UNIVERSITÀ DEGLI STUDI DELL'AQUILA

Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica

Doctorate Course: Matematica e Modelli

XXXVIII Cycle

Ph.D. Thesis

Asymptotic analysis of different kinds of multi-agent systems with time-delayed coupling, non-universal interaction and leadership

MATH-03/A

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Academic Year 2024/2025

A Rebecca e Aurora
perchè sappiano che per loro tutto è possibile.

Acknowledgments

I would like to express my deepest gratitude to my Ph.D. supervisor, Cristina, for her helpfulness and professionalism, and for guiding me through this journey with patience and competence. Her constant encouragement always pushed me to do my best, and I am truly grateful for all the time she devoted to me. I thank Cristina for all the opportunities she has given me, which allowed me to grow both as a mathematician and as a person. I was fortunate to have a supervisor who genuinely cared about my education and my work.

Another important thank you goes to my co-supervisor, Young-Pil, for the work we did together, which gave me the opportunity to actively pursue international collaboration. These experiences not only enhanced my scientific development but also strengthened my ability to work in multidisciplinary and multicultural teams. Thank you for welcoming me as part of your group.

Many thanks to my family, whose continued support made it possible for me to reach this milestone, thank you from the bottom of my heart.

To my relatives, friends, and colleagues, thank you for always being close to me. A special thanks to Elisa, for all the experiences we shared during the Ph.D., and to Vanessa, who became a sister to me in another country. Thank you both for being a constant example and source of inspiration.

Finally, I would like to thank the referees and the committee for the time they devoted to reading my thesis and for being part of my Ph.D. defense.

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Abstract

This thesis aims to investigate the asymptotic behavior of solutions to opinion formation and flocking models. In particular, we study the effects of time delays, which represent either the reaction time of agents or the time required for information to propagate between them. Even arbitrarily small delays can induce instability phenomena in the system's dynamics, making the stability analysis of such models a crucial topic of study.
We establish consensus results for both the Hegselmann–Krause opinion formation model and the Cucker–Smale flocking model, examining several distinct scenarios. Specifically, we consider cases involving non-universal interactions between agents and communication failure. Furthermore, we extend our analysis to settings that include leadership structures and investigate the associated control problems arising in these frameworks.
In addition, we study a Kuramoto model with non-universal interactions and time-delay effects, for which we provide an exponential decay estimate toward synchronization.

Chapter 1

Introduction

This Thesis is devoted to analyzing different kinds of multi-agent systems and their controllability. An important feature often analyzed is the possible emergence of self-organization leading the group's agents to globally collective behaviors.
Here, we are interested in the celebrated Hegselmann-Krause model for opinion formation, originally proposed in [53]. Moreover, we study the second-order version of such a model, the Cucker-Smale model [36] describing flocking phenomena, and the Kuramoto model [58] for synchronization.
This research aims to study the asymptotic behaviour of interacting systems in different frameworks. In particular, we study systems where the *interaction between the particles is delayed*, due to the effects on process information, considering a response time in the model. This provides a natural extension of the models cited above.
Moreover, we consider a scenario of *non-universal interaction* and/or *lack of communication*. These are all assumptions suggested by real-world problems, like the presence of *leaders*, who can control the dynamics of the particles.
We pursue two main objectives: (i) to define a model representing a real-world situation, and (ii) to study the asymptotic behavior of the associated dynamical system. The latter is described by a system of ordinary differential equations modeling the interactions among a finite number of agents.

1.1 An Overview of Recent Results for Multi-agent Systems

Due to applications in various scientific fields, multi-agent systems have become, in recent years, a very attractive research topic. They naturally appear e.g. in biology [12, 17, 36], ecology [71], economics [17, 60], social sciences [6, 7, 8, 19, 39, 59, 66], physics [40, 73], control theory [2, 38, 43, 64, 65, 74], engineering and robotics [11, 37]. For other applications, see also [54, 57, 75].
In the context of multi-agent systems, we focused our study on the analysis of the well-known Hegselmann-Krause model, proposed by R. Hegselmann and U. Krause in 2002 to study the opinion formation in an interactive group [53]. They were interested in investigating different frameworks where the interaction led to a unique consensus, polarization, or fragmentation, through computer simulations. Since then, several generalizations have been proposed (see e.g. [6, 7, 13, 14, 20, 56]). Later on, the second-order version of this model was proposed by F. Cucker and S. Smale in 2007 to describe flocking phenomena, like schooling of fish, flocking of

birds, or swarming of bacteria [36].
Another well-known system that we describe is the Kuramoto model, first proposed by Y. Kuramoto to describe synchronization phenomena [58]. It provides a simple yet powerful framework for studying the emergence of synchronization in large ensembles of coupled oscillators. Over the years, the Kuramoto model has found applications in various fields, including neuroscience [10, 77] and power grids [26, 42], among others. Indeed, synchronization phenomena appear in nature, ranging from the rhythmic flashing of fireflies to the coordinated beating of the heart muscle cells. The mathematical tractability and versatility of the Kuramoto model make it a valuable tool for theoretical investigations as well as for interpreting experimental observations of synchronization phenomena.

One of the most natural extensions of this model concerns the analysis of the time-delayed interactions. Incorporating time-delayed mathematical formulations is crucial for capturing the dynamics of real-world systems accurately. Time delays are prevalent in various physical and biological systems, stemming from finite signal propagation speeds, processing times, or communication latencies. Neglecting these delays can lead to inaccurate predictions and overlook essential aspects of system behavior. Of course, the presence of a delay in the interaction can make the problem more difficult to analyze, since the delay, even if it is small, can destroy some geometric property of the system, as symmetry. In fact, in the Hegselmann-Krause model with positive and symmetric interaction, it is easier to show the convergence to a consensus due to this feature. Adding the delay in this system destroys the symmetric structure and, in turn, the asymptotic analysis requires a finer argument.
Opinion formation models in the presence of time delay effects have already been studied by several authors, see e.g. [28, 34, 35, 43, 50, 52, 63, 64]. Concerning the Cucker-Smale model, delayed interactions have also been considered in many papers, see e.g. [21, 24, 25, 27, 33, 44, 51, 67, 68, 70]. In most of these papers, a smallness assumption on the size of the time delay is assumed to prove the asymptotic consensus. However, very recently, Rodriguez Cartabia proved in [70] the asymptotic flocking for the Cucker-Smale model with constant time delay without assuming any restrictions on the time delay size. Moreover, a similar result was achieved by J. Haskovek in [50] for a consensus result for the Hegselmann-Krause model.

In this thesis, we present some recent results we obtained ([18, 22, 30, 31, 32]), in which generalizations of first and second-order alignment models like Hegselmann-Krause, Cucker-Smale, and Kuramoto models involving time delay effects are considered. In particular, extending some argument used in [50] and [70], we were able to prove an exponential decay estimate to consensus for the Hegselmann-Krause type model and to synchronization for the Kuramoto type model. Moreover, we proved the exhibition of flocking for a Cucker-Smale type model. All of these results are obtained without any smallness assumptions over the time delay size.

In Chapter 2, we are interested in describing the opinion dynamics of two different populations that interact thanks to the presence of a small group of leaders with a time-delayed coupling [32]. It is natural to assume that, while one can consider almost instantaneous the influence among agents in the same population, a certain time lag appears in the interaction among individuals of different populations. A more general model would include time delays (eventually smaller) also in the interactions within the same population. Here, for simplicity, we choose to consider delay effects only in the interactions among agents of different populations. An analogous analysis could also be performed in the more general situation with multiple delays. This framework can be translated into a network topology over the structure of the model: we do not have a universal interaction between the agents, since a follower of the first population does not interact directly with a follower of the second population. To deal with the network structure, we modify the mathematical tools used in [70]. We adapt the step-by-step argument in [50], using the connection of the graph.

In Chapter 3, inspired by this result, we present a Kuramoto-type model with non-universal interaction and time-delayed coupling [18], to establish synchronization. First, we show a uniform-in-time bound estimate of the phase diameter. In particular, we will prove that the diameter is bounded by a constant lower than $\pi/2$ for sufficiently large times. Then, we can use such a bound to deduce an asymptotic synchronization result. Moreover, we consider the case of an all-to-all connection, i.e., the oscillators are all connected. Thanks to the larger number of connections between the agents, we can obtain a stronger result, namely an exponential asymptotic synchronization estimate.

In Chapter 4, trying to relax the hypothesis of a strong connection on the graph associated with the system, in [30] we analyze a situation in which two agents in the same population always have a *common influencer*. In this contest, we considered different frameworks with leadership and, in particular, we considered the case of a leader with a controlled trajectory.

A possible scenario that can also occur in the analysis of such models is the one in which the system's particles sometimes suspend the interactions they have with the agents they are linked to. As a consequence, there is a temporary lack of connection between the system's elements that, of course, hinders the convergence to consensus, for the first-order model, or the flocking for the second-order one. Then, it is important to find conditions guaranteeing the system's alignment. In [9], the convergence to consensus and the asymptotic flocking for a class of Cucker-Smale systems under communication failures, namely with interaction weights possibly degenerating among the system's agents, have been proved under suitable assumptions in the case of symmetric interaction coefficients. The convergence to consensus for a first-order alignment system involving weights depending on the couple of agents that can eventually degenerate has also been proved in [3] under the so-called *Persistence Excitation Condition.* In the case of nonsymmetric interaction coefficients, the exponential convergence to consensus for the Hegselmann-Krause model with time delay and possible communication failures has been obtained in [34].
In Chapter 5, we investigate another generalization of the Hegselmann-Krause model, in which we consider, other than time-delayed interaction and graph topology, also a lack of communication [31]. We establish the convergence to consensus for this model. We complete the study considering the same framework for the second-order model, proving the exhibition of flocking.

In Chapter 6, we consider a time-delayed variant of the Hegselmann-Krause opinion formation model featuring a small group of leaders and a large group of non-leaders [22]. In this model, leaders influence all agents but only interact among themselves. At the same time, non-leaders update their opinions via interactions with their peers and the leaders, with time delays accounting for communication and decision-making lags. We prove the exponential convergence to consensus of the particle system, without imposing smallness assumptions on the delay parameters. Furthermore, we analyze the mean-field limit in two regimes: (i) with a fixed number of leaders and an infinite number of non-leaders, and (ii) with both populations tending to infinity, obtaining existence, uniqueness, and exponential decay estimates for the corresponding macroscopic models. Moreover, we analyze the stability of the mean-field system, which rigorously establishes the mean-field limit procedure.

For well-posedness results for alignment models in the presence of time delay effects, we refer to classical texts on functional differential equations [48, 49]. Here, we will focus on the asymptotic behavior of the solutions.

1.2 The Hegselmann-Krause model

From now on, we shall denote with $|\cdot|$ and $\langle\cdot,\cdot\rangle$ the usual norm and scalar product in $\mathbb{R}^d$, respectively, and let us denote with $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. Consider a finite set of $N \in \mathbb{N}$ agents, with $N \geq 2$. Let $x_i(t) \in \mathbb{R}^d$ be the opinion of the $i-$th agent at time t. Then, the classical Hegselmann-Krause reads as follows:

$$\frac{d}{dt}x_i(t) = \sum_{j:\ j\neq i} a_{ij}(t)(x_j(t) - x_i(t)), \quad t \geq 0,\ i = 1,\dots,N. \tag{1.2.1}$$

Typically, the weights $a_{ij}(t)$ are defined as

$$a_{ij}(t) := \frac{1}{N-1}\psi(|x_j(t) - x_i(t)|), \tag{1.2.2}$$

where we call $\psi : \mathbb{R} \to \mathbb{R}$ *influence function* and is a nonnegative continuous function depending on the distance between two agents. In the classical framework, we require that the influence function is nonincreasing. In this way, each particle can influence only a group of particles within a certain radius of confidence.
However, in the following discussion, we can deal with more general weights, defined as

$$a_{ij}(t) := \frac{1}{N-1}\psi(x_i(t), x_j(t)), \tag{1.2.3}$$

where $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a continuous, positive and bounded function, with

$$K :=\| \psi \|_\infty . \tag{1.2.4}$$

So, the influence function does not depend anymore on the distance between the states, but could be a generic function of the opinions. Moreover, the monotonicity assumption is no longer required. This implies that a larger class of influence functions is included in our analysis, like the oscillatory or Gaussian ones.
As we specified above, in this thesis, we deal with different types of the Hegselmann-Krause opinion formation model. In the more general framework, we describe the Hegselmann-Krause model with time-delayed coupling as follows:

$$\frac{d}{dt}x_i(t) = \sum_{j:\ j\neq i} a_{ij}(t)(x_j(t-\tau(t)) - x_i(t)), \quad t \geq 0,\ i = 1,\dots,N, \tag{1.2.5}$$

with the communication rates given by

$$a_{ij}(t) := \frac{1}{N-1}\psi(x_i(t), x_j(t-\tau(t))), \tag{1.2.6}$$

and a continuous and time variable time-delay function $\tau : [0, +\infty) \to [0, +\infty)$, such that

$$0 \le \tau(t) \le \bar{\tau}, \quad \forall\, t \ge 0, \tag{1.2.7}$$

for some positive constant $\bar{\tau}$. In our studies, we consider constant time delays or state-dependent time delays too. The general case with the time-dependent function $\tau(t)$ is in [31].
The most general case we consider in this thesis is a Hegselmann-Krause type model with pair-dependent time-variable time delay, with non-universal interaction and communication failure. The system reads as

$$\frac{d}{dt}x_i(t) = \sum_{j:\ j \ne i} \chi_{ij}\alpha_{ij}(t)a_{ij}(t)(x_j(t-\tau_{ij}(t)) - x_i(t)), \quad t \ge 0,\ i = 1, \dots, N, \tag{1.2.8}$$

where the time delay function $\tau_{ij} : [0, +\infty) \to [0, +\infty)$ is continuous and satisfy (1.2.7) with a suitable positive constant $\bar{\tau}$. Here, the terms χ_{ij} are so defined

$$\chi_{ij} = \begin{cases} 1, & \text{if } j \text{ transmits information to } i, \\ 0, & \text{otherwise,} \end{cases} \tag{1.2.9}$$

and, the communication rates a_{ij} are of the form

$$a_{ij}(t) := \frac{1}{N-1}\psi(x_i(t), x_j(t-\tau_{ij}(t))), \quad t > 0,\ \forall i, j = 1, \dots, N, \tag{1.2.10}$$

where the influence function $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is positive, bounded and continuous with (6.1.14) as upper bound. Moreover, the weight functions $\alpha_{ij} : [0, +\infty) \to [0, 1]$ are $\mathcal{L}^1$-measurable and describe the lack of communication. α_{ij} satisfy the following Persistence Excitation Condition (cf. [3, 9]):

(PE) there exist two positive constants T and $\tilde{\alpha}$ such that

$$\int_t^{t+T} \alpha_{ij}(s)ds \ge \tilde{\alpha}, \quad \forall t \ge 0, \tag{1.2.11}$$

for all $i, j = 1, \dots, N$ such that $\chi_{ij} = 1$.

Without loss of generality, we can assume that the positive constant $\tilde{\alpha}$ appearing in (1.2.11) satisfies $\tilde{\alpha}K < 1$. Let us note that (1.2.11) becomes relevant when T is large and $\tilde{\alpha}$ is small. In this case, the agents could eventually suspend their interaction for long enough. We also point out that, in the case in which $\alpha_{ij}(t) = 1$, for a.e. $t \ge 0$ and for any $i, j = 1, \dots, N$, i.e., in the case in which the agents do not interrupt their exchange of information, the condition (1.2.11) is of course satisfied.
Due to the presence of the time delay, the initial conditions are functions defined in the interval $[-\bar{\tau}, 0]$. The initial conditions

$$x_i(s) = x_i^0(s), \quad \forall s \in [-\bar{\tau}, 0], \forall i = 1, \dots, N, \tag{1.2.12}$$

are assumed to be continuous functions.
Notice that the functions $(\chi_{ij})_{ij}$ describe a $(0, 1)-$adjacency matrix and define a graph topology over the model structure. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph consisting of a finite set $\mathcal{V} = \{1, ..., N\}$ of vertices and a set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ of arcs. We assume that the agents are located at the vertices

and interact with each other via the underlying network topology. For each vertex i, we denote by $\mathcal{N}_i$ the set of vertices that directly influence the vertex i, namely

$$\mathcal{N}_i := \{j = 1, \dots, N : \chi_{ij} = 1\}. \tag{1.2.13}$$

The set $\mathcal{N}_i$ can also be defined in the following way: $j \in \mathcal{N}_i$ if and only if $(i, j) \in \mathcal{E}$. Also, we denote with

$$N_i := |\mathcal{N}_i|. \tag{1.2.14}$$

Throughout the dissertation, we will exclude self-loops, i.e., we assume that $i \notin \mathcal{N}_i$ for all $1 \leq i \leq N$. A *path* in a digraph $\mathcal{G}$ from i_0 to i_p is a finite sequence $i_0, i_1, \dots, i_p$ of distinct vertices such that each successive pair of vertices is an arc of $\mathcal{G}$. The integer p is called *length* of the path. If there exists a path from i to j, then vertex j is said to be *reachable* from vertex i and we define the distance from i to j, in notation $\mathrm{dist}(i, j)$, as the length of the shortest path from i to j. A digraph $\mathcal{G}$ is said to be *strongly connected* if each vertex is reachable from any other vertex. We assume that our digraph $\mathcal{G}$ is strongly connected. We define the *depth* γ of the digraph as follows:

$$\gamma := \max_{i,j=1,\dots,N} \mathrm{dist}(i, j). \tag{1.2.15}$$

Thus, any particle can be connected to the other individuals of the system via no more than γ intermediate agents. By definition, since $i \notin \mathcal{N}_i$, for all $i = 1, \dots, N$, we have that $\gamma \leq N - 1$. Also, since the digraph is strongly connected, $\gamma \geq 1$.

For existence results about the above model, we refer to the classical books [48, 49]. Here, we want to establish the convergence to consensus for the Hegselmann-Krause type model (1.2.8). Let us define the *diameter function* as

$$d(t) := \max_{i,j=1,\dots,N} |x_i(t) - x_j(t)|. \tag{1.2.16}$$

We have the following definition.

Definition 1.2.1. *We say that a solution converges to consensus if*

$$\lim_{t\to\infty} d(t) = 0.$$

1.3 The Cucker-Smale model

Consider a finite set of $N \in \mathbb{N}$ particles, with $N \geq 2$. Let $x_i(t) \in \mathbb{R}^d$ and $v_i(t) \in \mathbb{R}^d$ denote the position and the velocity of the i-th particle at time t, respectively. The classical Cucker-Smale model takes the following form

$$\begin{cases} \frac{d}{dt} x_i(t) = v_i(t), & t > 0, \ \forall i = 1, \dots, N, \\ \frac{d}{dt} v_i(t) = \sum\limits_{j: j \neq i} b_{ij}(t)(v_j(t) - v_i(t)), & t > 0, \ \forall i = 1, \dots, N, \end{cases} \tag{1.3.17}$$

where the communication weights are defined by

$$b_{ij}(t) := \frac{1}{N-1} \tilde{\psi}(|x_i(t) - x_j(t)|), \quad t > 0, \forall i, j = 1, \dots, N, \tag{1.3.18}$$

and the influence function $\tilde{\psi} : \mathbb{R} \to \mathbb{R}$ is a non-negative, continuous and non-increasing function. As we made for the first-order model, we defined a Cucker-Smale type model that describes a more general situation, in which there is non-universal interaction between the agents, time variable time delays, and communication failure. The system reads as follows:

$$\begin{cases} \frac{d}{dt}x_i(t) = v_i(t), & t > 0,\ \forall i = 1, \dots, N, \\ \frac{d}{dt}v_i(t) = \sum\limits_{j:j\neq i} \chi_{ij} c_{ij}(t)(v_j(t-\tau_{ij}(t)) - v_i(t)), & t > 0,\ \forall i = 1, \dots, N, \end{cases} \tag{1.3.19}$$

where the time delay functions $\tau_{ij} : [0, +\infty) \to [0, +\infty)$ are state-dependent continuous functions, satisfy (1.2.7), and the terms χ_{ij} are defined as in (1.2.9).
Here, the communication rates c_{ij} are of the form

$$c_{ij}(t) := \frac{1}{N-1}\alpha_{ij}(t)\tilde{\psi}(|x_i(t) - x_j(t-\tau_{ij}(t))|), \quad t > 0, \forall i, j = 1, \dots, N, \tag{1.3.20}$$

where $\tilde{\psi} : [0, +\infty) \to \mathbb{R}$ is a positive, bounded, and continuous function, with

$$\tilde{K} := \|\tilde{\psi}\|_\infty, \tag{1.3.21}$$

and the weight functions $\alpha_{ij} : [0, +\infty) \to [0, 1]$ are $\mathcal{L}^1$-measurable and satisfy the Persistence Excitation Condition **(PE)**. Again, without loss of generality, we can assume that the positive constant $\tilde{\alpha}$ appearing in (1.2.11) satisfies $\tilde{\alpha}\tilde{K} < 1$.
The initial conditions

$$x_i(s) = x_i^0(s), \quad v_i(s) = v_i^0(s), \quad \forall s \in [-\bar{\tau}, 0], \forall i = 1, \dots, N, \tag{1.3.22}$$

are assumed to be continuous functions.
For this general framework, we want to describe the exhibition of flocking. We define the space and velocity diameters as follows

$$d_X(t) := \max_{i,j=1,\dots,N} |x_i(t) - x_j(t)|, \quad \forall t \geq -\bar{\tau}, \tag{1.3.23}$$

$$d_V(t) := \max_{i,j=1,\dots,N} |v_i(t) - v_j(t)|, \quad \forall t \geq -\bar{\tau}. \tag{1.3.24}$$

Definition 1.3.1. *We say that a solution* $\{(x_i, v_i)\}_{i=1,\dots,N}$ *to system* (1.3.19) *exhibits asymptotic flocking if it satisfies the two following conditions:*

1. *there exists a positive constant* d^* *such that*

$$\sup_{t \geq -\bar{\tau}} d_X(t) \leq d^*;$$

2. $\lim\limits_{t\to\infty} d_V(t) = 0.$

1.4 The Kuramoto model

Consider a finite number $N \in \mathbb{N}$, $N \geq 2$, of coupled oscillators. The classical formulation of the Kuramoto model takes the following form

$$\frac{d}{dt}\theta_i(t) = \Omega_i + \frac{1}{N} \sum_{j:\ j\neq i} K_{ij} \sin(\theta_j(t) - \theta_i(t)), \quad i = 1, \dots, N,\ t > 0, \tag{1.4.25}$$

where $\theta_i = \theta_i(t)$ represents the phase of $i-$th Kuramoto oscillator at time $t > 0$. Here, $\{\Omega_i\}_{i=1}^N$ are the natural frequencies of the oscillators, which are assumed to be random variables extracted from a given distribution $g = g(\Omega)$, and $\{K_{ij}\}_{ij}$ are positive constants describing the coupling strength between the oscillators.
In the present work, as we discussed above, we analyze the emergence of synchronization in the Kuramoto model on directed graphs, considering the effects of time delays in the coupling between oscillators. More precisely, we investigate the conditions under which synchronization emerges in such systems and explore the influence of network topology and time delays on the synchronization dynamics. To present our model, we consider a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consisting of a finite set $\mathcal{V} = \{1, ..., N\}$ of vertices and a set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ of arcs. We assume that Kuramoto oscillators are located at the vertices and interact with each other via the underlying network topology. For each vertex i, let $\mathcal{N}_i$ be the set of vertices that directly influence the vertex i, as defined in (1.2.13), and we denote by N_i the number of oscillators in the set $\mathcal{N}_i$, as in (1.2.14). Then, our main system reads

$$\frac{d}{dt}\theta_i(t) = \Omega_i + \frac{\kappa}{N-1}\sum_{k\neq i}\chi_{ij}\sin(\theta_k(t-\tau_{ij}) - \theta_i(t)), \quad i = 1, \dots, N, \quad t > 0, \tag{1.4.26}$$

where $\tau_{ij} > 0$ denotes communication time delay in the information flow from vertex j to vertex i, and, for simplicity, we assumed them to be positive constants. Notice that the result can be extended to the case of a time variable time delay. Once again, the network topology is given by its $(0, 1)$-adjacency matrix (χ_{ij}). Notice that self-time delay is not allowed, i.e. $\tau_{ii} = 0$ and we exclude a self-loop, i.e. $i \notin \mathcal{N}_i$ for all $i = 1, \dots, N$. Moreover, $\kappa > 0$ is the coupling strength. The classical Kuramoto model, extensively studied in literature (see, e.g. [1, 5, 15, 16, 23, 29, 47, 61, 72]), corresponds to the cases where $\chi_{ij} = 1$ and $\tau_{ij} = 0$ for all $i, j = 1, \dots, N$. However, in our formulation (1.4.26), we introduce two additional structures, namely the time delay effect and network topology.
Taking into account the network topologies in the interaction is very natural. The connectivity structure dictates which oscillators directly influence each other's phases. In (1.4.26), the network connectivity determines the phase relations between connected oscillators and influences the overall synchronization behavior. For recent results and more detailed background regarding these extensions, we refer to [26, 27, 41, 42, 45, 76, 78, 79] and references therein.

We supply the system (1.4.26) with the initial data:

$$\theta_i(s) = \theta_i^0(s) \quad \text{for} \quad s \in [-\tau, 0], \tag{1.4.27}$$

where $\theta_i^0 \in \mathcal{C}_b^1(-\tau, 0) \cap \mathcal{C}[-\tau, 0]$, $i = 1, \dots, N$, and $\tau := \max_{i,j=1,\dots,N} \tau_{ij}$.
For a solution $\theta(t) := (\theta_1(t), \cdots, \theta_N(t))$ to (1.4.26), we denote the phase and velocity diameters by

$$d_\theta(t) := \max_{1\le i,j\le N}|\theta_i(t) - \theta_j(t)| \quad \text{and} \quad d_\omega(t) := \max_{1\le i,j\le N}|\omega_i(t) - \omega_j(t)|, \tag{1.4.28}$$

respectively, where $\omega_i(t) := \dot\theta_i(t) := \frac{d\theta_i(t)}{dt}$ with $\omega_i(0) := \lim_{t\to 0-}\dot\theta_i(t)$. We provide a notion of our complete frequency synchronization in the definition below.

Definition 1.4.1. *Let $\theta(t) := (\theta_1(t), \dots, \theta_N(t))$ be the global classical solution to the system (1.4.26)-(1.4.27). Then the system exhibits the complete frequency synchronization if and only if the velocity diameter tends to 0, as the time goes to infinity:*

$$\lim_{t\to\infty} d_\omega(t) = 0.$$

Chapter 2

Opinion dynamics of two populations

In this chapter, we are interested in studying the convergence to consensus for a Hegselmann-Krause type opinion formation model involving two populations with time-delayed coupling. It is natural to assume a time-delay effect in this type of interaction. Indeed, while one can consider almost instantaneous the influence among agents in the same population, a certain time lag appears in the interaction among individuals of different populations. A more general model would include time delays (eventually smaller) also in the interactions within the same population. Here, for simplicity, we choose to consider delay effects only in the interactions among agents of different populations. An analogous analysis could also be performed in the more general situation with multiple delays.
Consider two finite sets of N and M agents respectively, with $N, M \in \mathbb{N}$, $N, M \geq 2$. Without loss of generality, we assume $M \leq N$. Let $x_i(t) \in \mathbb{R}^d, i = 1, \dots, N$, be the opinion of the i-th particle of the first family at time t and $y_i(t) \in \mathbb{R}^d$, $i = 1, ..., M$, be the opinion of the i-th particle of the second family at time t. We consider that a (small) group of agents of the first family interacts with another (small) group of the second family, with a time delay appearing as the time needed for an agent of a population to receive information from agents of the other one. The time delay is assumed to be a positive constant, $\tau > 0$. Given $h, k \in \mathbb{N}$, $h < M$ and $k < N$, the opinions of the two populations evolve following the Hegselmann-Krause opinion formation model:

$$
\begin{aligned}
&\frac{d}{dt}x_i(t) = \sum_{j\neq i} a_{ij}(t)(x_j(t) - x_i(t)) + \sum_{j=1}^{h} \epsilon_{ij}(t)(y_j(t-\tau) - x_i(t)), \quad t > 0,\ i = 1, ..., k,\\
&\frac{d}{dt}x_i(t) = \sum_{j\neq i} a_{ij}(t)(x_j(t) - x_i(t)), \quad t > 0,\ i = k+1, ..., N,\\
&\frac{d}{dt}y_i(t) = \sum_{j\neq i} b_{ij}(t)(y_j(t) - y_i(t)) + \sum_{j=1}^{k} \eta_{ij}(t)(x_j(t-\tau) - y_i(t)), \quad t > 0,\ i = 1, ..., h,\\
&\frac{d}{dt}y_i(t) = \sum_{j\neq i} b_{ij}(t)(y_j(t) - y_i(t)), \quad t > 0,\ i = h+1, ..., M,
\end{aligned}
\tag{2.0.1}
$$

with the interaction weights $a_{ij}(t), t \geq 0$, of the form:

$$
\begin{aligned}
a_{ij}(t) &:= \frac{1}{N+h-1}\psi(x_i(t), x_j(t)), \qquad i=1,...,k,\ j=1,\dots,N,\\
a_{ij}(t) &:= \frac{1}{N-1}\psi(x_i(t), x_j(t)), \qquad i=k+1,...,N,\ j=1,\dots,N,
\end{aligned}
\tag{2.0.2}
$$

and the weights $b_{ij}(t), t \geq 0$, of the form:

$$
\begin{aligned}
b_{ij}(t) &:= \frac{1}{M+k-1}\psi^*(y_i(t), y_j(t)), \qquad i=1,...,h,\ j=1,\dots,M,\\
b_{ij}(t) &:= \frac{1}{M-1}\psi^*(y_i(t), y_j(t)), \qquad i=h+1,...,M,\ j=1,\dots,M.
\end{aligned}
\tag{2.0.3}
$$

Here, $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $\psi^* : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ are continuous, positive and bounded functions. Moreover, the interaction coefficients $\epsilon_{ij}(t)$ and $\eta_{ij}(t)$, for $t \geq 0$, among individuals of different populations have the form:

$$
\begin{aligned}
\epsilon_{ij}(t) &:= \frac{1}{N+h-1}\phi(x_i(t), y_j(t-\tau)), \qquad i=1,...,k,\ j=1,\dots,h,\\
\eta_{ij}(t) &:= \frac{1}{M+k-1}\phi^*(y_i(t), x_j(t-\tau)), \qquad i=1,...,h,\ j=1,\dots,k,
\end{aligned}
\tag{2.0.4}
$$

where $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $\phi^* : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ are continuous, positive and bounded functions. Note that the different normalization factors in the above coefficients correspond, for each group, to the number of agents involved in the interaction. We emphasize that, in our model, the influence functions do not necessarily depend on the distance between the agents; instead, in most of the related literature. Moreover, we do not require symmetry or monotonicity assumptions.
Let us denote

$$
\Lambda := \max\left\{\|\psi\|_\infty, \|\psi^*\|_\infty, \|\phi\|_\infty, \|\phi^*\|_\infty\right\}.
$$

Let us assume the initial conditions:

$$
\begin{cases}
x_i(t) = x_i^0(t), \quad i=1,...,k,\ t\in[-\tau,0],\\
x_i(0) = x_i^0, \quad i=k+1,...,N,\\
y_i(t) = y_i^0(t), \quad i=1,...,h,\ t\in[-\tau,0],\\
y_i(0) = y_i^0, \quad i=h+1,...,M,
\end{cases}
\tag{2.0.5}
$$

where $x_i^0(\cdot), i=1,\dots,k$, $y_i^0(\cdot), i=1,\dots,h$, are continuous functions defined on $[-\tau,0]$, $x_i^0 \in \mathbb{R}^d$, $i=k+1,\dots,N$, $y_i^0 \in \mathbb{R}^d$, $i=h+1,\dots,M$.

For well-posedness results for model (2.0.1)-(2.0.5), we refer to classical texts on functional differential equations [48, 49]. Related to the system (1.2.8), we are here in the case with $\alpha_{ij}(t)=1$, $\tau_{ij}(t)=\tau>0$ for all $i,j=1,\dots,N+M$, and $\gamma=3$. We will focus on The asymptotic behavior of the solutions. For this aim, we assume the continuity of the involved influence functions only. Of course, the continuity alone does not ensure uniqueness. For solutions to (2.0.1)-(2.0.5), we want to prove the convergence to consensus. For this case, let us define the *diameter* of each population as

$$d_X(t) := \max_{i,j=1,\dots,N} |x_i(t) - x_j(t)|, \quad d_Y(t) := \max_{i,j=1,\dots,M} |y_i(t) - y_j(t)|.$$

Moreover, let us define the *global diameter* as

$$d(t) := \max \Big\{ d_X(t), d_Y(t), \max_{i=1,\dots,N} \max_{j=1,\dots,M} |x_i(t) - y_j(t)| \Big\}.$$

To achieve the consensus result, for this particular model, we have to show the convergence state in Definition 1.2.1 with the global diameter defined above.

This kind of model can have applications in social sciences, economics, politics, and ecology. Indeed, it is reasonable to try reaching a global consensus among individuals of different countries, or different groups of individuals in the same country, about important questions such as, e.g., ecological behaviors, climate change's reasons, appropriate strategies to reduce CO_2 emissions, etc. The proof of a consensus result for model (2.0.1) can be considered as a first insight for more quantitative studies aiming to design appropriate control strategies.

For other consensus results for opinion formation models on a network in the presence of time delays, see [62]. However, we deal here with more general interaction coefficients. In particular, the influence functions depend on both arguments, x_i, x_j, and not necessarily on their distance. Moreover, we do not require any lower bounds, Lipschitz continuity, or monotonicity assumptions.

2.1 Preliminaries

In this section, we present some preliminary results useful for studying the consensus behavior.

Firstly, for any fixed a vector $v \in \mathbb{R}^d$, let us define the following quantities:

$$\begin{aligned} m_0 := \min \Big\{ &\min_{i=1,\dots,k} \min_{t\in[-\tau,0]} \langle x_i(t), v\rangle, \min_{i=k+1,\dots,N} \langle x_i(0), v\rangle, \\ &\min_{i=1,\dots,h} \min_{t\in[-\tau,0]} \langle y_i(t), v\rangle, \min_{i=h+1,\dots,M} \langle y_i(0), v\rangle \Big\}, \end{aligned} \tag{2.1.6}$$

and

$$\begin{aligned} M_0 := \max \Big\{ &\max_{i=1,\dots,k} \max_{t\in[-\tau,0]} \langle x_i(t), v\rangle, \max_{i=k+1,\dots,N} \langle x_i(0), v\rangle, \\ &\max_{i=1,\dots,h} \max_{t\in[-\tau,0]} \langle y_i(t), v\rangle, \max_{i=h+1,\dots,M} \langle y_i(0), v\rangle \Big\}. \end{aligned} \tag{2.1.7}$$

Note that m_0 and M_0 should be m_0^v and M_0^v. For simplicity of notation, we omit the dependency on the vector v. The following estimates hold.

Lemma 2.1.1. *Let* $(x_i(t), y_j(t))$, $i = 1\dots, N$, $j = 1,\dots, M$, *be a global classical solution of the system* (6.1.15)-(2.0.5). *Then, for all* $v \in \mathbb{R}^d$ *we have that*

$$m_0 \le \langle x_i(t), v\rangle \le M_0, \tag{2.1.8}$$

and

$$m_0 \le \langle y_j(t), v\rangle \le M_0, \tag{2.1.9}$$

for all $t \ge -\tau$, $i = 1,\dots, k$, $j = 1,\dots, h$, *and for all* $t \ge 0$, $i = k+1,\dots, N$, $j = h+1,\dots, M$.

Proof. Fix a vector $v \in \mathbb{R}^d$ and let m_0, M_0 be the constants defined in (2.1.6), (2.1.7). By definition of m_0 and M_0, we have that the inequalities (2.1.8) and (2.1.9) are trivially satisfied for $t \in [-\tau, 0]$, $i = 1, \dots, k$ and $j = 1, \dots, h$. Then, we want to prove (2.1.8) and (2.1.9) for $t \geq 0$. Let us prove (2.1.8); (2.1.9) follows analougously.

For a fixed parameter $\epsilon > 0$, let us define the following set:

$$\begin{aligned} T^\epsilon := \{t > 0 \ : \ &\langle x_i(s), v\rangle < M_0 + \epsilon, \ \forall i = 1, \dots, N, \\ &\langle y_i(s), v\rangle < M_0 + \epsilon, \ \forall i = 1, \dots, M, \ \forall s \in [0, t)\}. \end{aligned}$$

By continuity, $T^\epsilon \neq \emptyset$. Let us call $S^\epsilon := \sup T^\epsilon$. We want to prove that $S^\epsilon = +\infty$. Let us suppose by contraddiction that $S^\epsilon < +\infty$. Then,

$$\max_{i=1,\dots,N} \langle x_i(t), v\rangle < M_0 + \epsilon, \ \forall t \in [0, S^\epsilon), \tag{2.1.10}$$

and

$$\lim_{t \to S^{\epsilon-}} \max_{i=1,\dots,N} \langle x_i(t), v\rangle = M_0 + \epsilon. \tag{2.1.11}$$

For all $t \in [0, S^\epsilon)$ and $i = k+1, \dots, N$ we have

$$\begin{aligned} \frac{d}{dt}\langle x_i(t), v\rangle &= \sum_{j \neq i} a_{ij}(t)\langle x_j(t) - x_i(t), v\rangle \\ &\leq \sum_{j \neq i} a_{ij}(t)(M_0 + \epsilon - \langle x_i(t), v\rangle) \\ &\leq \Lambda(M_0 + \epsilon - \langle x_i(t), v\rangle). \end{aligned} \tag{2.1.12}$$

Applying the Grönwall's Lemma over $t \in [0, S^\epsilon)$, we find

$$\begin{aligned} \langle x_i(t), v\rangle &\leq e^{-\Lambda t}\langle x_i(0), v\rangle + (M_0 + \epsilon)(1 - e^{-\Lambda t}) \\ &\leq M_0 + \epsilon - \epsilon e^{-\Lambda t} \leq M_0 + \epsilon - \epsilon e^{-\Lambda S^\epsilon}, \end{aligned} \tag{2.1.13}$$

for all $t \in [0, S^\epsilon)$ and $i = k+1, \dots, N$. Therefore, we deduce that

$$\max_{i=k+1,\dots,N} \langle x_i(t), v\rangle < M_0 + \epsilon - \epsilon e^{-\Lambda S^\epsilon}, \ \forall t \in (0, S^\epsilon).$$

Consider now $t \in [0, S^\epsilon)$ and $i = 1, \dots, k$. Then, we have

$$\begin{aligned} \frac{d}{dt}\langle x_i(t), v\rangle &= \sum_{j \neq i} a_{ij}(t)\langle x_j(t) - x_i(t), v\rangle + \sum_{j=1}^{h} \epsilon_{ij}(t)\langle y_j(t-\tau) - x_i(t), v\rangle \\ &\leq \sum_{j \neq i} a_{ij}(t)(M_0 + \epsilon - \langle x_i(t), v\rangle) + \sum_{j=1}^{h} \epsilon_{ij}(t)(M_0 + \epsilon - \langle x_i(t), v\rangle) \\ &\leq \Lambda(M_0 + \epsilon - \langle x_i(t), v\rangle). \end{aligned} \tag{2.1.14}$$

Therefore, we can find analogously that

$$\max_{i=1,\dots,k} \langle x_i(t), v\rangle < M_0 + \epsilon - \epsilon e^{-\Lambda S^\epsilon}, \ \forall t \in [0, S^\epsilon).$$

Then, we have that

$$\max_{i=1,\dots,N}\langle x_i(t),v\rangle < M_0+\epsilon-\epsilon e^{-\Lambda S^\epsilon},\ \forall t\in[0,S^\epsilon).$$

Passing to the limit for $t\to S^{\epsilon-}$, we find

$$\lim_{t\to S^{\epsilon-}}\max_{i=1,\dots,N}\langle x_i(t),v\rangle \le M_0+\epsilon-\epsilon e^{-\Lambda S^\epsilon}<M_0+\epsilon,$$

and this gives a contradiction. Then, we have that $S^\epsilon=+\infty$. Hence, by arbitrarity of ϵ, we have that

$$\max_{i=1,\dots,N}\langle x_i(t),v\rangle \le M_0,\ \forall t\ge 0,\ v\in\mathbb{R}^d.$$

Therefore, we have that.

$$\langle x_i(t),v\rangle \le M_0,\ \forall\ t\ge -\tau,\ i=1,\dots,k,$$

and

$$\langle x_i(t),v\rangle \le M_0,\ \forall t\ge 0,\ i=k+1,\dots,N.$$

To prove the other inequality, we observe that, by the proven estimate,

$$\begin{aligned}
-\langle x_i(t),v\rangle &= \langle x_i(t),-v\rangle\\
&\le \max\Big\{\max_{i=1,\dots,k}\max_{t\in[-\tau,0]}\langle x_i(t),-v\rangle,\ \max_{i=k+1,\dots,N}\langle x_i(0),-v\rangle,\\
&\qquad \max_{i=1,\dots,h}\max_{t\in[-\tau,0]}\langle y_i(t),-v\rangle,\ \max_{i=h+1,\dots,M}\langle y_i(0),-v\rangle\Big\}\\
&= -\min\Big\{\min_{i=1,\dots,k}\min_{t\in[-\tau,0]}\langle x_i(t),v\rangle,\ \min_{i=k+1,\dots,N}\langle x_i(0),v\rangle,\\
&\qquad \min_{i=1,\dots,h}\min_{t\in[-\tau,0]}\langle y_i(t),v\rangle,\ \min_{i=h+1,\dots,M}\langle y_i(0),v\rangle\Big\} = -m_0.
\end{aligned} \tag{2.1.15}$$

This concludes the proof. □

The above lemma allows us to deduce a bound on the states.

Lemma 2.1.2. *Let $(x_i(t),y_j(t)),\ i=1\dots,N,\ j=1,\dots,M$, be a global classical solution of the system* (2.0.1)-(2.0.5)*. Then,*

$$|x_i(t)|\le C_0,\quad |y_j(t)|\le C_0, \tag{2.1.16}$$

$\forall\, t\ge-\tau$, for $i=1,\dots,k,\ j=1,\dots,h$, and $\forall\, t\ge 0$, for $i=k+1,\dots,N,\ j=h+1,\dots,M$, where C_0 is given by

$$C_0:=\max\Big\{\max_{i=1,\dots,k}\max_{t\in[-\tau,0]}|x_i(t)|,\ \max_{i=k+1,\dots,N}|x_i(0)|,\ \max_{j=1,\dots,h}\max_{t\in[-\tau,0]}|y_j(s)|,\ \max_{j=h+1,\dots,M}|y_j(0)|\Big\}.$$

Proof. We prove the first inequality of (2.1.16). The second one for $|y_j(t)|,\ j=1,\dots,M$, follows analogously. For $i=1,\dots,N$ and $t\ge-\tau$, if $|x_i(t)|=0$, the result is trivial. Let us suppose $|x_i(t)|>0$ and define the vector

$$v := \frac{x_i(t)}{|x_i(t)|} \ .$$

Then, applying (2.1.8) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|x_i(t)| = \langle x_i(t), v\rangle \leq M_0 \\
&= \max\Big\{ \max_{i=1,\dots,k} \max_{t\in[-\tau,0]} \langle x_i(t), v\rangle, \max_{i=k+1,\dots,N} \langle x_i(0), v\rangle, \max_{i=1,\dots,h} \max_{t\in[-\tau,0]} \langle y_i(t), v\rangle, \\
&\qquad \max_{i=h+1,\dots,M} \langle y_i(0), v\rangle \Big\} \\
&\leq \max\Big\{ \max_{i=1,\dots,k} \max_{t\in[-\tau,0]} |x_i(t)||v|, \max_{i=k+1,\dots,N} |x_i(0)||v|, \max_{j=1,\dots,h} \max_{t\in[-\tau,0]} |y_j(s)||v|, \\
&\qquad \max_{j=h+1,\dots,M} |y_j(0)||v| \Big\} = C_0,
\end{aligned} \tag{2.1.17}$$

being v a unit vector. So, the first inequality of (2.1.16) is proven. □

Remark 2.1.3. *From Lemma 2.1.2, since the influence functions ψ and ψ^* are continuous, we deduce that*

$$\begin{aligned}
\psi(x_i(t), x_j(t)) \geq \psi_0 := \min_{|z_1|,|z_2|\leq C_0} \psi(z_1, z_2) > 0, \\
\psi^*(y_l(t), y_r(t)) \geq \psi_0^* := \min_{|z_1|,|z_2|\leq C_0} \psi^*(z_1, z_2) > 0,
\end{aligned} \tag{2.1.18}$$

for each $t \geq 0$, $i, j = 1, ..., N$ and $l, r = 1, ..., M$. Moreover, since the functions ϕ and ϕ^* are continuous too, again we deduce that

$$\begin{aligned}
\phi(x_i(t), y_j(t-\tau)) \geq \phi_0 := \min_{|z_1|,|z_2|\leq C_0} \phi(z_1, z_2) > 0, \\
\phi^*(y_l(t), x_r(t-\tau)) \geq \phi_0^* := \min_{|z_1|,|z_2|\leq C_0} \phi^*(z_1, z_2) > 0,
\end{aligned} \tag{2.1.19}$$

for each $t \geq 0$, $i, r = 1, ..., k$ and $j, l = 1, ..., h$.

From Remark 2.1.3, we can define the positive constant

$$\Gamma := \min\Big\{\psi_0, \psi_0^*, \phi_0, \phi_0^*\Big\}. \tag{2.1.20}$$

Now, fix $v \in \mathbb{R}^d$ and let m_0, M_0 be as in (2.1.6) and (2.1.7) respectively. Since, up to changes of influence function, system (2.0.1) is invariant by translation, without loss of generality, we may assume

$$0 < m_0 \leq M_0.$$

Inspired by [50], we can prove the next lemma. Note that [50] deals with a Hegselmann-Krause model with all-to-all connection, namely, each agent is influenced and influences any other agent. In our model, we have four different agent groups: two populations and, in each population, leaders and non-leaders. This requires finer and trickier analysis: our estimates can be deduced through careful arguments involving the different agents' groups in the appropriate order.

Lemma 2.1.4. *Let $(x_i(t), y_j(t))$, with $i = 1,\dots,N$ and $j = 1,\dots,M$, be a global classical solution of the system* (2.0.1)-(2.0.5)*. Then, for $t \in [5\tau, 6\tau]$, we have*

$$m_0 + \frac{\Gamma_1}{2}(M_0 - m_0) \le \langle x_i(t), v\rangle \le M_0 - \frac{\Gamma_1}{2}(M_0 - m_0), \tag{2.1.21}$$

and

$$m_0 + \frac{\Gamma_1}{2}(M_0 - m_0) \le \langle y_j(t), v\rangle \le M_0 - \frac{\Gamma_1}{2}(M_0 - m_0), \tag{2.1.22}$$

for a suitable constant $\Gamma_1 \in (0,1)$.

Proof. Let us proceed by steps.

Step 1: Suppose that exists $L \in \{k+1,\dots,N\}$ such that $\langle x_L(0), v\rangle = m_0$. Then, since $|\langle \dot{x}_i(t), v\rangle| \le 2\Lambda M_0,\ \forall t \ge 0$, we have that

$$m_0 \le \langle x_L(t), v\rangle \le \frac{M_0 + m_0}{2},\ t \in [0,\sigma],$$

where σ is a positive number such that

$$\sigma \le \min\Big\{\tau, \frac{M_0 - m_0}{4\Lambda M_0}\Big\}. \tag{2.1.23}$$

Consider $i \in \{k+1,\dots,N\} \setminus \{L\}$ and $t \in [0,\sigma]$. Then,

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t), v\rangle &= \sum_{\substack{j\neq i\\ j\neq L}} a_{ij}(t)\langle x_j(t) - x_i(t), v\rangle + a_{iL}(t)\langle x_L(t) - x_i(t), v\rangle\\
&\le \sum_{\substack{j\neq i\\ j\neq L}} a_{ij}(t)(M_0 - \langle x_i(t), v\rangle) + a_{iL}(t)\Big(\frac{M_0+m_0}{2} - \langle x_i(t), v\rangle\Big)\\
&= \Big(\sum_{j\neq i} a_{ij}(t) - a_{iL}(t)\Big)(M_0 - \langle x_i(t), v\rangle) + a_{iL}(t)\Big(\frac{M_0+m_0}{2} - \langle x_i(t), v\rangle\Big) \qquad (2.1.24)\\
&\le (\Lambda - a_{iL}(t))(M_0 - \langle x_i(t), v\rangle) + a_{iL}(t)\Big(\frac{M_0+m_0}{2} - \langle x_i(t), v\rangle\Big)\\
&= \Lambda(M_0 - \langle x_i(t), v\rangle) - a_{iL}(t)\frac{M_0 - m_0}{2}\\
&\le \Big(\Lambda M_0 - \frac{\Gamma}{N-1}\frac{M_0 - m_0}{2}\Big) - \Lambda\langle x_i(t), v\rangle.
\end{aligned}$$

Integrating over $[0,t]$ with $t \in [0,\sigma]$, we find

$$\langle x_i(t), v\rangle \le e^{-\Lambda t}\langle x_i(0), v\rangle + \Big(M_0 - \frac{1}{N-1}\frac{\Gamma}{\Lambda}\frac{M_0 - m_0}{2}\Big)(1 - e^{-\Lambda t}).$$

Taking $t = \sigma$ in the above inequality, we can find

$$\langle x_i(\sigma), v\rangle \le M_0 - (1 - e^{-\Lambda\sigma})\frac{1}{N-1}\frac{\Gamma}{\Lambda}\frac{M_0 - m_0}{2}.$$

Denoting

$$\delta_-^1 := (1 - e^{-\Lambda\sigma})\frac{1}{2(N-1)}\frac{\Gamma}{\Lambda}\Big(1 - \frac{m_0}{M_0}\Big), \tag{2.1.25}$$

we have the inequality

$$\langle x_i(\sigma), v\rangle \leq (1-\delta_-^1)M_0, \ \forall i \in \{k+1,\dots,N\}\setminus\{L\}. \tag{2.1.26}$$

Consider now $t \in [\sigma, 6\tau]$.

$$\begin{aligned}\frac{d}{dt}\langle x_i(t), v\rangle &= \sum_{j\neq i} a_{ij}(t)\langle x_j(t)-x_i(t), v\rangle \\ &\leq \Lambda(M_0 - \langle x_i(t), v\rangle), \ \forall i \in \{k+1,\dots,N\}\setminus\{L\}.\end{aligned} \tag{2.1.27}$$

Integrating over $[\sigma, t]$ with $t \in [\sigma, 6\tau]$, we have

$$\begin{aligned}\langle x_i(t), v\rangle &\leq e^{-\Lambda(t-\sigma)}\langle x_i(\sigma), v\rangle + M_0(1-e^{-\Lambda(t-\sigma)}) \\ &\leq e^{-\Lambda(t-\sigma)}(1-\delta_-^1)M_0 + M_0(1-e^{-\Lambda(t-\sigma)}) \\ &= M_0\left(1-\delta_-^1 e^{-\Lambda(t-\sigma)}\right) \\ &\leq M_0\left(1-\delta_-^1 e^{-6\Lambda\tau}\right),\end{aligned}$$

Then,

$$\langle x_i(t), v\rangle \leq M_0(1-\delta_-^1 e^{-6\Lambda\tau}), \ \forall i \in \{k+1,\dots,N\}\setminus\{L\}, \ \forall t \in [\sigma, 6\tau]. \tag{2.1.28}$$

Consider now $i \in \{1,\dots,k\}$ and fix $i_1 \in \{k+1,\dots,N\}\setminus\{L\}$. Taking $t \in [\sigma, 6\tau]$, we have

$$\begin{aligned}\frac{d}{dt}\langle x_i(t), v\rangle &= \sum_{\substack{j\neq i\\ j\neq i_1}} a_{ij}(t)\langle x_j(t)-x_i(t), v\rangle + \sum_{j=1}^{h}\epsilon_{ij}(t)\langle y_j(t-\tau)-x_i(t), v\rangle \\ &+ a_{ii_1}(t)\langle x_{i_1}(t)-x_i(t), v\rangle \\ &\leq \Big(\sum_{j\neq i} a_{ij}(t) - a_{ii_1}(t)\Big)(M_0-\langle x_i(t), v\rangle) + \frac{h}{N+h-1}\Lambda(M_0-\langle x_i(t), v\rangle) \\ &+ a_{ii_1}(t)(M_0(1-\delta_-^1 e^{-6\tau\Lambda}) - \langle x_i(t), v\rangle) \\ &\leq \frac{N-1}{N+h-1}\Lambda(M_0-\langle x_i(t), v\rangle) + \frac{h}{N+h-1}\Lambda(M_0-\langle x_i(t), v\rangle) \\ &- \frac{1}{N+h-1}\Gamma M_0\delta_-^1 e^{-6\tau\Lambda} \\ &= \Lambda M_0\Big(1-\frac{1}{N+h-1}\frac{\Gamma}{\Lambda}\delta_-^1 e^{-6\tau\Lambda}\Big) - \Lambda\langle x_i(t), v\rangle.\end{aligned} \tag{2.1.29}$$

Integrating over $[\sigma, t]$ with $t \in [\sigma, 6\tau]$ we find

$$\begin{aligned}\langle x_i(t), v\rangle &\leq e^{-\Lambda(t-\sigma)}\langle x_i(\sigma), v\rangle + M_0\Big(1-\frac{1}{N+h-1}\frac{\Gamma}{\Lambda}\delta_-^1 e^{-6\tau\Lambda}\Big)(1-e^{-\Lambda(t-\sigma)}) \\ &\leq M_0\Big(1-\frac{1}{N+h-1}\frac{\Gamma}{\Lambda}\delta_-^1 e^{-6\tau\Lambda}(1-e^{-\Lambda(t-\sigma)})\Big).\end{aligned} \tag{2.1.30}$$

Shrinking to $t \in [2\tau, 6\tau]$ we find that

$$\langle x_i(t), v\rangle \leq M_0\Big(1-\frac{1}{N+h-1}\frac{\Gamma}{\Lambda}\delta_-^1 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})\Big), \ \forall i \in \{1,\dots,k\}. \tag{2.1.31}$$

Using the state $i_1 \in \{k+1,\dots,N\} \setminus \{L\}$, we can find an upper bound for i_L too. Indeed, for $t \in [\sigma, 6\tau]$, analogously to (2.1.27), we obtain

$$\begin{aligned} \frac{d}{dt}\langle x_L(t), v\rangle &= \sum_{\substack{j\neq L\\ j\neq i_1}} a_{Lj}\langle x_j(t)-x_L(t),v\rangle + a_{Li_1}\langle x_{i_1}(t)-x_L(t),v\rangle \\ &\leq \Lambda M_0\Big(1-\frac{1}{N-1}\frac{\Gamma}{\Lambda}\delta_-^1 e^{-6\Lambda\tau}\Big) - \Lambda\langle x_L(t),v\rangle. \end{aligned} \tag{2.1.32}$$

Integrating over $[\sigma, t]$ with $t \in [\sigma, 6\tau]$, we find

$$\langle x_L(t),v\rangle \leq e^{-\Lambda(t-\sigma)}\langle x_L(\sigma),v\rangle + M_0\Big(1-\frac{1}{N-1}\frac{\Gamma}{\Lambda}\delta_-^1 e^{-6\Lambda\tau}\Big)(1-e^{-\Lambda(t-\sigma)}). \tag{2.1.33}$$

Shrinking to $t \in [2\tau, 6\tau]$, we have the estimate

$$\langle x_L(t),v\rangle \leq M_0\Big(1-\frac{1}{N-1}\frac{\Gamma}{\Lambda}\delta_-^1 e^{-6\Lambda\tau}(1-e^{-\Lambda\tau})\Big). \tag{2.1.34}$$

Consider now $i \in \{1,\dots,h\}$ and fix $i_2 \in \{1,\dots,k\}$. Taking $t \in [3\tau, 6\tau]$, we have

$$\begin{aligned} \frac{d}{dt}\langle y_i(t),v\rangle &= \sum_{j\neq i} b_{ij}(t)\langle y_j(t)-y_i(t),v\rangle + \sum_{\substack{j=1\\ j\neq i_2}}^{k}\eta_{ij}(t)\langle x_j(t-\tau)-y_i(t),v\rangle \\ &\qquad + \eta_{ii_2}(t)\langle x_{i_2}(t-\tau)-y_i(t),v\rangle \\ &\leq \frac{M-1}{M+k-1}\Lambda(M_0-\langle y_i(t),v\rangle) + \Big(\sum_{j=1}^{k}\eta_{ij}(t)-\eta_{ii_2}(t)\Big)(M_0-\langle y_i(t),v\rangle) \\ &\qquad + \eta_{ii_2}(t)\Big[M_0\Big(1-\frac{1}{N+h-1}\frac{\Gamma}{\Lambda}\delta_-^1 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})\Big)-\langle y_i(t),v\rangle\Big] \\ &\leq \Lambda M_0\Big(1-\frac{1}{(N+h-1)(M+k-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^2\delta_-^1 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})\Big) - \Lambda\langle y_i(t),v\rangle. \end{aligned} \tag{2.1.35}$$

Integrating over $[3\tau, t]$, with $t \in [3\tau, 6\tau]$, we find that

$$\begin{aligned} \langle y_i(t),v\rangle &\leq e^{-\Lambda(t-3\tau)}\langle y_i(3\tau),v\rangle \\ &\quad + M_0\Big(1-\frac{1}{(N+h-1)(M+k-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^2\delta_-^1 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})\Big)(1-e^{-\Lambda(t-3\tau)}) \\ &\leq M_0\Big(1-\frac{1}{(N+h-1)(M+k-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^2\delta_-^1 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})(1-e^{-\Lambda(t-3\tau)})\Big). \end{aligned} \tag{2.1.36}$$

Shrinking to $t \in [4\tau, 6\tau]$ we find that

$$\langle y_i(t),v\rangle \leq M_0\Big(1-\frac{1}{(N+h-1)(M+k-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^2\delta_-^1 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})^2\Big),\ \forall i \in \{1,\dots,h\}. \tag{2.1.37}$$

Finally, consider $i \in \{h+1,\dots,M\}$ and fix $j_1 \in \{1,\dots,h\}$. Taking $t \in [4\tau, 6\tau]$, with analogous computation, we find

$$\begin{aligned}\frac{d}{dt}\langle y_i(t), v\rangle &= \sum_{\substack{j\neq i\\ j\neq j_1}} b_{ij}(t)\langle y_j(t) - y_i(t), v\rangle + b_{ij_1}(t)\langle y_{j_1}(t) - y_i(t), v\rangle\\ &\leq \Lambda M_0\Big(1 - \frac{1}{(M-1)(N+h-1)(M+k-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^3 \delta_-^1 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})^2\Big) - \Lambda\langle y_i(t), v\rangle.\end{aligned} \tag{2.1.38}$$

Integrating over $[4\tau, t]$ with $t \in [4\tau, 6\tau]$, we have that

$$\begin{aligned}\langle y_i(t), v\rangle &\leq e^{-\Lambda(t-4\tau)}\langle y_i(4\tau), v\rangle\\ &+ M_0\Big[1 - \frac{1}{(M-1)(N+h-1)(M+k-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^3\delta_-^1 e^{-6\Lambda\tau}(1-e^{-\Lambda\tau})^2\Big](1-e^{-\Lambda(t-4\tau)})\\ &\leq M_0\Big[1 - \frac{1}{(M-1)(N+h-1)(M+k-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^3\delta_-^1 e^{-6\Lambda\tau}(1-e^{-\Lambda\tau})^2(1-e^{-\Lambda(t-4\tau)})\Big].\end{aligned}$$

Shrinking to $t \in [5\tau, 6\tau]$, we find that

$$\langle y_i(t), v\rangle \leq M_0\Big[1 - \frac{1}{(M-1)(N+h-1)(M+k-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^3\delta_-^1 e^{-6\Lambda\tau}(1-e^{-\Lambda\tau})^3\Big], \quad \forall i \in \{h+1,\dots,M\}. \tag{2.1.39}$$

Then, the inequality (2.1.39) holds for all the non-leaders of the second population, for $t \in [5\tau, 6\tau]$. Moreover, one can notice that the right-hand side of (2.1.39) is larger than the right-hand side of (2.1.37). Thus, the estimate (2.1.39) holds for all the states of the second population. Since the right-hand side of (2.1.39) is larger than that one of (2.1.31) and (2.1.34), we have that (2.1.39) holds for all the states of the first population too, for $t \in [5\tau, 6\tau]$.

