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Analysis and control of some evolutive models

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Abstract

The aim of this thesis is to investigate the asymptotic behavior of solutions to opinion formation and flocking models and to abstract evolution equations. Most of the models we will deal with involve time delay effects. Time delays, even arbitrarily small, may induce instability phenomena. Hence, the stability analysis for delayed systems is an important issue to deepen.

In this thesis, we will establish consensus results for the Hegselmann-Krause opinion formation model and the Cucker-Smale flocking model, investigating different scenarios. Namely, we will analyze situations where lack of connections, non-universal interactions or repulsive dynamics may occur among the system's agents.

Also, we will provide suitable decay estimates for solutions to linear evolution equations with time-dependent time delays and to semilinear evolution equations with memory and time-dependent time delay feedback.

Introduction

This thesis is devoted to the study of the asymptotic behavior of solutions to multiagent systems, namely the Hegselmann-Krause opinion formation and its second-order version, the Cucker-Smale flocking model, and to abstract evolution equations.

In particular, we investigate the convergence to consensus for solutions to Hegselmann-Krause type models and the asymptotic flocking for solutions to Cucker-Smale type models.

On the other hand, we establish exponential decay estimates for solutions to some abstract evolution equations and for the correspondent energies.

Most of the models we deal with involve time delay effects. The analysis of time delays both in multiagent systems and in evolution equations has aroused a substantial interest in the scientific community. Indeed, time delays often appear in many biological, sociological, and engineering applications.

In many cases, even when the time delay is arbitrarily small, the presence of time delays may destabilize the system. Therefore, the stability for systems with time lags is an important issue to deal with.

For a detailed insight into delay differential equations, we refer to [61, 62]. Also, we refer to [73] for the discussion of several applications of delay differential equations, especially to population dynamics.

0.1 Multiagent systems

In these last years, multiagent systems have caught the attention of many researchers, due to their wide application to several scientific disciplines, such as biology [23, 46], economics [2, 67], robotics [21, 68], control theory [14, 20, 105, 94, 93, 8], social sciences [16, 99, 29, 3, 74].

Among them, there is the celebrated Hegselmann-Krause model [66], proposed by Hegselmann and Krause in 2002. Later on, the second-order version of the Hegselmann-Krause model was introduced by Cucker and Smale in [46] for the description of flocking phenomena (for instance, flocking of birds, schooling of fish or swarming of bacteria). Typically, for the solution of such models, the convergence to consensus, in the case of the Hegselmann-Krause model, and the exhibition of asymptotic flocking, in the case of the Cucker-Smale model, are investigated.

In the analysis of such models, it is important to introduce time delay effects. Indeed, one has to take into account certain time lags due to the propagation of the information or to reaction times.

The presence of a delay makes the models more difficult to deal with since a delay, even small, can destroy some geometric features typical of the undelayed models. In particular, for Hegselmann-Krause models with always positive symmetric interactions, it is easy to show that the system converges to consensus due to symmetry reasons. If we add a delay in such models, then the symmetry is broken and, in turn, the asymptotic analysis requires finer arguments. On the other hand, despite mathematical difficulties to overcome, the presence of time delays, which naturally appear in applications, allows us to better describe the real features of the models.

The analysis of the Hegselmann-Krause model and the Cucker-Smale model in presence of time delays (that can be constant or, more realistically, time-dependent), has been carried out by many authors, [35, 36, 37, 38, 53, 63, 64, 65, 76, 77, 87, 98]. Most of them require a smallness condition on the time delay size in order to prove the asymptotic consensus. However, very recently, Rodriguez Cartabia proved in [101] the asymptotic flocking for the Cucker-Smale model with constant time delay without assuming any restrictions on the time delay size (see also [64] for a consensus result for the Hegselmann-Krause model).

In this thesis, we will present some results taken from recent papers ([41, 40, 42, 39]) in which first and second-order Cucker-Smale models involving time delay effects are considered. In particular, generalizing and extending the arguments in [101], we are able to establish the exponential consensus for the Hegselmann-Krause model with time-variable time delay and the exponential flocking for the Cucker-Smale model with time-variable (see [41, 40]), without assuming the time delay size to be small. Smallness conditions on the time delay size are not required either in the results from [42, 39].

For the two aforementioned systems, we will examine different scenarios. First of all, we will investigate the situation in which the agents involved in the opinion formation or flocking process suspend their interaction at certain times. Namely, weight functions that are pair-dependent and that can eventually degenerate are included in the considered models. Then, it is important to find conditions ensuring the convergence to consensus despite the lack of connection among the agents. Particular attention has been paid in these last times to the analysis of the asymptotic behavior of solutions to first and second-order Cucker-Smale models under communication failures (see [19, 12]).

Also, it could happen that the agents are not able to exchange information with all the other components of the system. In this case, we are in the presence of a non-universal interaction, so that the agents are able to influence only the opinions or the velocities of the particles they are linked to. To deal with this kind of interaction, a graph topology over the structure of the model has to be considered (see [27]).

In this thesis, we will establish consensus estimates for first and second-order alignment models with time delay, non-universal interaction, and communication failures by assuming that the digraph describing the interaction among the agents is strongly connected and that the weight functions satisfy a so-called Persistence Excitation Condition. These results are mainly contained in [39].

Another scenario we will consider is the one in which the agents involved in the opinion formation or flocking process have positive-negative interaction, namely the system's particles attract each other in certain time intervals and repeal each other in other ones. Of course, the fact that the agents repeal each other at certain prevents the asymptotic consensus for solutions to the Hegselmann-Krause model and to the Cucker-Smale model.

In this case, in order to get the asymptotic consensus or the asymptotic flocking, one has to compensate the behavior of the solutions to the considered model in the *bad* intervals, i.e. the intervals in which the agents repeal each other, with the *good* behavior of the solutions in the intervals in which the influence among the agents is positive. Under suitable assumptions, we will establish the asymptotic consensus for both models with attractive-repulsive interaction. To this aim, some restrictions on the length of the *bad* time intervals will be required. The consensus results related to first and second-order Cucker-Smale models with attractive-repulsive interaction are contained in [43].

Finally, in all the results we will present, on the influence function, that describes the interactions among the agents involved in the opinion formation or flocking process, monotonicity assumptions, which are usually required when dealing with such models, are removed, namely the influence function is assumed to be just positive bounded and continuous.

0.1.1 The Hegselmann-Krause model

Consider a finite set of $N \in \mathbb{N}$ agents, with $N \geq 2$. Let $x_i(t) \in \mathbb{R}^d$ be the opinion of the i-th agent at time t. Then, the undelayed Hegselmann-Krause model reads as follows:

$$\frac{d}{dt}x_i(t) = \sum_{j:j\neq i} b_{ij}(t)(x_j(t) - x_i(t)), \quad t > 0, \ \forall i = 1, \ldots, N. \tag{0.1.1}$$

Generally, the communication rates b_{ij} are of the form

$$b_{ij}(t) := \frac{1}{N-1}\psi(|x_i(t) - x_j(t)|), \quad \forall t > 0, \forall i, j = 1, \ldots, N, \tag{0.1.2}$$

where the influence function $\psi : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function that is required to be nonincreasing. In this way, each agent is able to influence only the opinion of particles that belong to a certain radius of confidence.
However, in this thesis we will be able to deal with more general influence functions, namely the communication rates b_{ij} are given by the following expression

$$b_{ij}(t) := \frac{1}{N-1}\psi(x_i(t), x_j(t)), \quad \forall t > 0, \forall i, j = 1, \ldots, N, \tag{0.1.3}$$

and the influence function $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a positive continuous and bounded function and

$$K := \|\psi\|_\infty. \tag{0.1.4}$$

So, the influence function does not depend anymore on the distance among the agents' opinions but can be a generic function of the opinions. Moreover, monotonicity assumptions on the influence functions are no longer required. By doing so, a larger class of

influence functions is included in our analysis, for instance of type gaussian or oscillatory (see Chapter 1 for some numerical simulations involving not monotonic influence functions).
In this thesis, we will deal with different Hegselmann-Krause type models. First of all, we will focus on a Hegselmann-Krause model with time-variable time delays:

$$\frac{d}{dt}x_i(t) = \sum_{j:j\neq i} b_{ij}(t)(x_j(t-\tau(t)) - x_i(t)), \quad t > 0, \ \forall i = 1, \dots, N, \tag{0.1.5}$$

where the communication rates are defined as follows

$$b_{ij}(t) := \frac{1}{N-1}\psi(x_i(t), x_j(t-\tau(t))), \quad \forall t > 0, \forall i, j = 1, \dots, N, \tag{0.1.6}$$

and the time delay function $\tau : [0, +\infty) \to [0, +\infty)$ is continuous and satisfies

$$0 \le \tau(t) \le \bar{\tau}, \quad \forall t \ge 0, \forall i, j = 1, \dots, N, \tag{0.1.7}$$

for some positive constant $\bar{\tau}$.

We will also analyze a Hegselmann-Krause type model with pair-dependent and time-dependent time delay, communication failures and non-universal interaction:

$$\frac{d}{dt}x_i(t) = \sum_{j:j\neq i} \chi_{ij}\alpha_{ij}(t)b_{ij}(t)(x_j(t-\tau_{ij}(t)) - x_i(t)), \quad t > 0, \ \forall i = 1, \dots, N. \tag{0.1.8}$$

where the time delay functions $\tau_{ij} : [0, +\infty) \to [0, +\infty)$ are continuous and satisfy

$$0 \le \tau_{ij}(t) \le \bar{\tau}, \quad \forall t \ge 0, \forall i, j = 1, \dots, N, \tag{0.1.9}$$

for some positive constant $\bar{\tau}$.
Here, the communication rates b_{ij} are of the form

$$b_{ij}(t) := \frac{1}{N-1}\psi(x_i(t), x_j(t-\tau_{ij}(t))), \quad \forall t > 0, \forall i, j = 1, \dots, N. \tag{0.1.10}$$

The terms χ_{ij} are so defined

$$\chi_{ij} = \begin{cases} 1, & \text{if } j \text{ transmits information to } i, \\ 0, & \text{otherwise.} \end{cases} \tag{0.1.11}$$

Thus, there could be agents that could never communicate among themselves and, in this case, the interaction will be non-universal.
The weight functions $\alpha_{ij} : [0, +\infty) \to [0, 1]$ are $\mathcal{L}^1$-measurable and satisfy the following Persistence Excitation Condition (cf. [19, 12]):

(PE) there exist two positive constants T and $\tilde{\alpha}$ such that

$$\int_t^{t+T} \alpha_{ij}(s)ds \ge \tilde{\alpha}, \quad \forall t \ge 0, \tag{0.1.12}$$

for all $i, j = 1, \dots, N$ such that $\chi_{ij} = 1$. Without loss of generality, we can assume that $\tilde{\alpha}K \le 1$ and that $T \ge \bar{\tau}$.

So, the interaction is missing not only among agents that are not connected but also among agents that can generally communicate. Let us note that (0.1.12) becomes relevant when T is large and $\tilde{\alpha}$ is small. In this case, the agents could eventually suspend their interaction for long enough. We also point out that, in the case in which $\alpha_{ij}(t) = 1$, for a.e. $t \geq 0$ and for any $i, j = 1, \ldots, N$, i.e. in the case in which the agents do not interrupt their exchange of information, the condition (0.1.12) is of course satisfied.

To deal with the non-universal interaction, we will consider a graph topology over the model structure. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph consisting of a finite set $\mathcal{V} = \{1, ..., N\}$ of vertices and a set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ of arcs. We assume that the agents are located at the vertices and interact with each other via the underlying network topology. For each vertex i, we denote by $\mathcal{N}_i$ the set of vertices that directly influence the vertex i, namely

$$\mathcal{N}_i := \{j = 1, \ldots, N : \chi_{ij} = 1\}. \tag{0.1.13}$$

The set $\mathcal{N}_i$ can also be defined in the following way: $j \in \mathcal{N}_i$ if and only if $(i, j) \in \mathcal{E}$. Also, we denote with

$$N_i := |\mathcal{N}_i|. \tag{0.1.14}$$

We will exclude self loops, i.e. we assume that $i \notin \mathcal{N}_i$ for all $1 \leq i \leq N$. We also denote the network topology via its $(0, 1)$-adjacency matrix $(\chi_{ij})_{ij}$. A *path* in a digraph $\mathcal{G}$ from i_0 to i_p is a finite sequence $i_0, i_1, \ldots, i_p$ of distinct vertices such that each successive pair of vertices is an arc of $\mathcal{G}$. The integer p is called *length* of the path. If there exists a path from i to j, then vertex j is said to be *reachable* from vertex i and we define the distance from i to j, in notation $\operatorname{dist}(i, j)$, as the length of the shortest path from i to j. A digraph $\mathcal{G}$ is said to be *strongly connected* if each vertex is reachable from any other vertex. We assume that our digraph $\mathcal{G}$ is strongly connected. We define the *depth* γ of the digraph as follows:

$$\gamma := \max_{i,j=1,\ldots,N} \operatorname{dist}(i, j). \tag{0.1.15}$$

Thus, any particle can be connected to the other individuals of the system via no more than γ intermediate agents. By definition, since $i \notin \mathcal{N}_i$, for all $i = 1, \ldots, N$, we have that $\gamma \leq N - 1$. Also, since the digraph is strongly connected, the depth $\gamma \geq 1$.

Due to the presence of the time delay, the initial conditions for both systems (0.1.5) and (0.1.8) are functions defined in the time interval $[-\bar{\tau}, 0]$. The initial conditions

$$x_i(s) = x_i^0(s), \quad \forall s \in [-\bar{\tau}, 0], \forall i = 1, \ldots, N, \tag{0.1.16}$$

are assumed to be continuous functions.

Finally, we will consider the following Hegselmann-Krause model with attractive-repulsive interaction:

$$\frac{d}{dt} x_i(t) = \sum_{j:j \neq i} \alpha(t) a_{ij}(t)(x_j(t) - x_i(t)), \quad t > 0, \ \forall i = 1, \ldots, N. \tag{0.1.17}$$

with initial conditions

$$x_i(0) = x_i^0 \in \mathbb{R}^d, \quad \forall i = 1, \ldots, N. \tag{0.1.18}$$

The communication rates are as in (0.1.3). Moreover, the weight function $\alpha : [0, +\infty) \to \{-1, 1\}$ is defined as follows

$$\alpha(0) = 1, \quad \alpha(t) = \begin{cases} 1, & t \in (t_{2n}, t_{2n+1}), \quad n \in \mathbb{N}_0, \\ -1, & t \in [t_{2n+1}, t_{2n+2}], \ n \in \mathbb{N}_0, \end{cases} \tag{0.1.19}$$

where $\{t_n\}_n$ is a sequence of nonnegative numbers such that $t_0 = 0$, $t_n \to \infty$ as $n \to \infty$ and

$$t_{2n+2} - t_{2n+1} < \frac{\ln 2}{K}, \quad \forall n \in \mathbb{N}_0. \tag{0.1.20}$$

