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SYZ mirror symmetry of solvmanifolds

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Abstract

We present an effective construction of non-Kähler supersymmetric mirror pairs in the sense of Lau, Tseng and Yau (Commun. Math. Phys. 340:145–170, 2015) starting from left-invariant affine structures on Lie groups. Applying this construction we explicitly find SYZ mirror symmetric partners of all known compact 6-dimensional completely solvable solvmanifolds that admit a semi-flat type IIA structure.

Keywords Mirror symmetry · SYZ conjecture · Lagrangian fibrations · Non-Kähler geometry · Affine structures · Solvmanifolds

Mathematics Subject Classification 53D37 · 53A15 · 22E25 · 53Z05

1 Introduction

The Strominger-Yau-Zaslow (SYZ) conjecture (see [24]) tries to describe mirror symmetry of Calabi-Yau manifolds in terms of dual Lagrangian torus fibrations.

In this paper we deal with a non-Kähler version of SYZ mirror symmetry where the correspondence between symplectic and complex structures of the partners is made explicit through Fourier-Mukai transform. The $SU(n)$-structures involved are *type IIA* and *type IIB* structures (see subsection 2.2 for the definition for $n = 3$). The role of the Dolbeault cohomology is replaced by the Bott-Chern cohomology on the complex side and by a refined version of the Tseng-Yau cohomology on the symplectic side. The procedure is thouroghly explained in [21]. Here we will mainly stick to the case of manifolds of real dimension 6, which is the ambient where originally mirror symmetry made its appearance.

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One of the aims of the present paper is to show that, unlike in the Kähler case, it is possible to find many interesting mirror pairs of compact 6-manifolds without the need of singularities in the fibrations. The first and only example of this kind known so far is the nilmanifold featured in [21].

Nilmanifolds and more generally solvmanifolds are of course a natural ambient to look at in the seek of such structures. For example there are plenty of explicit symplectic structures on non-abelian nilmanifolds and none of these can be Kähler.

Moreover an important special case of type IIA structure is given by *symplectic half-flat structures* and all 6-dimensional solvable Lie algebras admitting such structures are classified (see [14]).

Compact quotients of the corresponding simply connected Lie groups are the best known explicit examples of compact manifolds carrying a type IIA structure.

While reinterpreting the example given in [21], the aim of the present paper is to explicitly find SYZ mirror symmetric partners (in the sense of [21]) of all known compact 6-dimensional symplectic half-flat completely solvable solvmanifolds that admit a semi-flat structure.

The starting observation is that all such examples are indeed quotients of Lie groups having a particular structure of semi-direct product. This semi-direct product structure is intimately related to a left invariant affine structure on 3-dimensional solvable Lie groups, hence to a Lagrangian torus bundle on suitable quotients of it.

It is this semi-direct product structure that allows us to explicitly find the non-singular (i.e. semi-flat) dual torus fibrations, hence the mirror partner. This is explained in section 4.

The first main result is summarized in the following

Theorem 1 *Let* (X, ω, Ω) *be a compact solvmanifold endowed with a semi-flat left-invariant IIA structure. Then its SYZ mirror partner* $(\check{X}, \check{\omega}, \check{\Omega})$ *is a compact solvmanifold endowed with a semi-flat left-invariant IIB structure.*

In table 2 we list all the IIB mirror partners of completely solvable semi-flat 6-dimensional IIA Lie algebras.

The second main result is the explicit construction of all the mirror pairs and the relevant structures coming from left-invariant affine structures on completely solvable 3-dimensional unimodular Lie groups. This is carried out in section 6.

In table 2 we also write down the Tseng-Yau and Bott-Chern numbers of the algebras involved that realize the mirror symmetric *non-Kähler Hodge diamonds*.

As an upshot of our constructions we find a compact type IIA manifold X admitting two inequivalent Lagrangian torus fibrations giving rise to two non-isomorphic semi-flat mirror pairs: the complex IIB partners $\check{X}$ and $\check{X}'$ are not even diffeomorphic (see 6.1.2 and 6.2.2).

As a by-product we also find a new balanced metric on a nilmanifold which is missing from the classification given in [20], see subsection 6.3.2, Remark 15.

A systematic study of all the semi-flat $\mathrm{SU}(3)$-mirror pairs coming from left-invariant affine structures using the classification of [16] is carried over in a forthcoming paper.

Several related results should be mentioned. Firstly in [7] invariant symplectic structures on T^*G are constructed on a Lie group G carrying an invariant affine structure. The analogous construction of invariant complex structures for TG can be found in [5].

Moreover in [9] the authors list all the pairs of *nilpotent* 6-dimensional Lie algebras constructed via dual semi-direct product and show that they have isomorphic differential Gerstenhaber algebras realizing a sort of algebraic *weak* mirror symmetry.

2 Preliminaries

2.1 Affine structures and dual torus bundles

For details about affine structures see [18, 25], here we will just recall the notions relevant to our construction.

An affine structure on a n-manifold is an atlas whose transition functions are restrictions of affine maps.

Any affine structure on a n-manifold M defines a *developing map* $D : \widetilde{M} \to \mathbb{R}^n$, where $\widetilde{M} \to M$ is the universal covering, and a holonomy representation $h : \pi_1(M) \to \mathrm{Aff}(\mathbb{R}^n)$ (for the precise definition see [25] or [18]).
The affine structure is said to be *complete* if the developing map is a homeomorphism.
Viceversa starting from a pair (D, h) where $h : \pi_1(M) \to \mathrm{Aff}(\mathbb{R}^n)$ is a homomorphism and $D : \widetilde{M} \to \mathbb{R}^n$ is a homeomorphism equivariant with respect to the $\pi_1(M)$ and $h(\pi_1(M))$ actions one can recover a unique complete affine structure on M such that D is the induced developing map and h is the induced holonomy representation.

An affine structure is *integral* if the linear part of transition functions is integral (i.e. takes values in $\mathrm{GL}(n, \mathbb{Z})$). Any integral affine structure $\mathcal{A}$ on a manifold B defines a Lagrangian bundle $X \to B$ over B in the following way. Let $r_1, \ldots, r_n$ be local affine coordinates on $U \subseteq B$. Then for every $q \in U$ we can define Λ_q^* to be the integral lattice of $T_q^* B$ generated by $dr_1, \ldots, dr_n$. This definition does not depend on the choice of the local affine chart. The manifold T^*B/Λ^* will be an n-torus bundle over B. Furthermore the canonical symplectic structure of T^*B passes to the quotient and the fibers are indeed Lagrangian with respect to it.

Viceversa every Lagrangian torus bundle $X \to B$ induces an integral affine structure on B. Over the affine coordinate charts the torus bundle is locally isomorphic to one of the form $T^*B/\Lambda^* \to B$: this is a consequence of the famous *Arnol'd-Liouville Theorem* in classical mechanics which in particular establishes the existence of the so-called *action-angle* coordinates. We remark here that the action coordinates are exactly the coordinates associated to the developing map of the integral affine structure.

Again for details about Lagrangian torus bundles we refer to the classical paper of Duistermaat [13]. See also [8, section 3] for a good presentation of this topic.

Given any Lagrangian torus bundle $X \to B$ together with its action-angle coordinates $r_1, \ldots, r_n, \theta_1, \ldots, \theta_n$ we can define the *dual torus bundle* $\check{X} \to B$ simply by dualizing the transition functions. Locally this is isomorphic to the torus bundle $TB/\Lambda \to B$ obtained by considering the fiberwise lattice $\Lambda \subset TB$ locally generated by $\frac{\partial}{\partial r_1}, \ldots, \frac{\partial}{\partial r_n}$. If we denote by $\check{\theta}_k$ the fiber coordinates corresponding to the action coordinates r_k, we get local complex coordinates $z_k = \check{\theta}_k + i r_k$ on TB/Λ hence on $\check{X}$. With respect to this complex structure the fibers of $\check{X} \to B$ are totally real.

If we assume that the integral affine structure of X is *special*, that is the linear part of the transition functions lies in $\mathrm{SL}(n, \mathbb{Z})$, the complex $(n, 0)$-form on $\check{X}$

$$dz_1 \wedge \cdots \wedge dz_n$$

is globally defined. We will call it $\check{\Omega}$.

A symplectic manifold (X, ω) together with a Lagrangian torus fibration and its complex dual endowed with the holomorphic volume $(\check{X}, \check{\Omega})$ are said to form a *semi-flat mirror pair* in [21].

2.2 Non-Kähler SYZ mirror symmetry

Here we briefly describe the non Kähler version of SYZ mirror symmetry as presented in [21]. First recall that an $\mathrm{SU}(n)$-structure is determined on a real $2n$-manifold X by a pair of differential forms (ω, Ω), where

(1) Ω is a nowhere vanishing decomposable complex n-form such that setting

$$T^{0,1}X = \{v \in TX \otimes C : \iota_v \Omega = 0\}$$

and $T^{1,0}X = \overline{T^{0,1}X}$ we have a splitting

$$TX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$$

inducing an almost complex structure J.

(2) ω is a positive $(1, 1)$-form with respect to J.

We will denote by F the conformal factor defined by

$$\Omega \wedge \bar{\Omega} = i^n F \frac{\omega^n}{n!} .$$

It is easy to prove that the almost complex structure J defined by Ω is integrable if and only if Ω is closed.

In the 3-dimensional case we have the following

Definition 2 An SU(3)-manifold (X, ω, Ω) is said to be supersymmetric of type IIA if $d\omega = 0$ and $d \operatorname{Re} \Omega = 0$.

Definition 3 An SU(3)-manifold (X, ω, Ω) is said to be supersymmetric of type IIB if $d(\omega^2) = 0$ and $d\Omega = 0$.

Note that type IIB manifolds are *balanced* complex manifolds with holomorphically trivial canonical bundle while type IIA manifolds with constant F are often called *symplectic half-flat* manifolds.

Let $\pi : (X, \omega) \to B$ be a Lagrangian torus bundle

and let $\check{\pi} : (\check{X}, \check{\Omega}) \to B$ be its dual so that (X, ω) and $(\check{X}, \check{\Omega})$ form a semi-flat mirror pair.

