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Analysis of non-linear approximated value equation under multiple risk factors and stochastic intensities

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ABSTRACT

We develop a numerical method to approximate the adjusted value of a European contingent claim subject to multiple credit risks in a market model where the underlying's price is correlated with the stochastic default intensities of both parties of the contract. When the close-out value of the contract is chosen as a fraction of the adjusted value, the latter verifies a non-linear, not explicitly solvable BSDE. In a Markovian setting, this adjusted value is a deterministic function of the state variables verifying a non-linear PDE. Thus, we build a numerical method to approximate the solution of this non-linear PDE, as an alternative to the commonly used Monte Carlo simulations, which require large computational times, especially when the number of the state variables grows. We construct this approximated solution by the simple method of finite differences and we show the method to be accurate and efficient.

1. Introduction

In periods of financial distress or crisis, some classical financial models became inadequate to represent all the risk factors. As a matter of fact, in 2004, the Basel Committee signed the Basel II agreement regarding the capital requirements banks must meet to curb financial risks. In particular, Basel II set up the accounting standards regarding Counterparty Credit Risk (CCR), which is the risk that a counterparty might default before honouring its engagements, and it covers loans and repurchase agreement (Repo) transactions, and most importantly, over-the-counter (OTC) derivatives. In the last decade, the interest in CCR increased remarkably, and a theory of Value Adjustments was developed. The first to be introduced was the Credit Valuation Adjustment (CVA), as the difference between the risk-free value of a portfolio and the calculation taking into account the possibility of counterparty default, while the investor was always considered default-free.

Over the years, the role of the CVA has increased considerably, and it has become crucial in derivatives trading in OTC markets, stimulating much research in the field: see [16,20,26,27], and [8] for a list of frequently asked questions on the subject.

Before the financial crisis of 2007 – 08, most institutions did not correctly incorporate default risks in their risk system, indeed a statistical study showed that more than two-thirds of the losses for that period were due to incorrect evaluation of derivative products rather than an actual counterpart's default. Hence after the financial crisis, in 2009, the Basel Committee issued a new version of the act called Basel III, pushing financial institutions to incorporate default risks of either party, when evaluating products with cashflows in both directions.

A new measure called *Debt Valuation Adjustment* (DVA) was introduced as an accruement of the claim's value due to the investor's default risk.

To mitigate credit risk collateralization usually employed to balance the parties exposure to reciprocal default event, it is often possible to re-hypothecate the collateral for self-financing. The impact of collateralization on default risk, CVA and DVA has been analyzed in [11,10,15].

As investments/collateralizations are often funded also from external sources, further risks are involved and further adjustments had to be introduced. Funding Value Adjustment (FVA) and Liquidity Value Adjustment (LVA), which makes the pricing problem recursive and nonlinear, as those quantities are closely linked to the adjusted price itself.

We refer the reader to [13,15,18,19] for a detailed discussion.

Following those papers and exploiting the intensity approach [1,12,13], it can be show that the adjusted value of a financial contract, under a risk-neutral measure, can be characterized as the solution of a BSDE (Backward Stochastic Differential equation). This equation depends on the so-called "close-out value", which is a portion of a contractually agreed price to be paid as partial compensation when the default of one of the parties occurs. There are fundamentally two possibilities:

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either the close-out value is taken as a portion of the default-free price or of the defaultable contract price. The first choice usually determines a solvable linear BSDE, while the second, returns a non-linear BSDE, not explicitly solvable. The first setting was investigated in [1] with an expansion and approximation technique, which fails in presence of non linearity. Typically, this leaves only Monte Carlo simulation techniques, (for instance [33]), as the main choice for the numerical approximation of the solution, with usually very long computational times.

Recently, Deep Learning techniques have been successfully employed to solve BSDE's and hence employed to compute non-linear value adjustments, in [25] a deep BSDE Solver showed to be a highly efficient method to approximate the dynamics of defaultable portfolios. Nevertheless, those algorithms remain computationally costly in the learning phase.

Here, we propose a simple numerical method with low computational times and efficient results, in a Markovian setting when the derivative's value can be expressed as a deterministic function of the state variables. Such function satisfies a non-linear PDE with final condition given by the product's payoff (for a detailed discussion about the connection between CVA and the PDEs, see [17]), and we propose to discretize this associated PDE.

More precisely, we consider the case of a European claim subject to default, funding, and liquidity risks, when

- the default intensities are constant;
- the default intensities are driven by a CIR process.

Our main goal is to treat the second case, and we use the first to compare with the results obtained in [13] in the same setting. A fixed point approach was proposed in [31], though not succeeding to consider non-constant rates and intensities.

Still with constant intensities, we quote the recent work by [3,2] that extend the analysis to jump-diffusion models and apply the finite elements and contraction methods to approximate the PDE solutions. Stochastic intensities are rarely treated [23] (except for the Gaussian case, not fit to represent positive processes in [2]), even though they represent an important modeling choice.

Here, we consider both parties to be defaultable with stochastic intensities subject to funding and collateralization risks, and the close-out value to be a portion of the adjusted value itself. The intensities are represented by means of CIR processes to ensure their positivity.

Under those choices, the backward equation becomes non-linear reflecting in a 4 dimensional non-linear PDE. We used the simple method of finite differences [21,22,29] to approximate the PDE with a system of ODEs, which by one-step methods, such as the Euler one, can be solved.

We implemented the method by Matlab software, and we used Monte Carlo simulations as benchmark.

We achieved the same degree of accuracy as by Monte Carlo method was preserved by the method of finite differences. Computational times were remarkably shorter even when compared with the methods in [3, 2].

The paper is structured as follows. In the first section, we briefly describe the modeling of the problem, and we introduce the BSDE characterizing the adjusted contract's value. Next, we consider the specific case of a European call, and we obtain the associated non-linear PDE. The third section is dedicated to the method of finite differences and its implementation. In the last sections, we discuss the numerical results, and a sensitivity analysis is performed.