Step 2: Assume now that $L \in \{1,\dots,k\}$ is such that $\langle x_L(s), v\rangle = m_0$ for some $s \in [-\tau, 0]$. By continuity, then there exists a closed interval $[\alpha_L, \beta_L] \subset [-\tau, 0]$ such that

$$m_0 \leq \langle x_L(t), v\rangle \leq \frac{M_0 + m_0}{2},\ t \in [\alpha_L, \beta_L].$$

Eventually choosing a smaller σ in (2.1.23), we may assume $\beta_L - \alpha_L = \sigma$. Consider $i \in \{1,\dots,h\}$ and $t \in [\alpha_L + \tau, \beta_L + \tau]$. Then,

$$\begin{aligned}\frac{d}{dt}\langle y_i(t), v\rangle &= \sum_{j\neq i} b_{ij}(t)\langle y_j(t) - y_i(t), v\rangle + \sum_{\substack{j=1\\ j\neq L}}^{k} \eta_{ij}(t)\langle x_j(t-\tau) - y_i(t), v\rangle\\ &+ \eta_{iL}(t)\langle x_L(t-\tau) - y_i(t), v\rangle\\ &\leq \frac{M-1}{M+k-1}\Lambda(M_0 - \langle y_i(t), v\rangle) + \Big(\sum_{j=1}^{k}\eta_{ij} - \eta_{iL}\Big)(M_0 - \langle y_i(t), v\rangle)\\ &+ \eta_{iL}(t)\Big(\frac{M_0+m_0}{2} - \langle y_i(t), v\rangle\Big),\end{aligned}$$

and so,

$$\begin{aligned}\frac{d}{dt}\langle y_i(t),v\rangle &\leq \frac{M-1}{M+k-1}\Lambda(M_0-\langle y_i(t),v\rangle)+\frac{k}{M+k-1}(M_0-\langle y_i(t),v\rangle)\\&-\eta_{iL}(t)(M_0-\langle y_i(t),v\rangle)+\eta_{iL}(t)\Big(\frac{M_0+m_0}{2}-\langle y_i(t),v\rangle\Big)\\&=\Lambda(M_0-\langle y_i(t),v\rangle)-\eta_{iL}(t)\frac{M_0-m_0}{2}\\&\leq\Big(\Lambda M_0-\frac{1}{M+k-1}\Gamma\frac{M_0-m_0}{2}\Big)-\Lambda\langle y_i(t),v\rangle.\end{aligned} \tag{2.1.40}$$

Integrating (2.1.40) on $[\alpha_L+\tau,t]$ with $t\in[\alpha_L+\tau,\beta_L+\tau]$, we have that

$$\langle y_i(t),v\rangle\leq e^{-\Lambda(t-\alpha_L-\tau)}\langle y_i(\alpha_L+\tau),v\rangle+\Big(M_0-\frac{1}{M+k-1}\frac{\Gamma}{\Lambda}\frac{M_0-m_0}{2}\Big)(1-e^{-\Lambda(t-\alpha_L-\tau)}).$$

Putting $t=\beta_L+\tau$ in the equation above, we find

$$\langle y_i(\beta_L+\tau),v\rangle\leq M_0-\frac{1}{M+k-1}\frac{\Gamma}{\Lambda}\frac{M_0-m_0}{2}(1-e^{-\Lambda\sigma}). \tag{2.1.41}$$

From (2.1.41), denoting,

$$\delta_-^2:=\frac{1}{2(M+k-1)}\frac{\Gamma}{\Lambda}\Big(1-\frac{m_0}{M_0}\Big)(1-e^{-\Lambda\sigma}), \tag{2.1.42}$$

we deduce that

$$\langle y_i(\beta_L+\tau),v\rangle\leq(1-\delta_-^2)M_0,\ \forall i\in\{1,\dots,h\}. \tag{2.1.43}$$

Consider now $t\in[\beta_L+\tau,6\tau]$. Then,

$$\begin{aligned}\frac{d}{dt}\langle y_i(t),v\rangle &=\sum_{j\neq i}b_{ij}(t)\langle y_j(t)-y_i(t),v\rangle+\sum_{j=1}^{k}\eta_{ij}(t)\langle x_j(t-\tau)-y_i(t),v\rangle\\&\leq\Lambda(M_0-\langle y_i(t),v\rangle).\end{aligned} \tag{2.1.44}$$

Integrating on $[\beta_L+\tau,t]$ with $t\in[\beta_L+\tau,6\tau]$, we have that

$$\begin{aligned}\langle y_i(t),v\rangle&\leq e^{-\Lambda(t-\beta_L-\tau)}\langle y_i(\beta_L+\tau),v\rangle+M_0(1-e^{-\Lambda(t-\beta_L-\tau)})\\&\leq e^{-\Lambda(t-\beta_L-\tau)}(1-\delta_-^2)M_0+M_0(1-e^{-\Lambda(t-\beta_L-\tau)})\\&=M_0(1-\delta_-^2e^{-\Lambda(t-\beta_L-\tau)})\\&\leq M_0(1-\delta_-^2e^{-6\tau\Lambda}),\end{aligned} \tag{2.1.45}$$

where the last inequality is obtained by observing that $t-\beta_L-\tau\leq 6\tau$. Then,

$$\langle y_i(t),v\rangle\leq M_0(1-\delta_-^2e^{-6\tau\Lambda}),\ \forall i\in\{1,\dots,h\},\ t\in[\beta_L+\tau,6\tau],\ i\in\{1,\dots,h\}. \tag{2.1.46}$$

Consider now $i \in \{h+1,\dots,M\}$ and fix $i_1 \in \{1,\dots,h\}$. Taking $t \in [\beta_L+\tau, 6\tau]$, we have

$$\begin{aligned}\frac{d}{dt}\langle y_i(t),v\rangle &= \sum_{\substack{j\neq i\\ j\neq i_1}} b_{ij}(t)\langle y_j(t)-y_i(t),v\rangle + b_{ii_1}(t)\langle y_{i_1}(t)-y_i(t),v\rangle\\ &\leq \Big(\sum_{j\neq i} b_{ij}(t)-b_{ii_1}(t)\Big)(M_0-\langle y_i(t),v\rangle) + b_{ii_1}(t)[M_0(1-\delta_-^2 e^{-6\tau\Lambda})-\langle y_i(t),v\rangle]\\ &\leq \Lambda M_0\Big[1-\frac{1}{M-1}\frac{\Gamma}{\Lambda}\delta_-^2 e^{-6\tau\Lambda}\Big]-\Lambda\langle y_i(t),v\rangle.\end{aligned} \tag{2.1.47}$$

Applying the Grönwall inequality on $[\beta_L+\tau,t]$ with $t \in [\beta_L+\tau,6\tau]$, from (2.1.47) we find that

$$\begin{aligned}\langle y_i(t),v\rangle &\leq e^{-\Lambda(t-\beta_L-\tau)}\langle y_i(\beta_L+\tau),v\rangle + M_0\Big[1-\frac{1}{M-1}\frac{\Gamma}{\Lambda}\delta_-^2 e^{-6\tau\Lambda}\Big](1-e^{-\Lambda(t-\beta_L-\tau)})\\ &\leq M_0\Big[1-\frac{1}{M-1}\frac{\Gamma}{\Lambda}\delta_-^2 e^{-6\tau\Lambda}(1-e^{-\Lambda(t-\beta_L-\tau)})\Big].\end{aligned}$$

Shrinking to $t \in [2\tau,6\tau]$, noticing that $t-\beta_L-\tau \geq \tau$, we have that

$$\langle y_i(t),v\rangle \leq M_0\Big[1-\frac{1}{M-1}\frac{\Gamma}{\Lambda}\delta_-^2 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})\Big],\ \forall i \in \{h+1,\dots,M\}. \tag{2.1.48}$$

Consider now $i \in \{1,\dots,k\}$ and fix $i_2 \in \{1,\dots,h\}$. Notice that could be that $i_2 = i_1$. For $t \in [2\tau,6\tau]$, we have

$$\begin{aligned}\frac{d}{dt}\langle x_i(t),v\rangle &= \sum_{j\neq i} a_{ij}(t)\langle x_j(t)-x_i(t),v\rangle + \sum_{\substack{j=1\\ j\neq i_2}}^{h}\epsilon_{ij}(t)\langle y_j(t-\tau)-x_i(t),v\rangle\\ &\quad+\epsilon_{ii_2}(t)\langle y_{i_2}(t)-x_i(t),v\rangle\\ &\leq \Lambda M_0\Big[1-\frac{1}{N+h-1}\frac{\Gamma}{\Lambda}\delta_-^2 e^{-6\tau\Lambda}\Big]-\Lambda\langle x_i(t),v\rangle.\end{aligned}$$

Integrating over $[2\tau,t]$ with $t \in [2\tau,6\tau]$, we find

$$\langle x_i(t),v\rangle \leq e^{-\Lambda(t-2\tau)}\langle x_i(2\tau),v\rangle + M_0\Big(1-\frac{1}{N+h-1}\frac{\Gamma}{\Lambda}\delta_-^2 e^{-6\tau\Lambda}\Big)(1-e^{-\Lambda(t-2\tau)}).$$

Shrinking to $t \in [3\tau,6\tau]$ we have

$$\langle x_i(t),v\rangle \leq M_0\Big(1-\frac{1}{N+h-1}\frac{\Gamma}{\Lambda}\delta_-^2 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})\Big),\ \forall i \in \{1,\dots,k\}. \tag{2.1.49}$$

Finally, we consider $i \in \{k+1,\dots,N\}$ and fix $i_3 \in \{1,\dots,k\}$. Taking $t \in [3\tau,6\tau]$ we have

$$\begin{aligned}\frac{d}{dt}\langle x_i(t),v\rangle &= \sum_{\substack{j\neq i\\ j\neq i_3}} a_{ij}(t)\langle x_j(t)-x_i(t),v\rangle + a_{ii_3}(t)\langle x_{i_3}(t)-x_i(t),v\rangle\\ &\leq \Lambda M_0\Big[1-\frac{1}{(N-1)(N+h-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^2\delta_-^2 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})\Big]-\Lambda\langle x_i(t),v\rangle.\end{aligned}$$

Integrating the inequality above over $[3\tau, t]$ for $t \in [3\tau, 6\tau]$, we find that

$$\langle x_i(t), v\rangle \le e^{-\Lambda(t-3\tau)}\langle x_i(3\tau), v\rangle \\ + M_0\Big[1 - \frac{1}{(N-1)(N+h-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^2 \delta_-^2 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})\Big](1-e^{-\Lambda(t-3\tau)}).$$

Shrinking to $t \in [4\tau, 6\tau]$, we finally have that

$$\langle x_i(t), v\rangle \le M_0\Big(1 - \frac{1}{(N-1)(N+h-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^2 \delta_-^2 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})^2\Big), \ \forall i \in \{k+1, \dots, N\}. \tag{2.1.50}$$

As in the previous case, the estimate (2.1.50) holds for all the possible states of the system for $t \in [5\tau, 6\tau]$.

Step 3: From *Step 1* and *Step 2*, using the definitions (2.1.25) and (2.1.42), we have then

$$\langle x_i(t), v\rangle \le M_0\Big[1 - \frac{1}{2(N-1)(M-1)(M+k-1)(N+h-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^4 \times \\ \times e^{-6\tau\Lambda}(1-e^{-\Lambda\sigma})(1-e^{-\Lambda\tau})^3\Big(1-\frac{m_0}{M_0}\Big)\Big], \tag{2.1.51}$$

$\forall i \in \{1, \dots, N\}$, $t \in [5\tau, 6\tau]$, and

$$\langle y_i(t), v\rangle \le M_0\Big[1 - \frac{1}{2(N-1)(M-1)(M+k-1)(N+h-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^4 \times \\ \times e^{-6\tau\Lambda}(1-e^{-\Lambda\sigma})(1-e^{-\Lambda\tau})^3\Big(1-\frac{m_0}{M_0}\Big)\Big], \tag{2.1.52}$$

$\forall i \in \{1, \dots, M\}$, $t \in [5\tau, 6\tau]$. Analogous estimates can be obtained if m_0 or M_0 is attained by scalar products $\langle y_i, v\rangle$, $i = 1, \dots, M$. Since

$$(N-1)(M-1)(N+h-1)(M+k-1) \le 4N^4,$$

from (2.1.51) and (2.1.52), we obtain the second inequalities of (2.1.21) and (2.1.22), respectively, with

$$\Gamma_1 := \frac{1}{8N^4}\Big(\frac{\Gamma}{\Lambda}\Big)^4 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})^3(1-e^{-\Lambda\sigma}). \tag{2.1.53}$$

Step 4: Now, we focus on the lower bound in (2.1.21) and (2.1.22).
Assume that there exists $R \in \{k+1, \dots, N\}$ such that $\langle x_R(0), v\rangle = M_0$. Then, as before, we have that

$$\frac{M_0 + m_0}{2} \le \langle x_R(t), v\rangle \le M_0, \ t \in [0, \sigma],$$

with σ as in (2.1.23). Using similar arguments to the ones in *Step 1*, we find that, for $t \in [5\tau, 6\tau]$,

$$\langle x_i(t), v\rangle \ge m_0\Big[1 + \frac{1}{(M-1)(N+h-1)(M+k-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^3 \delta_+^1 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})^3\Big], \\ \forall i \in \{1, \dots, N\}, \tag{2.1.54}$$

and

$$\langle y_i(t), v\rangle \geq m_0\Big[1+\frac{1}{(M-1)(N+h-1)(M+k-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^3\delta_+^1 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})^3\Big], \quad \forall i\in\{1,\dots,M\}, \tag{2.1.55}$$

with

$$\delta_+^1 := \frac{1}{2(N-1)}\frac{\Gamma}{\Lambda}(1-e^{-\Lambda\sigma})\Big(\frac{M_0}{m_0}-1\Big). \tag{2.1.56}$$

Suppose, instead, that $\langle x_R(t), v\rangle = M_0$ for some $R\in\{1,\dots,k\}$ and for some $t\in[-\tau,0]$. Then, by continuity, there exists a closed interval $[\alpha_R,\beta_R]\subset[-\tau,0]$ such that

$$\frac{M_0+m_0}{2}\leq \langle x_R(t), v\rangle \leq M_0,\ t\in[\alpha_R,\beta_R].$$

Eventually choosing a smaller σ in (2.1.23) above, we may assume that $\beta_R-\alpha_R=\sigma$. Arguing analogously to *Step 2*, we can obtain, for $t\in[5\tau,6\tau]$,

$$\langle x_i(t), v\rangle \geq m_0\Big[1+\frac{1}{(N-1)(N+h-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^2\delta_+^2 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})^2\Big], \quad \forall i\in\{1,\dots,N\}, \tag{2.1.57}$$

and

$$\langle y_i(t), v\rangle \geq m_0\Big[1+\frac{1}{(N-1)(N+h-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^2\delta_+^2 e^{-6\tau\Lambda}(1-e^{-\Lambda\tau})^2\Big], \quad \forall i\in\{1,\dots,M\}, \tag{2.1.58}$$

with

$$\delta_+^2 := \frac{1}{2(M+k-1)}\frac{\Gamma}{\Lambda}(1-e^{-\Lambda\sigma})\Big(\frac{M_0}{m_0}-1\Big). \tag{2.1.59}$$

Now, note that the right-hand side of (2.1.54) and (2.1.55) are smaller than the right-hand side of (2.1.57) and (2.1.58) and so, using the definitions (2.1.56) and (2.1.59), we have that, for $t\in[5\tau,6\tau]$,

$$\langle x_i(t), v\rangle \geq m_0\Big[1+\frac{1}{2(N-1)(M-1)(M+k-1)(N+h-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^4\times \\ \times e^{-6\tau\Lambda}(1-e^{-\Lambda\sigma})(1-e^{-\Lambda\tau})^3\Big(\frac{M_0}{m_0}-1\Big)\Big],$$

$\forall i\in\{1,\dots,N\}$, and

$$\langle y_i(t), v\rangle \geq m_0\Big[1+\frac{1}{2(N-1)(M-1)(M+k-1)(N+h-1)}\Big(\frac{\Gamma}{\Lambda}\Big)^4\times \\ \times e^{-6\tau\Lambda}(1-e^{-\Lambda\sigma})(1-e^{-\Lambda\tau})^3\Big(\frac{M_0}{m_0}-1\Big)\Big],$$

$\forall i\in\{1,\dots,M\}$. Using the definition (2.1.53), from the last two inequalities we obtain the lower bounds in the lemma's statement. This completes the proof. □

2.2 Asymptotic consensus

In this section, we will show the asymptotic convergence to consensus of solutions to (2.0.1).

Definition 2.2.1. *For fixed $v \in \mathbb{R}^d$ unit vector, for all $n \in \mathbb{N}$, one can define the quantities M_n and m_n as follows:*

$$\begin{aligned} m_n := \min\Big\{ &\min_{i=1,\dots,k} \min_{t \in I_n} \langle x_i(t), v\rangle, \min_{i=k+1,\dots,N} \langle x_i(6n\tau), v\rangle, \\ &\min_{i=1,\dots,h} \min_{t \in I_n} \langle y_i(t), v\rangle, \min_{i=h+1,\dots,M} \langle y_i(6n\tau), v\rangle \Big\}, \end{aligned} \tag{2.2.60}$$

$$\begin{aligned} M_n := \max\Big\{ &\max_{i=1,\dots,k} \max_{t \in I_n} \langle x_i(t), v\rangle, \max_{i=k+1,\dots,N} \langle x_i(6n\tau), v\rangle, \\ &\max_{i=1,\dots,h} \max_{t \in I_n} \langle y_i(t), v\rangle, \max_{i=h+1,\dots,M} \langle y_i(6n\tau), v\rangle \Big\}, \end{aligned} \tag{2.2.61}$$

with $I_n = [(6n-1)\tau, 6n\tau]$. Notice that for $n = 0$ we recover (2.1.6) and (2.1.7).

Theorem 2.2.2. *Let $(x_i(t), y_j(t))$, $i = 1, \dots, N$, $j = 1, \dots, M$, be a global classical solution to the system (2.0.1) with continuous initial conditions (2.0.5).Then, $(x_i(t), y_j(t))$, $i = 1, \dots, N$, $j = 1, \dots, M$, achieve an asymptotic consensus in the sense of Definition 1.2.1.*

Proof. Using (2.2.60) and (2.2.61), we define the quantities $D_n := M_n - m_n$. Moreover, let us denote

$$\Gamma_{1n} := \frac{1}{8N^4}\Big(\frac{\Gamma}{\Lambda}\Big)^4 e^{-6\tau\Lambda}(1 - e^{-\Lambda\tau})^3(1 - e^{-\Lambda\sigma_n}),$$

where $\sigma_n := \min\{\tau, \frac{M_n - m_n}{4\Lambda M_0}\}$, for $n \geq 1$, and $\sigma_0 = \sigma$ as in (3.2.14). Then, $\Gamma_1 = \Gamma_{10}$ and $\Gamma_{1n} \in (0,1)$ if $M_n > m_n$. Now, we use Lemma 2.1.4 with $t \in I_n$, $n \in \mathbb{N}$. For $n = 1$, we have

$$D_1 = M_1 - m_1 \leq M_0 - \frac{\Gamma_{10}}{2}(M_0 - m_0) - m_0 - \frac{\Gamma_{10}}{2}(M_0 - m_0) = (M_0 - m_0)(1 - \Gamma_{10}) = D_0(1 - \Gamma_{10}).$$

So, we find that $D_1 \leq (1 - \Gamma_{10})D_0$. Iterating the process of Lemma 2.1.4 we can find

$$D_{n+1} \leq (1 - \Gamma_{1n})D_n, \ \forall n \in \mathbb{N}.$$

Let us denote

$$\tilde{\sigma}(D) := \min\Big\{\tau, \frac{D}{4\Lambda M_0}\Big\},$$

so that $\sigma_n = \tilde{\sigma}(D_n)$, $\forall n \in \mathbb{N}$, $n \geq 1$. Moreover, let be

$$\tilde{\Gamma}_1(D) := \frac{1}{8N^4}\Big(\frac{\Gamma}{\Lambda}\Big)^4 e^{-6\tau\Lambda}(1 - e^{-\Lambda\tau})^3(1 - e^{-\Lambda\tilde{\sigma}(D)}),$$

so that $\Gamma_{1n} = \tilde{\Gamma}_1(D_n)$, $\forall n \in \mathbb{N}$. Therefore, we have

$$D_{n+1} \leq (1 - \tilde{\Gamma}(D_n))D_n.$$

Then, $\{D_n\}_{n\in\mathbb{N}}$ is a non-negative and decreasing sequence. Let us calling D the limit of $\{D_n\}_{n\in\mathbb{N}}$ and, passing to the limit as n goes to $+\infty$ in the above estimate, we find

$$D \leq (1 - \tilde{\Gamma}_1(D))D,$$

that is true if and only if $\tilde{\Gamma}_1(D) \leq 0$. This gives $D = 0$ and, noticing that,

$$\langle x_i(t) - x_j(t), v\rangle \leq M_n - m_n = D_n$$

for all $i, j = 1, \ldots, N$, we have that $\langle x_i(t) - x_j(t), v\rangle \to 0$ as $t \to +\infty$ and for all $i, j = 1, \ldots, N$. The same holds for $\langle y_i(t) - y_j(t), v\rangle$, with $i, j = 1, \ldots, M$, and for $\langle x_i(t) - y_j(t), v\rangle$, with $i = 1, \ldots, N$ and $j = 1, \ldots, M$.
Notice that the result above can be obtained for each unit vector $v \in \mathbb{R}^d$. In particular, by considering the canonical basis of $\mathbb{R}^d$, $\{e_h\}_{h=1}^d$, and taking $v = e_h$ we have that

$$|\langle x_i(t) - x_j(t), e_h\rangle| \to 0,$$

as $t \to +\infty$, for all $i, j = 1, \ldots, N$ and $h = 1, \ldots, d$. The same happens to $|\langle y_i(t) - y_j(t), e_h\rangle|$, for all $i, j = 1, \ldots, M$, and to $|\langle x_i(t) - y_j(t), e_h\rangle|$, for all $i = 1, \ldots, N$ and $j = 1, \ldots, M$. Then, the system achieves asymptotic consensus. □

2.3 Numerical simulations

In this section, we present some numerical tests for the system (2.0.1) in the one-dimensional case, i.e., $d = 1$. We consider the weight functions $a_{ij}(t)$ and $b_{ij}(t)$ defined by

$$\psi(r, r') = \psi^*(r, r') := \tilde{\psi}(|r - r'|), \quad r, r' \in [0, +\infty).$$

Meanwhile, the weight functions $\epsilon_{ij}(t)$ and $\eta_{ij}(t)$ are assumed to be constant.

In particular, we consider the functions

$$\begin{aligned}
&\tilde{\psi}(r) := e^{-(r-1)^2}, \ r \in [0, +\infty), \\
&\epsilon_{ij}(t) := \frac{K_1}{N + h - 1}, \ \forall\, i \in \{1, \ldots, k\}, \ j \in \{1, \ldots, h\} \\
&\eta_{ij}(t) := \frac{K_2}{M + k - 1}, \ \forall\, i \in \{1, \ldots, h\}, \ j \in \{1, \ldots, k\},
\end{aligned} \tag{2.3.62}$$

with K_1, K_2 positive constants.

In Figure 2.1, the top two graphics illustrate a scenario where one population is larger than the other, yet the influence of the leaders from the smaller population overpowers that of the larger one ($K_1 \gg K_2$). As expected, the larger population tends to converge towards the consensus of the smaller population. The bottom two graphics depict a similar scenario but with equal influence from both sets of leaders ($K_1 = K_2$). In this case, it is observed that the larger population pulls the smaller one towards its consensus.

Moving to Figure 2.2, the top two graphics illustrate a scenario where the total number of agents in both populations is equal, but the distribution of leaders differs ($k = 4, \ h = 1$). Furthermore, the solitary leader in the second population holds more influence than the others ($K_2 \gg K_1$). It is noticeable that the system tends towards a consensus closer to the initial average of the population with only one leader. This is due to the different normalization factors of the weight functions. In the bottom two graphics of Figure 2.2, a similar scenario is presented, but with equally strong influence from leaders on both sides ($K_1 = K_2$).

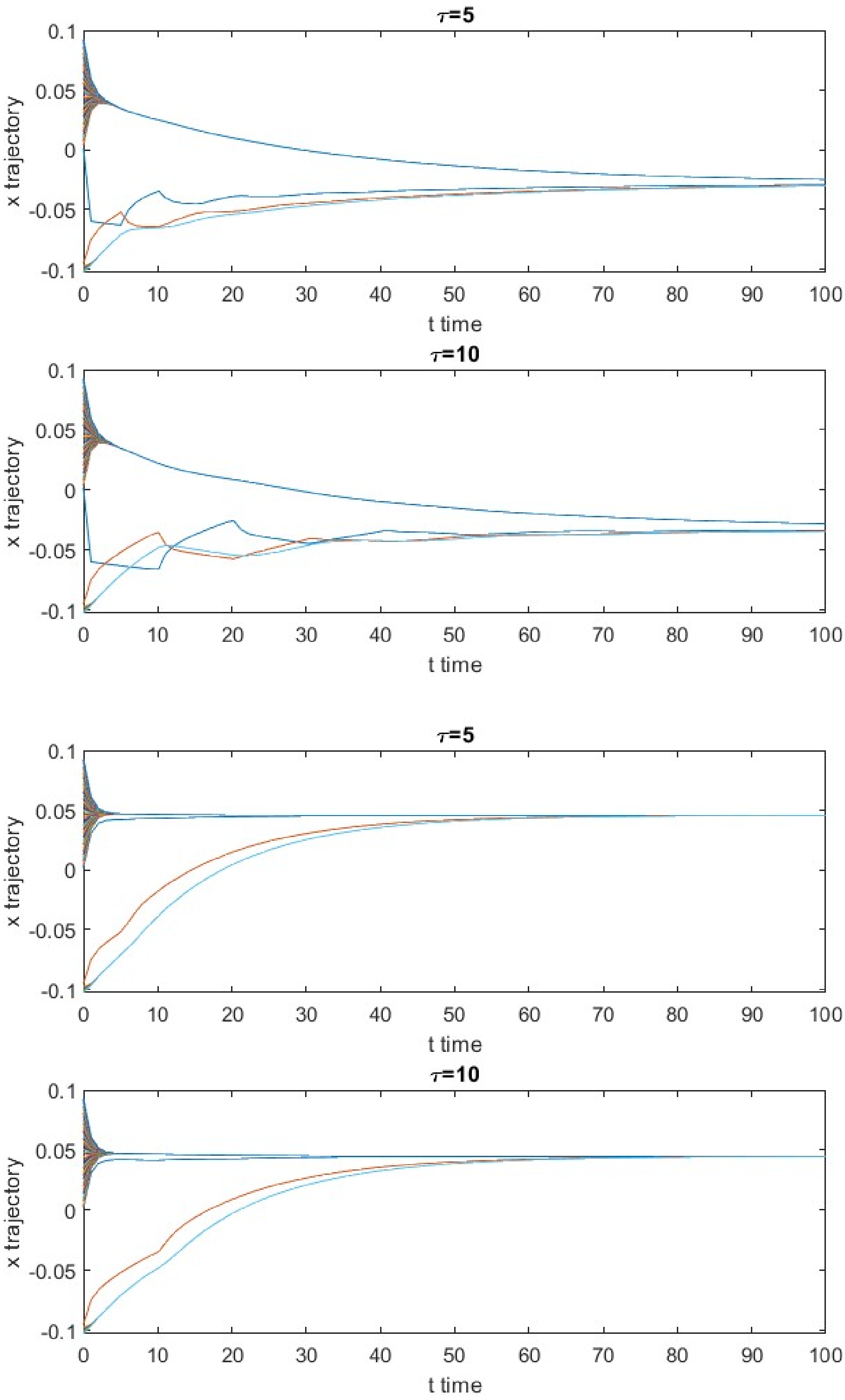


Figure 2.1: Time evolution of solutions with different time delays, number of agents $N = 50$, $M = 5$, number of leaders $k = h = 1$.

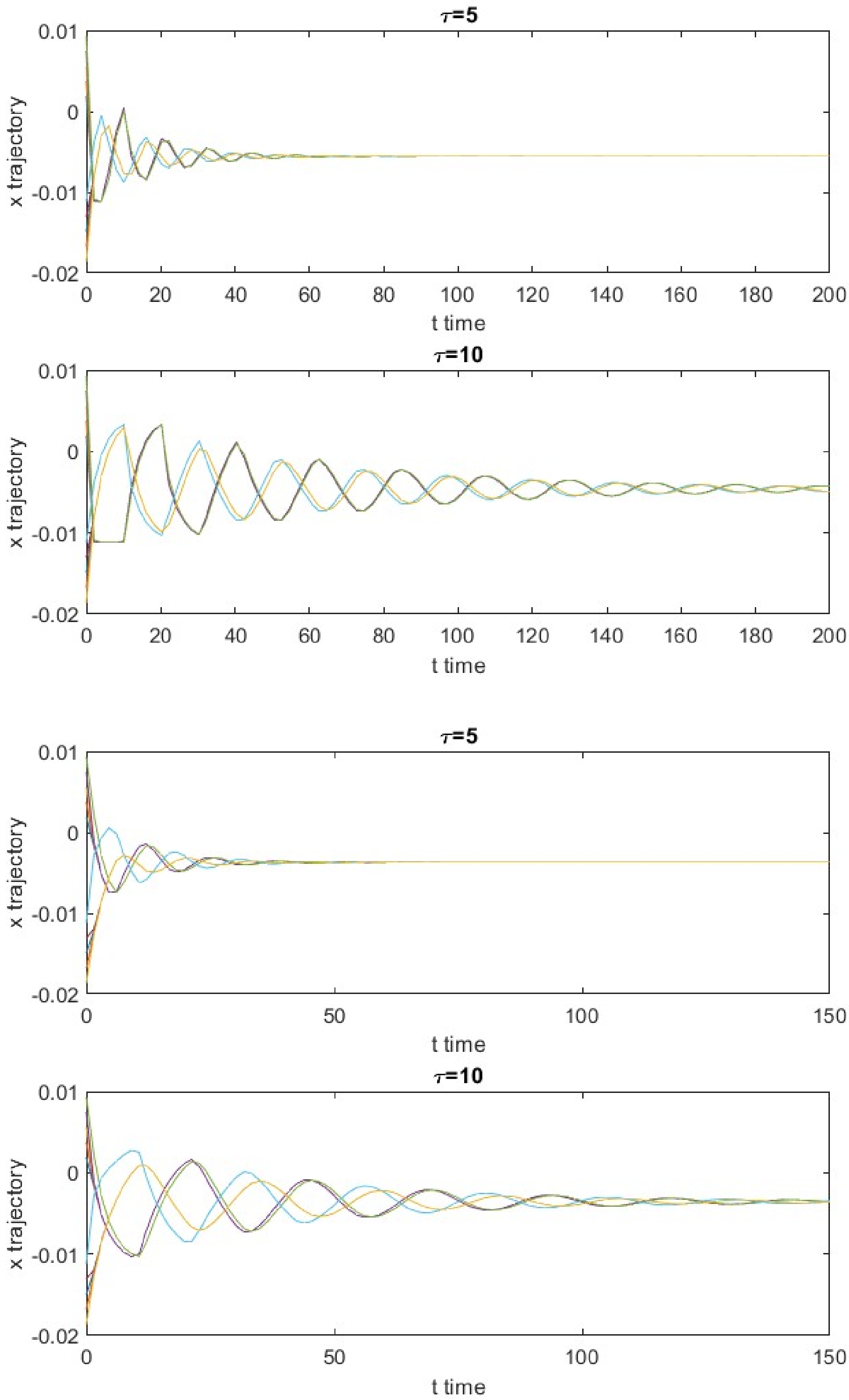


Figure 2.2: Time evolution of solutions with different time delays, number of agents $N = M = 5$, number of leaders $k = 4$, $h = 1$.

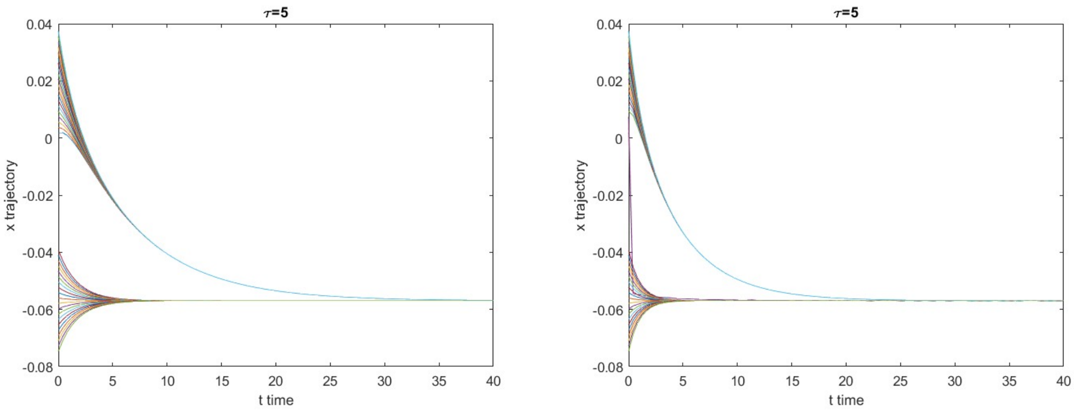


Figure 2.3: Opinion formation in an ecological discussion: to the left, number of agents $N = M = 20$, number of leaders $k = h = 20$; to the right, number of agents $N = M = 20$, number of leaders $k = 4,\ h = 20$.

Here, it is observed that the consensus converges towards a mean value of the initial states.

In this work, we used a Hegselmann-Krause type model to explore social dynamics and opinion formation in a set of two interacting populations, particularly in the context of discussing ecology strategies and sustainable development. We conducted simulations to investigate two different scenarios:

1. Equal Influence Scenario (Figure 2.3, Left): In this scenario, all agents are considered equal, meaning the total population coincides with the leaders' subgroup. However, only one group has a mild influence on the other group ($k = h = N = M$, $K_1 = 0.3$, $K_2 = 0$). This could represent a situation with only one group spreading ecological information on social media.

2. Asymmetric Influence Scenario (Figure 2.3, Right): In this scenario, one group has a significant influence over a subgroup of the other population ($K_1 = 30$, $K_2 = 0$, $k < h$). This could represent a scenario where a community of scientists interacts with a leading group in the other population, e.g., a politician's group.

We observed that with a fixed time delay ($\tau = 5$), consensus is reached more rapidly in the second scenario. This implies that exerting a strong influence on decision-makers who, in turn, influence the entire population, leads to faster consensus formation. Therefore, we conclude that the most effective strategy for raising awareness on ecological topics is to exert a strong influence on key decision-makers who can influence the entire population. This highlights the importance of targeting influential individuals or groups in shaping public opinion and fostering consensus on issues, e.g., related to ecology and sustainability.

Chapter 3

Asymptotic synchronization of Kuramoto oscillators

In this chapter, we focus on the emergence of frequency synchronization in (1.4.26) with some interaction network and time delay, namely, we identify the class of initial configurations leading to the fact that all oscillators tend to have the same frequency as time goes to infinity. To begin with, we refer to basic concepts related to the direct graphs introduced in Chapter 1 Section 1.2.

For a solution $\theta(t) := (\theta_1(t), \cdots, \theta_N(t))$ to (1.4.26), we want to prove the complete frequency synchronization in the sense of Definition 1.4.1. We consider the phase and velocity diameters as in (1.4.28). Along with those notations, we also set

$$D(\Omega) := \max_{1\le i,j\le N} |\Omega_i - \Omega_j|,$$

$$D_\theta(0) := \max_{1\le i,j\le N, s,t\in[-\tau,0]} |\theta_i(s)-\theta_j(t)|, \quad \text{and} \quad D_\omega(0) := \max_{1\le i,j\le N, s,t\in[-\tau,0]} |\omega_i(s)-\omega_j(t)|. \tag{3.0.1}$$

Let us now state our main asymptotic frequency synchronization results in a somewhat rough manner. The precise statements are given in Theorem 3.2.7 (strongly connected case) and Theorem 3.3.1 (all-to-all connected case).

Theorem 3.0.1. *Let* $\{\theta_i(t)\}_{i=1}^N$ *be a solution to the Kuramoto model* (1.4.26) *with initial data* (1.4.27)*. We have the following asymptotic frequency synchronization results.*

(i) (strongly connected case) Assume the digraph $\mathcal{G}$ *is strongly connected, i.e., each vertex can be reached from any vertex. Suppose that* $D_\theta(0) < \pi$*,* τ *is sufficiently small, and* κ *is large enough. Then the time-delayed Kuramoto oscillators* (1.4.26) *achieve the asymptotic complete frequency synchronization in the sense of Definition 1.4.1.*

(ii) (all-to-all connected case) Assume that all oscillators are connected, i.e., $\chi_{ij} = 1$ *for all* $i, j = 1, \ldots, N$*. Suppose that* $D_\theta(0) < \pi$*,* τ *is sufficiently small, and* κ *is large enough. Then the time-delayed Kuramoto oscillators* (1.4.26) *achieve the asymptotic complete frequency synchronization in the sense of Definition 1.4.1 exponentially fast.*

The initial step of our analysis involves establishing a uniform-in-time bound estimate of the phase diameter. Specifically, we demonstrate that for sufficiently large times, the oscillators will be confined within a region of a quarter circle under suitable assumptions on the initial configurations. This bound estimate plays a critical role in unveiling the dissipation structure of (1.4.26), and thus it serves as a foundational element for deriving the asymptotic frequency synchronization result. Notably, our strategy improves the previous work [41] where the initial phase diameter $D_\theta(0)$ is assumed to be less than $\pi/2$. In the case of an all-to-all connection, i.e., the oscillators are all connected, we achieve a more robust result, namely an exponential asymptotic synchronization estimate. These results extend previous results in the literature. In particular, compared to [27], we introduce considerations for the network structure and pair-dependent delays. Moreover, we significantly relax previous assumptions on the time delay size (see e.g. [27, 79]).

3.1 Uniform-in-time bound of phase diameter

In this section, we provide the uniform-in-time bound estimate of the phase diameter. Specifically, our goal of this section is to show that the phase diameter $d_\theta(t)$ is bounded by some $d_\infty \in (0, \frac{\pi}{2})$ for any t large enough under suitable assumptions on the initial configurations. For this, we first start with the estimate providing the bound on the difference between the time-delayed and non-time-delayed phases. Note that we can easily find

$$|\omega_i(t)| \leq |\Omega_i| + \kappa \leq \max_{1\leq i\leq N} |\Omega_i| + \kappa,$$

for all $i = 1, \cdots, N$. Let us denote

$$R_\omega := \max_{1\leq i\leq N} |\Omega_i| + \kappa, \tag{3.1.2}$$

then

$$|\omega_i(t)| \leq R_\omega, \quad \forall\, t \geq 0. \tag{3.1.3}$$

From (3.1.3), we deduce that for all $i = 1, \dots, N$,

$$|\theta_i(t) - \theta_i(s)| = \left| \int_s^t \omega_i(r)dr \right| \leq R_\omega |t-s|, \quad \forall\, s, t \geq 0.$$

In particular, for all $i, j = 1, \dots, N$, we have

$$|\theta_i(t - \tau_{ij}) - \theta_i(t)| \leq R_\omega \tau, \quad t \geq 0.$$

For the uniform-in-time bounded estimate of phase diameter $d_\theta(t)$, we need to use the dissipative structure of the system (1.4.26). For this, motivated by [76], we define an ensemble of oscillators as a convex combination denoted by

$$\mathcal{L}_l^k\,(C_{l,k}) := \sum_{i=l}^{k} c_i\theta_i, \quad 1 \leq l \leq k \leq N,$$

where all c_i are non-negative and $\sum_{i=l}^k c_i = 1$. We also introduce notions of root and general root in the definition below.

Definition 3.1.1 (Root and General Root)**.**

(i) We say θ_k is a root if it is not influenced by any other oscillators, i.e., $i \notin \mathcal{N}_k$ for all $i \neq k$.

(ii) An ensemble of oscillators $\mathcal{L}_l^k\,(C_{l,k})$ is a general root if it is not influenced by any oscillators excluded from the ensemble, i.e. $j \notin \mathcal{N}_i$ for all $i \in \{l,\dots,k\}$ and $j \in \{1,\dots,N\}\backslash\{l,\dots,k\}$.

For the analysis, we use the following algorithm, denoted by $\mathcal{A}$, proposed in [46] for constructing convex combinations of oscillators.

Step I For any time t, we reorder the oscillator indices such that the phases are increasing from minimum to maximum:

$$\theta_1(t) \le \theta_2(t) \le \cdots \le \theta_N(t).$$

For the next steps, we introduce the following sub-algorithms:

($\mathcal{A}_1$): If $\overline{\mathcal{L}}_k^N\left(\bar{C}_{k,N}\right)$ is not a general root, then we construct (from top to bottom)

$$\overline{\mathcal{L}}_{k-1}^N\left(\bar{C}_{k-1,N}\right) = \frac{\bar{a}_{k-1}\overline{\mathcal{L}}_k^N\left(\bar{C}_{k,N}\right) + \theta_{k-1}}{\bar{a}_{k-1}+1}.$$

($\mathcal{A}_2$): If $\underline{\mathcal{L}}_1^l\left(\underline{C}_{1,l}\right)$ is not a general root, then we construct (from bottom to top)

$$\underline{\mathcal{L}}_1^{l+1}\left(\underline{C}_{1,l+1}\right) = \frac{\underline{a}_{l+1}\underline{\mathcal{L}}_1^l\left(\underline{C}_{1,l}\right) + \theta_{l+1}}{\underline{a}_{l+1}+1}.$$

Step II Since $\mathcal{G}$ is strongly connected, we have that $\overline{\mathcal{L}}_1^N\left(\bar{C}_{1,N}\right)$ is a general root, and $\overline{\mathcal{L}}_k^N\left(\bar{C}_{k,N}\right)$ is not a general root for $k > 1$. Therefore, we may start from θ_N and follow the process $\mathcal{A}_1$ to construct $\overline{\mathcal{L}}_k^N\left(\bar{C}_{k,N}\right)$ until $k = 1$.

Step III Similarly, $\underline{\mathcal{L}}_1^N\left(\underline{C}_{1,N}\right)$ is a general root and $\underline{\mathcal{L}}_1^l\left(\underline{C}_{1,l}\right)$ is not a general root for $l < N$. Therefore, we may analogously start from θ_1 and follow the process $\mathcal{A}_2$ until $l = N$.

Remark 3.1.2. *The coefficients in the above constructions are determined inductively according to*

$$\overline{\mathcal{L}}_{k-1}^N\left(\bar{C}_{k-1,N}\right) \text{ with } \bar{a}_N = 0,\ \bar{a}_{k-1} = \eta\,(2N-k+2)\,(\bar{a}_k+1)\,, \quad 2 \le k \le N,$$
$$\underline{\mathcal{L}}_1^{k+1}\left(\underline{C}_{1,k+1}\right) \text{ with } \underline{a}_1 = 0,\ \underline{a}_{k+1} = \eta\,(k+1+N)\,(\underline{a}_k+1)\,, \quad 1 \le k \le N-1$$

or in summation form:

$$\bar{a}_{k-1} = \sum_{j=1}^{N-k+1} \eta^j P\,(2N-k+2, j)\,, \quad 2 \le k \le N,$$

$$\underline{a}_{k+1} = \sum_{j=1}^{k} \eta^j P\,(k+1+N, j)\,, \quad 1 \le k \le N-1,$$

where η is a positive parameter and $P(m,k)$ denotes the k-permutations of m, i.e. the number of permutations of k elements arranged in a specific order in a set of m elements:

$$P(m,k) := \frac{m!}{(m-k)!} = m\cdot(m-1)\cdots(m-k+1).$$

We now set

$$\bar{\theta}_k := \overline{\mathcal{L}}_k^N \left(\bar{C}_{k,N}\right), \quad \underline{\theta}_k := \underline{\mathcal{L}}_1^k \left(\underline{C}_{1,k}\right), \quad 1 \le k \le N,$$

and define a non-negative $q_\theta(t)$ quantity which will be used to control the phase diameter:

$$q_\theta := \bar{\theta}_1 - \underline{\theta}_N.$$

By using those newly defined notations, we state two lemmas on a monotone property of the interaction term and a relation between $q_\theta(t)$ and $d_\theta(t)$ whose proofs can be found in [76, Lemma 4.1] and [76, Lemma 4.2], respectively. Here, we notice that these lemmas depend only on the graph structure.

Lemma 3.1.3. *Consider the strongly connected network $\mathcal{G}$, with phases well-ordered according to $\mathcal{A}$. Moreover, we assume that the phase diameter and free parameter η satisfy*

$$d_\theta(t) < \zeta < \xi < \pi \quad \text{and} \quad \eta > \max\left\{\frac{1}{\sin\xi}, \frac{1}{\cos\left(R_\omega\tau\right)}, \frac{2}{1-\frac{\zeta}{\xi}}\right\}, \tag{3.1.4}$$

respectively. Then, we have

$$\sum_{i=n}^{N}\left(\eta^{i-n} \min_{\substack{j\in\mathcal{N}_i \\ j\le i}} \sin\left(\theta_j - \theta_i\right)\right) \le \sin\left(\theta_{\bar{k}_n} - \theta_N\right)$$

and

$$\sum_{i=1}^{n}\left(\eta^{n-i} \max_{\substack{j\in\mathcal{N}_i \\ j\ge i}} \sin\left(\theta_j^0 - \theta_i^0\right)\right) \ge \sin\left(\theta_{\underline{k}_n} - \theta_1\right),$$

where

$$\bar{k}_n := \min_{j\in\cup_{i=n}^N\mathcal{N}_i} j \quad \text{and} \quad \underline{k}_n := \max_{j\in\cup_{i=1}^n\mathcal{N}_i} j \quad \text{for } 1 \le n \le N.$$

Lemma 3.1.4. *Consider the strongly connected network $\mathcal{G}$, with the coefficients of the convex combinations satisfying the conditions according to algorithm $\mathcal{A}$. Then we have*

$$\beta d_\theta(t) \le q_\theta(t) \le d_\theta(t) \ \text{with}\ \beta = 1 - \frac{2}{\eta},$$

where η satisfies (3.1.4).

We next provide a result concerning the dynamics of $q_\theta(t)$, which will serve as a cornerstone in deriving our phase bound.

Lemma 3.1.5. *Let $\{\theta_i(t)\}_{i=1}^N$ be a solution to the Kuramoto model* (1.4.26) *on a strongly connected digraph $\mathcal{G}$, with initial conditions satisfying*

$$D_\theta(0) < \pi.$$

Let ζ and ξ such that $D_\theta(0) < \zeta < \xi < \pi$ and let d_∞, η be two parameters such that

$$d_\infty < \min\left\{\frac{\pi}{2}, d_\theta(0)\right\}, \qquad \eta > \max\left\{\frac{1}{\sin\xi}, \frac{1}{\cos\left(R_\omega\tau\right)}, \frac{2}{1-\frac{\zeta}{\xi}}\right\},$$

where R_ω is the bound on the velocities defined in (3.1.2). *Assume the following conditions hold:*

$$\tan(R_\omega\tau) < \frac{\beta d_\infty}{\left(1+\frac{\zeta}{\zeta-D_\theta(0)}\right)2(N-1)c},$$
$$d_\infty + R_\omega\tau < \frac{\pi}{2}, \quad \kappa > \left(1+\frac{\zeta}{\zeta-D_\theta(0)}\right)\frac{(D(\Omega)+2\kappa\sin(R_\omega\tau))\,(N-1)c}{2\cos(R_\omega\tau)}\frac{1}{\beta d_\infty},$$

where

$$c := \frac{\left(\sum_{j=1}^{N-1}\eta^j P\,(2N,j)+1\right)\xi}{\sin\xi}.$$

Then we have

$$d_\theta(t) < \xi, \quad \forall\, t \in [0,\infty)$$

and

$$\dot q_\theta(t) \le D(\Omega) + 2\kappa\sin(R_\omega\tau) - \frac{2\kappa\cos(R_\omega\tau)}{(N-1)c}q_\theta(t), \quad a.e.\ t\in[0,+\infty).$$

Remark 3.1.6. *From Lemma 3.1.5, by applying the Grönwall's lemma, we find*

$$q_\theta(t) \le q_\theta(0)\exp\left(-\frac{2\kappa\cos(R_\omega\tau)}{(N-1)c}t\right) + \frac{(D(\Omega)+2\kappa\sin(R_\omega\tau))(N-1)c}{2\kappa\cos(R_\omega\tau)}\left(1-\exp\left(-\frac{2\kappa\cos(R_\omega\tau)}{(N-1)c}t\right)\right).$$

In particular, we obtain

$$q_\theta(t) \le \max\left\{d_\theta(0), \frac{(D(\Omega)+2\kappa\sin(R_\omega\tau))(N-1)c}{2\kappa\cos(R_\omega\tau)}\right\}, \quad \forall\, t\in[0,\infty).$$

Proof of Lemma 3.1.5. Let us denote Ω_M and Ω_m the maximal and minimal natural frequency, respectively, i.e.

$$\Omega_M := \max_{i=1,\ldots,N}\Omega_i \quad \text{and} \quad \Omega_m := \min_{i=1,\ldots,N}\Omega_i.$$

We proceed with the proof in three steps.

Step I We first define a set $\mathcal{S} := \{T>0 : d_\theta(t)<\xi,\ \forall\, t\in[0,T)\}$. Since $D_\theta(0)<\xi$ and $d_\theta(t)$ is continuous, the set $\mathcal{S}$ is non-empty. Thus, we can set $T^* := \sup\mathcal{S}$. We claim that $T^* = +\infty$.

Suppose not, i.e. $T^* < +\infty$, then by the continuity of $d_\theta(t)$ we get

$$d_\theta(t) < \xi, \quad \forall t\in[0,T^*), \quad d_\theta(T^*) = \xi.$$

Next, we divide the time interval into sub-intervals in which no two oscillators overlap,

$$[0,T^*) = \bigcup_{l=1}^{r} J_l, \quad J_l = [t_{l-1}, t_l),$$

where the end-points t_l are the times at which such an overlap occurs. We then apply the well-ordering of phases in each interval:

$$\theta_1(t) \le \theta_2(t) \le \ldots \le \theta_N(t), \quad t\in J_l.$$

Step II We claim that for $1 \leq n \leq N$

$$\dot{\bar{\theta}}_n(t) \leq \Omega_M + \kappa \sin(R_\omega \tau) + \frac{\kappa \cos(R_\omega \tau)}{(N-1)(\bar{a}_n + 1)} \sum_{i=n}^{N} \eta^{i-n} \min_{\substack{j \in \mathcal{N}_i \\ j \leq i}} \sin\left(\theta_j(t) - \theta_i(t)\right). \tag{3.1.5}$$

For this, we use the inductive argument. Note that $\bar{\theta}_N = \theta_N$, and thus by mean-value theorem, we obtain

$$\begin{aligned}
\dot{\theta}_N(t) &= \Omega_N + \frac{\kappa}{N-1} \sum_{j \in \mathcal{N}_N} \sin\left(\theta_j(t - \tau_{jN}) - \theta_N(t)\right) \\
&= \Omega_N + \frac{\kappa}{N-1} \sum_{j \in \mathcal{N}_N} \sin\Big(\theta_j(t) - \theta_N(t) - \dot{\theta}_j(t^*_{jN})\tau_{jN}\Big) \\
&\leq \Omega_M + \frac{\kappa}{N-1} \sum_{j \in \mathcal{N}_N} \Big[\sin\left(\theta_j(t) - \theta_N(t)\right) \cos(\dot{\theta}_j(t^*_{jN})\tau_{jN}) - \cos\left(\theta_j(t) - \theta_N(t)\right) \sin(\dot{\theta}_j(t^*_{jN})\tau_{jN})\Big] \\
&\leq \Omega_M + \kappa \sin(R_\omega \tau) + \frac{\kappa}{N-1} \cos(R_\omega \tau) \min_{j \in \mathcal{N}_N} \sin\left(\theta_j(t) - \theta_N(t)\right)
\end{aligned}$$

for some $t^*_{jN} \in (t - \tau_{jN}, t)$ and all $t \in [0, T^*)$. This shows that (3.1.5) holds for $n = N$. Now we assume that (3.1.5) holds for $n \in [2, N-1]$. Note that

$$\begin{aligned}
&\frac{\kappa}{N-1} \sum_{j \in \mathcal{N}_{n-1}} \sin(\theta_j(t - \tau_{j(n-1)}) - \theta_{n-1}(t)) \\
&\quad \leq \kappa \sin(R_\omega \tau) + \frac{\kappa}{N-1} \left(\sum_{\substack{j \in \mathcal{N}_{n-1} \\ j \leq n-1}} + \sum_{\substack{j \in \mathcal{N}_{n-1} \\ j > n-1}} \right) \sin(\theta_j(t) - \theta_{n-1}(t)) \cos(\dot{\theta}_j(t^*_{j(n-1)})\tau_{j(n-1)}) \\
&\quad \leq \kappa \sin(R_\omega \tau) + \frac{\kappa}{N-1} \cos(R_\omega \tau) \min_{\substack{j \in \mathcal{N}_{n-1} \\ j \leq n-1}} \sin(\theta_j(t) - \theta_{n-1}(t)) + \frac{\kappa}{N-1} \sum_{\substack{j \in \mathcal{N}_{n-1} \\ j > n-1}} \sin(\theta_j(t) - \theta_{n-1}(t)).
\end{aligned}$$

By using the above together with

$$\frac{\bar{a}_{n-1}}{\bar{a}_n + 1} = \eta(2N - n + 2) = \eta N + \eta + \eta(N - n + 1),$$

we estimate

$$
\begin{aligned}
\dot{\bar\theta}_{n-1} &= \frac{d}{dt}\left(\frac{\bar a_{n-1}\bar\theta_n + \theta_{n-1}}{\bar a_{n-1}+1}\right) = \frac{\bar a_{n-1}}{\bar a_{n-1}+1}\dot{\bar\theta}_n + \frac{1}{\bar a_{n-1}+1}\dot\theta_{n-1} \\
&\le \frac{\bar a_{n-1}}{\bar a_{n-1}+1}\left(\Omega_M + \kappa\sin(R_\omega\tau) + \frac{\kappa\cos(R_\omega\tau)}{(N-1)(\bar a_n+1)}\sum_{i=n}^N \eta^{i-n}\min_{\substack{j\in\mathcal{N}_i\\ j\le i}}\sin\left(\theta_j(t)-\theta_i(t)\right)\right) \\
&\quad + \frac{1}{\bar a_{n-1}+1}\left(\Omega_M + \frac{\kappa}{N-1}\sum_{j\in\mathcal{N}_{n-1}}\sin(\theta_j(t-\tau_{j(n-1)})-\theta_{n-1}(t))\right) \\
&\le \Omega_M + \kappa\sin(R_\omega\tau) + \frac{\kappa\cos(R_\omega\tau)N}{(N-1)(\bar a_{n-1}+1)}\sum_{i=n}^N \eta^{i-n+1}\min_{\substack{j\in\mathcal{N}_i\\ j\le i}}\sin\left(\theta_j(t)-\theta_i(t)\right) \\
&\quad + \frac{\kappa\cos(R_\omega\tau)}{(N-1)(\bar a_{n-1}+1)}\left(\sum_{i=n}^N \eta^{i-n+1}\min_{\substack{j\in\mathcal{N}_i\\ j\le i}}\sin\left(\theta_j(t)-\theta_i(t)\right) + \min_{\substack{j\in\mathcal{N}_{n-1}\\ j\le n-1}}\sin(\theta_j(t)-\theta_{n-1}(t))\right) \\
&\quad + \frac{\kappa}{(N-1)(\bar a_{n-1}+1)}\left(\eta\,(N-n+1)\cos(R_\omega\tau)\sum_{i=n}^N \eta^{i-n}\min_{\substack{j\in\mathcal{N}_i\\ j\le i}}\sin\left(\theta_j-\theta_i\right) + \sum_{\substack{j\in\mathcal{N}_{n-1}\\ j>n-1}}\sin\left(\theta_j-\theta_{n-1}\right)\right).
\end{aligned}
$$

Here, the third term on the right-hand side is nonpositive, and the fourth term can be written as

$$
\frac{\kappa\cos(R_\omega\tau)}{(N-1)(\bar a_{n-1}+1)}\left(\sum_{i=n-1}^N \eta^{i-(n-1)}\min_{\substack{j\in\mathcal{N}_i\\ j\le i}}\sin\left(\theta_j(t)-\theta_i(t)\right)\right).
$$

Thus, to prove (3.1.5), it suffices to show that the last term is nonpositive.