We denote by $\mathcal{A}^k_B(X, \mathbb{C})$ the space of complex-valued k-forms on X which depend only on the base, also called *semi-flat forms*. An element $\phi \in \mathcal{A}^k_B(X, \mathbb{C})$ is locally written as

$$\phi = \sum_{I,J} a_{IJ}(r) d\theta_I \wedge dr_J$$

where $I = (i_1, \ldots, i_p)$, $J = (j_1, \ldots, j_q)$ are multi-indices and $p + q = k$, (r_i, θ_i) are action-angle coordinates and $a_{IJ}(r)$ are complex-valued functions on B.

Analogously we will denote by $\mathcal{A}^{p,q}_B(\check{X})$ the semi-flat (p, q)-forms on the SYZ-dual $\check{X}$ which are locally written as:

$$\check{\phi} = \sum_{I,J} a_{IJ}(r) dz_I \wedge d\bar{z}_J$$

In the 3-dimensional case of we can refine the previous definition of SU(3)-structures:

Definition 4 An $SU(3)$-manifold (X, ω, Ω) is said to be semi-flat supersymmetric of type IIA if $d\omega = 0$ and $d \operatorname{Re} \Omega = 0$ and both ω and Ω are in $\mathcal{A}^\bullet_B(X, \mathbb{C})$

Definition 5 An $SU(3)$-manifold (X, ω, Ω) is said to be semi-flat supersymmetric of type IIB if $d(\omega^2) = 0$ and $d\Omega = 0$ and both ω and Ω are in $\mathcal{A}^\bullet_B(X, \mathbb{C})$.

We further define the *polarization switch operator* $\mathcal{P}$ on $\mathcal{A}^{\bullet;\bullet}_B(\check{X})$ in the following way:

$$\check{\phi} = \sum_{I,J} a_{IJ}(r) dz_I \wedge d\bar{z}_J \;\mapsto\; \mathcal{P} \cdot \check{\phi} = \sum_{I,J} a_{IJ}(r) d\check{\theta}_I \wedge dr_J \,.$$

2.3 The Fourier-Mukai transform

Let (X, ω) and $(\check{X}, \check{\Omega})$ be a semi-flat mirror pair on a n-dimensional base B. Consider their fiber product over B:

$$\begin{array}{ccccc} & & X \times_B \check{X} & & \\ & \swarrow^{p} & & \searrow^{\check{p}} & \\ X & & & & \check{X} \\ & \searrow_{\pi} & & \swarrow_{\check{\pi}} & \\ & & B & & \end{array}$$

On the Poincaré line bundle over $X \times_B \check{X}$ there is a universal connection which locally is written as $d + i\left(\check{\theta}_k d\theta_k + \theta_k d\check{\theta}_k\right)$. Its curvature form is

$$F = 2i \sum_i^3 d\check{\theta}_i \wedge d\theta_i \,. \tag{1}$$

Let $\phi \in \mathcal{A}^\bullet_B(X)$ and $\check{\phi} \in \mathcal{A}^\bullet_B(\check{X})$. Their *Fourier-Mukai transforms* are defined as

$$\begin{aligned} \mathrm{FT} \cdot \check{\phi} &:= p_*\Big((\check{p}^*(\mathcal{P} \cdot \check{\phi})) \wedge \exp \frac{F}{2i}\Big) \\ \mathrm{FT} \cdot \phi &:= \mathcal{P}^{-1} \cdot \Big(\check{p}_*\big((p^*\phi) \wedge \exp \frac{-F}{2i}\big)\Big), \end{aligned} \tag{2}$$

where the pushforward maps p_*, $\check{p}_*$ are just the integration along the fibers.

The main results by Lau, Tseng, and Yau exploiting the Fourier-Mukai transform involve a refined version of the symplectic cohomology developed by Tseng and Yau in [26, 27].

Let $\Delta \subset TM$ be the Lagrangian distribution coming from the Lagrangian bundle structure of $(X, \omega) \to B$. In the presence of a metric we have also its orthogonal $\Delta^\perp$. This choice allows us to decompose the space of differential forms:

$$\mathcal{A}^\bullet(X) = \bigoplus_{p+q} \mathcal{A}^{p,q}_B(X)$$

where $\mathcal{A}^{p,q}_B$ ranges over p Δ-directions and over q $\Delta^\perp$-directions.

The Δ-refined Tseng-Yau cohomology of (X, ω) is

$$H^{p,q}_{B,TY}(X) := \frac{\mathrm{Ker}(d + d^\Lambda) \cap \mathcal{A}^{p,q}_B(X)}{\mathrm{Im}(dd^\Lambda) \cap \mathcal{A}^{p,q}_B(X)}$$

where $d^\Lambda = d\Lambda - \Lambda d$ and Λ is the adjoint of the Lefschetz operator $L = \omega \wedge \cdot$.

We also recall that the (semi-flat) Bott-Chern cohomology of $(\check{X}, \check{\Omega})$ is

$$H^{p,q}_{B,BC}(\check{X}) := \frac{\operatorname{Ker} d \ \cap \mathcal{A}^{p,q}_B(\check{X})}{\operatorname{Im}(\partial\bar{\partial}) \cap \mathcal{A}^{p,q}_B(\check{X})}.$$

Theorem 6 (Theorem 4.5 and Theorem 6.7 [21]) *Fourier-Mukai transform is an isomorphism of double complexes*

$$\left(\mathcal{A}^\bullet_B(X, \mathbb{C}), \frac{(-1)^n i}{2} d, \frac{(-1)^n i}{2} d^\Lambda\right) \simeq \left(\mathcal{A}^\bullet_B(\check{X}, \mathbb{C}), \bar{\partial}, \partial\right)$$

and at level of cohomologies gives

$$H^{n-p,q}_{B,TY}(X, \mathbb{C}) \simeq H^{p,q}_{B,BC}(\check{X})\,. \tag{3}$$

Theorem 7 (Theorem 5.1 [21]) *Let (X, ω) and $(\check{X}, \check{\Omega})$ be a 3-dimensional semi-flat mirror pair. Let $\check{\omega}$ be a real $(1, 1)$-form in $\mathcal{A}^{1,1}_B(\check{X})$ and set $\Omega = FT(e^{2\check{\omega}})$. Then*

(1) The triple $(\check{X}, \check{\omega}, \check{\Omega})$ defines a $SU(3)$-structure if and only if (X, ω, Ω) defines a $SU(3)$-structure.
(2) (X, ω, Ω) is supersymmetric of type IIA if and only if $(\check{X}, \check{\omega}, \check{\Omega})$ is supersymmetric of type IIB.

Note that we stated the last Theorem in the 3-dimensional case since in this paper we are not dealing with general type IIA and type IIB SU(n)-structures.

3 Known type IIA manifolds

Few non-Kähler type IIA manifolds are known. Main examples are symplectic half-flat nilmanifolds and solvmanifolds.

The first nilpotent example was found in [6], the case of nilpotent Lie algebras is considered in [10], while the case of solvable non-nilpotent Lie algebras is treated in [14]. Important contributions with explicit examples are [11], [12] and [28]. As far as we know there is still no *classification* up to isomorphism of symplectic half flat structures on any solvable Lie algebra.

In table 1 we give the complete list of non-abelian unimodular solvable Lie algebras admitting invariant symplectic half flat structures. For each of them we also provide an example of type IIA structure. At present to our knowledge it is not known if any of these Lie algebras admits a type IIA structure inequivalent to those in table 1.

We also specify if the algebras are completely solvable.

Following the usual convention we present a Lie algebra choosing a left-invariant coframe $e^1, \dots, e^6$ and listing their differential. As usual e^{ij} stands for $e^i \wedge e^j$.

More compact examples are provided by twistor spaces of compact 4-dimensional self-dual Einstein manifolds of negative scalar curvature, see [29]. For interesting non-compact examples see [22, 23].

In section 6 we will apply the construction of section 4 in order to obtain the mirror partners of the compact type IIA solvmanifolds arising from the completely solvable algebras of table 1 except case 3.

Case 3 is excluded because our construction cannot be applied (see beginning of section 6). We also leave out from our treatment the non-completely solvable cases for reasons concerning the cohomology of the quotients, see [1] and [2].

Table 1 Lie algebras with left-invariant type IIA structure

	Lie algebra	ω	Ω	c.s
1	$(0,0,0,0,e^{12},e^{13})$	$e^{14}+e^{26}+e^{35}$	$(e^1+ie^4)\wedge(e^2+ie^6)\wedge(e^3+ie^5)$	yes
2*	$(0,0,0,e^{12},e^{13},e^{23})$	$e^{61}+\lambda e^{52}+(1-\lambda)e^{34}$	$(e^6+ie^1)\wedge(e^5+i\lambda e^2)\wedge(e^3+i(1-\lambda)e^4)$	yes
3	$(0,-e^{13},-e^{12},0,-e^{46},-e^{45})$	$e^{14}+e^{23}+e^{56}$	$(1+i)(e^1+ie^4)\wedge(e^2+ie^3)\wedge(e^5+ie^6)$	yes
4	$(e^{15},-e^{25},-e^{35},e^{45},0,0)$	$e^{31}+e^{24}+e^{56}$	$(e^3+ie^1)\wedge(e^2+ie^4)\wedge(e^5+ie^6)$	yes
5**	$(\alpha e^{15}+e^{25},-e^{15}+\alpha e^{25},-\alpha e^{35}+e^{45},-e^{35}-\alpha e^{45},0,0)$	$e^{13}+e^{24}+e^{56}$	$(e^3+ie^1)\wedge(e^2+ie^4)\wedge(e^5+ie^6)$	no
6	$(e^{23},-e^{36},e^{26},e^{26}-e^{56},e^{36}+e^{46},0)$	$-2e^{16}+e^{34}-e^{25}$	$(-2e^1+ie^6)\wedge(e^3+ie^4)\wedge(e^5+ie^2)$	no
7	$(e^{16}+e^{35},-e^{26}+e^{45},e^{36},-e^{46},0,0)$	$e^{14}+e^{23}+e^{56}$	$(e^1+ie^4)\wedge(e^2+ie^3)\wedge(e^5+ie^6)$	yes
8	$(-e^{16}+e^{25},-e^{15}-e^{26},e^{36}-e^{45},e^{35}+e^{46},0,0)$	$e^{14}+e^{23}+e^{65}$	$(e^1+ie^4)\wedge(e^2+ie^3)\wedge(e^6+ie^5)$	no

* $\lambda \neq 0, 1$
** $\alpha \geq 0$

4 Solvmanifolds, semi-direct products and equivariant dual torus bundles

Let G be a simply connected n-dimensional Lie group endowed with a complete left-invariant affine structure with developing map $D : G \to \mathbb{R}^n$. Without loss of generality we may assume $D(1_G) = 0$.

