**Research Article**

Filomena De Filippis*, Francesco Leonetti, and Giulia Treu

# Nonoccurrence of Lavrentiev gap for a class of functionals with nonstandard growth

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**Abstract:** We consider the functional

$$\mathcal{F}(u) := \int_{\Omega} f(x, Du(x))\mathrm{d}x,$$

where $f(x,z)$ satisfies a $(p,q)$-growth condition with respect to $z$ and can be approximated by means of a suitable sequence of functions. We consider $B_R \Subset \Omega$ and the spaces

$$X = W^{1,p}(B_R, \mathbb{R}^N) \quad \text{and} \quad Y = W^{1,p}(B_R, \mathbb{R}^N) \cap W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N).$$

We prove that the lower semicontinuous envelope of $\mathcal{F}|_Y$ coincides with $\mathcal{F}$ or, in other words, that the Lavrentiev term is equal to zero for any admissible function $u \in W^{1,p}(B_R, \mathbb{R}^N)$. We perform the approximations by means of functions preserving the values on the boundary of $B_R$.

**Keywords:** regularity, minimizer, variational, integral, Lavrentiev, phenomenon

**MSC 2020:** 35B65, 35J60, 35J47

## 1 Introduction

Let $X$ be a first countable topological space and $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$. The sequentially lower semicontinuous (s.l.s.c.) envelope of $\mathcal{F}$ is defined as

$$\bar{\mathcal{F}}_X := \sup\{\mathcal{G} : X \to [0,+\infty] : \mathcal{G} \quad \text{s.l.s.c.}, \quad \mathcal{G} \leqslant \mathcal{F} \quad \text{on } X\}. \tag{1}$$

Analogously, if $Y$ is a dense subspace of $X$ the s.l.s.c. envelope of $\mathcal{F}$ with respect to $Y$ is

$$\bar{\mathcal{F}}_Y := \sup\{\mathcal{G} : X \to [0,+\infty] : \mathcal{G} \quad \text{s.l.s.c.}, \quad \mathcal{G} \leqslant \mathcal{F} \quad \text{on } Y\}. \tag{2}$$

We obviously have that

$$\bar{\mathcal{F}}_X(u) \leqslant \bar{\mathcal{F}}_Y(u), \qquad \text{for any } u \in X, \tag{3}$$

and the strict inequality may occur.

Buttazzo and Mizel in [15] introduced the notion of Lavrentiev term, namely: for every $u \in X$,

$$\mathcal{L}(u) := \begin{cases} \bar{\mathcal{F}}_Y - \bar{\mathcal{F}}_X, & \text{if } \bar{\mathcal{F}}_X < +\infty \\ 0, & \text{if } \bar{\mathcal{F}}_X = +\infty. \end{cases} \tag{4}$$

*** Corresponding author: Filomena De Filippis**, DISIM, Università degli Studi dell'Aquila, Via Vetoio snc, Coppito, 67100 L'Aquila, Italy, e-mail: filomena.defilippis@graduate.univaq.it
**Francesco Leonetti:** DISIM, Università degli Studi dell'Aquila, Via Vetoio snc, Coppito, 67100 L'Aquila, Italy, e-mail: francesco.leonetti@univaq.it
**Giulia Treu:** Dipartimento di Matematica, Università degli Studi di Padova, Via Trieste, 63-35121 Padova, Italy, e-mail: giulia.treu@unipd.it

Moreover, they also say that there is a Lavrentiev gap at $u$ whenever $\mathcal{L}(u) > 0$. In this article, we are concerned with the functional

$$\mathcal{F}(u) \coloneqq \int_\Omega f(x, Du(x))\mathrm{d}x,$$

and we aim to show that, under suitable assumptions, the Lavrentiev term is identically zero.

In the framework of the Calculus of Variations the first example in which it is shown that $\mathcal{L}(u)$ can be strictly greater than zero is due to Lavrentiev [33] and it has been simplified by Manià [34]. He considered the functional

$$\mathcal{F}(u) = \int_0^1 (u^3 - x)^2 |u'|^6 \mathrm{d}x,$$

the spaces

$$X = \{u \in W^{1,1}([0,1], \mathbb{R}) : \ u(0) = 0, \ u(1) = 1\}$$
$$Y = \{u \in W^{1,\infty}([0,1], \mathbb{R}) : \ u(0) = 0, \ u(1) = 1\},$$

and he proved that $\mathcal{L}(u_*) > 0$, where $u_*(x) = x^{\frac{1}{3}}$ is the minimum of $\mathcal{F}$ on $X$.

The problem of understanding how much the occurrence of this phenomenon is related to the special structure of the functional led the search of different kinds of examples. We list here the most significant ones underlying that in all these cases, the Lavrentiev term is computed on the minimum and it is strictly positive. As far as the one-dimensional case is concerned, Ball and Mizel [5] added a coercive term to the Manià functional. Zhikov [43] treated a functional depending only on the variables $x$ and $Du(x)$, in the case where $x \in \Omega \subset \mathbb{R}^2$. This example has been suitably generalized to the $n$-dimensional scalar case in [22]. It is worth to mention that, still in the vectorial case, there are examples exhibiting the phenomenon for functionals depending on $(u, Du)$ (see [2]) and also depending just on $Du$ [29]. We refer to the article by Belloni and Buttazzo ([6], 1995) for a more extensive list of references; as regards more recent examples we just mention [4,7,17,19,20,24,27,35].

Another important direction of research is devoted to identify assumptions on the Lagrangian $f$ that imply that the Lavrentiev term in (4) is zero for the minima. Moreover, Lavrentiev himself [33] took into account this problem in dimension one. Obviously, a consequence of results concerning regularity or higher integrability is the absence of the Lavrentiev phenomenon for minima. Another almost trivial remark is that, at least for the one-dimensional and the multidimensional scalar cases, if

$$a_1 + b_1 \, |\xi|^p \leqslant f(x, u, \xi) \leqslant a_2 + b_2 \, |\xi|^p,$$

where $a_i, b_i, i = 1, 2$, are the suitable positive constants, the dominated converge theorem implies that $\mathcal{L}(u)$ is identically equal to zero. In Section 4 of the article [6], there is a survey of many results regarding the absence of the Lavrentiev gap for the minima. We underline that in the one-dimensional case, Alberti and Serra Cassano [3] proved the nonoccurrence of the gap for any autonomous functional. An analogous result has been recently obtained for the multidimensional scalar case in [10] and [11]. Further results in this framework can be found in [9,38,39].

The problem of identifying classes of functionals such that $\mathcal{L}(u)$ is equal to zero for every function $u \in X$ is less studied. For some results in this direction, see, for example, [1,6,10–13,40].

In this article, we deal with functional of type

$$\mathcal{F}(u) \coloneqq \int_\Omega f(x, Du(x))\mathrm{d}x, \tag{5}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $u : \Omega \to \mathbb{R}^N$, $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a Carathéodory function, $n \geqslant 2$ and $N \geqslant 1$. The Lagrangian $f$ satisfies the so-called $(p, q)$-growth conditions, according to Marcellini's terminology [36]:

$$|z|^p \leqslant f(x, z) \leqslant L(1 + |z|^q), \quad L \in [1, +\infty). \tag{6}$$

In the realm of functions that satisfy (6), we find the double-phase functional

$$f(x,z) = |z|^p + a(x)|z|^q, \tag{7}$$

where $1 < p < q < +\infty$ and the coefficient $a(x)$ is $C^{0,\alpha}(\Omega)$ and non-negative, $\alpha \in (0,1]$. The example of Lavrentiev phenomenon presented by Zhikov [43] falls in this class. We remark that he considered the case: $n = 2, N = 1, \alpha = 1$, and $1 \leqslant p \leqslant 2$ and $q > 3$. In [22], the authors showed that when $p$ and $q$ are not so close, precisely when

$$\frac{q}{p} > \frac{n+\alpha}{n}, \tag{8}$$

it is possible to find an example presenting the Lavrentiev phenomenon. This example is a extension to $n \geqslant 2$ of the Zhikov's one. The main result in [22] shows that, when $q/p < (n+\alpha)/n$, if $u$ is a minimizer and $\mathcal{L}(u) = 0$, then $u \in W^{1,p}(B_R,\mathbb{R}^N) \cap W^{1,q}_{\mathrm{loc}}(B_R,\mathbb{R}^N)$. Moreover, they also proved that $\mathcal{L}(u) \equiv 0$ for some model cases.