We first observe from Lemma 3.1.3 that

$$
\sum_{i=n}^N \eta^{i-n}\min_{\substack{j\in\mathcal{N}_i\\ j\le i}}\sin\left(\theta_j-\theta_i\right) \le \sin\left(\theta_{\bar k_n}-\theta_N\right), \quad \bar k_n := \min_{j\in\cup_{i=n}^N\mathcal{N}_i} j.
$$

We then consider the case $\xi > \frac{\pi}{2}$. If $\theta_N - \theta_{\bar k_n} \le \frac{\pi}{2}$, then by noticing $\bar k_n \le n-1$ and $\eta\cos(R_\omega\tau) > 1$, we find

$$
\begin{aligned}
&\eta\,(N-n+1)\cos(R_\omega\tau)\sum_{i=n}^N \eta^{i-n}\min_{\substack{j\in\mathcal{N}_i\\ j\le i}}\sin\left(\theta_j-\theta_i\right) + \sum_{\substack{j\in\mathcal{N}_{n-1}\\ j>n-1}}\sin\left(\theta_j-\theta_{n-1}\right) \\
&\le \eta\,(N-n+1)\cos(R_\omega\tau)\sin\left(\theta_{\bar k_n}-\theta_N\right) + (N-n+1)\sin(\theta_N-\theta_{n-1}) \\
&\le 0.
\end{aligned}
$$

On the other hand, if $\frac{\pi}{2} < \theta_N - \theta_{\bar k_N} < \xi$, we use

$$
\eta > \frac{1}{\sin\xi} \quad \text{and} \quad \sin\left(\theta_N - \theta_{\bar k_n}\right) > \sin\xi
$$

to see $\eta\sin\left(\theta_{\bar k_N}-\theta_N\right) \le -1$, and this gives the nonpositivity of the last term. The case for $\xi \le \frac{\pi}{2}$ follows similarly.

We now use (3.1.5) and Lemma 3.1.3 for $n = 1$ and find that

$$\begin{aligned}
\dot{\bar{\theta}}_1 &\le \Omega_M + \kappa \sin(R_\omega \tau) + \frac{\kappa \cos(R_\omega \tau)}{(N-1)(\bar{a}_1+1)} \sum_{i=1}^{N} \eta^{i-1} \min_{\substack{j\in\mathcal{N}_i\\ j\le i}} \sin(\theta_j - \theta_i) \\
&\le \Omega_M + \kappa \sin(R_\omega \tau) + \frac{\kappa \cos(R_\omega \tau)}{(N-1)(\bar{a}_1+1)} \sin\left(\theta_{\bar{k}_1} - \theta_N\right) \\
&= \Omega_M + \kappa \sin(R_\omega \tau) + \frac{\kappa \cos(R_\omega \tau)}{(N-1)(\bar{a}_1+1)} \sin(\theta_1 - \theta_N),
\end{aligned}$$

where we used the strong connectivity of $\mathcal{G}$, and which completes the process for $\mathcal{A}_1$.

Step III We can similarly build from bottom to top with $\mathcal{A}_2$ to arrive at

$$\begin{aligned}
\dot{\underline{\theta}}_N(t) &\ge \Omega_m - \kappa \sin(R_\omega \tau) + \frac{\kappa \cos(R_\omega \tau)}{(N-1)(\underline{a}_N+1)} \sum_{i=1}^{N} \eta^{i-1} \min_{\substack{j\in\mathcal{N}_i\\ j\ge i}} \sin(\theta_j - \theta_i) \\
&\ge \Omega_m - \kappa \sin(R_\omega \tau) + \frac{\kappa \cos(R_\omega \tau)}{(N-1)(\underline{a}_N+1)} \sin\left(\theta_{\underline{k}_N} - \theta_1\right) \\
&= \Omega_m - \kappa \sin(R_\omega \tau) + \frac{\kappa \cos(R_\omega \tau)}{(N-1)(\bar{a}_1+1)} \sin(\theta_N - \theta_1),
\end{aligned}$$

from which we obtain

$$\begin{aligned}
\dot{q}_\theta(t) &\le D(\Omega) + 2\kappa \sin(R_\omega \tau) - \frac{2\kappa \cos(R_\omega \tau)}{(N-1)(\bar{a}_1+1)} \sin(\theta_N - \theta_1) \\
&\le D(\Omega) + 2\kappa \sin(R_\omega \tau) - \frac{2\kappa \cos(R_\omega \tau)}{N-1} \frac{1}{\sum_{j=1}^{N-1} \eta^j P(2N, j) + 1} \sin(\theta_N - \theta_1),
\end{aligned}$$

where we used

$$\bar{a}_1 = \sum_{j=1}^{N-1} \eta^j P(2N, j).$$

Since the function $\frac{\sin x}{x}$ is monotonically decreasing in $(0, \pi]$, we obtain

$$\sin(\theta_N - \theta_1) \ge \frac{\sin \xi}{\xi} (\theta_N - \theta_1).$$

Moreover, since $q_\theta(t) \le \theta_N(t) - \theta_1(t)$,

$$\dot{q}_\theta(t) \le D(\Omega) + 2\kappa \sin(R_\omega \tau) - \frac{2\kappa \cos(R_\omega \tau)}{(N-1)c} q(t) \tag{3.1.6}$$

for a.e. $t \in (0, T^*)$. Then, by Lemma 3.1.4 and Remark 3.1.6, we further have

$$\beta d_\theta(t) \le q_\theta(t) \le \max\left\{ D_\theta(0), \frac{(D(\Omega) + 2\kappa \sin(R_\omega \tau))(N-1)c}{2\kappa \cos(R_\omega \tau)} \right\}.$$

On the other hand, by the hypotheses, we get

$$\frac{D_\theta(0)}{\beta} < \frac{\zeta}{\beta} = \frac{\zeta}{1 - \frac{2}{\eta}} < \xi \quad \text{and} \quad \frac{(D(\Omega) + 2\kappa \sin(R_\omega \tau))(N-1)c}{2\beta\kappa \cos(R_\omega \tau)} < d_\infty < \zeta < \xi,$$

and hence,

$$\xi = d_\theta(T^*) \le \frac{1}{\beta}\max\left\{D_\theta(0), \frac{(D(\Omega)+2\kappa\sin(R_\omega\tau))(N-1)c}{2\kappa\cos(R_\omega\tau)}\right\} < \xi.$$

This is a contradiction, and thus $T^* = \infty$, i.e. $d_\theta(t) < \xi$ for all $[0,\infty)$. Moreover, (3.1.6) holds for a.e. $t \in (0,\infty)$. □

Lemma 3.1.7. *Let $\{\theta_i(t)\}_{i=1}^N$ be a solution to the Kuramoto model* (1.4.26) *on a strongly connected digraph $\mathcal{G}$ and assume the hypothesis of Lemma 3.1.5 holds. Then, there exists a finite time t_* such that*

$$d_\theta(t) \le d_\infty \quad \text{for } t \in [t_*, +\infty). \tag{3.1.7}$$

Proof. We first observe that

$$\frac{(D(\Omega)+2\kappa\sin(R_\omega\tau))(N-1)c}{2\kappa\cos(R_\omega\tau)} < \beta d_\infty < \beta d_\theta(0) \le q_\theta(0). \tag{3.1.8}$$

Now, we set

$$f(t) := q_\theta(0)\exp\left(-\frac{2\kappa\cos(R_\omega\tau)}{(N-1)c}t\right) + \frac{(D(\Omega)+2\kappa\sin(R_\omega\tau))(N-1)c}{2\kappa\cos(R_\omega\tau)}\left(1-\exp\left(-\frac{2\kappa\cos(R_\omega\tau)}{(N-1)c}t\right)\right).$$

Then $f(t)$ is continuous on $[0,\infty)$, monotonic, and

$$\lim_{t\to\infty} f(t) = \frac{(D(\Omega)+2\kappa\sin(R_\omega\tau))(N-1)c}{2\kappa\cos(R_\omega\tau)}.$$

Here, in the limit's formula, we dropped an intermediate step not clear to us. This together with (3.1.8) yields that there exists $t_* > 0$ such that

$$f(t) \le \beta d_\infty, \quad \forall\, t \in [t_*, \infty).$$

Then, we now use Lemma 3.1.5 and our Grönwall inequality to conclude

$$d_\theta(t) \le \frac{1}{\beta} f(t) \le d_\infty, \quad \forall\, t \in [t_*, \infty).$$

This completes the proof. □

3.2 Asymptotic synchronization: strongly connected case

In this section, we discuss the asymptotic frequency synchronization result for solutions to system (1.4.26), i.e., the frequency-diameter decays to zero as time tends to infinity. For this, by differentiating the system (1.4.26) with respect to t, we obtain

$$\begin{aligned} &\frac{d}{dt}\theta_i(t) = \omega_(t), \quad i = 1,\dots,N, \quad t > 0, \\ &\frac{d}{dt}\omega_i(t) = \frac{\kappa}{N-1}\sum_{k\ne i}\chi_{ik}\cos(\theta_k(t-\tau_{ik}) - \theta_i(t))(\omega_k(t-\tau_{ik}) - \omega_i(t)). \end{aligned} \tag{3.2.9}$$

In order to prove the diameter decay estimate for solutions to the above second-order system, we need some preliminary results.

In the rest of this section, we assume that the hypotheses of Lemma 3.1.5 are satisfied. Then, (3.1.7) holds for $t \ge t_* > 0$. Without loss of generality, we may assume $t_* > \tau$, and we deduce the following estimates.

Remark 3.2.1. *If* $\{\theta_i(t)\}_{i=1}^N$ *is a global solution of* (1.4.26)*, then*

$$\begin{aligned} |\theta_i(t-\tau_{ik})-\theta_j(t)| &\le |\theta_i(t-\tau_{ik})-\theta_i(t)|+|\theta_i(t)-\theta_j(t)| \\ &\le R_\omega\tau + d_\theta(t) \\ &\le R_\omega\tau + d_\infty < \frac{\pi}{2},\ \forall\, t\ge t_*. \end{aligned}$$

Thus, if we denote $\xi_* = \cos\Big(R_\omega\tau + d_\infty\Big)$*, we have that*

$$\cos(\theta_i(t-\tau_{ik})-\theta_j(t)) \ge \xi_* > 0,\ \forall\, t\ge t_*. \tag{3.2.10}$$

Let us denote

$$\tau_j := \max_{i\in\mathcal{N}_j}\{\tau_{ji}\} \quad \text{and} \quad \tau_0 := \min_{i=1,\dots,N}\{\tau_i\}.$$

Definition 3.2.2. *We define two numbers,* M_0 *and* m_0*, as*

$$M_0 := \max_{i=1,\dots,N}\ \max_{t\in I_0^i}\ \omega_i(t) \quad \textit{and} \quad m_0 := \min_{i=1,\dots,N}\ \min_{t\in I_0^i}\ \omega_i(t), \tag{3.2.11}$$

respectively, where $I_0^i := [t_*-\tau_i, t_*]$.

First, we show that the velocities remain bounded. Indeed, we have the following lemma.

Lemma 3.2.3. *Let* $\{\theta_i(t)\}_{i=1}^N$ *a global classical solution of* (1.4.26)*. Then, for all* $i=1,\dots,N$*, we have*

$$m_0 \le \omega_i(t) \le M_0, \tag{3.2.12}$$

for all $t\ge t_*-\tau_i$*, with* m_0 *and* M_0 *given by* (3.2.11)*.*

Proof. Fix $\epsilon>0$ and let us set

$$\mathcal{T}^\epsilon := \Big\{ t>t_* \ :\ \max_{i=1,\dots,N}\omega_i(s) < M_0+\epsilon \quad \forall s\in[t_*,t)\Big\}.$$

Since the inequality (3.2.12) is trivial in $[t_*-\tau_i,t_*]$, $i=1,\dots,N$, by continuity we deduce that $\mathcal{T}^\epsilon\neq\emptyset$. Let us denote $\mathcal{S}^\epsilon := \sup\mathcal{T}^\epsilon$. By definition of $\mathcal{T}^\epsilon$, trivially $\mathcal{S}^\epsilon>0$. We claim that $\mathcal{S}^\epsilon=+\infty$. Assume by contradiction that $\mathcal{S}^\epsilon<+\infty$. By definition of $\mathcal{S}^\epsilon$ we have that

$$\max_{i=1,\dots,N}\omega_i(t) < M_0+\epsilon, \quad \forall t\in[t_*,\mathcal{S}^\epsilon),$$

and

$$\lim_{t\to\mathcal{S}^{\epsilon}-}\ \max_{i=1,\dots,N}\omega_i(t) = M_0+\epsilon\,.$$

For every $i=1,...,N$, $\forall t\in(t_*,\mathcal{S}^\epsilon)$, we obtain

$$\frac{d}{dt}\omega_i(t) = \frac{\kappa}{N-1}\sum_{k\neq i}\chi_{ik}\cos(\theta_k(t-\tau_{ik})-\theta_i(t))(\omega_k(t-\tau_{ik})-\omega_i(t)).$$

Note that, from (3.2.10), $\cos(\theta_k(t-\tau_{ik})-\theta_i(t))>0$, for all $i,k=1,\dots,N$, and for every $t\ge t_*$. Moreover, if $t\in(t_*,\mathcal{S}^\epsilon)$, then $t-\tau_{ik}\in(t_*-\tau,\mathcal{S}^\epsilon)$ and thus,

$$\omega_k(t-\tau_{ik}) < M_0+\epsilon, \quad \forall k=1,...,N.$$

Then, it follows from (3.2.9) that

$$\frac{d}{dt}\omega_i(t) \leq \frac{\kappa}{N-1}\sum_{k\neq i}\chi_{ik}\cos(\theta_k(t-\tau_{ik})-\theta_i(t))(M_0+\epsilon-\omega_i(t)) \leq \kappa(M_0+\epsilon-\omega_i(t)),$$

where we used that also $\omega_i(t) < M_0+\epsilon$, for all $i=1,...,N$, and thus, $M_0+\epsilon-\omega_i(t)\geq 0$. Moreover, we used the fact that $\sum_{k\neq i}\chi_{ik}\leq N-1$, for all $i=1,\dots,N$.

Using the Grönwall's lemma, we find that

$$\begin{aligned}\omega_i(t) &\leq e^{-\kappa(t-t_*)}\omega_i(t_*)+\kappa(M_0+\epsilon)\int_{t_*}^{t} e^{-\kappa(t-s)}\,ds\\ &= e^{-\kappa(t-t_*)}\omega_i(t_*)+(M_0+\epsilon)(1-e^{-\kappa(t-t_*)})\\ &\leq e^{-\kappa(t-t_*)}M_0+M_0+\epsilon-e^{-\kappa(t-t_*)}M_0-\epsilon e^{-\kappa(t-t_*)}\\ &= M_0+\epsilon-\epsilon e^{-\kappa(t-t_*)}\leq M_0+\epsilon-\epsilon e^{-\kappa(\mathcal{S}^\epsilon-t_*)}.\end{aligned}$$

This implies

$$\max_{i=1,...,N}\omega_i(t)\leq M_0+\epsilon-\epsilon e^{-\kappa(\mathcal{S}^\epsilon-t_*)},\quad \forall\, t\in(t_*,\mathcal{S}^\epsilon).$$

Taking the limit for $t\to\mathcal{S}^{\epsilon-}$, we have that

$$\lim_{t\to\mathcal{S}^{\epsilon-}}\max_{i=1,...,N}\omega_i(t)\leq M_0+\epsilon-\epsilon e^{-\kappa(\mathcal{S}^\epsilon-t_*)}<M_0+\epsilon,$$

and this gives a contradiction. Then, $\mathcal{S}^\epsilon=+\infty$, and subsequently, we arrive at

$$\max_{i=1,...,N}\omega_i(t)<M_0+\epsilon,\quad \forall\, t\geq t_*.$$

Since $\epsilon>0$ is arbitrary, we conclude that

$$\max_{i=1,...,N}\omega_i(t)\leq M_0,$$

and then

$$\omega_i(t)\leq M_0,\quad \forall\, t\geq t_*-\tau_i,\ \forall\, i=1,...,N.$$

Applying the above argument to $-\omega_i(t)$, $t\geq t_*$, we have

$$-\omega_i(t)\leq \max_{j=1,...,N}\max_{s\in[t_*-\tau_i,t_*]}\{-\omega_j(s)\}=-\min_{j=1,...,N}\min_{s\in[t_*-\tau_i,t_*]}\omega_j(s)=-m_0.$$

Hence we deduce

$$\omega_i(t)\geq m_0,\quad \forall\, t\geq t_*-\tau_i,\ \forall\, i=1,...,N.$$

This concludes the proof. □

Without loss of generality, being system (1.4.26) invariant by translation, we may assume

$$0<m_0\leq M_0.$$

Definition 3.2.4. *For all $n\in\mathbb{N}$ we define the quantities M_n and m_n as*

$$M_n:=\max_{i=1,...,N}\max_{t\in I_n^i}\omega_i(t)\quad \textit{and}\quad m_n:=\min_{i=1,...,N}\min_{t\in I_n^i}\omega_i(t), \tag{3.2.13}$$

respectively, where $I_n^i:=[2\gamma n\tau+t_*-\tau_i, 2\gamma n\tau+t_*]$.

Note that, for $n = 0$, from (6.2.36) we obtain the constants already defined in (3.2.11).

Remark 3.2.5. *Arguing as in Lemma 3.2.3 we have that*

$$m_n \leq \omega_i(t) \leq M_n,$$

for all $n \in \mathbb{N}$ *and* $t \geq 2\gamma n\tau + t_* - \tau_i$.

Now, recalling the definition of R_ω (3.1.2), we define the quantities:

$$\sigma := \min\Big\{\tau_0, \frac{M_0 - m_0}{4\kappa R_\omega}\Big\}, \tag{3.2.14}$$

and

$$\Gamma := \Big(\frac{\xi_*}{N-1}\Big)^\gamma e^{-\kappa 2\gamma\tau}(1 - e^{-\frac{\kappa\tau}{N-1}})^{\gamma-1}(1 - e^{-\frac{\kappa}{N-1}\sigma}). \tag{3.2.15}$$

The following lemma extends to the network structure an argument of [50], as we did in Chapter 2.

Lemma 3.2.6. *Let* $\{\theta_i(t)\}_{i=1}^N$ *a global classical solution of* (1.4.26)*. Then,*

$$m_0 + \frac{\Gamma}{2}(M_0 - m_0) \leq \omega_i(t) \leq M_0 - \frac{\Gamma}{2}(M_0 - m_0), \quad t \in [t_* + (2\gamma - 1)\tau, t_* + 2\gamma\tau],$$

for all $i \in \{1, \dots, N\}$*, where* Γ *is defined by* (3.2.15)*.*

Proof. Let $L \in \{1, \dots, N\}$ such that $\omega_L(s) = m_0$ for some $s \in [t_* - \tau_L, t_*]$. Since it is true that $\omega_L(t) \leq R_\omega$, then we have that $|\dot\omega_L(t)| \leq 2\kappa R_\omega$. Therefore, one can find a closed interval $[\alpha_L, \beta_L] \subset [t_* - \tau_L, t_*]$ of length σ, defined as in (3.2.14), such that

$$m_0 \leq \omega_L(t) \leq \frac{M_0 + m_0}{2}, \ t \in [\alpha_L, \beta_L].$$

Let $i_1 \in \{1, \dots, N\} \setminus \{L\}$ such that $\chi_{i_1 L} = 1$ and $\tau_{i_1 L} = \tau_L$. Consider $t \in [\alpha_L + \tau_L, \beta_L + \tau_L]$. From the equation (3.2.9) we have:

$$\begin{aligned}\frac{d}{dt}\omega_{i_1}(t) = &\frac{\kappa}{N-1}\sum_{\substack{j\neq i_1\\ j\neq L}} \chi_{i_1 j}\cos(\theta_j(t - \tau_{i_1 j}) - \theta_{i_1}(t))(\omega_j(t - \tau_{i_1 j}) - \omega_{i_1}(t))\\ &+ \frac{\kappa}{N-1}\cos(\theta_L(t - \tau_{i_1 L}) - \theta_{i_1}(t))(\omega_L(t - \tau_{i_1 L}) - \omega_{i_1}(t)).\end{aligned}$$

Notice that

$$\sum_{j\neq i}\chi_{ij}\cos(\theta_j(t - \tau_{ij}) - \theta_i(t)) \leq \sum_{j\neq i}\chi_{ij} = N_i.$$

Then, using Remark 3.2.1, for $t \in [\alpha_L + \tau_L, \beta_L + \tau_L]$, we estimate

$$\begin{aligned}
\frac{d}{dt}\omega_{i_1}(t) &\leq \frac{\kappa}{N-1}\sum_{\substack{j\neq i_1\\ j\neq L}} \chi_{i_1 j}\cos(\theta_j(t-\tau_{i_1 j})-\theta_{i_1}(t))(M_0-\omega_{i_1}(t))\\
&\quad + \frac{\kappa}{N-1}\cos(\theta_L(t-\tau_{i_1 L})-\theta_{i_1}(t))\Big(\frac{M_0+m_0}{2}-\omega_{i_1}(t)\Big)\\
&\leq \frac{\kappa}{N-1}(N_{i_1}-\cos(\theta_L(t-\tau_{i_1 L})-\theta_{i_1}(t)))(M_0-\omega_{i_1}(t))\\
&\quad + \frac{\kappa}{N-1}\cos(\theta_L(t-\tau_{i_1 L})-\theta_{i_1}(t))\Big(\frac{M_0+m_0}{2}-\omega_{i_1}(t)\Big)\\
&= \frac{\kappa}{N-1}(N_{i_1}-\cos(\theta_L(t-\tau_{i_1 L})-\theta_{i_1}(t)))(M_0-\omega_{i_1}(t))\\
&\quad + \frac{\kappa}{N-1}\cos(\theta_L(t-\tau_{i_1 L})-\theta_{i_1}(t))\Big(\frac{M_0+m_0}{2}-\omega_{i_1}(t)\Big) \pm \frac{\kappa}{N-1}\cos(\theta_L(t-\tau_{i_1 L})-\theta_{i_1}(t))M_0\\
&= \frac{\kappa}{N-1}N_{i_1}M_0 - \cos(\theta_L(t-\tau_{i_1 L})-\theta_{i_1}(t))\frac{M_0-m_0}{2} - \frac{\kappa}{N-1}N_{i_1}\omega_{i_1}(t)\\
&\leq \frac{\kappa}{N-1}(N_{i_1}M_0 - \xi_*\frac{M_0-m_0}{2}) - \frac{\kappa}{N-1}N_{i_1}\omega_{i_1}(t).
\end{aligned}$$

Integrating the above inequality over $t \in [\alpha_L + \tau_L, \beta_L + \tau_L]$ gives

$$\omega_{i_1}(t) \leq e^{-\frac{\kappa}{N-1}N_{i_1}(t-\alpha_L-\tau_L)}\omega_{i_1}(\alpha_L+\tau_L) + \frac{1}{N_{i_1}}\Big(N_{i_1}M_0 - \xi_*\frac{M_0-m_0}{2}\Big)(1-e^{-\frac{\kappa}{N-i}N_{i_1}(t-\alpha_L-\tau_L)}).$$

Putting $t = \beta_L + \tau_L$ we find

$$\begin{aligned}
\omega_{i_1}(\beta_L+\tau_L) &\leq e^{-\frac{\kappa}{N-1}N_{i_1}\sigma}\omega_{i_1}(\alpha_L+\tau_L) + \frac{1}{N_{i_1}}\Big(N_{i_1}M_0 - \xi_*\frac{M_0-m_0}{2}\Big)(1-e^{-\frac{\kappa}{N-1}N_{i_1}\sigma})\\
&\leq M_0 - (1-e^{-\frac{\kappa}{N-1}N_{i_1}\sigma})\xi_*\frac{M_0-m_0}{2N_{i_1}}.
\end{aligned} \tag{3.2.16}$$

Denoting

$$\delta_- := \frac{\xi_*}{2(N-1)}(1-e^{-\frac{\kappa}{N-1}\sigma})\Big(1-\frac{m_0}{M_0}\Big),$$

from (3.2.16) we have that

$$\omega_{i_1}(\beta_L+\tau_L) \leq (1-\delta_-)M_0. \tag{3.2.17}$$

Consider now $t \in [\beta_L + \tau_L, t_* + 2\gamma\tau]$. Note that

$$\begin{aligned}
\frac{d}{dt}\omega_{i_1}(t) &= \frac{\kappa}{N-1}\sum_{j\neq i_1}\chi_{i_1 j}\cos(\theta_j(t-\tau_{i_1 j})-\theta_{i_1}(t))(\omega_j(t-\tau_{i_1 j})-\omega_{i_1}(t))\\
&\leq \frac{\kappa}{N-1}N_{i_1}(M_0-\omega_{i_1}(t)).
\end{aligned}$$

Integrating over $[\beta_L + \tau_L, t]$ with $t \in [\beta_L + \tau_L, t_* + 2\gamma\tau]$, we obtain

$$\begin{aligned}
\omega_{i_1}(t) &\leq e^{-\frac{\kappa}{N-1}N_{i_1}(t-\beta_L-\tau_L)}\omega_{i_1}(\beta_L+\tau_L) + (1-e^{-\frac{\kappa}{N-1}N_{i_1}(t-\beta_L-\tau_L)})M_0\\
&\leq e^{-\frac{\kappa}{N-1}N_{i_1}(t-\beta_L-\tau_L)}(1-\delta_-)M_0 + (1-e^{-\frac{\kappa}{N-1}N_{i_1}(t-\beta_L-\tau_L)})M_0\\
&\leq M_0(1-e^{-\frac{\kappa}{N-1}N_{i_1}2\gamma\tau}\delta_-),
\end{aligned}$$

where we used (3.2.17). Using $1 \le N_{i_1} \le N-1$, we find

$$\omega_{i_1}(t) \le M_0(1 - e^{-\kappa 2\gamma\tau}\delta_-), \ t \in [\beta_L + \tau_L, t_* + 2\gamma\tau]. \tag{3.2.18}$$

Consider now $i_2 \in \{1,\dots,N\} \setminus \{i_1\}$ such that $\chi_{i_2 i_1} = 1$ and consider $t \in [t_* + 2\tau, t_* + 2\gamma\tau]$. Again, from equation (3.2.9) we have

$$\begin{aligned}
\frac{d}{dt}\omega_{i_2}(t) &= \frac{\kappa}{N-1}\sum_{\substack{j\ne i_2\\ j\ne i_1}} \chi_{i_2 j}\cos(\theta_j(t-\tau_{i_2 j}) - \theta_{i_2}(t))(\omega_j(t-\tau_{i_2 j}) - \omega_{i_2}(t))\\
&\quad + \frac{\kappa}{N-1}\cos(\theta_{i_1}(t-\tau_{i_2 i_1}) - \theta_{i_2}(t))(\omega_{i_1}(t-\tau_{i_2 i_1}) - \omega_{i_2}(t))\\
&\le \frac{\kappa}{N-1}(N_{i_2} - \cos(\theta_{i_1}(t-\tau_{i_2 i_1}) - \theta_{i_2}(t)))(M_0 - \omega_{i_2}(t))\\
&\quad + \frac{\kappa}{N-1}\cos(\theta_{i_1}(t-\tau_{i_2 i_1}) - \theta_{i_2}(t))[M_0(1 - e^{-\kappa 2\gamma\tau}\delta_-) - \omega_{i_2}(t)]\\
&\le \frac{\kappa}{N-1}M_0(N_{i_2} - e^{-\kappa 2\gamma\tau}\xi_*\delta_-) - \frac{\kappa}{N-1}N_{i_2}\omega_{i_2}(t),
\end{aligned}$$

where we used (3.2.18). Integrating over $[t_* + 2\tau, t]$ with $t \in [t_* + 2\tau, t_* + 2\gamma\tau]$ deduces

$$\omega_{i_2}(t) \le M_0\left[1 - e^{-\kappa 2\gamma\tau}\xi_*\delta_-\left(\frac{1 - e^{-\frac{\kappa}{N-1}(t - t_* - 2\tau)}}{N-1}\right)\right].$$

Then, for $t \in [t_* + 3\tau, t_* + 2\gamma\tau]$, we have

$$\omega_{i_2}(t) \le M_0\Big(1 - e^{-\kappa 2\gamma\tau}\xi_*\delta_-\Big(\frac{1 - e^{-\frac{\kappa\tau}{N-1}}}{N-1}\Big)\Big).$$

Iterating this process, along the path starting from i_1, we obtain the following upper bound:

$$\omega_{i_n}(t) \le M_0\left[1 - e^{-\kappa 2\gamma\tau}\delta_-\xi_*^{n-1}\left(\frac{1 - e^{-\frac{\kappa\tau}{N-1}}}{N-1}\right)^{n-1}\right],$$

with n such that $2 \le n \le \gamma$ and $t \in [t_* + (2n-1)\tau, t_* + 2\gamma\tau]$.

Note that, being the digraph strongly connected, from i_1 one can reach any other state along a path of length less than or equal to the depth γ. Therefore, for all $i = 1,\dots,N$, we have

$$\omega_i(t) \le M_0\left[1 - e^{-\kappa 2\gamma\tau}\delta_-\xi_*^{\gamma-1}\left(\frac{1 - e^{-\frac{\kappa\tau}{N-1}}}{N-1}\right)^{\gamma-1}\right], \quad t \in [t_* + (2\gamma-1)\tau, t_* + 2\gamma\tau]. \tag{3.2.19}$$

Now, consider the state $R \in \{1,\dots,N\}$ such that $\omega_R(s) = M_0$ for some $s \in [t_* - \tau_R, t_*]$. Again, we can find a closed interval $[\alpha_R, \beta_R] \subset [t_* - \tau_R, t_*]$ such that

$$\frac{m_0 + M_0}{2} \le \omega_R(t) \le M_0, \ t \in [\alpha_R, \beta_R].$$

Using arguments analogous to the previous ones, we find a lower bound for all $\omega_i(t)$, $i = 1,\dots,N$:

$$\omega_i(t) \ge m_0\left[1 + e^{-\kappa 2\gamma\tau}\delta_+\xi_*^{\gamma-1}\left(\frac{1 - e^{-\frac{\kappa\tau}{N-1}}}{N-1}\right)^{\gamma-1}\right], \quad t \in [t_* + (2\gamma-1)\tau, t_* + 2\gamma\tau], \tag{3.2.20}$$

with
$$\delta_+ := \frac{\xi_*}{2(N-1)}(1-e^{-\frac{\kappa}{N-1}\sigma})\Big(\frac{M_0}{m_0}-1\Big).$$
Finally, recalling definition (3.2.15), from (3.2.19) and (3.2.20), we can find the final estimate
$$m_0 + \frac{\Gamma}{2}(M_0-m_0) \le \omega_i(t) \le M_0 - \frac{\Gamma}{2}(M_0-m_0),$$
for $i = 1,\dots,N$ and $t \in [t_* + (2\gamma-1)\tau, t_* + 2\gamma\tau]$. □

We are now able to prove the main result of this paper.

Theorem 3.2.7. *Let* $\{\theta_i(t)\}_{i=1}^N$ *be a solution to the Kuramoto model* (1.4.26) *on a strongly connected digraph* $\mathcal{G}$*, with initial data* (1.4.27)*. Assume that the hypotheses of Lemma 3.1.5 are satisfied. Then, the Kuramoto oscillators with delayed coupling achieve the asymptotic complete frequency synchronization in the sense of Definition 1.4.1.*

Proof. We define the quantities $D_n := M_n - m_n$ and
$$\Gamma_n := \Big(\frac{\xi_*}{N-1}\Big)^{\gamma} e^{-\kappa 2\gamma\tau}(1-e^{-\frac{\kappa\tau}{N-1}})^{\gamma-1}(1-e^{-\frac{\kappa\sigma_n}{N-1}}),$$
where $\sigma_n := \min\{\tau_0, \frac{M_n-m_n}{4\kappa R_\omega}\}$. Then, Γ_0 is the constant Γ defined in (3.2.15) and $\Gamma_n \in (0,1)$ for all $n \ge 0$. Now, we use Lemma 3.2.6 on the intervals $J_n := [t_* + (2\gamma n-1)\tau, t_* + 2\gamma n\tau]$, $n \in \mathbb{N}$. For $t \in J_1$ we get
$$\begin{aligned} D_1 &= M_1 - m_1 \\ &= \max_{i=1,\dots,N}\max_{t\in I_1^i}\omega_i(t) - \min_{i=1,\dots,N}\min_{t\in I_1^i}\omega_i(t) \\ &\le M_0 - \frac{\Gamma_0}{2}(M_0-m_0) - m_0 - \frac{\Gamma_0}{2}(M_0-m_0) \\ &= (M_0-m_0)(1-\Gamma_0) \\ &= D_0(1-\Gamma_0). \end{aligned}$$
Thus, we find that $D_1 \le (1-\Gamma_0)D_0$. Iterating the argument gives
$$D_{n+1} \le (1-\Gamma_n)D_n, \ \forall n \in \mathbb{N}.$$
Let us denote
$$\tilde\sigma(D) := \min\Big\{\tau_0, \frac{D}{4\kappa R_\omega}\Big\},$$
so that $\sigma_n = \tilde\sigma(D_n)$ for $n \in \mathbb{N}$. Similarly, let us denote
$$\tilde\Gamma(D) := \Big(\frac{\xi_*}{N-1}\Big)^{\gamma} e^{-\kappa 2\gamma\tau}(1-e^{-\frac{\kappa\tau}{N-1}})^{\gamma-1}\left(1-e^{-\frac{\kappa\tilde\sigma(D)}{N-1}}\right)$$
so that $\Gamma_n = \tilde\Gamma(D_n)$ for $n \in \mathbb{N}$. Then we find
$$D_{n+1} \le (1-\tilde\Gamma(D_n))D_n,$$

and subsequently, $\{D_n\}_{n\in\mathbb{N}}$ is a non-negative and decreasing sequence. Passing to the limit $n \to \infty$ in the above estimate, and denoting D the limit of $\{D_n\}_{n\in\mathbb{N}}$, we have

$$D \leq (1 - \tilde{\Gamma}(D))D,$$

that is true only if $\tilde{\Gamma}(D) \leq 0$. This gives $D = 0$ and, noticing that, from Lemma 3.2.3 and Remark 3.2.5, for further times to respect J_n, we have $\omega_i(t) - \omega_j(t) \leq M_n - m_n = D_n$ for all $i, j = 1, \ldots, N$, we conclude that

$$|\omega_i(t) - \omega_j(t)| \to 0,$$

as $t \to +\infty$ and for all $i, j = 1, \ldots, N$. Then, the system achieves asymptotic synchronization. □

3.3 Asymptotic synchronization: all-to-all connected case

In this section, we consider the Kuramoto oscillators with delayed coupling in the case of all-to-all connection, i.e. $\chi_{ij} = 1$, for all $i, j = 1, \ldots, N$. In this case, the system (1.4.26) reduces to

$$\frac{d}{dt}\theta_i(t) = \Omega_i + \frac{\kappa}{N-1}\sum_{k\neq i}\sin(\theta_k(t - \tau_{ik}) - \theta_i(t)), \quad i = 1, \ldots, N, \quad t > 0, \tag{3.3.21}$$

subject to the initial data (1.4.27).

Different from the strongly connected case, in the all-to-all connected case, we prove the exponential frequency synchronization of system (3.3.21), i.e., the frequency-diameter decays to zero exponentially fast as time goes to infinity.

To be more specific, our main result of this section is the following.

Theorem 3.3.1. *Let $\{\theta_i(t)\}_{i=1}^N$ be a global-in-time solution to the Kuramoto model* (3.3.21) *with initial data* (1.4.27). *Assume that the hypotheses of Lemma 3.1.5 are satisfied. Then, the time-delayed Kuramoto oscillators achieve the asymptotic complete frequency synchronization in the sense of Definition 1.4.1 exponentially fast. More precisely, we have*

$$d_\omega(t) \leq \mathcal{C}e^{-\tilde{\gamma}t} \quad \textit{for all} \quad t \geq 0,$$

where $\tilde{\gamma}$ and $\mathcal{C}$ are positive constants depending on R_ω, κ, and τ.

To prove Theorem (3.3.1), we need some preliminary lemmas (cf. [27, 70]). In the whole section, we assume that the hypotheses of Lemma 3.1.5 are satisfied.

First, extending (3.0.1), we define, for $n \in \mathbb{N}$,

$$D^*_\theta(n) := \max_{\substack{1\leq i,j\leq N,\\ s,t\in[t_*+(n-1)\tau, t_*+n\tau]}} |\theta_i(s) - \theta_j(t)|,$$
$$D^*_\omega(n) := \max_{\substack{1\leq i,j\leq N,\\ s,t\in[t_*+(n-1)\tau, t_*+n\tau]}} |\omega_i(s) - \omega_j(t)|,$$

where t_* is the time in (3.1.7).

The following lemma generalizes Lemma 3.2.3. We omit the proof since it is analogous to the previous one.

Lemma 3.3.2. *Let $\{\theta_i(t)\}_{i=1}^N$ be a solution to the Kuramoto model* (3.3.21) *with initial conditions* (1.4.27)*. Then, for any $T \geq t_*$, we have*

$$\min_{j=1,...,N} \min_{s\in[T-\tau,T]} \omega_j(s) \leq \omega_i(t) \leq \max_{j=1,...,N} \max_{s\in[T-\tau,T]} \omega_j(s)$$

for all $t \geq T-\tau$ and $i = 1, ..., N$.

From Lemma 3.3.2, we deduce the properties below.

Lemma 3.3.3. *For every $n \in \mathbb{N}$ and $i, j = 1, ..., N$, we get*

$$|\omega_i(s) - \omega_j(t)| \leq D_\omega^*(n), \quad \forall s, t \geq t_* + (n-1)\tau.$$

Proof. Fix $n \in \mathbb{N}$ and $i, j = 1, ..., N$. Without loss of generality, taken $s, t \geq t_* + (n-1)\tau$, we can suppose $\omega_i(s) > \omega_j(t)$. Then, using the Lemma 3.3.2 with $T = t_* + n\tau$ and Cauchy–Schwartz inequality, we have

$$\begin{aligned}
|\omega_i(s) - \omega_j(t)| &= \omega_i(s) - \omega_j(t) \\
&\leq \max_{l=1,...,N} \max_{r\in[t_*+(n-1)\tau, t_*+n\tau]} \omega_l(r) - \min_{l=1,...,N} \min_{r\in[t_*+(n-1)\tau, t_*+n\tau]} \omega_l(r) \\
&\leq \max_{l,k=1,...,N} \max_{r,\sigma\in[t_*+(n-1)\tau, t_*+n\tau]} |\omega_l(r) - \omega_k(\sigma)| = D_\omega^*(n).
\end{aligned}$$

This completes the proof. □

Remark 3.3.4. *Note that, from Lemma 3.3.3, for $n = 0$, we find $|\omega_i(s) - \omega_j(t)| \leq D_\omega^*(0)$. Moreover,*

$$D_\omega^*(n+1) \leq D_\omega^*(n), \ \forall\, n \in \mathbb{N}.$$

Lemma 3.3.5. *For all $i, j = 1, ..., N$ and $n \in \mathbb{N}$, we have that*

$$\omega_i(t) - \omega_j(t) \leq e^{-\kappa(t-t_0)}(\omega_i(t_0) - \omega_j(t_0)) + (1 - e^{-\kappa(t-t_0)})D_\omega^*(n), \tag{3.3.22}$$

for all $t \geq t_0 \geq t_* + n\tau$. Moreover, for all $s_0, t_0 \in [t_* + n\tau, t_* + (n+2)\tau]$, we obtain

$$\omega_i(t_* + (n+2)\tau) - \omega_j(t_* + (n+2)\tau) \leq e^{-2\kappa\tau}(\omega_i(t_0) - \omega_j(s_0)) + (1 - e^{-2\kappa\tau})D_\omega^*(n). \tag{3.3.23}$$

Proof. Fix $n \in \mathbb{N}$ and let us denote

$$M_n^* := \max_{i=1,...,N} \max_{t\in[t_*+(n-1)\tau, t_*+n\tau]} \omega_i(t) \quad \text{and} \quad m_n^* := \min_{i=1,...,N} \min_{t\in[t_*+(n-1)\tau, t_*+n\tau]} \omega_i(t).$$

Note that $M_n^* - m_n^* \leq D_\omega^*(n)$. For all $i = 1, ..., N$ and $t \geq t_0 \geq t_* + n\tau$, we write

$$\begin{aligned}
\frac{d}{dt}\omega_i(t) &= \frac{\kappa}{N-1} \sum_{k\neq i} \cos(\theta_k(t-\tau_{ik}) - \theta_i(t))(\omega_k(t-\tau_{ik}) - \omega_i(t)) \\
&\leq \frac{\kappa}{N-1} \sum_{k\neq i} \cos(\theta_k(t-\tau_{ik}) - \theta_i(t))(M_{n-1}^* - \omega_i(t)),
\end{aligned}$$

where we used the fact that $t - \tau_{ik} \geq t_* + (n-1)\tau$. By Lemma 3.3.2, since $t \geq t_* + n\tau$, we get $\omega_i(t) \leq M_{n-1}^*$, that implies $M_{n-1}^* - \omega_i(t) \geq 0$. Then we write

$$\frac{d}{dt}\omega_i(t) \leq \kappa(M_{n-1}^* - \omega_i(t)).$$

Applying Grönwall's lemma, we find

$$\omega_i(t) \le e^{-\kappa(t-t_0)}\omega_i(t_0) + (1 - e^{-\kappa(t-t_0)})M^*_{n-1}. \tag{3.3.24}$$

Similarly, for all $i = 1, ..., N$ and $t \ge t_0 \ge t_* + n\tau$, we deduce

$$\omega_i(t) \ge e^{-\kappa(t-t_0)}\omega_i(t_0) + (1 - e^{-\kappa(t-t_0)})m^*_{n-1}. \tag{3.3.25}$$

Therefore, for $i, j = 1, ..., N$ and $t \ge t_0 \ge t_* + n\tau$, we have

$$\begin{aligned}
\omega_i(t) - \omega_j(t) &\le e^{-\kappa(t-t_0)}\omega_i(t_0) + (1 - e^{-\kappa(t-t_0)})M^*_{n-1} - e^{-\kappa(t-t_0)}\omega_j(t_0) - (1 - e^{-\kappa(t-t_0)})m^*_{n-1} \\
&= e^{-\kappa(t-t_0)}(\omega_i(t_0) - \omega_j(t_0)) + (1 - e^{-\kappa(t-t_0)})(M^*_{n-1} - m^*_{n-1}) \\
&\le e^{-\kappa(t-t_0)}(\omega_i(t_0) - \omega_j(t_0)) + (1 - e^{-\kappa(t-t_0)})D^*_\omega(n).
\end{aligned}$$

This gives the first assertion (3.3.22).

For (3.3.23), from (3.3.24) with $t_0 \in [t_* + n\tau, t_* + (n+2)\tau]$ and $t := t_* + (n+2)\tau$ we find

$$\begin{aligned}
\omega_i(t_* + (n+2)\tau) &\le e^{-\kappa(t_* + (n+2)\tau - t_0)}\omega_i(t_0) + (1 - e^{-\kappa(t_* + (n+2)\tau - t_0)})M^*_{n-1} \\
&= e^{-\kappa(t_* + (n+2)\tau - t_0)}(\omega_i(t_0) - M^*_{n-1}) + M^*_{n-1} \\
&\le e^{-2\kappa\tau}(\omega_i(t_0) - M^*_{n-1}) + M^*_{n-1} \\
&= e^{-2\kappa\tau}\omega_i(t_0) + (1 - e^{-2\kappa\tau})M^*_{n-1}.
\end{aligned} \tag{3.3.26}$$

Similarly, from (3.3.25), for any $j = 1, \ldots, N$ and $s_0 \in [t_* + n\tau, t_* + (n+2)\tau]$ we have

$$\omega_i(t_* + (n+2)\tau) \ge e^{-2\kappa\tau}\omega_i(s_0) + (1 - e^{-2\kappa\tau})m^*_{n-1}. \tag{3.3.27}$$

Combining (3.3.26) and (3.3.27), we arrive at (3.3.23), and this completes the proof. □

Lemma 3.3.6. *For all $n \in \mathbb{N}$, we have that*

$$D^*_\omega(n+1) \le e^{-\kappa\tau}d_\omega(t_* + n\tau) + (1 - e^{-\kappa\tau})D^*_\omega(n).$$

Proof. Fix $n \in \mathbb{N}$ and let $i, j = 1, ..., N$, $s, t \in [t_* + n\tau, t_* + (n+1)\tau]$, such that $D^*_\omega(n+1) = |\omega_i(s) - \omega_j(t)|$. Notice that if $|\omega_i(s) - \omega_j(t)| = 0$ then it is obvious that

$$0 = D^*_\omega(n+1) \le e^{-\kappa\tau}d_\omega(t_* + n\tau) + (1 - e^{-\kappa\tau})D^*_\omega(n).$$

Then, we now suppose $|\omega_i(s) - \omega_j(t)| > 0$ and, without loss of generality, we may assume $\omega_i(s) > \omega_j(t)$. From (3.3.24) with $t_0 = t_* + n\tau$, we get

$$\begin{aligned}
\omega_i(s) &\le e^{-\kappa(s - t_* - n\tau)}\omega_i(t_* + n\tau) + (1 - e^{-\kappa(s - t_* - n\tau)})M^*_n \\
&= e^{-\kappa(s - t_* - n\tau)}(\omega_i(t_* + n\tau) - M^*_n) + M^*_n.
\end{aligned}$$

Thus, since $s \le t_* + (n+1)\tau$, we obtain

$$\omega_i(s) \le e^{-\kappa\tau}(\omega_i(t_* + n\tau) - M^*_n) + M^*_n = e^{-\kappa\tau}\omega_i(t_* + n\tau) + (1 - e^{-\kappa\tau})M^*_n.$$

Analogously, we have that

$$\omega_j(t) \ge e^{-\kappa\tau}\omega_j(t_* + n\tau) + (1 - e^{-\kappa\tau})m^*_n.$$

Therefore, we conclude that

$$\begin{aligned}
D^*_\omega(n+1) &\le e^{-\kappa\tau}(\omega_i(t_* + n\tau) - \omega_j(t_* + n\tau)) + (1 - e^{-\kappa\tau})D^*_\omega(n) \\
&\le e^{-\kappa\tau}d_\omega(t_* + n\tau) + (1 - e^{-\kappa\tau})D^*_\omega(n).
\end{aligned}$$

□

Before proving our main result, we need a further lemma.

Lemma 3.3.7. *There exists a constant* $C \in (0,1)$ *such that*

$$d_\omega(t_* + n\tau) \leq C D^*_\omega(n-2),$$

for all $n \geq 2$.

Proof. If $d_\omega(t_* + n\tau) = 0$, then the assetion is obvious for all $C \in (0,1)$. Then, let us suppose $d_\omega(t_* + n\tau) > 0$. Fix $i, j = 1, ..., N$ such that $d_\omega(t_* + n\tau) = |\omega_i(t_* + n\tau) - \omega_j(t_* + n\tau)|$. Without lost of generality, we may assume $\omega_i(t_* + n\tau) > \omega_j(t_* + n\tau)$. As before, we consider the following quantities:

$$M^*_{n-1} = \max_{l=1,...,N} \max_{s \in [t_* + (n-2)\tau, t_* + (n-1)\tau]} \omega_l(s)$$

and

$$m^*_{n-1} = \min_{l=1,...,N} \min_{s \in [t_* + (n-2)\tau, t_* + (n-1)\tau]} \omega_l(s)\,.$$

Then we get $M^*_{n-1} - m^*_{n-1} \leq D^*_\omega(n-1)$.

Let us distinguish two cases.

Case I: Suppose that there exist $t_0, s_0 \in [t_* + (n-2)\tau, t_* + n\tau]$ such that $\omega_i(t_0) - \omega_j(s_0) < 0$. Then, due to (3.3.23) we have

$$\begin{aligned} d_\omega(t_* + n\tau) &\leq e^{-2\kappa\tau}(\omega_i(t_0) - \omega_j(s_0)) + (1 - e^{-2\kappa\tau}) D^*_\omega(n-2) \\ &\leq (1 - e^{-2\kappa\tau}) D^*_\omega(n-2). \end{aligned}$$

Then, the assertion follows in this case.

Case II: Suppose that $\omega_i(t) - \omega_j(s) \geq 0$ for all $t, s \in [t_* + (n-2)\tau, t_* + n\tau]$. Then, for $t \in [t_* + (n-1)\tau, t_* + n\tau]$ we obtain

$$\begin{aligned} \frac{d}{dt}(\omega_i(t) - \omega_j(t)) &= \frac{\kappa}{N-1}\sum_{k \neq i} \cos(\theta_k(t - \tau_{ik}) - \theta_i(t))(\omega_k(t - \tau_{ik}) - \omega_i(t)) \\ &\quad - \frac{\kappa}{N-1}\sum_{k \neq j} \cos(\theta_k(t - \tau_{jk}) - \theta_j(t))(\omega_k(t - \tau_{jk}) - \omega_j(t)) \\ &= \frac{\kappa}{N-1}\sum_{k \neq i} \cos(\theta_k(t - \tau_{ik}) - \theta_i(t))(\omega_k(t - \tau_{ik}) - M^*_{n-1} + M^*_{n-1} - \omega_i(t)) \\ &\quad + \frac{\kappa}{N-1}\sum_{k \neq j} \cos(\theta_k(t - \tau_{jk}) - \theta_j(t))(\omega_j(t) - m^*_{n-1} + m^*_{n-1} - \omega_k(t - \tau_{jk})) \\ &:= S_1 + S_2. \end{aligned} \tag{3.3.28}$$

Since $t \in [t_* + (n-1)\tau, t_* + n\tau]$, we get $t - \tau_{ij} \in [t_* + (n-2)\tau, t_* + (n-1)\tau]$ for all $i, j = 1, \ldots, N$. Thus, it follows from Lemma 3.3.2 with $T = t_* + (n-2)\tau$ that

$$m^*_{n-1} \leq \omega_k(t) \leq M^*_{n-1} \quad \text{and} \quad m^*_{n-1} \leq \omega_k(t - \tau_{ik}) \leq M^*_{n-1}$$

for all $i, k = 1, ..., N$. Then, we estimate

$$\begin{aligned} S_1 &= \frac{\kappa}{N-1}\sum_{k \neq i} \cos(\theta_k(t - \tau_{ik}) - \theta_i(t))(\omega_k(t - \tau_{ik}) - M^*_{n-1}) \\ &\quad + \frac{\kappa}{N-1}\sum_{k \neq i} \cos(\theta_k(t - \tau_{ik}) - \theta_i(t))(M^*_{n-1} - \omega_i(t)) \\ &\leq \frac{\kappa}{N-1}\xi_* \sum_{k \neq i}(\omega_k(t - \tau_{ik}) - M^*_{n-1}) + \kappa(M^*_{n-1} - \omega_i(t)), \end{aligned} \tag{3.3.29}$$

and

$$\begin{aligned} S_2 := &\frac{\kappa}{N-1}\sum_{k\neq j}\cos(\theta_k(t-\tau_{jk})-\theta_j(t))(\omega_j(t)-m^*_{n-1}) \\ &+\frac{\kappa}{N-1}\sum_{k\neq j}\cos(\theta_k(t-\tau_{jk})-\theta_j(t))(m^*_{n-1}-\omega_k(t-\tau_{jk})) \\ &\leq \kappa(\omega_j(t)-m^*_{n-1})+\frac{\kappa}{N-1}\xi_*\sum_{k\neq j}(m^*_{n-1}-\omega_k(t-\tau_{jk})), \end{aligned} \tag{3.3.30}$$

where ξ_* is the positive constant given in (3.2.10). Putting (3.3.29) and (3.3.30) in (3.3.28), we have

$$\begin{aligned} \frac{d}{dt}(\omega_i(t)-\omega_j(t)) \leq\ & \kappa(M^*_{n-1}-m^*_{n-1})-\kappa(\omega_i(t)-\omega_j(t)) \\ &+\frac{\kappa}{N-1}\xi_*\sum_{k\neq i}(\omega_k(t-\tau_{ik})-M^*_{n-1}) \\ &+\frac{\kappa}{N-1}\xi_*\sum_{k\neq j}(m^*_{n-1}-\omega_k(t-\tau_{jk})). \end{aligned}$$

Notice that, since $\omega_k(t-\tau_{ik})-M^*_{n-1}\leq 0$ for all $i,k=1,\dots,N$ and $t\in[t_*+(n-1)\tau,t_*+n\tau]$ and $j\neq i$, we can write

$$\sum_{k\neq i}(\omega_k(t-\tau_{ik})-M^*_{n-1})\leq \omega_j(t-\tau_{ij})-M^*_{n-1}.$$

Analogously,

$$\sum_{k\neq j}(m^*_{n-1}-\omega_k(t-\tau_{jk}))\leq m^*_{n-1}-\omega_i(t-\tau_{ji}).$$

Therefore, we obtain

$$\begin{aligned} \frac{d}{dt}(\omega_i(t)-\omega_j(t)) &\leq \kappa\Big(1-\frac{\xi_*}{N-1}\Big)(M^*_{n-1}-m^*_{n-1})-\kappa(\omega_i(t)-\omega_j(t)) \\ &\quad+\frac{\kappa}{N-1}\xi_*(\omega_j(t-\tau_{ij})-\omega_i(t-\tau_{ji})) \\ &\leq \kappa\Big(1-\frac{\xi_*}{N-1}\Big)(M^*_{n-1}-m^*_{n-1})-\kappa(\omega_i(t)-\omega_j(t)), \end{aligned}$$

where we used the assumption $\omega_i(t)-\omega_j(s)\geq 0$ for all $t,s\in[t_*+(n-2)\tau,t_*+n\tau]$.