In this thesis, we will establish the convergence to consensus for the aforementioned Hegselmann-Krause type models. To this aim, we define the diameter $d(\cdot)$ of the solution as

$$d(t) := \max_{i,j=1,\dots,N} |x_i(t) - x_j(t)|.$$

Definition 0.1.1. We say that a solution $\{x_i\}_{i=1,\dots,N}$ to system (0.1.1), (0.1.5), (0.1.8) or (0.1.17) converges to *consensus* if

$$\lim_{t \to +\infty} d(t) = 0.$$

0.1.2 The Cucker-Smale model

Consider a finite set of $N \in \mathbb{N}$ particles, with $N \geq 2$. Let $(x_i(t)) \in \mathbb{R}^d$ and $(v_i(t)) \in \mathbb{R}^d$ denote the position and the velocity of the i-th particle at time t, respectively. Then, the undelayed Cucker-Smale model takes the following form:

$$\begin{cases} \frac{d}{dt} x_i(t) = v_i(t), & t > 0, \ \forall i = 1, \dots, N, \\ \frac{d}{dt} v_i(t) = \sum\limits_{j:j \neq i} a_{ij}(t)(v_j(t) - v_i(t)), & t > 0, \ \forall i = 1, \dots, N, \end{cases} \tag{0.1.21}$$

where the communication rates a_{ij} of the form

$$a_{ij}(t) := \frac{1}{N-1} \tilde{\psi}(|x_i(t) - x_j(t - \tau(t))|), \quad \forall t > 0, \forall i, j = 1, \dots, N, \tag{0.1.22}$$

and the influence function $\tilde{\psi} : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous nonincreasing function. Also in this case, we will be able to remove monotonicity assumptions on the influence function. Indeed, we will require the influence function to be only positive and bounded. However, in order to establish the asymptotic flocking for solutions to the Cucker-Smale model, the influence function must depend on the distance among the agents' positions. We will denote by

$$\tilde{K} := \|\tilde{\psi}\|_\infty. \tag{0.1.23}$$

As for the first-order model, we will investigate a Cucker-Smale model with time variable time delays:

$$\begin{cases} \frac{d}{dt} x_i(t) = v_i(t), & t > 0, \ \forall i = 1, \dots, N, \\ \frac{d}{dt} v_i(t) = \sum\limits_{j:j \neq i} a_{ij}(t)(v_j(t - \tau(t)) - v_i(t)), & t > 0, \ \forall i = 1, \dots, N, \end{cases} \tag{0.1.24}$$

where the weights a_{ij} of the form

$$a_{ij}(t) := \frac{1}{N-1}\tilde{\psi}(|x_i(t) - x_j(t-\tau(t))|), \quad \forall t > 0, \forall i, j = 1, \ldots, N, \tag{0.1.25}$$

and the time delay function $\tau : [0, \infty) \to [0, \infty)$ is continuous and satisfies (0.1.7). Moreover, we will focus on a Cucker-Smale model with pair-dependent and time-dependent time delay, communication failures and non-universal interaction:

$$\begin{cases} \frac{d}{dt}x_i(t) = v_i(t), & t > 0, \ \forall i = 1, \ldots, N, \\ \frac{d}{dt}v_i(t) = \sum\limits_{j:j\neq i} \chi_{ij}\alpha_{ij}(t)a_{ij}(t)(v_j(t-\tau_{ij}(t)) - v_i(t)), & t > 0, \ \forall i = 1, \ldots, N, \end{cases} \tag{0.1.26}$$

where the time delay functions $\tau_{ij} : [0, +\infty) \to [0, +\infty)$ are continuous and satisfy (0.1.9), the terms χ_{ij} are as in (0.1.11) and the weight functions $\alpha_{ij} : [0, +\infty) \to [0, 1]$ are $\mathcal{L}^1$-measurable and satisfy the Persistence Excitation Condition **(PE)**. Moreover, the communication rates a_{ij} are of the form

$$a_{ij}(t) := \frac{1}{N-1}\tilde{\psi}(|x_i(t) - x_j(t-\tau_{ij}(t)|), \quad \forall t > 0, \forall i, j = 1, \ldots, N. \tag{0.1.27}$$

Due to the non-universal interaction, also in this case a graph topology will be considered over the structure of the model.
Furthermore, both for systems (0.1.24) and (0.1.26), since time delays are involved, the initial conditions

$$x_i(s) = x_i^0(s), \quad v_i(s) = v_i^0(s), \quad \forall s \in [-\bar{\tau}, 0], \forall i = 1, \ldots, N, \tag{0.1.28}$$

are assumed to be continuous functions.

Finally, a Cucker-Smale model with attractive repulsive interaction will be considered:

$$\begin{cases} \frac{d}{dt}x_i(t) = v_i(t), & t > 0, \ \forall i = 1, \ldots, N, \\ \frac{d}{dt}v_i(t) = \sum\limits_{j:j\neq i} \alpha(t)a_{ij}(t)(v_j(t) - v_i(t)), & t > 0, \ \forall i = 1, \ldots, N, \end{cases} \tag{0.1.29}$$

with initial conditions

$$\begin{cases} x_i(0) = x_i^0 \in \mathbb{R}^d, & \forall i = 1, \ldots, N, \\ v_i(0) = v_i^0 \in \mathbb{R}^d, & \forall i = 1, \ldots, N. \end{cases} \tag{0.1.30}$$

Here, the communication rates are as in (0.1.22). Furthermore, the weight function $\alpha : [0, +\infty) \to \{-1, 1\}$ is defined as in (0.1.19), where the sequence of nonnegative numbers $\{t_n\}$ satisfies

$$t_{2n+2} - t_{2n+1} < \frac{\ln 2}{\tilde{K}}, \quad \forall n \in \mathbb{N}_0. \tag{0.1.31}$$

For the second-order model, in order to study the exhibition of asymptotic flocking, we define the space and velocity diameters

$$d_X(t) := \max_{i,j=1,\ldots,N} |x_i(t) - x_j(t)|,$$

$$d_V(t) := \max_{i,j=1,\ldots,N} |v_i(t) - v_j(t)|.$$

Definition 0.1.2. (Unconditional flocking) We say that a solution $\{(x_i, v_i)\}_{i=1,\ldots,N}$ to system (0.1.21), (0.1.24), (0.1.26) or (0.1.29) exhibits *asymptotic flocking* if it satisfies the two following conditions:

1. there exists a positive constant d^* such that
$$\sup_{t\geq 0} d_X(t) \leq d^*,$$
in the case of the undelayed systems (0.1.21) and (0.1.29), or
$$\sup_{t\geq -\bar{\tau}} d_X(t) \leq d^*,$$
in the case of the delayed systems (0.1.24) and (0.1.26);

2. $\lim_{t\to\infty} d_V(t) = 0$.

0.2 Abstract evolution equations

The study of evolution equations in presence of delay or memory terms attracted, in recent years, the interest of many researchers. The presence of a time delay makes the problems more difficult to deal with and, of course, it is important to include in the models time delays/memory terms to take into account time lags, such as reaction times, maturation times, times needed to receive some information, etc., commonly present in real life phenomena.

On the other hand, it is well-known that a time delay may induce instability phenomena. In particular, for the damped wave equation, it has been proven that an arbitrarily small delay can make the model unstable even if it is uniformly asymptotically stable in absence of delay effects (see e.g. [49, 50, 81, 106]). Nevertheless, suitable feedback laws can ensure the delayed model has the same stability properties as the undelayed one (see [81, 106]).

Stability results for abstract evolution equations with delay have been already studied in [83, 84, 72]. In [83, 84] it is analyzed the case of a single constant delay and also the delay damping coefficient is assumed to be constant. The analysis has then been extended in [72] by considering, as here, (multiple) time-dependent time delays.

However, in this thesis we will work in a more general setting. Indeed, in [72], the classical set of assumptions usually employed to deal with wave-type equations in presence of time variable time delays (see e.g. [85, 33, 55]) is used. In particular, it is required that the time delay function $\tau \in W^{1,\infty}(0, +\infty)$ and that $\tau'(t) \leq c < 1$. On the contrary, in the results we will present in this thesis, that are taken from [44] and [45], we will only assume that the time delay function is continuous and bounded from above.

Also, in the case of semilinear wave equations with memory damping, usually an extra frictional not delayed damping is needed (see also [72, 83, 84]). In this thesis, we will consider wave-type equations with viscoelastic damping, delay feedback and source term, establishing well-posedness and stability, for small initial data, without adding any extra frictional not delayed dampings.

0.2.1 Linear evolution equations

Let us consider the following abstract model:

$$\begin{aligned} &U'(t) = AU(t) + k(t)BU(t-\tau(t)), \quad t \in (0,\infty), \\ &U(t) = f(t) \quad t \in [-\bar{\tau}, 0], \end{aligned} \tag{0.2.32}$$

where the operator A generates an exponentially stable C_0-semigroup $(S(t))_{t\geq 0}$ in a Hilbert space H, and B is a continuous linear operator of H into itself. The time delay function $\tau : [0,+\infty) \to (0,+\infty)$ belongs to $C(0,+\infty)$ and we assume that

$$0 \leq \tau(t) \leq \bar{\tau}, \quad \forall\, t \geq 0, \tag{0.2.33}$$

for some positive constant $\bar{\tau}$. Moreover, the delay damping coefficient $k : [-\bar{\tau}, +\infty) \to \mathbb{R}$ belongs to $\mathcal{L}^1_{loc}([-\bar{\tau},+\infty);\mathbb{R})$. We denote with $U_0 := f(0)$.
By the assumptions on the operator A, there exist two positive constants M and ω such that

$$\|S(t)\|_{\mathcal{L}(H)} \leq Me^{-\omega t}, \quad \forall t \geq 0. \tag{0.2.34}$$

Moreover, on the delay feedback coefficient, we assume that the integrals on intervals of length $\bar{\tau}$ are uniformly bounded, namely,

$$\int_{t-\bar{\tau}}^{t} |k(s)|ds \leq K, \quad \forall\, t \geq 0, \tag{0.2.35}$$

for some $K > 0$.

In this thesis, under some mild assumptions on the involved functions and parameters, we will establish the well-posedness of the problem (0.2.32), and we will obtain exponential decay estimates for its solutions.

A concrete model that can be rewritten in the form (0.2.32) is, e.g., the wave equation with frictional damping and delay feedback. In the case of constant delay feedback coefficient and constant time delay, this model has been first studied in [95]. Under a suitable smallness condition on the delay term coefficient, an exponential decay estimate has been proven. This result has then been extended to linear wave equations with internal delay feedback and boundary dissipative condition in [11]. In [7, 58] it is instead analyzed the case of the wave equations with delay feedback and viscoelastic damping.

For other stability estimates in the presence of time delay effects, for specific models, mainly in the case of constant time delay and constant delay damping coefficient, we quote, among the others, [1, 10, 13, 9, 47, 86, 102, 79]. We mention also the recent papers [71] and [26] dealing with delayed Korteweg-de Vries-Burgers and higher-order dispersive equations, respectively.

0.2.2 Semilinear evolution equations with memory and time delay

Let H be a Hilbert space and let A be a positive self-adjoint operator with dense domain $D(A)$ in H. Let us consider the system:

$$\begin{aligned}
&u_{tt}(t) + Au(t) - \int_0^{+\infty} \beta(s)Au(t-s)ds + k(t)BB^*u_t(t-\tau(t)) = \nabla\psi(u(t)),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (0,+\infty),\\
&u(t) = u_0(t), \quad t \in (-\infty, 0],\\
&u_t(t) = g(t), \quad t \in [-\bar{\tau}, 0],
\end{aligned} \tag{0.2.36}$$

where $\bar{\tau}$ is a fixed positive constant and the function $\tau : [0,+\infty) \to [0,+\infty)$ represents the time dependent time delay. We assume that the time delay is a continuous function satisfying

$$\tau(t) \leq \bar{\tau}, \quad \forall\, t \geq 0. \tag{0.2.37}$$

In (0.2.36), B is a bounded linear operator of H into itself, B^* denotes its adjoint. Let us denote

$$\|B\|_{\mathcal{L}(H)} = \|B^*\|_{\mathcal{L}(H)} = b. \tag{0.2.38}$$

Also, $(u_0(\cdot), g(\cdot))$ are the initial data taken in suitable spaces and we denote with $u_1 := g(0)$.

Moreover, on the delay damping coefficient $k : [-\bar{\tau}, +\infty) \to \mathbb{R}$ we assume that $k(\cdot) \in L^1_{loc}([-\bar{\tau}, +\infty))$ and the integral on time intervals of length $\bar{\tau}$ is uniformly bounded, i.e. there exists a positive constant K such that

$$\int_{t-\bar{\tau}}^{t} |k(s)|ds < K, \quad \forall t \geq 0. \tag{0.2.39}$$

The memory kernel $\beta : [0,+\infty) \to [0,+\infty)$ satisfies the following classical assumptions:

(i) $\beta \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$;

(ii) $\beta(0) = \beta_0 > 0$;

(iii) $\int_0^{+\infty} \beta(t)dt = \tilde{\beta} < 1$;

(iv) $\beta'(t) \leq -\delta\beta(t)$, for some $\delta > 0$.

On the nonlinear term, as in [6, 89], we assume that $\psi : D(A^{\frac{1}{2}}) \to \mathbb{R}$ is a functional having Gâteaux derivative $D\psi(u)$ at every $u \in D(A^{\frac{1}{2}})$. Moreover, we assume the following hypotheses:

(H1) For every $u \in D(A^{\frac{1}{2}})$, there exists a constant $c(u) > 0$ such that

$$|D\psi(u)(v)| \leq c(u)||v||_H \qquad \forall v \in D(A^{\frac{1}{2}}).$$

Then, ψ can be extended to the whole H and we denote by $\nabla\psi(u)$ the unique vector representing $D\psi(u)$ in the Riesz isomorphism, i.e.

$$\langle \nabla\psi(u), v\rangle_H = D\psi(u)(v), \qquad \forall v \in H;$$

(H2) for all $r > 0$ there exists a constant $L(r) > 0$ such that

$$||\nabla\psi(u) - \nabla\psi(v)||_H \leq L(r)||A^{\frac{1}{2}}(u-v)||_H,$$

for all $u, v \in D(A^{\frac{1}{2}})$ satisfying $||A^{\frac{1}{2}}u||_H \leq r$ and $||A^{\frac{1}{2}}v||_H \leq r$.

(H3) $\psi(0) = 0$, $\nabla\psi(0) = 0$ and there exists a strictly increasing continuous function h such that

$$||\nabla\psi(u)||_H \leq h(||A^{\frac{1}{2}}u||_H)||A^{\frac{1}{2}}u||_H, \tag{0.2.40}$$

for all $u \in D(A^{\frac{1}{2}})$.

In this thesis, we will study well-posedness and exponential stability, for small initial data, for model (0.2.36). The results we will establish extend the ones in [89], where the time delay is assumed to be constant (see also [31] for the constant delay case). The extension is not trivial since the classical step-by-step argument, often used to deal with time delay models, does not work in this case.

Other models with memory damping and time delay effects have been studied in the recent literature. The first result is due to [69], in the linear setting. In that paper, a standard frictional damping, not delayed, is included in the model to compensate for the destabilizing effect of the delay feedback. As later understood, the viscoelastic damping alone can counter the destabilizing delay effect, under suitable assumptions, without the need for any artificial extra dampings. This has been shown, e.g., in [7, 47, 58, 107]. The case of intermittent delay feedback has been studied in [87] while the paper [80] analyzes a plate equation with memory, source term, delay feedback and, in the same spirit of [69], an extra not delayed frictional damping. Models for wave-type equations with memory damping have been previously studied by several authors in the undelayed case (see e.g. [5, 6, 28]). See also [4] for results on the Timoshenko model, also in the undelayed case, and extensions to the time delay framework (see e.g. [102, 13]).

More rich is the literature in the case of frictional/structural damping, instead of a memory term, which compensates for the destabilizing effect of time delays and, for specific models, various stability results have been quite recently obtained under suitable assumptions (see e.g. [1, 10, 13, 34, 22, 81, 86, 72, 26, 96, 106]).

0.3 Outline

This thesis is organized as follows. In Chapter 1 we prove the exponential convergence to consensus for the Hegselmann-Krause model with time-variable time delays. The consensus result we establish in this chapter improves several previous related works, due to the very general setting we consider. Indeed, no smallness assumptions are required on the time delay size. Moreover, the influence function is a generic function of the agents' opinions which is assumed to be only positive, bounded and continuous, and no symmetry or monotonicity requirements have to be satisfied. Then, we introduce the continuum model associated with the particle system under consideration, obtained as the mean-field limit of the particle system when the number of agents goes to infinity, and we state a consensus theorem for the PDE model. Finally, extensions of the results presented in the chapter to a Hegselmann-Krause model with distributed delay are provided.

In Chapter 2, we establish the exponential flocking for the Cucker-Smale model with time-variable time delays. Again, the main result of this chapter is proved without assuming any restrictions on the time delay size and without requiring any monotonicity properties on the influence function.

In Chapter 3, we focus on first and second-order Cucker-Smale models with non-universal interaction, pair and time-dependent time delays and communication failures. Due to the presence of a non-universal interaction, we consider a network topology over the structure of the model. The asymptotic consensus is established for both systems by assuming that the digraph that describes the interaction among the agents is strongly connected and that the weight functions, that are related to the possible lack of connection among the agents, satisfy a Persistence Excitation condition.

