi^{123\bar2}$	0	0	0	0
$\psi^{123\bar3}$	0	0	0	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{123\bar1\bar2}$	0	0	0	0
$\psi^{123\bar1\bar3}$	0	0	0	0
$\psi^{123\bar2\bar3}$	0	0	0	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{1\bar1}$	0	0	0	0
$\psi^{1\bar2}$	0	0	0	0
$\psi^{1\bar3}$	$-\frac{i}{2}\psi^{12\bar1}$	0	$-\frac{i}{2}\psi^{12\bar1}$	0
$\psi^{2\bar1}$	0	0	0	0
$\psi^{2\bar2}$	0	0	0	0
$\psi^{2\bar3}$	$\frac{i}{2}\psi^{12\bar2}$	0	$\frac{i}{2}\psi^{12\bar2}$	0
$\psi^{3\bar1}$	0	$-\frac{i}{2}\psi^{1\bar1\bar2}$	$-\frac{i}{2}\psi^{1\bar1\bar2}$	0
$\psi^{3\bar2}$	0	$\frac{i}{2}\psi^{2\bar1\bar2}$	$\frac{i}{2}\psi^{2\bar1\bar2}$	0
$\psi^{3\bar3}$	$\frac{i}{2}(\psi^{13\bar2}+\psi^{23\bar1})$	$\frac{i}{2}(\psi^{1\bar2\bar3}+\psi^{2\bar1\bar3})$	$\frac{i}{2}(\psi^{13\bar2}+\psi^{23\bar1}+\psi^{1\bar2\bar3}+\psi^{2\bar1\bar3})$	$-\frac{i}{2}\psi^{12\bar1\bar2}$

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{12\bar1}$	0	0	0	0
$\psi^{12\bar2}$	0	0	0	0
$\psi^{12\bar3}$	0	0	0	0
$\psi^{13\bar1}$	0	0	0	0
$\psi^{13\bar2}$	0	$-\frac{i}{2}\psi^{12\bar1\bar2}$	$-\frac{i}{2}\psi^{12\bar1\bar2}$	0
$\psi^{13\bar3}$	$\frac{i}{2}\psi^{123\bar1}$	$-\frac{i}{2}\psi^{12\bar1\bar3}$	$\frac{i}{2}(\psi^{123\bar1}-\psi^{12\bar1\bar3})$	0
$\psi^{23\bar1}$	0	$-\frac{i}{2}\psi^{12\bar1\bar2}$	$-\frac{i}{2}\psi^{12\bar1\bar2}$	0
$\psi^{23\bar2}$	0	0	0	0
$\psi^{23\bar3}$	$\frac{i}{2}\psi^{123\bar2}$	$\frac{i}{2}\psi^{12\bar2\bar3}$	$\frac{i}{2}(\psi^{123\bar2}+\psi^{12\bar2\bar3})$	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
$\psi^{12\bar1\bar2}$	0	0	0	0
$\psi^{12\bar1\bar3}$	0	0	0	0
$\psi^{12\bar2\bar3}$	0	0	0	0
$\psi^{13\bar1\bar2}$	0	0	0	0
$\psi^{13\bar1\bar3}$	0	0	0	0
$\psi^{13\bar2\bar3}$	$\frac{i}{2}\psi^{123\bar1\bar2}$	$-\frac{i}{2}\psi^{12\bar1\bar2\bar3}$	$\frac{i}{2}(\psi^{123\bar1\bar2}-\psi^{12\bar1\bar2\bar3})$	0
$\psi^{23\bar1\bar2}$	0	0	0	0
$\psi^{23\bar1\bar3}$	$\frac{i}{2}\psi^{123\bar1\bar2}$	$-\frac{i}{2}\psi^{12\bar1\bar2\bar3}$	$\frac{i}{2}(\psi^{123\bar1\bar2}-\psi^{12\bar1\bar2\bar3})$	0
$\psi^{23\bar2\bar3}$	0	0	0	0

$$
\begin{aligned}
&H^{1,0}_{BC}=\langle\psi^1,\psi^2\rangle \quad , \quad H^{2,0}_{BC}=\langle\psi^{12}\rangle\\
&H^{1,1}_{BC}=\langle\psi^{1\bar1},\psi^{1\bar2},\psi^{2\bar1},\psi^{2\bar2}\rangle\\
&H^{2,1}_{BC}=\langle\psi^{12\bar1},\psi^{12\bar2},\psi^{12\bar3},\psi^{13\bar1},\psi^{23\bar2},\psi^{13\bar2}-\psi^{23\bar1}\rangle\\
&H^{2,2}_{BC}=\langle\psi^{12\bar1\bar3},\psi^{12\bar2\bar3},\psi^{13\bar1\bar2},\psi^{13\bar1\bar3},\psi^{13\bar2\bar3}-\psi^{23\bar1\bar3},\psi^{23\bar1\bar2},\psi^{23\bar2\bar3}\rangle\\
&H^{3,1}_{BC}=\langle\psi^{123\bar1},\psi^{123\bar2},\psi^{123\bar3}\rangle \quad , \quad H^{3,2}_{BC}=\langle\psi^{123\bar1\bar2},\psi^{123\bar1\bar3},\psi^{123\bar2\bar3}\rangle
\end{aligned}
\tag{A.25}
$$