Applying now the Grönwall's lemma over $[t,t_*+n\tau]$, with $t\in[t_*+(n-1)\tau,t_*+n\tau]$, we deduce

$$\begin{aligned} \omega_i(t)-\omega_j(t)\leq\ & e^{-\kappa(t-t_*-(n-1)\tau)}(\omega_i(t_*+(n-1)\tau)-\omega_j(t_*+(n-1)\tau)) \\ &+\Big(1-\frac{\xi_*}{N-1}\Big)(M^*_{n-1}-m^*_{n-1})(1-e^{-\kappa(t-t_*-(n-1)\tau)}). \end{aligned}$$

For $t=t_*+n\tau$, we get

$$\begin{aligned} d_\omega(t_*+n\tau) &\leq e^{-\kappa\tau}(\omega_i(t_*+(n-1)\tau)-\omega_j(t_*+(n-1)\tau)) \\ &\quad+\Big(1-\frac{\xi_*}{N-1}\Big)(M^*_{n-1}-m^*_{n-1})(1-e^{-\kappa\tau}) \\ &\leq \Big(1-\frac{\xi_*}{N-1}(1-e^{-\kappa\tau})\Big)D^*_\omega(n-2), \end{aligned}$$

where we used that $M^*_{n-1} - m^*_{n-1} \leq D^*_\omega(n-1)$ and the monotonicity property of $D^*_\omega(n)$.

Finally, we set

$$C := \max\left\{1 - e^{-2\kappa\tau}, 1 - \frac{\xi_*}{N-1}(1 - e^{-\kappa\tau})\right\} > 0$$

to conclude the desired result. □

Now, we are ready to prove the exponential synchronization estimate for the system (3.3.21) with (1.4.27).

Proof of Theorem 3.3.1. Let $\{\theta_i(t)\}_{i=1}^N$ a global classical solution of (3.3.21). We claim that

$$D^*_\omega(n+1) \leq \tilde{C} D^*_\omega(n-2), \quad \forall\, n \geq 2,$$

for a suitable constant $\tilde{C} \in (0,1)$. Indeed, for $n \geq 2$, applying Lemmas 3.3.6 and 3.3.7, we obtain

$$\begin{aligned} D^*_\omega(n+1) &\leq e^{-\kappa\tau} d_\omega(t_* + n\tau) + (1 - e^{-\kappa\tau}) D^*_\omega(n) \\ &\leq e^{-\kappa\tau} C D^*_\omega(n-2) + (1 - e^{-\kappa\tau}) D^*_\omega(n) \\ &\leq e^{-\kappa\tau} C D^*_\omega(n-2) + (1 - e^{-\kappa\tau}) D^*_\omega(n-2) \\ &= D^*_\omega(n-2)[1 - e^{-\kappa\tau}(1 - C)]. \end{aligned}$$

Denoting

$$\tilde{C} := [1 - e^{-\kappa\tau}(1 - C)],$$

we have the claim. This implies that

$$D^*_\omega(3n) \leq \tilde{C}^n D^*_\omega(0), \quad \forall\, n \geq 1.$$

Then, we have that

$$D^*_\omega(3n) \leq e^{-3n\tau\gamma} D^*_\omega(0), \quad \forall\, n \in \mathbb{N},$$

with

$$\gamma := \frac{1}{3\tau} \ln\left(\frac{1}{\tilde{C}}\right).$$

Now, fix $i, j = 1, ..., N$. For all $t \geq t_* - \tau$, we get $t \in [t_* + 3n\tau - \tau, t_* + 3n\tau + 2\tau]$ for some $n \in \mathbb{N}$. Then, by applying Lemma 3.3.3, we can write

$$|\omega_i(t) - \omega_j(t)| \leq D^*_\omega(3n) \leq e^{-3n\tau\gamma} D^*_\omega(0).$$

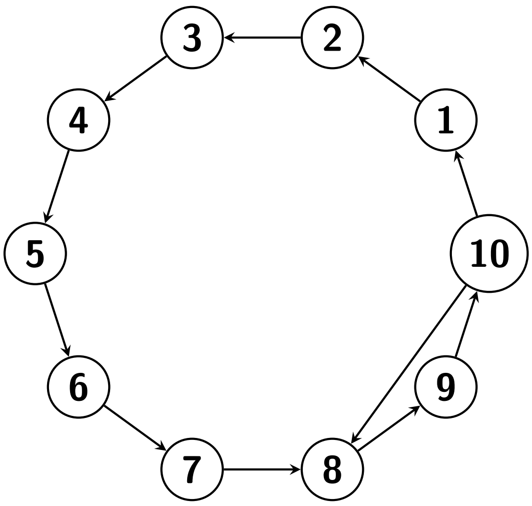
Since $t \leq t_* + 3n\tau + 2\tau$, we have that

$$|\omega_i(t) - \omega_j(t)| \leq e^{-\gamma t} e^{\gamma(t_* + 2\tau)} D^*_\omega(0).$$

Finally, passing to the maximum on $i, j = 1, ..., N$, we find that

$$d_\omega(t) \leq e^{-\gamma(t - t_* - 2\tau)} D^*_\omega(0),$$

and this completes the proof. □



$$\chi = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 1&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0&1&0 \end{pmatrix}$$

Figure 3.1: The strongly connected digraph and its adjacency matrix used in our simulations.

3.4 Numerical experiments

In this section, we present numerical experiments concerning the dynamics of solutions to the system (1.4.26) depending on the network structure and magnitude of time delays.

3.4.1 Strongly connected case

We perform simulations on the strong digraph consisting of 10 oscillators according to Figure 3.1, with time delays τ_{ij} generated according to random uniform distributions in $[0, 1]$ and $[0, 0.1]$ and coupling constant $\frac{\kappa}{N-1} = 1$.

The natural frequencies $\Omega = (\Omega_1, \ldots, \Omega_{10})$ are generated according to a zero-centered uniform distribution, to three decimal places,

$$\Omega \approx (-0.0687, 0.511, 0.826, -0.604, -0.371, -0.582, -0.146, 0.889, 0.535, 0.006).$$

The initial data are set to be constant and drawn from a random uniform distribution in the half circle:

$$\theta_0(s) \approx (1.632, 0.202, 0.801, 1.059, 1.391, 3.133, 0.637, 2.580, 1.264, 2.789), \quad s \in -[\tau, 0].$$

We make use of the JiTCDDE package from the JiTC*DE toolbox [4] for Python, which provides an extension to the commonly used SciPy ODE, allowing for the simulation of delay differential equations.

As illustrated in Figure 3.2, we observe the oscillatory behavior of frequencies when $\tau = 1$, and we cannot have the complete frequency synchronization behavior of solutions. On the other hand, when the magnitude of the time delay is small enough, we see the complete frequency synchronization as depicted in Figure 3.3. We also compare the time evolution of frequency diameters with $\tau = 1$ and $\tau = 0.1$ in Figure 3.4.

3.4.2 All-to-all connected case

In this subsection, we perform simulations for the all-to-all coupled system of 10 oscillators, with time delays τ_{ij} generated according to random uniform distributions in $[0, 1]$ and $[0, 0.1]$ and

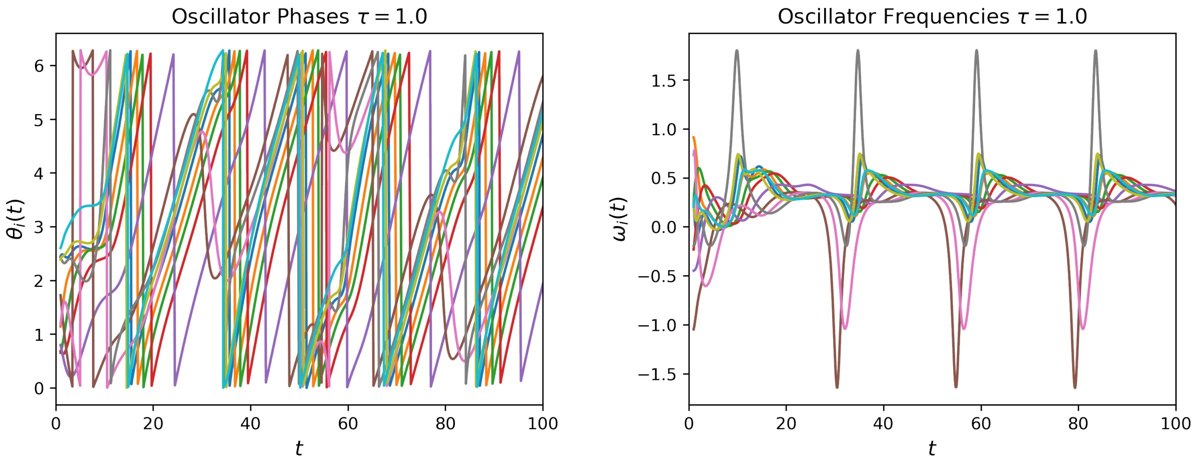


Figure 3.2: Time evolution of phases and frequencies in the case of $R_\omega \tau \approx 1.803 > \frac{\pi}{2}$

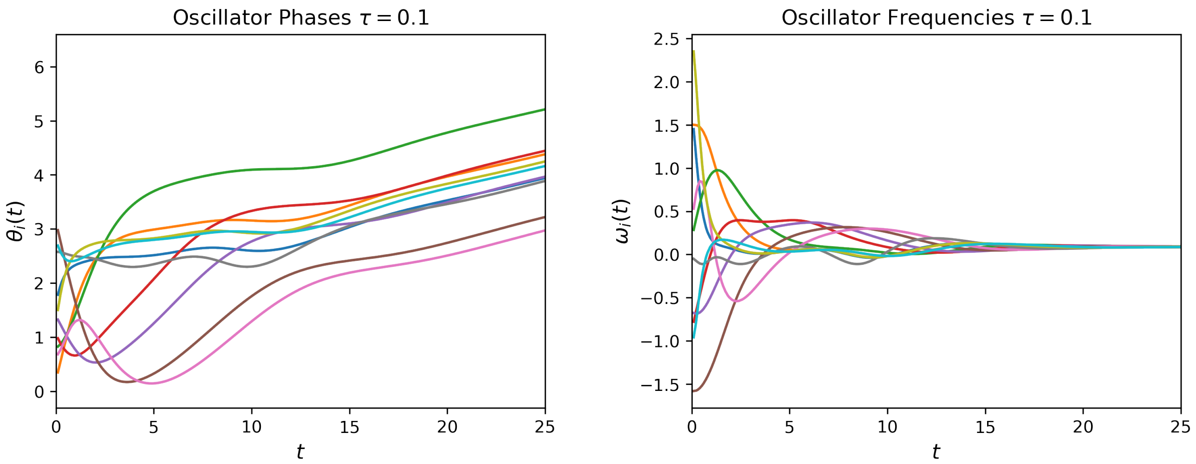


Figure 3.3: Time evolution of phases and frequencies in the case of $R_\omega \tau \approx 0.230 < \frac{\pi}{2}$

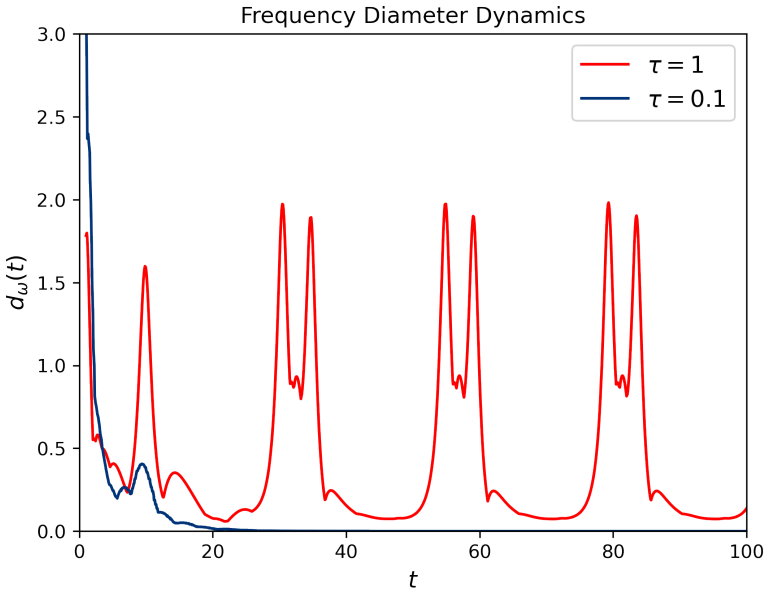


Figure 3.4: Comparison of frequency diameter in the large and small time delay regimes

coupling constant $\kappa = 1$. The natural frequencies were generated according to a zero-centered uniform distribution, to three decimal places.

$$\Omega \approx (-0.349, -0.635, 0.248, -0.639, -0.537, 0.145, -0.433, 0.236, 0.584, -0.114).$$

The initial data was set to be constant and drawn from a random uniform distribution in the half circle:

$$\theta_0(s) \approx (2.050, 1.169, 0.065, 2.948, 2.737, 0.669, 0.080, 2.860, 0.704, 2.228), \quad s \in -[\tau, 0].$$

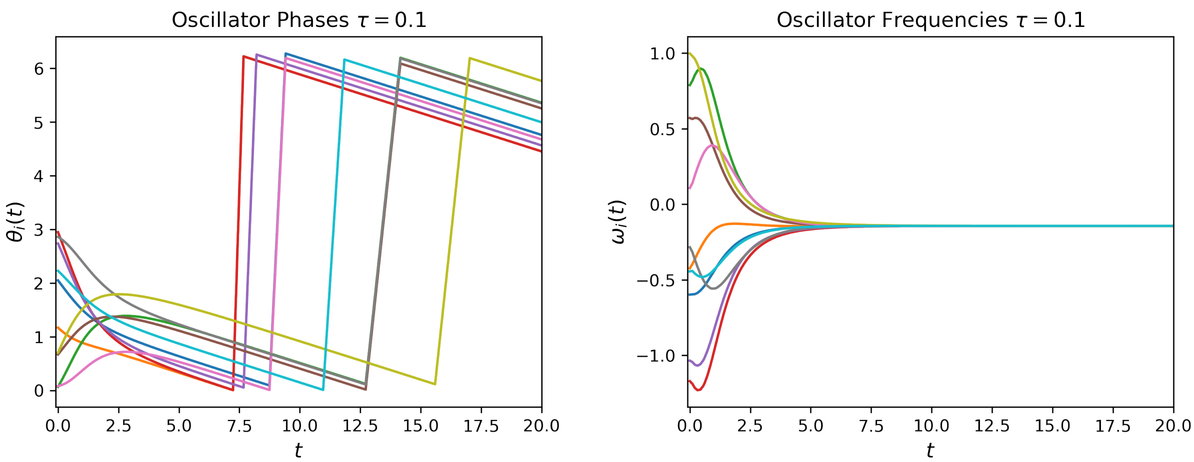


Figure 3.5: Frequency synchronization in the case of $R_\omega \tau \approx 0.161 < \frac{\pi}{2}$

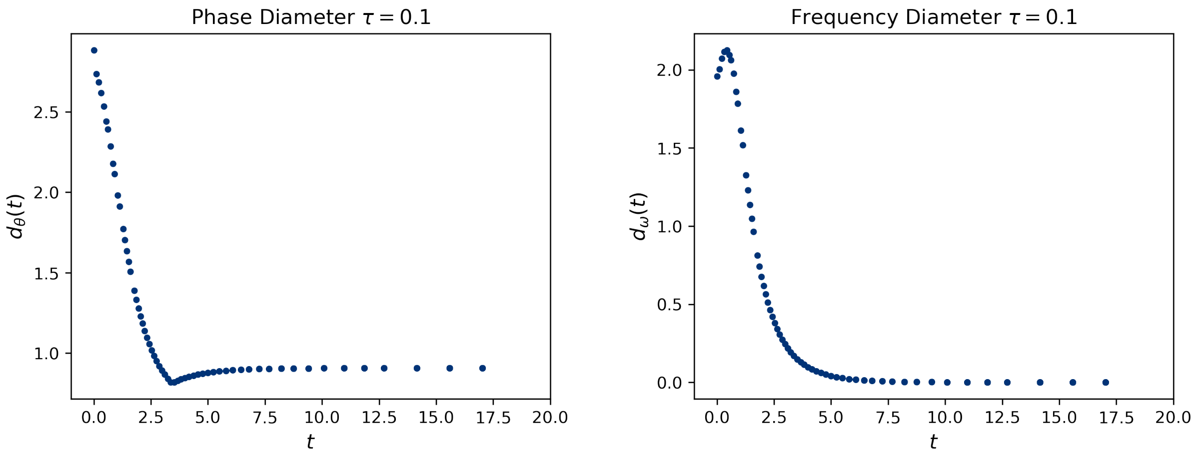


Figure 3.6: Time evolution of phase and frequency diameters

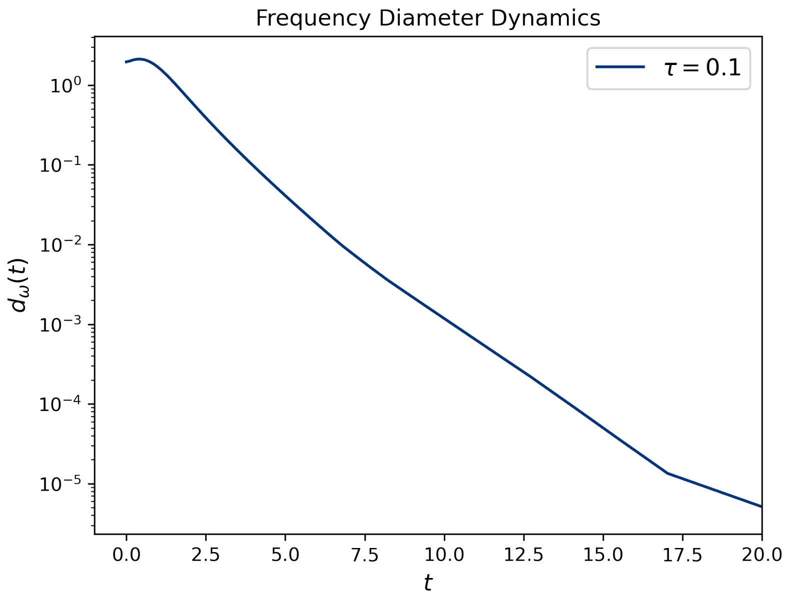


Figure 3.7: Logarithmic plot of frequency diameter exhibiting exponential decay

Chapter 4

Opinion Dynamics under Common Influencer Assumption

In this chapter, we examine scenarios where interactions are not universal but where a common influencer always exists between any two different agents. As always, we focus on studying the convergence to consensus for some Hegselmann-Krause opinion formation models with time-delayed coupling in this framework. We assume that the time delay between two agents is dependent on the agents themselves.

Consider a finite set of $N \in \mathbb{N},\ N \geq 3$, agents. Let $x_i(t) \in \mathbb{R}^d$ be the opinion of the i-th agent at time t. We assume that there is a time lag in the interaction between the agents, described by positive constants $\tau_{ij},\ i,j = 1,\dots,N$. Moreover, we assume that the interaction is non-universal. Then, the interaction between the particles of the system is described by the following Hegselmann-Krause type model:

$$\frac{d}{dt}x_i(t) = \frac{1}{N-1}\sum_{j \neq i} \chi_{ij} a_{ij}(t)(x_j(t-\tau_{ij}) - x_i(t)), \quad t > 0,\ i = 1, ..., N, \tag{4.0.1}$$

with weights $a_{ij}(t)$ of the form

$$a_{ij}(t) := \psi_{ij}(x_i(t), x_j(t-\tau_{ij})), \quad i,j = 1, ..., N, \tag{4.0.2}$$

where the influence functions $\psi_{ij} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ are positive, bounded and continuous. We denote

$$K_{ij} := \| \psi_{ij} \|_\infty, \quad \forall i,j = 1,\dots,N,$$

and let

$$K := \max_{i,j=1,\dots,N} K_{ij}. \tag{4.0.3}$$

The function χ_{ij} for $i,j = 1,\dots,N,\ j \neq i$, is defined as in (1.2.9). Moreover, let us assume the following initial conditions:

$$x_i(t) = x_i^0(t), \quad i = 1,\dots,N,\ t \in [-\tau, 0], \tag{4.0.4}$$

where

$$\tau := \max_{i,j=1,\dots,N} \tau_{ij},$$

and $x_i^0 : [-\tau, 0] \to \mathbb{R}^d$, $i = 1, \dots, N$, are continuous functions. Here, we are interested in convergence to consensus results.

We make the following assumption about the structure of the system:

(CI) For all $i, j \in \{1, \dots, N\}$ there exists $k \in \{1, \dots, N\} \setminus \{i, j\}$ such that $\chi_{ik} = \chi_{jk} = 1$ and $\tau_{ik} = \tau_{jk}$. This means that for each pair of agents (x_i, x_j) there's another agent x_k, that we call *common influencer of* x_i *and* x_j, that transmits information to both of them with the same time delay.

Note that the assumption **(CI)** does not imply that the underlying graph is strongly connected, except for a system of only 3 agents. Then, our arguments here are different from the ones in Chapter 2 or Chapter 3, where the convergence to consensus is investigated for a strongly connected digraph (see [31, 32, 58]). Our analysis is more in the spirit of [34], which deals with the all-to-all case, i.e., each agent influences and is influenced by any other agent. We use some arguments developed, but due to the lack of connections among some agent pairs, a more delicate and finer analysis is required.

After preliminary estimates, we will prove the convergence to consensus for solutions to system (6.1.15). As an example, falling in the previous setting, we will consider a model for a population with a leader subset. When the leaders' numbers are 1 or 2, the common influencer assumption is not satisfied. So, we will study these cases using different appropriate arguments. In the case of a unique leader, we will also analyze a control problem (cf. [43, 64, 74]).

4.1 Preliminaries

In this section, we present some preliminary results concerning system (6.1.15). We omit the proofs since they are analogous to the ones of Lemma 2.2, Lemma 2.3, and Lemma 2.5 of [34].

Lemma 4.1.1. *Let* $\{x_i(t)\}_{i=1}^N$ *be a solution to* (6.1.15)-(4.0.4). *Then, for each* $v \in \mathbb{R}^d$ *and* $T \geq 0$, *we have*

$$\min_{j=1,\dots,N} \min_{s\in[T-\tau,T]} \langle x_j(s), v\rangle \leq \langle x_i(t), v\rangle \leq \max_{j=1,\dots,N} \max_{s\in[T-\tau,T]} \langle x_j(s), v\rangle, \tag{4.1.5}$$

for all $t \geq T - \tau$, $i = 1, \dots, N$.

We now introduce the following notation.

Definition 4.1.2. *We define*

$$D_0 := \max_{i,j=1,\dots,N} \max_{s,t\in[-\tau,0]} |x_i(s) - x_j(t)|, \tag{4.1.6}$$

and in general for all $n \in \mathbb{N}$,

$$D_n := \max_{i,j=1,\dots,N} \max_{s,t\in[n\tau-\tau,n\tau]} |x_i(s) - x_j(t)|. \tag{4.1.7}$$

Let us denote with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Lemma 4.1.3. *Let* $\{x_i(t)\}_{i=1}^N$ *be a solution to* (6.1.15)-(4.0.4). *For each* $n \in \mathbb{N}_0$ *and* $\forall\ i, j = 1, ..., N$ *we get*

$$|x_i(t) - x_j(t)| \leq D_n, \quad \forall\ t \geq n\tau - \tau. \tag{4.1.8}$$

Remark 4.1.4. *Let us note that from Lemma 4.1.3 it follows, in particular,*

$$|x_i(s) - x_j(t)| \leq D_0, \quad \forall s, t \geq -\tau, \ \forall i, j = 1, \ldots, N.$$

Moreover, it holds that

$$D_{n+1} \leq D_n, \quad \forall n \in \mathbb{N}_0. \tag{4.1.9}$$

As in [34], one can find a bound on $|x_i(t)|$ uniform with respect to t and $i = 1, ..., N$. We have the following lemma (cf. [34, Lemma 2.5]).

Lemma 4.1.5. *Let $\{x_i(t)\}_{i=1}^N$ be a solution to* (6.1.15)-(4.0.4)*. For all $i = 1, ..., N$ we have*

$$|x_i(t)| \leq M_0, \quad \forall \ t \geq 0, \tag{4.1.10}$$

where

$$M_0 := \max_{i=1,\ldots,N} \max_{s\in[-\tau,0]} |x_i(0)|. \tag{4.1.11}$$

Remark 4.1.6. *From Lemma 4.1.5, since the influence functions ψ_{ij} are continuous, we deduce that*

$$\psi_{ij}(x_i(t), x_j(t-\tau_{ij})) \geq \psi_0 := \min_{i,j=1,\ldots,N} \min_{|z_1|,|z_2|\leq M_0} \psi_{ij}(z_1, z_2) > 0, \tag{4.1.12}$$

for each $t \geq 0$ and $i, j = 1, ..., N$.

Now, we need the following estimates.

Lemma 4.1.7. *Let $\{x_i(t)\}_{i=1}^N$ be a solution to* (6.1.15)-(4.0.4)*. For all $i, j = 1, \ldots, N$, unit vector $v \in \mathbb{R}^d$ and $n \in \mathbb{N}_0$ we have that*

$$\langle x_i(t) - x_j(t), v\rangle \leq e^{-K(t-t_0)}\langle x_i(t_0) - x_j(t_0), v\rangle + (1 - e^{-K(t-t_0)})D_n, \tag{4.1.13}$$

for all $t \geq t_0 \geq n\tau$. Moreover, for all $n \in \mathbb{N}_0$ we get

$$D_{n+1} \leq e^{-K\tau} d(n\tau) + (1 - e^{-K\tau})D_n. \tag{4.1.14}$$

Proof Let us fix $v \in \mathbb{R}^d$ such that $|v| = 1$ and $n \in \mathbb{N}_0$. We denote by

$$M_n := \max_{i=1,\ldots,N} \max_{t\in[n\tau-\tau,n\tau]} \langle x_i(t), v\rangle,$$

and

$$m_n := \min_{i=1,\ldots,N} \min_{t\in[n\tau-\tau,n\tau]} \langle x_i(t), v\rangle.$$

Note that $M_n - m_n \leq D_n$. Now, fix $i = 1, \ldots, N$ and $t \geq t_0 \geq n\tau$. Then, by definition of the system (6.1.15) and applying Lemma 4.1.1 with $T = n\tau$, we have that

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t), v\rangle &= \frac{1}{N-1}\sum_{j\neq i} \chi_{ij} a_{ij}(t)\langle x_j(t-\tau_{ij}) - x_i(t), v\rangle \\
&\leq \frac{1}{N-1}\sum_{j\neq i} a_{ij}(t)(M_n - \langle x_i(t), v\rangle) \\
&\leq K(M_n - \langle x_i(t), v\rangle)
\end{aligned}$$

where we used that $t - \tau_{ij} \geq n\tau - \tau$ and that

$$\sum_{j \neq i} \chi_{ij} \leq N - 1.$$

By applying Grönwall's Lemma, we find that

$$\langle x_i(t), v\rangle \leq e^{-K(t-t_0)}\langle x_i(t_0), v\rangle + (1 - e^{-K(t-t_0)})M_n. \tag{4.1.15}$$

On the other hand, for $i = 1, \dots, N$ and $t \geq t_0 \geq n\tau$, following a similar argument, we have that

$$\frac{d}{dt}\langle x_j(t), v\rangle \geq K(m_n - \langle x_j(t), v\rangle)$$

and applying again Grönwall's Lemma,

$$\langle x_j(t), v\rangle \geq e^{-K(t-t_0)}\langle x_j(t_0), v\rangle + (1 - e^{-K(t-t_0)})m_n. \tag{4.1.16}$$

From (6.1.24) and (6.1.25), we have that

$$\begin{aligned}\langle x_i(t) - x_j(t), v\rangle &\leq e^{-K(t-t_0)}\langle x_i(t_0) - x_j(t_0), v\rangle + (1 - e^{-K(t-t_0)})(M_n - m_n)\\ &\leq e^{-K(t-t_0)}\langle x_i(t_0) - x_j(t_0), v\rangle + (1 - e^{-K(t-t_0)})D_n.\end{aligned}$$

So, we have (4.1.13). The inequality (4.1.14) follows as in the second part of the proof of Lemma 2.6 in [34]. □

The following result is crucial to proving the exponential convergence to the consensus estimate.

Lemma 4.1.8. *Let* $\{x_i(t)\}_{i=1}^N$ *be a solution to* (6.1.15)-(4.0.4) *and let us assume* **(CI)**. *Then, there exists a constant* $C \in (0, 1)$ *such that*

$$d(n\tau) \leq CD_{n-2}, \tag{4.1.17}$$

for all $n \geq 2$.

Proof Notice that if $d(n\tau) = 0$, the result is trivial. Suppose $d(n\tau) > 0$ and let be $i, j = 1, \dots, N$ such that $d(n\tau) = |x_i(n\tau) - x_j(n\tau)|$. We set

$$v := \frac{x_i(n\tau) - x_j(n\tau)}{|x_i(n\tau) - x_j(n\tau)|}.$$

Then, $d(n\tau) = \langle x_i(n\tau) - x_j(n\tau), v\rangle$. Consider,

$$M_{n-1} := \max_{i=1,\dots,N} \max_{s \in [(n-2)\tau, (n-1)\tau]} \langle x_i(s), v\rangle$$

and

$$m_{n-1} := \min_{i=1,\dots,N} \min_{s \in [(n-2)\tau, (n-1)\tau]} \langle x_i(s), v\rangle.$$

It holds that $M_{n-1} - m_{n-1} \leq D_{n-1}$.
From (6.1.15), for $t \in [(n-1)\tau, n\tau]$, we can write

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t) - x_j(t), v\rangle &= \frac{1}{N-1}\sum_{l:l\neq i}\chi_{il}a_{il}(t)\langle x_l(t-\tau_{il}) - x_i(t), v\rangle \\
&\quad - \frac{1}{N-1}\sum_{l:l\neq j}\chi_{jl}a_{jl}(t)\langle x_l(t-\tau_{jl}) - x_j(t), v\rangle \\
&= \frac{1}{N-1}\sum_{l:l\neq i}\chi_{il}a_{il}(t)(\langle x_l(t-\tau_{il}), v\rangle - M_{n-1}) \\
&\quad + \frac{1}{N-1}\sum_{l:l\neq i}\chi_{il}a_{il}(t)(M_{n-1} - \langle x_i(t), v\rangle) \\
&\quad - \frac{1}{N-1}\sum_{l:l\neq j}\chi_{jl}a_{jl}(t)(\langle x_l(t-\tau_{jl}), v\rangle - m_{n-1}) \\
&\quad - \frac{1}{N-1}\sum_{l:l\neq j}\chi_{jl}a_{jl}(t)(m_{n-1} - \langle x_j(t), v\rangle).
\end{aligned} \tag{4.1.18}$$

Let us define the sums S_1 and S_2 and, using Remark 6.1.5, we have

$$\begin{aligned}
S_1 &:= \frac{1}{N-1}\sum_{l:l\neq i}\chi_{il}a_{il}(t)(\langle x_l(t-\tau_{il}), v\rangle - M_{n-1}) \\
&\quad + \frac{1}{N-1}\sum_{l:l\neq i}\chi_{il}a_{il}(t)(M_{n-1} - \langle x_i(t), v\rangle) \\
&\leq \frac{\psi_0}{N-1}\sum_{l:l\neq i}\chi_{il}(\langle x_l(t-\tau_{il}), v\rangle - M_{n-1}) + K(M_{n-1} - \langle x_i(t), v\rangle),
\end{aligned} \tag{4.1.19}$$

and

$$\begin{aligned}
S_2 &:= \frac{1}{N-1}\sum_{l:l\neq j}\chi_{jl}a_{jl}(t)(m_{n-1} - \langle x_l(t-\tau_{jl}), v\rangle) \\
&\quad + \frac{1}{N-1}\sum_{l:l\neq j}\chi_{jl}a_{jl}(t)(\langle x_j(t), v\rangle - m_{n-1}) \\
&\leq \frac{\psi_0}{N-1}\sum_{l:l\neq j}\chi_{jl}(m_{n-1} - \langle x_l(t-\tau_{jl}), v\rangle) + K(\langle x_j(t), v\rangle - m_{n-1}),
\end{aligned} \tag{4.1.20}$$

where we used the fact that, being $t \in [(n-1)\tau, n\tau]$, it holds that
$t - \tau_{ij} \in [(n-2)\tau, n\tau]$, $\forall ij = 1, \dots, N$, and then

$$m_{n-1} \leq \langle x_l(t-\tau_{il}), v\rangle \leq M_{n-1}, \quad m_{n-1} \leq \langle x_l(t-\tau_{jl}), v\rangle \leq M_{n-1},$$

for all $l = 1, \dots, N$. Then, using (4.1.19) and (4.1.20) in (4.1.18), we have that

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t) - x_j(t), v\rangle &\leq K(M_{n-1} - m_{n-1}) - K\langle x_i(t) - x_j(t), v\rangle \\
&\quad + \frac{\psi_0}{N-1}\sum_{l:l\neq i}\chi_{il}(\langle x_l(t-\tau_{il}), v\rangle - M_{n-1}) \\
&\quad + \frac{\psi_0}{N-1}\sum_{l:l\neq j}\chi_{jl}(m_{n-1} - \langle x_l(t-\tau_{jl}), v\rangle).
\end{aligned}$$

Now, we use the assumption **(CI)**. Let x_k, for some $k \in \{1, \dots, N\}$, be a common influencer between x_i and x_j. Using that, for all $l = 1, \dots, N$,

$$\langle x_l(t - \tau_{il}), v\rangle - M_{n-1} \le 0$$

and

$$m_{n-1} - \langle x_l(t - \tau_{jl}), v\rangle \le 0,$$

we find that

$$\begin{aligned}
&\frac{d}{dt}\langle x_i(t) - x_j(t), v\rangle \le K(M_{n-1} - m_{n-1}) - K\langle x_i(t) - x_j(t), v\rangle \\
&+ \frac{\psi_0}{N-1}(\langle x_k(t - \tau_{ik}), v\rangle - M_{n-1} + m_{n-1} - \langle x_k(t - \tau_{jk}), v\rangle) \\
&= \left(K - \frac{\psi_0}{N-1}\right)(M_{n-1} - m_{n-1}) - K\langle x_i(t) - x_j(t), v\rangle.
\end{aligned}$$

Applying the Grönwall's Lemma in $[(n-1)\tau, t]$, with $t \in [(n-1)\tau, n\tau]$, we find that

$$\begin{aligned}
\langle x_i(t) - x_j(t), v\rangle &\le e^{-K(t - n\tau + \tau)}\langle x_i(n\tau - \tau) - x_j(n\tau - \tau), v\rangle \\
&\quad + \left(1 - \frac{\psi_0}{K(N-1)}\right)(M_{n-1} - m_{n-1})(1 - e^{-K(t - n\tau + \tau)}).
\end{aligned}$$

Since this is valid $\forall\ t \in [(n-1)\tau, n\tau]$, let us consider $t = n\tau$. Then,

$$\begin{aligned}
d(n\tau) &\le e^{-K\tau}\langle x_i(n\tau - \tau) - x_j(n\tau - \tau), v\rangle \\
&\quad + \left(1 - \frac{\psi_0}{K(N-1)}\right)(M_{n-1} - m_{n-1})(1 - e^{-K\tau}) \\
&\le e^{-K\tau}|x_i(n\tau - \tau) - x_j(n\tau - \tau)||v| \\
&+ \left(1 - \frac{\psi_0}{K(N-1)}\right)(M_{n-1} - m_{n-1})(1 - e^{-K\tau}) \\
&\le D_{n-1}\left[e^{-K\tau} + 1 - \frac{\psi_0}{K(N-1)}(1 - e^{-K\tau})\right] \\
&\le D_{n-2}\left[1 - \frac{\psi_0}{K(N-1)}(1 - e^{-K\tau})\right].
\end{aligned} \tag{4.1.21}$$

Therefore, (4.1.17) follows with

$$C := 1 - \frac{\psi_0}{K(N-1)}(1 - e^{-K\tau}). \tag{4.1.22}$$

□

4.2 Consensus under Common Influencer assumption

We are ready to give our convergence to the consensus estimate, using the same argument that we used in Chapter 3 for the all-to-all connected case.

Theorem 4.2.1. *Assume* **(CI)***. Then, every solution* $\{x_i(t)\}_{i=1}^N$ *to* (6.1.15)-(4.0.4) *satisfies the exponential decay estimate*

$$d(t) \leq D_0 e^{-\gamma(t-2\tau)} \text{ for all } t \geq 0, \tag{4.2.23}$$

for a suitable positive constant γ *depending on* N *and on the value* M_0 *defined in* (4.1.11).

Proof Let $\{x_i(t)\}_{i=1}^N$ be the solution of (6.1.15)-(4.0.4). We claim that

$$D_{n+1} \leq \tilde{C} D_{n-2}, \ \forall n \geq 2, \tag{4.2.24}$$

for some constant $\tilde{C} \in (0,1)$. Using (4.1.9),(4.1.14) and (4.1.17) we get

$$\begin{aligned} D_{n+1} &\leq e^{-K\tau} d(n\tau) + (1 - e^{-K\tau}) D_n \\ &\leq e^{-K\tau} C D_{n-2} + (1 - e^{-K\tau}) D_n \\ &\leq e^{-K\tau} C D_{n-2} + (1 - e^{-K\tau}) D_{n-2} \\ &= \left(1 - e^{-K\tau}(1 - C)\right) D_{n-2}, \end{aligned}$$

where C is defined in (6.1.28). So, setting

$$\tilde{C} := 1 - e^{-K\tau}(1 - C),$$

we have the claim. This implies that

$$D_{3n} \leq \tilde{C}^n D_0, \ \forall n \geq 1. \tag{4.2.25}$$

From (6.1.32), we have that

$$D_{3n} \leq e^{-3n\gamma\tau} D_0, \ \forall n \in \mathbb{N}_0.$$

where

$$\gamma := \frac{1}{3\tau} \ln\left(\frac{1}{\tilde{C}}\right).$$

Now, fix $i, j = 1, \ldots, N$ and $t \geq 0$. Then one can find $t \in [3n\tau - \tau, 3n\tau + 2\tau]$ for some $n \in \mathbb{N}_0$. Therefore, using Lemma 4.1.3, we find that

$$|x_i(t) - x_j(t)| \leq D_{3n} \leq e^{-3n\gamma\tau} D_0.$$

Thus, being $t \leq 3n\tau + 2\tau$, we get

$$|x_i(t) - x_j(t)| \leq e^{-\gamma t} e^{2\gamma\tau} D_0,$$

and, finally, we find that

$$d(t) \leq e^{-\gamma(t-2\tau)} D_0, \quad \forall t \geq 0.$$

So, (4.2.23) is proved. □

Remark 4.2.2. *Note that the convergence consensus estimate* (4.2.23) *depends on the time delay size. In particular, a large time delay slows the convergence to consensus.*

As a particular case of such a model, we can consider a system in which there is a (eventually small) group of m leaders, $m \in \mathbb{N}$. These agents influence all other agents in the population, but they are influenced only by the other leaders. Let $y_i(t) \in \mathbb{R}^d, i = 1, \dots, m$, be the opinion of the i-th leader at time t and $x_i(t) \in \mathbb{R}^d,\ i = 1, \dots, N$ be the opinion of the i-th non-leader at time t. We assume a non-universal interaction between the non-leaders. We still consider a time delay in the interaction between the agents to appear as the time needed to discuss and make a decision. Then, the opinions of the population evolve following the Hegselmann-Krause opinion formation model:

$$
\begin{aligned}
\frac{d}{dt}y_i(t) &= \frac{1}{m-1}\sum_{j\neq i} a_{ij}(t)(y_j(t-\tilde{\tau}_j) - y_i(t)), \quad t>0,\ i=1,...,m,\\
\frac{d}{dt}x_i(t) &= \frac{1}{N+m-1}\sum_{j\neq i} \chi_{ij} b_{ij}(t)(x_j(t-\tau_{ij}) - x_i(t))\\
&\qquad +\frac{1}{N+m-1}\sum_{j=1}^{m} c_{ij}(t)(y_j(t-\tilde{\tau}_j) - x_i(t)),\\
&\qquad\qquad t>0,\ i=1,...,N,
\end{aligned}
\tag{4.2.26}
$$

with the interaction weights $a_{ij}(t), b_{ij}(t), c_{ij}(t)$ $t \geq 0$, defined analogously to (4.0.2), namely

$$a_{ij}(t) := \tilde{\psi}_{ij}(y_i(t), y_j(t-\tilde{\tau}_j)), \quad i,j=1,...,m, \tag{4.2.27}$$

$$b_{ij}(t) := \psi_{ij}(x_i(t), x_j(t-\tau_{ij})), \quad i,j=1,...,N, \tag{4.2.28}$$

$$c_{ij}(t) := \psi^*_{ij}(x_i(t), y_j(t-\tilde{\tau}_j)), \quad i=1,\dots,m,\ j=1,...,N, \tag{4.2.29}$$

being the influence functions $\psi, \tilde{\psi}, \psi^*$ positive, continuous and bounded. Let us assume the initial conditions

$$y_i(t) = y_i^0(t), \quad i=1,\dots,m,\ t\in[-\tau,0], \tag{4.2.30}$$

and

$$x_i(t) = x_i^0(t), \quad i=1,\dots,N,\ t\in[-\tau,0], \tag{4.2.31}$$

where

$$\tau = \max\left\{\max_{i,j=1,\dots,N}\tau_{ij},\ \max_{j=1,\dots,m}\tilde{\tau}_j\right\},$$

and $y_i^0(\cdot), i = 1, \dots, m$, and $x_i^0(\cdot), i = 1, \dots N$, are continuous functions.

Remark 4.2.3. *Note that system* (6.0.1) *satisfies the assumption* ***(CI)*** *if* $m \geq 3$. *Then, in that case, the agents exponentially converge to an asymptotic consensus.*

The special cases with only one or two leaders deserve different analyses that will be developed in the next two sections.

4.3 A HK-model with one leader

Here, we will focus on the case in which only one leader, influencing the other agents but not influenced by anyone, is present in the agents' group. First, we consider the leader to have a fixed opinion; then, we will deal with a controlled leader.

4.3.1 A unique leader with constant trajectory

In this case, the system reads as:

$$\begin{aligned}
&\frac{d}{dt}y_0(t) = 0, \quad t > 0,\\
&\frac{d}{dt}x_i(t) = \frac{1}{N}\sum_{j\neq i}\chi_{ij}b_{ij}(t)(x_j(t-\tau_{ij}) - x_i(t)) + \frac{c_{i0}(t)}{N}(y^0 - x_i(t)),\\
&\qquad\qquad\qquad\qquad\qquad\qquad t > 0,\ i = 1, ..., N,
\end{aligned} \tag{4.3.32}$$

where $b_{ij}(t)$, $i, j = 1, \dots, N$, are defined as in (6.1.1) and $c_{0i}(t) = \phi_{0i}(x_i(t), y^0)$, with $\phi_{0i} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ continuous, positive and bounded. Let us assume the initial conditions

$$y_0(0) = y^0 \quad \text{and} \quad x_i(t) = x_i^0(t), \quad t \in [-\tau, 0],\ \forall\, i = 1, \dots, N, \tag{4.3.33}$$

being, as before, $\tau = \max_{i,j=1,...,N} \tau_{ij}$. We want to study the convergence to consensus of system (4.3.32). The *diameter function* for solutions to (4.3.32) can be defined as

$$d(t) := \max\left\{\max_{i=1,...,N}|x_i(t) - y^0|, \max_{i,j=1,...,N}|x_i(t) - x_j(t)|\right\}.$$

Note that, in this case, assumption **(CI)** is not satisfied. Indeed, for each $i = 1, \dots, N$, the pair (x_i, y_0) does not admit a common influencer.
Let us define the following quantities:

$$M_T := \max\left\{\max_{j=1,...,N}\max_{s\in[T-\tau,T]}\langle x_j(s), v\rangle, \langle y^0, v\rangle\right\}, \tag{4.3.34}$$

and

$$m_T := \min\left\{\min_{j=1,...,N}\min_{s\in[T-\tau,T]}\langle x_j(s), v\rangle, \langle y^0, v\rangle\right\}. \tag{4.3.35}$$

Moreover, let us define

$$\tilde{K} := \max\left\{\max_{i,j=1,...,N}\|\psi_{ij}\|_\infty, \max_{i=1,...,N}\|\phi_{0i}\|_\infty\right\}. \tag{4.3.36}$$

Similarly to the previous case, we can state the following results.

Lemma 4.3.1. *Let* $\left(\{x_i(t)\}_{i=1}^N, y_0\right)$ *be a solution to* (4.3.32)-(4.3.33). *Then, for each* $v \in \mathbb{R}^d$ *and* $T \geq 0$, *we have*

$$m_T \leq \langle x_i(t), v\rangle \leq M_T, \tag{4.3.37}$$

for all $t \geq T - \tau$, $i = 1, \dots, N$.

Remark 4.3.2. *Notice that, from the definition* (4.3.34) *and* (4.3.35), *it is trivial that*

$$m_T \leq \langle y^0, v\rangle \leq M_T.$$

Let us specify the notation in Definition 4.1.2 for this case.

Definition 4.3.3. *We define*

$$D_0 := \max\Big\{ \max_{i,j=1,\dots,N} \max_{s,t\in[-\tau,0]} |x_i(s)-x_j(t)|, \\ \max_{i=1,\dots,N} \max_{s\in[-\tau,0]} |x_i(s)-y^0|\Big\}, \tag{4.3.38}$$

and, in general for all $n \in \mathbb{N}$,

$$D_n := \max\Big\{ \max_{i,j=1,\dots,N} \max_{s,t\in[n\tau-\tau,n\tau]} |x_i(s)-x_j(t)|, \\ \max_{i=1,\dots,N} \max_{s\in[n\tau-\tau,n\tau]} |x_i(s)-y^0|\Big\}. \tag{4.3.39}$$

As before, we need preliminary estimates to prove the consensus result.

Lemma 4.3.4. *Let $\left(\{x_i(t)\}_{i=1}^N, y_0\right)$ be a solution to* (4.3.32)*-*(4.3.33)*. For each $n \in \mathbb{N}_0$ and $\forall\, i,j = 1,...,N$, we get*

$$|x_i(t)-x_j(t)| \le D_n, \quad \forall\, t \ge n\tau-\tau, \tag{4.3.40}$$

and, similarly, $\forall i = \dots, N$,

$$|x_i(t)-y^0| \le D_n, \quad \forall\, t \ge n\tau-\tau. \tag{4.3.41}$$

Let us note that, as in Remark 6.1.3, Lemma 4.3.4 implies that $\{D_n\}_{n\in\mathbb{N}_0}$ is a non-increasing sequence.

Lemma 4.3.5. *Let $\left(\{x_i(t)\}_{i=1}^N, y_0\right)$ be a solution to* (4.3.32)*-*(4.3.33)*. For all $i = 1,...,N$, we have*

$$|x_i(t)| \le C_0, \quad \forall\, t \ge 0, \tag{4.3.42}$$

and, in particular,

$$|y^0| \le C_0, \quad \forall\, t \ge 0, \tag{4.3.43}$$

where

$$C_0 := \max\left\{ \max_{i=1,\dots,N} \max_{s\in[-\tau,0]} |x_i(0)|, |y^0| \right\}. \tag{4.3.44}$$

Remark 4.3.6. *As in Remark 6.1.5, from Lemma 4.3.5, since the influence functions ψ_{ij} and ϕ_{0i} are continuous and positive, we deduce that*

$$\psi_{ij}(x_i(t), x_j(t-\tau_{ij})) \ge \psi_0 := \min_{i,j=1,\dots,N} \min_{|z_1|,|z_2|\le C_0} \psi_{ij}(z_1,z_2) > 0, \tag{4.3.45}$$

for each $t \ge 0$ and $i,j = 1,...,N$, and

$$\phi_{0i}(x_i(t), y^0) \ge \phi_0 := \min_{i=1,\dots,N} \min_{|z_1|,|z_2|\le C_0} \phi_{0i}(z_1,z_2) > 0, \tag{4.3.46}$$

for each $t \ge 0$ and $i = 1,...,N$.

Let us denote

$$\tilde{\psi}_0 := \min\{\psi_0, \phi_0\}. \tag{4.3.47}$$

Lemma 4.3.7. *Let* $\left(\{x_i(t)\}_{i=1}^N, y_0\right)$ *be a solution to* (4.3.32)-(4.3.33). *For all* $i, j = 1, \ldots, N$, *unit vector* $v \in \mathbb{R}^d$ *and* $n \in \mathbb{N}_0$ *we have that*

$$\langle x_i(t) - x_j(t), v\rangle \leq e^{-\tilde{K}(t-t_0)}\langle x_i(t_0) - x_j(t_0), v\rangle + (1 - e^{-\tilde{K}(t-t_0)})D_n, \tag{4.3.48}$$

for all $t \geq t_0 \geq n\tau$, $i, j = 1, \ldots, N$, *and*

$$\langle x_i(t) - y^0, v\rangle \leq e^{-\tilde{K}(t-t_0)}\langle x_i(t_0) - y^0, v\rangle + (1 - e^{-\tilde{K}(t-t_0)})D_n, \tag{4.3.49}$$

for all $t \geq t_0 \geq n\tau$. *Moreover,* (4.1.14) *holds true for all* $n \in \mathbb{N}_0$.

Proof The estimate (4.3.48) can be proved analogously to (4.1.13). So, let us prove (4.3.49). Let us fix $v \in \mathbb{R}^d$ such that $|v| = 1$ and $n \in \mathbb{N}_0$. We denote by

$$M_n := \max\left\{\max_{i=1,\ldots,N}\max_{t\in[n\tau-\tau,n\tau]}\langle x_i(t), v\rangle, \langle y^0, v\rangle\right\}$$

and

$$m_n := \min\left\{\min_{i=1,\ldots,N}\min_{t\in[n\tau-\tau,n\tau]}\langle x_i(t), v\rangle, \langle y^0, v\rangle\right\}.$$

Note that $M_n - m_n \leq D_n$. Now, fix $i = 1, \ldots, N$ and $t \geq t_0 \geq n\tau$. Then, as before, we have that

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t), v\rangle &= \frac{1}{N}\sum_{j\neq i}\chi_{ij}b_{ij}(t)\langle x_j(t-\tau_{ij}) - x_i(t), v\rangle + \frac{c_{i0}(t)}{N}\langle y^0 - x_i(t), v\rangle \\
&\leq \frac{1}{N}(N-1)\tilde{K}\left(M_n - \langle x_i(t), v\rangle\right) + \frac{\tilde{K}}{N}\left(M_n - \langle x_i(t), v\rangle\right) \\
&\leq \tilde{K}(M_n - \langle x_i(t), v\rangle).
\end{aligned}$$

By applying Grönwall's Lemma, we find that

$$\langle x_i(t), v\rangle \leq e^{-\tilde{K}(t-t_0)}\langle x_i(t_0), v\rangle + (1 - e^{-\tilde{K}(t-t_0)})M_n. \tag{4.3.50}$$

On the other hand, notice that it holds that

$$\frac{d}{dt}\langle y^0, v\rangle = 0 \geq \tilde{K}(m_n - \langle y^0, v\rangle).$$

Then, applying again Grönwall's Lemma,

$$\langle y^0, v\rangle \geq e^{-\tilde{K}(t-t_0)}\langle y^0, v\rangle + (1 - e^{-\tilde{K}(t-t_0)})m_n. \tag{4.3.51}$$

From (4.3.50) and (4.3.51), we have find (4.3.49). □

Finally, we have the following result.

Lemma 4.3.8. *Let* $\left(\{x_i(t)\}_{i=1}^N, y_0\right)$ *be a solution to* (4.3.32)-(4.3.33). *There exists a constant* $C \in (0, 1)$ *such that*

$$d(n\tau) \leq CD_{n-2}, \tag{4.3.52}$$

for all $n \geq 2$.

Proof Let us assume that $d(n\tau) = |x_i(n\tau) - y^0|$ for a given $i = 1, \dots, N$. We set

$$v := \frac{x_i(n\tau) - y^0}{|x_i(n\tau) - y^0|}.$$

Then, $d(n\tau) = \langle x_i(n\tau) - y^0, v\rangle$. Consider,

$$M_{n-1} := \max\left\{\max_{i=1,\dots,N}\max_{s\in[(n-2)\tau,(n-1)\tau]}\langle x_i(s), v\rangle, \langle y^0, v\rangle\right\}$$

and

$$m_{n-1} := \min\left\{\min_{i=1,\dots,N}\min_{s\in[(n-2)\tau,(n-1)\tau]}\langle x_i(s), v\rangle, \langle y^0, v\rangle\right\}.$$

It holds that $M_{n-1} - m_{n-1} \le D_{n-1}$.
From (4.3.32), for $t \in [(n-1)\tau, n\tau]$, we can write

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t) - y^0, v\rangle &= \frac{1}{N}\sum_{j\neq i}\chi_{ij}b_{ij}(t)\langle x_j(t-\tau_{ij}) - x_i(t), v\rangle + \frac{c_{i0}(t)}{N}\langle y^0 - x_i(t), v\rangle \\
&= \frac{1}{N}\sum_{j\neq i}\chi_{ij}b_{ij}(t)\left(\langle x_j(t-\tau_{ij}), v\rangle - M_{n-1}\right) + \frac{1}{N}\sum_{j\neq i}\chi_{ij}b_{ij}(t)\left(M_{n-1} - x_i(t), v\rangle\right) \\
&\quad + \frac{c_{i0}(t)}{N}\left(\langle y^0, v\rangle - M_{n-1}\right) + \frac{c_{i0}(t)}{N}\left(M_{n-1} - x_i(t), v\rangle\right) \\
&\le \frac{N-1}{N}\tilde{K}(M_{n-1} - \langle x_i(t), v\rangle) + \frac{\tilde{K}}{N}(M_{n-1} - \langle x_i(t), v\rangle) - \frac{\tilde{\psi}_0}{N}(\langle y^0, v\rangle - M_{n-1}).
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t) - y^0, v\rangle &\le \tilde{K}(M_{n-1} - \langle x_i(t), v\rangle) + \frac{\tilde{\psi}_0}{N}(\langle y^0, v\rangle - M_{n-1}) \\
&\quad + c_{i0}(t)(\langle y^0, v\rangle - m_{n-1} + m_{n-1} - \langle y^0, v\rangle) \\
&\le \tilde{K}(M_{n-1} - \langle x_i(t), v\rangle) + \frac{\tilde{\psi}_0}{N}(\langle y^0, v\rangle - M_{n-1}) \\
&\quad + \tilde{K}(\langle y^0, v\rangle - m_{n-1}) + \tilde{\psi}_0(m_{n-1} - \langle y^0, v\rangle) \\
&\le \tilde{K}(M_{n-1} - \langle x_i(t), v\rangle) + \frac{\tilde{\psi}_0}{N}(\langle y^0, v\rangle - M_{n-1}) \\
&\quad + \tilde{K}(\langle y^0, v\rangle - m_{n-1}) + \frac{\tilde{\psi}_0}{N}(m_{n-1} - \langle y^0, v\rangle) \\
&\le \left(\tilde{K} - \frac{\tilde{\psi}_0}{N}\right)(M_{n-1} - m_{n-1}) - \tilde{K}\langle x_i(t) - y^0, v\rangle,
\end{aligned} \tag{4.3.53}$$

where we used the fact that $m_{n-1} - \langle y^0, v\rangle \le 0$.
Notice that, repeating the argument in the proof of Lemma 4.1.8, from the estimate above, applying Grönwall's Lemma, we can obtain a similar estimate as (4.1.21). Finally, assuming that $d(n\tau) = |x_i(n\tau) - x_j(n\tau)|$ for some $i, j = 1, \dots, N$, and using similar steps as above, we can find that

$$\frac{d}{dt}\langle x_i(t) - x_j(t), v\rangle \le \left(\tilde{K} - \frac{\tilde{\psi}_0}{N}\right)(M_{n-1} - m_{n-1}) - \tilde{K}\langle x_i(t) - x_j(t), v\rangle.$$

Therefore, we have the result with

$$C := 1 - \frac{\tilde{\psi}_0}{\tilde{K}N}(1 - e^{-\tilde{K}\tau}). \tag{4.3.54}$$

Then, the exponential consensus result follows, arguing as in the previous section. □

Theorem 4.3.9. *Every solution* $\left(\{x_i(t)\}_{i=1}^N, y_0\right)$ *to* (4.3.32)-(4.3.33) *satisfies the exponential decay estimate*

$$d(t) \le D_0 e^{-\tilde{\gamma}(t-2\tau)}, \quad \forall\, t \ge 0, \tag{4.3.55}$$

for a suitable positive constant $\tilde{\gamma}$ *depending on* N *and on the value* C_0 *defined in* (4.3.44).