Note that such a group is necessarily *solvable*. Indeed left invariant complete affine structures on Lie groups correspond to simply transitive subgroups of $\mathrm{Aff}(\mathbb{R}^n)$ (see [16]) and these are solvable (see [3]).

Let us call α the faithful affine representation $\alpha : G \to \mathrm{Aff}(\mathbb{R}^n)$ given by $\alpha(g) = D \circ L_g \circ D^{-1}$ where $L_g : G \to G$ is the left-multiplication by g. Let $\rho : G \to \mathrm{GL}(n, \mathbb{R})$ be its linear part. Of course ρ need not be faithful.

Choosing a lattice (i.e. a co-compact discrete subgroup) $\Gamma \leqslant G$ whose left multiplications, read through D, are *integral* affine gives a well defined integral affine structure $\mathcal{A}$ to the set of right cosets B. (Throughout the paper we will always use the non-customary notation G/Γ for the quotient space with respect to action of Γ on G by left multiplication, that is the set of *right* cosets). Of course, for such a Γ to exist, the group G needs to be unimodular. The holonomy representation of this structure is the restriction of α to Γ.

According to the construction defined in section 2.1 we thus have a well defined lattice $\Lambda^* \subset T^*B$, a Lagrangian torus fibration $X = T^*B/\Lambda^* \to B$ and its dual torus fibration $\check{X} = TB/\Lambda \to B$.

We will work for simplicity on the latter. Let $\pi : G \to G/\Gamma$ be the canonical projection. Let us identify $B \times \mathbb{R}^n$ with TB via the map

$$(\Gamma h, v) \mapsto d(\pi \circ L_h \circ D^{-1})_0\, v\,.$$

Using this identification we define an action of $G \ltimes_\rho \mathbb{R}^n$ on TB by

$$(g, y)(\Gamma h, v) = (\Gamma h g^{-1}, y + \rho(g)v).$$

The lattice $\Lambda \subset TB$ is defined as follows:

$$\Lambda_{\Gamma h} = \mathbb{Z}\{d(\pi \circ D^{-1})_{D(h)} e_i : i = 1, \ldots, n\}$$

Of course here e_i is thought of as an element of $T_h G = T_h \mathbb{R}^n = \mathbb{R}^n$. Note that the lattice is well defined exactly because $\rho(\gamma) \in \mathrm{GL}(n, \mathbb{Z})$ for every $\gamma \in \Gamma$.

Now we claim that the previous action descends to the quotient TB/Λ.

In order to prove it we must show that for every $g \in G$, $y \in \mathbb{R}^n$, $h \in G$, $w \in T_{\Gamma h}B$ and $\lambda \in \Lambda_{\Gamma h}$ we have

$$(g, y)(w + \lambda) - (g, y)w \in \Lambda_{\Gamma h g^{-1}}.$$

Now let $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$:

$$\begin{aligned}
(g, y)(w + \lambda) &= (g, y)(d(\pi \circ L_h \circ D^{-1})_0\, v + \sum_i m_i d(\pi \circ D^{-1})_{D(h)} e_i) \\
&= (g, y)(d(\pi \circ L_h \circ D^{-1})_0\, (v + \sum_i m_i (d(D \circ L_{h^{-1}} \circ D^{-1})_{D(h)} e_i)) \\
&= d(\pi \circ L_{hg^{-1}} \circ D^{-1})_0 (y + \rho(g)v + \sum_i m_i d(\alpha(gh^{-1}))_{D(hg^{-1})} e_i) \\
&= d(\pi \circ L_{hg^{-1}} \circ D^{-1})_0 (y + \rho(g)v + \sum_i m_i (d(D \circ L_{gh^{-1}} \circ D^{-1}))_{D(hg^{-1})} e_i) \\
&= (g, y)w + \sum_i m_i d(\pi \circ D^{-1})_{D(hg^{-1})} e_i\,.
\end{aligned}$$

Finally we note that the action $G \ltimes_\rho \mathbb{R}^n \curvearrowright TB/\Lambda$ is clearly transitive, and the stabilizer at $(\Gamma, 0)$ is exactly $\Gamma \ltimes_\rho \mathbb{Z}^n$.
Dualizing everything we obtain the following

Theorem 8 *Let $X \to G/\Gamma$ be the torus bundle induced by the integral affine structure defined by the triple (G, Γ, D). Let $\check{X} \to G/\Gamma$ be its dual. Then*

(1) The total space X is acted on transitively by the semidirect product $G \ltimes_{\rho^*} (\mathbb{R}^n)^*$ with stabilizer $\Gamma \ltimes_{\rho^*} (\mathbb{Z}^n)^*$, where $\rho^* : G \to \mathrm{Aff}((\mathbb{R}^n)^*)$ is the dual representation induced by ρ.
(2) The total space $\check{X}$ is acted on transitively by the semidirect product $G \ltimes_\rho \mathbb{R}^n$ with stabilizer $\Gamma \ltimes_\rho \mathbb{Z}^n$.

Remark 9 From the argument preceding Theorem 8 it is apparent that the same construction can be applied in the slightly more general case in which $\alpha_{|\Gamma}$ lies in the automorphism group of a lattice Ξ in $\mathbb{R}^n$ conjugated to $\mathbb{Z}^n$. In this case the stabilizers mentioned in Theorem 8 will be $\Gamma \ltimes_{\rho^*} \Xi^*$ and $\Gamma \ltimes_\rho \Xi$.

4.1 SU(n)-manifolds from affine structures

As above we assume that G is a simply connected n-dimensional unimodular Lie group endowed with a complete left-invariant affine structure with developing map $D : G \to \mathbb{R}^n$.

Let us endow T^*G with the Lie group structure induced by the identification $T^*G = G \ltimes_{\rho^*} (\mathbb{R}^n)^*$. Let $\Gamma \leqslant G$ be a lattice whose left multiplications, read through D, are *integral* affine.

The canonical symplectic structure ω on T^*G passes to the quotient $X = T^*B/\Lambda^*$ and the induced projection $X \to B$ (that we are going to call again π) becomes a Lagrangian torus bundle.

In the previous section we proved that X itself is a solvmanifold under the action of $T^*G = G \ltimes_{\rho^*} (\mathbb{R}^n)^*$.

Proposition 10 *The canonical symplectic form ω on $T^*G = G \ltimes_{\rho^*} (\mathbb{R}^n)^*$ is left-invariant.*

Proof The important fact is that the local action-angle coordinates $r_1, \dots, r_n, \theta_1, \dots, \theta_n$ coming from Arnol'd-Liouville theorem become globally defined functions once lifted to the universal cover $\tilde{X}$ which coincide with $G \ltimes_\rho \mathbb{R}^n$.

On $\tilde{X}$ globally $\omega = \sum_i d\theta_i \wedge dr_i$.

Let $h = (g, v) \in G \ltimes_{\rho^*} (\mathbb{R}^n)^*$. Let $\mathcal{L}_g$ be the $n \times n$ matrix representing $(L_g)_*$ in the global frame $\frac{\partial}{\partial r_1}, \dots, \frac{\partial}{\partial r_n}$ of $G = \tilde{B}$. Now from the definition of the group law on $G \ltimes_{\rho^*} (\mathbb{R}^n)^*$ one gets $L_h^*(dr_i) = \sum_j (\mathcal{L}_g)^{-1}_{ji} dr_j$ and $L_h^*(d\theta_i) = \sum_j (\mathcal{L}_g)_{ij} d\theta_j$. From this the result immediately follows since the matrix representing L_h^* w.r.t. $dr_1, \dots, dr_n, d\theta_1, \dots, d\theta_n$ is of the kind $\begin{bmatrix} (A^T)^{-1} & 0 \\ 0 & A \end{bmatrix}$ hence symplectic. □

Dualizing each ingredient we get a left-invariant holomorphic structure on $TG = G \ltimes_\rho \mathbb{R}^n$ hence on $\check{X} = (G \ltimes_\rho \mathbb{R}^n)/(\Gamma \ltimes_\rho \mathbb{Z}^n)$.

Proposition 11 *The tangent bundle $TG = G \ltimes_\rho \mathbb{R}^n$ has a canonical left-invariant integrable complex structure.*

Proof On TG we have the global coordinates $r_1, \dots, r_n, \check{\theta}_1, \dots, \check{\theta}_n$ where the $\check{\theta}_k$'s are the dual coordinates corresponding to the θ_k's. The coordinates $z_k = \check{\theta}_k + i r_k$, $k = 1, \dots, n$ give TG an integrable complex structure. Now we prove that the complex volume $\check{\Omega} = \bigwedge_{k=1}^n dz_k$ is left-invariant on $G \ltimes_\rho \mathbb{R}^n$.

Let $\check{h} = (g, \check{v}) \in G \ltimes_\rho \mathbb{R}^n$. From the definition of the group law on $G \ltimes_{\rho^*} \mathbb{R}^n$ one gets $L_h^*(d\check{\theta}_i) = \sum_j (\mathcal{L}_g)^{-1}_{ji} d\check{\theta}_j$. Thus the matrix representing L_h^* w.r.t. $dr_1, \dots, dr_n, d\check{\theta}_1, \dots, d\check{\theta}_n$ is of the kind $\begin{bmatrix} (A^T)^{-1} & 0 \\ 0 & (A^T)^{-1} \end{bmatrix}$ hence complex. Moreover this lies in $\mathrm{SL}(n, \mathbb{C})$ since every complete affine structure on a compact solvmanifold is in fact *special*, that is the linear part of the transition functions lies indeed in $\mathrm{SL}(n, \mathbb{Z})$ (see [17, Theorem 2]). □

The left-invariant symplectic structure on X and the left-invariant complex structure on $\check{X}$ defined above only depend on the affine structure and not on the choice of D. Though the choice of the developing map D allows us also to define distinguished left-invariant $\mathrm{SU}(n)$-structures on X and $\check{X}$.