In Chapter 4, we still deal with a Hegselmann-Krause model with time delay and communication failures. With respect to Chapter 3, we work in the case of all-to-all interaction, namely each agent can exchange information with all the other components of the system, and we assume that the time delay functions and the weight functions are not pair-dependent. Although the analysis we carry out in this chapter is less general with respect to the one in Chapter 3, the consensus result we establish in this chapter improves the one in Chapter 3 in the case of universal interaction. Indeed, we provide exponential decay estimates for solutions to the considered Hegselmann-Krause model that are independent of the number of agents, whereas in Chapter 3 the constants that appear in the proof of the consensus result depend on the number of agents. This allows us to obtain consensus estimates for the related PDE model.

In Chapter 5, we deal with first and second-order Cucker-Smale models with attractive repulsive interaction. We provide conditions ensuring that both models achieve the asymptotic consensus. The asymptotic consensus is proven by compensating for the *bad* behavior of the solutions in the intervals of negative interaction with the behavior of the solutions in the intervals of positive interaction.

In Chapter 6, we consider a linear evolution equation with time-dependent time delay. We establish well-posedness and exponential stability for the considered abstract model. This is done by dealing with a very general time delay function, namely the time delay function is just a continuous function bounded from above. The results that hold for the linear model are then extended to a nonlinear model. Finally, applications of the results established in this chapter are provided.

Finally, in Chapter 7 a semilinear evolution equation with memory and time-dependent time delay feedback is analyzed. Under suitable assumptions on the delay feedback coefficient and on the nonlinear term, we prove well-posedness and exponential stability for solutions to the considered model corresponding to sufficiently small initial data. Also, applications to the wave equation with memory and different source terms are discussed.

Chapter 1

The Hegselmann-Krause model with time variable time delays

In this chapter, we will establish the exponential consensus for solutions to the Hegselmann-Krause model with time variable time delay (0.1.5). All the results contained in this chapter are taken from [41].

The consensus result we will prove is the following.

Theorem 1.0.1. *Assume that* $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *is a positive, bounded, continuous function and that* $\tau : [0, +\infty) \to [0, +\infty)$ *is a continuous functions for which* (0.1.7) *holds. Moreover, let* $x_i^0 : [-\bar{\tau}, 0] \to \mathbb{R}^d$ *be a continuous function, for any* $i = 1, \dots, N$. *Then, for every solution* $\{x_i\}_{i=1,\dots,N}$ *to* (0.1.5) *under the initial conditions* (0.1.16)*, the diameter* $d(\cdot)$ *satisfies the exponential decay estimate*

$$d(t) \leq \left(\max_{i,j=1,\dots,N} \max_{r,s \in [-\bar{\tau},0]} |x_i(r) - x_j(s)| \right) e^{-\gamma(t-2\bar{\tau})}, \quad \forall t \geq 0, \tag{1.0.1}$$

for a suitable positive constant γ*, independent of* N.

1.1 Preliminary results

In order to prove the consensus result 1.0.1, we need some auxiliary lemmas. We assume that the hypotheses of Theorem 1.0.1 are satisfied. Let $\{x_i\}_{i=1,\dots,N}$ be solution to (0.1.5) under the initial conditions (0.1.16).

The following results generalize and extend the ones developed in [101] in the case of a Cucker-Smale model with constant time delay. In particular, to deal with time-dependent time delays, in the next lemma, we combine arguments from [101] with a continuity argument used in [37] for a Hegselmann-Krause model with time-dependent time delay.

Lemma 1.1.1. *For each* $v \in \mathbb{R}^d$ *and* $T \geq 0$*, we have that*

$$\min_{j=1,\dots,N} \min_{s \in [T-\bar{\tau},T]} \langle x_j(s), v \rangle \leq \langle x_i(t), v \rangle \leq \max_{j=1,\dots,N} \max_{s \in [T-\bar{\tau},T]} \langle x_j(s), v \rangle, \tag{1.1.1}$$

for all $t \geq T - \bar{\tau}$ *and for all* $i = 1, \dots, N$.

Proof. Fix $T \geq 0$. First of all, we note that the inequalities in (1.1.1) are satisfied for every $t \in [T - \bar{\tau}, T]$.
Now, given a vector $v \in \mathbb{R}^d$, we set

$$M_T = \max_{j=1,\dots,N} \max_{s \in [T-\bar{\tau}, T]} \langle x_j(s), v \rangle.$$

For all $\epsilon > 0$, let us define

$$K^\epsilon := \left\{ t > T : \max_{i=1,\dots,N} \langle x_i(s), v \rangle < M_T + \epsilon, \, \forall s \in [T, t) \right\}.$$

By continuity, we have that $K^\epsilon \neq \emptyset$. Thus, denoted with

$$S^\epsilon := \sup K^\epsilon,$$

it holds that $S^\epsilon > T$.
We claim that $S^\epsilon = +\infty$. Indeed, suppose by contradiction that $S^\epsilon < +\infty$. Note that by definition of S^ϵ it turns out that

$$\max_{i=1,\dots,N} \langle x_i(t), v \rangle < M_T + \epsilon, \quad \forall t \in (T, S^\epsilon), \tag{1.1.2}$$

and

$$\lim_{t \to S^{\epsilon -}} \max_{i=1,\dots,N} \langle x_i(t), v \rangle = M_T + \epsilon. \tag{1.1.3}$$

For all $i = 1, \dots, N$ and $t \in (T, S^\epsilon)$, we compute

$$\frac{d}{dt} \langle x_i(t), v \rangle = \frac{1}{N-1} \sum_{j: j \neq i} \psi(x_i(t), x_j(t - \tau(t))) \langle x_j(t - \tau(t)) - x_i(t), v \rangle.$$

Notice that, being $t \in (T, S^\epsilon)$, then $t - \tau(t) \in (T - \bar{\tau}, S^\epsilon)$ and

$$\langle x_j(t - \tau(t)), v \rangle < M_T + \epsilon, \quad \forall j = 1, \dots, N. \tag{1.1.4}$$

Moreover, (1.1.2) implies that

$$\langle x_i(t), v \rangle < M_T + \epsilon,$$

so that

$$M_T + \epsilon - \langle x_i(t), v \rangle \geq 0.$$

Combining this last fact with (1.1.4), we can write

$$\begin{aligned} \frac{d}{dt} \langle x_i(t), v \rangle &\leq \frac{1}{N-1} \sum_{j: j \neq i} \psi(x_i(t), x_j(t - \tau(t)))(M_T + \epsilon - \langle x_i(t), v \rangle) \\ &\leq K(M_T + \epsilon - \langle x_i(t), v \rangle), \quad \forall t \in (T, S^\epsilon). \end{aligned} \tag{1.1.5}$$

Then, from Gronwall's inequality we get

$$\begin{aligned}\langle x_i(t), v\rangle &\leq e^{-K(t-T)}\langle x_i(T), v\rangle + K(M_T+\epsilon)\int_T^t e^{-K(t-s)}ds\\ &= e^{-K(t-T)}\langle x_i(T), v\rangle + (M_T+\epsilon)e^{-Kt}(e^{Kt}-e^{KT})\\ &= e^{-K(t-T)}\langle x_i(T), v\rangle + (M_T+\epsilon)(1-e^{-K(t-T)})\\ &\leq e^{-K(t-T)}M_T + M_T+\epsilon - M_T e^{-K(t-T)} - \epsilon e^{-K(t-T)}\\ &= M_T+\epsilon-\epsilon e^{-K(t-T)}\\ &= M_T+\epsilon-\epsilon e^{-K(S^\epsilon-T)},\end{aligned}$$

for all $t\in(T,S^\epsilon)$. We have so proved that, $\forall i=1,\dots,N$,

$$\langle x_i(t), v\rangle \leq M_T+\epsilon-\epsilon e^{-K(S^\epsilon-T)}, \quad \forall t\in(T,S^\epsilon).$$

Thus, we get

$$\max_{i=1,\dots,N}\langle x_i(t), v\rangle \leq M_T+\epsilon-\epsilon e^{-K(S^\epsilon-T)}, \quad \forall t\in(T,S^\epsilon). \tag{1.1.6}$$

Letting $t\to S^{\epsilon-}$ in (1.1.6), from (1.1.3) we have that

$$M_T+\epsilon \leq M_T+\epsilon-\epsilon e^{-K(S^\epsilon-T)} < M_T+\epsilon,$$

which is a contradiction. Thus, $S^\epsilon=+\infty$, which means that

$$\max_{i=1,\dots,N}\langle x_i(t), v\rangle < M_T+\epsilon, \quad \forall t>T.$$

From the arbitrariness of ϵ we can conclude that

$$\max_{i=1,\dots,N}\langle x_i(t), v\rangle \leq M_T, \quad \forall t>T,$$

from which

$$\langle x_i(t), v\rangle \leq M_T, \quad \forall t>T,\ \forall i=1,\dots,N,$$

which proves the second inequality in (1.1.1). Now, to prove the other inequality, let $v\in\mathbb{R}^d$ and define

$$m_T=\min_{j=1,\dots,N}\ \min_{s\in[T-\bar\tau,T]}\langle x_j(s), v\rangle.$$

Then, for all $i=1,\dots,N$ and $t>T$, by applying the second inequality in (1.1.1) to the vector $-v\in\mathbb{R}^d$, we get

$$\begin{aligned}-\langle x_j(t), v\rangle = \langle x_i(t), -v\rangle &\leq \max_{j=1,\dots,N}\ \max_{s\in[T-\bar\tau,T]}\langle x_j(s), -v\rangle\\ &= -\min_{j=1,\dots,N}\ \min_{s\in[T-\bar\tau,T]}\langle x_j(s), v\rangle = -m_T,\end{aligned}$$

from which

$$\langle x_j(t), v\rangle \geq m_T.$$

Thus, also the first inequality in (1.1.1) is fulfilled. □

We now introduce some notation.

Definition 1.1.1. We define

$$D_0 = \max_{i,j=1,\ldots,N} \max_{s,t\in[-\bar{\tau},0]} |x_i(s) - x_j(t)|,$$

and, in general, we define the sequence

$$D_n := \max_{i,j=1,\ldots,N} \max_{s,t\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} |x_i(s) - x_j(t)|, \quad \forall\, n \in \mathbb{N}. \tag{1.1.7}$$

Let us denote with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Lemma 1.1.2. *For each $n \in \mathbb{N}_0$ and $i, j = 1, \ldots, N$, we get*

$$|x_i(s) - x_j(t)| \le D_n, \quad \forall s, t \ge n\bar{\tau} - \bar{\tau}. \tag{1.1.8}$$

Proof. Fix $n \in \mathbb{N}_0$ and $i, j = 1, \ldots, N$. Given $s, t \ge n\bar{\tau} - \bar{\tau}$, if $|x_i(s) - x_j(t)| = 0$ then of course $D_n \ge 0 = |x_i(s) - x_j(t)|$. Thus, we can assume $|x_i(s) - x_j(t)| > 0$ and we set

$$v = \frac{x_i(s) - x_j(t)}{|x_i(s) - x_j(t)|}.$$

It turns out that v is a unit vector and, by using (1.1.1) with $T = n\bar{\tau}$ and the Cauchy-Schwarz inequality, we can write

$$\begin{aligned}
|x_i(s) - x_j(t)| &= \langle x_i(s) - x_j(t), v\rangle = \langle x_i(s), v\rangle - \langle x_j(t), v\rangle \\
&\le \max_{l=1,\ldots,N} \max_{r\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} \langle x_l(r), v\rangle - \min_{l=1,\ldots,N} \min_{r\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} \langle x_l(r), v\rangle \\
&\le \max_{l,k=1,\ldots,N} \max_{r,\sigma\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} \langle x_l(r) - x_k(\sigma), v\rangle \\
&\le \max_{l,k=1,\ldots,N} \max_{r,\sigma\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} |x_l(r) - x_k(\sigma)||v| = D_n,
\end{aligned}$$

which proves (1.1.8). □

Remark 1.1.3. Let us note that from (1.1.8), in particular, it follows that

$$|x_i(s) - x_j(t)| \le D_0, \quad \forall s, t \ge -\bar{\tau}. \tag{1.1.9}$$

Moreover, for the sequence $\{D_n\}_n$ defined in (1.1.7), it holds

$$D_{n+1} \le D_n, \quad \forall n \in \mathbb{N}_0. \tag{1.1.10}$$

With an analogous argument, one can find a bound on $|x_i(t)|$, uniform with respect to t and $i = 1, \ldots, N$. Indeed, we have the following lemma.

Lemma 1.1.4. *For every $i = 1, \dots, N$, we have that*

$$|x_i(t)| \le M^0, \quad \forall t \ge -\bar{\tau}, \tag{1.1.11}$$

where

$$M^0 := \max_{i=1,\dots,N} \max_{s\in[-\bar{\tau},0]} |x_i(s)|.$$

Proof. Given $i = 1, \dots, N$ and $t \ge -\bar{\tau}$, if $|x_i(t)| = 0$ then trivially $M^0 \ge 0 = |x_i(t)|$. On the contrary, if $|x_i(t)| > 0$, we define

$$v = \frac{x_i(t)}{|x_i(t)|},$$

which is a unit vector for which we can write

$$|x_i(t)| = \langle x_i(t), v\rangle.$$

Then, by applying (1.1.1) for $T = 0$ and by using the Cauchy-Schwarz inequality we get

$$\begin{aligned} |x_i(t)| &\le \max_{j=1,\dots,N} \max_{s\in[-\bar{\tau},0]} \langle x_j(s), v\rangle \le \max_{j=1,\dots,N} \max_{s\in[-\bar{\tau},0]} |x_j(s)||v| \\ &= \max_{j=1,\dots,N} \max_{s\in[-\bar{\tau},0]} |x_j(s)| = M^0, \end{aligned}$$

which proves (1.1.11). □

Remark 1.1.5. From the estimate (1.1.11), since the influence function ψ is continuous, we deduce that

$$\psi(x_i(t), x_j(t-\tau(t))) \ge \psi_0 := \min_{|y|,|z|\le M^0} \psi(y,z) > 0, \tag{1.1.12}$$

for all $t \ge 0$, for all $i, j = 1, \dots, N$.

The following lemma extends and improves an analogous result in [101] for the Cucker-Smale model with constant time delay.