4.3.2 A unique leader with controlled trajectory

Consider now an HK-model with a unique leader following a controlled trajectory:

$$\begin{aligned}
&\frac{d}{dt}y_0(t) = u(t) \in \mathbb{R}^d, \quad t > 0,\\
&\frac{d}{dt}x_i(t) = \frac{1}{N}\sum_{j\neq i}\chi_{ij}b_{ij}(t)(x_j(t-\tau_{ij}) - x_i(t)) + \frac{c_{i0}(t)}{N}(y_0(t-\tau_{i0}) - x_i(t)),\\
&\qquad\qquad\qquad\qquad\qquad\qquad t > 0,\ i = 1, ..., N,
\end{aligned} \tag{4.3.56}$$

where $b_{ij}(t)$, $i, j = 1, \dots, N$, are defined as in (6.1.1) and $c_{0i}(t) = \phi_{0i}(x_i(t), y_0(t))$, with $\phi_{0i} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ continuous, positive and bounded, as before. Let us assume the initial conditions

$$y_0(0) = y^0(t) \quad \text{and} \quad x_i(t) = x_i^0(t), \quad t \in [-\tau, 0],\ \forall\, i = 1, \dots, N, \tag{4.3.57}$$

being, as before, $\tau = \max\left\{\max_{i,j=1,\dots,N}\tau_{ij}, \max_{i=1,\dots,N}\tau_{i0}\right\}$. Here, we want to study the convergence to any consensus state for solutions to system (4.3.56). In this case, the definition of the *diameter function* can be written as

$$d(t) := \max\left\{\max_{i=1,\dots,N}|x_i(t) - y_0(t)|, \max_{i,j=1,\dots,N}|x_i(t) - x_j(t)|\right\}.$$

Note that, also in this configuration, assumption **(CI)** is not satisfied.
Consider, again, the appropriate definitions of m_T and M_T:

$$M_T := \max\left\{\max_{j=1,\dots,N}\max_{s\in[T-\tau,T]}\langle x_j(s), v\rangle, \max_{s\in[T-\tau,T]}\langle y_0(s), v\rangle\right\},$$

$$m_T := \min\left\{\min_{j=1,\dots,N}\min_{s\in[T-\tau,T]}\langle x_j(s), v\rangle, \min_{s\in[T-\tau,T]}\langle y_0(s), v\rangle\right\}.$$

and let $\tilde{K}$ be the constant defined in (4.3.36).

Definition 4.3.10. *Fix a constant* $M > 0$. *We say that a* $\mathcal{L}^1$*-measurable strategy* $u(\cdot)$ *is admissible if* $\|u\|_\infty \le M$.

We will prove that, given any state $\bar{x} \in \mathbb{R}^d$, there exists an admissible strategy that steers each agent to $\bar{x}$. Firstly, in the following lemma, we prove that there exists an admissible strategy that steers the leader to $\bar{x} \in \mathbb{R}^d$ and the other agents stay close enough to $\bar{x}$.

Lemma 4.3.11. *Let us consider the system* (4.3.56)-(4.3.57) *and let* $\bar{x} \in \mathbb{R}^d$*. Then, there exists an admissible control strategy* $u : [0, +\infty) \to \mathbb{R}^d$ *such that* y_0 *reaches* $\bar{x}$ *in finite time, i.e. there exists* $t_0 > 0$ *such that* $y_0(t) = \bar{x}$ *for all* $t \geq t_0$*. Moreover, if we denote*

$$R := \max\left\{ \max_{i=1,\dots,N} \max_{s\in[-\tau,0]} |x_i(s) - \bar{x}|,\ \max_{s\in[-\tau,0]} |y_0(s) - \bar{x}| \right\}, \tag{4.3.58}$$

then

$$\max_{i=1,\dots,N} |x_i(t) - \bar{x}| \leq R, \tag{4.3.59}$$

for all $t \geq 0$*.*

Proof Let us define, for $t \geq 0$, the admissible control

$$u(t) := \begin{cases} M \frac{\bar{x}-y^0}{|\bar{x}-y^0|}, & \text{if } y_0(t) \neq \bar{x}, \\ 0, & \text{if } y_0(t) = \bar{x}, \end{cases} \tag{4.3.60}$$

where $y^0 = y_0(0)$. Consider the unit vector $v \in \mathbb{R}^d$ defined as

$$v := \frac{y^0 - \bar{x}}{|y^0 - \bar{x}|},$$

so that $\langle y^0 - \bar{x}, v\rangle = |y^0 - \bar{x}|$. Therefore, from the definition of the system (4.3.56) and (4.3.60), we have for $t \geq 0$ that

$$\frac{d}{dt}\langle y_0(t) - \bar{x}, v\rangle = \frac{M}{|\bar{x} - y^0|}\langle \bar{x} - y^0, v\rangle = -\frac{M}{|\bar{x} - y^0|}|\bar{x} - y^0| = -M.$$

Therefore,

$$\langle y_0(t) - \bar{x}, v\rangle \leq |y_0(t) - \bar{x}| \leq |y^0 - \bar{x}| - Mt, \quad t \in [0, t_0],$$

where

$$t_0 := \frac{|y^0 - \bar{x}|}{M}.$$

It is immediate to see that

$$\langle y_0(t) - \bar{x}, v\rangle = |y_0(t) - \bar{x}|,$$

then the first part of the lemma is proven. Notice now that by definition $|y^0 - \bar{x}| \leq R$, then

$$|y_0(t) - \bar{x}| \leq R,$$

for all $t \geq 0$. Now, let $v \in \mathbb{R}^d$ be a unit vector and let us define the set

$$\mathcal{K}_\epsilon := \left\{ t \in [0, +\infty) \middle|\ \max_{i=1,\dots,N} \langle x_i(s) - \bar{x}, v\rangle < R + \epsilon,\ \forall s \in [0, t) \right\}.$$

By continuity, $\mathcal{K} \neq \emptyset$. Let $S := \sup \mathcal{K}_\epsilon$. We claim that $S = +\infty$. In order to prove it, suppose by contradiction that $S < +\infty$. Then,

$$\lim_{t\to S^-} \max_{i=1,\dots,N} \langle x_i(s) - \bar{x}, v\rangle = R + \epsilon.$$

Consider $t \in [0, S)$. Then, from (4.3.56), we have that

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t) - \bar{x}, v\rangle &= \frac{1}{N}\sum_{j\neq i}\chi_{ij}b_{ij}(t)\langle x_j(t-\tau_{ij}) - x_i(t), v\rangle \\
&\qquad + \frac{c_{i0}(t)}{N}\langle y_0(t-\tau_{i0}) - x_i(t), v\rangle \\
&= \frac{1}{N}\sum_{j\neq i}\chi_{ij}b_{ij}(t)\left(\langle x_j(t-\tau_{ij}) - \bar{x}, v\rangle - \langle x_i(t) - \bar{x}, v\rangle\right) \\
&\qquad + \frac{c_{i0}(t)}{N}\left(\langle y_0(t-\tau_{i0}) - \bar{x}, v\rangle - \langle x_i(t) - \bar{x}, v\rangle\right) \\
&\le \tilde{K}(R + \epsilon - \langle x_i(t) - \bar{x}, v\rangle),
\end{aligned}$$

where we used the fact that, if $t \in [0, S)$, then $t - \tau_{ij} \in [-\tau, S)$, for all $i = 1, \dots, N,\ j = 0, \dots, N$, and, by definition of $\mathcal{K}_\epsilon$ we have

$$\langle x_j(t-\tau_{ij}) - \bar{x}, v\rangle \le R + \epsilon,$$

and

$$\langle y_0(t-\tau_{i0}) - \bar{x}, v\rangle \le R + \epsilon.$$

Using the Grönwall inequality, we get

$$\begin{aligned}
\langle x_i(t) - \bar{x}, v\rangle &\le e^{-\tilde{K}t}\langle x_i(0) - \bar{x}, v\rangle + (R+\epsilon)(1 - e^{-\tilde{K}t}) \\
&\le e^{-\tilde{K}t}R + R + \epsilon - e^{-\tilde{K}t}R - \epsilon e^{-\tilde{K}t} \\
&= R + \epsilon - \epsilon e^{-\tilde{K}t} \\
&\le R + \epsilon - \epsilon e^{-\tilde{K}S} < R + \epsilon.
\end{aligned}$$

Therefore, sending $t \to S^-$, we finally find

$$\lim_{t\to S^-}\max_{i=1,\dots,N}\langle x_i(t) - \bar{x}, v\rangle < R + \epsilon,$$

that gives the contradiction. Thus, $S = +\infty$ and

$$\max_{i=1,\dots,N}\langle x_i(t) - \bar{x}, v\rangle < R + \epsilon,$$

for all $t \ge 0$ and $v \in \mathbb{R}^d$. Because of the arbitrarily of $v \in \mathbb{R}^d$ we can choose

$$v := \frac{x_i(t) - \bar{x}}{|x_i(t) - \bar{x}|},$$

so that $\langle x_i(t) - \bar{x}, v\rangle = |x_i(t) - \bar{x}|$. Hence, we have that

$$\max_{i=1,\dots,N}|x_i(t) - \bar{x}| < R + \epsilon.$$

Finally, sending $\epsilon \to 0$, we have the result. □

We are now able to prove the following result.

Theorem 4.3.12. *Let us consider the system* (4.3.56)-(4.3.57) *and let* $\bar{x} \in \mathbb{R}^d$. *Then, there exists an admissible control strategy* $u : [0, +\infty) \to \mathbb{R}^d$ *such that the leader* y_0 *reaches* $\bar{x}$ *in finite time, and*

$$d(t) \leq C^* e^{-\gamma^* t}, \quad \forall\, t \geq 0, \tag{4.3.61}$$

for suitable positive constants γ^* *and* C^*.

Proof Notice that with the preliminary lemma we prove that the leader reaches in a finite time t_0 the state $\bar{x} \in \mathbb{R}^d$ and stays there for all $t \geq t_0$, and the other agents maintain the distance from $\bar{x}$ less than or equal to R. Now, to prove the consensus result, one can apply the results in Section 4.3.1, since

$$\frac{d}{dt} y_0(t) = 0,$$

for all $t \geq t_0$. □

Remark 4.3.13. *Note that the above result extends the one in [74], which proved the consensus in the absence of time delays, and the one in [64], considering delay effects, since now we do not need any restrictions on the time delay size. We point out that [64, 74] assume the coefficients* $b_{ij}(t)$ *are compactly supported, while here they are always positive. However, it is easy to see from the proof that our result is valid also in the case* $b_{ij}(t)$ *is compactly supported. What we need is, indeed, that the leader's influence is always positive, only, exactly as in [64, 74].*

4.4 A HK-model with two leaders

Now, we want to analyze the case of a Hegselmann-Krause model in the presence of two leaders. The system reads as follows:

$$\begin{aligned}
\frac{d}{dt} y_i(t) &= a_{ij}(t)(y_j(t - \tilde{\tau}_j) - y_i(t)), \quad t > 0,\ i = 1, 2, \\
\frac{d}{dt} x_i(t) &= \frac{1}{N+1} \sum_{j \neq i} \chi_{ij} b_{ij}(t)(x_j(t - \tau_{ij}) - x_i(t)) \\
&\quad + \frac{1}{N+1} \sum_{j=1}^{2} c_{ij}(t)(y_j(t - \tilde{\tau}_j) - x_i(t)), \quad t > 0,\ i = 1, ..., N,
\end{aligned} \tag{4.4.62}$$

with the interaction weights $a_{ij}(t), b_{ij}(t), c_{ij}(t),\ t \geq 0$, defined as in (4.2.27), (6.1.1), (4.2.29). Also, in this case, the assumption **(CI)** is not satisfied. Indeed, the two leaders do not admit a common influencer. Let us assume the initial conditions

$$\begin{aligned}
y_i(t) &= y_i^0(t), \quad t \in [-\tau, 0],\ i = 1, 2, \\
x_i(t) &= x_i^0(t), \quad t \in [-\tau, 0],\ \forall\, i = 1, \dots, N,
\end{aligned} \tag{4.4.63}$$

being $\tau = \max\{\tilde{\tau}_1, \tilde{\tau}_2, \max_{i,j=1,\dots,N} \tau_{ij}\}$. We want to study the convergence to consensus of system (4.4.62). The *diameter function* can be written, in this case, as

$$d(t) := \max \left\{ \max_{\substack{i=1,\dots,N \\ j=1,2}} |x_i(t) - y_j(t)|, \max_{i,j=1,\dots,N} |x_i(t) - x_j(t)|, |y_1(t) - y_2(t)| \right\}.$$

Again, let us define the following quantities:

$$M_T := \max\left\{\max_{j=1,\dots,N}\max_{s\in[T-\tau,T]}\langle x_j(s),v\rangle, \max_{i=1,2}\max_{t\in[-\tau,0]}\langle y_i(t),v\rangle\right\},$$

and

$$m_T := \min\left\{\min_{j=1,\dots,N}\min_{s\in[T-\tau,T]}\langle x_j(s),v\rangle, \min_{i=1,2}\min_{t\in[-\tau,0]}\langle y_i(t),v\rangle\right\},$$

and let

$$\tilde{K} := \max\left\{\max_{i=1,2}\|\tilde{\psi}_{ij}\|_\infty, \max_{i,j=1,\dots,N}\|\psi_{ij}\|_\infty, \max_{i=1,\dots,N}\|\psi^*_{i1}\|_\infty, \max_{i=1,\dots,N}\|\psi^*_{i2}\|_\infty\right\}.$$

Analogously to the one-leader case, we can prove the following preliminary lemmas.

Lemma 4.4.1. *Let* $\left(\{x_i(t)\}_{i=1}^N, \{y_j(t)\}_{j=1}^2\right)$ *be a solution to* (4.4.62)-(4.4.63)*. Then, for each* $v\in\mathbb{R}^d$ *and* $T\geq 0$*, we have*

$$m_T \leq \langle x_i(t),v\rangle \leq M_T, \tag{4.4.64}$$

and

$$m_T \leq \langle y_j(t),v\rangle \leq M_T, \tag{4.4.65}$$

for all $t\geq T-\tau$*,* $i=1,\dots,N$ *and* $j=1,2$*.*

Let us introduce the appropriate notation:

Definition 4.4.2. *We define*

$$\begin{aligned} D_0 := \max\Big\{ &\max_{i,j=1,\dots,N}\max_{s,t\in[-\tau,0]}|x_i(s)-x_j(t)|, \max_{\substack{i=1,\dots,N\\ j=1,2}}\max_{s,t\in[-\tau,0]}|x_i(s)-y_j(t)|, \\ &\max_{s,t\in[-\tau,0]}|y_1(s)-y_2(t)|\Big\}, \end{aligned} \tag{4.4.66}$$

and in general for all $n\in\mathbb{N}$*,*

$$\begin{aligned} D_n := \max\Big\{ &\max_{i,j=1,\dots,N}\max_{s,t\in[n\tau-\tau,n\tau]}|x_i(s)-x_j(t)|, \\ &\max_{\substack{i=1,\dots,N\\ j=1,2}}\max_{s,t\in[n\tau-\tau,n\tau]}|x_i(s)-y_j(t)|, \max_{s,t\in[n\tau-\tau,n\tau]}|y_1(s)-y_2(t)|\Big\}. \end{aligned} \tag{4.4.67}$$

Lemma 4.4.3. *Let* $\left(\{x_i(t)\}_{i=1}^N, \{y_j(t)\}_{j=1}^2\right)$ *be a solution to* (4.4.62)-(4.4.63)*. For each* $n\in\mathbb{N}_0$*, we get*

$$\begin{aligned} &|x_i(t)-x_j(t)| \leq D_n, \quad \forall\, i,j=1,...,N, \\ &|x_i(t)-y_j(t)| \leq D_n, \quad \forall\, i=1,\dots,N,\ j=1,2, \\ &|y_1(t)-y_2(t)| \leq D_n, \end{aligned} \tag{4.4.68}$$

for all $t\geq n\tau-\tau$*.*

Again, as a consequence of Lemma 4.4.3 we have that $\{D_n\}_{n\in\mathbb{N}_0}$ is a non-increasing sequence.

Lemma 4.4.4. *Let* $\left(\{x_i(t)\}_{i=1}^N, \{y_j(t)\}_{j=1}^2\right)$ *be a solution to* (4.4.62)-(4.4.63). *Then, for all* $i = 1, ..., N$ *and* $j = 1, 2$, *we have*

$$\begin{aligned} |x_i(t)| &\leq \tilde{C}_0, \\ |y_j(t)| &\leq \tilde{C}_0 \end{aligned} \tag{4.4.69}$$

for all $t \geq 0$, *where*

$$\tilde{C}_0 := \max\left\{ \max_{i=1,\dots,N} \max_{s\in[-\tau,0]} |x_i(0)|, \max_{i=1,2} \max_{s\in[-\tau,0]} |y_i(0)| \right\}.$$

From Lemma 4.4.4 we deduce again a positive lower bound, $\tilde{\psi}_0$, as in (4.1.12), for the interaction rates. Moreover, we can prove the following estimates.

Lemma 4.4.5. *Let* $\left(\{x_i(t)\}_{i=1}^N, \{y_j(t)\}_{j=1}^2\right)$ *be a solution to* (4.4.62)-(4.4.63). *For any unit vector* $v \in \mathbb{R}^d$ *and* $n \in \mathbb{N}_0$ *we have that*

$$\begin{aligned} \langle x_i(t) - x_j(t), v\rangle &\leq e^{-\tilde{K}(t-t_0)}\langle x_i(t_0) - x_j(t_0), v\rangle + (1 - e^{-\tilde{K}(t-t_0)})D_n, \\ &\qquad i, j = 1, \dots, N \\ \langle x_i(t) - y_j(t), v\rangle &\leq e^{-\tilde{K}(t-t_0)}\langle x_i(t_0) - y_j(t_0), v\rangle + (1 - e^{-\tilde{K}(t-t_0)})D_n, \\ &\qquad i = 1, \dots, N,\ j = 1, 2, \\ \langle y_1(t) - y_2(t), v\rangle &\leq e^{-\tilde{K}(t-t_0)}\langle y_1(t_0) - y_2(t_0), v\rangle + (1 - e^{-\tilde{K}(t-t_0)})D_n, \end{aligned} \tag{4.4.70}$$

for all $t \geq t_0 \geq n\tau$. *Moreover,* (4.1.14) *holds true for all* $n \in \mathbb{N}_0$.

Proof Notice that the first two inequalities in (4.4.70) can be obtained by arguing as in Section 4.1. For the last inequality in (4.4.70), we want to emphasize that the equations for the leaders $y_1(t)$ and $y_2(t)$ are independent from the other agents $\{x_i(t)\}_{i=1}^N$. Therefore, it represents a Hegselmann-Krause 2×2 system itself. So, the estimate follows as in [34]. □

Lemma 4.4.6. *Let* $\left(\{x_i(t)\}_{i=1}^N, \{y_j(t)\}_{j=1}^2\right)$ *be a solution to* (4.4.62)-(4.4.63). *Then, there exists a constant* $C \in (0, 1)$ *such that*

$$d(n\tau) \leq CD_{n-2}, \tag{4.4.71}$$

for all $n \geq 2$.

Proof Let us suppose, as before, without loss of generality, that $d(n\tau) > 0$. Let us assume that $d(n\tau) = |x_i(n\tau) - y_j(n\tau)|$, for some $i = 1, \dots, N$ and $j = 1, 2$. In particular, suppose $d(n\tau) = |x_i(n\tau) - y_1(n\tau)|$. Let us define the unit vector $v \in \mathbb{R}^d$ as

$$v := \frac{x_i(n\tau) - y_1(n\tau)}{|x_i(n\tau) - y_1(n\tau)|}.$$

Then we have that $d(n\tau) = \langle x_i(n\tau) - y_1(n\tau), v\rangle$. Consider, as before,

$$M_{n-1} = \max\left\{ \max_{l=1,\dots,N} \max_{s\in[(n-2)\tau,(n-1)\tau]} \langle x_l(s), v\rangle, \max_{l=1,2} \max_{s\in[(n-2)\tau,(n-1)\tau]} \langle y_l(s), v\rangle \right\}$$

and

$$m_{n-1} = \min\left\{ \min_{l=1,\dots,N} \min_{s\in[(n-2)\tau,(n-1)\tau]} \langle x_l(s), v\rangle, \min_{l=1,2} \min_{s\in[(n-2)\tau,(n-1)\tau]} \langle y_l(s), v\rangle \right\}.$$

Again, we have that $M_{n-1} - m_{n-1} \leq D_{n-1}$. For $t \in [(n-1)\tau, n\tau]$ we have that

$$\begin{aligned}
&\frac{d}{dt}\langle x_i(t) - y_1(t), v\rangle = \frac{1}{N+1}\sum_{k\neq i}\chi_{ik}b_{ik}(t)\langle x_k(t-\tau_{ik}) - x_i(t), v\rangle \\
&\quad + \frac{1}{N+1}\sum_{k=1}^{2} c_{ik}(t)\langle y_k(t-\tilde{\tau}_k) - x_i(t), v\rangle - a_{12}(t)\langle y_2(t-\tilde{\tau}_2) - y_1(t), v\rangle \\
&= \frac{1}{N+1}\sum_{k\neq i}\chi_{ik}b_{ik}(t)(\langle x_k(t-\tau_{ik}), v\rangle - M_{n-1} + M_{n-1} - \langle x_i(t), v\rangle) \\
&\quad + \frac{1}{N+1}\sum_{k=1}^{2} c_{ik}(t)(\langle y_k(t-\tilde{\tau}_k), v\rangle - M_{n-1} + M_{n-1} - \langle x_i(t), v\rangle) \\
&\quad - a_{12}(t)(\langle y_2(t-\tilde{\tau}_2), v\rangle - m_{n-1} + m_{n-1} - \langle y_1(t), v\rangle) \\
&=: S_1 + S_2.
\end{aligned} \tag{4.4.72}$$

Since $t \in [(n-1)\tau, n\tau]$, then $t - \tilde{\tau}_k, t - \tau_{ij} \in [(n-2)\tau, n\tau]$ for all $k = 1, 2$, $i, j = 1, \ldots, N$. Then, we can write

$$\begin{aligned}
S_1 &= \frac{1}{N+1}\sum_{k\neq i}\chi_{ik}b_{ik}(t)(\langle x_k(t-\tau_{ik}), v\rangle - M_{n-1}) \\
&\quad + \frac{1}{N+1}\sum_{k\neq i}\chi_{ik}b_{ik}(t)(M_{n-1} - \langle x_i(t), v\rangle) \\
&\quad + \frac{1}{N+1}\sum_{k=1}^{2} c_{ik}(t)(\langle y_k(t-\tilde{\tau}_k), v\rangle - M_{n-1}) \\
&\quad + \frac{1}{N+1}\sum_{k=1}^{2} c_{ik}(t)(M_{n-1} - \langle x_i(t), v\rangle).
\end{aligned} \tag{4.4.73}$$

So, we find that

$$\begin{aligned}
S_1 &\leq \frac{\tilde{\psi}_0}{N+1}\sum_{k\neq i}\chi_{ik}(\langle x_k(t-\tau_{ik}), v\rangle - M_{n-1}) + \frac{N-1}{N+1}\tilde{K}(M_{n-1} - \langle x_i(t), v\rangle) \\
&\quad + \frac{\tilde{\psi}_0}{N+1}\sum_{k=1}^{2}(\langle y_k(t-\tilde{\tau}_k), v\rangle - M_{n-1}) + \frac{2}{N+1}\tilde{K}(M_{n-1} - \langle x_i(t), v\rangle) \\
&= \tilde{K}(M_{n-1} - \langle x_i(t), v\rangle) + \frac{\tilde{\psi}_0}{N+1}\sum_{k\neq i}\chi_{ik}(\langle x_k(t-\tau_{ik}), v\rangle - M_{n-1}) \\
&\quad + \frac{\tilde{\psi}_0}{N+1}\sum_{k=1}^{2}(\langle y_k(t-\tilde{\tau}_k), v\rangle - M_{n-1}).
\end{aligned} \tag{4.4.74}$$

Analogously, we can estimate

$$\begin{aligned}
S_2 &:= a_{12}(t)(m_{n-1} - \langle y_2(t-\tilde{\tau}_2), v\rangle + a_{12}(t)(\langle y_1(t), v\rangle - m_{n-1}) \\
&\leq \tilde{\psi}_0(m_{n-1} - \langle y_2(t-\tilde{\tau}_2), v\rangle + \tilde{K}(\langle y_1(t), v\rangle - m_{n-1}).
\end{aligned} \tag{4.4.75}$$

Putting (4.4.74) and (4.4.75) in (4.4.72), we can write

$$\begin{aligned}
&\frac{d}{dt}\langle x_i(t)-y_1(t),v\rangle \le \tilde{K}(M_{n-1}-m_{n-1})-\tilde{K}\langle x_i(t)-y_1(t),v\rangle \\
&+\frac{\tilde{\psi}_0}{N+1}\sum_{k\neq i}\chi_{ik}(\langle x_k(t-\tau_{ik}),v\rangle - M_{n-1})+\frac{\tilde{\psi}_0}{N+1}\sum_{k=1}^{2}(\langle y_k(t-\tilde{\tau}_k),v\rangle - M_{n-1}) \\
&\qquad\qquad\qquad\qquad + \tilde{\psi}_0(m_{n-1}-\langle y_2(t-\tilde{\tau}_2),v\rangle \\
&\le \tilde{K}(M_{n-1}-m_{n-1})-\tilde{K}\langle x_i(t)-y_1(t),v\rangle \\
&\qquad +\frac{\tilde{\psi}_0}{N+1}\sum_{k=1}^{2}(\langle y_k(t-\tilde{\tau}_k),v\rangle - M_{n-1}) + \tilde{\psi}_0(m_{n-1}-\langle y_2(t-\tilde{\tau}_2),v\rangle).
\end{aligned} \tag{4.4.76}$$

Notice that, since $\langle y_k(t-\tilde{\tau}_k),v\rangle - M_{n-1}\le 0$ for $k=1,2$ and $t\in[(n-1)\tau,n\tau]$, we have that

$$\sum_{k=1}^{2}(\langle y_k(t-\tilde{\tau}_k),v\rangle - M_{n-1}) \le \langle y_2(t-\tilde{\tau}_2),v\rangle - M_{n-1}.$$

Therefore, we have that

$$\begin{aligned}
&\frac{d}{dt}\langle x_i(t)-y_1(t),v\rangle \le \tilde{K}(M_{n-1}-m_{n-1})-\tilde{K}\langle x_i(t)-y_1(t),v\rangle \\
&+\frac{\tilde{\psi}_0}{N+1}(\langle y_2(t-\tilde{\tau}_2),v\rangle - M_{n-1}+m_{n-1}-\langle y_2(t-\tilde{\tau}_2),v\rangle) \\
&=\left(\tilde{K}-\frac{\tilde{\psi}_0}{N+1}\right)(M_{n-1}-m_{n-1})-\tilde{K}\langle x_i(t)-y_1(t),v\rangle,
\end{aligned} \tag{4.4.77}$$

where we used that $\tilde{\psi}_0(m_{n-1}-\langle y_2(t-\tilde{\tau}_2),v\rangle) \le \frac{\tilde{\psi}_0}{N+1}(m_{n-1}-\langle y_2(t-\tilde{\tau}_2),v\rangle)$.
Applying now the Grönwall inequality over $[(n-1)\tau,t]$, with $t\in((n-1)\tau,n\tau]$, we have

$$\begin{aligned}
\langle x_i(t)-y_1(t),v\rangle &\le e^{-\tilde{K}(t-(n-1)\tau)}\langle x_i((n-1)\tau)-y_1((n-1)\tau),v\rangle \\
&+\left(1-\frac{\tilde{\psi}_0}{\tilde{K}(N+1)}\right)(M_{n-1}-m_{n-1})(1-e^{-\tilde{K}(t-(n-1)\tau)}).
\end{aligned} \tag{4.4.78}$$

For $t=n\tau$, we have

$$\begin{aligned}
d(n\tau) &\le e^{-\tilde{K}\tau}\langle x_i((n-1)\tau)-y_1((n-1)\tau),v\rangle \\
&\quad+\left(1-\frac{\tilde{\psi}_0}{\tilde{K}(N+1)}\right)(M_{n-1}-m_{n-1})(1-e^{-\tilde{K}\tau}) \\
&\le \left(1-\frac{\tilde{\psi}_0}{\tilde{K}(N+1)}(1-e^{-\tilde{K}\tau})\right)D_{n-2},
\end{aligned} \tag{4.4.79}$$

where we used that $M_{n-1}-m_{n-1}\le D_{n-1}$ and the monotonicity property of D_n.
Notice that if $d(n\tau)=|x_i(n\tau)-x_j(n\tau)|$, for some fixed $i,j=1,\dots,N$, using a similar argument, we can find again that

$$\frac{d}{dt}\langle x_i(t)-x_j(t),v\rangle \le \left(\tilde{K}-\frac{\tilde{\psi}_0}{N+1}\right)(M_{n-1}-m_{n-1})-\tilde{K}\langle x_i(t)-x_j(t),v\rangle,$$

for $t \in [(n-1)\tau, n\tau]$.
Finally, assume that $d(n\tau) = |y_1(n\tau) - y_2(n\tau)|$. Let us define again an unit vector $v \in \mathbb{R}^d$ as

$$v := \frac{y_1(n\tau) - y_2(n\tau)}{|y_1(n\tau) - y_2(n\tau)|}.$$

Then, we can write $d(n\tau) = \langle y_1(n\tau) - y_2(n\tau), v\rangle$. Consider $t \in [(n-2)\tau, n\tau]$. Let us distinguish two cases.
Case 1. There exists $t_0 \in [(n-2)\tau, n\tau]$ such that $\langle y_1(t_0) - y_2(t_0), v\rangle < 0$. Applying Lemma 4.4.5, we get

$$\begin{aligned} d(n\tau) &\le e^{-\tilde{K}(n\tau - t_0)}\langle y_1(t_0) - y_2(t_0), v\rangle + (1 - e^{-\tilde{K}(n\tau - t_0)})D_{n-2} \\ &\le (1 - e^{-\tilde{K}(n\tau - t_0)})D_{n-2} \\ &\le (1 - e^{-2\tilde{K}\tau})D_{n-2}, \end{aligned} \tag{4.4.80}$$

and we have the statement.
Case 2. Assume that $\langle y_1(t) - y_2(t), v\rangle \ge 0, \ \forall t \in [(n-2)\tau, n\tau]$. Consider $t \in [(n-1)\tau, n\tau]$. From (4.4.62) we have that

$$\begin{aligned} \frac{d}{dt}\langle y_1(t) - y_2(t), v\rangle &= a_{12}(t)\langle y_2(t - \tilde{\tau}_2) - y_1(t), v\rangle - a_{21}(t)\langle y_1(t - \tilde{\tau}_1) - y_2(t), v\rangle \\ &= a_{12}(t)\left(\langle y_2(t - \tilde{\tau}_2, v\rangle - M_{n-1}\right) + a_{12}(t)\left(M_{n-1} - \langle y_1(t), v\rangle\right) \\ &+ a_{21}(t)\left(m_{n-1} - \langle y_1(t - \tilde{\tau}_1), v\rangle\right) + a_{21}(t)\left(\langle y_2(t), v\rangle - m_{n-1}\right) \\ &\le \tilde{\psi}_0\left(\langle y_2(t - \tilde{\tau}_2), v\rangle - M_{n-1}\right) + \tilde{K}\left(M_{n-1} - \langle y_1(t), v\rangle\right) \\ &+ \tilde{\psi}_0\left(m_{n-1} - \langle y_1(t - \tilde{\tau}_1), v\rangle\right) + \tilde{K}\left(\langle y_2(t), v\rangle - m_{n-1}\right). \end{aligned}$$

Then,

$$\begin{aligned} \frac{d}{dt}\langle y_1(t) - y_2(t), v\rangle &= \left(\tilde{K} - \tilde{\psi}_0\right)\left(M_{n-1} - m_{n-1}\right) - \tilde{K}\langle y_1(t) - y_2(t), v\rangle \\ &- \tilde{\psi}_0(\langle y_1(t - \tilde{\tau}_1), v\rangle - \langle y_2(t - \tilde{\tau}_2), v\rangle) \\ &\le \left(\tilde{K} - \tilde{\psi}_0\right)\left(M_{n-1} - m_{n-1}\right) - \tilde{K}\langle y_1(t) - y_2(t), v\rangle, \end{aligned}$$

where, since $t - \tilde{\tau}_1, t - \tilde{\tau}_2 \ge (n-2)\tau$, we use the initial assumption. Therefore, applying the Grönwall inequality, we find that, for $t = n\tau$,

$$d(n\tau) \le \left(1 - \frac{\tilde{\psi}_0}{\tilde{K}}(1 - e^{-\tilde{K}\tau})\right) D_{n-2}.$$

Thus, the result follows for a suitable constant C. □

Again, from this result, we can deduce the exponential consensus estimate for the case of two leaders.

Theorem 4.4.7. *Every solution* $\left(\{x_i(t)\}_{i=1}^N, \{y_j(t)\}_{j=1}^2\right)$ *to* (4.4.62)-(4.4.63) *satisfies the exponential decay estimate*

$$d(t) \le D(0)e^{-\bar{\gamma}(t-2\tau)} \text{ for all } t \ge 0,$$

for a suitable positive constant $\bar{\gamma}$.

4.5 Numerical tests

In this section, we present some numerical simulations illustrating the theoretical results. To produce our simulations, we used the MATLAB environment. The solutions of the system are computed thanks to the MATLAB function *dde23*, which computes the solution of delay differential equations (DDEs) with constant time delays. Moreover, for our simulations, the time step is 0.3, and we consider the size of the followers' population as $N = 10$.
In Figure 4.1, we plot a case in which the common influencer assumption **(CI)** holds. In particular, we assume

$$\chi_{j1} = 1, \quad \text{for} \quad j = 2, 3, 4, 10,$$
$$\chi_{j10} = 1, \quad \text{for} \quad j = 1, 2, 6, 7, 8, 9,$$
$$\chi_{ij} = 1, \quad \text{for} \quad i, j = 2, 3, 4, 5, 6, 7, 8, 9,$$

and $\chi_{ij} = 0$ in the remaining cases. We consider weight functions $a_{ij}(t)$ in the form

$$a_{ij}(t) := \psi(r, r') := \tilde{\psi}(|r - r'|), \quad r, r' \in [0, +\infty),$$

where

$$\tilde{\psi}(r) := e^{-(r-1)^2}, \ r \in [0, +\infty), \tag{4.5.81}$$

and as time delays $\tau_{ij} = 5$ for all $i, j = 1, \ldots, N$.

From Figure 4.1, we can see that in the case of the common influencer assumption, the consensus is finally achieved, but first, the agents tend to form subgroups, depending on their mutual relations.
To underline the dependence of the rate decay γ on the norm of the initial data (i.e., the constant M_0 given by (4.1.11)), we performed in Figure 4.1 two simulations for the system under the common influencer assumption with different initial data. One can notice that the convergence to the consensus is achieved, with different rates in the two cases. Note that, in both cases, the time delay reduces the convergence rate (cf. Remark 4.2.2).

In Figure 4.3, we present the case with one leader with a controlled trajectory. Here, we use the weight functions defined in (4.5.81) and a constant time delay $\tau_{ij} = 1$, for all $i = 1, \ldots, N, \ j = 0, \ldots, N$. In Figure 4.2, we consider the case of one leader with a constant trajectory. We consider the weight functions $b_{ij}(t)$ defined by (4.5.81) again for all $i, j = 1, \ldots, N$. Meanwhile, the weight functions $c_{i0}(t)$ are assumed to be the same for all $i = 1, \ldots, N$ and it is defined by a constant function. In particular, we assume

$$c_{i0}(t) := \frac{\tilde{K}}{N}, \ \forall \ i \in \{1, \ldots, N\}, \tag{4.5.82}$$

with $\tilde{K}$ is a positive constant. Moreover, for simplicity, we assume a constant time delay $\tau_{ij} = 5$, for all $i, j = 1, \ldots, N$.

Finally, in Figure 4.4, we consider the case with two leaders. Again, we choose the functions $a_{ij}(t)$ for $i, j = 1, 2$, and $b_{ij}(t)$ for $i, j = 1, \ldots, N$, equal to the function $\tilde{\psi}(r)$ in (4.5.81). Meanwhile, the function $c_{ij}(t)$ is supposed constant and equal to $\frac{\tilde{K}}{N+1}$. The time delays $\tilde{\tau}_j$ for $j = 1, 2$ is constant and $\tilde{\tau}_j = 5$. For simplicity, we assumed $\chi_{ij} = 1$ in all the figures above.

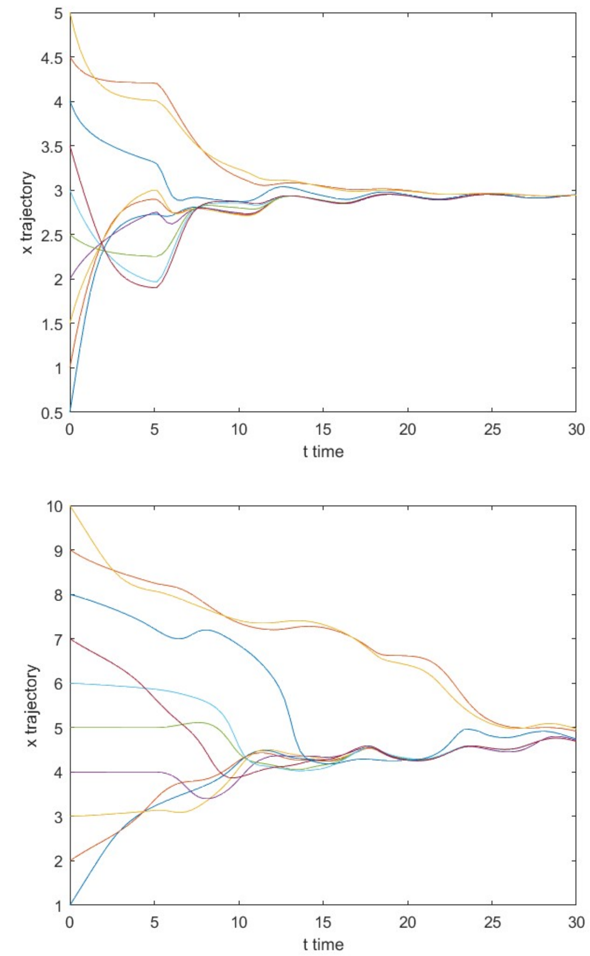


Figure 4.1: HK-model under common influencer assumption

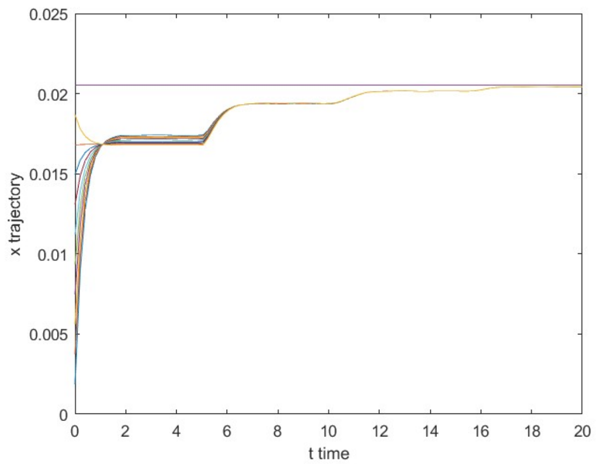


Figure 4.2: HK-model with a unique leader with constant trajectory

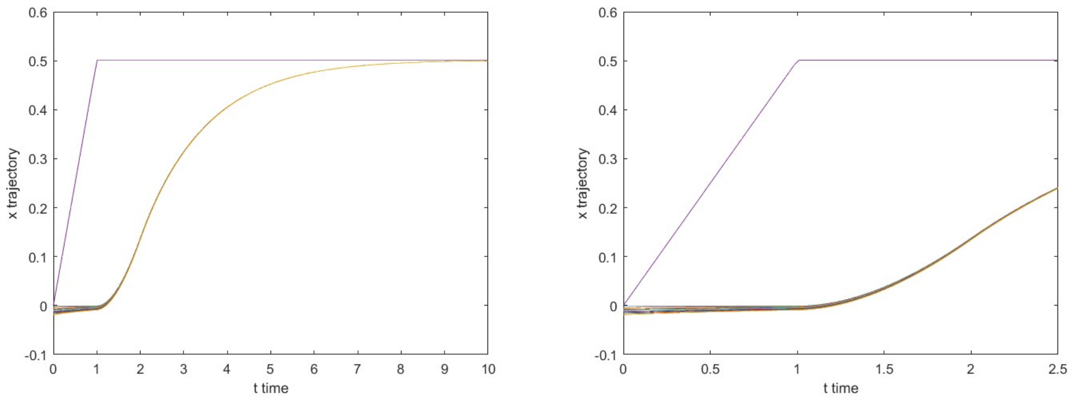


Figure 4.3: HK-model with a unique leader with control

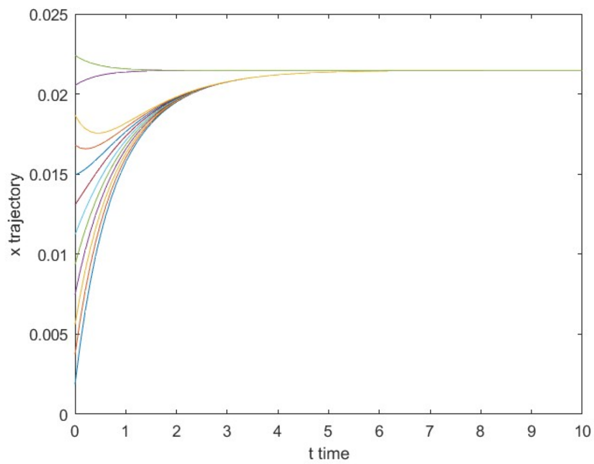


Figure 4.4: HK-model with two leaders

Chapter 5

Opinion dynamic with communication failure

In this chapter, we analyze the asymptotic behavior of the solutions to first and second-order alignment models, i.e., Hegselmann-Krause and Cucker-Smale models, in the presence of (pair and time-dependent) time delays, non-universal interaction, and possible lack of interaction among connected agents during the evolution. Namely, in these models, time- and pair-dependent weight functions that can degenerate are considered. In this case, the interaction can be missing sometimes, not only among agents that are not linked to each other but also among agents that are generally able to exchange information. Under a Persistence Excitation Condition, we establish the exponential consensus and flocking for the Hegselmann-Krause opinion formation model and the Cucker-Smale model whenever the digraph that describes the interaction among the agents is strongly connected.

This is done by dealing with a general influence function (no symmetry or monotonicity assumptions are needed) and without requiring any smallness assumptions on the time delay size. Our result seems very general and greatly improves previous related works [3, 9, 34]. Indeed, in [3] only the first-order model is analyzed, with a less general influence function, in the case of all-to-all interactions. Furthermore, time delay effects are not considered, and the convergence to consensus is not achieved exponentially fast. Here, we deal with non-universal interaction, in the presence of time delays, pair and time-dependent, and very general influence functions. This generality requires finer and more sophisticated arguments. Concerning [34], where only the first-order model is analyzed, the main novelties are the more general weight and time delay functions (now pair-dependent). Moreover, here, we work in the network topology setting. For the first-order model, we also mention [69] where a consensus result for a linear version of the model is obtained, under a weaker network topology assumption. Finally, the paper [9] deals with the second-order model too. However, the analysis requires symmetry conditions on the weight functions and all-to-all interaction. Moreover, no time delays are included.

5.1 The first-order alignment model

Consider a finite set of $N \in \mathbb{N}$ agents, with $N \geq 2$. Let $x_i(t) \in \mathbb{R}^d$ be the opinion of the i-th agent at time t. The interactions between the elements of the system are described by the following Hegselmann-Krause type model:

$$\frac{d}{dt}x_i(t) = \sum_{j:j\neq i}\chi_{ij}b_{ij}(t)(x_j(t-\tau_{ij}(t)) - x_i(t)), \quad t>0,\ \forall i=1,\dots,N, \tag{5.1.1}$$

where the time delay functions $\tau_{ij} : [0,+\infty) \to [0,+\infty)$ are assumed to be continuous and satisfy the following:

$$0 \le \tau_{ij}(t) \le \tau, \quad \forall t \ge 0,\ \forall i,j=1,\dots,N, \tag{5.1.2}$$

for some positive constant τ, and the terms χ_{ij} are defined by (1.2.9). Moreover, the communication rates b_{ij} are of the form

$$b_{ij}(t) := \frac{1}{N-1}\alpha_{ij}(t)\psi(x_i(t), x_j(t-\tau_{ij}(t))), \quad t>0,\ \forall i,j=1,\dots,N, \tag{5.1.3}$$

where the influence function $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is positive, bounded and continuous with constant (6.1.14), and the weight functions $\alpha_{ij} : [0,+\infty) \to [0,1]$ are $\mathcal{L}^1$-measurable and satisfy the Persistence Excitation Condition (1.2.11). Remember that we can assume that the positive constant $\tilde{\alpha}$ appearing in (1.2.11) satisfies $\tilde{\alpha}K < 1$.
The initial conditions

$$x_i(s) = x_i^0(s), \quad \forall s \in [-\tau, 0],\ \forall i=1,\dots,N, \tag{5.1.4}$$

are assumed to be continuous functions.
We set

$$C_0 := \max_{i=1,\dots,N}\ \max_{s\in[-\tau,0]}|x_i(s)|, \tag{5.1.5}$$

$$\psi_0 := \min_{|y|,|z|\le C_0}\psi(y,z). \tag{5.1.6}$$

We will consider a graph topology over the model structure, as we described in Section 1.2. Given the definition of the diameter of the system (6.0.5), we want to prove the convergence toward consensus of this type of system, in the sense of Definition 1.2.1.
We will prove the following exponential convergence to the consensus result.

Theorem 5.1.1. *Assume* (5.1.2) *and that the digraph $\mathcal{G}$ over the structure of the model is strongly connected. Let $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a positive, bounded, continuous function. Assume that the weight functions $\alpha_{ij} : [0,+\infty) \to [0,1]$ are $\mathcal{L}^1$-measurable and satisfy* **(PE)**. *Let $x_i^0 : [-\tau, 0] \to \mathbb{R}^d$ be a continuous function, for any $i = 1,\dots,N$. Then, every solution $\{x_i\}_{i=1,\dots,N}$ to* (5.1.1) *with the initial conditions* (5.1.4) *satisfies the following exponential decay estimate*

$$d(t) \le C_1\left(\max_{i,j=1,\dots,N}\ \max_{r,s\in[-\tau,0]}|x_i(r) - x_j(s)|\right)e^{-C_2 t}, \quad \forall t \ge 0, \tag{5.1.7}$$

where C_1, C_2 are the positive constants defined as

$$C_1 := \frac{1}{1 - e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)}\left(\frac{\psi_0\tilde{\alpha}}{N-1}\right)^{\gamma}}, \tag{5.1.8}$$

$$C_2 := \frac{1}{\gamma(T+\tau)+\tau}\ln\left(\frac{1}{1 - e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)}\left(\frac{\psi_0\tilde{\alpha}}{N-1}\right)^{\gamma}}\right), \tag{5.1.9}$$

being $\gamma > 0$ the depth of the digraph, T and $\tilde{\alpha}$ the positive constants in (1.2.11)*, and ψ_0 the positive constant in* (5.1.6)*.*

Remark 5.1.2. *Let us note that the velocity decay C_2 in (5.1.9) decreases for increasing values of γ, τ, and T and decreases for decreasing values of $\tilde{\alpha}$. Indeed, for fixed values of τ, T and $\tilde{\alpha}$, we have*

$$\frac{1}{\gamma(T+\tau)+\tau} \to 0, \quad \textit{as } \gamma \to +\infty.$$

Moreover, being $N \geq 2$ and $\tilde{\alpha}K < 1$, we have $\frac{\psi_0\tilde{\alpha}}{N-1} \leq \psi_0\tilde{\alpha} \leq K\tilde{\alpha} < 1$, from which

$$\left(\frac{\psi_0\tilde{\alpha}}{N-1}\right)^\gamma \to 0, \quad \textit{as } \gamma \to +\infty.$$

Also, $e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)} \to 0$, as $\gamma \to +\infty$. Thus, $\ln\left(\frac{1}{1-e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)}\left(\frac{\psi_0\tilde{\alpha}}{N-1}\right)^\gamma}\right) \to 0$, as $\gamma \to +\infty$. Therefore, $C_2 \to 0$, as $\gamma \to +\infty$.
Analogously, $C_2 \to 0$, as $\tau \to +\infty$ or $T \to +\infty$. Finally, for $\tilde{\alpha} \to 0$, it is easy to see that $C_2 \to 0$. So, C_2 decreases as γ, T, τ grow and as $\tilde{\alpha}$ decays. This is expected since, for large values of γ and T and for small values of $\tilde{\alpha}$, the connection among the agents can be very weak. Furthermore, increasing time lags in the interaction among the agents slows down the convergence to consensus for the Hegselmann-Krause model.

5.1.1 Preliminary lemmas

Let $\{x_i\}_{i=1,\dots,N}$ be solution to (5.1.1) under the initial conditions (5.1.4). We assume that the hypotheses of Theorem 5.1.1 are satisfied. We present some auxiliary lemmas.

Definition 5.1.3. *Given a vector $v \in \mathbb{R}^d$, for all $n \in \mathbb{N}_0$ we define*

$$I_n := [n(\gamma(T+\tau)+\tau) - \tau, n(\gamma(T+\tau)+\tau)]$$

$$m_n^v := \min_{j=1,\dots,N} \min_{s\in I_n} \langle x_j(s), v\rangle,$$

$$M_n^v := \max_{j=1,\dots,N} \max_{s\in I_n} \langle x_j(s), v\rangle.$$

Also, we define, for all $n \in \mathbb{N}_0$,

$$\tilde{m}_n^v := \min_{j=1,\dots,N} \langle x_j(n(\gamma(T+\tau)+\tau)), v\rangle,$$

$$\tilde{M}_n^v := \max_{j=1,\dots,N} \langle x_j(n(\gamma(T+\tau)+\tau)), v\rangle.$$

Lemma 5.1.4. *For each vector $v \in \mathbb{R}^d$, we have that*

$$m_0^v \leq \langle x_i(t), v\rangle \leq M_0^v, \tag{5.1.10}$$

for all $t \geq -\tau$ and for any $i = 1, \dots, N$.

Proof. First of all, we note that the inequalities in (5.1.10) are satisfied for every $t \in [-\tau, 0]$. Now, let $v \in \mathbb{R}^d$. For all $\epsilon > 0$, we define

$$K^\epsilon := \left\{t > 0 : \max_{i=1,\dots,N} \langle x_i(s), v\rangle < M_0^v + \epsilon,\ \forall s \in [0,t)\right\},$$

and

$$S^\epsilon := \sup K^\epsilon.$$

By continuity, we have that $K^\epsilon \neq \emptyset$ and $S^\epsilon > 0$.
We claim that $S^\epsilon = +\infty$. Indeed, suppose by contradiction that $S^\epsilon < +\infty$. By definition of S^ϵ, it turns out that

$$\max_{i=1,\dots,N} \langle x_i(t), v\rangle < M_0^v + \epsilon, \quad \forall t \in (0, S^\epsilon), \tag{5.1.11}$$

$$\lim_{t\to S^{\epsilon-}} \max_{i=1,\dots,N} \langle x_i(t), v\rangle = M_0^v + \epsilon. \tag{5.1.12}$$

For all $i = 1, \dots, N$, for $t \in (0, S^\epsilon)$, we have that

$$\frac{d}{dt}\langle x_i(t), v\rangle = \frac{1}{N-1}\sum_{j:j\neq i} \chi_{ij}\alpha_{ij}(t)\psi(x_i(t), x_j(t-\tau_{ij}(t)))\langle x_j(t-\tau_{ij}(t)) - x_i(t), v\rangle.$$

Now, being $t \in (0, S^\epsilon)$, it holds that $t - \tau_{ij}(t) \in (-\tau, S^\epsilon)$. Then, from (5.1.11)

$$\langle x_j(t-\tau_{ij}(t)), v\rangle < M_0^v + \epsilon, \quad \forall j = 1, \dots, N, \tag{5.1.13}$$

where we have used the fact that the second inequality in (5.1.10) is satisfied in $[-\tau, 0]$.
Therefore, using (6.1.14), (5.1.11), (5.1.13) and recalling that $\chi_{ij}, \alpha_{ij} \leq 1$, for a.e. $t \in (0, S^\epsilon)$ we can write

$$\begin{aligned}\frac{d}{dt}\langle x_i(t), v\rangle &\leq \frac{1}{N-1}\sum_{j:j\neq i} \chi_{ij}\alpha_{ij}(t)\psi(x_i(t), x_j(t-\tau_{ij}(t)))(M_0^v + \epsilon - \langle x_i(t), v\rangle)\\ &\leq K(M_0^v + \epsilon - \langle x_i(t), v\rangle).\end{aligned}$$

Thus, Gronwall's inequality yields

$$\begin{aligned}\langle x_i(t), v\rangle &\leq e^{-Kt}\langle x_i(0), v\rangle + K(M_0^v + \epsilon)\int_0^t e^{-K(t-s)}ds\\ &= e^{-Kt}\langle x_i(0), v\rangle + (M_0^v + \epsilon)e^{-Kt}(e^{Kt} - 1)\\ &= e^{-Kt}\langle x_i(0), v\rangle + (M_0^v + \epsilon)(1 - e^{-Kt})\\ &\leq e^{-Kt}M_0^v + M_0^v + \epsilon - M_0^v e^{-Kt} - \epsilon e^{-Kt}\\ &= M_0^v + \epsilon - \epsilon e^{-Kt}\\ &\leq M_0^v + \epsilon - \epsilon e^{-KS^\epsilon},\end{aligned}$$

for all $t \in (0, S^\epsilon)$. We have so proved that, $\forall i = 1, \dots, N$,

$$\langle x_i(t), v\rangle \leq M_0^v + \epsilon - \epsilon e^{-KS^\epsilon}, \quad \forall t \in (0, S^\epsilon).$$

Thus, we get

$$\max_{i=1,\dots,N} \langle x_i(t), v\rangle \leq M_0^v + \epsilon - \epsilon e^{-KS^\epsilon}, \quad \forall t \in (0, S^\epsilon). \tag{5.1.14}$$

Letting $t \to S^{\epsilon-}$ in (5.1.14), from (5.1.12) we have that

$$M_0^v + \epsilon \leq M_0^v + \epsilon - \epsilon e^{-KS^\epsilon} < M_0^v + \epsilon,$$

which is a contradiction. Thus, $S^\epsilon = +\infty$ and

$$\max_{i=1,\dots,N} \langle x_i(t), v\rangle < M_0^v + \epsilon, \quad \forall t > 0.$$

From the arbitrariness of ϵ we can conclude that

$$\max_{i=1,\dots,N} \langle x_i(t), v\rangle \leq M_0^v, \quad \forall t > 0,$$

from which

$$\langle x_i(t), v\rangle \leq M_0^v, \quad \forall t > 0, \forall i = 1, \dots, N.$$

So, the second inequality in (5.1.10) is proven.
Now, to show that the other inequality holds, fix $v \in \mathbb{R}^d$. Then, for all $i = 1, \dots, N$ and $t > 0$, by applying the second inequality in (5.1.10) to the vector $-v \in \mathbb{R}^d$ we get

$$\begin{aligned} -\langle x_i(t), v\rangle = \langle x_i(t), -v\rangle &\leq \max_{j=1,\dots,N} \max_{s\in[-\tau,0]} \langle x_j(s), -v\rangle \\ &= -\min_{j=1,\dots,N} \min_{s\in[-\tau,0]} \langle x_j(s), v\rangle = -m_0^v, \end{aligned}$$

from which

$$\langle x_i(t), v\rangle \geq m_0^v, \quad \forall t \geq 0, \forall i = 1, \dots, N.$$

Thus, also the first inequality in (5.1.10) is fulfilled. □

Using the same arguments employed in the proof of the previous lemma, one can prove the following more general result.

Lemma 5.1.5. *For each vector $v \in \mathbb{R}^d$ and for all $n \in \mathbb{N}_0$, we have that*

$$m_n^v \leq \langle x_i(t), v\rangle \leq M_n^v, \tag{5.1.15}$$

for all $t \geq n(\gamma(T+\tau)+\tau) - \tau$ and for any $i = 1, \dots, N$.

Now, we define the following quantities.

Definition 5.1.6. *For all $n \in \mathbb{N}_0$, we define*

$$D_n := \max_{i,j=1,\dots,N} \max_{r,s\in I_n} |x_i(r) - x_j(s)|.$$

Let us note that, for $n = 0$,

$$D_0 := \max_{i,j=1,\dots,N} \max_{r,s\in I_0} |x_i(r) - x_j(s)| = \max_{i,j=1,\dots,N} \max_{r,s\in[-\tau,0]} |x_i(r) - x_j(s)|.$$

So, the exponential decay estimate in (5.1.7) can be written as

$$d(t) \leq e^{-C_2 t} C_1 D_0, \quad \forall t \geq 0,$$

where $C_1 > 0$ and $C_2 > 0$ are the constants in (5.1.8) and (5.1.9), respectively.

Lemma 5.1.7. *For each $n \in \mathbb{N}_0$, we have that*

$$|x_i(s) - x_j(t)| \leq D_n, \tag{5.1.16}$$

for all $s, t \geq n(\gamma(T+\tau)+\tau) - \tau$ and for any $i, j = 1, \dots, N$.