Let us start from the symplectic side X. For $i = 1 \dots, n$ define e_i to be the global left-invariant 1-form on $T^*G = G \ltimes_{\rho^*} (\mathbb{R}^n)^*$ such that $e_{k|e} = d\theta_{k|e}$ and $e_{k+n|e} = dr_{k|e}$. Note that in these coordinates the canonical symplectic form is $\omega = \sum_k e_k \wedge e_{n+k}$. Now set $\Omega := \bigwedge_{k=1}^n (e_k + i e_{k+n})$. Clearly (ω, Ω) defines a symplectic $\mathrm{SU}(n)$-structure on X.

The construction is analogous on the complex side. For $i = 1 \dots, n$ define $\check{e}_i$ to be the global left-invariant 1-form on $TG = G \ltimes_\rho \mathbb{R}^n$ such that $\check{e}_{k|e} = d\check{\theta}_{k|e}$ and $\check{e}_{k+n|e} = dr_{k|e}$. Note that in these coordinates the canonical complex n-form takes the expression $\check{\Omega} := \bigwedge_{k=1}^n (\check{e}_k + i\check{e}_{k+n})$.

Now set $\check{\omega} := \sum_{k=1}^n (\check{e}_k \wedge \check{e}_{k+n})$. Clearly $(\check{\omega}, \check{\Omega})$ defines a complex $\mathrm{SU}(n)$-structure on $\check{X}$.

In dimension 3 we have the following lemma that gives the link with Theorem 7.

Lemma 12 *Let (X, ω) and $(\check{X}, \check{\Omega})$ be the 3-dimensional semi-flat mirror pair induced by (G, Γ, D). Let $\check{\omega}$ be the $(1, 1)$ form on $\check{X}$ as above. Then the form Ω defined above is the Fourier-Mukai transform of $e^{2\check{\omega}}$.*

Proof It is enough to express the two relevant forms in action-angle coordinates. First set $S = \mathcal{L}^{-T}\mathcal{L}^{-1}$ and $\eta_j = \sum_k S_{jk} dr_k$.

Let us also introduce the following basis of $(1, 0)$-forms $\psi^k = \check{e}^k + i\check{e}^{k+3}$. They are related to the differential of complex coordinates via $\psi^k = \sum_{k=1}^3 \mathcal{L}^{-1}_{kj} dz_j = \sum_{k=1}^3 \mathcal{L}^{-1}_{kj} (d\check{\theta}_j + i dr_j)$. Then we have

$$\begin{aligned} \check{\omega} &= \frac{i}{2} \sum_{i=1}^3 \psi^{k\bar{k}} = \frac{i}{2} \sum_{k=1}^3 \left(dz_k \wedge S_{kj} d\bar{z}_j \right) = \sum_{k=1}^3 d\check{\theta}_k \wedge \eta_k \\ \Omega &= \bigwedge_k (d\theta_k + i\eta_k) \end{aligned}$$

Now we can do the following straightforward computation

$$\begin{aligned} FT(e^{2\check{\omega}}) &= p_* \left(p^* \left(\mathcal{P} \cdot e^{2\check{\omega}} \right) \wedge e^{\frac{F}{2i}} \right) = p_* \left(e^{i \sum_{k=1}^3 d\check{\theta}_k \wedge \eta_k} \wedge e^{\sum_{k=1}^3 d\check{\theta}_k \wedge d\theta_k} \right) \\ &= p_* \left(e^{\sum_{k=1}^3 d\check{\theta}_k \wedge (d\theta_k + i\eta_k)} \right) = p_* \left(\bigwedge_{k=1}^3 d\check{\theta}_k \wedge (d\theta_k + i\eta_k) \right) \\ &= \bigwedge_{k=1}^3 (d\theta_k + i\eta_k) = \Omega\,. \end{aligned}$$

□

5 Affine 3-dimensional solvmanifolds

Up to isomorphisms there are only four simply connected unimodular solvable Lie groups of dimension 3 (see [4]):

- The abelian Lie group $(\mathbb{R}^3, +)$;
- The 3-dimensional real Heisenberg group $\mathcal{H}_3(\mathbb{R})$, that is the group of upper uni-triangular 3-by-3 real matrices.
- $\mathrm{Sol}_3 = E(1, 1)$: The group of rigid motions of the Minkowski plane. Explicitly it is $\mathbb{R}^3$ with the product

  $$(x, y, z)\star(x', y', z') = (x + e^z x', y + e^{-z} y', z + z').$$

  The group may also be seen as $\mathbb{R} \ltimes_\mu \mathbb{R}^2$ where $\mu(z)(x', y') = (e^z x', e^{-z} y')$. This is *completely solvable*.
  A matrix representation is the following

  $$\begin{pmatrix} e^z & 0 & 0 & x \\ 0 & e^{-z} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- $\widetilde{E(2)}$: The universal cover of the group of rigid motions of the Euclidean plane. Explicitly it is $\mathbb{R}^3$ with the product

  $$(x, y, z)\star(x', y', z') = (x + x' \cos z - y' \sin z, y + y' \cos z + x' \sin z, z + z').$$

  The group may also be seen as $\mathbb{R}\ltimes_\mu\mathbb{R}^2$ where $\mu(z)(x', y') = (x' \cos z - y' \sin z, y' \cos z + x' \sin z)$. This is *non completely solvable*.

The lattices of such solvable groups are classified up to conjugacy in [4].

Complete left-invariant affine structures on 3-dimensional simply connected unimodular solvable Lie groups are classified in [16].

Now we present the complete left-invariant affine structures giving rise to type IIA completely solvable solvmanifolds.

5.1 $(\mathbb{R}^3, +)$

The standard trivial affine structure of $\mathbb{R}^3$, together with the standard lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$ gives rise, via the construction of section 4, to the trivial flat SU(3)- structure on the 6-dimensional torus T^6. However it is possible to twist the affine structure of $\mathbb{R}^3$ to get a non-trivial compact type IIA manifold.

Consider the affine structure $\mathcal{A}_{(\mathbb{R}^3,\bowtie)}$ given by the following developing map:

$$\mathrm{D} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 + x_1 x_2 \end{pmatrix} \tag{4}$$

The corresponding representation $\alpha : (\mathbb{R}^3, +) \to \mathrm{Aff}(\mathbb{R}^3)$ is given by

$$\alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & x_1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ x_3 + x_1 x_2 \end{pmatrix} \tag{5}$$

Choosing again the standard lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$ we get the following affine holonomy of $T^3 = \mathbb{R}^3/\mathbb{Z}^3$:

$$\alpha \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ n_2 & n_1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \\ n_3 + n_1 n_2 \end{pmatrix} \tag{6}$$

In the next section we will prove that this affine structure gives rise to the type IIA nilmanifold corresponding to the algebra $(0, 0, 0, 0, e^{12}, e^{13})$.

5.2 $\mathcal{H}_3(\mathbb{R})$

Consider first the developing map

$$\begin{aligned} \mathrm{D} : \mathcal{H}_3(\mathbb{R}) &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} &\longmapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned} \tag{7}$$

For $g = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$ we compute $\alpha = \mathrm{D} \circ L_g \circ \mathrm{D}^{-1}$:

$$\alpha(g)(v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x_1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{8}$$

Choosing the standard lattice $\mathcal{H}_3(\mathbb{Z}) \subset \mathcal{H}_3(\mathbb{R})$ of matrices with integral entries we get the following affine holonomy of the Heisenberg manifold $\mathcal{H}_3(\mathbb{R})/\mathcal{H}_3(\mathbb{Z})$:

$$\alpha \begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & n_1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}. \tag{9}$$

In the next section we will prove that this affine structure, denoted with $\mathcal{A}_{(\mathcal{H}_3(\mathbb{R}),0)}$, gives rise to the type IIA nilmanifold corresponding again to the algebra $(0, 0, 0, 0, e^{12}, e^{13})$, but with a choice of Lagrangian fibration different from the one obtained from the twisted affine structure on the torus.

Consider now the following family of developing maps parametrised by $\lambda \in \mathbb{R} \setminus \{0, 1\}$:

$$\begin{aligned} D : \mathcal{H}_3(\mathbb{R}) &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} &\longmapsto \begin{pmatrix} x_1 \\ \lambda x_2 \\ (\lambda - 1)x_3 + x_1 x_2 \end{pmatrix} \end{aligned} \tag{10}$$

In this case for the affine representation α we get:

$$\alpha(g)(v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & x_1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} x_1 \\ \lambda x_2 \\ (\lambda - 1)x_3 + x_1 x_2 \end{pmatrix} \tag{11}$$

Choosing again the standard lattice $\mathcal{H}_3(\mathbb{Z}) \subset \mathcal{H}_3(\mathbb{R})$ we get the following affine holonomy of the Heisenberg manifold $\mathcal{H}_3(\mathbb{R})/\mathcal{H}_3(\mathbb{Z})$:

$$\alpha \begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ n_2 & n_1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} n_1 \\ \lambda n_2 \\ (\lambda - 1) n_3 + n_1 n_2 \end{pmatrix} \tag{12}$$

In the next section we will prove that this family of affine structures, denoted with $\mathcal{A}_{(\mathcal{H}_3(\mathbb{R}),\lambda)}$, gives rise to three inequivalent type IIA nilmanifolds, all of them with underlying algebra $(0, 0, 0, e^{12}, e^{13}, e^{23})$.

5.3 $E(1, 1)$

Choose as developing map

$$\begin{aligned} D : E(1,1) &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} e^{x_1} & 0 & 0 & x_2 \\ 0 & e^{-x_1} & 0 & x_3 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} &\longmapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned} \tag{13}$$

Thus for $g = \begin{pmatrix} e^{x_1} & 0 & 0 & x_2 \\ 0 & e^{-x_1} & 0 & x_3 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$ we have

$$\alpha(g)(v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{x_1} & 0 \\ 0 & 0 & e^{-x_1} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{14}$$

Let t be a real number such that $e^t + e^{-t}$ is an integer bigger than 2. We call Γ_t the subgroup of $E(1, 1)$ made by the elements of the form

$$\gamma = \begin{pmatrix} e^{tn_1} & 0 & 0 & n_2 + e^t n_3 \\ 0 & e^{-tn_1} & 0 & n_2 + e^{-t} n_3 \\ 0 & 0 & 1 & tn_1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with $n_1, n_2, n_3 \in \mathbb{Z}$. It is easy to verify that Γ_t is a lattice of $E(1, 1)$. If we compute again the integral affine representation $\alpha(\gamma)(v)$ we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{tn_1} & 0 \\ 0 & 0 & e^{-tn_1} \end{pmatrix}$$

as linear part which does not lie in $\mathrm{GL}(3, \mathbb{Z})$. Nevertheless it is conjugate to an element of $\mathrm{GL}(3, \mathbb{Z})$ as the following identity shows

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & 1 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & e^t + e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & 1 & e^{-t} \end{pmatrix}^{-1}. \tag{15}$$

Therefore, though the linear part has not integer entries, it represents an automorphism of the lattice

$$\Xi_t = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\e^t\\e^{-t}\end{pmatrix} \right\rangle_{\mathbb{Z}}$$

inside $\mathbb{R}^3$. This is exactly the situation described in Remark 9.