Lemma 1.1.6. *For all $i, j = 1, \dots, N$, unit vector $v \in \mathbb{R}^d$ and $n \in \mathbb{N}_0$ we have that*

$$\langle x_i(t) - x_j(t), v\rangle \le e^{-K(t-t_0)}\langle x_i(t_0) - x_j(t_0), v\rangle + (1 - e^{-K(t-t_0)})D_n, \tag{1.1.13}$$

for all $t \ge t_0 \ge n\bar{\tau}$, where D_n is as in (1.1.7). Moreover, for all $n \in \mathbb{N}_0$, we get

$$D_{n+1} \le e^{-K\bar{\tau}} d(n\bar{\tau}) + (1 - e^{-K\bar{\tau}})D_n. \tag{1.1.14}$$

Proof. Fix $n \in \mathbb{N}_0$ and $v \in \mathbb{R}^d$ such that $|v| = 1$. We set

$$M_n = \max_{i=1,\dots,N} \max_{t\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} \langle x_i(t), v\rangle,$$

$$m_n = \min_{i=1,\dots,N} \min_{t\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} \langle x_i(t), v\rangle.$$

Then, it is easy to see that $M_n - m_n \leq D_n$. Now, for all $i = 1, \dots, N$ and $t \geq t_0 \geq n\bar{\tau}$ we have that

$$\begin{aligned}\frac{d}{dt}\langle x_i(t), v\rangle &= \sum_{j:j\neq i} b_{ij}(t)\langle x_j(t-\tau(t)) - x_i(t), v\rangle \\ &= \frac{1}{N-1}\sum_{j:j\neq i} \psi(x_i(t), x_j(t-\tau(t)))(\langle x_j(t-\tau(t)), v\rangle - \langle x_i(t), v\rangle) \\ &\leq \frac{1}{N-1}\sum_{j:j\neq i} \psi(x_i(t), x_j(t-\tau(t)))(M_n - \langle x_i(t), v\rangle).\end{aligned}$$

Note that, being $t \geq n\bar{\tau}$, $\langle x_i(t), v\rangle \leq M_n$ from (1.1.1). Therefore, we have that $M_n - \langle x_i(t), v\rangle \geq 0$ and we can write

$$\frac{d}{dt}\langle x_i(t), v\rangle \leq \frac{1}{N-1}K\sum_{j:j\neq i}(M_n - \langle x_i(t), v\rangle) = K(M_n - \langle x_i(t), v\rangle).$$

Thus, from the Gronwall's inequality it comes that

$$\begin{aligned}\langle x_i(t), v\rangle &\leq e^{-K(t-t_0)}\langle x_i(t_0), v\rangle + \int_{t_0}^{t} KM_n e^{-K(t-t_0)+K(s-t_0)}ds \\ &= e^{-K(t-t_0)}\langle x_i(t_0), v\rangle + e^{-K(t-t_0)}M_n(e^{K(t-t_0)} - 1),\end{aligned}$$

that is

$$\langle x_i(t), v\rangle \leq e^{-K(t-t_0)}\langle x_i(t_0), v\rangle + (1 - e^{-K(t-t_0)})M_n. \tag{1.1.15}$$

On the other hand, for all $i = 1, \dots, N$ and $t \geq t_0 \geq n\bar{\tau}$ it holds that

$$\begin{aligned}\frac{d}{dt}\langle x_i(t), v\rangle &= \frac{1}{N-1}\sum_{i:j\neq i} \psi(x_i(t), x_j(t-\tau(t)))(\langle x_j(t-\tau(t)), v\rangle - \langle x_i(t), v\rangle) \\ &\geq \frac{1}{N-1}\sum_{j:j\neq i} \psi(x_i(t), x_j(t-\tau(t)))(m_n - \langle x_i(t), v\rangle).\end{aligned}$$

Note that, from (1.1.1), $\langle x_i(t), v\rangle \geq m_n$ since $t \geq n\bar{\tau}$. Thus, $m_n - \langle x_i(t), v\rangle \leq 0$ and, by recalling that ψ is bounded, we get

$$\frac{d}{dt}\langle x_i(t), v\rangle \geq K(m_n - \langle x_i(t), v\rangle).$$

Hence, by using Gronwall's inequality, it turns out that

$$\langle x_i(t), v\rangle \geq e^{-K(t-t_0)}\langle x_i(t_0), v\rangle + (1 - e^{-K(t-t_0)})m_n. \tag{1.1.16}$$

Therefore, for all $i, j = 1, \dots, N$ and $t \geq t_0 \geq n\bar{\tau}$, by using (1.1.15) and (1.1.16) and by recalling that $M_n - m_n \leq D_n$, we finally get

$$\begin{aligned}\langle x_i(t) - x_j(t), v\rangle &= \langle x_i(t), v\rangle - \langle x_j(t), v\rangle \\ &\leq e^{-K(t-t_0)}\langle x_i(t_0), v\rangle + (1 - e^{-K(t-t_0)})M_n \\ &\quad - e^{-K(t-t_0)}\langle x_j(t_0), v\rangle - (1 - e^{-K(t-t_0)})m_n \\ &= e^{-K(t-t_0)}\langle x_i(t_0) - x_j(t_0), v\rangle + (1 - e^{-K(t-t_0)})(M_n - m_n) \\ &\leq e^{-K(t-t_0)}\langle x_i(t_0) - x_j(t_0), v\rangle + (1 - e^{-K(t-t_0)})D_n,\end{aligned}$$

i.e. (1.1.13) holds true.

Now, we prove (1.1.14). Given $n \in \mathbb{N}_0$, let $i, j = 1, \dots, N$ and $s, t \in [n\bar{\tau}, n\bar{\tau} + \bar{\tau}]$ be such that $D_{n+1} = |x_i(s) - x_j(t)|$. Note that, if $|x_i(s) - x_j(t)| = 0$, then obviously

$$0 = D_{n+1} \leq e^{-K\bar{\tau}}d(n\bar{\tau}) + (1 - e^{-K\bar{\tau}})D_n.$$

So, we can assume $|x_i(s) - x_j(t)| > 0$. Let us define the unit vector

$$v = \frac{x_i(s) - x_j(t)}{|x_i(s) - x_j(t)|}.$$

Hence, we can write

$$D_{n+1} = \langle x_i(s) - x_j(t), v\rangle = \langle x_i(s), v\rangle - \langle x_j(t), v\rangle.$$

Now, by using (1.1.15) with $t_0 = n\bar{\tau}$, we have that

$$\begin{aligned}\langle x_i(s), v\rangle &\leq e^{-K(s-n\bar{\tau})}\langle x_i(n\bar{\tau}), v\rangle + (1 - e^{-K(s-n\bar{\tau})})M_n \\ &= e^{-K(s-n\bar{\tau})}(\langle x_i(n\bar{\tau}), v\rangle - M_n) + M_n.\end{aligned}$$

Thus, since $s \leq n\bar{\tau} + \bar{\tau}$ and $\langle x_i(n\bar{\tau}), v\rangle - M_n \leq 0$ from (1.1.1), we get

$$\begin{aligned}\langle x_i(s), v\rangle &\leq e^{-K\bar{\tau}}(\langle x_i(n\bar{\tau}), v\rangle - M_n) + M_n \\ &\leq e^{-K\bar{\tau}}\langle x_i(n\bar{\tau}), v\rangle + (1 - e^{-K\bar{\tau}})M_n.\end{aligned} \tag{1.1.17}$$

Similarly, by taking into account (1.1.1) and (1.1.16), we have that

$$\langle x_j(t), v\rangle \geq e^{-K\bar{\tau}}\langle x_j(n\bar{\tau}), v\rangle + (1 - e^{-K\bar{\tau}})m_n. \tag{1.1.18}$$

Therefore, combining (1.1.17) and (1.1.18), we can write

$$\begin{aligned}D_{n+1} &\leq e^{-K\bar{\tau}}\langle x_i(n\bar{\tau}), v\rangle + (1 - e^{-K\bar{\tau}})M_n - e^{-K\bar{\tau}}\langle x_j(n\bar{\tau}), v\rangle - (1 - e^{-K\bar{\tau}})m_n \\ &= e^{-K\bar{\tau}}\langle x_i(n\bar{\tau}) - x_j(n\bar{\tau}), v\rangle + (1 - e^{-K\bar{\tau}})(M_n - m_n).\end{aligned}$$

Then, by recalling that $M_n - m_n \leq D_n$ and by using the Cauchy-Schwarz inequality, we can conclude that

$$\begin{aligned}D_{n+1} &\leq e^{-K\bar{\tau}}|x_i(n\bar{\tau}) - x_j(n\bar{\tau})||v| + (1 - e^{-K\bar{\tau}})D_n \\ &\leq e^{-K\bar{\tau}}d(n\bar{\tau}) + (1 - e^{-K\bar{\tau}})D_n.\end{aligned}$$

□

1.2 Proof of the consensus estimate

Finally, we need the following crucial result.

Lemma 1.2.1. *There exists a constant $C \in (0,1)$, independent of $N \in \mathbb{N}$, such that*

$$d(n\bar{\tau}) \leq CD_{n-2}, \tag{1.2.1}$$

for all $n \geq 2$, and the sequence $\{D_n\}_n$ is as in (1.1.7).

Proof. Trivially, if $d(n\bar{\tau}) = 0$, then of course inequality (1.2.1) holds for any constant $C \in (0,1)$. So, suppose $d(n\bar{\tau}) > 0$. Let $i, j = 1, \ldots, N$ be such that $d(n\bar{\tau}) = |x_i(n\bar{\tau}) - x_j(n\bar{\tau})|$. We set

$$v = \frac{x_i(n\bar{\tau}) - x_j(n\bar{\tau})}{|x_i(n\bar{\tau}) - x_j(n\bar{\tau})|}.$$

Then, v is a unit vector for which we can write

$$d(n\bar{\tau}) = \langle x_i(n\bar{\tau}) - x_j(n\bar{\tau}), v\rangle.$$

Let us define

$$M_{n-1} = \max_{l=1,\ldots,N} \max_{s \in [n\bar{\tau}-2\bar{\tau}, n\bar{\tau}-\bar{\tau}]} \langle x_l(s), v\rangle,$$
$$m_{n-1} = \min_{l=1,\ldots,N} \min_{s \in [n\bar{\tau}-2\bar{\tau}, n\bar{\tau}-\bar{\tau}]} \langle x_l(s), v\rangle.$$

So, $M_{n-1} - m_{n-1} \leq D_{n-1}$. Now, we distinguish two different situations.

Case I. Assume that there exists $t_0 \in [n\bar{\tau} - 2\bar{\tau}, n\bar{\tau}]$ such that

$$\langle x_i(t_0) - x_j(t_0), v\rangle < 0.$$

Then, from (1.1.13) with $n\bar{\tau} \geq t_0 \geq n\bar{\tau} - 2\bar{\tau}$, we have that

$$\begin{aligned} d(n\bar{\tau}) &\leq e^{-K(n\bar{\tau}-t_0)}\langle x_i(t_0) - x_j(t_0), v\rangle + (1 - e^{-K(n\bar{\tau}-t_0)})D_{n-2} \\ &\leq (1 - e^{-K(n\bar{\tau}-t_0)})D_{n-2} \leq (1 - e^{-2K\bar{\tau}})D_{n-2}. \end{aligned}$$

Case II. Suppose that

$$\langle x_i(t) - x_j(t), v\rangle \geq 0, \quad \forall t \in [n\bar{\tau} - 2\bar{\tau}, n\bar{\tau}]. \tag{1.2.2}$$

Then, for every $t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]$ we have that

$$\begin{aligned} \frac{d}{dt}\langle x_i(t) - x_j(t), v\rangle &= \frac{1}{N-1}\sum_{l:l\neq i} \psi(x_i(t), x_l(t-\tau(t)))\langle x_l(t-\tau(t)) - x_i(t), v\rangle \\ &\quad - \frac{1}{N-1}\sum_{l:l\neq j} \psi(x_i(t), x_l(t-\tau(t)))\langle x_l(t-\tau(t)) - x_j(t), v\rangle \\ &= \frac{1}{N-1}\sum_{l:l\neq i} \psi(x_i(t), x_l(t-\tau(t)))(\langle x_l(t-\tau(t)), v\rangle - M_{n-1} + M_{n-1} - \langle x_i(t), v\rangle) \\ &\quad + \frac{1}{N-1}\sum_{l:l\neq j} \psi(x_i(t), x_l(t-\tau(t)))(\langle x_j(t), v\rangle - m_{n-1} + m_{n-1} - \langle x_l(t-\tau(t)), v\rangle) \\ &:= S_1 + S_2. \end{aligned}$$

Now, being $t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]$, it holds that $t - \tau(t) \in [n\bar{\tau} - 2\bar{\tau}, n\bar{\tau}]$. Therefore, both $t, t - \tau(t) \geq n\bar{\tau} - 2\bar{\tau}$ and from (1.1.1) we have that

$$m_{n-1} \leq \langle x_k(t), v\rangle \leq M_{n-1}, \quad m_{n-1} \leq \langle x_k(t - \tau(t)), v\rangle \leq M_{n-1}, \quad \forall k = 1, \dots, N. \tag{1.2.3}$$

Therefore, using (1.1.11), we get

$$\begin{aligned} S_1 = &\frac{1}{N-1}\sum_{l:l\neq i}\psi(x_i(t), x_l(t-\tau(t)))(\langle x_l(t-\tau(t)), v\rangle - M_{n-1}) \\ &\quad + \frac{1}{N-1}\sum_{l:l\neq i}\psi(x_i(t), x_l(t-\tau(t)))(M_{n-1} - \langle x_i(t), v\rangle) \\ &\leq \frac{1}{N-1}\psi_0\sum_{l:l\neq i}(\langle x_l(t-\tau(t)), v\rangle - M_{n-1}) + K(M_{n-1} - \langle x_i(t), v\rangle), \end{aligned}$$

and

$$\begin{aligned} S_2 = &\frac{1}{N-1}\sum_{l:l\neq j}\psi(x_i(t), x_l(t-\tau(t)))(\langle x_j(t), v\rangle - m_{n-1}) \\ &\quad + \frac{1}{N-1}\sum_{l:l\neq j}\psi(x_i(t), x_l(t-\tau(t)))(m_{n-1} - \langle x_l(t-\tau(t)), v\rangle) \\ &\leq K(\langle x_j(t), v\rangle - m_{n-1}) + \frac{1}{N-1}\psi_0\sum_{l:l\neq j}(m_{n-1} - \langle x_l(t-\tau(t)), v\rangle). \end{aligned}$$

Combining this last fact with (1.2.3), it comes that

$$\begin{aligned} \frac{d}{dt}\langle x_i(t) - x_j(t), v\rangle &\leq K(M_{n-1} - m_{n-1} - \langle x_i(t) - x_j(t), v\rangle) \\ &\quad + \frac{1}{N-1}\psi_0\sum_{l:l\neq i,j}(\langle x_l(t-\tau(t)), v\rangle - M_{n-1} + m_{n-1} - \langle x_l(t-\tau(t)), v\rangle) \\ &\quad + \frac{1}{N-1}\psi_0(\langle x_j(t-\tau(t)), v\rangle - M_{n-1} + m_{n-1} - \langle x_i(t-\tau(t)), v\rangle) \\ &= K(M_{n-1} - m_{n-1}) - K\langle x_i(t) - x_j(t), v\rangle + \frac{N-2}{N-1}\psi_0(-M_{n-1} + m_{n-1}) \\ &\quad + \frac{1}{N-1}\psi_0(\langle x_j(t-\tau(t)), v\rangle - M_{n-1} + m_{n-1} - \langle x_i(t-\tau(t)), v\rangle). \end{aligned}$$

Therefore, since from (1.2.2) $\langle x_i(t-\tau(t)) - x_j(t-\tau(t)), v\rangle \geq 0$, we get

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t) - x_j(t), v\rangle &\leq K(M_{n-1} - m_{n-1}) - K\langle x_i(t) - x_j(t), v\rangle \\
&\quad + \frac{N-2}{N-1}\psi_0(-M_{n-1} + m_{n-1}) + \frac{1}{N-1}\psi_0(-M_{n-1} + m_{n-1}) \\
&\quad - \frac{1}{N-1}\psi_0\langle x_i(t-\tau(t)) - x_j(t-\tau(t)), v\rangle \\
&\leq K(M_{n-1} - m_{n-1}) - K\langle x_i(t) - x_j(t), v\rangle + \psi_0(-M_{n-1} + m_{n-1}) \\
&= (K - \psi_0)(M_{n-1} - m_{n-1}) - K\langle x_i(t) - x_j(t), v\rangle.
\end{aligned}$$

Hence, from Gronwall's inequality it comes that

$$\begin{aligned}
\langle x_i(t) - x_j(t), v\rangle &\leq e^{-K(t-n\bar\tau+\bar\tau)}\langle x_i(n\bar\tau - \bar\tau) - x_j(n\bar\tau - \bar\tau), v\rangle \\
&\quad + (K - \psi_0)(M_{n-1} - m_{n-1}) \int_{n\bar\tau-\bar\tau}^{t} e^{-K(t-s)} ds,
\end{aligned}$$

for all $t \in [n\bar\tau - \bar\tau, n\bar\tau]$. In particular, for $t = n\bar\tau$ it comes that

$$\begin{aligned}
d(n\bar\tau) &\leq e^{-K\bar\tau}\langle x_i(n\bar\tau - \bar\tau) - x_j(n\bar\tau - \bar\tau), v\rangle + \frac{K-\psi_0}{K}(M_{n-1} - m_{n-1})(1 - e^{-K\bar\tau}) \\
&\leq e^{-K\bar\tau}|x_i(n\bar\tau - \bar\tau) - x_j(n\bar\tau - \bar\tau)||v| + \frac{K-\psi_0}{K}(M_{n-1} - m_{n-1})(1 - e^{-K\bar\tau}) \\
&\leq e^{-K\bar\tau}d(n\bar\tau - \bar\tau) + \frac{K-\psi_0}{K}(M_{n-1} - m_{n-1})(1 - e^{-K\bar\tau}).
\end{aligned}$$