2. Evaluation of European claims under the intensity approach

We consider a finite time interval $[0,T]$ and a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The market is described by the risk-free interest rate process r_t, determining the money market account and by an adapted process S_t, representing the asset price (*underlying*), and we assume to be in absence of arbitrage with $\mathbb{P}$ representing a risk-neutral measure selected by some criterion. We denote by $\{\mathcal{F}_t\}_{t\in[0,T]}$ the market filtration generated by the processes S_t, r_t and other possible stochastic factors. The process S_t has the following dynamics under the measure $\mathbb{P}$:

$$dS_t = r_t S_t dt + \sigma(t, S_t) dW_t \tag{1}$$

where W_t is a Brownian motion, and $\sigma(t,x)$ is Lipschitz continuous in x uniformly in time and with sub-linear growth.[1]

We consider two financial entities (1 =Counterparty, 2 =Investor) exchanging some European claim with maturity T and payoff $\Phi(S_T)$, where Φ is a function as regular as needed, not necessarily non-negative.

We take the perspective of the investor with the objective to compute the contract value, that we denote by $\bar{V}_t$, taking into account all the cashflows (see [13,14] for more details).

Both parties might default, and we denote by τ_1, τ_2 the random variables representing their default times. We assume that they cannot occur jointly, which is, for example, satisfied in all intensity models of credit risk.

In general, these r.v.'s are not necessarily stopping times with respect to the filtration $\mathcal{F}_t$, generated by the market observable. To price the defaultable contract, we first need to extend $\mathcal{F}_t$, to $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}^1_t \vee \mathcal{H}^2_t$, where $\mathcal{H}^i_t = \sigma(\{1_{\{\tau_i \le s\}}, s \le t\})$, $i = 1,2$, which is the smallest filtration making the random variables τ_i stopping times. We assume that τ_1, τ_2 are the first jump times of two Cox processes with stochastic $\mathcal{F}$-predictable positive intensities λ^1, λ^2 (see [13,30]). More precisely

- $\tau_i = \inf\{t \ge 0 | \int_0^t \lambda^i_u du > \xi_i\}$, where ξ_i are two independent standard exponential random variables.

The so defined default times result being *conditionally independent* with respect to $\mathcal{F}$, that is

$$\mathbb{P}[\tau_1 > t_1, \tau_2 > t_2 | \mathcal{F}_t] = \mathbb{P}[\tau_1 > t_1 | \mathcal{F}_t]\mathbb{P}[\tau_2 > t_2 | \mathcal{F}_t], \quad \forall t_1, t_2 \in [0,t],$$

so that the probability of simultaneous default is 0.

As a consequence, the conditional distribution of the "first to default" time $\tau = min(\tau_1, \tau_2)$ has the representation

$$\mathbb{P}[\tau > t | \mathcal{F}_s] = \mathrm{e}^{-\int_0^t \lambda_u du}, \qquad \lambda = \lambda^1 + \lambda^2, \quad s \ge t$$

and we denote $\bar{\tau} = min(\tau, T)$.

We denote $\mathbb{E}^{\mathcal{A}}[\cdot] = \mathbb{E}[\cdot|\mathcal{A}]$, $\mathcal{A} = \mathcal{G}, \mathcal{F}$, and we assume that so called H-hypothesis (H) every $\mathcal{F}_t$-martingale remains a $\mathcal{G}_t$-martingale (see [24, 30]).

By following [13], we know the $\mathcal{G}_t$-adapted value process of the defaultable derivative, $\bar{V}_t$, is given by the sum of the discounted default-free price and the adjustments due to default, funding, and collateralization risks, and it is characterized as the solution of the following BSDE

$$\begin{aligned} \bar{V}_t = & \mathbb{E}^{\mathcal{G}}_t\left[1_{\{\tau>T\}}\mathrm{e}^{-\int_t^T r_s ds}\Phi(S_T) + \int_t^{\bar{\tau}} \mathrm{e}^{-\int_t^u r_s ds}\pi_u du\right] \\ & - \mathbb{E}^{\mathcal{G}}_t\left[\int_t^{\bar{\tau}} \mathrm{e}^{-\int_t^u r_s ds}\left(c_u - r_u\right)C_u du\right] \\ & - \mathbb{E}^{\mathcal{G}}_t\left[\int_t^{\bar{\tau}} \mathrm{e}^{-\int_t^u r_s ds}(\tilde{f}_u - r_u)(\bar{V}_u - C_u) du\right] \\ & - \mathbb{E}^{\mathcal{G}}_t\left[\int_t^{\bar{\tau}} \mathrm{e}^{-\int_t^u r_s ds}(r_u - \bar{h}_u)\bar{H}_u du\right] \end{aligned} \tag{2}$$

[1] Such hypotheses guarantee the existence and uniqueness of a strong solution for equation (1).

Table 1
Summary of cashflows and their measurability properties.

Symbol	Role	Assumption
$\Phi()$	Payoff at maturity	Lipschitz function of S_T
π	Contract dividends	$\mathcal{F}$-predictable
C	Collateral process	$\mathcal{F}$-predictable
$\bar{H}$	Hedging process	$\mathcal{G}$-predictable
ϵ	Close-out value	$\mathcal{F}$-predictable
c	Collateral rate	$\mathcal{F}$-predictable
$\bar{f}$	Funding rate	$\mathcal{G}$-predictable
$\bar{h}$	Hedging rate	$\mathcal{G}$-predictable
LGD_i, $i=1,2$	Loss Given Default	Constant

$$+\mathbb{E}_t^{\mathcal{G}}\left[e^{-\int_t^\tau r_s ds}1_{\{t\le\tau\le T\}}\Big(\epsilon_\tau - 1_{\{\tau_1<\tau_2\}}LGD_1(\epsilon_\tau - C_\tau)^+ + 1_{\{\tau_2<\tau_1\}}LGD_2(\epsilon_\tau - C_\tau)^-\Big)\right].$$

Some terms in the above are predefined by the contract's agreement, others depend on the price evolution. We summarize their meaning and measurability properties in Table 1. The close-out value ϵ_u is usually taken as the default-free price or as the adjusted price of the defaultable claim: the first choice gives a solvable linear BSDE, while the second ($\epsilon_u = \bar{V}_u$) determines a non-linear BSDE, not explicitly solvable.

It is to be noted that the default times are not market observable, thus the theoretical price represented by (2) must be projected on to the market filtration $\mathcal{F}_t$.