Proof. Fix $n \in \mathbb{N}_0$. Let $i, j = 1, \dots, N$ and $s, t \geq n(\gamma(T+\tau)+\tau)-\tau$. Then, if $|x_i(s)-x_j(t)| = 0$, (5.1.16) is obviously satisfied. So we can assume $|x_i(s)-x_j(t)| > 0$. Let us define the unit vector

$$v = \frac{x_i(s) - x_j(t)}{|x_i(s) - x_j(t)|}.$$

Then, using (5.1.15) and Cauchy-Schwarz inequality, we have that

$$|x_i(s) - x_j(t)| = \langle x_i(s) - x_j(t), v\rangle = \langle x_i(s), v\rangle - \langle x_j(t), v\rangle \leq M_n^v - m_n^v$$

$$\leq \max_{k,l=1,\dots,N} \max_{r,\sigma \in I_n} |x_k(r) - x_l(\sigma)| = D_n.$$

□

Remark 5.1.8. *Note that* (5.1.16) *yields*

$$d(t) \leq D_n, \quad \forall t \geq n(\gamma(T+\tau)+\tau) - \tau. \tag{5.1.17}$$

Moreover, from (5.1.16) *it comes that*

$$D_{n+1} \leq D_n, \quad \forall n \in \mathbb{N}_0. \tag{5.1.18}$$

Next, we show that the agents' opinions are bounded by a constant that depends on the initial data.

Lemma 5.1.9. *For every* $i = 1, \dots, N$*, we have that*

$$|x_i(t)| \leq C_0, \quad \forall t \geq -\tau, \tag{5.1.19}$$

where C_0 *is the constant defined in* (5.1.5)*.*

Proof. Given $i = 1, \dots, N$ and $t \geq -\tau$, if $|x_i(t)| = 0$, then trivially $C_0 \geq |x_i(t)|$. On the contrary, if $|x_i(t)| > 0$, we define

$$v = \frac{x_i(t)}{|x_i(t)|},$$

which is a unit vector. Then, by applying (5.1.10) and by using the Cauchy-Schwarz inequality, we get

$$|x_i(t)| = \langle x_i(t), v\rangle \leq M_0^v = \max_{j=1,\dots,N} \max_{s\in[-\tau,0]} \langle x_j(s), v\rangle$$

$$\leq \max_{j=1,\dots,N} \max_{s\in[-\tau,0]} |x_j(s)||v| = \max_{j=1,\dots,N} \max_{s\in[-\tau,0]} |x_j(s)| = C_0,$$

and (5.1.19) is satisfied. □

Remark 5.1.10. *From the estimate* (5.1.19)*, since the influence function* ψ *is continuous, we deduce that*

$$\psi(x_i(t), x_j(t - \tau_{ij}(t))) \geq \psi_0, \tag{5.1.20}$$

for all $t \geq 0$*, for all* $i, j = 1, \dots, N$*, where* ψ_0 *is the positive constant in* (5.1.6)*.*

5.1.2 Consensus estimate

To prove the consensus result, we need the following crucial proposition, inspired by a previous argument in [50], and that we used in Chapter 2 and Chapter 3.

Proposition 5.1.11. *For all $v \in \mathbb{R}^d$, it holds*

$$m_0^v + \Gamma(\tilde{M}_0^v - m_0^v) \leq \langle x_i(t), v\rangle \leq M_0^v - \Gamma(M_0^v - \tilde{m}_0^v), \tag{5.1.21}$$

for all $t \in I_1$ and for all $i = 1, \dots, N$, where Γ is the positive constant defined as follows

$$\Gamma := e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)} \left(\frac{\psi_0 \tilde{\alpha}}{N-1}\right)^{\gamma}. \tag{5.1.22}$$

Remark 5.1.12. *Let us note that $\Gamma \in (0,1)$ since we have assumed $\tilde{\alpha}K < 1$. Moreover, by definition of Γ, the positive constants C_1, C_2 in* (5.1.7) *can be rewritten in the following way (see* (5.1.8) *and* (5.1.9)*):*

$$C_1 = \frac{1}{1-\Gamma},$$

$$C_2 = \frac{1}{\gamma(T+\tau)+\tau} \ln\left(\frac{1}{1-\Gamma}\right).$$

Proof. Fix $v \in \mathbb{R}^d$. Let $L = 1, \dots, N$ be such that $\langle x_L(0), v\rangle = \tilde{m}_0^v$. Note that from (5.1.10), $M_0^v \geq \tilde{m}_0^v$. Then, for a.e. $t \in [0, \gamma(T+\tau)+\tau]$, using (5.1.10) we have

$$\begin{aligned}
\frac{d}{dt}\langle x_L(t), v\rangle &= \sum_{j:j\neq L} \chi_{Lj}(t) b_{Lj}(t)(\langle x_j(t-\tau_{Lj}(t)), v\rangle - \langle x_L(t), v\rangle) \\
&\leq \sum_{j:j\neq L} \chi_{Lj} b_{Lj}(t)(M_0^v - \langle x_L(t), v\rangle) \\
&\leq \frac{K}{N-1} \sum_{j:j\neq L} (M_0^v - \langle x_L(t), v\rangle) = K(M_0^v - \langle x_L(t), v\rangle).
\end{aligned}$$

Thus, Gronwall's inequality yields

$$\begin{aligned}
\langle x_L(t), v\rangle &\leq e^{-Kt}\langle x_L(0), v\rangle + M_0^v(1-e^{-Kt}) \\
&= e^{-Kt}\tilde{m}_0^v + M_0^v(1-e^{-Kt}) \\
&= M_0^v - e^{-Kt}(M_0^v - \tilde{m}_0^v) \\
&\leq M_0^v - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v).
\end{aligned}$$

Hence,

$$\langle x_L(t), v\rangle \leq M_0^v - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v), \quad \forall t \in [0, \gamma(T+\tau)+\tau]. \tag{5.1.23}$$

Now, let $i_1 = 1, \dots, N \setminus \{L\}$ be such that $\chi_{i_1 L} = 1$. Such an index i_1 exists since the digraph is strongly connected. Then, for a.e. $t \in [\tau, \gamma(T+\tau)+\tau]$, from (5.1.23) we get

$$\begin{aligned}
\frac{d}{dt}\langle x_{i_1}(t), v\rangle &= \sum_{j\neq i_1, L} \chi_{i_1 j} b_{i_1 j}(t)(\langle x_j(t-\tau_{i_1 j}(t)), v\rangle - \langle x_{i_1}(t), v\rangle) \\
&\quad + b_{i_1 L}(t)(\langle x_L(t-\tau_{i_1 L}(t)), v\rangle - \langle x_{i_1}(t), v\rangle) \\
&\leq \sum_{j\neq i_1, L} \chi_{i_1 j} b_{i_1 j}(t)(M_0^v - \langle x_{i_1}(t), v\rangle) \\
&\quad + b_{i_1 L}(t)\left(M_0^v - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v) - \langle x_{i_1}(t), v\rangle\right) \\
&= (M_0^v - \langle x_{i_1}(t), v\rangle) \sum_{j\neq i_1, L} \chi_{i_1 j} b_{i_1 j}(t) \\
&\quad + b_{i_1 L}(t)\left(M_0^v - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v) - \langle x_{i_1}(t), v\rangle\right).
\end{aligned}$$

Note that

$$\sum_{j\neq i_1, L} \chi_{i_1 j} b_{i_1 j}(t) = \sum_{j\neq i_1} \chi_{i_1 j} b_{i_1 j}(t) - b_{i_1 L}(t) \leq \frac{K}{N-1}\sum_{j\neq i_1} \chi_{i_1 j} - b_{i_1 L}(t) = \frac{KN_{i_1}}{N-1} - b_{i_1 L}(t).$$

Thus, it comes that

$$\begin{aligned}
\frac{d}{dt}\langle x_{i_1}(t), v\rangle &\leq \frac{KN_{i_1}}{N-1}(M_0^v - \langle x_{i_1}(t), v\rangle) - b_{i_1 L}(t)(M_0^v - \langle x_{i_1}(t), v\rangle) \\
&\quad + b_{i_1 L}(t)\left(M_0^v - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v) - \langle x_{i_1}(t), v\rangle\right) \\
&= \frac{KN_{i_1}}{N-1}(M_0^v - \langle x_{i_1}(t), v\rangle) - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v) b_{i_1 L}(t) \\
&\leq \frac{KN_{i_1}}{N-1}(M_0^v - \langle x_{i_1}(t), v\rangle) - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\alpha_{i_1 L}(t)\frac{\psi_0}{N-1} \\
&= \frac{KN_{i_1}}{N-1}M_0^v - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\alpha_{i_1 L}(t)\frac{\psi_0}{N-1} - \frac{KN_{i_1}}{N-1}\langle x_{i_1}(t), v\rangle.
\end{aligned}$$

Hence, Gronwall's estimate yields

$$\begin{aligned}
\langle x_{i_1}(t), v\rangle &\leq e^{-\frac{KN_{i_1}}{N-1}(t-\tau)}\langle x_{i_1}(\tau), v\rangle + M_0^v(1 - e^{-\frac{KN_{i_1}}{N-1}(t-\tau)}) \\
&\quad - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\frac{\psi_0}{N-1}\int_\tau^t \alpha_{i_1 L}(s) e^{-\frac{KN_{i_1}}{N-1}(t-s)} ds \\
&\leq e^{-\frac{KN_{i_1}}{N-1}(t-\tau)} M_0^v + M_0^v(1 - e^{-\frac{KN_{i_1}}{N-1}(t-\tau)}) \\
&\quad - e^{-K(\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v) e^{-K\gamma(T+\tau)}\frac{\psi_0}{N-1}\int_\tau^t \alpha_{i_1 L}(s) ds \\
&= M_0^v - e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\frac{\psi_0}{N-1}\int_\tau^t \alpha_{i_1 L}(s) ds,
\end{aligned}$$

for all $t \in [\tau, \gamma(T+\tau)+\tau]$. In particular, for $t \in [T+\tau, \gamma(T+\tau)+\tau]$, we find

$$\langle x_{i_1}(t), v\rangle \leq M_0^v - e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\frac{\psi_0}{N-1}\tilde{\alpha}, \tag{5.1.24}$$

where here we have used the fact that, from (1.2.11),

$$\int_\tau^t \alpha_{i_1 L}(s)ds \geq \int_\tau^{T+\tau} \alpha_{i_1 L}(s)ds \geq \tilde{\alpha}.$$

Let us note that, if $\gamma = 1$, estimate (5.1.24) holds for each agent. If $\gamma > 1$, let us consider an index $i_2 \in \{1, \dots, N\} \setminus \{i_1\}$ such that $\chi_{i_2 i_1} = 1$. Then, for a.e. $t \in [T+2\tau, \gamma(T+\tau)+\tau]$, from (5.1.24) it comes that

$$\begin{aligned}
\frac{d}{dt}\langle x_{i_2}(t), v\rangle = & \sum_{j\neq i_1, i_2} \chi_{i_2 j} b_{i_2 j}(t)(\langle x_j(t-\tau_{i_2 j}(t)), v\rangle - \langle x_{i_2}(t), v\rangle) \\
& + b_{i_2 i_1}(t)(\langle x_{i_1}(t-\tau_{i_2 i_1}(t)), v\rangle - \langle x_{i_2}(t), v\rangle) \\
\leq & (M_0^v - \langle x_{i_2}(t), v\rangle) \sum_{j\neq i_1, i_1} \chi_{i_2 j} b_{i_2 j}(t) \\
& + b_{i_2 i_1}(t)\left(M_0^v - e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\frac{\psi_0}{N-1}\tilde{\alpha} - \langle x_{i_2}(t), v\rangle\right).
\end{aligned}$$

Thus, arguing as above,

$$\begin{aligned}
\frac{d}{dt}\langle x_{i_2}(t), v\rangle \leq & \frac{KN_{i_2}}{N-1}(M_0^v - \langle x_{i_2}(t), v\rangle) - b_{i_2 i_1}(t)(M_0^v - \langle x_{i_2}(t), v\rangle) \\
& + b_{i_2 i_1}(t)\left(M_0^v - e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\frac{\psi_0}{N-1}\tilde{\alpha} - \langle x_{i_2}(t), v\rangle\right) \\
= & \frac{KN_{i_2}}{N-1}(M_0^v - \langle x_{i_2}(t), v\rangle) - b_{i_2 i_1}(t)e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\frac{\psi_0}{N-1}\tilde{\alpha} \\
\leq & \frac{KN_{i_2}}{N-1}(M_0^v - \langle x_{i_2}(t), v\rangle) - \alpha_{i_2 i_1}(t)e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\left(\frac{\psi_0}{N-1}\right)^2\tilde{\alpha} \\
= & \frac{KN_{i_2}}{N-1}M_0^v - \alpha_{i_2 i_1}(t)e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\left(\frac{\psi_0}{N-1}\right)^2\tilde{\alpha} - \frac{KN_{i_2}}{N-1}\langle x_{i_2}(t), v\rangle.
\end{aligned}$$

Again, using Gronwall's estimate, it comes that

$$\begin{aligned}
\langle x_{i_2}(t), v\rangle \leq & e^{-\frac{KN_{i_2}}{N-1}(t-T-2\tau)}\langle x_{i_2}(T+2\tau), v\rangle + M_0^v(1 - e^{-\frac{KN_{i_2}}{N-1}(t-T-2\tau)}) \\
& - e^{-K(2\gamma(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\left(\frac{\psi_0}{N-1}\right)^2\tilde{\alpha}\int_{T+2\tau}^t \alpha_{i_2 i_1}(s)e^{-\frac{KN_{i_2}}{N-1}(t-s)}ds \\
\leq & M_0^v - e^{-K(3\gamma(T+\tau)-T)}(M_0^v - \tilde{m}_0^v)\left(\frac{\psi_0}{N-1}\right)^2\tilde{\alpha}\int_{T+2\tau}^t \alpha_{i_2 i_1}(s)ds,
\end{aligned}$$

for all $t \in [T+2\tau, \gamma(T+\tau)+\tau]$. In particular, for $t \in [2T+2\tau, \gamma(T+\tau)+\tau]$, the condition (1.2.11) yields

$$\langle x_{i_2}(t), v\rangle \leq M_0^v - e^{-K(3\gamma(T+\tau)-T)}(M_0^v - \tilde{m}_0^v)\left(\frac{\psi_0}{N-1}\right)^2\tilde{\alpha}^2. \tag{5.1.25}$$

Finally, iterating the above procedure along the path $i_0, i_1, \dots, i_r$, $r \leq \gamma$, that starts from $i_0 = L$ we find the following upper bound

$$\langle x_{i_k}(t), v\rangle \leq M_0^v - e^{-K((k+1)\gamma(T+\tau)-\left(\sum_{l=0}^{k-1} l\right)(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\left(\frac{\psi_0\tilde{\alpha}}{N-1}\right)^k, \tag{5.1.26}$$

for all $1 \leq k \leq r$ and for all $t \in [k(T+\tau), \gamma(T+\tau)+\tau]$. In particular, if the path has length γ, for $k = \gamma$, since $\sum_{l=0}^{\gamma-1} l = \frac{\gamma(\gamma-1)}{2}$, inequality (5.1.26) reads as

$$\langle x_{i_\gamma}(t), v\rangle \leq M_0^v - e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)}(M_0^v - \tilde{m}_0^v)\left(\frac{\psi_0\tilde{\alpha}}{N-1}\right)^\gamma, \tag{5.1.27}$$

for all $t \in [\gamma(T+\tau), \gamma(T+\tau)+\tau]$.
Let us note that (5.1.27) holds for every agent in the path starting from $i_0 = L$ for $t \in [\gamma(T+\tau), \gamma(T+\tau)+\tau]$. Then, from the arbitrariness of the path and since the digraph is strongly connected, (5.1.27) holds for all the agents.

Now, let $R = 1, \dots, N$ be such that $\tilde{M}_0^v = \langle x_R(0), v\rangle$. Then, arguing as before, we get

$$\langle x_R(t), v\rangle \geq m_0^v(1 + e^{-K(\gamma(T+\tau)+\tau)}(\tilde{M}_0^v - m_0^v)), \quad \forall t \in [0, \gamma(T+\tau)+\tau]. \tag{5.1.28}$$

Employing the same arguments used above, we can conclude that

$$\langle x_i(t), v\rangle \geq m_0^v + e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)}(\tilde{M}_0^v - m_0^v)\left(\frac{\psi_0\tilde{\alpha}}{N-1}\right)^\gamma,$$

for all $t \in [\gamma(T+\tau), \gamma(T+\tau)+\tau]$ and for all $i = 1, \dots, N$. Finally, we can deduce that estimate (5.1.21) holds. □

The following proposition generalizes the previous one in successive time intervals. Its proof is analogous to the previous one, so we omit it.

Proposition 5.1.13. *Let $v \in \mathbb{R}^d$. For any $n \in \mathbb{N}_0$, it holds*

$$m_n^v + \Gamma(\tilde{M}_n^v - m_n^v) \leq \langle x_i(t), v\rangle \leq M_n^v - \Gamma(M_n^v - \tilde{m}_n^v), \tag{5.1.29}$$

for all $t \in I_{n+1}$ and for all $i = 1, \dots, N$, where Γ is the positive constant in (5.1.22).

Now, we can prove the consensus Theorem 5.1.1.

Proof of Theorem 5.1.1. Fix $v \in \mathbb{R}^d$. Let us define the quantities

$$\mathcal{D}_n^v := M_n^v - m_n^v, \quad \forall n \in \mathbb{N}_0,$$

where M_n^v, m_n^v are the constants introduced in Definition 5.1.3. Note that, for all $n \in \mathbb{N}_0$, we have $\mathcal{D}_n^v \geq 0$, being $M_n^v \geq m_n^v$.
Let $\Gamma \in (0,1)$ be the constant in (5.1.22). We claim that

$$\mathcal{D}_{n+1}^v \leq (1-\Gamma)\mathcal{D}_n^v, \quad \forall n \in \mathbb{N}_0. \tag{5.1.30}$$

Indeed, fix $n \in \mathbb{N}_0$. Let $i, j = 1, \dots, N$ and $s, t \in I_{n+1}$ be such that $\langle x_i(s), v\rangle = M_{n+1}^v$ and $\langle x_j(t), v\rangle = m_{n+1}^v$. Then, applying Lemma 5.1.13, we can write

$$\begin{aligned}\mathcal{D}_{n+1}^v &= M_{n+1}^v - m_{n+1}^v = \langle x_i(s), v\rangle - \langle x_j(t), v\rangle \\ &\leq M_n^v - m_n^v - \Gamma(M_n^v - \tilde{m}_n^v) - \Gamma(\tilde{M}_n^v - m_n^v).\end{aligned} \tag{5.1.31}$$

Now, we distinguish four cases.
Case I) Assume that $M_n^v - \tilde{m}_n^v = 0$ and $\tilde{M}_n^v - m_n^v = 0$. Then, since from (5.1.15)

$$m_n^v \le \tilde{m}_n^v \le \tilde{M}_n^v = m_n^v,$$

we get

$$m_n^v = \tilde{m}_n^v = M_n^v.$$

As a consequence, (5.1.31) becomes

$$\mathcal{D}_{n+1}^v \le 0 = (1-\Gamma)\mathcal{D}_n^v.$$

Case II) Assume that $M_n^v - \tilde{m}_n^v = 0$ and $\tilde{M}_n^v - m_n^v > 0$. Then, since from (5.1.15)

$$\tilde{m}_n^v \le \tilde{M}_n^v \le M_n^v = \tilde{m}_n^v,$$

we can write

$$\tilde{M}_n^v = M_n^v.$$

As a consequence, (5.1.31) becomes

$$\mathcal{D}_{n+1}^v \le M_n^v - m_n^v - \Gamma\tilde{M}_n^v + \Gamma m_n^v = (1-\Gamma)(M_n^v - m_n^v) = (1-\Gamma)\mathcal{D}_n^v.$$

Case III) Assume that $M_n^v - \tilde{m}_n^v > 0$ and $\tilde{M}_n^v - m_n^v = 0$. Then, from (5.1.15) we have

$$m_n^v \le \tilde{m}_n^v \le \tilde{M}_n^v = m_n^v,$$

from which

$$\tilde{m}_n^v = m_n^v.$$

As a consequence, (5.1.31) becomes

$$\mathcal{D}_{n+1}^v \le M_n^v - m_n^v - \Gamma M_n^v + \Gamma\tilde{m}_n^v = (1-\Gamma)(M_n^v - m_n^v) = (1-\Gamma)\mathcal{D}_n^v.$$

Case IV) Assume that $M_n^v - \tilde{m}_n^v > 0$ and $\tilde{M}_n^v - m_n^v > 0$. In this case, using the fact that $\tilde{M}_n^v \ge \tilde{m}_n^v$, from (5.1.31) we get

$$\mathcal{D}_{n+1}^v \le (1-\Gamma)(M_n^v - m_n^v) - \Gamma\tilde{M}_n^v + \Gamma\tilde{m}_n^v \le (1-\Gamma)(M_n^v - m_n^v) = (1-\Gamma)\mathcal{D}_n^v.$$

Hence, (5.1.30) is fulfilled.
As a consequence, since the positive constant Γ in (5.1.30) does not depend of the choice of the vector v, we find the following estimate :

$$D_{n+1} \le (1-\Gamma)D_n, \quad \forall n \in \mathbb{N}_0. \tag{5.1.32}$$

To see this, fix $n \in \mathbb{N}$. Let $i, j = 1, \dots, N$ and $s, t \in I_{n+1}$ be such that

$$D_{n+1} = |x_i(s) - x_j(t)|.$$

Let us define the unit vector

$$v = \frac{x_i(s) - x_j(t)}{|x_i(s) - x_j(t)|}.$$

Then, using (5.1.15) and (5.1.30),

$$\begin{aligned}
D_{n+1} &= \langle x_i(s) - x_j(t), v\rangle = \langle x_i(s), v\rangle - \langle x_j(t), v\rangle \\
&\le M^v_{n+1} - m^v_{n+1} = \mathcal{D}^v_{n+1} \\
&\le (1-\Gamma)\mathcal{D}^v_n = (1-\Gamma)(M^v_n - m^v_n) \\
&\le (1-\Gamma) \max_{k,l=1,\dots,N} \max_{r,w\in I_n} |x_k(r) - x_l(w)| = (1-\Gamma) D_n.
\end{aligned}$$

Thus, (5.1.32) holds.
Now, from (5.1.32) it comes that

$$D_n \le (1-\Gamma)^n D_0, \quad \forall n \in \mathbb{N}_0. \tag{5.1.33}$$

Let us note that (5.1.33) can be rewritten as

$$D_n \le e^{-nC_2(\gamma(T+\tau)+\tau)} D_0, \quad \forall n \in \mathbb{N}_0, \tag{5.1.34}$$

where C_2 is the positive constant in (5.1.9).
Now, let $t \ge 0$. Thus, $t \in [n(\gamma(T+\tau)+\tau), (n+1)(\gamma(T+\tau)+\tau)]$, for some $n \in \mathbb{N}_0$. Then, using (5.1.17) and (5.1.34), it comes that

$$d(t) \le D_n \le e^{-nC_2(\gamma(T+\tau)+\tau)} D_0 \le e^{-C_2(t-\gamma(T+\tau)-\tau)} D_0 = e^{-C_2 t} C_1 D_0,$$

where C_1 is the positive constant in (5.1.8). This concludes our proof. □

Remark 5.1.14. *Assume that $\chi_{ij} = 1$, for all $i, j = 1, \dots, N$, $i \ne j$, i.e. assume that the interaction is all-to-all. Then, if for all $i, j = 1, \dots, N$ we have $\alpha_{ij}(t) = \alpha(t)$, for a.e. $t \ge 0$, and $\tau_{ij}(t) = \tau(t)$, for all $t \ge 0$, where $\alpha(\cdot)$ and $\tau(\cdot)$ are suitable functions satisfying* (1.2.11) *and* (5.1.2) *respectively, the constants C_1, C_2 in Theorem 5.1.1 can be chosen independently of the number of agents (see [34]).*

5.2 The second-order alignment model

Consider a finite set of $N \in \mathbb{N}$ particles, with $N \ge 2$. Let $x_i(t) \in \mathbb{R}^d$ and $v_i(t) \in \mathbb{R}^d$ denote the position and the velocity of the i-th particle at time t, respectively. The interactions between the elements of the system are described by the following Cucker-Smale type model with pair and time variable time delays,

$$\begin{cases} \frac{d}{dt} x_i(t) = v_i(t), & t > 0,\ \forall i = 1, \dots, N, \\ \frac{d}{dt} v_i(t) = \sum\limits_{j:j\ne i} \chi_{ij} c_{ij}(t)(v_j(t-\tau_{ij}(t)) - v_i(t)), & t > 0,\ \forall i = 1, \dots, N, \end{cases} \tag{5.2.35}$$

where the time delay functions $\tau_{ij} : [0, +\infty) \to [0, +\infty)$ are as in (5.1.2) and the terms χ_{ij} are defined as in (1.2.9).
Here, the communication rates c_{ij} are of the form (1.3.20). Once again, the weight functions $\alpha_{ij} : [0, +\infty) \to [0, 1]$ are $\mathcal{L}^1$-measurable and satisfy the Persistence Excitation Condition **(PE)**.
The initial conditions

$$x_i(s) = x_i^0(s), \quad v_i(s) = v_i^0(s), \quad \forall s \in [-\tau, 0], \forall i = 1, \dots, N, \tag{5.2.36}$$

are assumed to be continuous functions.
We set

$$C_0^V := \max_{i=1,\dots,N} \max_{s\in[-\tau,0]} |v_i(s)|, \tag{5.2.37}$$

$$M_0^X := \max_{i=1,\dots,N} \max_{s,t\in[-\tau,0]} |x_i(s) - x_i(t)|. \tag{5.2.38}$$

We define the space and velocity diameters as in (1.3.23) and (1.3.24), respectively. We want to prove the unconditional flocking as in the Definition 1.3.1. Our main result is the following.

Theorem 5.2.1. *Assume* (5.1.2) *and that the digraph $\mathcal{G}$ is strongly connected. Let $\tilde{\psi} : [0, +\infty) \to \mathbb{R}$ be a positive, bounded, continuous function that satisfies*

$$\int_0^{+\infty} \left(\min_{r\in[0,t]} \tilde{\psi}(r) \right)^\gamma dt = +\infty, \tag{5.2.39}$$

where γ is the depth of the digraph. Assume that the weight functions $\alpha_{ij} : [0, +\infty) \to [0, 1]$ are $\mathcal{L}^1$-measurable and satisfy **(PE)**. *Moreover, let $x_i^0, v_i^0 : [-\tau, 0] \to \mathbb{R}^d$ be continuous functions, for any $i = 1, \dots, N$. Then, for every solution $\{(x_i, v_i)\}_{i=1,\dots,N}$ to* (5.2.35) *with the initial conditions* (5.2.36), *there exists a positive constant d^* such that*

$$\sup_{t\geq -\tau} d_X(t) \leq d^*, \tag{5.2.40}$$

and the following exponential decay estimate holds

$$d_V(t) \leq C_3 \left(\max_{i,j=1,\dots,N} \max_{r,s\in[-\tau,0]} |v_i(r) - v_j(s)| \right) e^{-C_4 t}, \quad \forall t \geq 0, \tag{5.2.41}$$

where C_3, C_4 are the positive constants defined as

$$C_3 := \frac{1}{1 - e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)} \left(\frac{\tilde{\alpha}}{N-1}\right)^\gamma \left(\min_{r\in[0,d^*]} \psi(r)\right)^\gamma} \tag{5.2.42}$$

$$C_4 := \frac{1}{\gamma(T+\tau)+\tau} \ln \left(\frac{1}{1 - e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)} \left(\frac{\tilde{\alpha}}{N-1}\right)^\gamma \left(\min_{r\in[0,d^*]} \psi(r)\right)^\gamma} \right), \tag{5.2.43}$$

being $\gamma > 0$ the depth of the digraph, T and $\tilde{\alpha}$ the positive constants in (1.2.11).

Remark 5.2.2. *Let us note that, if the function $\tilde{\psi}$ is nonincreasing and the interaction is universal, i.e. $\gamma = 1$, then the condition* (5.2.39) *reduces to*

$$\int_0^{+\infty} \tilde{\psi}(t)dt = +\infty,$$

which is the classical assumption to obtain unconditional flocking (see e.g. [70]). Since here we deal with an influence function not necessarily monotonic and the interaction is not universal, we require the stronger assumption (5.2.39) *(cf. [33] for the case of universal interaction).*

Remark 5.2.3. *Let us note that, analogously to the first-order model (see Remark 5.1.2), the decay velocity C_4 tends to 0 as γ, T or τ goes to $+\infty$ and as $\tilde{\alpha} \to 0$. Therefore, C_4 decreases for increasing values of γ, T and τ and for decreasing values of $\tilde{\alpha}$.*

5.2.1 Preliminary lemmas

Let $\{x_i, v_i\}_{i=1,\dots,N}$ be solution to (5.2.35) under the initial conditions (5.2.36). We assume that the hypotheses of Theorem 5.2.1 are satisfied. The following lemmas hold. We omit their proofs since they can be proved using the same arguments employed in Section 5.1.

Definition 5.2.4. *Given a vector $v \in \mathbb{R}^d$, for all $n \in \mathbb{N}_0$ we define*

$$r_n^v := \min_{j=1,\dots,N} \min_{s\in I_n} \langle v_j(s), v\rangle,$$
$$R_n^v := \max_{j=1,\dots,N} \max_{s\in I_n} \langle v_j(s), v\rangle,$$

where, as in the previous section,

$$I_n = [n(\gamma(T+\tau)+\tau)-\tau, n(\gamma(T+\tau)+\tau)].$$

Also, we define, for all $n \in \mathbb{N}_0$,

$$\tilde{r}_n^v := \min_{j=1,\dots,N} \langle v_j(n(\gamma(T+\tau)+\tau)), v\rangle,$$

$$\tilde{R}_n^v := \max_{j=1,\dots,N} \langle v_j(n(\gamma(T+\tau)+\tau)), v\rangle.$$

Lemma 5.2.5. *For each vector $v \in \mathbb{R}^d$ and for any $n \in \mathbb{N}_0$, we have that*

$$r_n^v \le \langle v_i(t), v\rangle \le R_n^v, \tag{5.2.44}$$

for all $t \ge n(\gamma(T+\tau)+\tau)-\tau$ and for any $i = 1,\dots,N$.

Definition 5.2.6. *For all $n \in \mathbb{N}_0$, we define*

$$F_n := \max_{i,j=1,\dots,N} \max_{r,s\in I_n} |v_i(r)-v_j(s)|.$$

Remark 5.2.7. *Let us note that*

$$F_0 := \max_{i,j=1,\dots,N} \max_{r,s\in I_0} |v_i(r)-v_j(s)| = \max_{i,j=1,\dots,N} \max_{r,s\in[-\tau,0]} |v_i(r)-v_j(s)|.$$

Then, the exponential decay estimate in (5.2.41) *can be written as*

$$d_V(t) \le e^{-C_4 t} C_3 F_0, \quad \forall t \ge 0,$$

where $C_3 > 0$ and $C_4 > 0$ are the constants defined in (5.2.42) *and* (5.2.43)*, respectively.*

Lemma 5.2.8. *For each $n \in \mathbb{N}_0$, we have that*

$$|v_i(s)-v_j(t)| \le F_n, \tag{5.2.45}$$

for all $s, t \ge n(\gamma(T+\tau)+\tau)-\tau$ and for any $i, j = 1,\dots,N$.

Remark 5.2.9. *Let us note that* (5.2.45) *yields*

$$d_V(t) \le F_n, \quad \forall t \ge n(\gamma(T+\tau)+\tau)-\tau. \tag{5.2.46}$$

Furthermore, from (5.2.45) *it follows that*

$$F_{n+1} \le F_n, \quad \forall n \in \mathbb{N}_0. \tag{5.2.47}$$

Also, arguing as in Section 5.1, we can find a bound on the velocities $|v_i(t)|$, which is uniform with respect to t and $i = 1, \dots, N$.

Lemma 5.2.10. *For every $i = 1, \dots, N$, we have that*

$$|v_i(t)| \le C_0^V, \quad \forall t \ge -\tau, \tag{5.2.48}$$

where C_0^V is the constant defined in (5.2.37)*.*

Now, we provide the following result, in which an estimate of the position diameters is established.

Lemma 5.2.11. *For every $i, j = 1, \dots, N$, we get*

$$|x_i(t) - x_j(t - \tau_{ij}(t))| \le \tau C_0^V + M_0^X + d_X(t), \quad \forall t \ge 0, \tag{5.2.49}$$

where C_0^V and M_0^X are the positive constants in (5.2.37) *and* (5.2.38)*, respectively.*

Proof. Given $i, j = 1, \dots, N$ and $t \ge 0$, we have

$$\begin{aligned} |x_i(t) - x_j(t - \tau_{ij}(t))| &\le |x_i(t) - x_j(t)| + |x_j(t) - x_j(t - \tau_{ij}(t))| \\ &\le d_X(t) + |x_j(t) - x_j(t - \tau_{ij}(t))|. \end{aligned} \tag{5.2.50}$$

Now, we estimate

$$|x_j(t) - x_j(t - \tau_{ij}(t))|.$$

If $t - \tau_{ij}(t) > 0$, from (5.1.2) and (5.2.48) we get

$$\begin{aligned} |x_j(t) - x_j(t - \tau_{ij}(t))| &\le \int_{t-\tau_{ij}(t)}^t |v_j(s)| ds \\ &\le C_0^V \tau_{ij}(t) \le \tau C_0^V. \end{aligned}$$

On the other hand, if $t - \tau_{ij}(t) \le 0$, then $t \le \tau$ and

$$\begin{aligned} |x_j(t) - x_j(t - \tau_{ij}(t))| &\le |x_j(0) - x_j(t - \tau_{ij}(t))| + \int_0^t |v_j(s)| ds \\ &\le M_0^X + t C_0^V \le M_0^X + \tau C_0^V. \end{aligned}$$

Therefore, in both cases,

$$|x_j(t) - x_j(t - \tau_{ij}(t))| \le M_0^X + \tau C_0^V,$$

from which (5.2.50) becomes

$$|x_i(t) - x_j(t - \tau_{ij}(t))| \le M_0^X + \tau C_0^V + d_X(t).$$

□

5.2.2 The flocking estimate

To prove the flocking result, we need, as before, a crucial proposition. First of all, we give the following definition.

Definition 5.2.12. *We define*

$$\tilde{\phi}(t) := \min\left\{\tilde{\psi}(r) : r \in \left[0, \tau C_0^V + M_X^0 + \max_{s\in[-\tau,t]} d_X(s)\right]\right\},$$

for all $t \geq -\tau$.

Remark 5.2.13. *Let us note that from* (5.2.49)

$$\tilde{\psi}(|x_i(t) - x_j(t-\tau_{ij}(t))|) \geq \tilde{\phi}(t), \quad \forall t \geq 0, \forall i, j = 1, \dots, N.$$

from which

$$c_{ij}(t) \geq \frac{1}{N-1}\alpha_{ij}(t)\tilde{\phi}(t), \quad \forall t \geq 0, \forall i, j = 1, \dots, N. \tag{5.2.51}$$

Proposition 5.2.14. *For all* $v \in \mathbb{R}^d$*, it holds*

$$r_0^v + \Gamma_1(\tilde{R}_0^v - r_0^v) \leq \langle v_i(t), v\rangle \leq R_0^v - \Gamma_1(R_0^v - \tilde{r}_0^v), \tag{5.2.52}$$

for all $t \in I_1$ *and for all* $i = 1, \dots, N$*, where* Γ_1 *is the positive constant defined as follows*

$$\Gamma_1 := e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)}\left(\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)\tilde{\alpha}}{N-1}\right)^{\gamma}. \tag{5.2.53}$$

Remark 5.2.15. *Let us note that* $\Gamma_1 \in (0,1)$ *since we have assumed* $\tilde{\alpha}\tilde{K} < 1$.

Proof. Fix $v \in \mathbb{R}^d$. Let $L = 1, \dots, N$ be such that $\langle v_L(0), v\rangle = \tilde{r}_0^v$. Note that from (5.2.44), $R_0^v \geq \tilde{r}_0^v$. Then, for a.e. $t \in [0, \gamma(T+\tau)+\tau]$, from (5.2.44)

$$\begin{aligned}\frac{d}{dt}\langle v_L(t), v\rangle &= \sum_{j:j\neq L}\chi_{Lj}c_{Lj}(t)(\langle v_j(t-\tau_{Lj}(t)), v\rangle - \langle v_L(t), v\rangle)\\ &\leq \sum_{j:j\neq L}\chi_{Lj}c_{Lj}(t)(R_0^v - \langle v_L(t), v\rangle)\\ &\leq \frac{\tilde{K}}{N-1}\sum_{j:j\neq L}(R_0^v - \langle v_L(t), v\rangle) = \tilde{K}(R_0^v - \langle v_L(t), v\rangle).\end{aligned}$$

Thus, Gronwall's inequality yields

$$\begin{aligned}\langle v_L(t), v\rangle &\leq e^{-\tilde{K}t}\langle v_L(0), v\rangle + R_0^v(1 - e^{-\tilde{K}t})\\ &= R_0^v - e^{-\tilde{K}t}(R_0^v - \tilde{r}_0^v)\\ &\leq R_0^v - e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v).\end{aligned}$$

Therefore, we have

$$\langle v_L(t), v\rangle \leq R_0^v - e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v), \quad \forall t \in [0, \gamma(T+\tau)+\tau]. \tag{5.2.54}$$

Now, let $i_1 = 1, \dots, N \setminus \{L\}$ be such that $\chi_{i_1 L} = 1$. Such an index i_1 exists since the digraph is strongly connected. Then, for a.e. $t \in [\tau, \gamma(T+\tau)+\tau]$, from (5.2.54) we get

$$\begin{aligned}
\frac{d}{dt}\langle v_{i_1}(t), v\rangle &= \sum_{j\neq i_1, L} \chi_{i_1 j} c_{i_1 j}(t)(\langle v_j(t-\tau_{i_1 j}(t)), v\rangle - \langle v_{i_1}(t), v\rangle) \\
&\quad + c_{i_1 L}(t)(\langle v_L(t-\tau_{i_1 L}(t)), v\rangle - \langle v_{i_1}(t), v\rangle) \\
&\leq \sum_{j\neq i_1, L} \chi_{i_1 j} c_{i_1 j}(t)(R_0^v - \langle v_{i_1}(t), v\rangle) \\
&\quad + c_{i_1 L}(t)\left(R_0^v - e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v) - \langle v_{i_1}(t), v\rangle\right) \\
&= (R_0^v - \langle v_{i_1}(t), v\rangle)\sum_{j\neq i_1, L} \chi_{i_1 j} c_{i_1 j}(t) \\
&\quad + c_{i_1 L}(t)\left(R_0^v - e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v) - \langle v_{i_1}(t), v\rangle\right).
\end{aligned}$$

Note that

$$\sum_{j\neq i_1, L} \chi_{i_1 j} c_{i_1 j}(t) = \sum_{j\neq i_1} \chi_{i_1 j} c_{i_1 j}(t) - c_{i_1 L}(t) \leq \frac{\tilde{K}}{N-1}\sum_{j\neq i_1, L} \chi_{i_1 j} - c_{i_1 L}(t) = \frac{\tilde{K}N_{i_1}}{N-1} - c_{i_1 L}(t).$$

Thus, from (5.2.51) it comes that

$$\begin{aligned}
\frac{d}{dt}\langle v_{i_1}(t), v\rangle &\leq \frac{\tilde{K}N_{i_1}}{N-1}(R_0^v - \langle v_{i_1}(t), v\rangle) - c_{i_1 L}(t)(R_0^v - \langle v_{i_1}(t), v\rangle) \\
&\quad + c_{i_1 L}(t)\left(R_0^v - e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v) - \langle v_{i_1}(t), v\rangle\right) \\
&\leq \frac{\tilde{K}N_{i_1}}{N-1}(R_0^v - \langle v_{i_1}(t), v\rangle) - \alpha_{i_1 L}(t)\frac{\tilde{\phi}(t)}{N-1}e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v) \\
&= \frac{\tilde{K}N_{i_1}}{N-1}R_0^v - \alpha_{i_1 L}(t)\frac{\tilde{\phi}(t)}{N-1}e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v) - \frac{\tilde{K}N_{i_1}}{N-1}\langle v_{i_1}(t), v\rangle.
\end{aligned}$$

Hence, Gronwall's estimate yields

$$\begin{aligned}
\langle v_{i_1}(t), v\rangle &\leq e^{-\frac{\tilde{K}N_{i_1}}{N-1}(t-\tau)}\langle v_{i_1}(\tau), v\rangle + R_0^v(1 - e^{-\frac{\tilde{K}N_{i_1}}{N-1}(t-\tau)}) \\
&\quad - e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\frac{1}{N-1}\int_\tau^t \tilde{\phi}(s)\alpha_{i_1 L}(s)e^{-\frac{\tilde{K}N_{i_1}}{N-1}(t-s)}ds \\
&\leq e^{-\frac{\tilde{K}N_{i_1}}{N-1}(t-\tau)}R_0^v + R_0^v(1 - e^{-\frac{\tilde{K}N_{i_1}}{N-1}(t-\tau)}) \\
&\quad - e^{-\tilde{K}(\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)e^{-\tilde{K}\gamma(T+\tau)}\frac{1}{N-1}\int_\tau^t \tilde{\phi}(s)\alpha_{i_1 L}(s)ds \\
&= R_0^v - e^{-\tilde{K}(2\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\frac{1}{N-1}\int_\tau^t \tilde{\phi}(s)\alpha_{i_1 L}(s)ds,
\end{aligned}$$

for all $t \in [\tau, \gamma(T+\tau)+\tau]$. Note that, since $\tilde{\phi}$ is a nonincreasing function,

$$\tilde{\phi}(t) \geq \tilde{\phi}(\gamma(T+\tau)+\tau), \quad \forall t \in [0, \gamma(T+\tau)+\tau]. \tag{5.2.55}$$

Then, we can write

$$\langle v_{i_1}(t), v\rangle \leq R_0^v - e^{-\tilde{K}(2\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{N-1}\int_\tau^t \alpha_{i_1 L}(s)ds,$$

for all $t \in [\tau, \gamma(T+\tau)+\tau]$. In particular, for $t \in [T+\tau, \gamma(T+\tau)+\tau]$, we find

$$\langle v_{i_1}(t), v\rangle \leq R_0^v - e^{-\tilde{K}(2\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{N-1}\tilde{\alpha}, \tag{5.2.56}$$

where here we have used the fact that (1.2.11) implies the following inequality

$$\int_\tau^t \alpha_{i_1 L}(s)ds \geq \int_\tau^{T+\tau} \alpha_{i_1 L}(s)ds \geq \tilde{\alpha}.$$

Now, if $\gamma = 1$, (5.2.56) holds for each agent. On the other hand, if $\gamma > 1$, let us consider an index $i_2 \in \{1, \dots, N\} \setminus \{i_1\}$ such that $\chi_{i_2 i_1} = 1$. Then, for a.e. $t \in [T+2\tau, \gamma(T+\tau)+\tau]$, from (5.2.56) it comes that

$$\begin{aligned}
\frac{d}{dt}\langle v_{i_2}(t), v\rangle &= \sum_{j \neq i_1, i_2} \chi_{i_2 j} c_{i_2 j}(t)(\langle v_j(t-\tau_{i_2 j}(t)), v\rangle - \langle v_{i_2}(t), v\rangle) \\
&\quad + c_{i_2 i_1}(t)(\langle v_{i_1}(t-\tau_{i_2 i_1}(t)), v\rangle - \langle v_{i_2}(t), v\rangle) \\
&\leq (R_0^v - \langle v_{i_2}(t), v\rangle)\sum_{j \neq i_1, i_2} \chi_{i_2 j} c_{i_2 j}(t) \\
&\quad + c_{i_2 i_1}(t)\left(R_0^v - e^{-\tilde{K}(2\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{N-1}\tilde{\alpha} - \langle v_{i_2}(t), v\rangle\right).
\end{aligned}$$

Hence, arguing as above, we obtain

$$\begin{aligned}
\frac{d}{dt}\langle v_{i_2}(t), v\rangle &\leq \frac{\tilde{K}N_{i_2}}{N-1}(R_0^v - \langle v_{i_2}(t), v\rangle) - c_{i_2 i_1}(t)(R_0^v - \langle v_{i_2}(t), v\rangle) \\
&\quad + c_{i_2 i_1}(t)\left(R_0^v - e^{-\tilde{K}(2\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{N-1}\tilde{\alpha} - \langle v_{i_2}(t), v\rangle\right) \\
&\leq \frac{\tilde{K}N_{i_2}}{N-1}(R_0^v - \langle v_{i_2}(t), v\rangle) - \alpha_{i_2 i_1}(t)e^{-\tilde{K}(2\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{(N-1)^2}\tilde{\phi}(t)\tilde{\alpha}.
\end{aligned}$$

Again, using Gronwall's estimate, it comes that

$$\begin{aligned}
\langle v_{i_2}(t), v\rangle &\leq e^{-\frac{\tilde{K}N_{i_2}}{N-1}(t-T-2\tau)}\langle v_{i_2}(T+2\tau), v\rangle + R_0^v(1 - e^{-\frac{\tilde{K}N_{i_2}}{N-1}(t-T-2\tau)}) \\
&\quad - e^{-\tilde{K}(2\gamma(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{(N-1)^2}\tilde{\alpha}\int_{T+2\tau}^t \tilde{\phi}(s)\alpha_{i_2 i_1}(s)e^{-\frac{\tilde{K}N_{i_2}}{N-1}(t-s)}ds \\
&\leq R_0^v - e^{-\tilde{K}(3\gamma(T+\tau)-T)}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{(N-1)^2}\tilde{\alpha}\int_{T+2\tau}^t \tilde{\phi}(s)\alpha_{i_2 i_1}(s)ds,
\end{aligned}$$

for all $t \in [T + 2\tau, \gamma(T + \tau) + \tau]$. In particular, for $t \in [2T + 2\tau, \gamma(T + \tau) + \tau]$, the condition (1.2.11) and the inequality (5.2.55) imply that

$$\langle v_{i_2}(t), v\rangle \leq R_0^v - e^{-\tilde{K}(3\gamma(T+\tau)-T)}(R_0^v - \tilde{r}_0^v)\left(\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)}{N-1}\right)^2 \tilde{\alpha}^2. \tag{5.2.57}$$

Finally, iterating the above procedure along the path $i_0, i_1, \ldots, i_r$, with $r \leq \gamma$, starting from $i_0 = L$ we find the following upper bound

$$\langle v_{i_k}(t), v\rangle \leq R_0^v - e^{-K((k+1)\gamma(T+\tau)-(T+\tau)(\sum_{l=0}^{k-1} l)+\tau)}(R_0^v - \tilde{r}_0^v)\left(\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)\tilde{\alpha}}{N-1}\right)^k, \tag{5.2.58}$$

for all $1 \leq k \leq r$ and for all $t \in [k(T + \tau), \gamma(T + \tau) + \tau]$. In particular, if the path has length γ, for $k = \gamma$, since $\sum_{l=0}^{\gamma-1} l = \frac{\gamma(\gamma-1)}{2}$, inequality (5.2.58) reads as

$$\langle v_{i_\gamma}(t), v\rangle \leq R_0^v - e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)}(R_0^v - \tilde{r}_0^v)\left(\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)\tilde{\alpha}}{N-1}\right)^\gamma, \tag{5.2.59}$$

for all $t \in [\gamma(T + \tau), \gamma(T + \tau) + \tau]$. Arguing as in Proposition 5.1.11, we can say that (5.2.59) holds for every $i = 1, \ldots, N$.
Now, let $R = 1, \ldots, N$ be such that $\tilde{R}_0^v = \langle v_R(0), v\rangle$. Then, arguing as before, we get

$$\langle v_R(t), v\rangle \geq r_0^v + e^{-K(\gamma(T+\tau)+\tau)}(\tilde{R}_0^v - r_0^v), \quad \forall t \in [0, \gamma(T + \tau) + \tau]. \tag{5.2.60}$$

Employing the same arguments used above, we can conclude that

$$\langle v_i(t), v\rangle \geq r_0^v + e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)}(\tilde{R}_0^v - r_0^v)\left(\frac{\tilde{\phi}(\gamma(T+\tau)+\tau)\tilde{\alpha}}{N-1}\right)^\gamma,$$

for all $t \in [\gamma(T + \tau), \gamma(T + \tau) + \tau]$ and for all $i = 1, \ldots, N$. Finally, we can deduce that estimate (5.2.52) holds. □

The following proposition extends the previous one in successive time intervals. We omit its proof since it is analogous to the previous one.

Proposition 5.2.16. *Let $v \in \mathbb{R}^d$. For any $n \in \mathbb{N}$, it holds*

$$r_n^v + \Gamma_{n+1}(\tilde{R}_n^v - r_n^v) \leq \langle v_i(t), v\rangle \leq R_n^v - \Gamma_{n+1}(R_n^v - \tilde{r}_n^v), \tag{5.2.61}$$

for all $t \in I_{n+1}$ and for all $i = 1, \ldots, N$, where Γ_{n+1} is the positive constant defined as

$$\Gamma_{n+1} := e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)}\left(\frac{\tilde{\phi}((n+1)(\gamma(T+\tau)+\tau))\tilde{\alpha}}{N-1}\right)^\gamma. \tag{5.2.62}$$

Remark 5.2.17. *Let us note that from* (5.2.61) *it comes that*

$$R_{n+1}^v - r_{n+1}^v \leq (1 - \Gamma_{n+1})(R_n^v - r_n^v), \quad \forall n \in \mathbb{N}_0. \tag{5.2.63}$$

where $\Gamma_{n+1} \in (0,1)$ *is the constant in* (5.2.62).
Indeed, given $n \in \mathbb{N}_0$*, let* $i,j = 1,\dots,N$ *and* $s,t \in I_{n+1}$ *be such that* $\langle v_i(s), v\rangle = R^v_{n+1}$ *and* $\langle v_j(t), v\rangle = r^v_{n+1}$*. Then, applying Lemma 5.2.16, we can write*

$$\begin{aligned} R^v_{n+1} - r^v_{n+1} &= \langle v_i(s), v\rangle - \langle v_j(t), v\rangle \\ &\le R^v_n - r^v_n - \Gamma_{n+1}(R^v_n - \tilde{r}^v_n) - \Gamma_{n+1}(\tilde{R}^v_n - r^v_n). \end{aligned} \tag{5.2.64}$$

Then, arguing as in the proof of Theorem 5.1.1, we get that estimate (6.1.29) *holds.*
Also, setting $C^* := e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)}\left(\frac{\tilde{\alpha}}{N-1}\right)^{\gamma}$*, it holds that*

$$\Gamma_{n+1} = C^*(\tilde{\phi}((n+1)(\gamma(T+\tau)+\tau)))^{\gamma}, \quad \forall n \in \mathbb{N}_0. \tag{5.2.65}$$

As a consequence, (6.1.29) *can be written as*

$$R^v_{n+1} - r^v_{n+1} \le (1 - C^*(\tilde{\phi}((n+1)(\gamma(T+\tau)+\tau)))^{\gamma})(R^v_n - r^v_n), \quad \forall n \in \mathbb{N}_0. \tag{5.2.66}$$

In particular, from (6.1.29) *and* (5.2.66)*, arguing as in Theorem 5.1.1, it comes that*

$$F_{n+1} \le (1-\Gamma_{n+1})F_n, \quad \forall n \in \mathbb{N}_0, \tag{5.2.67}$$

or, equivalently,

$$F_{n+1} \le (1 - C^*(\tilde{\phi}((n+1)(\gamma(T+\tau)+\tau)))^{\gamma})F_n, \quad \forall n \in \mathbb{N}_0, \tag{5.2.68}$$

Now, we can prove Theorem 5.2.1.