$\check{\mathfrak{g}}_Y=(0,0,0,0,12,13,15)$

$$\check\Omega=(e^3+ie^1)\wedge(e^5+ie^2)\wedge(e^6+ie^4)$$

$$
\begin{aligned}
&\psi^1=e^3+ie^1 \quad , \quad d\psi^1=0\\
&\psi^2=e^5+ie^2 \quad , \quad d\psi^2=-e^{13}=\frac{i}{2}\psi^{1\bar1}\\
&\psi^3=e^6+ie^4 \quad , \quad d\psi^3=-e^{15}-ie^{12}=\frac{i}{2}(\psi^{12}+\psi^{2\bar1})
\end{aligned}
\tag{A.26}
$$

	∂	$\bar\partial$	d	$\partial\bar\partial$
ψ^1	0	0	0	0
ψ^2	0	$\frac{i}{2}\psi^{1\bar1}$	$\frac{i}{2}\psi^{1\bar1}$	0
ψ^3	$\frac{i}{2}\psi^{12}$	$-\frac{i}{2}\psi^{2\bar1}$	$\frac{i}{2}(\psi^{12}+\psi^{2\bar1})$	0
$\psi^{\bar1}$	0	0	0	0
$\psi^{\bar2}$	$\frac{i}{2}\psi^{1\bar1}$	0	$\frac{i}{2}\psi^{1\bar1}$	0
$\psi^{\bar3}$	$\frac{i}{2}\psi^{1\bar2}$	$-\frac{i}{2}\psi^{\bar1\bar2}$	$\frac{i}{2}(\psi^{1\bar2}-\psi^{\bar1\bar2})$	0

	∂	$\bar\partial$	d	$\partial\bar\partial$
ψ^{12}	0	0	0	0
ψ^{13}	0	$-\frac{i}{2}\psi^{12\bar1}$	$-\frac{i}{2}\psi^{12\bar1}$	0
ψ^{23}	0	$-\frac{i}{2}\psi^{13\bar1}$	$-\frac{i}{2}\psi^{13\bar1}$	0

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
$\psi^{123\bar{1}}$	0	0	0	0
$\psi^{123\bar{2}}$	0	0	0	0
$\psi^{123\bar{3}}$	0	$\frac{i}{2}\psi^{123\bar{1}\bar{2}}$	$\frac{i}{2}\psi^{123\bar{1}\bar{2}}$	0

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
$\psi^{123\bar{1}\bar{2}}$	0	0	0	0
$\psi^{123\bar{1}\bar{3}}$	0	0	0	0
$\psi^{123\bar{2}\bar{3}}$	0	0	0	0

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
$\psi^{1\bar{1}}$	0	0	0	0
$\psi^{1\bar{2}}$	0	0	0	0
$\psi^{1\bar{3}}$	0	$\frac{i}{2}\psi^{1\bar{1}\bar{2}}$	$\frac{i}{2}\psi^{1\bar{1}\bar{2}}$	0
$\psi^{2\bar{1}}$	0	0	0	0
$\psi^{2\bar{2}}$	$-\frac{i}{2}\psi^{12\bar{1}}$	$\frac{i}{2}\psi^{1\bar{1}\bar{2}}$	$\frac{i}{2}(-\psi^{12\bar{1}} + \psi^{1\bar{1}\bar{2}})$	0
$\psi^{2\bar{3}}$	$-\frac{i}{2}\psi^{12\bar{2}}$	$\frac{i}{2}(\psi^{1\bar{1}\bar{3}} + \psi^{2\bar{1}\bar{2}})$	$\frac{i}{2}(-\psi^{12\bar{2}} + \psi^{1\bar{1}\bar{3}} + \psi^{2\bar{1}\bar{2}})$	0
$\psi^{3\bar{1}}$	$\frac{i}{2}\psi^{12\bar{1}}$	0	$\frac{i}{2}\psi^{12\bar{1}}$	0
$\psi^{3\bar{2}}$	$\frac{i}{2}(\psi^{12\bar{2}} + \psi^{13\bar{1}})$	$\frac{i}{2}\psi^{2\bar{1}\bar{2}}$	$\frac{i}{2}(\psi^{12\bar{2}} + \psi^{13\bar{1}} + \psi^{2\bar{1}\bar{2}})$	0
$\psi^{3\bar{3}}$	$\frac{i}{2}(\psi^{12\bar{3}} + \psi^{13\bar{2}})$	$\frac{i}{2}(\psi^{2\bar{1}\bar{3}} + \psi^{3\bar{1}\bar{2}})$	$\frac{i}{2}(\psi^{12\bar{3}} + \psi^{13\bar{2}} + \psi^{2\bar{1}\bar{3}} + \psi^{3\bar{1}\bar{2}})$	$-\frac{1}{2}\psi^{12\bar{1}\bar{2}}$