To do so, we employ the well-known Key Lemma and its extensions, (see for instance [5,6,20,30]).

Theorem 1 *(Key Lemma). For any $\mathcal{G}$-measurable random variable X and $t>0$, we have*

$$\mathbb{E}_t^{\mathcal{G}}\left[1_{\{t<\tau_i\le s\}}X\right] = 1_{\{\tau>t\}}\frac{\mathbb{E}_t^{\mathcal{F}}\left[1_{\{t<\tau_i\le s\}}X\right]}{\mathbb{E}_t^{\mathcal{F}}\left[1_{\{\tau_i>t\}}\right]}, i=1,2.$$

In particular, we have that for any $\mathcal{G}_t$-measurable random variable Y there exists an $\mathcal{F}_t$-measurable random variable Z such that $1_{\{\tau_i>t\}}Y = 1_{\{\tau_i>t\}}Z$, $i=1,2$.

- *If φ_u is a $\mathcal{G}$-adapted process, then*

$$\mathbb{E}^{\mathcal{G}_t}\left[\int_t^{\bar\tau}\varphi_u du\right] = 1_{\{\tau_i>t\}}\mathbb{E}^{\mathcal{F}_t}\left[\int_t^T e^{-\int_t^u \lambda_s^i ds}\bar\varphi_u du\right],\quad i=1,2,$$

where $\bar\varphi_u$ is an $\mathcal{F}_u$-adapted process such that $1_{\{\tau_i>u\}}\bar\varphi = 1_{\{\tau_i>u\}}\varphi$.

- *If φ_u is an $\mathcal{F}$-predictable process, we have*

$$\mathbb{E}_t^{\mathcal{G}}\left[1_{\{t<\tau<T\}}1_{\{\tau_1<\tau_2\}}\varphi_\tau\right] = 1_{\{\tau>t\}}\mathbb{E}_t^{\mathcal{F}}\left[\int_t^T e^{-\int_t^u(\lambda_s^1+\lambda_s^2)ds}\lambda_u^1\varphi_u du\right].$$

Projecting (2) on $\mathcal{F}_t$, and employing the previous lemma we may conclude that the $\mathcal{F}_t$-adapted adjusted price V_t, such that $1_{\{\tau>t\}}V_t = 1_{\{\tau>t\}}\bar{V}_t$ verifies the following $\mathcal{F}$-BSDE

$$\begin{aligned}V_t = \mathbb{E}_t^{\mathcal{F}}\Bigg[&e^{-\int_t^T(r_s+\lambda_s)ds}\Phi(S_T)\\ &+\int_t^T e^{-\int_t^u(r_s+\lambda_s)ds}\Big(\pi_u-(c_u-r_u)C_u-(f_u-r_u)(V_u-C_u)\\ &-(r_u-h_u)H_u+V_u\lambda_u - LGD_1\lambda_u^1(V_u-C_u)^+ + LGD_2\lambda_u^2(V_u-C_u)^-\Big)du\Bigg],\end{aligned}\tag{3}$$

where f_u, h_u and H_u are $\mathcal{F}$-adapted processes such that $1_{\{\tau>t\}}\xi_u = 1_{\{\tau>t\}}\bar{\xi}_u$ for $\xi = f,h,H$.

If $\mathcal{F}_t$ is generated by a (possibly multidimensional) Brownian motion driving the market assets prices, by the *martingale representation theorem*, taking for granted the necessary integrability conditions, we can rewrite (3) as

$$\begin{aligned}V_t = \Phi(S_T) + \int_t^T&\Big(\pi_u + (f_u-c_u)C_u - f_uV_u - (r_u-h_u)H_u\\ &- LGD_1\lambda_u^1(V_u-C_u)^+ + LGD_2\lambda_u^2(V_u-C_u)^-\Big)du - \int_t^T Z_u dW_u + \mathcal{M}_t,\end{aligned}\tag{4}$$

where W_t is a (vector) Brownian motion, Z_t an $\mathcal{F}$-adapted possibly square integrable (vector) process, and $\mathcal{M}_t$ is a martingale orthogonal to $\int_t^T Z_u\cdot dW_u$, possibly depending on further stochastic factors.

Missing a closed form solution for (4), one may try to construct an appropriate approximation procedure. In the literature, the most widespread method is Monte Carlo simulations (possibly coupled with deep learning-techniques as in [25]), which imply very long computational times. It is then worth looking for alternative, less costly methods, which is possible to attain when the underlying processes are diffusion, generating a Markovian vector.

3. Markovian BSDE and PDE for V_t

Under appropriate conditions, (3) has a unique adapted solution (V,Z) (see [13,34]), and in a Markovian setting this is a deterministic function of the state variables, which is the case when (S,λ^1,λ^2) are diffusion processes.

Using the flow notation, we assume that the market model is given by S satisfying:

$$S_s^{t,x} = x + \int_t^s r_u S_u^{t,x}du + \int_t^s \sigma S_u^{t,x}dW_u,\quad \sigma>0,\quad t\le s\le T,\tag{5}$$

with default intensities following CIR dynamics

$$\begin{aligned}\lambda_s^{1,t,y} &= y + \int_t^s \gamma_1(\psi_1-\lambda_u^{1,t,y})du + \int_t^s \eta_1\sqrt{\lambda_u^{1,t,y}}\,dB_u^1\\ \lambda_s^{2,t,z} &= z + \int_t^s \gamma_2(\psi_2-\lambda_u^{2,t,z})du + \int_t^s \eta_2\sqrt{\lambda_u^{2,t,z}}\,dB_u^2,\end{aligned}\tag{6}$$

with

$$W_t = \rho_1 B_t^1 + \rho_2 B_t^2 + \sqrt{1-\rho_1^2-\rho_2^2}B_t^3,\qquad \rho_1^2+\rho_2^2\le 1,\quad -1\le\rho_i\le 1\tag{7}$$

where (B_t^1, B_t^2, B_t^3) is 3-dimensional standard Brownian motion, and $\gamma_i,\psi_i,\eta_i\ge 0$, $i=1,2$, verify the *Feller condition*, $2\gamma_i\psi_i\ge\eta_i^2$, to ensure the processes' positivity.