Then, by recalling that $M_{n-1} - m_{n-1} \leq D_{n-1}$ we get

$$\begin{aligned}
d(n\bar\tau) &\leq e^{-K\bar\tau}d(n\bar\tau - \bar\tau) + \frac{K-\psi_0}{K}D_{n-1}(1 - e^{-K\bar\tau}) \\
&\leq e^{-K\bar\tau}d(n\bar\tau - \bar\tau) + \frac{K-\psi_0}{K}D_{n-1}(1 - e^{-K\bar\tau}).
\end{aligned}$$

Finally, by using (1.1.8) and (1.1.10) we have that that

$$\begin{aligned}
d(n\bar\tau) &\leq e^{-K\bar\tau}D_n + \frac{K-\psi_0}{K}D_{n-1}(1 - e^{-K\bar\tau}) \\
&\leq e^{-K\bar\tau}D_{n-2} + \frac{K-\psi_0}{K}D_{n-2}(1 - e^{-K\bar\tau}) \\
&= \left[1 - \frac{\psi_0}{K}(1 - e^{-K\bar\tau})\right] D_{n-2}.
\end{aligned} \tag{1.2.4}$$

Now, we set

$$C = \max\left\{1 - e^{-2K\bar\tau}, 1 - \frac{\psi_0}{K}(1 - e^{-K\bar\tau})\right\} \in (0,1). \tag{1.2.5}$$

Then, taking into account (1.2.4), we can conclude that C is the constant for which inequality (1.2.1) holds. □

Proof of Theorem 1.0.1. Let $\{x_i\}_{i=1,\ldots,N}$ be solution to (0.1.5), (0.1.16). We claim that

$$D_{n+1} \leq \tilde{C} D_{n-2}, \quad \forall n \geq 2, \tag{1.2.6}$$

for some constant $\tilde{C} \in (0,1)$. Indeed, given $n \geq 2$, from (1.1.10), (1.1.14) and (1.2.1) we have that

$$\begin{aligned} D_{n+1} &\leq e^{-K\bar{\tau}} d(n\bar{\tau}) + (1 - e^{-K\bar{\tau}}) D_n \\ &\leq e^{-K\bar{\tau}} C D_{n-2} + (1 - e^{-K\bar{\tau}}) D_n \\ &\leq e^{-K\bar{\tau}} C D_{n-2} + (1 - e^{-K\bar{\tau}}) D_{n-2} \\ &\leq (1 - e^{-K\bar{\tau}}(1 - C)) D_{n-2}, \end{aligned}$$

where the constant C is defined in (1.2.5). So, setting

$$\tilde{C} = 1 - e^{-K\bar{\tau}}(1 - C),$$

we can conclude that $\tilde{C} \in (0,1)$ is the constant for which (1.2.6) holds true.
This implies that

$$D_{3n} \leq \tilde{C}^n D_0, \quad \forall n \geq 1. \tag{1.2.7}$$

Indeed, by induction, if $n = 1$ we know from (1.2.6) that

$$D_3 \leq \tilde{C} D_0.$$

So, assume that (1.2.7) holds for $n \geq 1$ and we prove it for $n + 1$. By using again (1.2.6) and from the induction hypothesis it comes that

$$D_{3(n+1)} \leq \tilde{C} D_{3n} \leq \tilde{C} \tilde{C}^n D_0 = \tilde{C}^{n+1} D_0,$$

i.e. (1.2.7) is fulfilled.
Notice that (1.2.7) can be rewritten as

$$D_{3n} \leq e^{-3n\gamma\bar{\tau}} D_0, \quad \forall n \in \mathbb{N}_0, \tag{1.2.8}$$

with

$$\gamma = \frac{1}{3\bar{\tau}} \ln \left(\frac{1}{\tilde{C}} \right).$$

Now, fix $i, j = 1, \ldots, N$ and $t \geq 0$. Then, $t \in [3n\bar{\tau} - \bar{\tau}, 3n\bar{\tau} + 2\bar{\tau}]$, for some $n \in \mathbb{N}_0$. Therefore, by using (1.1.8) and (1.2.8), it turns out that

$$|x_i(t) - x_j(t)| \leq D_{3n} \leq e^{-3n\gamma\bar{\tau}} D_0.$$

Thus, being $t \leq 3n\bar{\tau} + 2\bar{\tau}$, then $-3n\bar{\tau} \leq -t + 2\bar{\tau}$ and we get

$$|x_i(t) - x_j(t)| \leq e^{-\gamma(t - 2\bar{\tau})} D_0.$$

Therefore,

$$d(t) \leq e^{-\gamma(t-2\bar{\tau})} D_0, \quad \forall t \geq 0,$$

and (1.0.1) is proved. □

1.3 The continuum model

In this section, we consider the continuum model obtained as the mean-field limit of the particle system when $N \to \infty$. Let $\mathcal{M}(\mathbb{R}^d)$ be the set of probability measures on the space $\mathbb{R}^d$. Then, the continuum model associated with the particle system (0.1.5) is given by

$$\begin{aligned} &\partial_t \mu_t + \operatorname{div}\left(F[\mu_{t-\tau(t)}]\mu_t\right) = 0, \quad t > 0, \\ &\mu_s = g_s, \quad x \in \mathbb{R}^d, \quad s \in [-\bar{\tau}, 0], \end{aligned} \tag{1.3.1}$$

where the velocity field F is defined as

$$F[\mu_{t-\tau(t)}](x) = \int_{\mathbb{R}^d} \psi(x, y)(y - x)\, d\mu_{t-\tau(t)}(y), \tag{1.3.2}$$

and $g_s \in \mathcal{C}([-\bar{\tau}, 0]; \mathcal{M}(\mathbb{R}^d))$.

We assume that the potential $\psi(\cdot,\cdot)$ in (1.3.2) is Lipschitz continuous, namely there exists $L > 0$ such that, for any $(x, y), (x', y') \in \mathbb{R}^{2d}$,

$$|\psi(x, y) - \psi(x', y')| \le L(|y - y'| + |x - x'|).$$

Definition 1.3.1. Let $T > 0$. We say that $\mu_t \in \mathcal{C}([0, T); \mathcal{M}(\mathbb{R}^d))$ is a measure-valued solution to (1.3.1) on the time interval $[0, T)$ if for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times [0, T))$ we have:

$$\int_0^T \int_{\mathbb{R}^d} \left(\partial_t \varphi + F[\mu_{t-\tau(t)}](x) \cdot \nabla_x \varphi\right) d\mu_t(x)dt + \int_{\mathbb{R}^d} \varphi(x, 0)dg_0(x) = 0. \tag{1.3.3}$$

Before stating the consensus result for solutions to model (1.3.1), we recall some basic tools on probability spaces and measures.

Definition 1.3.2. Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ be two probability measures on $\mathbb{R}^d$. We define the 1-Wasserstein distance between μ and ν as

$$d_1(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| d\pi(x, y),$$

where $\Pi(\mu, \nu)$ is the space of all couplings for μ and ν, namely all those probability measures on $\mathbb{R}^{2d}$ having as marginals μ and ν:

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x)d\pi(x, y) = \int_{\mathbb{R}^d} \varphi(x)d\mu(x), \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(y)d\pi(x, y) = \int_{\mathbb{R}^d} \varphi(y)d\nu(y),$$

for all $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$.

Let us introduce the space $\mathcal{P}_1$ of all probability measures with finite first-order moment. It is well-known that $(\mathcal{P}_1(\mathbb{R}^d), d_1(\cdot, \cdot))$ is a complete metric space.

Now, we define the position diameter for a compactly supported measure $g \in \mathcal{P}_1(\mathbb{R}^d)$ as follows:

$$d_X[g] := \operatorname{diam}(supp\ g).$$

Since the consensus result for the particle model (0.1.5) holds without any upper bounds on the time delay $\tau(\cdot)$, one can improve the consensus theorem for the PDE model (1.3.1) obtained in [37] removing the smallness assumption on the time delay $\tau(t)$. We omit the proof since, once we have the result for the particle system (0.1.5), the consensus estimate for the continuum model is obtained with arguments analogous to the ones in [37] and [88].

Theorem 1.3.1. *Let $\mu_t \in \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d))$ be a measure-valued solution to* (1.3.1) *with compactly supported initial datum $g_s \in \mathcal{C}([-\bar{\tau},0];\mathcal{P}_1(\mathbb{R}^d))$ and let F as in* (1.3.2)*. Then, there exists a constant $C>0$ such that*

$$d_X(\mu_t) \leq \left(\max_{s\in[-\bar{\tau},0]} d_X(g_s)\right) e^{-Ct}, \quad \forall t \geq 0.$$

1.4 The distributed time delay case

Now, we extend the results obtained for the Hegselmann-Krause model with a pointwise time delay to a model with distributed time delay. In particular, we consider the system

$$\frac{d}{dt}x_i(t) = \frac{1}{h(t)}\sum_{j:j\neq i}\int_{t-\tau_2(t)}^{t-\tau_1(t)} \beta(t-s)b_{ij}(t;s)(x_j(s)-x_i(t))ds, \quad t>0, \quad \forall i=1,\ldots,N, \tag{1.4.1}$$

where the time delay functions $\tau_1 : [0,+\infty) \to [0,+\infty)$, $\tau_2 : [0,+\infty) \to [0,+\infty)$ are continuous and satisfy

$$0 \leq \tau_1(t) < \tau_2(t) \leq \bar{\tau}, \quad \forall t \geq 0, \tag{1.4.2}$$

for some positive constant $\bar{\tau}$.
The communication rates $b_{ij}(t;s)$ are of the form

$$b_{ij}(t;s) := \frac{1}{N-1}\psi(x_i(t),x_j(s)), \quad \forall t\geq 0, \quad \forall i,j=1,\ldots,N, \tag{1.4.3}$$

where the influence function $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is positive, continuous and bounded. Moreover, $\beta : [0,\bar{\tau}] \to (0,+\infty)$ is a continuous weight function and

$$h(t) := \int_{\tau_1(t)}^{\tau_2(t)} \beta(s)ds, \quad \forall t \geq 0. \tag{1.4.4}$$

Note that, since we assume $\tau_1(t) < \tau_2(t)$ and $\beta(t) > 0$, $\forall t \geq 0$, then the function $h(t)$ is always positive.

As in Section 1.1, one can prove the following crucial lemma.

Lemma 1.4.1. *Let $\{x_i\}_{i=1,\ldots,N}$ be a solution to system* (1.4.1) *with the continuous initial conditions* (0.1.16)*. Then, for each vector $v \in \mathbb{R}^d$ and for any $T \geq 0$, we have that*

$$\min_{j=1,\ldots,N}\min_{s\in[T-\bar{\tau},T]}\langle x_j(s),v\rangle \leq \langle x_i(t),v\rangle \leq \max_{j=1,\ldots,N}\max_{s\in[T-\bar{\tau},T]}\langle x_j(s),v\rangle, \tag{1.4.5}$$

for all $t \geq T-\bar{\tau}$ and for all $i=1,\ldots,N$.

Proof. First of all, we note that, for each $v \in \mathbb{R}^d$ and $T \geq 0$, the inequalities in the statement are satisfied for every $t \in [T-\bar{\tau}, T]$.
Now, fix $T \geq 0$, a vector $v \in \mathbb{R}^d$ and a positive constant ϵ. Define the constant M_T and the set K^ϵ as in the proof of Lemma 1.1.1. Then, denoted as before $S^\epsilon := \sup K^\epsilon$, it holds that $S^\epsilon > T$.
We claim that $S^\epsilon = +\infty$. Indeed, suppose by contradiction that $S^\epsilon < +\infty$. Note that by definition of S^ϵ it turns out that

$$\max_{i=1,\dots,N} \langle x_i(t), v\rangle < M_T + \epsilon, \quad \forall t \in (T, S^\epsilon), \tag{1.4.6}$$

and

$$\lim_{t \to S^{\epsilon -}} \max_{i=1,\dots,N} \langle x_i(t), v\rangle = M_T + \epsilon. \tag{1.4.7}$$

For all $i = 1, \dots, N$ and $t \in (T, S^\epsilon)$, we compute

$$\begin{aligned}\frac{d}{dt}\langle x_i(t), v\rangle &= \frac{1}{h(t)} \sum_{j:j\neq i} \int_{t-\tau_2(t)}^{t-\tau_1(t)} \alpha(t-s) a_{ij}(t;s) \langle x_j(s) - x_i(t), v\rangle ds \\ &= \frac{1}{N-1}\frac{1}{h(t)} \sum_{j:j\neq i} \int_{t-\tau_2(t)}^{t-\tau_1(t)} \alpha(t-s)\psi(x_i(t), x_j(s))(\langle x_j(s), v\rangle - \langle x_i(t), v\rangle) ds.\end{aligned}$$

Notice that, being $t \in (T, S^\epsilon)$, then $t-\tau_2(t), t-\tau_1(t) \in (T-\bar{\tau}, S^\epsilon)$ and

$$\langle x_j(s), v\rangle < M_T + \epsilon, \quad \forall s \in [t-\tau_2(t), t-\tau_1(t)], \forall j = 1, \dots, N. \tag{1.4.8}$$

Moreover, (1.4.6) implies that

$$\langle x_i(t), v\rangle < M_T + \epsilon, \quad \forall t \in (T, S^\epsilon).$$

so that

$$M_T + \epsilon - \langle x_i(t), v\rangle \geq 0, \quad \forall t \in (T, S^\epsilon).$$

Combining this last fact with (1.4.8) and by recalling of (1.4.4), we can write

$$\begin{aligned}\frac{d}{dt}\langle x_i(t), v\rangle &\leq \frac{1}{N-1}\frac{1}{h(t)} \sum_{j:j\neq i} \int_{t-\tau_2(t)}^{t-\tau_1(t)} \alpha(t-s)\psi(x_i(t), x_j(s))(M_T + \epsilon - \langle x_i(t), v\rangle) ds \\ &\leq \frac{K}{N-1}\frac{1}{h(t)}(M_T + \epsilon - \langle x_i(t), v\rangle) \sum_{j:j\neq i} \int_{t-\tau_2(t)}^{t-\tau_1(t)} \alpha(t-s) ds \\ &= K\frac{1}{h(t)}(M_T + \epsilon - \langle x_i(t), v\rangle) \int_{t-\tau_2(t)}^{t-\tau_1(t)} \alpha(t-s) ds \\ &= K(M_T + \epsilon - \langle x_i(t), v\rangle),\end{aligned}$$

for all $t \in (T, S^\epsilon)$. Then, Gronwall's Lemma allows us to conclude the proof of the second inequality arguing analogously to the proof of Lemma 1.1.1. Also, the proof of the first inequality is obtained similarly with respect to the pointwise time delay case. We omit the details.

□

As before, one can define the quantities D_n, $n \in \mathbb{N}_0$, and prove the analogous, for solutions to the model with distributed time delay (1.4.1), of the lemmas in Section 1.1 and in Section 1.2. Then, the following exponential convergence to consensus holds.

Theorem 1.4.2. *Assume that $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a positive, bounded, continuous function and that the time delay functions $\tau_1 : [0,+\infty) \to [0,+\infty)$, $\tau_2 : [0,+\infty) \to [0,+\infty)$ are continuous functions and satisfy* (1.4.2)*. Let $\alpha : [0,\bar{\tau}] \to (0,+\infty)$ be a continuous function. Moreover, let $x_i^0 : [-\bar{\tau},0] \to \mathbb{R}^d$ be a continuous function, for any $i = 1,\dots,N$. Then, every solution $\{x_i\}_{i=1,\dots,N}$ to* (1.4.1)*, with the initial conditions* (0.1.16) *satisfies the exponential decay estimate*

$$d(t) \leq \left(\max_{i,j=1,\dots,N} \max_{r,s\in[-\bar{\tau},0]} |x_i(r) - x_j(s)| \right) e^{-\gamma(t-2\bar{\tau})}, \quad \forall t \geq 0,$$

for a suitable positive constant γ, independent of N.