As far as we are interested in results proving that the Lavrentiev term is identically zero, we have to mention [23] where they considered the case of a Lagrangian that satisfies, among other assumptions, also the following: the minimum point $\tilde{y}$ of $y \mapsto f(y,z)$ on small balls does not depend on $z$; we underline that they consider $X = W^{1,p}(B_R,\mathbb{R}^N)$ and $Y = W^{1,p}(B_R,\mathbb{R}^N) \cap W^{1,q}_{\mathrm{loc}}(B_R,\mathbb{R}^N)$. The same kind of hypothesis is used in [31] and [32], to prove again that $\mathcal{L}(u) \equiv 0$ taking into account suitable spaces similar to the previous ones. We also point out that in the recent work [14], under suitable assumptions, it is shown that (3) is zero when calculated on the minimum point. The same thing on minimizers happens in [8], where the authors consider a Lagrangian controlled by a function, which is convex and anisotropic in $z$, with $X$ a Sobolev-type space and $Y = C^\infty$. We remark once again that whenever one obtains a result on the regularity of the minimizers, one also rules out the Lavrentiev phenomenon. We point out the fact that regularity of the minimizers of functionals with $(p,q)$-growth has been widely studied. We refer to [41] for an extensive survey on this subject and to [37,42] for more recent references (see also [26] and [30]).

In [21], it has been proved that the Lavrentiev term $\mathcal{L}(u)$ is equal to zero when $u$ is a minimizer of $\mathcal{F}$ under a suitable set of assumptions on the Lagrangian. To be more precise, the authors assume that $f$ has the $(p,q)$-growth and also satisfies the following conditions on its derivative with respect to the gradient:

$$L^{-1}(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_1 - z_2|^2 \leqslant \left\langle \frac{\partial f}{\partial z}(x,z_1) - \frac{\partial f}{\partial z}(x,z_2), z_1 - z_2 \right\rangle, \tag{9}$$

$$\left| \frac{\partial f}{\partial z}(x,z) - \frac{\partial f}{\partial z}(y,z) \right| \leqslant L\,|x-y|^\alpha (1 + |z|^{q-1}). \tag{10}$$

Moreover, they also assume that there exists a sequence $f_k$ of convex functions approximating the Lagrangian $f$ from below, sharing the same hypothesis of $f$ with the peculiarity that (10) holds also with $p$ instead of $q$, for a new constant $L(p,k)$ depending also on $k$.

In this article, we consider the functional (5) with $(p,q)$-growth under the same set of assumptions as in [21] and we deal with the spaces

$$X = W^{1,p}(B_R,\mathbb{R}^N) \quad \text{and} \quad Y = W^{1,p}(B_R,\mathbb{R}^N) \cap W^{1,q}_{\mathrm{loc}}(B_R,\mathbb{R}^N). \tag{11}$$

We prove that $\mathcal{L}(u_*)$ is zero for any $u_* \in W^{1,p}(\Omega,\mathbb{R}^N)$. We remark that (9) implies that $z \mapsto f(x,z)$ is convex and then $\bar{\mathcal{F}}_X(u) = \mathcal{F}(u)$ for every $u \in X$.

The first step of the proof consists in considering a sequence $\{v_k\}_{k\in\mathbb{N}} \subset Y$ converging to $u_* \in X$ with respect to the strong topology of $X$. Then, we introduce the perturbed functional

$$\mathcal{G}_k(u) \coloneqq \int_{B_R} \left[ f(x,Du(x)) + \frac{1}{k}(1 + k^2\,|Du(x) - Dv_k(x)|^2)^{\frac{p}{2}} \right] \mathrm{d}x,$$

and we apply Remark 4.4 in [21] to $\mathcal{G}_k$, so that $\mathcal{G}_k$ admits a minimizer $u_k$ that belongs to $Y$. The proof is concluded showing that the sequence $u_k$ converges strongly to $u_*$ and approximates $u_*$ in energy, i.e.,

$$\mathcal{F}(u_k) \to \mathcal{F}(u_*).$$

In Section 4, we apply the penalizing method to prove that the gap is identically zero for other functionals: in particular, those previously considered in [25] and [16]. In the first case, a vectorial problem set in Morrey spaces is considered. In the second one, we are concerned with a multidimensional scalar problem whose Lagrangian is of sum type.

Let us end this introduction by remarking the difference between [21] and this article. In [21], it is shown that $\mathcal{L}(u) = 0$ when $u$ is a minimizer for $\mathcal{F}$; in this article, we show that $\mathcal{L}(u) = 0$ for every $u \in W^{1,p}$. Moreover, while in [21], the proof makes strongly use of the minimality property of the minimizer $u$, here we cannot proceed in the same way and we need to modify the functional to construct a sequence converging both strongly and in energy to the fixed function $u$.

## 2 Notation and preliminary results

We denote by $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, a bounded open set. We define the functional

$$\mathcal{F}(u, \Omega) = \int_{\Omega} f(x, Du(x))\mathrm{d}x,$$

where the Lagrangian $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is of Carathéodory type, $N \geqslant 1$. We adopt the usual definition of local minimizer.

**Definition 2.1.** A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local minimizer of $\mathcal{F}(\cdot,\Omega)$ if and only if $x \mapsto f(x, Du(x)) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_{\operatorname{supp}\varphi} f(x, Du(x))\mathrm{d}x \leqslant \int_{\operatorname{supp}\varphi} f(x, Du(x) + D\varphi(x))\mathrm{d}x,$$

for any $\varphi \in W^{1,1}(\Omega, \mathbb{R}^N)$ with $\operatorname{supp}\varphi \subset \Omega$.

In particular, we fix $x_0 \in \Omega$ and we indicate by $B_R \equiv B_R(x_0) \Subset \Omega$, $R \in (0, 1]$, the ball of center $x_0$ and radius $R$. From now on, we set

$$\mathcal{F}(u) = \mathcal{F}(u, B_R) = \int_{B_R} f(x, Du(x))\mathrm{d}x,$$

and we consider the spaces

$$X = W^{1,p}(B_R, \mathbb{R}^N) \quad \text{and} \quad Y = W^{1,p}(B_R, \mathbb{R}^N) \cap W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N).$$

For a Lagrangian satisfying the $(p, q)$-growth condition, our aim is to show that the term

$$\mathcal{L}(u) = \begin{cases} \bar{\mathcal{F}}_Y - \bar{\mathcal{F}}_X, & \text{if } \bar{\mathcal{F}}_X < +\infty, \\ 0, & \text{if } \bar{\mathcal{F}}_X = +\infty, \end{cases}$$

is equal to zero for all $u \in X$, where the relaxed functionals $\bar{\mathcal{F}}_X$ and $\bar{\mathcal{F}}_Y$ are defined in (1). For this purpose, we will use the following lemma.