To simplify our discussion we also assume that

- the claim pays no dividends, hence $\pi=0$;
- the rates r,f,c,h are deterministic, bounded functions of time;
- the collateral process is a fraction of the process V_u, namely $C_u = \alpha_u V_u$, where $0\le\alpha_u\le 1$ is a function of time;
- the process $H_t = H(t,S_t,V_t,Z_t)$, where $H(u,x,v,z)$ is a deterministic, Lipschitz-continuous function in v,z, uniformly in u. Besides $H(u,x,0,0)$ is continuous in x. This means that we have an explicit representation for the hedging process H_t (see [1,13,16,9]);

Here, we choose the two default intensities independent of each other to simplify calculations, but this assumption may be easily removed by adding a correlation parameter in the discussion that follows.

We remark that by taking $\gamma_i=\psi_i=\eta_i=0$, $i=1,2$, we can restrict to the case of deterministic intensities (as in [13]).

Using this representation, (4) becomes

$$\begin{aligned} dV_s^{t,x,y,z} =& \Bigg[(1-\alpha_t)\Big[f_t V_s^{t,x,y,z} + LGD_1 \lambda_s^{1,t,y} V_s^{t,x,y,z,+} - LGD_2 \lambda_s^{2,t,y} V_s^{t,x,y,z,-} \Big] \\ &+ \alpha_t c_t V_s^{t,x,y,z} + (r_t - h_t) H(t, S_s^{t,x}, V_s^{t,x,y,z}, Z_s^{t,x,y,z}) \Bigg] dt \\ &+ Z_s^{t,x,y,z} dW_t + d\mathcal{M}_t \\ V_T^{t,x} =& \Phi(S_T^{t,x}). \end{aligned} \tag{8}$$

As shown in [1], assuming a δ-hedging for this product, an appropriate change of probability may be applied to include the hedging function H in the dynamics, to rewrite (8) as

$$\begin{aligned} dV_s^{t,x,y,z} =& - \Bigg[(1-\alpha_t)\Big[- f_t V_s^{t,x,y,z} - LGD_1 \lambda_s^{1,t,y} V_s^{t,x,y,z,+} \\ &+ LGD_2 \lambda_s^{2,t,y} V_s^{t,x,y,z,-} \Big] - \alpha_t c_t V_s^{t,x,y,z} - (r_t - h_t) \frac{\partial V_s^{t,x,y,z}}{\partial S} S_s^{t,x} \Bigg] dt \\ &+ Z_s^{t,x,y,z} dW_t + d\mathcal{M}_t \\ V_T^{t,x} =& \Phi(S_T^{t,x}), \end{aligned} \tag{9}$$

when $x^+ = max(x,0)$, $x^- = max(-x,0)$.

Since the triple $(S^{t,x}, \lambda^{1,t,y}, \lambda^{2,t,z})$ is Markovian, $V_s^{t,x,y,z}$ is a deterministic function of the state variables, $u(s, S_s^{t,x}, \lambda_s^{1,t,y}, \lambda_s^{2,t,z})$, with $u(t,x,y,z) \in C^{1,2}([0,T] \times \mathbb{R}_+^3)$.

By applying Ito's formula, and comparing the two expressions, it can be shown that $u(t,x,y,z)$ verifies the non-linear PDE independent of the interest rate r.

$$\begin{cases} \mathcal{L}(u)(t,x,y,z) - (1-\alpha)\Big[LGD_1 y u(t,x,y,z)^+ - LGD_2 z u(t,x,y,z)^- \Big] = 0 \\ u(T,x,y,z) = \Phi(x), \end{cases} \tag{10}$$

where, with the parameters introduced by the model (5), (6), (7)

$$\begin{aligned} \mathcal{L}(u)(t,x,y,z) =& \partial_t u(t,x,y,z) + \gamma_1(\psi_1 - y)\partial_y u(t,x,y,z) \\ &+ \gamma_2(\psi_2 - z)\partial_z u(t,x,y,z) \\ &+ hx\partial_x u(t,x,y,z) + \frac{1}{2}\eta_1^2 y \partial_y^2 u(t,x,y,z) + \frac{1}{2}\eta_2^2 z \partial_z^2 u(t,x,y,z) \\ &+ \frac{1}{2}\sigma^2 x^2 \partial_x^2 u(t,x,y,z) + \rho_1 \sigma x \eta_1 \sqrt{y} \partial_{xy} u(t,x,y,z) \\ &+ \rho_2 \sigma x \eta_2 \sqrt{z} \partial_{xz} u(t,x,y,z) - \alpha c u(t,x,y,z) \\ &- (1-\alpha) f u(t,x,y,z). \end{aligned}$$

When considering a triple of stochastic processes, Monte Carlo simulations needed to approximate V_t in (3) are bound to be extremely costly in terms of machine time, thus we suggest a discretization of (10) that seems to work efficiently in terms of computational times and accuracy.

4. Problem discretization

The method of finite differences (see for instance [21,22,29,32,35,36,38]) is a numerical method to solve PDEs by approximating the spatial derivatives with finite differences, so generating a system of ODEs at each point of the discretization grid that can be solved by a suitable time integration method.

The spatial domain $\mathbb{R}_+^3$ is unbounded, so we need to restrict it to an appropriate bounded rectangle $[a_x, b_x] \times [a_y, b_y] \times [a_z, b_z] \subset \mathbb{R}_+^3$. This truncation requires defining appropriate boundary conditions, which can be done by identifying, when possible, the asymptotic behaviour of the solution. Here, we decided to exploit the knowledge of the Black & Scholes formula, with adjusted rates to include the default intensities

$$\begin{aligned} &u(t,a_x,y,z) = 0, && u(t,b_x,y,z) = \phi(t,b_x; r+\lambda,\sigma), \\ &u(t,x,a_y,z) = \phi(t,x; r+\lambda,\sigma), && u(t,x,b_y,z) = \phi(t,x; r+\lambda,\sigma), \\ &u(t,x,y,a_z) = \phi(t,x; r+\lambda,\sigma), && u(t,x,y,b_z) = \phi(t,x; r+\lambda,\sigma), \end{aligned} \tag{11}$$