In the next section we will prove that this affine structure, denoted with $\mathcal{A}_{(E(1,1),0)}$, gives rise to the type IIA solvmanifold corresponding to the algebra (15, −25, −35, 45, 0, 0) (see [14]).

5.3.1 Twisted developing map for $E(1, 1)$

Take now as developing map

$$\begin{aligned} D : E(1,1) &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} e^{x_1} & 0 & 0 & x_2 \\ 0 & e^{-x_1} & 0 & x_3 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} &\longmapsto \begin{pmatrix} x_1 + x_2 x_3 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned} \tag{16}$$

Again we compute

$$\alpha(g)(v) = \begin{pmatrix} 1 & x_3 e^{x_1} & x_2 e^{-x_1} \\ 0 & e^{x_1} & 0 \\ 0 & 0 & e^{-x_1} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} x_1 + x_2 x_3 \\ x_2 \\ x_3 \end{pmatrix} \tag{17}$$

Take $\gamma \in \Gamma_t$. If we compute $\alpha(\gamma)(v)$ we obtain as linear part

$$\begin{pmatrix} 1 & e^{tn_1}(n_2 + e^{-t} n_3) & e^{-tn_1}(n_2 + e^t n_3) \\ 0 & e^{tn_1} & 0 \\ 0 & 0 & e^{-tn_1} \end{pmatrix} \tag{18}$$

which, again, does not lie in $\mathrm{GL}(3, \mathbb{Z})$. Nevertheless the following identities on the generators of Γ_t

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & 1 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & e^t + e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & 1 & e^{-t} \end{pmatrix}^{-1} \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & 1 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 2 & e^t + e^{-t} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & 1 & e^{-t} \end{pmatrix}^{-1} \\ \begin{pmatrix} 1 & e^t & e^{-t} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & 1 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & e^t + e^{-t} & e^{2t} + e^{-2t} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^t \\ 0 & 1 & e^{-t} \end{pmatrix}^{-1} \end{aligned} \tag{19}$$

show that we can interpret the matrix (18) as an automorphism of the lattice Ξ_t.

In the next section we will prove that this affine structure, denoted with $\mathcal{A}_{(E(1,1),\bowtie)}$, gives rise to the type IIA solvmanifold corresponding to the algebra (16 + 35, −25 + 45, 36, −46, 0, 0) (see [14]).

Remark 13 For the purpose of this paper we just considered the completely solvable case, nevertheless the construction of section 4 can be successfully applied to the complete left-invariant affine structures of $E(2)$. This will be the subject of a forthcoming paper.

6 Semi-flat six-dimensional mirror pairs

In this section we apply Theorem 8 to build the six-dimensional Lie groups $G(\mathcal{A}) = G \ltimes_{\rho^*} (\mathbb{R}^3)^*$ and $\check{G}(\mathcal{A}) = G \ltimes_\rho \mathbb{R}^3$ where $\mathcal{A}$ is one of the affine structures presented in section 5. We describe the group law and its Lie (co)algebra. Also we relate the algebras obtained with the ones from the various classifications.

Note that we recover all of the type IIA completely solvable Lie algebras listed in table 1 except case 3. The corresponding Lie group (which is isomorphic to $E(1,1) \times E(1,1)$) does not admit a semidirect product decomposition $G \ltimes \mathbb{R}^3$ giving rise to a *Lagrangian* fibration with respect to the relevant symplectic structure.

We will not consider

6.1 Twisted affine structure of $\mathbb{T}^3$

6.1.1 $G(\mathcal{A}_{(\mathbb{R}^3,\bowtie)})$

The six-dimensional Lie group $G(\mathcal{A}_{(\mathbb{R}^3,\bowtie)})$ associated to the twisted affine structure of the abelian $\mathbb{R}^3$ is $\mathbb{R}^6$ with the multiplication

$$
\begin{aligned}
&(x_1, x_2, x_3, y_1, y_2, y_3)(x_1', x_2', x_3', y_1', y_2', y_3') \\
&\quad = (x_1 + x_1', x_2 + x_2', x_3 + x_3', y_1 + y_1' - x_2 y_3', y_2 + y_2' - x_1 y_3', y_3 + y_3')
\end{aligned}
$$

which gives the following basis of left-invariant 1-forms

$$
\begin{aligned}
&e^1 = dy_1 + x_2 dy_3, \quad e^2 = dy_2 + x_1 dy_3, \quad e^3 = dy_3 \\
&e^4 = dx_1, \quad e^5 = dx_2, \quad e^6 = dx_3
\end{aligned} \tag{20}
$$

with

$$
\begin{aligned}
&de^1 = -e^{35}, \quad de^2 = -e^{34}, \quad de^3 = 0 \\
&de^4 = 0, \quad de^5 = 0, \quad de^6 = 0
\end{aligned} \tag{21}
$$

The algebra obtained is isomorphic to (0, 0, 0, 0, 12, 13), see table 1.

The action-angle coordinates are

$$
\begin{cases} r_1 = x_1 \\ r_2 = x_2 \\ r_3 = x_3 + x_1 x_2 \end{cases}, \quad \begin{cases} \theta_1 = y_1 \\ \theta_2 = y_2 \\ \theta_3 = y_3 \end{cases} \tag{22}
$$

In these coordinates the coframe of left-invariant 1-forms rewrites as

$$\begin{aligned} e^1 &= d\theta_1 + r_2 d\theta_3 \\ e^2 &= d\theta_2 + r_1 d\theta_3 \\ e^3 &= d\theta_3 \\ e^4 &= dr_1 \\ e^5 &= dr_2 \\ e^6 &= dr_3 - r_2 dr_1 - r_1 dr_2 \end{aligned} \tag{23}$$

The induced left-invariant symplectic structure is $\omega = e^{14} + e^{25} + e^{36} = \sum_{i=1}^{3} d\theta_i \wedge dr_i$. Now consider the distinguished 3-form

$$\Omega = (e^1 + ie^4) \wedge (e^2 + ie^5) \wedge (e^3 + ie^6)$$

induced by the choice of the developing map.

One easily verifies that $d\,\mathrm{Re}\,\Omega = 0$ and this indeed corresponds to case 1 in table 1.

6.1.2 $\check{G}(\mathcal{A}_{(\mathbb{R}^3,\bowtie)})$

The dual six-dimensional Lie group $\check{G}(\mathcal{A}_{(\mathbb{R}^3,\bowtie)})$ associated to the twisted affine structure of the abelian $\mathbb{R}^3$ is $\mathbb{R}^6$ with the following multiplication

$$\begin{aligned} &(x_1, x_2, x_3, \check{y}_1, \check{y}_2, \check{y}_3)(x_1', x_2', x_3', \check{y}_1', \check{y}_2', \check{y}_3') \\ &\quad = (x_1 + x_1', x_2 + x_2', x_3 + x_3', \check{y}_1 + \check{y}_1', \check{y}_2 + \check{y}_2', \check{y}_3 + \check{y}_3' + x_2\check{y}_1' + x_1\check{y}_2') \end{aligned}$$

which gives the following basis of left-invariant 1-forms

$$\begin{aligned} &\check{e}^1 = d\check{y}_1, \quad \check{e}^2 = d\check{y}_2, \quad \check{e}^3 = d\check{y}_3 - x_1 d\check{y}_2 - x_2 d\check{y}_1 \\ &\check{e}^4 = dx_1, \quad \check{e}^5 = dx_2, \quad \check{e}^6 = dx_3 \end{aligned} \tag{24}$$

with differentials

$$\begin{aligned} &d\check{e}^1 = 0, \quad d\check{e}^2 = 0, \quad d\check{e}^3 = \check{e}^{24} + \check{e}^{15} \\ &d\check{e}^4 = 0, \quad d\check{e}^5 = 0, \quad d\check{e}^6 = 0 \end{aligned} \tag{25}$$

The dual action-angle coordinates are

$$\begin{cases} r_1 = x_1 \\ r_2 = x_2 \\ r_3 = x_3 + x_1 x_2 \end{cases}, \quad \begin{cases} \check{\theta}_1 = \check{y}_1 \\ \check{\theta}_2 = \check{y}_2 \\ \check{\theta}_3 = \check{y}_3 \end{cases} \tag{26}$$

In these coordinates the coframe of left-invariant 1-forms rewrites as

$$\begin{aligned} \check{e}^1 &= d\check{\theta}_1 \\ \check{e}^2 &= d\check{\theta}_2 \\ \check{e}^3 &= d\theta_3 - r_2 d\check{\theta}_1 - r_1 d\check{\theta}_2 \\ \check{e}^4 &= dr_1 \\ \check{e}^5 &= dr_2 \\ \check{e}^6 &= dr_3 - r_2 dr_1 - r_1 dr_2 \end{aligned} \tag{27}$$

The induced left-invariant complex structure is induced by $\check{\Omega} = \left(\check{e}^1 + i\check{e}^4\right) \wedge \left(\check{e}^2 + i\check{e}^5\right) \wedge \left(\check{e}^3 + i\check{e}^6\right) = \bigwedge_{k=1}^{3} \left(d\check{\theta}_k + i dr_k\right)$.

Now consider the distinguished 2-form

$$\check{\omega} = \check{e}^{14} + \check{e}^{25} + \check{e}^{36}$$

One easily verifies that $d\check{\omega}^2 = 0$. The algebra obtained is listed as $\mathfrak{h}_3 = (0, 0, 0, 0, 0, 12+34)$ in table 1 of [20].