**Lemma 2.2.** *Let $\mathcal{F} : W^{1,p}(B_R, \mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ be sequentially lower semicontinuous, and let $u \in W^{1,p}(B_R, \mathbb{R}^N)$ be such that $\mathcal{F}(u) < +\infty$. Then,*

$$\mathcal{L}(u) = 0$$

*if and only if there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \in Y$ such that $u_k \rightharpoonup u$ weakly in $W^{1,p}(B_R, \mathbb{R}^N)$ and*

$$\mathcal{F}(u_k) \to \mathcal{F}(u).$$

# 3 Main theorem

The most part of this section is devoted to the proof of the main result of this article, stated in Theorem 3.1. We shall assume the existence of a suitable sequence $f_k$ that approximates the density $f$ from below. This assumption will be replaced appropriately in Corollary 3.2. The example at the end of the section clarifies the relations between our assumptions and those of [8,18,23]. We shall assume that there exist constants $L \in [1, +\infty)$ and $\mu \in [0, 1]$ such that the density $f$ satisfies

$$z \mapsto f(x, z) \in C^1(\mathbb{R}^{N\times n}); \tag{AP1}$$

$$L^{-1}|z|^p \leqslant f(x, z) \leqslant L(1 + |z|^q); \tag{AP2}$$

$$L^{-1}(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_1 - z_2|^2 \leqslant \left\langle \frac{\partial f}{\partial z}(x, z_1) - \frac{\partial f}{\partial z}(x, z_2), z_1 - z_2 \right\rangle; \tag{AP3}$$

$$\left|\frac{\partial f}{\partial z}(x, z) - \frac{\partial f}{\partial z}(y, z)\right| \leqslant L\,|x - y|^\alpha(1 + |z|^{q-1}). \tag{AP4}$$

**Theorem 3.1.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $\alpha \in (0, 1]$ and $p$, $q$ be such that*

$$1 < p < q < p\left(\frac{n + \alpha}{n}\right).$$

*Let $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Carathéodory function satisfying* (AP1)–(AP4). *Assume also that there exists a sequence of Carathéodory function $f^l : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying* (AP1)–(AP4), *with the same $p, q, \alpha, L$, and $\mu$ as for f. Assume, moreover, that, for every $l \in \mathbb{N}$, there exists $c(l) \in (0, +\infty)$ such that*

$$\left|\frac{\partial f^l}{\partial z}(x, z)\right| \leqslant c(l)(1 + |z|^{p-1}); \tag{AP5}$$

$$\left|\frac{\partial f^l}{\partial z}(x, z) - \frac{\partial f^l}{\partial z}(y, z)\right| \leqslant c(l)|x - y|^\alpha(1 + |z|^{p-1}); \tag{AP6}$$

$$f^l(x, z) \leqslant f(x, z); \tag{AP7}$$

$$\lim_{l\to+\infty} f^l(x, z) = f(x, z); \tag{AP8}$$

$$f^l(x, z) \leqslant f^{l+1}(x, z). \tag{AP9}$$

*Let $u_*$ be a function in $W^{1,p}(B_R, \mathbb{R}^N)$ such that $\mathcal{F}(u_*) < +\infty$, where $B_R \Subset \Omega$, $R \in (0, 1]$. Then,*

$$\mathcal{L}(u_*) = 0.$$

**Proof.** By Lemma 2.2, we need to show that there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(B_R, \mathbb{R}^N) \cap W^{1,q}_{\text{loc}}(B_R, \mathbb{R}^N)$ such that

$$u_k \rightharpoonup u_*, \quad \text{weakly in } W^{1,p}(B_R, \mathbb{R}^N),$$

and

$$\mathcal{F}(u_k) \to \mathcal{F}(u_*).$$

We divide the proof into two steps.

*Step 1: we construct a suitable sequence*

$$\{u_k\}_{k\in\mathbb{N}} \subset (u_* + W^{1,p}_0(B_R, \mathbb{R}^N)) \cap W^{1,q}_{\text{loc}}(B_R, \mathbb{R}^N).$$

We start observing that by Theorem 3.10 in [28], it is possible to find a $\rho > 0$ and a function $\bar{u}_* \in W^{1,p}_0(B_{R+\rho}, \mathbb{R}^N)$: $\bar{u}_* = u_*$ in $B_R$. Moreover, $\bar{u}_*$ can be extended to all of $\mathbb{R}^n$ by setting $\bar{u}_* = 0$ outside $B_{R+\rho}$, and such an extension belongs to $W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$; we keep calling this extension $u_*$. Now, let $k \in \mathbb{N}$, and we define $v_k$, the regularized of $u_*$, in the following way:

$$v_k(x) = \int_{B\left(x,\frac{1}{k}\right)} u_*(y)k^n\psi(k(x-y))\mathrm{d}y,$$

where $\psi \in C_c^\infty(B(0,1), \mathbb{R})$, $\psi \geqslant 0$ and $\int_{B(0,1)}\psi(x)\mathrm{d}x = 1$. Then, $\operatorname{supp} v_k$ is compact in $\mathbb{R}^n$ and $v_k \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^N) \subset W^{1,p}(\Omega, \mathbb{R}^N)$. Since $v_k \to u_*$ strongly in $W^{1,p}(B_R, \mathbb{R}^N)$, up to subsequence, still denoted $v_k$, we have

$$||u_* - v_k||_{W^{1,p}(B_R)} \leqslant \frac{1}{k}. \tag{12}$$

We define

$$\mathcal{G}_k(u) \coloneqq \int_{B_R}\left[f(x, Du(x)) + \frac{1}{k}(1 + k^2 |Du(x) - Dv_k(x)|^2)^{\frac{p}{2}}\right]\mathrm{d}x,$$

we call $g_k(x,z) \coloneqq f(x,z) + \frac{1}{k}(1 + k^2 |z - Dv_k(x)|^2)^{\frac{p}{2}}$, and we observe that

- $g_k(x,z) \geqslant 0, \quad \forall (x,z) \in \Omega \times \mathbb{R}^{N\times n}$;
- $z \mapsto g_k(x,z) \in C^1(\mathbb{R}^{N\times n}), \quad \forall x \in \Omega$;
- $z \mapsto g_k(x,z)$ is convex $\forall x \in \Omega$ (inasmuch as sum of convex functions);
- $g_k(x,z) \geqslant L^{-1}|z|^p + \frac{1}{k}(1 + k^2 |z - Dv_k(x)|^2)^{\frac{p}{2}} > L^{-1}|z|^p, \forall\ (x,z) \in \Omega \times \mathbb{R}^{N\times n}$;
- $\mathcal{G}_k(u_*) = \mathcal{F}(u_*) + \frac{1}{k}\int_{B_R}(1 + k^2 |Du_*(x) - Dv_k(x)|^2)^{\frac{p}{2}}\mathrm{d}x < +\infty$.

Therefore, we can apply the direct method of the calculus of variations: for any $k \in \mathbb{N}$, there exists $u_k \in u_* + W_0^{1,p}(B_R, \mathbb{R}^N)$ such that

$$\min_{u \in u_* + W_0^{1,p}(B_R,\mathbb{R}^N)} \mathcal{G}_k(u) = \mathcal{G}_k(u_k).$$

Now, let us observe that $g_k : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a Carathéodory function, and let us define $h_k(x,z) \coloneqq \frac{1}{k}(1 + k^2 |z - Dv_k(x)|^2)^{\frac{p}{2}}$, then by the convexity of $z \mapsto h_k(x,z)$, we have

$$\begin{aligned}\left\langle \frac{\partial g_k}{\partial z}(x,z_1) - \frac{\partial g_k}{\partial z}(x,z_2), z_1 - z_2\right\rangle &= \left\langle \frac{\partial f}{\partial z}(x,z_1) - \frac{\partial f}{\partial z}(x,z_2), z_1 - z_2\right\rangle + \left\langle \frac{\partial h_k}{\partial z}(x,z_1) - \frac{\partial h_k}{\partial z}(x,z_2), z_1 - z_2\right\rangle \\ &\geqslant L^{-1}(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_1 - z_2|^2.\end{aligned}$$