where $\phi(t,x;\omega,\sigma)$ is the Black and Scholes's pricing function. The choice of the Black and Scholes's pricing function to set the Dirichlet boundary conditions is somewhat arbitrary. The rationale behind such choice is that it is exactly what we would have, when considering only the CVA without any other feature. We sub-divide the three space intervals into m uniform[2] sub-intervals by taking, $x_i = a_x + i\Delta x$, $y_i = a_y + i\Delta y$, $z_i = a_z + i\Delta z$ with $\Delta x = \frac{(b_x - a_x)}{m}$, $\Delta y = \frac{(b_y - a_y)}{m}$, $\Delta z = \frac{(b_z - a_z)}{m}$ for $i = 0,\dots,m$, and we apply the finite difference method to approximate the space partial derivatives,

$$\partial_x u(t,x_k,y_i,z_j) \approx \frac{u(t,x_{k+1},y_i,z_j) - u(t,x_k,y_i,z_j)}{\Delta x} \qquad k = 0,\dots,m-1, \quad i,j = 0,\dots,m$$

$$\partial_x^2 u(t,x_k,y_i,z_j) \approx \frac{u(t,x_{k+1},y_i,z_j) - 2u(t,x_k,y_i,z_j) + u(t,x_{k-1},y_i,z_j)}{\Delta x^2} \qquad k = 1,\dots,m-1, \quad i,j = 0,\dots,m$$

$$\partial_{xy}^2 u(t,x_k,y_i,z_j) \approx \frac{u(t,x_{k+1},y_{i+1},z_j) - u(t,x_k,y_{i+1},z_j) - u(t,x_{k+1},y_i,z_j) + u(t,x_k,y_i,z_j)}{\Delta x \Delta y} \qquad i,k = 0,\dots,m-1, \quad j = 0,\dots,m.$$

We write the equation at each point x_k, y_i, z_j, and we denote the piecewise approximation of $u(t,x,y,z)$ by $u_{k,i,j}(t) = u(t,x_k,y_i,z_j)$ for $x \in [x_k, x_{k+1})$, $y \in [y_i, y_{i+1})$, $z \in [z_j, z_{j+1})$ with $i,j,k = 0,\dots,m-1$. For fixed x_k, y_i, z_j we get the following non-linear ODE

$$u_{k,i,j}(t)' = \mathcal{DL}(u_{k,i,j})(t) - (1-\alpha)\big[LGD_1 y_i u_{k,i,j}(t)^+ - LGD_2 z_j u_{k,i,j}(t)^- \big)], \qquad k,i,j = 0,\dots,m, \tag{12}$$

$\mathcal{DL}(u_{k,i,j})$ is the discretized operator of $\mathcal{L}(u)$.

We can write the non-linear ODE system in matrix form

$$\bar{u}(t)' = \mathbf{A}(\bar{x},\bar{y},\bar{z})\bar{u}(t) - (1-\alpha)\big[LGD_1 \bar{y}\bar{u}(t)^+ - LGD_2 \bar{z}\bar{u}(t)^- \big], \tag{13}$$

where $\bar{u}(t)'$, $\bar{u}(t)$, and $\mathbf{A}(\bar{x},\bar{y},\bar{z})$ is a 3-dimensional tensor respectively, and $\bar{x},\bar{y},\bar{z}$ are the vectors in $\mathbb{R}^{m+1}$ given by

$$\bar{x} = (a_x, x_1, \dots, x_{m-1}, b_x), \quad \bar{y} = (a_y, y_1, \dots, y_{m-1}, b_y), \quad \bar{z} = (a_z, z_1, \dots, z_{m-1}, b_z)$$

with final condition $u(T,\bar{x},\bar{y},\bar{z}) = \bar{\Phi}(x)$ holds, where $\bar{\Phi}(x) = \big(\Phi_0(x), \Phi_1(x), \dots, \Phi_m(x)\big)$.

Accordingly with the choice described before, we pose the boundary conditions

$$\begin{aligned} u(t,x_0,y_i,z_j) =& 0 && i,j = 0,\dots,m, \\ u(t,x_m,y_i,z_j) =& \phi(t,x_m; r+\lambda,\sigma) && i,j = 0,\dots,m, \\ u(t,x_k,y_0,z_j) =& u(t,x_k,y_m,z_j) = \phi(t,x_k; r+\lambda,\sigma) && k,j = 1,\dots,m-1, \\ u(t,x_k,y_i,z_0) =& u(t,x_k,y_i,z_m) = \phi(t,x_k; r+\lambda,\sigma) && k,i = 1,\dots,m-1. \end{aligned}$$

To solve system (13), we use the explicit Euler scheme with N $(0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T)$ time sub-intervals of uniform length $\Delta t = t_{i+1} - t_i$ for $i = 0,\dots,N-1$, so we get

[2] There may be sub-regions of the spatial sub-intervals that may be more probable than others, so it would be worthwhile to perform non-uniform discretization, for details see [37,28].

$$\bar{u}(t_i) = \bar{u}(t_{i+1}) - \Delta t\Big[\mathbf{A}(\bar{x},\bar{y},\bar{z})\bar{u}(t_{i+1}) \\ - (1-\alpha)\big(LGD_1\bar{y}\bar{u}(t_{i+1})^+ - LGD_2\bar{z}\bar{u}(t_{i+1})^-\big)\Big] \ . \tag{14}$$

We are aware that the explicit Euler method could produce serious numerical instabilities, and therefore favouring an implicit scheme would be a better choice, since it has no limitations on the time integration step, even through implying quite lengthy computations. In the next section, we compare some numerical results from the explicit, implicit, and semi-explicit Euler schemes. In our setting, we are indeed able to achieve a good and competitive accuracy by the explicit scheme in highly shorter computational times.

For $t = 0$, we are interested in computing the value $u(0, x, y, z)$ for given x, y, z. To do so, we simply choose the closest points of the grid such that $x_k \approx x$, $y_i \approx y$, $z_j \approx z$ for some $k, i, j = 0, \ldots, m$ and we approximate the solution value by $u(0, x_k, y_i, z_j)$, or, as suggested in [37], the specific option value is determined by spline interpolation.