The related PDE model is now:

$$\begin{aligned} &\partial_t \mu_t + \operatorname{div} \left(\frac{1}{h(t)} \int_{t-\tau_2(t)}^{t-\tau_1(t)} \alpha(t-s) F[\mu_s] ds \, \mu_t \right) = 0, \quad t > 0, \\ &\mu_s = g_s, \quad x \in \mathbb{R}^d, \quad s \in [-\bar{\tau}, 0], \end{aligned} \tag{1.4.9}$$

where the velocity field F is given by

$$F[\mu_s](x) = \int_{\mathbb{R}^d} \psi(x,y)(y-x) \, d\mu_s(y), \tag{1.4.10}$$

and $g_s \in \mathcal{C}([-\bar{\tau},0]; \mathcal{M}(\mathbb{R}^d))$.

As before, we assume that the potential $\psi(\cdot,\cdot)$ in (1.4.10) is also Lipschitz continuous with respect to the two arguments.

Definition 1.4.1. Let $T > 0$. We say that $\mu_t \in \mathcal{C}([0,T); \mathcal{M}(\mathbb{R}^d))$ is a measure-valued solution to (1.4.9) on the time interval $[0,T)$ if for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times [0,T))$ we have:

$$\int_0^T \int_{\mathbb{R}^d} \left(\partial_t \varphi + \frac{1}{h(t)} \int_{t-\tau_2(t)}^{t-\tau_1(t)} \alpha(t-s) F[\mu_s](x) ds \cdot \nabla_x \varphi \right) d\mu_t(x) dt + \int_{\mathbb{R}^d} \varphi(x,0) dg_0(x) = 0.$$

Since the consensus result for the particle model (1.4.1) holds without any upper bounds on the time delays $\tau_1(\cdot), \tau_2(\cdot)$, one can improve the consensus theorem for the PDE model (1.4.9) of [87]. Indeed, in [87], where the author concentrates in the case $\tau_1(t) \equiv 0$, the consensus estimate is obtained under a smallness condition on the time delay. The proof is analogous, then we omit it.

Theorem 1.4.3. *Let $\mu_t \in \mathcal{C}([0,T]; \mathcal{P}_1(\mathbb{R}^d))$ be a measure-valued solution to* (1.4.9) *with compactly supported initial datum $g_s \in \mathcal{C}([-\bar{\tau},0]; \mathcal{P}_1(\mathbb{R}^d))$ and let F as in* (1.4.10)*. Then, there exists a constant $C > 0$ such that*

$$d_X(\mu_t) \leq \left(\max_{s\in[-\bar{\tau},0]} d_X(g_s) \right) e^{-Ct}, \quad \forall t \geq 0.$$

1.5 Numerical tests

In this section, we present some numerical tests for the particle system (0.1.5) with weights b_{ij} in (0.1.3) defined via functions

$$\psi(r, r') = \tilde{\psi}(|r - r'|) \,,$$

always positive but nonmonotonic.

In particular, we consider an oscillatory function

$$\tilde{\psi}(r) = \sin^2 r + \frac{1}{1 + r^2}, \quad r \in [0, +\infty), \tag{1.5.1}$$

and a translated gaussian function like

$$\tilde{\psi}(r) = e^{-(r-1)^2}, \quad r \in [0, +\infty). \tag{1.5.2}$$

These are significant examples since, besides the more studied case with $\tilde{\psi}$ monotonic, it is important to consider some oscillatory behaviors in the agents' interaction or interactions which are more relevant when the distance between the agents is close to a certain value.

In Figure 1.1 we see the evolution of agents' opinions in the case of the interaction potential of an oscillatory type defined in (1.5.1), respectively for $N = 4$ (in the top) and $N = 7$ (in the bottom), considering time delays $\tau = 3$ and $\tau = 10$. We see that, after an initial oscillatory behavior, the system tends towards consensus. In case of the larger time delay, in order to observe the consensus behavior we have to wait a larger time (we take the time $t \in [0, 60]$ in the case $\tau = 10$ while $t \in [0, 40]$ is enough for $\tau = 3$.).

In Figure 1.2 we observe the opinions' evolution in the case of the potential function (1.5.2). We consider different time delays and, as in the previous case, $N = 4$ or $N = 7$. Also in such a case, we can see that the system converges to consensus after an initial oscillatory behavior. In the case of a larger delay, the convergence to consensus can be observed after a larger time. In particular, in the case of $N = 7$ agents, we first observe the formation of two clusters. This is related to the form of the influence function.

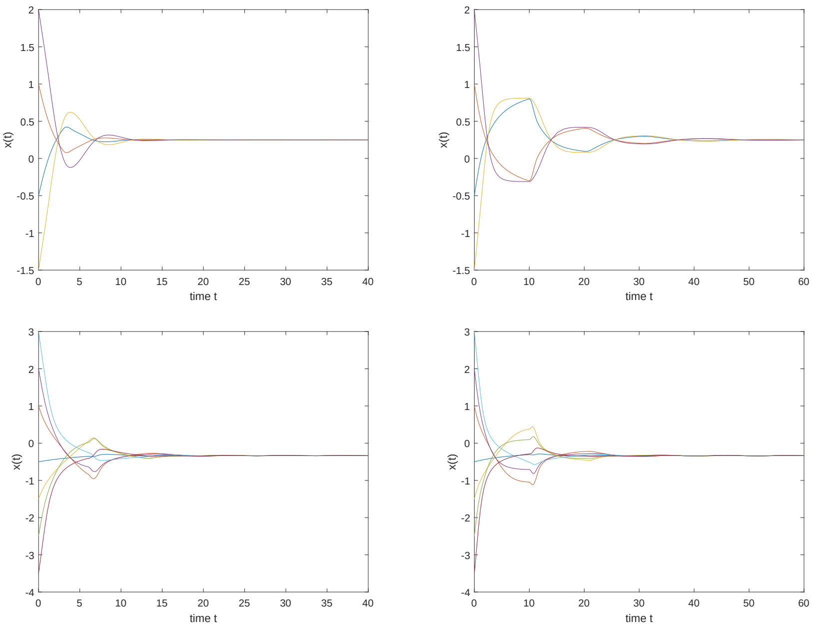

Figure 1.1: Communication rates (1.5.1): time evolution of solutions with different time delays and number N of agents; $\tau = 3, N = 4$ (top left), $\tau = 10, N = 4$ (top right), $\tau = 3, N = 7$ (bottom left), $\tau = 10, N = 7$ (bottom right).

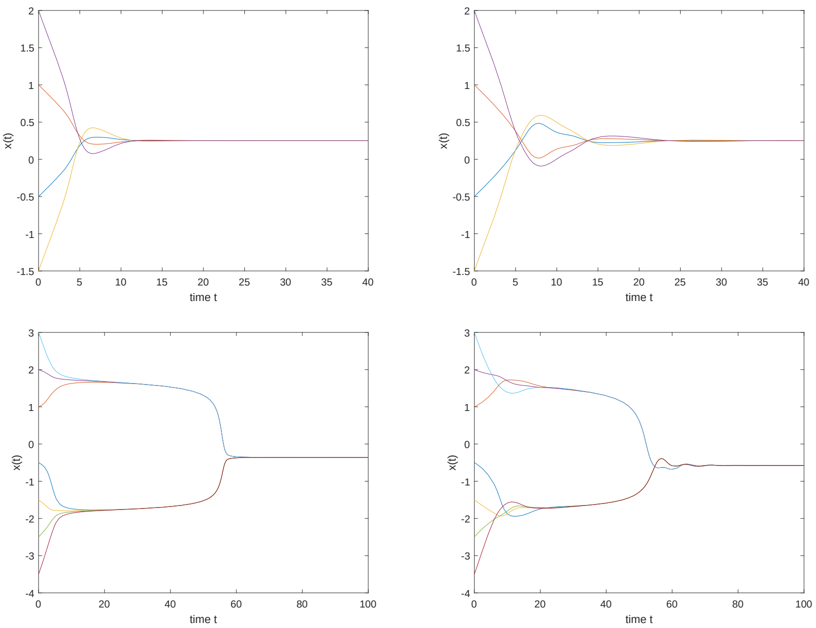

Figure 1.2: Communication rates (1.5.2): time evolution of solutions with different time delays and number N of agents; $\tau = 3, N = 4$ (top left), $\tau = 6, N = 4$ (top right), $\tau = 1, N = 7$ (bottom left), $\tau = 6, N = 7$ (bottom right).

Chapter 2

The Cucker-Smale model with time variable time delays

In this chapter, we will prove the exponential flocking for the Cucker-Smale model with time variable time delay (0.1.24). All the results we will present in this chapter are contained in [40].

Theorem 2.0.1. *Assume that* $\tilde{\psi} : \mathbb{R} \to \mathbb{R}$ *is a positive, bounded, continuous function that satisfies*

$$\int_0^{+\infty} \min_{r\in[0,x]} \tilde{\psi}(r)dx = +\infty. \tag{2.0.1}$$

Assuma that $\tau : [0, +\infty) \to [0, +\infty)$ *is a continuous functions for which* (0.1.7) *holds. Moreover, let* $x_i^0, v_i^0 : [-\bar{\tau}, 0] \to \mathbb{R}^d$ *be continuous functions, for any* $i = 1, \dots, N$. *Then, for every solution* $\{(x_i, v_i)\}_{i=1,\dots,N}$ *to* (0.1.24) *with the initial conditions* (0.1.28)*, there exists a positive constant* d^* *such that*

$$\sup_{t\geq-\bar{\tau}} d_X(t) \leq d^*, \tag{2.0.2}$$

and there exists another positive constant C*, independent of* N*, for which the following exponential decay estimate holds*

$$d_V(t) \leq \left(\max_{i,j=1,\dots,N} \max_{r,s\in[-\bar{\tau},0]} |v_i(r) - v_j(s)| \right) e^{-C(t-2\bar{\tau})}, \quad \forall t \geq -\bar{\tau}. \tag{2.0.3}$$

Remark 2.0.2. Let us note that, if the influence function ψ is nonincreasing, then the assumption (2.0.1) reduces to

$$\int_0^{+\infty} \tilde{\psi}(x)dx = +\infty. \tag{2.0.4}$$

The condition (2.0.4) is the one assumed in [101] in order to achieve the unconditional flocking for solutions of the Cucker-Smale model.

2.1 Preliminaries

We now present some auxiliary lemmas that generalize and extend the analogous results in [101]. We omit some of their proof since they can obtained with analogous arguments to the ones employed in Chapter 1. We assume that the hypotheses of Theorem 2.0.1 are satisfied. Let $\{x_i, v_i\}_{i=1,\dots,N}$ be solution to (0.1.24) under the initial conditions (0.1.28).

Lemma 2.1.1. *For each $v \in \mathbb{R}^d$ and $T \geq 0$, we have that*

$$\min_{j=1,\dots,N} \min_{s\in[T-\bar{\tau},T]} \langle v_j(s), v\rangle \leq \langle v_i(t), v\rangle \leq \max_{j=1,\dots,N} \max_{s\in[T-\bar{\tau},T]} \langle v_j(s), v\rangle, \tag{2.1.1}$$

for all $t \geq T-\bar{\tau}$ and $i = 1,\dots,N$.

We now introduce some notation.

Definition 2.1.1. We define

$$D_0 = \max_{i,j=1,\dots,N} \max_{s,t\in[-\bar{\tau},0]} |v_i(s) - v_j(t)|,$$

and in general, $\forall n \in \mathbb{N}$,

$$D_n := \max_{i,j=1,\dots,N} \max_{s,t\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} |v_i(s) - v_j(t)|.$$

Notice that inequality (2.0.3) can be written as

$$d_V(t) \leq D_0 e^{-C(t-2\bar{\tau})}, \quad \forall t \geq -\bar{\tau}.$$

Lemma 2.1.2. *For each $n \in \mathbb{N}_0$ we have that*

$$D_{n+1} \leq D_n. \tag{2.1.2}$$

Also, one can find a bound on $|v_i(t)|$, uniform with respect to t and $i = 1,\dots,N$.

Lemma 2.1.3. *For every $i = 1,\dots,N$, we have that*

$$|v_i(t)| \leq R_V^0, \quad \forall t \geq -\bar{\tau}, \tag{2.1.3}$$

where

$$R_V^0 := \max_{i=1,\dots,N} \max_{s\in[-\bar{\tau},0]} |v_i(s)|.$$

The previous lemma does not allow us to deduce a bound from below for the communication rates, as we did in Chapter 1. Indeed, in the case of the Cucker-Smale model, the communication rates depend on the distance between the agents' positions. To find a bound from below for the influence function, we need instead the following estimate.

Lemma 2.1.4. *For every $i, j = 1, \ldots, N$, we get*

$$|x_i(t-\tau(t)) - x_j(t)| \le 2\bar{\tau} R_V^0 + 4M_X^0 + d_X(t-\bar{\tau}), \quad \forall t \ge 0, \tag{2.1.4}$$

where

$$M_X^0 := \max_{i=1,\ldots,N} \max_{s\in[-\bar{\tau},0]} |x_i(s)|.$$

Proof. Given $i, j = 1, \ldots, N$ and $t \ge 0$, we have

$$\begin{aligned} |x_i(t-\tau(t)) - x_j(t)| \le & |x_i(t-\tau(t)) - x_i(t-\bar{\tau})| \\ & + |x_i(t-\bar{\tau}) - x_j(t-\bar{\tau})| + |x_j(t-\bar{\tau}) - x_j(t)| \\ \le & |x_i(t-\tau(t)) - x_i(t-\bar{\tau})| + d_X(t-\bar{\tau})|x_j(t-\bar{\tau}) - x_j(t)|. \end{aligned} \tag{2.1.5}$$

Now, assume $t > \bar{\tau}$. Then both $t-\bar{\tau}, t-\tau(t) > 0$ and from inequality (2.1.3) we get

$$\begin{aligned} |x_i(t-\tau(t)) - x_i(t-\bar{\tau})| &= \left| \int_{t-\bar{\tau}}^{t-\tau(t)} v_i(s)\, ds \right| \le \int_{t-\bar{\tau}}^{t-\tau(t)} |v_i(s)| ds \\ &\le R_V^0(t-\tau(t)-t+\bar{\tau}) \le \bar{\tau} R_V^0, \end{aligned}$$

and

$$|x_j(t-\bar{\tau}) - x_j(t)| = \left| -\int_{t-\bar{\tau}}^{t} v_j(s)\, ds \right| \le \int_{t-\bar{\tau}}^{t} |v_j(s)| ds \le \bar{\tau} R_V^0.$$

Thus, (2.1.5) becomes

$$|x_i(t-\tau(t)) - x_j(t)| \le 2\bar{\tau} R_V^0 + d_X(t-\bar{\tau}).$$

On the contrary, assume that $t \le \bar{\tau}$. Then $t-\bar{\tau} \le 0$ and from (2.1.3) we get

$$\begin{aligned} |x_j(t-\bar{\tau}) - x_j(t)| &= \left| x_j(t-\bar{\tau}) - x_j(0) - \int_0^t v_j(s)\, ds \right| \\ &\le |x_j(t-\bar{\tau}) - x_j(0)| + \int_0^t |v_j(s)| ds \\ &\le 2M_X^0 + tR_V^0 \le 2M_X^0 + \bar{\tau} R_V^0. \end{aligned}$$

Note that our assumption, $t \le \bar{\tau}$, does not imply that $t - \tau(t) \le 0$. So we can distinguish two cases.
If $t - \tau(t) > 0$, then

$$\begin{aligned} |x_i(t-\tau(t)) - x_i(t-\bar{\tau})| &= \left| x_i(0) + \int_0^{t-\tau(t)} v_i(s)\, ds - x_i(t-\bar{\tau}) \right| \\ &\le |x_i(0) - x_i(t-\bar{\tau})| + \int_0^{t-\tau(t)} |v_i(s)| ds \\ &\le 2M_X^0 + (t-\tau(t))R_V^0 \le 2M_X^0 + tR_V^0 \le 2M_X^0 + \bar{\tau} R_V^0, \end{aligned}$$

and (2.1.5) becomes

$$|x_i(t-\tau(t)) - x_j(t)| \leq 4M_X^0 + 2\bar{\tau}R_V^0 + d_X(t-\bar{\tau}).$$