We point out that since $v_k \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^N)$, there exists $M = M(k) \in [0, +\infty)$ such that $|Dv_k(x)| + |D^2 v_k(x)| \leqslant M$, for all $x \in \mathbb{R}^n$. Hence, we have

$$\begin{aligned}g_k(x,z) &= f(x,z) + \frac{1}{k}(1 + k^2 |z - Dv_k(x)|^2)^{\frac{p}{2}} \\ &\leqslant f(x,z) + \frac{2^{\frac{p}{2}}}{k}(1 + k^p |z - Dv_k(x)|^p) \\ &\leqslant L(1 + |z|^q) + \frac{2^{\frac{p}{2}}}{k} + 2^{\frac{p}{2}}k^{p-1}Lf(x, z - Dv_k(x)) \\ &\leqslant L(1 + |z|^q) + \frac{2^{\frac{p}{2}}}{k} + 2^{\frac{p}{2}}k^{p-1}L^2(1 + |z - Dv_k(x)|^q) \\ &\leqslant L(1 + |z|^q) + \frac{2^{\frac{p}{2}}}{k} + \tilde{c}_1(1 + |z|^q + |Dv_k(x)|^q) \\ &\leqslant L(1 + |z|^q) + \tilde{c}_2(1 + |z|^q) \\ &\leqslant (L + \tilde{c}_2)(1 + |z|^q),\end{aligned}$$

where $\tilde{c}_1 = \tilde{c}_1(k, p, q, L)$ and $\tilde{c}_2 = \tilde{c}_2(k, p, q, L, M)$ are suitable positive constants. Now, let us compute the derivative with respect to $x$ of $\frac{\partial h_k}{\partial z_i^\alpha}(x, z) = pk(1 + k^2 |z - Dv_k(x)|^2)^{\frac{p-2}{2}}(z_i^\alpha - D_i v_k^\alpha(x))$:

$$\frac{\partial^2 h_k}{\partial x_j \partial z_i^\alpha}(x, z) = pk\Big[k^2(p-2)(1 + k^2 |z - Dv_k(x)|^2)^{\frac{p-4}{2}} \times (z_i^\alpha - D_i v_k^\alpha(x)) \sum_{\beta=1}^{N} \sum_{r=1}^{n} (z_r^\beta - D_r v_k^\beta(x))(-D_j D_r v_k^\beta(x)) + (1 + k^2 |z - Dv_k(x)|^2)^{\frac{p-2}{2}}(-D_j D_i v_k^\alpha(x))\Big].$$

Since $x \mapsto \frac{\partial h_k}{\partial z}(x, z)$ is $C^1(\mathbb{R}^n)$, we can conclude that $\frac{\partial h_k}{\partial z}$ is $\alpha$-Hölder continuous with respect to $x$; more precisely, we find that

$$\left|\frac{\partial h_k}{\partial z}(x, z) - \frac{\partial h_k}{\partial z}(y, z)\right| \leqslant H |x - y|^\alpha (1 + |z|)^{p-2} \leqslant \tilde{H} |x - y|^\alpha (1 + |z|^{q-1}),$$

where $H$ and $\tilde{H}$ are the positive constants depending on $k, \Omega, p, q, n, N$, and $M$. Consequently,

$$\begin{aligned}\left|\frac{\partial g_k}{\partial z}(x, z) - \frac{\partial g_k}{\partial z}(y, z)\right| &\leqslant \left|\frac{\partial f}{\partial z}(x, z) - \frac{\partial f}{\partial z}(y, z)\right| + \left|\frac{\partial h_k}{\partial z}(x, z) - \frac{\partial h_k}{\partial z}(y, z)\right| \\ &\leqslant L |x - y|^\alpha (1 + |z|^{q-1}) + \tilde{H} |x - y|^\alpha (1 + |z|^{q-1}) \\ &\leqslant (L + \tilde{H})|x - y|^\alpha (1 + |z|^{q-1}).\end{aligned}$$

Now, let us define

$$g_k^l(x, z) \coloneqq f^l(x, z) + \frac{1}{k}(1 + k^2 |z - Dv_k(x)|^2)^{\frac{p}{2}},$$

where $\{f^l\}_{l\in\mathbb{N}}$ is the approximating sequence defined in the statement of the theorem. First of all, we observe that $g_k^l$ is $C^1$ with respect to $z$ and it satisfies (AP3) with the same constant $L^{-1}$. Moreover, we note that $g_k^l$ satisfies (AP2), (AP4), and (AP6), to be more precise,

$$L^{-1} |z|^p \leqslant g_k^l(x, z) \leqslant L_1(1 + |z|^q),$$

$$\left|\frac{\partial g_k^l}{\partial z}(x, z) - \frac{\partial g_k^l}{\partial z}(y, z)\right| \leqslant (L + \tilde{H})|x - y|^\alpha (1 + |z|^{q-1}),$$

$$\left|\frac{\partial g_k^l}{\partial z}(x, z) - \frac{\partial g_k^l}{\partial z}(y, z)\right| \leqslant (c(l) + 2^{p-1}H)|x - y|^\alpha (1 + |z|^{p-1}),$$

where $L_1$ depends on $L, k, p, q$, and $M$. As far as (AP5) is concerned, we observe that

$$\begin{aligned}\left|\frac{\partial g_k^l}{\partial z}(x, z)\right| &\leqslant \left|\frac{\partial f^l}{\partial z}(x, z)\right| + \left|\frac{\partial h}{\partial z}(x, z)\right| \\ &\leqslant c(l)(1 + |z|^{p-1}) + pk(1 + k^2 |z - Dv_k(x)|^2)^{\frac{p-2}{2}}|z - Dv_k(x)| \\ &\leqslant c(l)(1 + |z|^{p-1}) + pk(1 + k^2 |z - Dv_k|^2)^{\frac{p-2}{2}}(1 + k^2 |z - Dv_k|^2)^{\frac{1}{2}} \\ &\leqslant c(l)(1 + |z|^{p-1}) + \bar{c}_1(1 + |z|^{p-1} + |Dv_k(x)|^{p-1}) \\ &\leqslant c(l)(1 + |z|^{p-1}) + \bar{c}_2(1 + |z|^{p-1}) \\ &\leqslant (c(l) + \bar{c}_2)(1 + |z|^{p-1}),\end{aligned}$$

with $\bar{c}_1 = \bar{c}_1(p, k)$ and $\bar{c}_2 = \bar{c}_2(p, k, M)$ suitable positive constants. In the end, it is straightforward to check that $g_k^l$ satisfies (AP7), (AP8), and (AP9); more precisely,

$$g_k^l(x, z) \leqslant g_k(x, z);$$

$$\lim_{l\to+\infty} g_k^l(x,z) = g_k(x,z);$$

$$g_k^l(x,z) \leqslant g_k^{l+1}(x,z).$$

Observing that $\mathcal{G}_k(u_*) < +\infty$, we are at last in position to apply Remark 4.4 in [21] to the functional $\mathcal{G}_k$.

We recall [21, Remark 4.4] for the convenience of the reader: let $\psi$ be a function satisfying (AP1)–(AP4) with numbers $p, q, \alpha, L$, and $\mu$ as in Theorem 3.1. Let $\{\psi^l\}_{l\in\mathbb{N}}$ be a sequence of functions satisfying (AP1)–(AP9). If $u_0 \in W^{1,p}(B_R, \mathbb{R}^N)$ with $\mathcal{F}(u_0) < +\infty$, let $u \in u_0 + W_0^{1,p}(B_R, \mathbb{R}^N)$ be a minimizer of

$$\int_{B_R} \psi(x, Du(x))\mathrm{d}x,$$

then $u \in W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N)$.

Here, $g_k$ plays the role of $\psi$ and $\{g_k^l\}_{l\in\mathbb{N}}$ the one of the approximating sequence $\{\psi^l\}_{l\in\mathbb{N}}$. So we obtain that $u_k \in W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N)$.