6.2 Untwisted affine structure of $\mathcal{H}_3(\mathbb{R})$

6.2.1 $G(\mathcal{A}_{(\mathcal{H}_3(\mathbb{R}),0)})$

The six-dimensional Lie group $G(\mathcal{A}_{(\mathcal{H}_3(\mathbb{R}),0)})$ associated to the untwisted affine structure of the Heisenberg group $\mathcal{H}_3(\mathbb{R})$ is $\mathbb{R}^6$ with the following multiplication

$$\begin{aligned}
&(x_1, x_2, x_3, y_1, y_2, y_3)(x_1', x_2', x_3', y_1', y_2', y_3') \\
&\quad = (x_1 + x_1', x_2 + x_2', x_3 + x_3' + x_1 x_2', y_1 + y_1', y_2 + y_2' - x_1 y_3', y_3 + y_3')
\end{aligned}$$

which gives the following basis of left-invariant 1-forms

$$\begin{aligned}
&e^1 = dy_1, \quad e^2 = dy_2 + x_1 dy_3, \quad e^3 = dy_3 \\
&e^4 = dx_1, \quad e^5 = dx_2, \quad e^6 = dx_3 - x_1 dx_2
\end{aligned} \tag{28}$$

with

$$\begin{aligned}
&de^1 = 0, \quad de^2 = -e^{34}, \quad de^3 = 0 \\
&de^4 = 0, \quad de^5 = 0, \quad de^6 = -e^{45}.
\end{aligned} \tag{29}$$

The algebra obtained is isomorphic to (0, 0, 0, 0, 12, 13), see table 1.

The action-angle coordinates are

$$\begin{cases} r_1 = x_1 \\ r_2 = x_2 \\ r_3 = x_3 \end{cases}, \quad \begin{cases} \theta_1 = y_1 \\ \theta_2 = y_2 \\ \theta_3 = y_3 \end{cases} \tag{30}$$

In these coordinates the coframe of left-invariant 1-forms rewrites as

$$\begin{aligned}
e^1 &= d\theta_1 \\
e^2 &= d\theta_2 + r_1 d\theta_3 \\
e^3 &= d\theta_3 \\
e^4 &= dr_1 \\
e^5 &= dr_2 \\
e^6 &= dr_3 - r_1 dr_2
\end{aligned} \tag{31}$$

The induced left-invariant symplectic structure is $\omega = e^{14} + e^{25} + e^{36} = \sum_{i=1}^{3} d\theta_i \wedge dr_i$.

Now consider the distinguished 3-form

$$\Omega = (e^1 + ie^4) \wedge (e^2 + ie^5) \wedge (e^3 + ie^6)$$

induced by the choice of the developing map.

One easily verifies that $d\,\mathrm{Re}\,\Omega = 0$ and this again corresponds to case 1 in table 1.

As type IIA manifolds $G(\mathcal{A}_{(\mathcal{H}_3(\mathbb{R}),0)})$ and $G(\mathcal{A}_{(\mathbb{R}^3,\bowtie)})$ are equivariantly isomorphic but the Lagrangian fibrations are different. In other words the relevant compact six-dimensional type IIA manifold X admits two inequivalent Lagrangian torus fibrations giving rise to two non-isomorphic semi-flat mirror pairs. This is also reflected in the refined Tseng-Yau cohomology of the two cases.

We will see in the next subsection that the two mirror complex partners are even not diffeomorphic.

6.2.2 $\check{G}(\mathcal{A}_{(\mathcal{H}_3(\mathbb{R}),0)})$

The dual six-dimensional Lie group $\check{G}(\mathcal{A}_{(\mathcal{H}_3(\mathbb{R}),0)})$ associated to the untwisted affine structure of $\mathcal{H}_3(\mathbb{R})$ is $\mathbb{R}^6$ with the following multiplication

$$\begin{aligned}&(x_1, x_2, x_3, \check{y}_1, \check{y}_2, \check{y}_3)(x_1', x_2', x_3', \check{y}_1', \check{y}_2', \check{y}_3')\\&\quad = (x_1 + x_1', x_2 + x_2', x_3 + x_3' + x_1x_2', \check{y}_1 + \check{y}_1', \check{y}_2 + \check{y}_2', \check{y}_3 + \check{y}_3' + x_1\check{y}_2')\end{aligned}$$

which gives the following basis of left-invariant 1-forms

$$\begin{aligned}&\check{e}^1 = d\check{y}_1, \quad \check{e}^2 = d\check{y}_2, \quad \check{e}^3 = d\check{y}_3 - x_1 d\check{y}_2\\&\check{e}^4 = dx_1, \quad \check{e}^5 = dx_2, \quad \check{e}^6 = dx_3 - x_1 dx_2\end{aligned} \tag{32}$$

with

$$\begin{aligned}&d\check{e}^1 = 0, \quad d\check{e}^2 = 0, \quad d\check{e}^3 = \check{e}^{24}\\&d\check{e}^4 = 0, \quad d\check{e}^5 = 0, \quad d\check{e}^6 = -\check{e}^{45}\,.\end{aligned} \tag{33}$$

The dual action-angle coordinates are

$$\begin{cases} r_1 = x_1\\ r_2 = x_2\\ r_3 = x_3\end{cases}, \quad \begin{cases} \check{\theta}_1 = \check{y}_1\\ \check{\theta}_2 = \check{y}_2\\ \check{\theta}_3 = \check{y}_3\end{cases} \tag{34}$$

In these coordinates the coframe of left-invariant 1-forms rewrites as

$$\begin{aligned}\check{e}^1 &= d\check{\theta}_1\\ \check{e}^2 &= d\check{\theta}_2\\ \check{e}^3 &= d\check{\theta}_3 - r_1 d\check{\theta}_2\\ \check{e}^4 &= dr_1\\ \check{e}^5 &= dr_2\\ \check{e}^6 &= dr_3 - r_1 dr_2\end{aligned} \tag{35}$$

The left-invariant complex structure is induced by $\check{\Omega} = \left(\check{e}^1 + i\check{e}^4\right)\wedge\left(\check{e}^2 + i\check{e}^5\right)\wedge\left(\check{e}^3 + i\check{e}^6\right) = \bigwedge_{k=1}^3 \left(d\check{\theta}_k + i dr_k\right)$.

Now consider the distinguished 2-form

$$\check{\omega} = \check{e}^{14} + \check{e}^{25} + \check{e}^{36}\,.$$

One easily verifies that $d\check{\omega}^2 = 0$. The algebra obtained is listed as $\mathfrak{h}_6 = (0, 0, 0, 0, 12, 13)$ in table 1 of [20].

Remark 14 The mirror pair arising from this affine structure is special in many respects. First it corresponds to the only known six-dimensional example as presented in [21, section 7]. Here we additionally recognize that the total spaces T^*B/Λ^* and TB/Λ are both diffeomorphic to the nilmanifold G/Γ where G is the group of matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_2 & x_3 & 0 & 0 \\ 0 & 1 & x_4 & x_5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and Γ is the lattice given by the same matrices with integral entries.

Moreover as anticipated in subsection 6.2.1 $X = G/\Gamma$ provides an example of a type IIA manifold having two inequivalent torus Lagrangian fibrations with two different IIB partners.

6.3 Twisted affine structure of $\mathcal{H}_3(\mathbb{R})$

6.3.1 $G(\mathcal{A}_{(\mathcal{H}_3(\mathbb{R}),\bowtie,\lambda)})$

The six-dimensional Lie group $G(\mathcal{A}_{(\mathcal{H}_3(\mathbb{R}),\bowtie,\lambda)})$ associated to the twisted family of affine structures of the Heisenberg group $\mathcal{H}_3(\mathbb{R})$ is $\mathbb{R}^6$ with the following multiplication

$$(x_1, x_2, x_3, y_1, y_2, y_3)(x_1', x_2', x_3', y_1', y_2', y_3') = \\ (x_1 + x_1', x_2 + x_2', x_3 + x_3' + x_1x_2', y_1 + y_1' - x_2y_3', y_2 + y_2' - x_1y_3', y_3 + y_3')$$

A basis of left-invariant 1-forms is given by

$$\begin{aligned} f^1 &= dy_1 + x_2dy_3, \quad f^2 = dy_2 + x_1dy_3, \quad f^3 = dy_3 \\ f^4 &= dx_1, \quad f^5 = dx_2, \quad f^6 = dx_3 - x_1dx_2 \end{aligned} \tag{36}$$

with

$$\begin{aligned} df^1 &= -f^{35}, \quad df^2 = -f^{34}, \quad df^3 = 0 \\ df^4 &= 0, \quad df^5 = 0, \quad df^6 = -f^{45}. \end{aligned} \tag{37}$$

The algebra obtained is isomorphic to (0, 0, 0, 12, 13, 23), case 2 in table 1.

The action-angle coordinates are

$$\begin{cases} r_1 = x_1 \\ r_2 = \lambda x_2 \\ r_3 = (\lambda - 1)x_3 + x_1x_2 \end{cases}, \quad \begin{cases} \theta_1 = y_1 \\ \theta_2 = y_2 \\ \theta_3 = y_3 \end{cases} \tag{38}$$

In these coordinates the coframe of left-invariant 1-forms rewrites as

$$\begin{aligned} f^1 &= d\theta_1 + \frac{r_2}{\lambda} d\theta_3 \\ f^2 &= d\theta_2 + r_1 d\theta_3 \\ f^3 &= d\theta_3 \\ f^4 &= dr_1 \\ f^5 &= \frac{dr_2}{\lambda} \\ f^6 &= \frac{1}{\lambda - 1} dr_3 - \frac{r_2}{\lambda(\lambda-1)} dr_1 - \frac{r_1}{\lambda - 1} dr_2 \end{aligned} \tag{39}$$

The induced left-invariant symplectic structure is $\omega_\lambda = \sum_{i=1}^3 d\theta_i \wedge dr_i = f^{14} + \lambda f^{25} + (\lambda - 1) f^{36}$. Note that the frame $e_1, \ldots, e_6$ of section 4.1 is given by

$$\begin{aligned} e^1 &= f^1 \quad e^2 = f^2 \quad e^3 = f^3 \\ e^4 &= f^4 \quad e^5 = \lambda f^5 \quad e^6 = (\lambda - 1) f^6 \end{aligned}$$

One easily checks that the distinguished 3-form

$$\Omega_\lambda = (e^1 + ie^4) \wedge (e^2 + ie^5) \wedge (e^3 + ie^6)$$

induced by the choice of the developing map has closed real part for every $\lambda \in \mathbb{R} \setminus \{0, 1\}$.