We remark that for the solution of (14) to remain stable, the $min\big(\frac{\Delta t}{\Delta x^2}, \frac{\Delta t}{\Delta y^2}, \frac{\Delta t}{\Delta z^2}\big)$, which is called the *Courant-Friedricks-Levy* or CFL *number*, must remain below a critical value. Hence, if one wishes to increase the accuracy of (14) by using smaller Δx or $\Delta y, \Delta z$, also a smaller value of Δt is required to keep the CFL number below its critical value. Thus, there is a conflicting requirement between improving accuracy and maintaining stability (for more detail on the stability theory, see Chapter 9 of [32], or Chapter 9 of [29]), which may imply an increase in computational time. For completeness, we also give the expression using the implicit Euler method

$$\bar{u}(t_i) = \bar{u}(t_{i+1}) - \Delta t\Big[\mathbf{A}(\bar{x},\bar{y},\bar{z})\bar{u}(t_i) - (1-\alpha)\big(LGD_1\bar{y}\bar{u}(t_i)^+ - LGD_2\bar{z}\bar{u}(t_i)^-\big)\Big] \ , \tag{15}$$

and the semi-implicit Euler method [7,4]

$$\bar{u}(t_i) = \bar{u}(t_{i+1}) - \Delta t\Big[\mathbf{A}(\bar{x},\bar{y},\bar{z})\bar{u}(t_{i+1}) - (1-\alpha)\big(LGD_1\bar{y}\bar{u}(t_i)^+ - LGD_2\bar{z}\bar{u}(t_i)^-\big)\Big] \ . \tag{16}$$

5. Numerical results

In this section, we present some numerical results of our method for the European call price. First, we looked at the case with constant intensities to test the method's accuracy, comparing with the results obtained in [13] by Monte Carlo simulations, with the same set of parameters. In this case, only one state variable, represented by the underlying price, is present.

All the algorithms were implemented in MatLab(R2021a) on a Intel(R) Core(TM) i5-10210U CPU @ 1.60 GHz 2.11 GHz computer.

We consider a European call option with six months maturity, strike price $K = 90$, and we set (as in [13]) $r = 0.005$, $\sigma = 0.4$, $LGD_1 = 0.6$, $LGD_2 = 0.6$, $c = 0.002$, $f = r$, $\alpha = 0.5$ $\lambda_1 = 0.04$ and $\lambda_2 = 0.02$.

As the computational time was not reported in [13], we replicated their simulations, moreover a 95% confidence interval has been built, with $M = 10^6$ sample independent paths and with $N_t = 1000$ temporal nodes, obtaining the value 16.4494, in about 7 minutes of machine time (fairly close to 16.4534 in [13]). In Table 2, we report the results of our method with the relative computational times and we compare them with the results in [13]. We remark that with only 30 spatial nodes, we get about the same value as by Monte Carlo simulations, with almost nihil computational time. From Table 2, we achieve better performance and comparable accuracy also with respect to [3],[3] where the computational time is about 25 seconds.

Moreover, increasing the number of spatial nodes and of temporal nodes, the second and third digits stabilize, showing the convergence of the method. The first two decimal digits coincide with those obtained by [13], and thanks to the convergence, we probably achieve a better accuracy. Indeed, the digits seem to stabilize progressively.

Table 2
Prices of a European call with maturity 6 months and deterministic intensities with explicit scheme.

Monte Carlo simulations				
N_t	Seconds	confidence interval	Price	by Brigo
1000	416	(16.4405;16.4583)	16.4494	**16.4534**

Method of finite differences			
N_t	N_x	Seconds	Price
100	30	0.31	16.4545771
500	50	0.28	16.4272255
1000	90	0.57	16.4643242
5000	150	1.93	16.4555334
5000	200	1.52	16.4568071
10000	300	3.7	16.4574087
50000	500	18.48	16.4574889

In Table 3, we compare the explicit, semi-implicit and implicit[4] methods as the strike price varies ($K = 90; 100; 110$), with the benchmark values from [13] and from our Monte Carlo simulations. Finally in the Table 4, we run the same analysis for varying volatility ($\sigma = 0.3; 0.4; 0.6$).

Given a mesh dense enough, all Euler schemes produce faster results than Monte Carlo simulations, as shown in Table 3, and they approximate the benchmark very well. Furthermore, we observe that the explicit method achieves the same results as the implicit one, but in remarkably shorter times. This indicates that, in our particular setting, the explicit method might be preferable even through unstable. To emphasize the explicit technique, also includes the semi-implicit method is used. Again, the explicit method results considerably quicker than the semi-implicit one, marginally faster, yet less accurate, than the implicit one.

Since the explicit method, unlike the implicit one, imposes stability constraints on the time step, as the underlying's volatility increases, we expect the CFL constraint to become stricter. Actually, we observe instability also with the implicit scheme (see lines $1-5$ Table 4), due to a space step problem, while in the explicit scheme the problem is due to the time step. These considerations are worth investigation and they might bring up new lines of research for future work. Nevertheless, in Table 4, we show this has no impact on our issue as long as an appropriately dense mesh is selected.

In the case of stochastic intensities, we additionally set the following values for the parameters of the CIR processes

$\gamma_i = 0.02, \quad \psi_i = 0.161, \quad \eta_i = 0.08, \; i = 1, 2.$

In Table 5, we report the results of our method with the corresponding computational times. To the best of our knowledge, in the literature, we could not find numerical methods covering this general case, so we had to resort again to Monte Carlo simulations to provide a benchmark.

As shown in Table 5, Monte Carlo simulations give 16.4416 in about 9 minutes, while with 30 nodes for each space interval, we get a result close to the benchmark in less than a second. To stabilize the first two decimal digits, we increased the spatial nodes to 100, still with a very reasonable computational time. To achieve better accuracy, we increased the number of spatial and time nodes even further, inevitably paying a cost in terms of time machine. Certainly Monte Carlo simulations may be optimized, nevertheless, our approach provides consistent improvement in machine.

[3] All tests have been performed by using Matlab on an Intel(R) Xeon(R) CPU E3-1241 3.50 GHz computer.

[4] We use the Matlab function "fsolve" to implement the implicit and semi-implicit approaches.

Table 3
We compare explicit, semi-implicit and implicit methods with Monte Carlo simulations and with [13] results in the case of deterministic intensities.