Proof of Theorem 5.2.1. Let $\{(x_i, v_i)\}_{i=1,\dots,N}$ be solution to (5.2.35) under the initial conditions (5.2.36). Let us define

$$\tilde{\Gamma}_{n+1} = \frac{\Gamma_{n+1}}{\gamma(T+\tau)+\tau}, \quad \forall n \in \mathbb{N}_0.$$

Let us introduce the function $\mathcal{E} : [-\tau, +\infty) \to [0, +\infty)$,

$$\mathcal{E}(t) := \begin{cases} F_0, & t \in [-\tau, \gamma(T+\tau)+\tau], \\ \mathcal{E}(n(\gamma(T+\tau)+\tau))\left(1 - \tilde{\Gamma}_{n+1}(t - n(\gamma(T+\tau)+\tau))\right), & \\ & t \in (n(\gamma(T+\tau)+\tau), (n+1)(\gamma(T+\tau)+\tau)],\ n \ge 1. \end{cases}$$

By definition, $\mathcal{E}$ is continuous, nonnegative, and nonincreasing. Moreover, we claim that

$$F_n \le \mathcal{E}(t), \quad \forall t \in [-\tau, n(\gamma(T+\tau)+\tau)], \forall n \in \mathbb{N}_0. \tag{5.2.69}$$

We prove this by induction. For $n = 1$, from (5.2.47) we can immediately say that

$$F_1 \le F_0 = \mathcal{E}(t), \quad \forall t \in [-\tau, \gamma(T+\tau)+\tau].$$

Now, assume that (5.2.69) holds for some $n \ge 1$. We have to show that (5.2.69) is true also for $n+1$. From the induction hypothesis and by using again (5.2.47), we have that

$$F_{n+1} \le F_n \le \mathcal{E}(t),$$

for all $t \in [-\tau, n(\gamma(T+\tau)+\tau)]$. It lasts to prove that $F_{n+1} \leq \mathcal{E}(t)$, for all $t \in (n(\gamma(T+\tau)+\tau), (n+1)(\gamma(T+\tau)+\tau)]$. From (5.2.67), it comes that

$$\mathcal{E}(t) \geq \mathcal{E}((n+1)(\gamma(T+\tau)+\tau)) = \mathcal{E}(n(\gamma(T+\tau)+\tau))(1-\tilde{\Gamma}_{n+1}(\gamma(T+\tau)+\tau)) = (1-\Gamma_{n+1})F_n \geq F_{n+1},$$

for all $t \in (n(\gamma(T+\tau)+\tau), (n+1)(\gamma(T+\tau)+\tau)]$, where in the above inequalities we have used the fact that $\mathcal{E}$ is nonincreasing. Hence, (5.2.69) is proven.
Now, for almost all time (see [33] for further details)

$$\frac{d}{dt} \max_{s\in[-\tau,t]} d_X(s) \leq \left|\frac{d}{dt} d_X(t)\right| \leq d_V(t). \tag{5.2.70}$$

Next, let us define the function $\mathcal{W} : [-\tau, +\infty) \to [0, +\infty)$,

$$\mathcal{W}(t) := (\gamma(T+\tau)+\tau)\mathcal{E}(t) + C^* \int_0^{\tau C_0^V + M_0^X + \max_{s\in[-\tau,t+\gamma(T+\tau)+\tau]} d_X(s)} \left(\min_{\sigma\in[0,r]} \tilde{\psi}(\sigma)\right)^\gamma dr,$$

for all $t \geq -\tau$. By construction, $\mathcal{W}$ is continuous. Also, for each $n \geq 1$ and for a.e. $t \in (n(\gamma(T+\tau)+\tau), (n+1)(\gamma(T+\tau)+\tau))$, from (5.2.46), (5.2.69) and (5.2.70) it follows that

$$\begin{aligned}
\frac{d}{dt}\mathcal{W}(t) &= (\gamma(T+\tau)+\tau)\frac{d}{dt}\mathcal{E}(t) + C^*(\tilde{\phi}(t+\gamma(T+\tau)+\tau))^\gamma \frac{d}{dt} \max_{s\in[-\tau,t+(\gamma(T+\tau)+\tau)]} d_X(s) \\
&\leq -\mathcal{E}(n\gamma(T+\tau)+\tau)C^*(\tilde{\phi}((n+1)(\gamma(T+\tau)+\tau)))^\gamma \\
&\qquad + C^*(\tilde{\phi}(t+\gamma(T+\tau)+\tau))^\gamma d_V(t+(\gamma(T+\tau)+\tau)) \\
&\leq C^* F_n(-(\tilde{\phi}((n+1)(\gamma(T+\tau)+\tau)))^\gamma + (\tilde{\phi}((n+1)(\gamma(T+\tau)+\tau))^\gamma)) = 0.
\end{aligned}$$

Then,

$$\frac{d}{dt}\mathcal{W}(t) \leq 0, \quad \text{a.e. } t > \gamma(T+\tau)+\tau, \tag{5.2.71}$$

which implies

$$\mathcal{W}(t) \leq \mathcal{W}(\gamma(T+\tau)+\tau), \quad \forall t \geq \gamma(T+\tau)+\tau. \tag{5.2.72}$$

Now, by definition of $\mathcal{W}$, being $\mathcal{E}$ a nonnegative function, we have

$$C^* \int_0^{\tau C_0^V + M_0^X + \max_{s\in[-\tau,t+\gamma(T+\tau)+\tau]} d_X(s)} \left(\min_{\sigma\in[0,r]} \tilde{\psi}(\sigma)\right)^\gamma dr \leq \mathcal{W}(\gamma(T+\tau)+\tau),$$

for all $t \geq \gamma(T+\tau)+\tau$. Letting $t \to \infty$ in the above inequality, we can conclude that

$$C^* \int_0^{\tau C_0^V + M_0^X + \sup_{s\in[-\tau,+\infty)} d_X(s)} \left(\min_{\sigma\in[0,r]} \tilde{\psi}(\sigma)\right)^\gamma dr \leq \mathcal{W}(\gamma(T+\tau)+\tau). \tag{5.2.73}$$

Finally, since the function $\tilde{\psi}$ satisfies property (5.2.39), from (5.2.73), we can conclude that there exists a positive constant d^* such that

$$\tau C_0^V + M_0^X + \sup_{s\in[-\tau,+\infty)} d_X(s) \leq d^*. \tag{5.2.74}$$

Now, let us define

$$\hat{\phi} := \min_{r\in[0,d^*]} \tilde{\psi}(r).$$

Note that $\hat{\phi} > 0$. Also, (5.2.74) yields

$$\hat{\phi} \leq \tilde{\phi}(t), \quad \forall t \geq -\tau. \tag{5.2.75}$$

Then, from (5.2.68) and (5.2.75) we have

$$F_{n+1} \leq (1 - C^* \hat{\phi}^\gamma) F_n, \quad \forall n \in \mathbb{N}_0. \tag{5.2.76}$$

Thus, thanks to an induction argument, we can write

$$F_n \leq (1 - C^* \hat{\phi}^\gamma)^n F_0, \quad \forall n \in \mathbb{N}_0. \tag{5.2.77}$$

Note that $C^* \hat{\phi}^\gamma = e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\tau)+\tau)} \left(\frac{\tilde{\alpha}}{N-1}\right)^\gamma \left(\min_{r\in[0,d^*]} \tilde{\psi}(r)\right)^\gamma$. Thus, the positive constants C_3 and C_4 in (5.2.41) can be rewritten in the following way (see (5.2.42) and (5.2.43)):

$$C_3 = \frac{1}{1 - C^* \hat{\phi}^\gamma},$$

$$C_4 = \frac{1}{(\gamma(T+\tau)+\tau)} \ln\left(\frac{1}{1 - C^* \hat{\phi}^\gamma}\right).$$

As a consequence, inequality (5.2.77) can be rewritten as

$$F_n \leq e^{-nC_4(\gamma(T+\tau)+\tau)} F_0, \quad \forall n \in \mathbb{N}_0, \tag{5.2.78}$$

where C_4 is the positive constant in (5.2.43).
Finally, let $t \geq 0$. Then, $t \in [n(\gamma(T+\tau)+\tau), (n+1)(\gamma(T+\tau)+\tau)]$, for some $n \in \mathbb{N}_0$. Then, using (5.2.46) and (5.2.78)

$$d_V(t) \leq F_n \leq e^{-nC_4(\gamma(T+\tau)+\tau)} F_0 \leq e^{-C_4(t-\gamma(T+\tau)-\tau)} F_0 = e^{-C_4 t} C_3 F_0,$$

where C_3 is the positive constant in (5.2.42). This concludes our proof. □

5.3 Numerical simulations

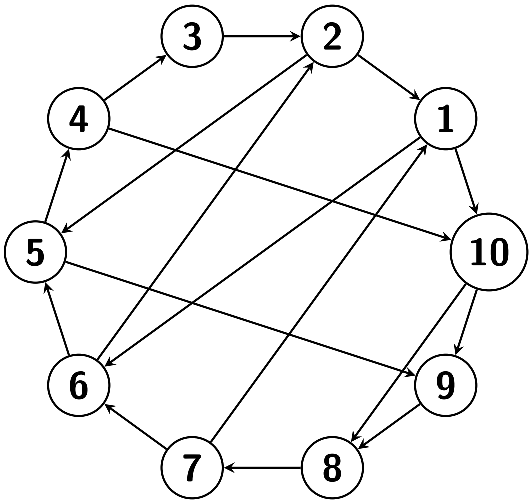
In this section, we present some numerical simulations for the first-order model (5.1.1) and the second-order model (5.2.35) in the one-dimensional case, i.e., $d = 1$, to give evidence to the theoretical results. We consider the influence functions in the definitions (6.0.2) and (1.3.20) defined by

$$\psi(r, r') = \tilde{\psi}(r, r') = \psi^*(|r - r'|), \quad r, r' \in [0, +\infty).$$

In particular, we assume that the function $\psi^*(\cdot)$ takes the form

$$\psi^*(r) := e^{-(r-1)^2}, \quad r \in [0, +\infty). \tag{5.3.79}$$

Meanwhile, for simplicity, the weight functions $\alpha_{ij}(\cdot)$ for all $i, j = 1, \dots, N$ coincide with a piecewise functions $\alpha(\cdot)$ equal to 1 or 0 alternately in time intervals of length 2. The initial conditions were set to be constant and drawn from a random distribution in the interval $[0, 1]$. To produce the tests, we used the MATLAB environment. The solutions of the systems are computed using the MATLAB functions *dde23*, which computes the solution of a given delay differential equation (DDE) with a constant time delay vector, and *ode45*, which performs the solution of an ordinary



$$\chi_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 5.1: Strongly connected digraph and its adjacency matrix, $\gamma = 6$

differential equation (ODE). In all our simulations, we consider a discretization of the time with a time step of 0.1 and the size of the population $N = 10$. For any agent $i \in \{1, \dots, N\}$, we assume the time delays τ_{ij} given by a constant vector in which each entrance takes value in the interval $[0, 1]$ as follows:

$$(\tau_{i1}, \dots, \tau_{iN}) = (0.3804,\ 0.5678,\ 0.0759,\ 0.0540,\ 0.5308,\ 0.7792,\ 0.9340,\ 0.1299,\ 0.5688,\ 0.4649).$$

In Figure 5.1, we draw the network structure that describes the interaction between the agents and the adjacency matrix associated with that structure. For this particular case, the depth of the graph is $\gamma = 6$. In this framework, in Figure 5.2 we perform the dynamics of the solution for the first-order Cucker-Smale, and in Figure 5.3 we compute the solutions for the second-order model. One can see that the opinion and velocity trajectories of all the agents converge to the same limit, ensuring consensus in the first case and flocking in the second case. Notice that we can see the effect of the presence of the function $\alpha(\cdot)$, which causes a lack of connection in some time intervals.

Another strongly connected digraph is considered in Figure 5.4, with depth $\gamma = 3$. Figure 5.5 and Figure 5.6 provide numerical tests for such a network structure for the first and second-order models, respectively.

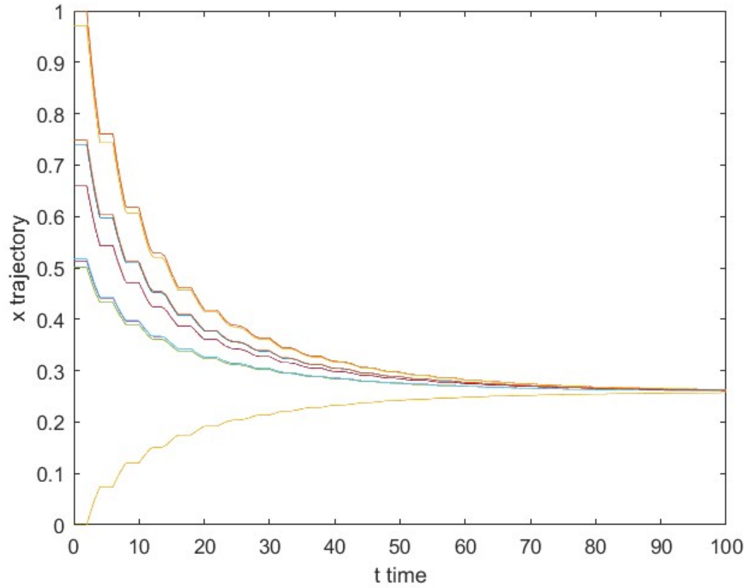


Figure 5.2: Numerical solution of the first-order model in the case $\gamma = 6$.

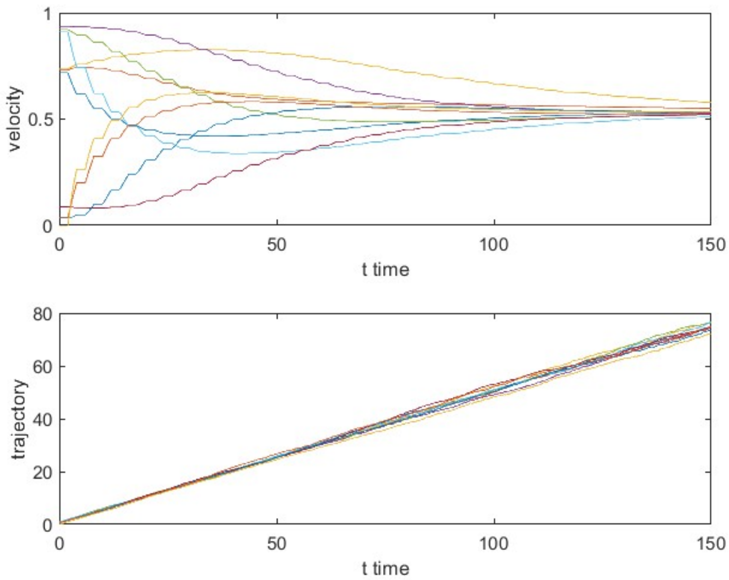
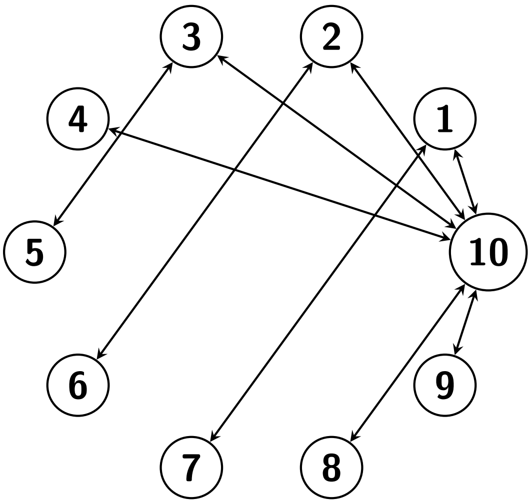


Figure 5.3: Numerical solution of the second-order model in the case $\gamma = 6$.



$$
\chi_{ij} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
$$

Figure 5.4: Strongly connected digraph and its adjacency matrix, $\gamma = 3$

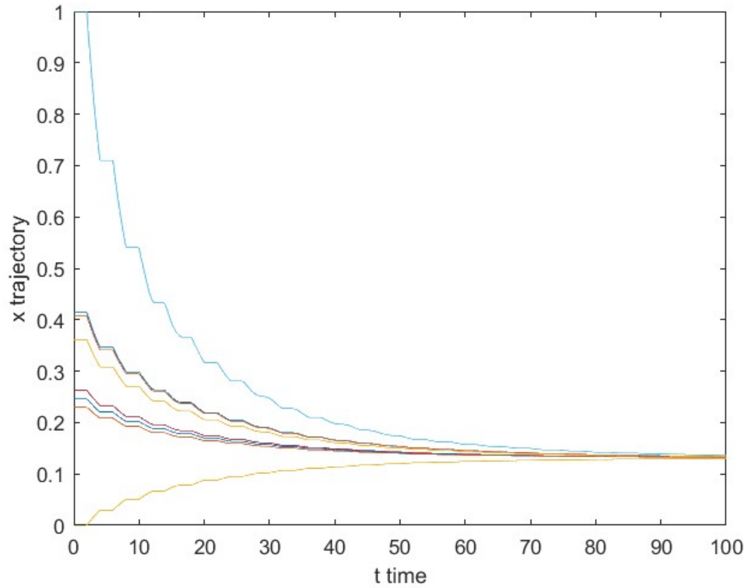


Figure 5.5: Numerical solution of the first-order model in the case $\gamma = 3$.

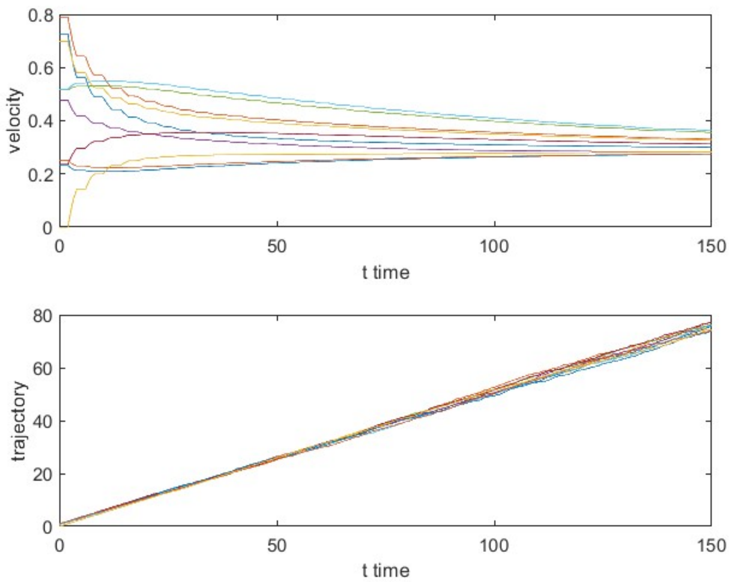


Figure 5.6: Numerical solution of the second-order model in the case $\gamma = 3$.

Chapter 6

Consensus, stability and mean-field limit

In this chapter, we investigate a time-delayed opinion formation model in which a population is divided into two groups: a small number of leaders and a larger group of followers (non-leaders). Specifically, we consider $m \in \mathbb{N}$ leaders and $N \in \mathbb{N}$ non-leaders, with

$$N > m \geq 2.$$

The defining feature of our setting is the asymmetric interaction structure: while leaders influence all agents in the system, they themselves are influenced only by their fellow leaders. This model's situations where opinion leaders, such as political figures, experts, or influencers, shape the views of the broader population but are insulated from the direct influence of non-leaders.

Let

$$y_i(t) \in \mathbb{R}^d, \quad i = 1, \dots, m,$$

denote the opinion of the i-th leader at time t, and let

$$x_i(t) \in \mathbb{R}^d, \quad i = 1, \dots, N,$$

denote the opinion of the i-th non-leader at time t. To account for the time required for discussion and decision-making, we introduce time delays into the interactions among agents. Consequently, the evolution of opinions is governed by the Hegselmann-Krause opinion formation model with delays:

$$\begin{aligned} \frac{d}{dt} y_i(t) &= \frac{1}{m} \sum_{j=1}^{m} \psi_{ij}^{\tau_1}(t)(y_j(t-\tau_1) - y_i(t)), \quad t > 0, \ i = 1, ..., m, \\ \frac{d}{dt} x_i(t) &= \frac{1}{N} \sum_{j=1}^{N} \phi_{ij}^{\tau_2}(t)(x_j(t-\tau_2) - x_i(t)) \\ &\qquad + \frac{1}{m} \sum_{j=1}^{m} \rho_{ij}^{\tau_1}(t)(y_j(t-\tau_1) - x_i(t)), \quad t > 0, \ i = 1, ..., N, \end{aligned} \tag{6.0.1}$$

where the interaction weights are defined by

$$\begin{aligned} \psi_{ij}^{\tau_1}(t) &:= \psi(y_i(t), y_j(t-\tau_1)), \\ \phi_{ij}^{\tau_2}(t) &:= \phi(x_i(t), x_j(t-\tau_2)), \\ \rho_{ij}^{\tau_1}(t) &:= \rho(x_i(t), y_j(t-\tau_1)), \end{aligned} \tag{6.0.2}$$

and $\psi, \phi, \rho : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ are assumed to be positive, Lipschitz continuous, and uniformly bounded. We want to emphasize that, to prove the convergence to consensus, we do not need the Lipschitz property for the communication rates ψ, ϕ, ρ. We will require the Lipschitz continuity only later on, to deal with the mean-field approximations. We denote the maximal delay by

$$\tau := \max\{\tau_1, \tau_2\},$$

and prescribe the initial data on the interval $[-\tau, 0]$:

$$y_i(t) = y_i^0(t), \quad i = 1, \dots, m, \ t \in [-\tau, 0], \tag{6.0.3}$$

and

$$x_i(t) = x_i^0(t), \quad i = 1, \dots, N, \ t \in [-\tau, 0], \tag{6.0.4}$$

where y_i^0 and x_i^0 are continuous functions from $[-\tau, 0]$ to $\mathbb{R}^d$.

Under the above framework, we aim to show that the system achieves consensus in the long run (cf. [30] for a related approach with different normalization factors).

To precisely formulate the consensus result, we introduce the following notions.

Definition 6.0.1. *The global diameter of the system is defined as*

$$d(t) := \max\Big\{ \max_{i,j=1,\dots,N} |x_i(t) - x_j(t)|, \max_{i,j=1,\dots,m} |y_i(t) - y_j(t)|, \max_{\substack{i=1,\dots,m \\ j=1,\dots,N}} |y_i(t) - x_j(t)| \Big\}. \tag{6.0.5}$$

Moreover, for each $n \in \mathbb{N}_0$, we define the diameter over the time interval $[n\tau - \tau, n\tau]$ by

$$D_n := \max_{s,t \in [n\tau-\tau, n\tau]} \Big\{ \max_{i,j=1,\dots,N} |x_i(s) - x_j(t)|, \max_{i,j=1,\dots,m} |y_i(s) - y_j(t)|, \max_{\substack{i=1,\dots,m \\ j=1,\dots,N}} |y_i(s) - x_j(t)| \Big\}.$$

We are now in a position to state our first main result regarding the exponential convergence to consensus.

Theorem 6.0.2. *Let $\{y_i(t)\}_{i=1}^m$ and $\{x_j(t)\}_{j=1}^N$ be the global-in-time classical solution to the system* (6.0.1) *with initial conditions* (6.0.3)–(6.0.4). *Then, there exists a constant $\gamma > 0$, independent of N and m, such that the global diameter decays exponentially:*

$$d(t) \le e^{-\gamma(t-2\tau)} D_0.$$

Similarly to the analysis of the previous chapters, the proof of Theorem 6.0.2 builds upon the analysis of the global opinion diameter functional $d(t)$, which measures the maximal deviation of opinions across all agents. A central challenge lies in controlling the evolution of $d(t)$ under delayed interactions, especially between leaders and followers. To address this, we derive a Grönwall-type inequality for $d(t)$ by decomposing the interaction terms and estimating their influence across time intervals of length τ. Through a stepwise contraction argument and bootstrapping, we establish that $d(t)$ decays exponentially after a transient layer, independently of the delay size. This approach allows us to obtain consensus convergence without imposing smallness assumptions on the delay parameters.

To understand the collective behavior of a large number of interacting agents, we next study the mean-field limit of the delayed particle system introduced in (6.0.1). The goal of this analysis is twofold: first, to derive macroscopic equations that describe the evolution of the system when the number of agents becomes large; and second, to establish convergence results toward consensus at the mean-field level.

We investigate two distinct asymptotic regimes that reflect different population structures.

Case (i): few leaders and many non-leaders. In this regime, the number of leaders m remains fixed, while the number of non-leaders N tends to infinity. This setup reflects situations in which a small number of influential individuals shape the dynamics of a much larger population. In the mean-field limit, the leaders retain their finite-dimensional dynamics, whereas the non-leaders are described by a probability density ν_t governed by a continuity equation. The limiting system reads:

$$\begin{aligned} &\frac{d}{dt}\bar{y}_i(t) = \frac{1}{m}\sum_{j=1}^{m} \bar{\psi}_{ij}^{\tau_1}(t)\big(\bar{y}_j(t-\tau_1) - \bar{y}_i(t)\big), \quad t>0, \quad i=1,\dots,m, \\ &\partial_t \nu_t + \nabla\cdot(\nu_t\, v_t^m) = 0, \quad x\in\mathbb{R}^d, \quad t>0, \end{aligned} \tag{6.0.6}$$

with the initial data

$$(\bar{y}_i(s), \nu_s) =: (\bar{y}_i^0(s), g_s), \quad i=1,\dots,m,\ x\in\mathbb{R}^d \quad \text{for } s\in[-\tau,0].$$

The interaction weight is given by $\bar{\psi}_{ij}^{\tau_1}(t) := \psi(\bar{y}_i(t), \bar{y}_j(t-\tau_1))$, and the velocity field for the non-leader density is given by

$$v_t^m(x) := \int_{\mathbb{R}^d} \phi(x,y)(y-x)\,\nu_{t-\tau_2}(dy) + \frac{1}{m}\sum_{j=1}^{m} \rho(x, \bar{y}_j(t-\tau_1))\big(\bar{y}_j(t-\tau_1) - x\big). \tag{6.0.7}$$

In this formulation, the leaders remain finite-dimensional agents evolving under delayed mutual interactions, while the non-leader population evolves continuously in time and space under the influence of both the leader group and its own internal dynamics. This hybrid description allows for a tractable yet rich model of hierarchical opinion dynamics.

Case (ii): infinite population limit for both leaders and non-leaders. In this fully macroscopic regime, both groups are described by probability densities. The mean-field limit then yields a pair of continuity equations for $\bar{\mu}_t$ (leaders) and $\bar{\nu}_t$ (non-leaders):

$$\begin{aligned} &\partial_t \bar{\mu}_t + \nabla\cdot(\bar{\mu}_t\,\bar{u}_t) = 0, \quad x\in\mathbb{R}^d, \quad t>0, \\ &\partial_t \bar{\nu}_t + \nabla\cdot(\bar{\nu}_t\,\bar{v}_t) = 0, \quad x\in\mathbb{R}^d, \quad t>0, \end{aligned} \tag{6.0.8}$$

subject to the initial data

$$(\bar{\mu}_s, \bar{\nu}_s) =: (\bar{f}_s, \bar{g}_s), \quad x\in\mathbb{R}^d \quad \text{for } s\in[-\tau,0]. \tag{6.0.9}$$

Here, the velocity fields are defined as

$$\bar{u}_t(x) := \int_{\mathbb{R}^d} \psi(x,y)(y-x)\,\bar{\mu}_{t-\tau_1}(dy), \tag{6.0.10}$$

$$\bar{v}_t(x) := \int_{\mathbb{R}^d} \phi(x,y)(y-x)\,\bar{\nu}_{t-\tau_2}(dy) + \int_{\mathbb{R}^d} \rho(x,y)(y-x)\,\bar{\mu}_{t-\tau_1}(dy). \tag{6.0.11}$$

This fully macroscopic description is particularly useful for analyzing large-scale patterns and stability properties of the system when the number of interacting agents is extremely high.

To study convergence and decay to consensus in these mean-field models, we introduce diameter-like quantities that measure the spread of the distributions.

In Case (i), we define:

$$d^{\nu}(t) := \max\Big\{ \sup_{x,y\in \mathrm{supp}(\nu_t)} |x-y|,\ \max_{i,j=1,\dots,m} |y_i(t)-y_j(t)|,\ \max_{i=1,\dots,m} \sup_{x\in \mathrm{supp}(\nu_t)} |y_i(t)-x| \Big\}.$$

In Case (ii), the diameter becomes

$$d^{\mu,\nu}(t) := \max\Big\{ \sup_{x,y\in \mathrm{supp}(\nu_t)} |x-y|,\ \sup_{x,y\in \mathrm{supp}(\mu_t)} |x-y|,\ \sup_{\substack{x\in \mathrm{supp}(\nu_t),\\ y\in \mathrm{supp}(\mu_t)}} |x-y| \Big\}.$$

To measure initial discrepancies, we define:

$$D_0^{\nu} := \max_{s,t\in[-\tau,0]} \max\Big\{ \sup_{\substack{x\in \mathrm{supp}(g_s),\\ y\in \mathrm{supp}(g_t)}} |x-y|,\ \max_{i,j=1,\dots,m} |y_i(s)-y_j(t)|,\ \max_{i=1,\dots,m} \sup_{x\in \mathrm{supp}(g_t)} |y_i(s)-x| \Big\},$$

$$D_0^{\mu,\nu} := \max_{s,t\in[-\tau,0]} \max\Big\{ \sup_{\substack{x\in \mathrm{supp}(g_s),\\ y\in \mathrm{supp}(g_t)}} |x-y|,\ \sup_{\substack{x\in \mathrm{supp}(f_s),\\ y\in \mathrm{supp}(f_t)}} |x-y|,\ \sup_{\substack{x\in \mathrm{supp}(f_s),\\ y\in \mathrm{supp}(g_t)}} |x-y| \Big\}.$$

We now recall the standard notions of push-forward and measure-valued solutions to make the above formulations precise.

Definition 6.0.3. *Let μ be a Borel measure in $\mathbb{R}^d$ and $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable map. The push-forward of μ by $\mathcal{T}$ is the measure $\mathcal{T}\#\mu$ defined by*

$$\mathcal{T}\#\mu(B) := \mu(\mathcal{T}^{-1}(B)),$$

for every Borel sets $B \subset \mathbb{R}^d$.

Definition 6.0.4. *Let $T > 0$ and let $\mathcal{P}(\mathbb{R}^d)$ denote a set of probability measures in $\mathbb{R}^d$. We say that $\mu_t \in C([0,T);\mathcal{P}(\mathbb{R}^d))$ is a measure-valued solution to a continuity equation of the form* (6.0.6) *or* (6.0.8) *if for every $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T))$, the following weak formulation holds:*

$$\int_0^T \int_{\mathbb{R}^d} (\partial_t \phi + v(x)\cdot \nabla_x \phi)\mu_t(dx)dt + \int_{\mathbb{R}^d} \phi(x,0)\mu_0(dx) = 0, \tag{6.0.12}$$

where v is the velocity field defined as (6.0.7), (6.0.10), *or* (6.0.11).

Theorem 6.0.5. *Assume that the initial data for the particle system* (6.0.1) *satisfy one of the following:*

Case (i): few leaders and many non-leaders: *The leader initial data are given by*

$$y_i^0 \in C([-\tau_1,0]), \quad i = 1,\dots,m,$$

and the non-leader initial distribution is

$$g \in C\big([-\tau_2,0];\mathcal{P}_\infty(\mathbb{R}^d)\big).$$

Here $\mathcal{P}_\infty(\mathbb{R}^d)$ denotes the space of all probability measures on $\mathbb{R}^d$ with bounded support.

Case (ii): infinite population limit for both leaders and non-leaders: *The initial densities satisfy*

$$f \in C\big([-\tau_1,0];\mathcal{P}_\infty(\mathbb{R}^d)\big) \quad \text{and} \quad g \in C\big([-\tau_2,0];\mathcal{P}_\infty(\mathbb{R}^d)\big).$$

Then, for any finite time $T > 0$, the corresponding mean-field model admits a unique solution of equations (6.0.6) *or* (6.0.8) *on the interval* $[0, T)$ *with the following regularity properties:*

- *In Case (i), the leader trajectories $\bar{y}_i(t)$ belong to $C^1([0,T))$, and the non-leader distribution ν belongs to $C([0,T);\mathcal{P}_\infty(\mathbb{R}^d))$, with uniformly compact support. Moreover, the solution satisfies the relation:*
$$\nu_t = X(t;\cdot)\#\nu_0, \tag{6.0.13}$$
where $X(t;\cdot)$ is the flow map generated by v_t^m.

- *In Case (ii), the measure-valued solutions $\bar{\mu}(t)$ and $\bar{\nu}(t)$ belong to $C([0,T);\mathcal{P}_\infty(\mathbb{R}^d))$, with uniformly compact support, and the solutions satisfy*
$$\bar{\mu}_t = X(t;\cdot)\#\bar{\mu}_0, \quad \bar{\nu}_t = Z(t;\cdot)\#\bar{\nu}_0,$$
where X and Z are the flow maps associated with $\bar{u}_t$ and $\bar{v}_t$, respectively.

Moreover, denoting by $d(t)$ the global diameter of the solution (i.e., $d^\nu(t)$ in Case (i) and $d^{\mu,\nu}(t)$ in Case (ii)) and by D_0 the corresponding initial discrepancy (i.e., D_0^ν in Case (i) and $D_0^{\mu,\nu}$ in Case (ii)), there exists a constant $\gamma > 0$ such that
$$d(t) \le e^{-\gamma(t-2\tau)}\, D_0, \quad \forall\, t \ge 0.$$

Theorem 6.0.5 addresses both the well-posedness and large-time behavior of the mean-field systems derived from the interacting particle dynamics. The proof proceeds in two steps. First, we establish the existence and uniqueness of measure-valued solutions by constructing them as push-forwards of initial measures under characteristic flows. The well-posedness is shown in the space of probability measures endowed with the p-Wasserstein distance. A key ingredient is a Lipschitz-type stability estimate, derived using optimal transport techniques, which ensures continuous dependence on the initial data. Second, to analyze the large-time behavior, we combine the exponential decay estimate for the particle system (Theorem 6.0.2) with a quantitative mean-field limit argument. In both regimes, (i) with finitely many leaders and infinitely many followers, and (ii) with both populations tending to infinity, we prove that the macroscopic dynamics inherit the consensus property from their particle counterparts. This two-step approach highlights the robustness of consensus formation under time delays and scaling limits.

6.1 Exponential consensus in the time-delayed particle system

In this section, we establish exponential convergence to consensus for the delayed leader-follower system defined in (6.0.1). Our analysis proceeds by constructing suitable upper and lower bounds for directional components of the trajectories and showing that the overall diameter contracts over time. We begin with a preliminary lemma that ensures directional components of agent positions remain uniformly bounded over delayed intervals.

For convenience, we set the uniform bound on interaction strengths:
$$K := \max\{\|\psi\|_{L^\infty}, \|\phi\|_{L^\infty}, \|\rho\|_{L^\infty}\}. \tag{6.1.14}$$

Lemma 6.1.1. *Let $\{x_i(t)\}_{i=1}^N$ and $\{y_j(t)\}_{j=1}^m$ be the solution to* (6.0.1) *with the initial conditions given by* (6.0.3)–(6.0.4)*. For any vector $v \in \mathbb{R}^d$ and time $T \ge 0$, define the quantities*
$$m_T := \min\left\{ \min_{j=1,\dots,N} \min_{s\in[T-\tau,T]} \langle x_j(s), v\rangle, \min_{j=1,\dots,m} \min_{s\in[T-\tau,T]} \langle y_j(s), v\rangle \right\},$$

and

$$M_T := \max\left\{\max_{j=1,\dots,N}\max_{s\in[T-\tau,T]}\langle x_j(s),v\rangle,\ \max_{j=1,\dots,m}\max_{s\in[T-\tau,T]}\langle y_j(s),v\rangle\right\}.$$

Then, for all $t \geq T-\tau$*:*

$$m_T \leq \langle x_i(t),v\rangle \leq M_T, \quad i=1,\dots,N, \tag{6.1.15}$$

and

$$m_T \leq \langle y_i(t),v\rangle \leq M_T, \quad i=1,\dots,m. \tag{6.1.16}$$

Proof. We prove (6.1.15); the proof of (6.1.16) follows analogously. Fix $v\in\mathbb{R}^d$ and $T\geq 0$. By definition, the inequalities in (6.1.15) are trivially satisfied for $t\in[T-\tau,T]$. For $t>T$, let $\epsilon>0$ be arbitrary and define the set

$$\mathcal{S}_\epsilon := \left\{t>T:\ \max_{j=1,\dots,N}\langle x_j(s),v\rangle < M_T+\epsilon,\ s\in[T,t)\right\}.$$

Since the trajectories are continuous, $\mathcal{S}_\epsilon$ is nonempty. Denote

$$S := \sup\mathcal{S}_\epsilon.$$

We now claim that $S=+\infty$. Assume, by contradiction, that $S<+\infty$. Then, for all $s\in(T,S)$ we have

$$\max_{j=1,\dots,N}\langle x_j(s),v\rangle < M_T+\epsilon,$$

and, by continuity,

$$\lim_{s\to S^-}\max_{j=1,\dots,N}\langle x_j(s),v\rangle = M_T+\epsilon.$$

Now, take any $t\in(T,S)$. Using the second equation in (6.0.1) and noting that $t-\tau_1,\ t-\tau_2\in(T-\tau,S)$, we differentiate to obtain

$$\begin{aligned}\frac{d}{dt}\langle x_i(t),v\rangle &= \frac{1}{N}\sum_{j=1}^N \phi_{ij}^{\tau_2}(t)\langle x_j(t-\tau_2)-x_i(t),v\rangle + \frac{1}{m}\sum_{j=1}^m \rho_{ij}^{\tau_1}(t)\langle y_j(t-\tau_1)-x_i(t),v\rangle\\ &\leq \frac{1}{N}\sum_{j=1}^N \phi_{ij}^{\tau_2}(t)\big(M_T+\epsilon-\langle x_i(t),v\rangle\big) + \frac{1}{m}\sum_{j=1}^m \rho_{ij}^{\tau_1}(t)\big(M_T+\epsilon-\langle x_i(t),v\rangle\big)\\ &\leq 2K\big(M_T+\epsilon-\langle x_i(t),v\rangle\big),\end{aligned}$$

where we used the bound $\phi_{ij}^{\tau_2}(t),\ \rho_{ij}^{\tau_1}(t)\leq K$. Then,

$$\frac{d}{dt}\langle x_i(t),v\rangle \leq 2K(M_T+\epsilon) - 2K\langle x_i(t),v\rangle,$$

and applying Grönwall's inequality on (T,t) yields

$$\langle x_i(t),v\rangle \leq e^{-2K(t-T)}\langle x_i(T),v\rangle + (M_T+\epsilon)\left(1-e^{-2K(t-T)}\right).$$

Since $\langle x_i(T),v\rangle\leq M_T$, by the definition of $\mathcal{S}_\epsilon$, we find

$$\max_{i=1,\dots,N}\langle x_i(t),v\rangle \leq M_T+\epsilon-\epsilon e^{-2K(S-T)}, \quad \forall t\in(T,S).$$

Taking the limit as $t \to S^-$, we get

$$\lim_{t\to S^-} \max_{i=1,\dots,N} \langle x_i(t), v\rangle \le M_T + \epsilon - \epsilon e^{-2K(S-T)} < M_T + \epsilon,$$

which contradicts the earlier limit. Therefore, we must have $S = +\infty$, and consequently,

$$\max_{i=1,\dots,N} \langle x_i(t), v\rangle < M_T + \epsilon, \quad \forall\, t \ge T.$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\max_{i=1,\dots,N} \langle x_i(t), v\rangle \le M_T, \quad \forall\, t \ge T.$$

In particular,

$$\langle x_i(t), v\rangle \le M_T, \quad \forall\, t \ge T,\ i = 1,\dots,N. \tag{6.1.17}$$

Now, we apply (6.1.17) with a vector $-v \in \mathbb{R}^d$ to get

$$\begin{aligned}
-\langle x_i(t), v\rangle = \langle x_i(t), -v\rangle &\le \max_{j=1,\dots,N} \max_{s\in[T-\tau,T]} \langle x_j(s), -v\rangle \\
&\le -\min_{j=1,\dots,N} \min_{s\in[T-\tau,T]} \langle x_j(s), v\rangle = -m_T.
\end{aligned}$$

Then, we have the second inequality

$$\langle x_i(t), v\rangle \ge m_T, \quad \forall\, t \ge T,\ i = 1,\dots,N.$$

Thus, (6.1.15) is established. □

6.1.1 Uniform boundedness and influence positivity

Building on Lemma 6.1.1, we now establish uniform bounds on the pairwise distances between agents' opinions. This is key to showing that the diameter of the system does not increase over time and will ultimately decay exponentially.

Lemma 6.1.2. *For each $n \in \mathbb{N}_0$, the following estimates hold:*

$$|x_i(s) - x_j(t)| \le D_n, \quad \forall\, s,t \ge n\tau - \tau,\ i,j = 1,...,N, \tag{6.1.18}$$

$$|y_i(s) - y_j(t)| \le D_n, \quad \forall\, s,t \ge n\tau - \tau,\ i,j = 1,\dots,m, \tag{6.1.19}$$

and

$$|y_i(s) - x_j(t)| \le D_n, \quad \forall\ s,t \ge n\tau - \tau,\ i = 1,\dots,m,\ j = 1,\dots,N. \tag{6.1.20}$$

Remark 6.1.3. *An immediate consequence of Lemma 6.1.2 is that the sequence $\{D_n\}_{n\in\mathbb{N}_0}$ is non-increasing:*

$$D_{n+1} \le D_n, \quad \text{for all } n \in \mathbb{N}_0.$$

Then, in particular,

$$|x_i(s) - x_j(t)| \le D_0, \quad \forall\ i,j = 1,...,N,$$

$$|y_i(s) - y_j(t)| \le D_0, \quad \forall\ i,j = 1,...,m,$$

and

$$|y_i(s) - x_j(t)| \le D_0, \quad \forall\ i = 1,...,m,\ j = 1,\dots,N,$$

for all $s,t \ge -\tau$.

Proof of Lemma 6.1.2. We present the proof of (6.1.18); the proofs of (6.1.19) and (6.1.20) are analogous. Fix $s, t \geq -\tau$ and suppose that $|x_i(s) - x_j(t)| > 0$ (the case when this difference is zero is trivial). Define the vector

$$v := x_i(s) - x_j(t).$$

By applying Lemma 6.1.1 and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|x_i(s) - x_j(t)|^2 &= \langle x_i(s) - x_j(t), v\rangle \\
&\leq \max\Big\{ \max_{i=1,\dots,m} \max_{s\in[n\tau-\tau,n\tau]} \langle y_i(s), v\rangle, \max_{i=1,\dots,N} \max_{s\in[n\tau-\tau,n\tau]} \langle x_i(s), v\rangle \Big\} \\
&\qquad - \min\Big\{ \min_{j=1,\dots,m} \min_{t\in[n\tau-\tau,n\tau]} \langle y_j(t), v\rangle, \min_{j=1,\dots,N} \min_{t\in[n\tau-\tau,n\tau]} \langle x_j(t), v\rangle \Big\} \\
&\leq \max\Big\{ \max_{i,j=1,\dots,m} \max_{s,t\in[n\tau-\tau,n\tau]} \langle y_i(s) - y_j(t), v\rangle, \max_{i,j=1,\dots,N} \max_{s,t\in[n\tau-\tau,n\tau]} \langle x_i(s) - x_j(t), v\rangle, \\
&\qquad \max_{i=1,\dots,m} \max_{j=1,\dots,N} \max_{s,t\in[n\tau-\tau,n\tau]} \langle y_i(s) - x_j(t), v\rangle \Big\} \\
&\leq \max\Big\{ \max_{i,j=1,\dots,m} \max_{s,t\in[n\tau-\tau,n\tau]} |y_i(s) - y_j(t)| \cdot |v|, \max_{i,j=1,\dots,N} \max_{s,t\in[n\tau-\tau,n\tau]} |x_i(s) - x_j(t)| \cdot |v|, \\
&\qquad \max_{i=1,\dots,m} \max_{j=1,\dots,N} \max_{s,t\in[n\tau-\tau,n\tau]} |y_i(s) - x_j(t)| \cdot |v|\rangle \Big\} \\
&\leq D_n^2.
\end{aligned}$$

This completes the proof. ☐

We next show that the agents' trajectories remain uniformly bounded in time.

Lemma 6.1.4. *For all $i = 1, ..., N$ and $j = 1, \dots, m$, the solutions of* (6.0.1) *satisfy*

$$|x_i(t)| \leq C_0 \quad \textit{and} \quad |y_i(t)| \leq C_0, \quad \forall\, t \geq 0,$$

where

$$C_0 := \max_{s\in[-\tau,0]} \left\{ \max_{i=1,\dots,N} |x_i(s)|, \max_{i=1,\dots,m} |y_i(s)| \right\}.$$

Proof. We prove the bound for $x_i(t)$; the corresponding estimate for $y_j(t)$ follows similarly. Fix $i \in \{1, \dots, N\}$ and $t \geq 0$. If $|x_i(t)| = 0$, the bound is trivial. Otherwise, set

$$v := x_i(t).$$

Then, using Lemma 6.1.1, we obtain

$$\begin{aligned}
|x_i(t)|^2 &= \langle x_i(t), v\rangle \\
&\leq \max\Big\{ \max_{j=1,\dots,m} \max_{s\in[-\tau,0]} \langle y_j(s), v\rangle, \max_{j=1,\dots,N} \max_{s\in[-\tau,0]} \langle x_j(s), v\rangle \Big\} \\
&\leq \max\Big\{ \max_{j=1,\dots,m} \max_{s\in[-\tau,0]} |y_j(s)| \cdot |v|, \max_{j=1,\dots,N} \max_{s\in[-\tau,0]} |x_j(s)| \cdot |v| \Big\} \\
&\leq C_0^2.
\end{aligned}$$

This completes the proof. ☐

From Lemma 6.1.4 we immediately obtain a useful lower bound for the influence functions. This bound is crucial, as it ensures that the interactions among the agents remain uniformly positive throughout the evolution of the system.

Remark 6.1.5. *Since the influence functions are continuous, we deduce that*

$$\begin{aligned}\psi(y_i(t), y_j(t-\tau_1)) &\geq \psi_0 := \min_{|z_1|,|z_2|\leq C_0} \psi(z_1, z_2) > 0,\\ \phi(x_i(t), x_j(t-\tau_2)) &\geq \phi_0 := \min_{|z_1|,|z_2|\leq C_0} \psi(z_1, z_2) > 0,\\ \rho(x_i(t), y_j(t-\tau_1)) &\geq \rho_0 := \min_{|z_1|,|z_2|\leq C_0} \rho(z_1, z_2) > 0,\end{aligned}$$

for each $t \geq 0$. This positivity is crucial since it prevents the influence terms from degenerating, thereby ensuring effective information exchange across the network.

6.1.2 Directional contraction estimates

With these lower bounds in hand, we now turn our attention to establishing contraction estimates that describe how the differences between agent states decay over time. These estimates will serve as the cornerstone of the consensus result.

Lemma 6.1.6. *For all unit vector $v \in \mathbb{R}^d$ and for every $n \in \mathbb{N}_0$, the following inequalities hold:*

$$\langle x_i(t)-x_j(t), v\rangle \leq e^{-2K(t-t_0)}\langle x_i(t_0)-x_j(t_0), v\rangle+(1-e^{-2K(t-t_0)})D_n, \quad \forall\, i,j = 1,\dots,N, \tag{6.1.21}$$

$$\langle y_i(t)-y_j(t), v\rangle \leq e^{-2K(t-t_0)}\langle y_i(t_0)-y_j(t_0), v\rangle+(1-e^{-2K(t-t_0)})D_n, \quad \forall\, i,j = 1,\dots,m, \tag{6.1.22}$$

and, $\forall\, i = 1,\dots,N,\ j = 1,\dots,m$,

$$\langle x_i(t) - y_j(t), v\rangle \leq e^{-2K(t-t_0)}\langle x_i(t_0) - y_j(t_0), v\rangle + (1 - e^{-2K(t-t_0)})D_n, \tag{6.1.23}$$

for all $t \geq t_0 \geq n\tau$.

Proof. We divide the proof into two steps. In the first step, we obtain contraction estimates for the differences among agents within the same group (both non-leaders and leaders), and in the second step, we treat the mixed case involving a non-leader and a leader.

Step 1. We first derive the contraction estimate for the non-leader agents. Fix a unit vector $v \in \mathbb{R}^d$ and a given $n \in \mathbb{N}_0$. To quantify the maximal and minimal projection values along v over the time interval $[n\tau - \tau,\, n\tau]$, define

$$M_n := \max_{s\in[n\tau-\tau,n\tau]} \left\{ \max_{j=1,\dots,N} \langle x_j(s), v\rangle,\ \max_{j=1,\dots,m} \langle y_j(s), v\rangle \right\},$$

and

$$m_n := \min_{s\in[n\tau-\tau,n\tau]} \left\{ \min_{j=1,\dots,N} \langle x_j(s), v\rangle,\ \min_{j=1,\dots,m} \langle y_j(s), v\rangle \right\}.$$

It is clear that $M_n - m_n \leq D_n$. Now, fix an index $i \in \{1,\dots,N\}$ and consider $t \geq t_0 \geq n\tau$.

By the definition of system (6.0.1) and applying Lemma 6.1.1, we have

$$\begin{aligned}\frac{d}{dt}\langle x_i(t), v\rangle &= \frac{1}{N}\sum_{j=1}^{N}\phi_{ij}^{\tau_2}(t)\langle x_j(t-\tau_2) - x_i(t), v\rangle + \frac{1}{m}\sum_{j=1}^{m}\rho_{ij}^{\tau_1}(t)\langle y_j(t-\tau_1) - x_i(t), v\rangle \\ &\le \frac{1}{N}\sum_{j=1}^{N}\phi_{ij}^{\tau_2}(t)\big(M_n - \langle x_i(t), v\rangle\big) + \frac{1}{m}\sum_{j=1}^{m}\rho_{ij}^{\tau_1}(t)\big(M_n - \langle x_i(t), v\rangle\big) \\ &\le 2K\big(M_n - \langle x_i(t), v\rangle\big),\end{aligned}$$

where we used that $\phi_{ij}^{\tau_2}(t)$ and $\rho_{ij}^{\tau_1}(t)$ are bounded by K, and that $t-\tau_1,\, t-\tau_2 \ge n\tau - \tau$. By applying the Grönwall's lemma, we find

$$\langle x_i(t), v\rangle \le e^{-2K(t-t_0)}\langle x_i(t_0), v\rangle + (1 - e^{-2K(t-t_0)})M_n. \tag{6.1.24}$$

Similarly, for any $j \in \{1, \dots, N\}$ and $t \ge t_0 \ge n\tau$, we derive the lower bound

$$\langle x_j(t), v\rangle \ge e^{-2K(t-t_0)}\langle x_j(t_0), v\rangle + (1 - e^{-2K(t-t_0)})m_n. \tag{6.1.25}$$

Subtracting (6.1.25) from (6.1.24) yields

$$\begin{aligned}\langle x_i(t) - x_j(t), v\rangle &\le e^{-2K(t-t_0)}\langle x_i(t_0) - x_j(t_0), v\rangle + (1 - e^{-2K(t-t_0)})(M_n - m_n) \\ &\le e^{-2K(t-t_0)}\langle x_i(t_0) - x_j(t_0), v\rangle + (1 - e^{-2K(t-t_0)})D_n.\end{aligned}$$

Thus, we have obtained the contraction estimate for the non-leader agents as stated in (6.1.21). By an analogous argument applied to the leader dynamics (using the first equation of (6.0.1) and the corresponding influence function ψ), one obtains a similar contraction estimate (6.1.22) for the leader agents.

Step 2. We now consider the mixed case involving a non-leader and a leader. For a leader $y_j(t)$, we use the first equation in system (6.0.1) and apply Lemma 6.1.1 to obtain

$$\begin{aligned}\frac{d}{dt}\langle y_j(t), v\rangle &= \frac{1}{m}\sum_{l=1}^{m}\psi_{jl}^{\tau_1}(t)\langle y_l(t-\tau_1) - y_j(t), v\rangle \\ &\ge \frac{1}{m}\sum_{l=1}^{m}\psi_{jl}^{\tau_1}(t)(m_n - \langle y_j(t), v\rangle) \\ &\ge K(m_n - \langle y_j(t), v\rangle) \\ &\ge 2K(m_n - \langle y_j(t), v\rangle),\end{aligned}$$

where we used $m_n - \langle y_j(t), v\rangle \le 0$ for all $j = 1, \dots, m$ and for all $t \ge n\tau$. Applying Grönwall's inequality then gives

$$\langle y_j(t), v\rangle \ge e^{-2K(t-t_0)}\langle y_j(t_0), v\rangle + (1 - e^{-2K(t-t_0)})m_n. \tag{6.1.26}$$

Subtracting (6.1.26) from the bound for $\langle x_i(t), v\rangle$ in (6.1.24) yields

$$\begin{aligned}\langle x_i(t) - y_j(t), v\rangle &\le e^{-2K(t-t_0)}\langle x_i(t_0) - y_j(t_0), v\rangle + (1 - e^{-2K(t-t_0)})(M_n - m_n) \\ &\le e^{-2K(t-t_0)}\langle x_i(t_0) - y_j(t_0), v\rangle + (1 - e^{-2K(t-t_0)})D_n.\end{aligned}$$

This completes the proof of inequality (6.1.23). □

Remark 6.1.7. *It is worth noting that for pairs of leaders, one may also derive a sharper contraction estimate that depends only on the leader group:*

$$\langle y_i(t)-y_j(t),v\rangle \le e^{-K(t-t_0)}\langle y_i(t_0)-y_j(t_0),v\rangle+(1-e^{-K(t-t_0)})\max_{h,k=1,\dots,m}\max_{r,s\in[n\tau-\tau,n\tau]}|y_h(r)-y_k(s)|,$$

for all $i,j=1,\dots,m$. However, for the overall consensus result, it is essential to work with unified estimates (as in Lemma 6.1.6) that simultaneously control all interactions in the mixed leader-follower system.

6.1.3 Recursive control of diameter

To estimate the evolution of the system diameter in discrete time, we now investigate how the maximum distance between any two agents at time $n\tau$ relates to earlier diameters. The following lemma provides a key step toward establishing exponential contraction of the global diameter.

Lemma 6.1.8. *There exists a constant $C\in(0,1)$ such that*

$$d(n\tau)\le CD_{n-2},$$

for all $n\ge 2$.

Proof. We prove the lemma by considering three distinct cases, each corresponding to a different configuration in which the diameter $d(n\tau)$ is achieved.

Case 1. Assume that

$$d(n\tau)=|x_i(n\tau)-x_j(n\tau)|$$

for some $i,j=1,\dots,N$. Since the case $|x_i(n\tau)-x_j(n\tau)|=0$ is trivial, we suppose $|x_i(n\tau)-x_j(n\tau)|>0$. In this case, we first normalize the difference by setting

$$v:=\frac{x_i(n\tau)-x_j(n\tau)}{|x_i(n\tau)-x_j(n\tau)|}.$$

To capture the spread of agent opinions over a preceding time interval, we now introduce the quantities

$$M_{n-1}:=\max_{s\in[(n-2)\tau,(n-1)\tau]}\left\{\max_{j=1,\dots,N}\langle x_j(s),v\rangle,\ \max_{j=1,\dots,m}\langle y_j(s),v\rangle\right\},$$

and

$$m_{n-1}:=\min_{s\in[(n-2)\tau,(n-1)\tau]}\left\{\min_{j=1,\dots,N}\langle x_j(s),v\rangle,\ \min_{j=1,\dots,m}\langle y_j(s),v\rangle\right\}.$$

It is clear that

$$M_{n-1}-m_{n-1}\le D_{n-1}.$$

Next, we analyze the evolution of the projection differences along v during the time interval $t\in[(n-1)\tau,n\tau]$. Using the system dynamics, we write the time derivative of the projection difference between agents x_i and x_j as follows:

$$\begin{aligned}
&\frac{d}{dt}\langle x_i(t)-x_j(t),v\rangle\\
&=\frac{1}{N}\sum_{l=1}^{N}\phi_{il}^{\tau_2}(t)\langle x_l(t-\tau_1)-x_i(t),v\rangle+\frac{1}{m}\sum_{l=1}^{m}\rho_{il}^{\tau_1}(t)\langle y_l(t-\tau_2)-x_i(t),v\rangle\\
&\quad-\frac{1}{N}\sum_{l=1}^{N}\phi_{jl}^{\tau_2}(t)\langle x_l(t-\tau_1)-x_j(t),v\rangle-\frac{1}{m}\sum_{l=1}^{m}\rho_{jl}^{\tau_1}(t)\langle y_l(t-\tau_2)-x_j(t),v\rangle.
\end{aligned}$$

We now regroup the terms by introducing the shift M_{n-1} (which represents an upper bound on the projections during $[(n-2)\tau,(n-1)\tau]$). In particular, we rewrite the above derivative as

$$
\begin{aligned}
&\frac{d}{dt}\langle x_i(t)-x_j(t),v\rangle\\
&=\frac{1}{N}\sum_{l=1}^{N}\phi_{il}^{\tau_2}(t)(\langle x_l(t-\tau_1),v\rangle-M_{n-1})+\frac{1}{N}\sum_{l=1}^{N}\phi_{il}^{\tau_2}(t)(M_{n-1}-\langle x_i(t),v\rangle)\\
&\quad+\frac{1}{m}\sum_{l=1}^{m}\rho_{il}^{\tau_1}(t)(\langle y_l(t-\tau_2),v\rangle-M_{n-1})+\frac{1}{m}\sum_{l=1}^{m}\rho_{il}^{\tau_1}(t)(M_{n-1}-\langle x_i(t),v\rangle)\\
&\quad+\frac{1}{N}\sum_{l=1}^{N}\phi_{jl}^{\tau_2}(t)(m_{n-1}-\langle x_l(t-\tau_1),v\rangle)+\frac{1}{N}\sum_{l=1}^{N}\phi_{jl}^{\tau_2}(t)(\langle x_j(t),v\rangle-m_{n-1})\\
&\quad+\frac{1}{m}\sum_{l=1}^{m}\rho_{jl}^{\tau_1}(t)(m_{n-1}-\langle y_l(t-\tau_2),v\rangle)+\frac{1}{m}\sum_{l=1}^{m}\rho_{jl}^{\tau_1}(t)(\langle x_j(t),v\rangle-m_{n-1}).
\end{aligned}
\tag{6.1.27}
$$

At this point, it is convenient to introduce two auxiliary sums, S_1 and S_2, corresponding respectively to the contributions involving the upper bound M_{n-1} and the lower bound m_{n-1}. Using Remark 6.1.5 and the fact that the weights are bounded by K, we obtain

$$
\begin{aligned}
S_1:=&\frac{1}{N}\sum_{l=1}^{N}\phi_{il}^{\tau_2}(t)(\langle x_l(t-\tau_1),v\rangle-M_{n-1})+\frac{1}{N}\sum_{l=1}^{N}\phi_{il}^{\tau_2}(t)(M_{n-1}-\langle x_i(t),v\rangle)\\
&+\frac{1}{m}\sum_{l=1}^{m}\rho_{il}^{\tau_1}(t)(\langle y_l(t-\tau_2),v\rangle-M_{n-1})+\frac{1}{m}\sum_{l=1}^{m}\rho_{il}^{\tau_1}(t)(M_{n-1}-\langle x_i(t),v\rangle)\\
\le&\frac{\phi_0}{N}\sum_{l=1}^{N}(\langle x_l(t-\tau_1),v\rangle-M_{n-1})+\frac{\rho_0}{m}\sum_{l=1}^{m}(\langle y_l(t-\tau_2),v\rangle-M_{n-1})\\
&+2K(M_{n-1}-\langle x_i(t),v\rangle).
\end{aligned}
$$

Similarly, we define

$$
\begin{aligned}
S_2:=&\frac{1}{N}\sum_{l=1}^{N}\phi_{jl}^{\tau_2}(t)(m_{n-1}-\langle x_l(t-\tau_1),v\rangle)+\frac{1}{N}\sum_{l=1}^{N}\phi_{jl}^{\tau_2}(t)(\langle x_j(t),v\rangle-m_{n-1})\\
&+\frac{1}{m}\sum_{l=1}^{m}\rho_{jl}^{\tau_1}(t)(m_{n-1}-\langle y_l(t-\tau_2),v\rangle)+\frac{1}{m}\sum_{l=1}^{m}\rho_{jl}^{\tau_1}(t)(\langle x_j(t),v\rangle-m_{n-1})\\
\le&\frac{\phi_0}{N}\sum_{l=1}^{N}(m_{n-1}-\langle x_l(t-\tau_1),v\rangle)+\frac{\rho_0}{m}\sum_{l=1}^{m}(m_{n-1}-\langle y_l(t-\tau_2),v\rangle)\\
&+2K(\langle x_j(t),v\rangle-m_{n-1}),
\end{aligned}
$$

where we used the fact that, being $t\in[(n-1)\tau,n\tau]$, it holds that $t-\tau_1,t-\tau_2\in[(n-2)\tau,n\tau]$.