On the other hand, if $t-\tau(t) \leq 0$, we have

$$|x_i(t-\tau(t)) - x_i(t-\bar{\tau})| \leq 2M_X^0,$$

and we can write

$$|x_i(t-\tau(t)) - x_j(t)| \leq 4M_X^0 + \bar{\tau}R_V^0 + d_X(t-\bar{\tau}).$$

We have so proved that, in all cases,

$$|x_i(t-\tau(t)) - x_j(t)| \leq 4M_X^0 + 2\bar{\tau}R_V^0 + d_X(t-\bar{\tau}),$$

which proves (2.1.4). □

In the following, given $t \geq -\bar{\tau}$, $i,j = 1,\dots,N$ and a vector $v \in \mathbb{R}^d$, we shall denote with

$$d_V^{(ij)_v}(t) := \langle v_i(t) - v_j(t), v\rangle.$$

Lemma 2.1.5. *For all $i,j = 1,\dots,N$, unit vector $v \in \mathbb{R}^d$ and $n \in \mathbb{N}_0$, we have that*

$$d_V^{(ij)_v}(t) \leq e^{-\tilde{K}(t-t_0)} d_V^{(ij)_v}(t_0) + (1-e^{-\tilde{K}(t-t_0)})D_n, \tag{2.1.6}$$

for all $t \geq t_0 \geq n\bar{\tau}$. Moreover, for each $n \in \mathbb{N}_0$ it holds

$$D_{n+1} \leq e^{-\tilde{K}\bar{\tau}} d_V(n\bar{\tau}) + (1-e^{-\tilde{K}\bar{\tau}})D_n. \tag{2.1.7}$$

Proof. Given $n \in \mathbb{N}_0$, for each $v \in \mathbb{R}^d$ unit vector, let denote with

$$M = \max_{l=1,\dots,N} \max_{s\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} \langle v_l(s), v\rangle,$$

$$m = \min_{l=1,\dots,N} \min_{s\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} \langle v_l(s), v\rangle.$$

Then $M - m \leq D_n$. We claim that, for all $i,j = 1,\dots,N$, $t \geq t_0 \geq n\bar{\tau}$,

$$\begin{aligned} \langle v_i(t), v\rangle &\leq e^{-\tilde{K}(t-t_0)}\langle v_i(t_0), v\rangle + (1-e^{-\tilde{K}(t-t_0)})M, \\ \langle v_j(t), v\rangle &\geq e^{-\tilde{K}(t-t_0)}\langle v_j(t_0), v\rangle + (1-e^{-\tilde{K}(t-t_0)})m. \end{aligned} \tag{2.1.8}$$

So, fix $i,j = 1,\dots,N$ and $t \geq t_0 \geq n\bar{\tau}$. Then, being $t \geq 0$, we have

$$\begin{aligned} \frac{d}{dt}\langle v_i(t), v\rangle &= \sum_{l:l\neq i} a_{il}(t)\langle v_l(t-\tau(t)) - v_i(t), v\rangle \\ &= \sum_{l:l\neq i} a_{il}(t)(\langle v_l(t-\tau(t)), v\rangle - \langle v_i(t), v\rangle) \end{aligned} \tag{2.1.9}$$

We recall that $a_{il}(t) = \frac{1}{N-1}\tilde{\psi}(|x_i(t) - x_l(t-\tau(t))|)$. Thus, being $\tilde{\psi}$ a bounded function, we can write $a_{il}(t) \leq \frac{\tilde{K}}{N-1}$. Furthermore, $t \geq n\bar{\tau}$, which implies that $t - \tau(t) \geq n\bar{\tau} - \bar{\tau}$. Then, by virtue of (2.1.1), we have that

$$m \leq \langle v_k(t-\tau(t)), v\rangle \leq M, \quad m \leq \langle v_k(t), v\rangle \leq M, \quad \forall k = 1, \dots, N.$$

So, combining all these facts, (2.1.9) becomes

$$\begin{aligned}
\frac{d}{dt}\langle v_i(t), v\rangle &= \sum_{l:l\neq i} a_{il}(t)(\langle v_l(t-\tau(t)), v\rangle - M + M - \langle v_i(t), v\rangle) \\
&\leq \sum_{l:l\neq i} a_{il}(t)(M - \langle v_i(t), v\rangle) \\
&\leq \frac{\tilde{K}}{N-1}\sum_{l:l\neq i}(M - \langle v_i(t), v\rangle) \\
&= \tilde{K}(M - \langle v_i(t), v\rangle).
\end{aligned}$$

Then, from the Gronwall's inequality with $t \geq t_0$ we get

$$\begin{aligned}
\langle v_i(t), v\rangle &\leq e^{-\int_{t_0}^t \tilde{K}ds}\langle v_i(t_0), v\rangle + \int_{t_0}^t \tilde{K}Me^{-(\int_{t_0}^t \tilde{K}dv - \int_{t_0}^s \tilde{K}dv)}ds \\
&= e^{-\tilde{K}(t-t_0)}\langle v_i(t_0), v\rangle + Me^{-\tilde{K}(t-t_0)}(e^{\tilde{K}(t-t_0)} - 1) \\
&= e^{-\tilde{K}(t-t_0)}\langle v_i(t_0), v\rangle + (1 - e^{-\tilde{K}(t-t_0)})M.
\end{aligned}$$

Hence, it holds

$$\langle v_i(t), v\rangle \leq e^{-\tilde{K}(t-t_0)}\langle v_i(t_0), v\rangle + (1 - e^{-\tilde{K}(t-t_0)})M, \tag{2.1.10}$$

for every $i = 1, \dots, N$, $t \geq t_0 \geq n\bar{\tau}$ and unit vector $v \in \mathbb{R}^d$, which proves the first inequality in (2.1.8).

Now, to prove the second inequality in (2.1.8), let $j = 1, \dots, N$, $t \geq t_0 \geq n\bar{\tau}$ and a unit vector $v \in \mathbb{R}^d$. Then, we can apply (2.1.10) to the unit vector $-v \in \mathbb{R}^d$ and we get

$$\langle v_j(t), -v\rangle \leq e^{-\tilde{K}(t-t_0)}\langle v_j(t_0), -v\rangle + (1 - e^{-\tilde{K}(t-t_0)})\left(\max_{l=1,\dots,N}\max_{s\in[n\bar{\tau}-\bar{\tau}, n\bar{\tau}]}\langle v_l(s), -v\rangle\right),$$

from which

$$\begin{aligned}
\langle v_j(t), v\rangle &\geq e^{-\tilde{K}(t-t_0)}\langle v_j(t_0), v\rangle + (1 - e^{-\tilde{K}(t-t_0)})\left(-\max_{l=1,\dots,N}\max_{s\in[n\bar{\tau}-\bar{\tau}, n\bar{\tau}]}\langle v_l(s), -v\rangle\right) \\
&= e^{-\tilde{K}(t-t_0)}\langle v_j(t_0), v\rangle + (1 - e^{-\tilde{K}(t-t_0)})m.
\end{aligned}$$

Therefore (2.1.8) holds true.
Now, from (2.1.8), for each $i,j = 1,\dots,N$, $v \in \mathbb{R}^d$ unit vector and $t \geq t_0 \geq n\bar{\tau}$, we have

$$\begin{aligned}
d_V^{(ij)_v}(t) &= \langle v_i(t) - v_j(t), v\rangle = \langle v_i(t)(t), v\rangle - \langle v_j(t), v\rangle \\
&\leq e^{-\tilde{K}(t-t_0)}\langle v_i(t_0), v\rangle + (1 - e^{-\tilde{K}(t-t_0)})M - e^{-\tilde{K}(t-t_0)}\langle v_j(t_0), v\rangle - (1 - e^{-\tilde{K}(t-t_0)})m \\
&= e^{-\tilde{K}(t-t_0)}\langle v_i(t_0) - v_j(t_0), v\rangle + (1 - e^{-\tilde{K}(t-t_0)})(M - m) \\
&= e^{-\tilde{K}(t-t_0)} d_V^{(ij)_v}(t_0) + (1 - e^{-\tilde{K}(t-t_0)})(M - m).
\end{aligned}$$

Then, by recalling that $M - m \leq D_n$, we finally get

$$d_V^{(ij)_v}(t) \leq e^{-\tilde{K}(t-t_0)} d_V^{(ij)_v}(t_0) + (1 - e^{-\tilde{K}(t-t_0)})D_n,$$

which proves (2.1.6).
Finally, we prove (2.1.7). Let $i,j = 1,\dots,N$ and $t_1, t_2 \in [n\bar{\tau}, n\bar{\tau} + \bar{\tau}]$ be such that

$$D_{n+1} = |v_i(t_1) - v_j(t_2)|.$$

Note that, if $D_{n+1} = 0$, then trivially

$$e^{-\tilde{K}\bar{\tau}} d_v(n\bar{\tau}) + (1 - e^{-\tilde{K}\bar{\tau}})D_n \geq 0 = D_{n+1}.$$

So we can assume $D_{n+1} > 0$ and we define the unit vector

$$v = \frac{v_i(t_1) - v_j(t_2)}{|v_i(t_1) - v_j(t_2)|}.$$

By applying (2.1.8) with $t_0 = n\bar{\tau} \leq t_1, t_2$, we get

$$\begin{aligned}
\langle v_i(t_1), v\rangle &\leq e^{-\tilde{K}(t_1 - n\bar{\tau})}\langle v_i(n\bar{\tau}), v\rangle + (1 - e^{-\tilde{K}(t_1 - n\bar{\tau})})M \\
&= e^{-\tilde{K}(t_1 - n\bar{\tau})}(\langle v_i(n\bar{\tau}), v\rangle - M) + M \\
&\leq e^{-\tilde{K}\bar{\tau}}(\langle v_i(n\bar{\tau}), v\rangle - M) + M \\
&= e^{-\tilde{K}\bar{\tau}}\langle v_i(n\bar{\tau}), v\rangle + (1 - e^{-\tilde{K}\bar{\tau}})M,
\end{aligned}$$

where we used the fact that $t_1 \leq n\bar{\tau} + \bar{\tau}$ and $\langle v_i(n\bar{\tau}), v\rangle - M \leq 0$, and

$$\begin{aligned}
\langle v_j(t_2), v\rangle &\geq e^{-\tilde{K}(t_2 - n\bar{\tau})}\langle v_j(n\bar{\tau}), v\rangle + (1 - e^{-\tilde{K}(t_2 - n\bar{\tau})})m \\
&= e^{-\tilde{K}(t_2 - n\bar{\tau})}(\langle v_j(n\bar{\tau}), v\rangle - m) + m \\
&\geq e^{-\tilde{K}\bar{\tau}}(\langle v_j(n\bar{\tau}), v\rangle - m) + m \\
&= e^{-\tilde{K}\bar{\tau}}\langle v_j(n\bar{\tau}), v\rangle + (1 - e^{-\tilde{K}\bar{\tau}})m,
\end{aligned}$$

where we used the fact that $t_2 \leq n\bar{\tau} + \bar{\tau}$ and $\langle v_j(n\bar{\tau}), v\rangle - m \geq 0$.
As a consequence, it holds

$$\begin{aligned}
D_{n+1} &= \langle v_i(t_1) - v_j(t_2), v\rangle = \langle v_i(t_1), v\rangle - \langle v_j(t_2), v\rangle \\
&\leq e^{-\tilde{K}\bar{\tau}}\langle v_i(n\bar{\tau}), v\rangle + (1 - e^{-\tilde{K}\bar{\tau}})M - e^{-\tilde{K}\bar{\tau}}\langle v_j(n\bar{\tau}), v\rangle - (1 - e^{-\tilde{K}\bar{\tau}})m \\
&= e^{-\tilde{K}\bar{\tau}}\langle v_i(n\bar{\tau}) - v_j(n\bar{\tau}), v\rangle + (1 - e^{-\tilde{K}\bar{\tau}})(M - m) \\
&\leq e^{-\tilde{K}\bar{\tau}}|v_i(n\bar{\tau}) - v_j(n\bar{\tau})||v| + (1 - e^{-\tilde{K}\bar{\tau}})(M - m) \\
&\leq e^{-\tilde{K}\bar{\tau}}d_V(n\bar{\tau}) + (1 - e^{-\tilde{K}\bar{\tau}})D_n,
\end{aligned}$$

which concludes our proof. □

2.2 Proof of the flocking estimate

Now, we give the following definition.

Definition 2.2.1. We define

$$\tilde{\phi}(t) := \min\left\{e^{-\tilde{K}\bar{\tau}}\tilde{\psi}_t, \frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}}\right\},$$

where

$$\tilde{\psi}_t = \min\left\{\tilde{\psi}(r) : r \in \left[0, 2\bar{\tau}R_V^0 + 4M_X^0 + \max_{s\in[-\bar{\tau},t]} d_X(s)\right]\right\},$$

for all $t \geq -\bar{\tau}$.

By definition, being $\tilde{\psi}$ a positive function, we have that $\tilde{\psi}_t > 0$, for all $t \geq -\bar{\tau}$. Thus, the function $\tilde{\phi}$ is positive too.

Remark 2.2.1. Let us note that from estimate (2.1.4), for all $t \geq 0$ and $i, j = 1, \dots, N$, it holds that

$$\tilde{\psi}(|x_i(t) - x_j(t - \tau(t))|) \geq \tilde{\psi}_{t-\bar{\tau}},$$

from which

$$\tilde{\psi}(|x_i(t) - x_j(t - \tau(t))|) \geq e^{K\bar{\tau}}\tilde{\phi}(t - \bar{\tau}). \tag{2.2.1}$$

Lemma 2.2.2. *For each integer $n \geq 2$, we have that*

$$D_{n+1} \leq \left(1 - e^{-K\bar{\tau}}\int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds\right) D_{n-2}. \tag{2.2.2}$$

Proof. We first show that, for each $n \geq 2$,

$$d_V(n\bar{\tau}) \leq \left(1 - \int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds\right) D_{n-2}. \tag{2.2.3}$$

To this aim, let $n \geq 2$. Note that, if $d_V(n\bar{\tau}) = 0$, by definition of ϕ we have that

$$\begin{aligned}\left(1 - \int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds\right) D_{n-2} &\geq \left(1 - \int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}} ds\right) D_{n-2} \\ &= \left(1 - \frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}}(n\bar{\tau} - \bar{\tau} - n\bar{\tau} + 2\bar{\tau})\right) D_{n-2} \\ &= \left(1 - e^{-2\tilde{K}\bar{\tau}}\right) D_{n-2} \geq 0 = d_V(n\bar{\tau}).\end{aligned}$$

So we can assume $d_V(n\bar{\tau}) > 0$. Moreover, let $i, j = 1, \ldots, N$ be such that

$$d_V(n\bar{\tau}) = |v_i(n\bar{\tau}) - v_j(n\bar{\tau})|.$$

We set

$$v = \frac{v_i(n\bar{\tau}) - v_j(n\bar{\tau})}{|v_i(n\bar{\tau}) - v_j(n\bar{\tau})|}.$$

Then v is a unit vector for which we can write

$$d_V(n\bar{\tau}) = \langle v_i(n\bar{\tau}) - v_j(n\bar{\tau}), v\rangle = d_V^{(ij)_v}(n\bar{\tau}).$$

At this point, we distinguish two cases.