K = 90							
P.Brigo	**16.4534**						
N_t	Confidence Interval			**Price MC**		Seconds	
1000	(16.4405;16.4583)			**16.4494**		416	
N_t	N_x	Seconds	**Explicit**	Seconds	**Semi-Implicit**	Seconds	**Implicit**
100	30	0.579	**16.454577**	0.723	**16.454279**	1.209	**16.454577**
1000	90	0.544	**16.464324**	11.56	**16.464295**	12.64	**16.464324**
3000	150	1.16	**16.455686**	81.5	**16.455677**	81.88	**16.455686**
K = 100							
P.Brigo	**11.2858**						
N_t	Confidence Interval			**Price MC**		Seconds	
1000	(11.3064;11.3160)			**11.3112**		437.36	
N_t	N_x	Seconds	**Explicit**	Seconds	**Semi-Implicit**	Seconds	**Implicit**
100	30	0.137	**11.207271**	0.71	**11.206931**	0.90	**11.207271**
1000	90	0.67	**11.290589**	12.3	**11.290555**	15.5	**11.290589**
3000	150	1.954	**11.296083**	87.6	**11.296071**	89.1	**11.296083**
K = 110							
P.Brigo	**7.4999**						
N_t	Confidence Interval			**Price MC**		Seconds	
1000	(7.5416;7.5463)			**7.5439**		448.22	
N_t	N_x	Seconds	**Explicit**	Seconds	**Semi-Implicit**	Seconds	**Implicit**
100	30	0.250	**7.498880**	1.12	**7.498544**	1.23	**7.498880**
1000	90	0.460	**7.519851**	25.18	**7.519817**	23.02	**7.519851**
3000	150	1.648	**7.513870**	84.5	**7.513859**	83.45	**7.513870**

Table 4
Comparison between explicit, semi-implicit and implicit methods for various volatilities in the case of deterministic intensities.

$\sigma = 60\%$							
N_t	Confidence Interval			**Price MC**		Seconds	
1000	(21.22770;21.26145)			**21.2445788**		459.6	
N_t	N_x	Seconds	**Explicit**	Seconds	**Semi-Implicit**	Seconds	**Implicit**
100	30	0.166	**8.46E+24**	0.89	**0.000000**	0.9	**0.000000**
1000	90	0.5	**20.854963**	20.74	**10.104217**	26.16	**20.854963**
2000	90	0.8	**21.328809**	19.93	**21.328788**	21.16	**21.328809**
3000	150	1.15	**20.854866**	117.09	**10.003868**	105.93	**10.003868**
6000	150	2.10	**21.323061**	155.5	**21.323054**	140.8	**21.323061**
$\sigma = 40\%$							
N_t	Confidence Interval			**Price MC**		Seconds	
1000	(16.4387;16.4567)			**16.44777**		433.06	
N_t	N_x	Seconds	**Explicit**	Seconds	**Semi-Implicit**	Seconds	**Implicit**
100	30	0.150	**16.454577**	0.59	**16.454279**	0.79	**16.454577**
1000	90	0.420	**16.464324**	10.69	**16.464295**	11.49	**16.464324**
3000	150	1.160	**16.455686**	78.5	**16.455677**	76.57	**16.455686**
$\sigma = 30\%$							
N_t	Confidence Interval			**Price MC**		Seconds	
1000	(14.0337; 14.0457)			**14.03972**		476.8	
N_t	N_x	Seconds	**Explicit**	Seconds	**Semi-Implicit**	Seconds	**Implicit**
100	30	0.158	**14.054061**	0.57	**14.053846**	0.83	**14.054061**
1000	90	0.420	**14.071745**	11.69	**14.071724**	12.15	**14.071745**
3000	150	1.18	**14.060971**	75.87	**14.060971**	75.21	**14.060978**

The Table 6 compare the results of Table 2 with a symmetric finite difference for the approximation of the first spatial derivative. This new discretization may certainly give benefits in terms faster, but there is no significant gain in accuracy.

6. Sensitivity analysis

In this section, we run a short sensitivity analysis for our method in the case of stochastic intensities. This is done employing the explicit

Table 5
Prices of a European call with maturity 6 months and stochastic intensities, $\rho_1 = \rho_2 = 0$ with explicit scheme.

Monte Carlo simulation			
N_t	Seconds	Price	confidence interval
1000	527	16.4416729	(16.433;16.4506)
Method of finite differences			
N_t	N_{xyz}	Seconds	Price
100	30;30;30	0.66	16.42897944
500	50;50;50	4.16	16.42209551
1000	90;90;90	17.68	16.46349986
1500	100;100;100	46.32	16.45064167
2000	120;120;120	173.43	16.45825752
5000	150;150;150	487.6	16.45475883

Table 6
Prices of a European call with maturity 6 months and stochastic intensities, $\rho_1 = \rho_2 = 0$ with classic and symmetric finite difference for the approximation of the first spatial derivatives.

		Forward finite difference		Symmetric finite difference	
N_t	N_{xyz}	Seconds	Price	Seconds	Price
50	10	0.226	16.16472574	0.150	16.41309
100	30	0.66	16.42897944	0.320	16.48178
500	50	4.16	16.42209551	1.630	16.46791
1000	90	17.68	16.46349986	12.800	16.51000
1500	100	46.32	16.45064167	32.090	16.49799

Euler scheme. Indeed, again a similar accuracy is achieved by both the explicit and the implicit scheme, but with much larger computational times for the second. Indeed a better accuracy can be obtained by increasing the number of spatial and temporal nodes. This is achieved using the explicit scheme (last line of Table 5), but it becomes prohibitive timewise when applying the implicit scheme. We further remark that when using the explicit scheme, the increase of computational times is due solely to the thickening of the spatial nodes, while they remain stable (about 1 second) as the number of temporal nodes increases.

In Table 7, we compare the explicit, semi-implicit and implicit Euler schemes. Especially, when using an semi-implicit and implicit schemes, computational times grow considerably when increasing the number of spatial nodes. Hence we were forced to keep the number of spatial nodes equal to 15 with consequently far less accuracy.

The Table 7 emphasizes that, despite the explicit technique's potential instability, it is precise and extraordinarily fast in solving this particular problem.