*Step 2: we show that $u_k \to u_*$ strongly in $W^{1,p}(B_R, \mathbb{R}^N)$ and $\mathcal{F}(u_k) \to \mathcal{F}(u_*)$.*

Let us consider the following chain of inequalities:

$$\begin{aligned}\mathcal{F}(u_k) \leqslant \mathcal{G}_k(u_k) \leqslant \mathcal{G}_k(u_*) &\leqslant \mathcal{F}(u_*) + \frac{2^{\frac{p}{2}}\operatorname{meas}(B_R)}{k} + 2^{\frac{p}{2}}k^{p-1}\|Du_* - Dv_k\|^p_{L^p(B_R)} \\ &\underbrace{\leqslant}_{(12)} \mathcal{F}(u_*) + \frac{2^{\frac{p}{2}}\operatorname{meas}(B_R)}{k} + \frac{2^{\frac{p}{2}}}{k}.\end{aligned} \tag{13}$$

Therefore,

$$\limsup_{k\to+\infty}\mathcal{F}(u_k) \leqslant \limsup_{k\to+\infty}\left(\mathcal{F}(u_*) + \frac{2^{\frac{p}{2}}\operatorname{meas}(B_R)}{k} + \frac{2^{\frac{p}{2}}}{k}\right) = \mathcal{F}(u_*). \tag{14}$$

Bearing in mind the positivity of $\mathcal{F}$, we observe that

$$\begin{aligned}k^{p-1}\int_{B_R}|Du_k(x) - Dv_k(x)|^p\mathrm{d}x &\leqslant \int_{B_R}\frac{1}{k}(1 + k^2\,|Du_k(x) - Dv_k(x)|^2)^{\frac{p}{2}} \\ &= \mathcal{G}_k(u_k) - \mathcal{F}(u_k) \\ &\underbrace{\leqslant}_{(13)} \mathcal{F}(u_*) + \frac{2^{\frac{p}{2}}\operatorname{meas}(B_R)}{k} + \frac{2^{\frac{p}{2}}}{k},\end{aligned}$$

namely,

$$\int_{B_R}|Du_k(x) - Dv_k(x)|^p\mathrm{d}x \leqslant \frac{\mathcal{F}(u_*)}{k^{p-1}} + \frac{2^{\frac{p}{2}}\operatorname{meas}(B_R)}{k^p} + \frac{2^{\frac{p}{2}}}{k^p}.$$

So we have that $\|Du_k - Dv_k\|_{L^p(B_R)} \to 0$ as $k \to +\infty$. We keep in mind (12) that guarantees $\|Dv_k - Du_*\|_{L^p(B_R)} \to 0$, then $Du_k \to Du_*$ in $L^p(B_R)$. Since $u_k = u_*$ on $\partial\Omega$, we have $u_k \to u_*$ in $W^{1,p}(B_R, \mathbb{R}^N)$. Consequently, by the lower semicontinuity of $\mathcal{F}$, we can write

$$\mathcal{F}(u_*) \leqslant \liminf_{k\to+\infty}\mathcal{F}(u_k).$$

Taking into account (14), we obtain

$$\limsup_{k\to+\infty}\mathcal{F}(u_k) \leqslant \mathcal{F}(u_*) \leqslant \liminf_{k\to+\infty}\mathcal{F}(u_k),$$

i.e.,

$$\mathcal{F}(u_k) \to \mathcal{F}(u_*).$$

This result holds true for all $u_* \in W^{1,p}(B_R, \mathbb{R}^N)$ such that $\mathcal{F}(u_*) < +\infty$. This ends the proof. □

In [21, Section 5], it has been proved that there is a special case in which it is possible to construct the approximating sequence $f^l$ that fulfills the assumptions of Theorem 3.1. We recall here that this is the case where $f$ is radially symmetric with respect to the second variable, i.e.,

$$f(x,z) = \tilde{f}(x,|z|), \quad \forall (x,z) \in \Omega \times \mathbb{R}^{N\times n}, \tag{S}$$

for some $\tilde{f} : \Omega \times [0,+\infty) \to \mathbb{R}$, and besides assumptions (AP1), (AP3), and (AP4), $f$ satisfies also

$$L^{-1}(\mu^2 + |z|^2)^{\frac{p}{2}} \leqslant f(x,z) \leqslant L(1+|z|^q), \tag{AP2'}$$

where $\mu$ is the same as in (AP3). We note that if we consider a function $f$ that satisfies (AP2) and (AP3) then, if $p \geqslant 2$, it obviously satisfies (AP2) and (AP3) with $\mu = 0$. In the case where $p < 2$, we can consider $\hat{f} = f + c$, where $c$ is a suitable constant, and now, $\hat{f}$ satisfies (AP2') and (AP3). Moreover, changing $f$ with $\hat{f}$ does not affect our problem. Then, we can state the following corollary.

**Corollary 3.2.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $\alpha \in (0,1]$ and $p$, $q$ be such that*

$$1 < p < q < p\left(\frac{n+\alpha}{n}\right).$$

*Let $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Carathéodory function such that there exist constants $L \in [1,+\infty)$ and $\mu \in [0,1]$: $f$ satisfies* (AP1), (AP2), (AP3), (AP4), *and* (S). *Let $u_*$ be a function in $W^{1,p}(B_R, \mathbb{R}^N)$ such that $\mathcal{F}(u_*) < +\infty$. Then,*

$$\mathcal{L}(u_*) = 0.$$

**Remark 3.3.** We note that the proof of Theorem 5.1 in [21] gives us a constructive way to define the sequence $\{f^l\}_{l\in\mathbb{N}}$.

**Remark 3.4.** Let us highlight the fact that we show more than what is needed to have the absence of the gap $\mathcal{L}$ between the spaces $X$ and $Y$. Indeed, for every $u \in W^{1,p}(B_R, \mathbb{R}^N)$, we find a sequence $u_k$ that coincides with $u$ on the boundary of $B_R$, i.e.,

$$u_k \in (u + W_0^{1,p}(B_R, \mathbb{R}^N)) \cap W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N),$$

and converges *strongly* to $u$ in $W^{1,p}(B_R, \mathbb{R}^N)$.

We conclude this section giving an example of integrand $f$ that satisfies the hypothesis of Theorem 3.1.

**Example 3.5.** For $\Omega$ and $p, q, \alpha$ as in Theorem 3.1, we define a density $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ as

$$f(x,z) \coloneqq g(x,z) + (\mu^2 + |z|^2)^{\frac{p}{2}},$$

with $g(x,z) \coloneqq \tilde{g}(x, z_n^N)$ and $\tilde{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$\tilde{g}(x, z_n^N) \coloneqq \begin{cases} |x_1|^\alpha (z_n^N - |x_1|^\alpha)^q, & \text{if} \quad z_n^N > |x_1|^\alpha, \\ 0, & \text{if} \quad z_n^N \leqslant |x_1|^\alpha, \end{cases}$$

where $x = (x_1, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$ and $z = \begin{pmatrix} z_1^1 & \ldots & z_n^1 \\ \vdots & \ddots & \vdots \\ z_1^N & \ldots & z_n^N \end{pmatrix} \in \mathbb{R}^{N\times n}$. The function $\tilde{g}$ behaves as in Figure 1.

The Lagrangian $f$ satisfies hypothesis (AP1)–(AP4) of Theorem 3.1, the calculations follow from those of [21, Section 6]. We point out that $f$ still satisfies neither hypothesis (4) of [23, Theorem 3.1] nor hypothesis (1.8) of

[18, Theorem 1.1], see [21, Section 6]. Moreover, the density $f$ does not verify Assumption (2) of [8] that we write for the convenience of the reader: there exist $0 < \nu, \beta < 1 < \tilde{\nu} < +\infty$ such that

$$\nu M(x, \beta z) \leqslant f(x, z) \leqslant \tilde{\nu}(M(x, z) + 1), \quad \forall (x, z) \in \Omega \times \mathbb{R}^{N\times n}, \tag{15}$$

where $M : \Omega \times \mathbb{R}^{N\times n} \to [0, +\infty)$ is a weak $N$-function so, in particular,

$$M(x, z) = M(x, -z), \tag{16}$$

for almost every $x \in \Omega$ and every $z \in \mathbb{R}^{N\times n}$. Let us see in detail that $f$ does not verify (15). If $z_n^N \leqslant |x_1|^\alpha$ then

$$\nu M(x, \beta z) \leqslant f(x, z) = (\mu^2 + |z|^2)^{\frac{p}{2}}.$$

Accordingly

$$M(x, \xi) \leqslant \frac{1}{\nu}\left(\mu^2 + \frac{1}{\beta^2}\, |\xi|^2\right)^{\frac{p}{2}},$$

for all $\xi \in \mathbb{R}^{N\times n}$ such that $\xi_n^N \leqslant \beta\, |x_1|^\alpha$. Now if $\xi_n^N > \beta\, |x_1|^\alpha$, then, by (15) and (16), we still obtain

$$M(x, \xi) = M(x, -\xi) \leqslant \frac{1}{\nu}\left(\mu^2 + \frac{1}{\beta^2}\, |\xi|^2\right)^{\frac{p}{2}}.$$

Hence, for $z_n^N > |x_1|^\alpha$,

$$\begin{aligned} f(x, z) &= |x_1|^\alpha (z_n^N - |x_1|^\alpha)^q + (\mu^2 + |z|^2)^{\frac{p}{2}} \\ &\leqslant \tilde{\nu}(M(x, z) + 1) \\ &\leqslant \tilde{\nu}\left[\frac{1}{\nu}\left(\mu^2 + \frac{1}{\beta^2}\, |z|^2\right)^{\frac{p}{2}} + 1\right]. \end{aligned}$$

Recalling that $q > p$, the aforementioned inequality is not in force if $z_n^N \to +\infty$; thus, Assumption (15) is not valid for $f$.