Combining the estimates from S_1 and S_2 in (6.1.27), we deduce that

$$\begin{aligned}\frac{d}{dt}\langle x_i(t) - x_j(t), v\rangle &\le 2K(M_{n-1} - m_{n-1}) - 2K\langle x_i(t) - x_j(t), v\rangle \\ &\quad + \frac{\phi_0}{N}\sum_{l=1}^{N}(\langle x_l(t-\tau_1), v\rangle - M_{n-1}) + \frac{\rho_0}{m}\sum_{l=1}^{m}(\langle y_l(t-\tau_2), v\rangle - M_{n-1}) \\ &\quad + \frac{\phi_0}{N}\sum_{l=1}^{N}(m_{n-1} - \langle x_l(t-\tau_1), v\rangle) + \frac{\rho_0}{m}\sum_{l=1}^{m}(m_{n-1} - \langle y_l(t-\tau_2), v\rangle) \\ &\le 2K(M_{n-1} - m_{n-1}) - 2K\langle x_i(t) - x_j(t), v\rangle + (\phi_0 + \rho_0)\,(-M_{n-1} + m_{n-1}).\end{aligned}$$

For notational simplicity, we set

$$\Lambda := \min\{\psi_0, \phi_0, \rho_0\}.$$

Then, the inequality simplifies to

$$\frac{d}{dt}\langle x_i(t) - x_j(t), v\rangle \le 2\,(K - \Lambda)\,(M_{n-1} - m_{n-1}) - 2K\langle x_i(t) - x_j(t), v\rangle.$$

Applying the Grönwall's lemma on the time interval $[(n-1)\tau, t]$ with $t \in [(n-1)\tau, n\tau]$, we find that

$$\begin{aligned}\langle x_i(t) - x_j(t), v\rangle &\le e^{-2K(t-n\tau+\tau)}\langle x_i(n\tau-\tau) - x_j(n\tau-\tau), v\rangle \\ &\quad + \left(1 - \frac{\Lambda}{K}\right)(M_{n-1} - m_{n-1})(1 - e^{-2K(t-n\tau+\tau)}).\end{aligned}$$

Since this is valid for all $t \in [(n-1)\tau, n\tau]$, taking $t = n\tau$, we obtain

$$\begin{aligned}&\langle x_i(n\tau) - x_j(n\tau), v\rangle \\ &\quad\le e^{-2K\tau}\langle x_i(n\tau-\tau) - x_j(n\tau-\tau), v\rangle + \left(1 - \frac{\Lambda}{K}\right)(M_{n-1} - m_{n-1})(1 - e^{-2K\tau}) \\ &\quad\le e^{-2K\tau}|x_i(n\tau-\tau) - x_j(n\tau-\tau)||v| + \left(1 - \frac{\Lambda}{K}\right)(M_{n-1} - m_{n-1})(1 - e^{-2K\tau}) \\ &\quad\le D_{n-1}\left[e^{-2K\tau} + \left(1 - \frac{\Lambda}{K}\right)(1 - e^{-2K\tau})\right] \\ &\quad\le D_{n-2}\left[1 - \frac{\Lambda}{K}(1 - e^{-2K\tau})\right],\end{aligned}$$

where we used Remark 6.1.3. Consequently, we deduce that

$$d(n\tau) \le D_{n-2}\left[1 - \frac{\Lambda}{K}(1 - e^{-2K\tau})\right].$$

Thus, we obtain the desired estimate for the case when the maximum diameter is determined by non-leader agents.

Case 2. Now, assume

$$d(n\tau) = |y_i(n\tau) - y_j(n\tau)|,$$

for some $i, j = 1, \dots, m$. As in Case 1, we begin by normalizing the difference. Define

$$v := \frac{y_i(n\tau) - y_j(n\tau)}{|y_i(n\tau) - y_j(n\tau)|}.$$

For $t \in [(n-1)\tau, n\tau]$, similarly as in Case 1, using the definition of the system, we write the time derivative of the projection difference between the leaders y_i and y_j as

$$\begin{aligned}
\frac{d}{dt}\langle y_i(t) - y_j(t), v\rangle &= \frac{1}{m}\sum_{l=1}^{m} \psi_{il}^{\tau_1}(t)\langle y_l(t-\tau_2) - y_i(t), v\rangle - \frac{1}{m}\sum_{l=1}^{m} \psi_{jl}^{\tau_1}(t)\langle y_l(t-\tau_2) - y_j(t), v\rangle \\
&= \frac{1}{m}\sum_{l=1}^{m} \psi_{il}^{\tau_1}(t)(\langle y_l(t-\tau_2), v\rangle - M_{n-1}) + \frac{1}{m}\sum_{l=1}^{m} \psi_{il}^{\tau_1}(t)(M_{n-1} - \langle y_i(t), v\rangle) \\
&\quad + \frac{1}{m}\sum_{l=1}^{m} \psi_{jl}^{\tau_1}(t)(m_{n-1} - \langle y_l(t-\tau_2), v\rangle) + \frac{1}{m}\sum_{l=1}^{m} \psi_{jl}^{\tau_1}(t)(\langle y_j(t), v\rangle - m_{n-1}) \\
&\le \frac{\psi_0}{m}\sum_{l=1}^{m}(\langle y_l(t-\tau_2), v\rangle - M_{n-1}) + K(M_{n-1} - \langle y_i(t), v\rangle) \\
&\quad + \frac{\psi_0}{m}\sum_{l=1}^{m}(m_{n-1} - \langle y_l(t-\tau_2), v\rangle) + K(\langle y_j(t), v\rangle - m_{n-1}).
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{d}{dt}\langle y_i(t) - y_j(t), v\rangle &\le K(M_{n-1} - m_{n-1}) - K\langle y_i(t) - y_j(t), v\rangle \\
&\quad + \frac{\psi_0}{m}\sum_{l=1}^{m}(\langle y_l(t-\tau_2), v\rangle - M_{n-1}) + \frac{\psi_0}{m}\sum_{l=1}^{m}(m_{n-1} - \langle y_l(t-\tau_2), v\rangle) \\
&\le K(M_{n-1} - m_{n-1}) - K\langle y_i(t) - y_j(t), v\rangle + \Lambda(-M_{n-1} + m_{n-1}) \\
&= (K - \Lambda)(M_{n-1} - m_{n-1}) - K\langle y_i(t) - y_j(t), v\rangle.
\end{aligned}$$

Applying the Grönwall's lemma over the interval $[(n-1)\tau, t]$ with $t \in [(n-1)\tau, n\tau]$, we find that

$$\begin{aligned}
\langle y_i(t) - y_j(t), v\rangle &\le e^{-K(t-n\tau+\tau)}\langle y_i(n\tau-\tau) - y_j(n\tau-\tau), v\rangle \\
&\quad + \left(1 - \frac{\Lambda}{K}\right)(M_{n-1} - m_{n-1})(1 - e^{-K(t-n\tau+\tau)}).
\end{aligned}$$

Taking $t = n\tau$, this simplifies to

$$\begin{aligned}
&\langle y_i(n\tau) - y_j(n\tau), v\rangle \\
&\quad \le e^{-K\tau}\langle y_i(n\tau-\tau) - y_j(n\tau-\tau), v\rangle + \left(1 - \frac{\Lambda}{K}\right)(M_{n-1} - m_{n-1})(1 - e^{-K\tau}) \\
&\quad \le e^{-K\tau}|y_i(n\tau-\tau) - y_j(n\tau-\tau)||v| + \left(1 - \frac{\Lambda}{K}\right)(M_{n-1} - m_{n-1})(1 - e^{-K\tau}) \\
&\quad \le D_{n-1}\left[e^{-K\tau} + 1 - \frac{\Lambda}{K}(1 - e^{-K\tau})\right] \\
&\quad \le D_{n-2}\left[1 - \frac{\Lambda}{K}(1 - e^{-K\tau})\right].
\end{aligned}$$

Thus, we conclude that,

$$d(n\tau) \le D_{n-2}\left[1 - \frac{\Lambda}{K}(1 - e^{-K\tau})\right].$$

This completes the derivation for Case 2.

Case 3. Now, assume that there exist indices $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, m\}$

$$d(n\tau) = |x_i(n\tau) - y_j(n\tau)|.$$

In this mixed case, the maximum diameter is achieved by a non-leader and a leader. As before, we begin by normalizing the difference; define

$$v := \frac{x_i(n\tau) - y_j(n\tau)}{|x_i(n\tau) - y_j(n\tau)|}.$$

Then, the distance can be expressed in the direction v as

$$|x_i(n\tau) - y_j(n\tau)| = \langle x_i(n\tau) - y_j(n\tau), v\rangle.$$

For $t \in [(n-1)\tau, n\tau]$, by using almost the same arguments used in the previous cases, we deduce

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t) - y_j(t), v\rangle &\leq 2K(M_{n-1} - m_{n-1}) - 2K\langle x_i(t) - y_j(t), v\rangle \\
&\quad + \frac{\Lambda}{N}\sum_{l=1}^{N}(\langle x_l(t-\tau_2), v\rangle - M_{n-1}) + \frac{\Lambda}{m}\sum_{l=1}^{m}(\langle y_l(t-\tau_1), v\rangle - M_{n-1}) \\
&\quad + \frac{\Lambda}{m}\sum_{l=1}^{m}(m_{n-1} - \langle y_l(t-\tau_1), v\rangle) \\
&\leq (2K - \Lambda)(M_{n-1} - m_{n-1}) - 2K\langle x_i(t) - y_j(t), v\rangle,
\end{aligned}$$

where we used that for all $l = 1, \dots, N$,

$$\langle x_l(t-\tau_2), v\rangle - M_{n-1} \leq 0.$$

Again, analogously, we obtain

$$\begin{aligned}
d(n\tau) &\leq e^{-2K\tau}\langle x_i(n\tau - \tau) - y_j(n\tau - \tau), v\rangle + \left(1 - \frac{\Lambda}{2K}\right)(M_{n-1} - m_{n-1})(1 - e^{-2K\tau}) \\
&\leq e^{-2K\tau}|x_i(n\tau - \tau) - y_j(n\tau - \tau)||v| + \left(1 - \frac{\Lambda}{2K}\right)(M_{n-1} - m_{n-1})(1 - e^{-2K\tau}) \\
&\leq D_{n-1}\left[e^{-2K\tau} + \left(1 - \frac{\Lambda}{2K}\right)(1 - e^{-2K\tau})\right] \\
&\leq D_{n-2}\left[1 - \frac{\Lambda}{2K}(1 - e^{-2K\tau})\right].
\end{aligned}$$

Finally, to complete the proof of the lemma, we define

$$C := 1 - \frac{\Lambda}{2K}\left(1 - e^{-K\tau}\right), \tag{6.1.28}$$

which yields the desired estimate. □

Remark 6.1.9. *From the proof of Lemma 6.1.8, it becomes evident that the normalization chosen for the weight functions in* (6.0.2) *plays a crucial role in the dynamics. Roughly speaking, this normalization ensures that the influence exerted by the leaders constitutes half of the total influence on any non-leader. This balanced distribution of influence is essential for deriving the homogeneous contraction estimates that lead to consensus.*

6.1.4 Exponential consensus: Proof of Theorem 6.0.2

Now, we prove the consensus result stated in Theorem 6.0.2.

Let $(x_i(t), y_j(t))$, with $i = 1, \ldots, N$ and $j = 1, \ldots, m$, be the solution of (6.0.1) with the initial conditions (6.0.3) and (6.0.4). Our goal is to show that the diameter of the system decays exponentially. To achieve this, we first claim that there exists a constant $\tilde{C} \in (0,1)$ such that

$$D_{n+1} \leq \tilde{C} D_{n-2}, \quad \forall\, n \geq 2. \tag{6.1.29}$$

We start by observing that from Lemma 6.1.6 the following estimate can be deduced:

$$D_{n+1} \leq e^{-2K\tau} d(n\tau) + (1 - e^{-2K\tau}) D_n.$$

To illustrate this, assume that for some $n \in \mathbb{N}_0$, there exist $s, t \in [n\tau, n\tau + \tau]$ and indices $i, j \in \{1, \ldots, N\}$ such that

$$D_{n+1} = |x_i(s) - x_j(t)|.$$

Assume that $|x_i(s) - x_j(t)| > 0$, since the case $|x_i(s) - x_j(t)| = 0$ is trivial. Define the unit vector

$$v := \frac{x_i(s) - x_j(t)}{|x_i(s) - x_j(t)|}.$$

Then, by definition,

$$D_{n+1} = \langle x_i(s) - x_j(t), v \rangle.$$

Using (6.1.24) with $t_0 = n\tau$, we obtain

$$\begin{aligned} \langle x_i(s), v \rangle &\leq e^{-2K(s-n\tau)} \langle x_i(n\tau), v \rangle + (1 - e^{-2K(s-n\tau)}) M_n \\ &= e^{-2K(s-n\tau)} (\langle x_i(n\tau), v \rangle - M_n) + M_n \\ &\leq e^{-2K\tau} \langle x_i(n\tau), v \rangle + (1 - e^{-2K\tau}) M_n. \end{aligned} \tag{6.1.30}$$

Analogously, using (6.1.25) we have

$$\langle x_j(t), v \rangle \geq e^{-2K\tau} \langle x_j(n\tau), v \rangle + (1 - e^{-2K\tau}) m_n. \tag{6.1.31}$$

Subtracting (6.1.31) from (6.1.30) yields

$$\begin{aligned} D_{n+1} &\leq e^{-2K\tau} \langle x_i(n\tau) - x_j(n\tau), v \rangle + (1 - e^{-2K\tau}) D_n \\ &\leq e^{-2K\tau} d(n\tau) + (1 - e^{-2K\tau}) D_n. \end{aligned}$$

A similar reasoning applies if D_{n+1} is defined by the other two possible forms (involving leader-leader or leader-non-leader differences).

From this and Lemma 6.1.8, we get that

$$\begin{aligned} D_{n+1} &\leq e^{-2K\tau} d(n\tau) + (1 - e^{-2K\tau}) D_n \\ &\leq e^{-2K\tau} C D_{n-2} + (1 - e^{-2K\tau}) D_n \\ &\leq e^{-2K\tau} C D_{n-2} + (1 - e^{-2K\tau}) D_{n-2} \\ &= \left(1 - e^{-2K\tau}(1 - C)\right) D_{n-2}, \end{aligned}$$

where C is defined in (6.1.28). Thus, setting

$$\tilde{C} := 1 - e^{-2K\tau} \frac{\Lambda}{2K} (1 - e^{-K\tau}),$$

we obtain the claim (6.1.29).

This recursive inequality implies that

$$D_{3n} \le \tilde{C}^n D_0, \quad \forall n \ge 1. \tag{6.1.32}$$

Since $\tilde{C} \in (0,1)$, we can rewrite (6.1.32) as

$$D_{3n} \le e^{-3n\gamma\tau} D_0,$$

where

$$\gamma := \frac{1}{3\tau} \ln\left(\frac{1}{\tilde{C}}\right) = -\frac{1}{3\tau} \ln\left(1 - e^{-2K\tau}\frac{\Lambda}{2K}(1 - e^{-K\tau})\right).$$

Finally, fix any time $t \ge 0$. Then there exists $n \in \mathbb{N}_0$ such that $t \in [3n\tau - \tau, 3n\tau + 2\tau]$. By Lemma 6.1.2 and the definition of the global diameter (6.0.5), we have

$$d(t) \le D_{3n} \le e^{-3n\gamma\tau} D_0.$$

Since $t \le 3n\tau + 2\tau$, it follows that

$$d(t) \le e^{-\gamma(t-2\tau)} D_0.$$

This completes the proof of Theorem 6.0.2.

6.2 Global existence of measure-valued solutions of the mean-field models

In this section, we establish the global-in-time existence and uniqueness of measure-valued solutions to the mean-field systems (6.0.6) and (6.0.8), which arise as formal limits of the particle system (6.0.1) when the number of agents tends to infinity.

We begin with the analysis of the first model (6.0.6), which describes a system with a finite number of leaders interacting with a continuum of followers.

6.2.1 Few leaders and many followers system

We consider the mean-field equation (6.0.6), which models the collective behavior of a large population of followers influenced by a finite number of leaders. The leaders follow prescribed trajectories $\{\bar{y}_j(t)\}_{j=1}^m$, while the followers evolve according to a transport equation driven by the interaction terms. The velocity field $v_t^m(x)$ is defined by the interaction kernel (6.0.7), which combines leader-follower and follower-follower interactions.

To ensure the well-posedness of the transport equation, we require that the influence functions ϕ and ρ satisfy the aforenoted regularity and boundedness conditions. We denoted by L_ϕ and L_ρ the Lipschitz constants of ϕ and ρ, respectively. We also write $\bar{\rho}_j^{\tau_1}(x) := \rho(x, \bar{y}_j(t-\tau_1))$ for the interaction term between the followers and the j-th leader.

We now prove that the velocity field $v_t^m(x)$ is globally Lipschitz and bounded under the assumption that the follower density ν_t has compact support.

Lemma 6.2.1. *Let $\nu_t \in C([0,T]; \mathcal{P}(\mathbb{R}^d))$ be a family of probability measures with compact support, i.e.,*

$$\textit{supp}\ \nu_t \subset B^d(0,R), \quad \forall t \in [0,T],$$

where $B^d(0,R)$ denotes the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^d$. Then the velocity field $v_t^m(x)$ defined by (6.0.7) satisfies the following properties:

(i) (Lipschitz continuity) There exists a constant $\tilde{K} > 0$ *such that*

$$|v_t^m(x) - v_t^m(\tilde{x})| \leq \tilde{K}|x - \tilde{x}|, \quad \forall\, x, \tilde{x} \in B^d(0, R),\ t \in [0, T]. \tag{6.2.33}$$

(ii) (Uniform boundedness) There exists a constant $\tilde{C} > 0$ *such that*

$$|v_t^m(x)| \leq \tilde{C}, \quad \forall\, x \in B^d(0, R),\ t \in [0, T]. \tag{6.2.34}$$

Proof. Fix $x, \tilde{x} \in B^d(0, R)$. We split the difference $|v_t^m(x) - v_t^m(\tilde{x})|$ into two parts:

$$\begin{aligned}
|v_t^m(x) - v_t^m(\tilde{x})| &\leq \Big| \int_{\mathbb{R}^d} \phi(x, y)(y - x)\nu_{t-\tau_2}(dy) - \int_{\mathbb{R}^d} \phi(\tilde{x}, y)(y - \tilde{x})\nu_{t-\tau_2}(dy) \Big| \\
&\quad + \Big| \frac{1}{m} \sum_{j=1}^m \bar{\rho}_j^{\tau_1}(x)(\bar{y}_j(t - \tau_1) - x) - \frac{1}{m} \sum_{j=1}^m \bar{\rho}_j^{\tau_1}(\tilde{x})(\bar{y}_j(t - \tau_1) - \tilde{x}) \Big| \\
&=: I + II.
\end{aligned}$$

Using the Lipschitz continuity of $\phi(x, y)$ and the compact support of the measure ν_t, we find that

$$\begin{aligned}
I &\leq \Big| \int_{\mathbb{R}^d} (\phi(x, y) - \phi(\tilde{x}, y))\, y\nu_{t-\tau_2}(dy) \Big| + \Big| \int_{\mathbb{R}^d} \phi(x, y)x\nu_{t-\tau_2}(dy) - \int_{\mathbb{R}^d} \phi(\tilde{x}, y)\tilde{x}\nu_{t-\tau_2}(dy) \Big| \\
&\leq RL_\phi|x - \tilde{x}| + \Big| \int_{\mathbb{R}^d} (\phi(x, y) - \phi(\tilde{x}, y))\, x\nu_{t-\tau_2}(dy) \Big| + \Big| \int_{\mathbb{R}^d} \phi(\tilde{x}, y)(x - \tilde{x})\nu_{t-\tau_2}(dy) \Big| \\
&\leq (K + 2RL_\phi)|x - \tilde{x}|.
\end{aligned}$$

Similarly, we estimate II as

$$\begin{aligned}
II &\leq \frac{1}{m} \sum_{j=1}^m |\bar{\rho}_j^{\tau_1}(x) - \bar{\rho}_j^{\tau_1}(\tilde{x})||\bar{y}_j(t - \tau_1)| + \frac{1}{m} \sum_{j=1}^m |\bar{\rho}_j^{\tau_1}(x) - \bar{\rho}_j^{\tau_1}(\tilde{x})||x| + \frac{1}{m} \sum_{j=1}^m |\bar{\rho}_j^{\tau_1}(\tilde{x})||x - \tilde{x}| \\
&\leq [L_\rho(C_0 + R) + K]|x - \tilde{x}|,
\end{aligned}$$

where $C_0 > 0$ is the bound on leader trajectories from Lemma 6.1.4. Combining the bounds yields (6.2.33) with $\tilde{K} := 2RL_\phi + 2K + L_\rho(C_0 + R)$.

To prove (6.2.34), we estimate directly:

$$\begin{aligned}
|v_t^m(x)| &\leq \Big| \int_{\mathbb{R}^d} \phi(x, y)(y - x)\nu_{t-\tau_2}(dy) \Big| + \frac{1}{m} \Big| \sum_{j=1}^m \bar{\rho}_j^{\tau_1}(t)(\bar{y}_j(t - \tau_1) - x) \Big| \\
&\leq 2KR + K(C_0 + R),
\end{aligned}$$

which gives (6.2.34) with $\tilde{C} := K(3R + C_0)$. □

Now, we are in a position to prove the global existence and uniqueness of solutions stated in Theorem 6.0.5 for the mean-field system (6.0.6).

Proof of Theorem 6.0.5: existence and uniqueness in Case (i). We begin by considering the ODE system describing the evolution of the leaders $\{y_j(t)\}_{j=1}^m$. Since the interaction functions are Lipschitz continuous and bounded, standard results from the theory of delay differential equations (see, e.g., [48, 49]) ensure the existence and uniqueness of solutions. Specifically, by applying the Banach fixed-point theorem on small time intervals and iterating the solution step-by-step, we can construct a unique global-in-time solution for the leader dynamics.

We now turn to the second component of the system (6.0.6), which is a continuity equation driven by a delayed, nonlocal velocity field. To obtain local-in-time existence and uniqueness of measure-valued solutions, we apply Lemma 6.2.1, which provides Lipschitz and boundedness estimates on the velocity field $v_t^m(x)$, together with [17, Theorem 3.10], which guarantees well-posedness under such conditions, provided that the solution remains compactly supported.

Thus, to extend this local-in-time solution to a global one, it is sufficient to control the growth of the support of ν_t. We do this by estimating the maximal spatial extension of the support. First, since the leaders' dynamics do not directly depend on the follower particles $\{x_i\}_{i=1}^N$, we focus on the leader dynamics and define the bound

$$C_0^y := \max_{s\in[-\tau,0]} \max_{j=1,\dots,m} |y_j(s)|.$$

By Lemma 6.1.4, we then have the uniform-in-time bound

$$|y_j(t)| \le C_0^y, \quad \text{for all } t \ge 0 \text{ and } j = 1, \dots, m.$$

Next, define the maximal radius of the support of the measure ν_t as

$$R_X(t) := \max\left\{ \max_{s\in[-\tau,t]} \sup_{x\in\overline{\text{supp } \nu_s}} |x|,\ C_0^y \right\}. \tag{6.2.35}$$

We now perform a continuity argument to control $R_X(t)$. Similarly to Lemma 6.1.1, for a fixed $\epsilon > 0$, we define a set

$$\mathcal{T}^\epsilon := \{t > 0 : R_X(s) < R_X(0) + \epsilon,\ \forall s \in [0, t)\}.$$

By continuity of trajectories, $\mathcal{T}^\epsilon$ is nonempty. Let $T_\epsilon := \sup \mathcal{T}^\epsilon$. Our goal is to show that $T_\epsilon \ge \tau^*$, where $\tau^* := \min\{\tau_1, \tau_2\}$. Suppose, for contradiction, that $T_\epsilon < \tau^*$. Then, we find

$$\lim_{t\to T_\epsilon^-} R_X(t) = R_X(0) + \epsilon \tag{6.2.36}$$

and

$$R_X(t) < R_X(0) + \epsilon, \quad \forall\ t < T_\epsilon$$

Consider the system of characteristics $X(t;x) : [0, T_\epsilon] \times \mathbb{R}^d \to \mathbb{R}^d$ associated to the continuity equation in (6.0.6), given by

$$\begin{cases} \frac{d}{dt}X(t;x) = v_t(X(t;x)), \\ X(0;x) = x \end{cases}$$

for $x \in \mathbb{R}^d$. Then, applying Lemma 6.2.1, this system admits a unique solution on the time interval $[0, T_\epsilon]$. Note that the measure-valued solution is transported by the characteristic flow, namely

$$\nu_t = X(t;\cdot)\#\nu_0, \qquad t \in [0, T_\epsilon].$$

In particular, if $x \in \text{supp } \nu_0$ then $X(t;x) \in \text{supp } \nu_t$ for all $t \in [0, T_\epsilon]$.

Let us simplify notation by writing $X(t;x)$ as $X(t)$ and denote $\bar{\rho}_j^{\tau_1}(X(t)) := \rho(X(t), \bar{y}_j(t-\tau_1))$. Then, we estimate

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|X(t)|^2 &= \langle \dot{X(t)}, X(t)\rangle \\
&= \int_{\mathbb{R}^d} \phi(X(t),y)\langle y - X(t), X(t)\rangle \nu_{t-\tau_2}(dy) + \frac{1}{m}\sum_{j=1}^m \bar{\rho}_j^{\tau_1}(X(t))\langle \bar{y}_j(t-\tau_1) - X(t), X(t)\rangle \\
&= \int_{\mathbb{R}^d} \phi(X(t),y)\langle y, X(t)\rangle \nu_{t-\tau_2}(dy) - \int_{\mathbb{R}^d} \phi(X(t),y)|X(t)|^2 \nu_{t-\tau_2}(dy) \\
&\quad + \frac{1}{m}\sum_{j=1}^m \bar{\rho}_j^{\tau_1}(X(t))\left(\langle \bar{y}_j(t-\tau_1), X(t)\rangle - |X(t)|^2\right)
\end{aligned}$$

Using the definition (6.2.35) and the fact that $|y|, |\bar{y}_j| \leq R_X(t)$, we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|X(t)|^2 \\
&\quad \leq |X(t)| \left(\int_{\mathbb{R}^d} \phi(X(t),y)(R_X(t) - |X(t)|)\nu_{t-\tau_2}(dy) + \frac{1}{m}\sum_{j=1}^m \bar{\rho}_j^{\tau_1}(X(t))(R_X(t) - |X(t)| \right).
\end{aligned}$$

Since $R_X(t) - |X(t)| \geq 0$ for $t < T_\epsilon$, and ϕ and $\bar{\rho}_j^{\tau_1}$ are bounded by K, we find

$$\frac{d}{dt}|X(t)| \leq 4K(R_X(0) + \epsilon - |X(t)|).$$

By Grönwall's inequality, this implies that $|X(t)| < R_X(0) + \epsilon$ on $[0, T_\epsilon]$, contradicting (6.2.36). Thus, $T_\epsilon \geq \tau^*$. Since $\epsilon > 0$ was arbitrary, we conclude that the support of ν_t remains uniformly bounded on $[0, \tau^*]$, and the solution can be extended beyond τ^*. Repeating this argument iteratively on time intervals of length τ^*, we construct a unique global-in-time solution.

Finally, as recalled above, following [17], we obtain that the measure-valued solution satisfies the weak formulation (6.0.12), and that the corresponding push-forward relation (6.0.13) holds. □

6.2.2 Infinite population limit for both leaders and followers

We now consider the case in which both populations, leaders and followers, consist of infinitely many agents. For the mean-field system (6.0.8), we establish the existence and uniqueness of measure-valued solutions, using arguments analogous to those employed in the previous subsection.

As before, we assume that the interaction kernels $\psi(x,y)$, $\phi(x,y)$, and $\rho(x,y)$ appearing in the fluxes (6.0.10) and (6.0.11) are positive, bounded, and Lipschitz continuous. Let us denote by

$$L := \max\{L_\psi, L_\phi, L_\rho\},$$

where L_ψ, L_ϕ, L_ρ are the respective Lipschitz constants.

We begin with a regularity estimate on the velocity fields induced by the measure solutions.

Lemma 6.2.2. *Consider the system* (6.0.8) *subject to the initial data* (6.0.9)*. Given a time* $T > 0$*, suppose that* $\bar{\mu}, \bar{\nu} \in C([0,T); \mathcal{P}_\infty(\mathbb{R}^d))$ *are measures with uniformly compact supports:*

$$\text{supp } \bar{\mu}_t \subset B^d(0, R_1), \quad \text{supp } \bar{\nu}_t \subset B^d(0, R_2), \quad \forall\, t \in [0,T),$$

where $B^d(0, R_i)$ *denotes the ball of radius* $R_i > 0$ *centered at the origin in* $\mathbb{R}^d$ *for* $i = 1, 2$.

Then, the velocity fields $\bar{u}_t$ *and* $\bar{v}_t$ *defined in* (6.0.10)–(6.0.11) *satisfy the following estimates:*

- *(Lipschitz continuity) There exist constants* $K_1, K_2 > 0$ *such that*

$$|\bar{u}_t(x) - \bar{u}_t(\tilde{x})| \le K_1 |x - \tilde{x}|, \quad |\bar{v}_t(z) - \bar{v}_t(\tilde{z})| \le K_2 |z - \tilde{z}|,$$

  for all $x, \tilde{x} \in B^d(0, R_1)$*,* $z, \tilde{z} \in B^d(0, R_2)$*, and* $t \in [0,T]$*.*

- *(Uniform boundedness) There exist constants* $C_1, C_2 > 0$ *such that*

$$|\bar{u}_t(x)| \le C_1, \quad |\bar{v}_t(z)| \le C_2,$$

  for all $x \in B^d(0, R_1)$*,* $z \in B^d(0, R_2)$*, and* $t \in [0,T]$*.*

Proof. Arguing in the same way as in the proof of Lemma 6.2.1, we estimate the differences in the velocity fields with constants:

$$K_1 := K + 2R_1 L, \qquad K_2 = 2K + L(R_1 + 3R_2).$$

The uniform boundedness of the velocity fields follows similarly with constants:

$$C_1 := 2KR_1, \qquad C_2 = K(R_1 + 3R_2).$$

This completes the proof. □

With the above estimates in hand, we now prove the existence and uniqueness result for the mean-field system (6.0.8).

Proof of Theorem 6.0.5: existence and uniqueness in Case (ii). We proceed in parallel to the argument for Case (i) in the previous subsection. Applying Lemma 6.2.2 and the framework established in [17], we first deduce the global-in-time existence and uniqueness of a measure-valued solution $\bar{\mu} \in C([0,T); \mathcal{P}_\infty(\mathbb{R}^d))$.

We then define the support control quantity

$$\tilde{R}_X(t) := \max_{-\tau \le s \le t} \left\{ \max_{x \in \text{supp } \bar{\mu}_s} |x|, \max_{z \in \text{supp } \bar{\nu}_s} |z| \right\},$$

and show that $\tilde{R}_X(t)$ remains uniformly bounded in time. The regularity of the velocity fields and compact support of the initial data then yield global-in-time existence and uniqueness of the second component $\bar{\nu} \in C([0,T); \mathcal{P}_\infty(\mathbb{R}^d))$, thereby completing the proof. □

6.3 Stability and consensus in the mean-field regime

In this section, we analyze the stability and asymptotic behavior of solutions to the mean-field systems introduced in Section 6.2. We begin by establishing a stability estimate with respect to initial data, measured in the Wasserstein distance. This will be followed by an investigation of long-time consensus behavior.

6.3.1 Wasserstein stability estimate

We establish a continuous dependence result for solutions of the mean-field model (6.0.6) in terms of their initial data, using the Wasserstein distance framework. We begin by recalling the definition of the Wasserstein distance.

Definition 6.3.1. *Let $\nu_t^1, \nu_t^2 \in \mathcal{P}(\mathbb{R}^d)$ two probability measures in $\mathbb{R}^d$. We define the Wasserstein distance of order $1 \leq p < \infty$ between ν_t^1 and ν_t^2 the quantity*

$$d_p(\nu_t^1, \nu_t^2) := \inf_{\pi \in \Pi(\nu_t^1, \nu_t^2)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| \pi(dx, dy) \right)^{\frac{1}{p}},$$

and for $p = \infty$, limiting case $p \to +\infty$,

$$d_\infty(\nu_t^1, \nu_t^2) := \inf_{\pi \in \Pi(\nu_t^1, \nu_t^2)} \left\{ \sup_{(x,y) \in supp(\pi)} |x-y| \right\}.$$

The following lemma provides a stability estimate for solutions of (6.0.6) with respect to initial perturbations.

Lemma 6.3.2. *Let $(\bar{y}^1, \nu_t^1)$ and $(\bar{y}^2, \nu_t^2)$ be two solutions of (6.0.6) constructed in Theorem 6.0.5, corresponding to initial data $(\bar{y}^{1,0}, g^1)$ and $(\bar{y}^{2,0}, g^2)$, respectively. Then, there exists a constant $\tilde{C} > 0$, depending on T but independent of p, such that for all $t \in [0, T]$ and $p \in [1, \infty)$,*

$$d_p(\nu_t^1, \nu_t^2) + \left(\frac{1}{m} \sum_{j=1}^m |\bar{y}_j^1(t) - \bar{y}_j^2(t)|^p \right)^{\frac{1}{p}}$$
$$\leq \tilde{C} \left[\sup_{s \in [-\tau, 0]} d_p(g_s^1, g_s^2) + \sup_{s \in [-\tau, 0]} \left(\frac{1}{m} \sum_{j=1}^m |\bar{y}_j^{1,0}(s) - \bar{y}_j^{2,0}(s)|^p \right)^{\frac{1}{p}} \right].$$

In the case $p = \infty$, the following bound holds:

$$d_\infty(\nu_t^1, \nu_t^2) + \max_{i=1,\dots,m} |\bar{y}_i^1(t) - \bar{y}_i^2(t)| \leq \tilde{C} \left(\sup_{s \in [-\tau, 0]} d_\infty(g_s^1, g_s^2) + \sup_{s \in [-\tau, 0]} \max_{i=1,\dots,m} |\bar{y}_i^{1,0}(s) - \bar{y}_i^{2,0}(s)| \right).$$

Proof. We begin by constructing the system of characteristics associated with each solution. For $i = 1, 2$, define $X^i(t; x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ as the flow map solving

$$\begin{cases} \dfrac{d}{dt} X^i(t; x) = v_t^{m,i}(X^i(t; x)), \quad x \in \mathbb{R}^d, \\ X^i(0; x) = x, \end{cases}$$

where $v_t^{m,i}$ is given by the formula (6.0.7). By Theorem 6.0.5, the flows X^i are well-defined on the interval $[0, T]$. By standard properties of transport by characteristics, we have $\nu_t^i = X^i(t; \cdot) \# \nu_0^i$, for all $t \in [0, T]$ and $i = 1, 2$. As before, we define the quantity $R_X^i(t)$ as in (6.2.35). Let $\mathcal{S}_0 : \mathbb{R}^d \to \mathbb{R}^d$ be the optimal transport map pushing ν_0^1 to ν_0^2 with respect to the p-Wasserstein distance, i.e.,

$$\nu_0^2 = \mathcal{S}_0 \# \nu_0^1, \quad d_p(\nu_0^1, \nu_0^2) = \left(\int_{\mathbb{R}^d} |x - \mathcal{S}_0(x)|^p \nu_0^1(dx) \right)^{\frac{1}{p}}.$$

Therefore, defining a map $\mathcal{T}^t := X^2(t;\cdot)\circ \mathcal{S}_0 \circ (X^1(t;\cdot))^{-1}$, for $t\in[0,T]$, we have

$$\mathcal{T}^t \# \nu_t^1 = \nu_t^2, \tag{6.3.37}$$

and thus

$$d_p(\nu_t^1,\nu_t^2) \le \left(\int_{\mathbb{R}^d} |x-\mathcal{T}^t(x)|^p \nu_t^1(dx)\right)^{\frac{1}{p}} =: \theta_p(t).$$

Using the identity $\mathcal{T}^t \circ X^1(t;\cdot) = X^2(t;\cdot)\circ\mathcal{S}_0$, we rewrite $\theta_p(t)$ as

$$\theta_p(t) = \left(\int_{\mathbb{R}^d} |X^1(t;x) - X^2(t;\mathcal{S}_0(x))|^p \nu_0^1(dx)\right)^{\frac{1}{p}}.$$

To incorporate the time-delay structure, we extend $\mathcal{T}^s$ on $[-\tau,0]$ as the optimal transport map between g_s^1 and g_s^2, and define

$$\theta_p(s) := d_p(g_s^1, g_s^2) = \left(\int_{\mathbb{R}^d} |x-\mathcal{T}^s(x)|^p g_s^1(dx)\right)^{\frac{1}{p}}, \quad s\in[-\tau,0].$$

Next, we estimate the time derivative of $\theta_p(t)^p$. For $t\in(0,T)$, using Hölder inequality,

$$\begin{aligned}
&\frac{d}{dt}\theta_p(t)^p = p\theta_p(t)^{p-1}\frac{d}{dt}\theta_p(t)\\
&\le p\int_{\mathbb{R}^d} |X^1(t;x)-X^2(t;\mathcal{S}_0(x))|^{p-1} |v_s^{m,1}(X^1(t;x)) - v_s^{m,2}(X^2(t;\mathcal{S}_0(x)))|\nu_0^1(dx)\\
&= p\int_{\mathbb{R}^d} |x-\mathcal{T}^t(x)|^{p-1}|v_t^{m,1}(x) - v_t^{m,2}(\mathcal{T}^t(x))|\nu_t^1(dx)\\
&\le p\theta_p(t)^{p-1}\left(\int_{\mathbb{R}^d} |v_t^{m,1}(x)-v_t^{m,2}(\mathcal{T}^t(x))|^p\nu_t^1(dx)\right)^{\frac{1}{p}},
\end{aligned}$$

and thus

$$\frac{d}{dt}\theta_p(t) \le \left(\int_{\mathbb{R}^d} |v_t^{m,1}(x)-v_t^{m,2}(\mathcal{T}^t(x))|^p\nu_t^1(dx)\right)^{\frac{1}{p}}.$$

Now, we estimate the velocity difference as

$$\begin{aligned}
&v_t^{m,1}(x) - v_t^{m,2}(\mathcal{T}^t(x))\\
&\quad= \int_{\mathbb{R}^d}\phi(x,y)(y-x)\nu_{t-\tau_2}^1(dy) - \int_{\mathbb{R}^d}\phi(\mathcal{T}^t(x),y)(y-\mathcal{T}^t(x))\nu_{t-\tau_2}^2(dy)\\
&\qquad+\frac{1}{m}\sum_{j=1}^m \rho(x,\bar{y}_j^1(t-\tau_1))(\bar{y}_j^1(t-\tau_1)-x) - \frac{1}{m}\sum_{j=1}^m \rho(\mathcal{T}^t(x),\bar{y}_j^2(t-\tau_1))(\bar{y}_j^2(t-\tau_1)-\mathcal{T}^t(x))\\
&\quad=: I + II.
\end{aligned}$$

Here, using (6.3.37),

$$\begin{aligned}
I &= \int_{\mathbb{R}^d}\phi(x,y)(y-x)\nu_{t-\tau_2}^1(dy) - \int_{\mathbb{R}^d}\phi(\mathcal{T}^t(x),\mathcal{T}^{t-\tau_2}(y))(\mathcal{T}^{t-\tau_2}(y)-\mathcal{T}^t(x))\nu_{t-\tau_2}^1(dy)\\
&= \int_{\mathbb{R}^d}\big(\phi(x,y)-\phi(\mathcal{T}^t(x),\mathcal{T}^{t-\tau_2}(y))\big)(y-x)\nu_{t-\tau_2}^1(dy)\\
&\quad+\int_{\mathbb{R}^d}\phi(\mathcal{T}^t(x),\mathcal{T}^{t-\tau_2}(y))((y-x)-(\mathcal{T}^{t-\tau_2}(y)-\mathcal{T}^t(x)))\nu_{t-\tau_2}^1(dy).
\end{aligned}$$

Using the Lipschitz continuity and boundedness of the function ϕ, we find

$$\begin{aligned}&\int_{\mathbb{R}^d} |\phi(x,y) - \phi(\mathcal{T}^t(x), \mathcal{T}^{t-\tau_2}(y))||y-x|\nu^1_{t-\tau_2}(dy)\\ &\quad \le L_\phi(|x| + R^1_X(t)) \left[|x - \mathcal{T}^t(x)| + \int_{\mathbb{R}^d} |y - \mathcal{T}^{t-\tau_2}(y)|\nu^1_{t-\tau_2}(dy)\right]\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^d} |\phi(\mathcal{T}^t(x), \mathcal{T}^{t-\tau_2}(y))||y - x - (\mathcal{T}^{t-\tau_2}(y) - \mathcal{T}^t(x))|\nu^1_{t-\tau_2}(dy)\\ &\quad \le K|x - \mathcal{T}^t(x)| + K\int_{\mathbb{R}^d} |y - \mathcal{T}^{t-\tau_2}(y)|\nu^1_{t-\tau_2}(dy).\end{aligned}$$

This implies

$$|I| \le \big(L_\phi(|x| + R^1_X(t)) + K\big)\left[|x - \mathcal{T}^t(x)| + \int_{\mathbb{R}^d} |y - \mathcal{T}^{t-\tau_2}(y)|\nu^1_{t-\tau_2}(dy)\right].$$

Similarly, we estimate

$$\begin{aligned}|II| &\le \frac{1}{m}\sum_{j=1}^m \big|\rho(x, \bar y^1_j(t-\tau_1)) - \rho(\mathcal{T}^t(x), \bar y^2_j(t-\tau_1))\big||\bar y^1_j(t-\tau_1) - x|\\ &\quad + \frac{1}{m}\sum_{j=1}^m \rho(\mathcal{T}^t(x), \bar y^2_j(t-\tau_1))\big|(\bar y^1_j(t-\tau_1) - x) - (\bar y^2_j(t-\tau_1) - \mathcal{T}^t(x))\big|\\ &\le \big(L_\rho(C_0 + |x|) + K\big)\left[|x - \mathcal{T}^t(x)| + \frac{1}{m}\sum_{j=1}^m |\bar y^1_j(t-\tau_1) - \bar y^2_j(t-\tau_1)|\right].\end{aligned}$$

Combining the estimates for I and II, we deduce

$$\begin{aligned}&|v^{m,1}_t(x) - v^{m,2}_t(\mathcal{T}^t(x))|\\ &\quad \le C(1+|x|)\left[|x - \mathcal{T}^t(x)| + \int_{\mathbb{R}^d} |y - \mathcal{T}^{t-\tau_2}(y)|\nu^1_{t-\tau_2}(dy) + \frac{1}{m}\sum_{j=1}^m |\bar y^1_j(t-\tau_1) - \bar y^2_j(t-\tau_1)|\right]\end{aligned}$$

for some constant $C > 0$ independent of p. Using Hölder's inequality, we get

$$\int_{\mathbb{R}^d} |y - \mathcal{T}^{t-\tau_2}(y)|\nu^1_{t-\tau_2}(dy) \le \left(\int_{\mathbb{R}^d} |y - \mathcal{T}^{t-\tau_2}(y)|^p\nu^1_{t-\tau_2}(dy)\right)^{\frac{1}{p}}$$

and

$$\frac{1}{m}\sum_{j=1}^m |\bar y^1_j(t-\tau_1) - \bar y^2_j(t-\tau_1)| \le \left(\frac{1}{m}\sum_{j=1}^m |\bar y^1_j(t-\tau_1) - \bar y^2_j(t-\tau_1)|^p\right)^{\frac{1}{p}} =: \xi_p(t-\tau_1).$$

This, together with the boundedness of the support of ν^1_t, yields

$$\frac{d}{dt}\theta_p(t) \le C\theta_p(t) + C\theta_p(t-\tau_2) + C\xi_p(t-\tau_1)$$

for some constant $C > 0$ independent of p. To close the estimate, we estimate $\xi_p(t)$ using the leader dynamics. Arguing similarly to above, we first get

$$\begin{aligned}
&\left| \frac{1}{m} \sum_{j=1}^{m} \psi(\bar{y}_i^1(t), \bar{y}_j^1(t-\tau_1))(\bar{y}_j^1(t-\tau_1) - \bar{y}_i^1(t)) - \frac{1}{m} \sum_{j=1}^{m} \psi(\bar{y}_i^2(t), \bar{y}_j^2(t-\tau_1))(\bar{y}_j^2(t-\tau_1) - \bar{y}_i^2(t)) \right| \\
&\quad \le \frac{1}{m} \sum_{j=1}^{m} \big|\psi(\bar{y}_i^1(t), \bar{y}_j^1(t-\tau_1)) - \psi(\bar{y}_i^2(t), \bar{y}_j^2(t-\tau_1))\big| |\bar{y}_j^1(t-\tau_1) - \bar{y}_i^1(t)| \\
&\qquad + \frac{1}{m} \sum_{j=1}^{m} \psi(\bar{y}_i^2(t), \bar{y}_j^2(t-\tau_1)) \big|(\bar{y}_j^1(t-\tau_1) - \bar{y}_i^1(t)) - (\bar{y}_j^2(t-\tau_1) - \bar{y}_i^2(t))\big| \\
&\quad \le \left(2C_0^y L_\psi + K\right) \left[|\bar{y}_i^1(t) - \bar{y}_i^2(t)| + \left(\frac{1}{m} \sum_{j=1}^{m} |\bar{y}_j^1(t-\tau_1) - \bar{y}_j^2(t-\tau_1)|^p \right)^{\frac{1}{p}} \right].
\end{aligned}$$

This yields

$$\frac{d}{dt}\xi_p(t) \le C\left(\xi_p(t) + \xi_p(t-\tau_1)\right),$$

where $C > 0$ is a constant independent of p. Following arguments in [24, 28], we set

$$\omega_p(t) := e^{-Ct}\xi_p(t).$$

From the above, we obtain

$$\frac{d}{dt}\omega_p(t) \le Ce^{-C\tau_1}\omega_p(t-\tau_1).$$

Now, consider $t \in [0, \tau_1]$. Therefore, we get

$$\frac{d}{dt}\omega_p(t) \le Ce^{-Ct} \sup_{s\in[-\tau_1,0]} \xi_p(s).$$

Integrating the above estimate, we find

$$\xi_p(t) \le C \sup_{s\in[-\tau_1,0]} \xi_p(s),$$

for a suitable constant $C > 0$. This further gives

$$\frac{d}{dt}\theta_p(t) \le C\left(\theta_p(t) + \theta_p(t-\tau_2) + \sup_{s\in[-\tau_1,0]} \xi_p(s)\right),$$

and again, following a similar argument, we deduce

$$\theta_p(t) \le C\left(\sup_{s\in[-\tau_2,0]} \theta_p(s) + \sup_{s\in[-\tau_1,0]} \xi_p(s)\right),$$

for some constant $C > 0$ independent of p. Hence, we have

$$\begin{aligned}
&d_p(\nu_t^1, \nu_t^2) + \left(\frac{1}{m} \sum_{j=1}^{m} |\bar{y}_j^1(t) - \bar{y}_j^2(t)|^p \right)^{\frac{1}{p}} \\
&\quad \le C\left(\sup_{s\in[-\tau,0]} d_p(g_s^1, g_s^2) + \sup_{s\in[-\tau,0]} \left(\frac{1}{m} \sum_{j=1}^{m} |\bar{y}_j^{1,0}(s) - \bar{y}_j^{2,0}(s)|^p \right)^{\frac{1}{p}} \right).
\end{aligned}$$

This completes the proof. □

We conclude this section by stating the stability result for the system (6.0.8). The proof follows from a direct adaptation of the arguments developed in Lemma 6.3.2, using the same strategy based on Wasserstein distance estimates and characteristic flows. Since no essential new difficulties arise in this case, we omit the detailed proof.

Lemma 6.3.3. *Let $T > 0$, and let $(\bar{\mu}_t^1, \bar{\nu}_t^1)$ and $(\bar{\mu}_t^2, \bar{\nu}_t^2)$ be two measure-valued solutions of* (6.0.8) *on the time interval $[0, T]$, constructed according to Theorem 6.0.5. Then, there exists a constant $\bar{C} > 0$, depending on T but independent of $p \in [1, \infty]$, such that*

$$d_p(\bar{\mu}_t^1, \bar{\mu}_t^2) + d_p(\bar{\nu}_t^1, \bar{\nu}_t^2) \le \bar{C} \sup_{s\in[-\tau,0]} d_p(f_s^1, f_s^2) + \bar{C} \sup_{s\in[-\tau,0]} d_p(g_s^1, g_s^2)$$

for all $t \in [0, T)$.

6.3.2 Mean-field limit and emergence of consensus

In this part, we provide the details on the proof of the consensus estimate in Theorem 6.0.5 establishing the consensus behavior of measure-valued solutions to the mean-field systems (6.0.6) and (6.0.8), based on a rigorous passage from the particle model (6.0.1). The key ingredient in the argument is the stability results obtained earlier, which allow us to control the distance between the empirical measure solutions of the particle system and the limiting measure-valued solutions.

We divide the argument into two cases corresponding to the systems (6.0.6) and (6.0.8).

Proof. Case (i). Let $(\bar{y}_i^0(s), g_s) \in C([-\tau, 0]; \mathbb{R}^d) \times C([-\tau, 0]; \mathcal{P}_\infty(\mathbb{R}^d))$ be given initial data. For each $N \in \mathbb{N}$, we construct a particle approximation of g_s by defining

$$g_s^N := \frac{1}{N} \sum_{i=1}^N \delta_{x_i^{N,0}(s)}, \quad s \in [-\tau, 0],$$

where $x_i^{N,0} \in C([-\tau, 0]; \mathbb{R}^d)$ are chosen such that

$$\max_{s\in[-\tau,0]} d_\infty(g_s, g_s^N) \to 0 \quad \text{as } N \to \infty.$$

Likewise, we choose $y_i^{N,0} \in C([-\tau, 0]; \mathbb{R}^d)$ satisfying

$$\max_{s\in[-\tau,0]} \max_{i=1,\dots,m} |y_i^{N,0}(s) - \bar{y}_i^0(s)| \to 0 \quad \text{as } N \to \infty.$$

Remark 6.3.4. *In principle, one could simply set $y_i^{N,0} = \bar{y}_i^0$ for all i and N. However, we keep the above more general approximation procedure, since in the treatment of Case (ii) below, we do not rely on such an identification and instead work with general approximating sequences. Adopting the same framework here makes the two cases fully parallel and simplifies the presentation.*

Let $\{y_i^N(t)\}_{i=1}^m$ and $\{x_j^N(t)\}_{j=1}^N$ denote the solution to the particle system (6.0.1) corresponding to these initial data. Then, by Theorem 6.0.2 and the definition of the diameter $d(t)$ in Definition 6.0.1, we have

$$d(t) \le e^{-\gamma(t-2\tau)} D_0, \tag{6.3.38}$$

for all $t \in [0, T)$.

We now define the empirical measure

$$\nu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i^N(t)},$$

which is the measure-valued solution to the system (6.0.6) in the sense of Definition 6.0.4. By Lemma 6.3.2, there exists a constant $C > 0$, independent of N, such that

$$d_\infty(\nu_t^N, \nu_t) + \max_{i=1,\dots,m} |y_i^N(t) - \bar{y}_i(t)| \le C\left(\sup_{s\in[-\tau,0]} d_\infty(g_s^N, g_s) + \sup_{s\in[-\tau,0]} \max_{i=1,\dots,m} |y_i^{N,0}(s) - \bar{y}_i^0(s)|\right).$$

This implies that $d(t) \to d^\nu(t)$ and $D_0 \to D_0^\nu$ as $N \to \infty$, and thus, passing to the limit in (6.3.38), we obtain

$$d^\nu(t) \le e^{-\gamma(t-2\tau)} D_0^\nu.$$

Case (ii). The argument is analogous. For given initial data $(\bar{f}_s, \bar{g}_s) \in C([-\tau, 0]; \mathcal{P}_\infty(\mathbb{R}^d)) \times C([-\tau, 0]; \mathcal{P}_\infty(\mathbb{R}^d))$, we consider approximations

$$\bar{f}_s^m := \frac{1}{m}\sum_{i=1}^m \delta_{\bar{y}_i^{m,0}(s)} \quad \text{and} \quad g^m \in C([-\tau, 0]; \mathcal{P}_\infty(\mathbb{R}^d))$$

with $\bar{y}_i^{m,0}(s) \in C([-\tau, 0]; \mathbb{R}^d)$ satisfying

$$\max_{s\in[-\tau,0]} d_\infty(\bar{f}_s^m, \bar{f}_s) + \max_{s\in[-\tau,0]} d_\infty(g_s^m, \bar{g}_s) \to 0 \quad \text{as } m \to \infty.$$

Let $\{y_i^m\}_{i=1}^m$ and ν^m denote the solutions to the system (6.0.6) corresponding to this initial data. Then, applying the result of Case (i), we obtain

$$d^{\nu^m}(t) \le e^{-\gamma(t-2\tau)} D_0^{\nu^m}.$$

Next, define the empirical measure

$$\bar{\mu}_t^m := \frac{1}{m}\sum_{i=1}^m \delta_{\bar{y}_i^m(t)},$$

so that the pair $(\bar{\mu}_t^m, \nu_t^m)$ solves the system (6.0.8). Then, applying the stability estimate in Lemma 6.3.3, we get

$$d_\infty(\bar{\mu}_t^m, \bar{\mu}_t) + d_\infty(\nu_t^m, \bar{\nu}_t) \le C\left(\sup_{s\in[-\tau,0]} d_\infty(\bar{f}_s^m, \bar{f}_s) + \sup_{s\in[-\tau,0]} d_\infty(g_s^m, g_s)\right).$$

As before, taking the limit as $m \to \infty$, we conclude

$$d^{\bar{\mu},\bar{\nu}}(t) \le e^{-\gamma(t-2\tau)} D_0^{\bar{\mu},\bar{\nu}}.$$

This completes the proof. □

Chapter 7

Conclusion and future directions

The contribution of this thesis lies at the intersection of mathematical analysis, control theory, and network science. It focuses on multi-agent systems with time delays and non-universal interaction frameworks, that model consensus formation, synchronization, and alignment in social, biological, and engineered networks. Through rigorous proofs supported by numerical validation, this work established precise convergence and stability conditions for delayed, directed, and leader-driven dynamics.

7.0.1 Key Scientific Contributions

The results presented in this thesis consistently advance understanding of how delays, topology, and partial information affect collective outcomes.
In [32], we developed and analyzed Hegselmann-Krause type models for two interacting populations and leader-follower structures, and proved asymptotic consensus under heterogeneous, time-delayed couplings, and we validated the results numerically. From the knowledge of the author, this is the first work introducing in the HK opinion formation model a non-universal interaction, including time delayed coupling. Therefore, it represents a novelty in the field of multi-agent systems, which is particularly important for the applications.
In [18], we generalize the Kuramoto model with a network structure. We established uniform phase-diameter bounds and asymptotic frequency synchronization for oscillators on directed graphs with non-universal coupling, and we demonstrated exponential convergence in the complete graph case. This result extends the result of [40] to a more general network structure (i.e., to a strongly connected digraph) and with a non-universal interaction.
In [30], we extended the analysis to cases with non-universal interaction patterns and either a common influencer or multiple independent leaders, and established conditions guaranteeing convergence to consensus even when delays depend on agent pairs. We focused on the fact that the **(CI)** condition does not imply that the underlying graph is strongly connected, except for a system of only 3 agents, and, more than this, does not imply the presence of leadership. In particular, the presence of one or more leaders could be a subcase of the common influencer assumption. The common influencer framework makes the model more heterogeneous, and for this reason, it extends the previous work we presented. Indeed, in the **(CI)** framework, all agents are "peers," but must share common influencers; meanwhile, with leaders, we have an explicit hierarchy: a few leaders influence followers. Moreover, in the first case we have symmetric interaction between the agents, while with leaders the influence is anti-symmetric. The analysis here is enriched with the case of the control trajectory of a unique leader, that is a

particular case in which the **(CI)** assumption does not hold.
In [31], we derived exponential consensus and flocking results for first- and second-order systems subject to intermittent communication and delay, under persistence-of-excitation and connectivity assumptions. This work extends previous result, like [9], to the case of delayed and non-universal interaction, with communication failures.
Finally, in [22], we obtained the exponential convergence to consensus of the Hegselmann-Krause with leadership and time-delayed interaction. Moreover, we established the well-posedness of the continuum description, and we rigourously justified the mean-field limit procedure. The validity of this work rests on several key aspects. First, many previous opinion dynamics models incorporating delays required restrictive hypotheses, typically “small delay” conditions, to guarantee convergence to consensus. This paper overcomes that limitation by proving convergence without imposing any smallness assumptions on the delay, thereby substantially broadening the applicability of delayed interaction models. Secondly, the introduction of a leader-follower hierarchy with asymmetric interactions adds a layer of realism. In many real social systems (such as contexts involving experts, influencers, or opinion’s of leaders), th information flow is strongly asymmetric: leaders influence many individuals while being only marginally influenced themselves. Modeling this structure explicitly makes the dynamics more relevant for empirical and sociological scenarios. Finally, the paper’s derivation and analysis of the mean-field limit provide valuable insights into large scale behavior. By characterizing the continuum limit of systems with finitely or infinitely many leaders and followers, we established a theoretical framework capable of describing the emergent behavior of very large populations.

7.0.2 Future Research Directions

A natural continuation of the research presented in this thesis is to incorporate stochastic and adaptive interactions into the existing models. This extension would generalize the current deterministic framework to account for random delays, communication noise, and adaptive coupling weights that evolve in response to the system’s dynamics. In the mean-field and PDE limit setting, a promising direction is to extend the results in [22] to second-order models, deriving the corresponding continuum equations for large-population limits that include leader dynamics and time delays.
Another line of research is to consider the mean-field limit with network topology. This analysis could provide new insights into the emergence of clustering phenomena and phase transitions in delayed multi-agent systems. This is a more challenging scenario, because sparse or local connectivity may produce persistent multi-cluster patterns instead of full consensus, and correlations induced by the graph can persist as $N \to \infty$, potentially preventing convergence of the empirical measure. Moreover, the classical mean-field approach, which for the Hegselmann–Krause model leads to a Vlasov-type transport equation, fails for the network system.
Another significant research avenue concerns the optimal control of collective dynamics. This would enable the design of minimal cost intervention strategies accounting for time delays and network heterogeneity, with potential applications to social, biological, and engineered systems.

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