This type IIA algebra indeed corresponds to case 2 in table 1 and appears for the first time in [11]. According to the value of the parameter λ we obtain non-equivalent IIA algebras, see the discussion in remark 15.

6.3.2 $\check{G}(\mathcal{A}_{(\mathcal{H}_3(\mathbb{R}),\bowtie,\lambda)})$

The dual six-dimensional Lie group $\check{G}(\mathcal{A}_{(\mathcal{H}_3(\mathbb{R}),\bowtie,\lambda)})$ associated to the twisted family of affine structures of the Heisenberg group $\mathcal{H}_3(\mathbb{R})$ is $\mathbb{R}^6$ with the following multiplication

$$\begin{aligned} &(x_1, x_2, x_3, \check{y}_1, \check{y}_2, \check{y}_3)(x_1', x_2', x_3', \check{y}_1', \check{y}_2', \check{y}_3') \\ &\quad = (x_1 + x_1', x_2 + x_2', x_3 + x_3' + x_1 x_2', \check{y}_1 + \check{y}_1', \check{y}_2 + \check{y}_2', \check{y}_3 + \check{y}_3' + x_1 \check{y}_2' + x_2 y_1') \end{aligned}$$

A basis of left-invariant 1-forms is given by

$$\begin{aligned} \check{f}^1 &= d\check{y}_1, \quad \check{f}^2 = d\check{y}_2, \quad \check{f}^3 = d\check{y}_3 - x_2 d\check{y}_1 - x_1 d\check{y}_2 \\ \check{f}^4 &= dx_1, \quad \check{f}^5 = dx_2, \quad \check{f}^6 = dx_3 - x_1 dx_2 \end{aligned} \tag{40}$$

with

$$\begin{aligned} d\check{f}^1 &= 0, \quad d\check{f}^2 = 0, \quad d\check{f}^3 = \check{f}^{24} + \check{f}^{15} \\ d\check{f}^4 &= 0, \quad d\check{f}^5 = 0, \quad d\check{f}^6 = -\check{e}^{45} \end{aligned} \tag{41}$$

The algebra obtained is isomorphic to $\mathfrak{h}_4 = (0, 0, 0, 0, 12, 14 + 23)$ in [20].

The dual action-angle coordinates are

$$\begin{cases} r_1 = x_1 \\ r_2 = \lambda x_2 \\ r_3 = (\lambda - 1) x_3 + x_1 x_2 \end{cases}, \qquad \begin{cases} \check{\theta}_1 = \check{y}_1 \\ \check{\theta}_2 = \check{y}_2 \\ \check{\theta}_3 = \check{y}_3 \end{cases} \tag{42}$$

In these coordinates the coframe of left-invariant 1-forms rewrites as

$$\begin{aligned}
\check{f}^1 &= d\check{\theta}_1 \\
\check{f}^2 &= d\check{\theta}_2 \\
\check{f}^3 &= d\check{\theta}_3 - \frac{r_2}{\lambda} d\check{\theta}_1 - r_1 d\check{\theta}_2 \\
\check{f}^4 &= dr_1 \\
\check{f}^5 &= \frac{dr_2}{\lambda} \\
\check{f}^6 &= \frac{1}{\lambda - 1} dr_3 - \frac{r_2}{\lambda(\lambda - 1)} dr_1 - \frac{r_1}{\lambda - 1} dr_2
\end{aligned} \tag{43}$$

The induced left-invariant complex structure is $\check{\Omega}_\lambda = \bigwedge_{k=1}^3 \left(d\check{\theta}_k + i dr_k\right) = \left(\check{f}^1 + i\check{f}^4\right) \wedge \left(\check{f}^2 + i\lambda\check{f}^5\right) \wedge \left(\check{f}^3 + i(\lambda-1)\check{f}^6\right)$. Again the frame $\check{e}_1, \ldots, \check{e}_6$ of section 4.1 is given by

$$\begin{aligned}
&\check{e}^1 = \check{f}^1 \quad \check{e}^2 = \check{f}^2 \quad \check{e}^3 = \check{f}^3 \\
&\check{e}^4 = \check{f}^4 \quad \check{e}^5 = \lambda\check{f}^5 \quad \check{e}^6 = (\lambda - 1)\check{f}^6
\end{aligned} \tag{44}$$

Now consider the distinguished 2-form

$$\check{\omega}_\lambda = \check{e}^{14} + \check{e}^{25} + \check{e}^{36}$$

One easily verifies that $d\check{\omega}_\lambda^2 = 0$.

Remark 15 According to the value of the parameter λ we obtain non-equivalent IIB algebras. More precisely the Bott-Chern numbers distinguish 3 different cases:

- $\lambda = -1$. This type IIB algebra is missing in the classification of [20].
- $\lambda = \frac{1}{2}, 2$.
- $\lambda \neq -1, \frac{1}{2}, 2$.

6.4 Untwisted affine structure of $E(1,1)$

6.4.1 $G(\mathcal{A}_{E(1,1),0})$

The six-dimensional Lie group $G(\mathcal{A}_{E(1,1),0})$ associated to the untwisted affine structure of the group $E(1,1)$ is $\mathbb{R}^6$ with the following multiplication

$$\begin{aligned}
&(x_1, x_2, x_3, y_1, y_2, y_3)(x_1', x_2', x_3', y_1', y_2', y_3') \\
&\quad = (x_1 + x_1', x_2 + e^{x_1}x_2', x_3 + e^{-x_1}x_3', y_1 + y_1', y_2 \\
&\quad\quad + e^{-x_1}y_2', y_3 + e^{x_1}y_3')
\end{aligned}$$

which gives the following basis of left-invariant 1-forms

$$\begin{aligned}
&e^1 = dy_1, \quad e^2 = e^{x_1}dy_2, \quad e^3 = e^{-x_1}dy_3 \\
&e^4 = dx_1, \quad e^5 = e^{-x_1}dx_2, \quad e^6 = e^{x_1}dx_3
\end{aligned} \tag{45}$$

with

$$\begin{aligned}
&de^1 = 0, \quad de^2 = -e^{24}, \quad de^3 = e^{34} \\
&de^4 = 0, \quad de^5 = -e^{45}, \quad de^6 = e^{46}
\end{aligned} \tag{46}$$

The algebra obtained is isomorphic to (15, −25, −35, 45, 0, 0), see table 1.

The action-angle coordinates are

$$\begin{cases} r_1 = x_1 \\ r_2 = x_2 \\ r_3 = x_3 \end{cases}, \quad \begin{cases} \theta_1 = y_1 \\ \theta_2 = y_2 \\ \theta_3 = y_3 \end{cases} \tag{47}$$

In these coordinates the coframe of left-invariant 1-forms rewrites as

$$\begin{aligned} e^1 &= d\theta_1 \\ e^2 &= e^{r_1} d\theta_2 \\ e^3 &= e^{-r_1} d\theta_3 \\ e^4 &= dr_1 \\ e^5 &= e^{-r_1} dr_2 \\ e^6 &= e^{r_1} dr_3 \end{aligned} \tag{48}$$

The induced left-invariant symplectic structure is $\omega = e^{14} + e^{25} + e^{36} = \sum_{i=1}^{3} d\theta_i \wedge dr_i$.

Now consider the distinguished 3-form

$$\Omega = (e^1 + ie^4) \wedge (e^2 + ie^5) \wedge (e^3 + ie^6)$$

induced by the choice of the developing map.

One easily verifies that $d\mathrm{Re}\,\Omega = 0$ and this indeed corresponds to case 4 in table 1

6.4.2 $\check{G}(\mathcal{A}_{(E(1,1),0)})$

The dual six-dimensional Lie group $\check{G}(\mathcal{A}_{E(1,1),0)})$ associated to the untwisted affine structure of the completely solvable $E(1,1)$ is $\mathbb{R}^6$ with the following multiplication

$$\begin{aligned} &(x_1, x_2, x_3, \check{y}_1, \check{y}_2, \check{y}_3)(x_1', x_2', x_3', \check{y}_1', \check{y}_2', \check{y}_3') \\ &\quad = (x_1 + x_1', x_2 + e^{x_1} x_2', x_3 + e^{-x_1} x_3', \check{y}_1 + \check{y}_1', \check{y}_2 \\ &\quad\ + e^{x_1} \check{y}_2', \check{y}_3 + e^{-x_1} \check{y}_3') \end{aligned}$$

which gives the following basis of left-invariant 1-forms

$$\begin{aligned} &\check{e}^1 = d\check{y}_1, \quad \check{e}^2 = e^{-x_1} d\check{y}_2, \quad \check{e}^3 = e^{x_1} d\check{y}_3 \\ &\check{e}^4 = dx_1, \quad \check{e}^5 = e^{-x_1} dx_2, \quad \check{e}^6 = e^{x_1} dx_3 \end{aligned} \tag{49}$$

with

$$\begin{aligned} &d\check{e}^1 = 0, \quad d\check{e}^2 = \check{e}^{24}, \quad d\check{e}^3 = -\check{e}^{34} \\ &d\check{e}^4 = 0, \quad d\check{e}^5 = -e^{45}, \quad d\check{e}^6 = \check{e}^{46} \end{aligned} \tag{50}$$

Note that the algebra obtained is again isomorphic to (15, −25, −35, 45, 0, 0).

The dual action-angle coordinates are

$$\begin{cases} r_1 = x_1 \\ r_2 = x_2 \\ r_3 = x_3 \end{cases}, \quad \begin{cases} \check{\theta}_1 = \check{y}_1 \\ \check{\theta}_2 = \check{y}_2 \\ \check{\theta}_3 = \check{y}_3 \end{cases} \tag{51}$$

In these coordinates the coframe of left-invariant 1-forms rewrites as

$$\begin{aligned} \check{e}^1 &= d\check{\theta}_1 \\ \check{e}^2 &= e^{-r_1} d\check{\theta}_2 \\ \check{e}^3 &= e^{r_1} d\check{\theta}_3 \\ \check{e}^4 &= dr_1 \\ \check{e}^5 &= e^{-r_1} dr_2 \\ \check{e}^6 &= e^{r_1} dr_3 \end{aligned} \tag{52}$$

The induced left-invariant complex structure is induced by $\check{\Omega} = \left(\check{e}^1 + i\check{e}^4\right) \wedge \left(\check{e}^2 + i\check{e}^5\right) \wedge \left(\check{e}^3 + i\check{e}^6\right) = \bigwedge_{k=1}^3 \left(d\check{\theta}_k + i dr_k\right)$.