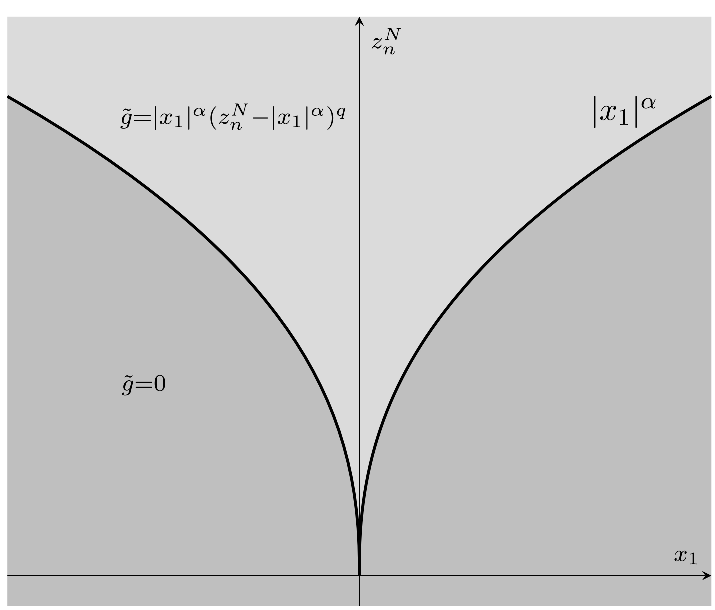


**Figure 1:** Strictly above $z_n^N = |x_1|^\alpha$, the function $\tilde{g}$ is equal to $|x_1|^\alpha(z_n^N - |x_1|^\alpha)^q$, while below $\tilde{g}$ is zero.

# 4 Application of the penalization technique to other functionals

In this section, we want to examine if a suitable penalization can be applied to other functionals in order to obtain $\mathcal{L}(u) \equiv 0$ between suitable spaces $X$ and $Y$. In particular, we analyze two situations. The first one concerns a functional in the vectorial case, with the density $f = f(x, z)$ satisfying the hypotheses of the article [25]. In this setting, the approximating sequence belongs to the Morrey space. The second framework regards the scalar case and deals with densities like $f(x, u, z) = d(z) + h(x, u)$ that verify the assumptions of [16]. Here, the approximating sequence has bounded gradients.

## 4.1 Vectorial functionals and Morrey space

In the following, we take into account again

$$\int_\Omega f(x, Du(x))\mathrm{d}x,$$

with $u : \Omega \to \mathbb{R}^N$, where $\Omega \subset \mathbb{R}^n$ is open and bounded, $n \geqslant 2$ and $N \geqslant 1$. In this section, we need to specify the integration domains; for this reason, we use the following notation:

$$\mathcal{F}(u, A) \coloneqq \int_A f(x, Du(x))\mathrm{d}x.$$

We will deal with the notions of Morrey space and Morrey–Sobolev space; we recall here the definitions.

**Definition 4.1.** (Morrey space) For each $p \in [1, \infty)$ and $0 \leqslant \lambda \leqslant n$, we define the Morrey space

$$L^{p,\lambda}(\Omega, \mathbb{R}^N) \coloneqq \left\{ u \in L^p(\Omega, \mathbb{R}^N) : \sup_{x \in \Omega, \rho > 0} \frac{1}{\rho^\lambda} \int_{\Omega \cap B_\rho(x)} |u|^p \mathrm{d}x < \infty \right\}.$$

**Definition 4.2.** (Sobolev-Morrey space) For each $p \in [1, +\infty)$ and $0 \leqslant \lambda \leqslant n$, we say that a mapping $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ belongs to the Sobolev-Morrey space $W^{1,(p,\lambda)}(\Omega, \mathbb{R}^N)$ if $u \in L^{p,\lambda}(\Omega, \mathbb{R}^N)$ and $Du \in L^{p,\lambda}(\Omega, \mathbb{R}^{N\times n})$.

We consider $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ as in Theorem 1.1 of Fey-Foss [25], namely: there exist numbers $1 < p < q$, $0 < \lambda < n$, $L \geqslant 1$, and a convex function $\tilde{f} : [0, +\infty) \to [0, +\infty)$ satisfying, for all $R > 0$,

$$\begin{cases} \tilde{f} \in C^1([0, +\infty)) \cap C^2((0, +\infty)), \tilde{f}'' \in L^1((0, R)), \\ (p-1)\dfrac{\tilde{f}'(t)}{t} \leqslant \tilde{f}''(t) \leqslant (q-1)\dfrac{\tilde{f}'(t)}{t}, \quad t > 0, \\ \tilde{f}(0) = \tilde{f}'(0) = 0, \quad \tilde{f}(1) > 0, \end{cases} \tag{17}$$

such that for every $\varepsilon > 0$ and $x \in \Omega$, there exists $\sigma_\varepsilon(x) \in [0, +\infty)$:

$$|f(x, z) - \tilde{f}(|z|)| < \varepsilon \tilde{f}(|z|), \tag{18}$$

whenever $|z| > \sigma_\varepsilon(x)$. Moreover, there exist $a(x) \in [0, +\infty)$ such that

$$|f(x, z)| \leqslant L\, |z|^q + a(x), \tag{19}$$

for every $x \in \Omega$ and $z \in \mathbb{R}^{N\times n}$. In addition, $\sigma_\varepsilon \in L^{q,\lambda}(\Omega, \mathbb{R})$ and $a \in L^{1,\lambda}(\Omega, \mathbb{R})$. We strengthen the hypothesis of asymptotically convexity of $f$ given by (18) assuming that $z \mapsto f(x, z)$ is convex. Moreover, we impose $f(x, z) \geqslant 0$ for all $x \in \Omega$ and $z \in \mathbb{R}^{N\times n}$. Note that the previous assumptions imply, by Lemma 3.1 in [25], that the function $\tilde{f}$ is increasing and

$$\tilde{f}(t_1 + t_2) \leqslant 2^q(\tilde{f}(t_1) + \tilde{f}(t_2)), \quad \text{for all } t_1, t_2 \geqslant 0, \tag{20}$$

$$\tilde{f}(ct) \leqslant c^q \tilde{f}(t), \quad \text{for all } t \geqslant 0 \quad \text{and} \quad c \geqslant 1, \tag{21}$$

$$\tilde{f}(t) \leqslant \tilde{f}(1)(1+t^q), \quad \text{for all } t \geqslant 0. \tag{22}$$

Let us add another assumption, which is a slight modification of [25, Lemma 3.1 (iv)],

$$L^{-1}t^p \leqslant \tilde{f}(t). \tag{23}$$

We aim to approximate every $u_* \in W^{1,p}(B_R, \mathbb{R}^N)$ with a sequence in the Morrey space $\{u_k\}_{k\in\mathbb{N}} \in (u_* + W_0^{1,p}(B_\rho, \mathbb{R}^N)) \cap W^{1,(p,\lambda)}_{\mathrm{loc}}(B_\rho, \mathbb{R}^N)$, $\rho < R$, showing also the convergence in energy. For this purpose, let us take any $u_* \in W^{1,p}(B_R)$ such that $\mathcal{F}(u_*, B_R) < +\infty$. First of all, we prove that for any $\rho \in (0, R)$, there exists a suitable sequence $\{v_k\}_{k\in\mathbb{N}} \in \mathrm{Lip}(\overline{B}_\rho, \mathbb{R}^N)$ such that $v_k \to u_*$ strongly in $W^{1,p}(B_\rho, \mathbb{R}^N)$ and

$$\int_{B_\rho} \tilde{f}(|Du_*(x) - Dv_k(x)|)\mathrm{d}x \leqslant \frac{1}{k^p}. \tag{24}$$