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
$\psi^{12\bar{1}}$	0	0	0	0
$\psi^{12\bar{2}}$	0	0	0	0
$\psi^{12\bar{3}}$	0	$-\frac{i}{2}\psi^{12\bar{1}\bar{2}}$	$-\frac{i}{2}\psi^{12\bar{1}\bar{2}}$	0
$\psi^{13\bar{1}}$	0	0	0	0
$\psi^{13\bar{2}}$	0	0	0	0
$\psi^{13\bar{3}}$	0	$-\frac{i}{2}(\psi^{12\bar{1}\bar{3}} - \psi^{13\bar{1}\bar{3}})$	$-\frac{i}{2}(\psi^{12\bar{1}\bar{3}} - \psi^{13\bar{1}\bar{3}})$	0
$\psi^{23\bar{1}}$	0	0	0	0
$\psi^{23\bar{2}}$	$\frac{i}{2}\psi^{123\bar{1}}$	$-\frac{i}{2}\psi^{13\bar{1}\bar{2}}$	$\frac{i}{2}(\psi^{123\bar{1}} - \psi^{13\bar{1}\bar{2}})$	0
$\psi^{23\bar{3}}$	$\frac{i}{2}\psi^{123\bar{2}}$	$-\frac{i}{2}(\psi^{13\bar{1}\bar{3}} - \psi^{23\bar{1}\bar{2}})$	$\frac{i}{2}(\psi^{123\bar{2}} - \psi^{13\bar{1}\bar{3}} + \psi^{23\bar{1}\bar{2}})$	0

	∂	$\bar{\partial}$	d	$\partial\bar{\partial}$
$\psi^{12\bar{1}\bar{2}}$	0	0	0	0
$\psi^{12\bar{1}\bar{3}}$	0	0	0	0
$\psi^{12\bar{2}\bar{3}}$	0	0	0	0
$\psi^{13\bar{1}\bar{2}}$	0	0	0	0
$\psi^{13\bar{1}\bar{3}}$	0	0	0	0
$\psi^{13\bar{2}\bar{3}}$	0	$\frac{i}{2}\psi^{12\bar{1}\bar{2}\bar{3}}$	$\frac{i}{2}\psi^{12\bar{1}\bar{2}\bar{3}}$	0
$\psi^{23\bar{1}\bar{2}}$	0	0	0	0
$\psi^{23\bar{1}\bar{3}}$	$\frac{i}{2}\psi^{123\bar{1}\bar{2}}$	0	$\frac{i}{2}\psi^{123\bar{1}\bar{2}}$	0
$\psi^{23\bar{2}\bar{3}}$	$\frac{i}{2}\psi^{123\bar{1}\bar{3}}$	$\frac{i}{2}\psi^{12\bar{1}\bar{2}\bar{3}}$	$\frac{i}{2}(\psi^{123\bar{1}\bar{3}} - \psi^{12\bar{1}\bar{2}\bar{3}})$	0

$$
\begin{aligned}
&H_{BC}^{1,0} = \langle \psi^{1} \rangle \quad , \quad H_{BC}^{2,0} = \langle \psi^{12} \rangle \\
&H_{BC}^{1,1} = \langle \psi^{1\bar{1}}, \psi^{1\bar{2}}, \psi^{2\bar{1}}, -\psi^{1\bar{3}} + \psi^{2\bar{2}} + \psi^{3\bar{1}} \rangle \\
&H_{BC}^{2,1} = \langle \psi^{12\bar{1}}, \psi^{12\bar{2}}, \psi^{13\bar{1}}, \psi^{13\bar{2}}, \psi^{23\bar{1}} \rangle \\
&H_{BC}^{2,2} = \langle \psi^{12\bar{1}\bar{3}}, \psi^{12\bar{2}\bar{3}}, \psi^{13\bar{1}\bar{2}}, \psi^{13\bar{1}\bar{3}}, \psi^{23\bar{1}\bar{2}} \rangle \\
&H_{BC}^{3,1} = \langle \psi^{123\bar{1}}, \psi^{123\bar{2}} \rangle \quad , \quad H_{BC}^{3,2} = \langle \psi^{123\bar{1}\bar{2}}, \psi^{123\bar{1}\bar{3}}, \psi^{123\bar{2}\bar{3}} \rangle
\end{aligned} \tag{A.27}
$$

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