From Table 5 one might conclude that the introduction of randomness for the intensities did really affect the price. To understand whether this was due to the particular choice of parameters or it was a general feature, fixing 100 spatial nodes, we performed a short sensitivity analysis, with respect to the intensity parameters, maturity, and strike price. In Table 8 we consider a European call option with different maturity (six months, nine months, and one year) and different strike prices and we compared the results with the constant intensities case (taking the initial value of the CIR processes), to underline the effect of introducing randomness for the intensities. We used the explitic scheme for this comparison.

As expected, the price appears to be decreasing with respect to the strike price, and increasing with respect to maturity. Table 8 shows also that the randomness of the intensities affects the price up to the first decimal digit when maturity increases, confirming it might be significant to consider stochastic intensities models for longer maturities.

Fixing $S = 100, K = 90, T = 0.5, \alpha = 0.5, \rho_1 = \rho_2 = 0$, we also explored the sensitivity of the model varying the intensities parameters of λ^1 and λ^2. Being the derivative a call, the most relevant effect comes, as it is to be expected, by the parameters (regression speed and long term average) of the counterparty's default intensity, while the investor's intensity parameters influence the price almost irrelevantly (Table 9). Finally, in Table 10, we show how the volatility affects the explicit method's convergence.

7. Conclusions

In this work, we developed a simple approximation procedure for the adjusted value of a derivative contract subject to counterparty risk, collateralization and founding costs, assuming a diffusion model for the

Table 7
Prices of a European call with different volatility, with explicit, semi-implicit and implicit schemes in the stochastic case.

K = 90							
$\sigma = 60\%$							
N_t	Confidence Interval			**Price MC**		Seconds	
1000	(21.1139; 21.1475)			**21.130690**		598.69	
N_t	N_{xyz}	Seconds	**Explicit**	Seconds	**Semi-Implicit**	Seconds	**Implicit**
50	10	0.15	**20.8416**	19.80	**20.9650**	37.76	**20.7396**
100	15	0.24	**21.4150**	320.23	**21.5437**	759.70	**21.3704**
500	15	0.74	**21.3971**	1154.24	**21.3977**	2757.91	**21.3882**
$\sigma = 40\%$							
N_t	Confidence Interval			**Price MC**		Seconds	
1000	(16.433;16.4506)			**16.4416729**		527	
N_t	N_{xyz}	Seconds	**Explicit**	Seconds	**Semi-Implicit**	Seconds	**Implicit**
50	10	0.20	**16.1647**	21.20	**16.2614**	42.23	**16.1024**
100	15	0.21	**16.5132**	330.05	**16.6135**	599.97	**16.4902**
500	15	0.67	**16.5039**	1165.80	**16.6044**	1872.27	**16.4993**
$\sigma = 30\%$							
N_t	Confidence Interval			**Price MC**		Seconds	
1000	(14.0511;14.0631)			**14.05715**		587.5	
N_t	N_{xyz}	Seconds	**Explicit**	Seconds	**Semi-Implicit**	Seconds	**Implicit**
50	10	0.14	**13.9204**	20.08	**14.0037**	44.38	**13.8786**
100	15	0.21	**14.2040**	322.50	**14.2903**	354.83	**14.1904**
500	15	0.93	**14.1985**	1174.17	**14.2849**	5186.40	**14.1957**

Table 8
Prices of a European call with explicit scheme with different maturities (6 months, 9 months, and 1 year) and strike prices (90, 100, 110), in the deterministic and stochastic case.

6 months		*9 months*		*1 year*	
K = 90					
16.4518463	16.450641	18.6724165	18.6835031	20.4575975	20.554191
K = 100					
11.292111115	11.291427	13.7820539	13.7948775	15.76652711	15.868368
K = 110					
7.51052959	7.5102467	10.0105913	10.0248779	12.0356766	12.142071

Table 9
Sensitivity analysis for different regression speeds and fixed long term averages with explicit scheme.

$\eta_1 = 0.08$					
$\psi_1 = 0.161$		$\psi_1 = 0.25$		$\psi_1 = 0.4$	
γ_1	Price	γ_1	Price	γ_1	Price
0.02	16.4510	0.02	16.4499	0.02	16.4480
0.03	16.4502	0.03	16.4486	0.03	16.4458
0.05	16.4488	0.05	16.4459	0.05	16.4415
0.1	16.4451	0.1	16.4397	0.1	16.4307
0.2	16.4380	0.2	16.4274	0.2	16.4096
$\eta_2 = 0.08$					
$\psi_2 = 0.161$		$\psi_2 = 0.25$		$\psi_2 = 0.4$	
γ_2	Price	γ_2	Price	γ_2	Price
0.02	16.45102548	0.02	16.451025498	0.02	16.4510255148
0.03	16.45102549	0.03	16.451025512	0.03	16.4510255277
0.05	16.45102551	0.05	16.451025528	0.05	16.4510255387
0.1	16.45102553	0.1	16.451025541	0.1	16.4510255434
0.2	16.45102554	0.2	16.451025544	0.2	16.4510255436

Table 10
Sensitivity analysis for different volatilities with explicit scheme.

$\gamma_1 = \gamma_2 = 0.1$ $\psi_1 = \psi_2 = 0.05$			
K = 90			
$\sigma = 40\%$	$N_t = 1500$	$N_t = 2500$	$N_t = 3000$
η_1/η_2	0.08	0.1	0.2
0.08	16.42465705	16.42431168	16.34466761
0.1	16.42466191	16.42431655	16.34467243
0.2	NaN	16.42160683	16.34200124
$\sigma = 60\%$	$N_t = 2500$	$N_t = 3000$	$N_t = 4500$
η_1/η_2	0.08	0.1	0.2
0.08	21.39176544	21.39161946	21.28965302
0.1	21.39177163	21.39162566	21.28964689
0.2	NaN	21.38742986	21.28551264

default intensities and close-out values as a portion of the adjusted price itself. This generates a non-linear BSDE, with an associated non-linear PDE characterizing the price.

By the simple method of finite differences applied to this PDE, we showed that accurate approximations could be achieved in very manageable computational times, differently from what happens when employing Monte Carlo simulations. We ran a short sensitivity analysis to estimate the effects of the introduction of stochastic intensities, and we plan to dedicate future work to the error estimates.

Data availability

No data was used for the research described in the article.

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