Indeed, we start defining

$$S_1 \coloneqq B_R \cap \left\{x : |Du_*(x)| \leqslant \sigma_{\frac{1}{2}}(x)\right\}$$

and

$$S_2 \coloneqq B_R \cap \left\{x : |Du_*(x)| > \sigma_{\frac{1}{2}}(x)\right\}.$$

Now, using (22) and (18) with $\varepsilon = \frac{1}{2}$, we obtain

$$\frac{1}{2}\int_{B_R} \tilde{f}(|Du_*(x)|)\mathrm{d}x = \frac{1}{2}\left(\int_{S_1} \tilde{f}(|Du_*(x)|)\mathrm{d}x + \int_{S_2} \tilde{f}(|Du_*(x)|)\mathrm{d}x\right) \leqslant \frac{1}{2}\left(\tilde{f}(1)\int_{S_1}(\sigma_{\frac{1}{2}}(x))^q\mathrm{d}x + \mathrm{meas}(B_R)\right) + \int_{B_R} f(x, Du_*(x))\mathrm{d}x.$$

Recalling that $\sigma_\varepsilon \in L^{q,\lambda}$ and $\int_{B_R} f(x, Du_*(x))\mathrm{d}x < +\infty$, we obtain

$$\int_{B_R} \tilde{f}(|Du_*(x)|)\mathrm{d}x < +\infty.$$

Now, as in the proof of Theorem 3.1, for every $0 < \delta < \frac{R+\rho}{2}$, we mollify $u_*$ and we obtain $w_\delta \in \mathrm{Lip}(\overline{B}_\rho, \mathbb{R}^N)$ such that $w_\delta \to u_*$ strongly in $W^{1,p}(B_\rho, \mathbb{R}^N)$. Then, Jensen's inequality and Fubini's theorem lead to

$$\int_{B_\rho} \tilde{f}(|Dw_\delta(x)|)\mathrm{d}x \leqslant \int_{B_{\rho+\delta}} \tilde{f}(|Du_*(x)|)\mathrm{d}x. \tag{25}$$

We pick a subsequence $\{w_{\delta_j}\}_{j\in\mathbb{N}}$ such that

$$Dw_{\delta_j}(x) \to Du_*(x), \quad \text{for a.e.} \, x \in B_\rho.$$

Using (25) together with Fatou's lemma, we obtain

$$\begin{aligned}
\int_{B_\rho} \tilde{f}(|Du_*(x)|)\mathrm{d}x &\leqslant \liminf_{j\to+\infty} \int_{B_\rho} \tilde{f}(|Dw_{\delta_j}(x)|)\mathrm{d}x \\
&\leqslant \limsup_{j\to+\infty} \int_{B_\rho} \tilde{f}(|Dw_{\delta_j}(x)|)\mathrm{d}x \\
&\leqslant \limsup_{j\to+\infty} \int_{B_{\rho+\delta_j}} \tilde{f}(|Du_*(x)|)\mathrm{d}x \\
&= \lim_{j\to+\infty} \int_{B_{\rho+\delta_j}} \tilde{f}(|Du_*(x)|)\mathrm{d}x \\
&= \int_{B_\rho} \tilde{f}(|Du_*(x)|)\mathrm{d}x,
\end{aligned}$$

since $\delta_j \to 0$ as $j \to +\infty$ and $\tilde{f}(|Du_*|) \in L^1(B_R)$. Namely,

$$\int_{B_\rho} \tilde{f}(|Dw_{\delta_j}(x)|)\mathrm{d}x \to \int_{B_\rho} \tilde{f}(|Du_*(x)|)\mathrm{d}x.$$

At this point, we can apply the generalized dominate convergence theorem to obtain

$$\int_{B_\rho} \tilde{f}(|Du_*(x) - Dw_{\delta_j}(x)|)\mathrm{d}x \to 0.$$

Then, there exists $\{w_{\delta_{j_k}}\}_{k\in\mathbb{N}}$ such that

$$\int_{B_\rho} \tilde{f}(|Du_*(x) - Dw_{\delta_{j_k}}(x)|)\mathrm{d}x \leqslant \frac{1}{k^p}.$$

Hence, the sequence $\{v_k\}_{k\in\mathbb{N}}$ we were looking for is defined by

$$v_k \coloneqq w_{\delta_{j_k}}.$$

At this point, we consider the following perturbed functional:

$$\tilde{\mathcal{G}}_k(u) \coloneqq \int_{B_\rho} [f(x, Du(x)) + k\tilde{f}(|Du(x) - Dv_k(x)|)]\mathrm{d}x.$$

Let us set

$$\tilde{g}_k(x, z) \coloneqq f(x, z) + k\tilde{f}(|z - Dv_k(x)|).$$

It is easy to see that, by the direct method of the calculus of variations, for any $k \in \mathbb{N}$ there exists $u_k \in u_* + W_0^{1,p}(B_\rho, \mathbb{R}^N)$ such that

$$\min_{u\in u_*+W_0^{1,p}(B_\rho,\mathbb{R}^N)} \tilde{\mathcal{G}}_k(u) = \tilde{\mathcal{G}}_k(u_k).$$

To obtain Morrey regularity for the minimizer $u_k$, we aim to verify that $\tilde{g}_k$ satisfies the same hypothesis of $f$, in order to apply Theorem 1.1 in [25]. Recalling that $\tilde{f}$ is increasing and using (20), we achieve the existence of a constant $L_k \geqslant 1$ and a function $a_k \in L^{1,\lambda}$ both depending on $k$ such that

$$\tilde{g}_k(x, z) \leqslant L_k\ |z|^q + a_k(x),$$

for all $x \in \Omega$ and $z \in \mathbb{R}^{N\times n}$. Now, by (21), we obtain that for all $\tilde{\varepsilon} \in (0, 1]$,

$$(1 - \tilde{\varepsilon})^q\tilde{f}(|z|) \leqslant \tilde{f}((1 - \tilde{\varepsilon})|z|) \leqslant \tilde{f}(|z - Dv_k(x)|) \leqslant \tilde{f}((1 + \tilde{\varepsilon})|z|) \leqslant (1 + \tilde{\varepsilon})^q f(|z|),$$

whenever $|z| > \frac{|Dv_k(x)|}{\tilde{\varepsilon}}$. Then, for all $\varepsilon, \tilde{\varepsilon} \in (0, 1]$, there is $\tilde{\sigma}_{\varepsilon,\tilde{\varepsilon}}(x) \coloneqq \sigma_\varepsilon(x) + \frac{|Dv_k(x)|}{\tilde{\varepsilon}}$ such that if $|z| > \tilde{\sigma}_{\varepsilon,\tilde{\varepsilon}}(x)$, we have

$$\begin{aligned}\tilde{g}_k(x, z) &= f(x, z) + k\tilde{f}(|Du(x) - Dv_k(x)|)\\ &\leqslant (1 + \varepsilon)\tilde{f}(|z|) + k(1 + \tilde{\varepsilon})^q\tilde{f}(|z|)\\ &= [1 + \varepsilon + k(1 + \tilde{\varepsilon})^q]\tilde{f}(|z|),\end{aligned}$$

and, in a similar fashion,

$$\tilde{g}_k(x, z) \geqslant [(1 - \varepsilon) + k(1 - \tilde{\varepsilon})^q]\tilde{f}(|z|).$$

Taking

$$\tilde{\varepsilon} = \min\left\{1 - (1 - \varepsilon)^{\frac{1}{q}}, (1 + \varepsilon)^{\frac{1}{q}} - 1\right\},$$

we obtain

$$(1-\varepsilon)(1+k)\tilde{f}(|z|) \leqslant \tilde{g}_k(x,z) \leqslant (1+\varepsilon)(1+k)\tilde{f}(|z|).$$

Let us set

$$\tilde{f}_k(t) := (1+k)\tilde{f}(t).$$

It is straightforward to check that $\tilde{f}_k$ is convex and satisfies (17). Therefore, by [25, Theorem 1.1], the minimizer $u_k$ enjoys

$$u_k \in (u_* + W_0^{1,p}(B_\rho, \mathbb{R}^N)) \cap W^{1,(p,\lambda)}_{\mathrm{loc}}(B_\rho, \mathbb{R}^N).$$

Now, bearing in mind (23) and (24) and following Step 2 of Theorem 3.1, we obtain that $u_k \to u_*$ strongly in $W^{1,p}(B_\rho, \mathbb{R}^N)$ and $\mathcal{F}(u_k, B_\rho) \to \mathcal{F}(u_*, B_\rho)$. We have just proved the following theorem.