Now consider the distinguished 2-form

$$\check{\omega} = \check{e}^{14} + \check{e}^{25} + \check{e}^{36}$$

One easily verifies that $d\check{\omega}^2 = 0$ and this case corresponds to $\mathfrak{g}_1$ in [15, Theorem 2.8].

6.5 Twisted affine structure of $E(1,1)$

6.5.1 $G(\mathcal{A}_{(E(1,1),\bowtie)})$

The six-dimensional Lie group $G(\mathcal{A}_{(E(1,1),\bowtie)})$ associated to the twisted affine structure of the group $E(1,1)$ is $\mathbb{R}^6$ with the following multiplication

$$\begin{aligned} &(x_1, x_2, x_3, y_1, y_2, y_3)(x_1', x_2', x_3', y_1', y_2', y_3') \\ &\quad = (x_1 + x_1', x_2 + e^{x_1} x_2', x_3 + e^{-x_1} x_3', y_1 + y_1', y_2 \\ &\qquad + e^{-x_1} y_2' - x_3 y_1', y_3 + e^{x_1} y_3' - x_2 y_1') \end{aligned}$$

which gives the following basis of left-invariant 1-forms

$$\begin{aligned} &e^1 = dy_1, \quad e^2 = e^{x_1} dy_2 + x_3 e^{x_1} dy_1, \quad e^3 = e^{-x_1} dy_3 + x_2 e^{-x_1} dy_1 \\ &e^4 = dx_1, \quad e^5 = e^{-x_1} dx_2, \quad e^6 = e^{x_1} dx_3 \end{aligned} \tag{53}$$

with

$$\begin{aligned} &de^1 = 0, \quad de^2 = -e^{24} - e^{16}, \quad de^3 = e^{34} - e^{15} \\ &de^4 = 0, \quad de^5 = -e^{45}, \quad de^6 = e^{46} \end{aligned} \tag{54}$$

The algebra obtained is isomorphic to $(16 + 35, -26 + 45, 36, -46, 0, 0)$, see table 1.

The action-angle coordinates are

$$\begin{cases} r_1 = x_1 + x_2 x_3 \\ r_2 = x_2 \\ r_3 = x_3 \end{cases}, \quad \begin{cases} \theta_1 = y_1 \\ \theta_2 = y_2 \\ \theta_3 = y_3 \end{cases} \tag{55}$$

In these coordinates the coframe of left-invariant 1-forms rewrites as

$$\begin{aligned} e^1 &= d\theta_1 \\ e^2 &= e^{r_1 - r_2 r_3} d\theta_2 + r_3 e^{r_3 - r_1 r_2} d\theta_1 \\ e^3 &= e^{-r_1 + r_2 r_3} d\theta_3 + r_2 e^{-r_3 + r_1 r_2} d\theta_1 \\ e^4 &= dr_1 - r_2 dr_1 - r_1 dr_2 \\ e^5 &= e^{-r_1 + r_2 r_3} dr_2 \\ e^6 &= e^{r_1 - r_2 r_3} dr_3 \end{aligned} \tag{56}$$

The induced left-invariant symplectic structure is $\omega = e^{14} + e^{25} + e^{36} = \sum_{i=1}^{3} d\theta_i \wedge dr_i$. Now the distinguished 3-form

$$\Omega = (e^1 + ie^4) \wedge (e^2 + ie^5) \wedge (e^3 + ie^6)$$

induced by the choice of the developing map has closed real part. This type IIA algebra indeed corresponds to case 7 in table 1 and appears for the first time in [28].

6.5.2 $\check{G}(\mathcal{A}_{(E(1,1),\bowtie)})$

The dual six-dimensional Lie group $\check{G}(\mathcal{A}_{(E(1,1),\bowtie)})$ associated to the twisted affine structure of the group $E(1,1)$ is $\mathbb{R}^6$ with the following multiplication

$$\begin{aligned} &(x_1, x_2, x_3, \check{y}_1, \check{y}_2, \check{y}_3)(x_1', x_2', x_3', \check{y}_1', \check{y}_2', \check{y}_3') \\ &\quad = (x_1 + x_1', x_2 + e^{x_1} x_2', x_3 + e^{-x_1} x_3', \check{y}_1 + \check{y}_1' + x_3 e^{x_1} y_2' \\ &\qquad + x_2 e^{-x_1} y_3', \check{y}_2 + e^{x_1} \check{y}_2', \check{y}_3 + e^{-x_1} \check{y}_3') \end{aligned}$$

which gives the following basis of left-invariant 1-forms

$$\begin{aligned} &\check{e}^1 = d\check{y}_1 - x_3 d\check{y}_2 - x_2 d\check{y}_3, \quad \check{e}^2 = e^{-x_1} d\check{y}_2, \quad \check{e}^3 = e^{x_1} d\check{y}_3 \\ &\check{e}^4 = dx_1, \quad \check{e}^5 = e^{-x_1} dx_2, \quad \check{e}^6 = e^{x_1} dx_3 \end{aligned} \tag{57}$$

with

$$\begin{aligned} &d\check{e}^1 = -\check{e}^{35} - \check{e}^{26}, \quad d\check{e}^2 = \check{e}^{24}, \quad d\check{e}^3 = -\check{e}^{34} \\ &d\check{e}^4 = 0, \quad d\check{e}^5 = -\check{e}^{45}, \quad d\check{e}^6 = \check{e}^{46} \end{aligned} \tag{58}$$

The dual action-angle coordinates are

$$\begin{cases} r_1 = x_1 + x_2 x_3 \\ r_2 = x_2 \\ r_3 = x_3 \end{cases}, \quad \begin{cases} \check{\theta}_1 = \check{y}_1 \\ \check{\theta}_2 = \check{y}_2 \\ \check{\theta}_3 = \check{y}_3 \end{cases} \tag{59}$$

In these coordinates the coframe of left-invariant 1-forms rewrites as

$$\begin{aligned} \check{e}^1 &= d\check{\theta}_1 - r_3 d\check{\theta}_2 - r_2 d\check{\theta}_3 \\ \check{e}^2 &= e^{-r_1 + r_2 r_3} d\check{\theta}_2 \\ \check{e}^3 &= e^{r_1 - r_2 r_3} d\check{\theta}_3 \\ \check{e}^4 &= dr_1 - r_2 dr_1 - r_1 dr_2 \\ \check{e}^5 &= e^{-r_1 + r_2 r_3} dr_2 \\ \check{e}^6 &= e^{r_1 - r_2 r_3} dr_3 \end{aligned} \tag{60}$$

Table 2 Mirror lie algebras

Aff. structure	Lie algebra	Mirror lie algebra	$h^{1,0}_{TY}/h^{2,0}_{BC}$	$h^{0,1}_{TY}/h^{3,1}_{BC}$	$h^{2,0}_{TY}/h^{1,0}_{BC}$	$h^{1,1}_{TY}/h^{2,1}_{BC}$	$h^{0,2}_{TY}/h^{3,2}_{BC}$	$h^{2,1}_{TY}/h^{1,1}_{BC}$	$h^{1,2}_{TY}/h^{3,2}_{BC}$
$\mathcal{A}_{(\mathbb{R}^3,\bowtie)}$	$(0, 0, 0, 0, e^{12}, e^{13})$	$(0, 0, 0, 0, 0, e^{12} + e^{34})$	1	3	2	6	3	4	7
$\mathcal{A}_{(\mathcal{H}_3(\mathbb{R}),0)}$	$(0, 0, 0, 0, e^{12}, e^{13})$	$(0, 0, 0, 0, e^{12}, e^{13})$	2	2	2	6	3	5	6
$\mathcal{A}_{(\mathcal{H}_3(\mathbb{R}),\bowtie,\lambda=-1)}$	$(0, 0, 0, e^{12}, e^{13}, e^{23})$	$(0, 0, 0, 0, e^{12}, e^{14} + e^{23})$	1	2	2	6	3	4	7
$\mathcal{A}_{(\mathcal{H}_3(\mathbb{R}),\bowtie,\lambda=\frac{1}{2},2)}$	$(0, 0, 0, e^{12}, e^{13}, e^{23})$	$(0, 0, 0, 0, e^{12}, e^{14} + e^{23})$	1	2	2	6	3	5	6
$\mathcal{A}_{(\mathcal{H}_3(\mathbb{R}),\bowtie,\lambda\neq-1,\frac{1}{2},2)}$	$(0, 0, 0, e^{12}, e^{13}, e^{23})$	$(0, 0, 0, 0, e^{12}, e^{14} + e^{23})$	1	2	2	6	3	4	6
$\mathcal{A}_{(E(1,1),0)}$	$(e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0)$	$(e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0)$	1	1	1	3	1	3	3
$\mathcal{A}_{(E(1,1),\bowtie)}$	$(e^{16} + e^{35}, -e^{26} + e^{45}, e^{36}, -e^{46}, 0, 0)$	$(e^{24} + e^{35}, e^{26}, e^{36}, -e^{46}, -e^{56}, 0)$	1	1	0	2	1	2	1

The left-invariant complex structure is induced by $\check{\Omega} = (\check{e}^1+i\check{e}^4)\wedge(\check{e}^2+i\check{e}^5)\wedge(\check{e}^3+i\check{e}^6) = \bigwedge_{k=1}^3 (d\check{\theta}_k + i dr_k)$.

Now consider the distinguished 2-form

$$\check{\omega} = \check{e}^{14} + \check{e}^{25} + \check{e}^{36}$$

One easily verifies that $d\check{\omega}^2 = 0$ and this case corresponds to $\mathfrak{g}_5$ in [15, Theorem 2.8].

7 Table of mirror pairs

In table 2 we present all the mirror pairs of solvable Lie algebras constructed in the previous section and the dimension of their (refined) Tseng-Yau and Bott-Chern cohomology groups. This computation is valid also for all the corresponding compact solvmanifolds except for the complex side of the pair arising from $\mathcal{A}_{(E(1,1),\bowtie)}$, see [1, 2] where also the complete solvability plays a role.

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