**Theorem 4.3.** *Let $f : \Omega \times \mathbb{R}^{N\times n} \to [0,+\infty)$ be a Carathéodory function such that $z \mapsto f(x,z)$ is convex and* (18), (19) *are satisfied with $\tilde{f}$ as in* (17) *and* (23). *Let $u_*$ be a function in $W^{1,p}(B_R, \mathbb{R}^N)$ such that $\mathcal{F}(u_*, B_R) < +\infty$, where $B_R \subset \Omega$. Fix $\rho < R$. Then, there exists a sequence*

$$\{u_k\}_{k\in\mathbb{N}} \subset (u_* + W_0^{1,p}(B_\rho, \mathbb{R}^N)) \cap W^{1,(p,\lambda)}_{\mathrm{loc}}(B_\rho, \mathbb{R}^N)$$

*such that $u_k \to u$ strongly in $W^{1,p}(B_\rho, \mathbb{R}^N)$ and*

$$\mathcal{F}(u_k, B_\rho) \to \mathcal{F}(u_*, B_\rho).$$

An example of function that satisfies the hypotheses of Theorem 4.3 is

$$f(x,z) := \tilde{f}(|z| + b(x)),$$

for a suitable $b(x) \geqslant 0$, where

$$\tilde{f}(t) := t^p \ln(e+t).$$

## 4.2 Scalar functionals and bounded gradients

In this last section, we consider an example in the multidimensional scalar case. We deal with the following functional:

$$\mathcal{F}(u) := \int_\Omega f(x, u(x), Du(x))\mathrm{d}x,$$

with $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is open and bounded, $n \geqslant 2$. We assume that the density $f$ has the following form:

$$f(x,u,z) := d(z) + h(x,u),$$

and satisfies the hypotheses of Theorem 1.1 in [16], which we recall here for the convenience of the reader. Let $p$ and $q$ be such that

$$1 < p < q < p\left(\frac{n+1}{n}\right). \tag{26}$$

The function $d : \mathbb{R}^n \to [0,+\infty)$ has the $(p,q)$-growth (AP2) and is $p$-uniformly convex at infinity, i.e., there exist $\nu, R > 0$ such that

$$f\left(\frac{z_1+z_2}{2}\right) \leqslant \frac{1}{2}f(z_1) + \frac{1}{2}f(z_2) - \nu(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_1 - z_2|^2 \tag{27}$$

when the line segment joining $z_1$ and $z_2$ is all outside $B(0,R)$. The function $h:\Omega\times\mathbb{R}\to[0,+\infty)$ is of Carathéodory type and $x\mapsto h(x,0)\in L^1(\Omega,\mathbb{R})$. Moreover, there exists a function $a\in L^\infty_{\mathrm{loc}}(\Omega,\mathbb{R})$ such that

$$|h(x,u_1)-h(x,u_2)|\leqslant a(x)|u_1-u_2|,\quad \text{for a.e. } x\in\Omega,\text{ for all } u_1,u_2\in\mathbb{R}. \tag{28}$$

We stress that, unlike [16], we require that $d$ is also convex everywhere. Moreover, here we ask that $h(x,u)\geqslant 0$. Our target is to approximate every $u_*\in W^{1,p}(B_R,\mathbb{R})$ with a sequence $\{u_k\}_{k\in\mathbb{N}}\subset(u_*+W_0^{1,p}(B_R,\mathbb{R}))\cap W^{1,\infty}_{\mathrm{loc}}(B_R,\mathbb{R})$ and show the approximation in energy. So, let $u_*\in W^{1,p}(B_R,\mathbb{R})$ be a function such that $\mathcal{F}(u_*)<+\infty$.

We consider the following penalization of $\mathcal{F}$:

$$\tilde{\tilde{\mathcal{G}}}_k(u):=\int_{B_R}[f(x,u(x),Du(x))+k|u(x)-u_*(x)|]\mathrm{d}x.$$

By the direct method of the calculus of variations, we obtain, for every $k\in\mathbb{N}$, a minimizer $u_k\in u_*+W_0^{1,p}(B_R,\mathbb{R})$ of $\tilde{\tilde{\mathcal{G}}}_k$. Now, in order to apply [16, Theorem 1.1] to $\tilde{\tilde{\mathcal{G}}}_k$, we just need to observe that $h(x,u)+k|u(x)-u_*(x)|$ keeps checking (28) and that $x\mapsto h(x,0)+k|u_*(x)|\in L^1(B_R,\mathbb{R})$. Then, the minimizer $u_k$ enjoys

$$u_k\in(u_*+W_0^{1,p}(B_R,\mathbb{R}))\cap W^{1,\infty}_{\mathrm{loc}}(B_R,\mathbb{R}).$$

At this point, following Step 2 of Theorem 3.1, we obtain that $u_k\to u_*$ strongly in $L^1(B_R,\mathbb{R})$. But, by (AP2), we have

$$L^{-1}\|Du_k\|^p_{L^p(B_R)}\leqslant\mathcal{F}(u_k)\leqslant\tilde{\tilde{\mathcal{G}}}_k(u_k)\leqslant\tilde{\tilde{\mathcal{G}}}_k(u_*)=\mathcal{F}(u_*),$$

then, by the previous inequality, up to not relabeled subsequences, we may suppose that there exists $u_\infty\in u_*+W_0^{1,p}(B_R,\mathbb{R})$ such that

$$u_k\to u_\infty,\quad\text{strongly in } L^p(B_R,\mathbb{R}),$$
$$Du_k\rightharpoonup Du_\infty,\quad\text{weakly in } L^p(B_R,\mathbb{R}^n).$$

Therefore, due to the uniqueness of the limit, we can say that $u_\infty=u_*$ and

$$Du_k\rightharpoonup Du_*,\quad\text{weakly in } L^p(B_R,\mathbb{R}).$$

Thus, the following theorem holds true.

**Theorem 4.4.** *Let $p$ and $q$ be as in* (26). *Let $d:\mathbb{R}^n\to[0,+\infty)$ be a function that verifies* (AP2) *and* (27), *and let $h:\Omega\times\mathbb{R}\to[0,+\infty)$ be a Carathéodory function such that $x\mapsto h(x,0)\in L^1(\Omega,\mathbb{R})$ and* (28) *is satisfied. Let us set*

$$\mathcal{F}(u):=\int_{B_R}[d(Du(x))+h(x,u(x))]\mathrm{d}x.$$

*Let $u_*$ be a function in $W^{1,p}(B_R,\mathbb{R}^N)$ such that $\mathcal{F}(u_*)<+\infty$, where $B_R\subset\Omega$. Then, there exists a sequence*

$$\{u_k\}_{k\in\mathbb{N}}\subset(u_*+W_0^{1,p}(B_R,\mathbb{R}))\cap W^{1,\infty}_{\mathrm{loc}}(B_R,\mathbb{R})$$

*such that $u_k\rightharpoonup u$ weakly in $W^{1,p}(B_R,\mathbb{R})$ and*

$$\mathcal{F}(u_k)\to\mathcal{F}(u_*).$$

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