Case I. Assume that there exists $t_0 \in [n\bar{\tau} - 2\bar{\tau}, n\bar{\tau}]$ such that $d_V^{(ij)_v}(t_0) < 0$. Note that

$$\left(1 - \int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds\right) \geq 1 - e^{-2K\bar{\tau}}.$$

Then, by using (2.1.6) with $n\bar{\tau} \geq t_0 \geq n\bar{\tau} - 2\bar{\tau}$, we have

$$\begin{aligned}d_V^{(ij)_v}(n\bar{\tau}) &\leq e^{-\tilde{K}(n\bar{\tau}-t_0)} d_V^{(ij)_v}(t_0) + (1 - e^{-\tilde{K}(n\bar{\tau}-t_0)})D_{n-2} \\ &< (1 - e^{-\tilde{K}(n\bar{\tau}-t_0)})D_{n-2} \leq (1 - e^{-2\tilde{K}\bar{\tau}})D_{n-2} \leq \left(1 - \int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds\right) D_{n-2}.\end{aligned}$$

Case II. Assume that $d_V^{(ij)_v}(t) \geq 0$, for every $t \in [n\bar{\tau} - 2\bar{\tau}, n\bar{\tau}]$. We set

$$M = \max_{l=1,\ldots,N} \max_{s\in[n\bar{\tau}-2\bar{\tau}, n\bar{\tau}-\bar{\tau}]} \langle v_l(s), v\rangle,$$

$$m = \min_{l=1,\ldots,N} \min_{s\in[n\bar{\tau}-2\bar{\tau}, n\bar{\tau}-\bar{\tau}]} \langle v_l(s), v\rangle.$$

Then, $M - m \leq D_{n-1}$. Notice that, from (2.2.1), for each $l, k = 1, \ldots, N$ and $t \geq 0$,

$$a_{lk}(t) \geq \frac{e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t - \bar{\tau})}{N - 1}. \tag{2.2.4}$$

Thus, for every $t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]$, it comes that

$$\begin{aligned}
\frac{d}{dt} d_V^{(ij)_v}(t) &= \sum_{l:l\neq i} a_{il}(t)\langle v_l(t-\tau(t)) - v_i(t), v\rangle + \sum_{l:l\neq i} a_{jl}(t)\langle v_j(t) - v_l(t-\tau(t)), v\rangle \\
&= \sum_{l:l\neq i} a_{il}(t)(\langle v_l(t-\tau(t)), v\rangle - M + M - \langle v_i(t), v\rangle) \\
&\quad + \sum_{l:l\neq i} a_{jl}(t)(\langle v_j(t), v\rangle - m + m - \langle v_l(t-\tau(t)), v\rangle) \\
&:= S_1 + S_2.
\end{aligned}$$

We recall that ψ is bounded and that, from (2.1.1),

$$m \leq \langle v_k(s), v\rangle \leq M, \quad \forall s \geq n\bar{\tau} - 2\bar{\tau}, \forall k = 1, \dots, N.$$

Combining these facts with (2.2.4), for every $t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]$, it holds that $t, t-\tau(t) \geq n\bar{\tau} - 2\bar{\tau}$ and we can write

$$\begin{aligned}
S_1 &= \sum_{l:l\neq i} a_{il}(t)(\langle v_l(t-\tau(t)), v\rangle - M) + \sum_{l:l\neq i} a_{il}(t)(M - \langle v_i(t), v\rangle) \\
&\leq \frac{e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t-\bar{\tau})}{N-1} \sum_{l:l\neq i}(\langle v_l(t-\tau(t)), v\rangle - M) + \frac{\tilde{K}}{N-1}\sum_{l:l\neq i}(M - \langle v_i(t), v\rangle) \\
&= \frac{e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t-\bar{\tau})}{N-1} \sum_{l:l\neq i}(\langle v_l(t-\tau(t)), v\rangle - M) + \tilde{K}(M - \langle v_i(t), v\rangle),
\end{aligned}$$

and

$$\begin{aligned}
S_2 &= \sum_{l:l\neq j} a_{jl}(t)(\langle v_j(t), v\rangle - m) + \sum_{l:l\neq j} a_{jl}(t)(m - \langle v_l(t-\tau(t)), v\rangle) \\
&\leq \frac{\tilde{K}}{N-1}\sum_{l:l\neq j}(\langle v_j(t), v\rangle - m) + \frac{e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t-\bar{\tau})}{N-1}\sum_{l:l\neq j}(m - \langle v_l(t-\tau(t)), v\rangle) \\
&= \tilde{K}(\langle v_j(t), v\rangle - m) + \frac{e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t-\bar{\tau})}{N-1}\sum_{l:l\neq j}(m - \langle v_l(t-\tau(t)), v\rangle).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
S_1+S_2 \le\ & \tilde{K}(M-\langle v_i(t),v\rangle+\langle v_j(t),v\rangle-m)\\
&+\frac{e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t-\bar{\tau})}{N-1}\sum_{l:l\neq i,j}(\langle v_l(t-\tau(t)),v\rangle-M+m-\langle v_l(t-\tau(t)),v\rangle)\\
&+\frac{e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t-\bar{\tau})}{N-1}(\langle v_j(t-\tau(t)),v\rangle-M+m-\langle v_i(t-\tau(t)),v\rangle)\\
=\ & \tilde{K}(M-m-d_V^{(ij)_v}(t))+\frac{e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t-\bar{\tau})}{N-1}(N-2)(m-M)\\
&+\frac{e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t-\bar{\tau})}{N-1}(m-M-d_V^{(ij)_v}(t-\tau(t))).
\end{aligned}$$

Note that, being $t\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]$, it holds that $t-\tau(t)\in[n\bar{\tau}-2\bar{\tau},n\bar{\tau}]$. Therefore, from our assumption, we have $d_V^{(ij)_v}(t-\tau(t))\ge 0$, from which follows that

$$\frac{e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t-\bar{\tau})}{N-1}(m-M-d_V^{(ij)_v}(t-\tau(t))\le \frac{e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t-\bar{\tau})}{N-1}(m-M).$$

Thus, taking into account of the fact that $M-m\le D_{n-1}$, we get

$$\begin{aligned}
\frac{d}{dt}d_V^{(ij)_v}(t) &\le \tilde{K}(M-m-d_V^{(ij)_v}(t))+\frac{e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t-\bar{\tau})}{N-1}(N-2)(m-M)+\frac{e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t-\bar{\tau})}{N-1}(m-M)\\
&=\tilde{K}(M-m-d_V^{(ij)_v}(t))+e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t-\bar{\tau})(m-M)\\
&=(\tilde{K}-e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t-\bar{\tau}))(M-m)-\tilde{K}d_V^{(ij)_v}(t)\\
&\le(\tilde{K}-e^{\tilde{K}\bar{\tau}}\tilde{\phi}(t-\bar{\tau}))D_{n-1}-\tilde{K}d_V^{(ij)_v}(t),
\end{aligned}$$

for every $t\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]$. Then, from the Gronwall's inequality, for every $t\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]$, we have

$$\begin{aligned}
d_V^{(ij)_v}(t) &\le e^{-\int_{n\bar{\tau}-\bar{\tau}}^t\tilde{K}ds}d_V^{(ij)_v}(n\bar{\tau}-\bar{\tau})+D_{n-1}\int_{n\bar{\tau}-\bar{\tau}}^t(\tilde{K}-e^{\tilde{K}\bar{\tau}}\tilde{\phi}(s-\bar{\tau}))e^{-\left(\int_{n\bar{\tau}-\bar{\tau}}^t\tilde{K}dv-\int_{n\bar{\tau}-\bar{\tau}}^s\tilde{K}dv\right)}ds\\
&=e^{-\tilde{K}(t-n\bar{\tau}+\bar{\tau})}d_V^{(ij)_v}(n\bar{\tau}-\bar{\tau})+D_{n-1}\int_{n\bar{\tau}-\bar{\tau}}^t(\tilde{K}-e^{K\bar{\tau}})\tilde{\phi}(s-\bar{\tau})e^{-\tilde{K}(t-s)}ds\\
&=e^{-\tilde{K}(t-n\bar{\tau}+\bar{\tau})}d_V^{(ij)_v}(n\bar{\tau}-\bar{\tau})+D_{n-1}\left(e^{-\tilde{K}t}[e^{\tilde{K}s}]_{n\bar{\tau}-\bar{\tau}}^t-e^{\tilde{K}\bar{\tau}}\int_{n\bar{\tau}-\bar{\tau}}^t e^{-\tilde{K}(t-s)}\tilde{\phi}(s-\bar{\tau})ds\right)\\
&=e^{-\tilde{K}(t-n\bar{\tau}+\bar{\tau})}d_V^{(ij)_v}(n\bar{\tau}-\bar{\tau})+D_{n-1}\left(1-e^{-\tilde{K}(t-n\bar{\tau}+\bar{\tau})}-e^{\tilde{K}\bar{\tau}}\int_{n\bar{\tau}-\bar{\tau}}^t e^{-\tilde{K}(t-s)}\tilde{\phi}(s-\bar{\tau})ds\right).
\end{aligned}$$

In particular, for $t = n\bar{\tau}$ it holds

$$\begin{aligned} d_V^{(ij)_v}(n\bar{\tau}) &\leq e^{-\tilde{K}\bar{\tau}} d_V^{(ij)_v}(n\bar{\tau}-\bar{\tau}) + D_{n-1}\left(1 - e^{-\tilde{K}\bar{\tau}} - e^{\tilde{K}\bar{\tau}}\int_{n\bar{\tau}-\bar{\tau}}^{n\bar{\tau}} e^{-\tilde{K}(n\bar{\tau}-s)}\tilde{\phi}(s-\bar{\tau})ds\right) \\ &\leq e^{-\tilde{K}\bar{\tau}} d_V^{(ij)_v}(n\bar{\tau}-\bar{\tau}) + D_{n-1}\left(1 - e^{-\tilde{K}\bar{\tau}} - e^{\tilde{K}\bar{\tau}}\int_{n\bar{\tau}-\bar{\tau}}^{n\bar{\tau}} \tilde{\phi}(s-\bar{\tau})ds\right). \end{aligned}$$

Notice that $d_V^{(ij)_v}(n\bar{\tau}-\bar{\tau}) \leq D_{n-1}$ and that

$$e^{\tilde{K}\bar{\tau}}\int_{n\bar{\tau}-\bar{\tau}}^{n\bar{\tau}} \tilde{\phi}(s-\bar{\tau})ds \geq \int_{n\bar{\tau}-\bar{\tau}}^{n\bar{\tau}} \tilde{\phi}(s-\bar{\tau})ds.$$

So we can write

$$\begin{aligned} d_V^{(ij)_v}(n\bar{\tau}) &\leq e^{-\tilde{K}\bar{\tau}} D_{n-1} + D_{n-1}\left(1 - e^{-\tilde{K}\bar{\tau}} - \int_{n\bar{\tau}-\bar{\tau}}^{n\bar{\tau}} \tilde{\phi}(s-\bar{\tau})ds\right) \\ &= D_{n-1}\left(1 - \int_{n\bar{\tau}-\bar{\tau}}^{n\bar{\tau}} \tilde{\phi}(s-\bar{\tau})ds\right). \end{aligned}$$

Then, with a change of variable, we get

$$d_V^{(ij)_v}(n\bar{\tau}) \leq D_{n-1}\left(1 - \int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds\right),$$

and, being $D_{n-1} \leq D_{n-2}$, we can conclude that

$$d_V^{(ij)_v}(n\bar{\tau}) \leq \left(1 - \int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds\right) D_{n-2}.$$

Therefore, (2.2.3) holds true.
Now, we are able to prove (2.2.2). Indeed, for each $n \geq 2$, from (2.1.7) and (2.2.2), it immediately follows that

$$\begin{aligned} D_{n+1} &\leq e^{-\tilde{K}\bar{\tau}} d_V(n\bar{\tau}) + (1 - e^{-\tilde{K}\bar{\tau}})D_n \\ &\leq e^{-\tilde{K}\bar{\tau}}\left(1 - \int_{n\bar{\tau}-2\bar{\tau}}^{n\tau_1-\bar{\tau}} \tilde{\phi}(s)ds\right) D_{n-2} + (1 - e^{-\tilde{K}\bar{\tau}})D_{n-2} \\ &= \left(1 - e^{-\tilde{K}\bar{\tau}}\int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds\right) D_{n-2}. \end{aligned}$$

□

Proof of Theorem 2.0.1. Let $\{(x_i, v_i)\}_{i=1,\dots,N}$ be solution to (0.1.24) under the initial conditions (0.1.28). Following [101], we introduce the function $\mathcal{D} : [-\bar{\tau}, \infty) \to [0, \infty)$, defined as

$$\mathcal{D}(t) := \begin{cases} D_0, & t \in [-\bar{\tau}, 2\bar{\tau}] \\ \mathcal{D}(n\bar{\tau})\left(1 - e^{-\tilde{K}\bar{\tau}}\int_{n\bar{\tau}}^{t} \tilde{\phi}(s)ds\right)^{\frac{1}{3}}, & t \in (n\bar{\tau}, n\bar{\tau}+\bar{\tau}],\ n \geq 2 \end{cases}.$$

By construction, $\mathcal{D}$ is continuous and nonincreasing. Moreover, we claim that

$$D_n \leq \mathcal{D}(t), \tag{2.2.5}$$

for all $n \in \mathbb{N}_0$ and $t \in [-\bar{\tau}, n\bar{\tau}]$. To prove this, we first show that, for each $n \geq 3$,

$$1 - e^{-\tilde{K}\bar{\tau}} \int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds \leq \frac{\mathcal{D}(n\bar{\tau}+\bar{\tau})}{\mathcal{D}(n\bar{\tau}-2\bar{\tau})}. \tag{2.2.6}$$

So, let $n \geq 3$. We split

$$1 - e^{-\tilde{K}\bar{\tau}} \int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds$$
$$= \left(1 - e^{-\tilde{K}\bar{\tau}} \int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds\right)^{\frac{1}{3}} \left(1 - e^{-\tilde{K}\bar{\tau}} \int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds\right)^{\frac{1}{3}} \left(1 - e^{-\tilde{K}\bar{\tau}} \int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds\right)^{\frac{1}{3}}.$$

Now, it is easy to see that $\tilde{\phi}$ is a nonincreasing function. Thus, for each $m \geq n$,

$$\int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds \geq \int_{m\bar{\tau}-2\bar{\tau}}^{m\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds.$$

So we can write

$$1 - e^{-\tilde{K}\bar{\tau}} \int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds$$
$$\leq \left(1 - e^{-\tilde{K}\bar{\tau}} \int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds\right)^{\frac{1}{3}} \left(1 - e^{-\tilde{K}\bar{\tau}} \int_{n\bar{\tau}-\bar{\tau}}^{n\bar{\tau}} \tilde{\phi}(s)ds\right)^{\frac{1}{3}} \left(1 - e^{-\tilde{K}\bar{\tau}} \int_{n\bar{\tau}}^{n\bar{\tau}+\bar{\tau}} \tilde{\phi}(s)ds\right)^{\frac{1}{3}}$$
$$= \frac{\mathcal{D}(n\bar{\tau}-\bar{\tau})}{\mathcal{D}(n\bar{\tau}-2\bar{\tau})} \frac{\mathcal{D}(n\bar{\tau})}{\mathcal{D}(n\bar{\tau}-\bar{\tau})} \frac{\mathcal{D}(n\bar{\tau}+\bar{\tau})}{\mathcal{D}(n\bar{\tau})} = \frac{\mathcal{D}(n\bar{\tau}+\bar{\tau})}{\mathcal{D}(n\bar{\tau}-2\bar{\tau})},$$

from which (2.2.6) holds true.
At this point, we are able to prove (2.2.5). By induction, if $n \leq 2$, from Lemma 2.1.2 we can immediately say that

$$D_n \leq D_0 = \mathcal{D}(t),$$

for all $t \in [-\bar{\tau}, 2\bar{\tau}]$. So we can assume that (2.2.5) holds for each $2 < m \leq n$ and prove it for $n+1$. From the induction hypothesis and by using again Lemma 2.1.2, we have

$$D_{n+1} \leq D_n \leq \mathcal{D}(t),$$

for all $t \in [-\bar{\tau}, n\bar{\tau}]$. On the other hand, for all $t \in (n\bar{\tau}, n\bar{\tau}+\bar{\tau}]$, being $n > 2$, from (2.2.2) we get

$$D_{n+1} \leq \left(1 - e^{-\tilde{K}\bar{\tau}} \int_{n\bar{\tau}-2\bar{\tau}}^{n\bar{\tau}-\bar{\tau}} \tilde{\phi}(s)ds\right) D_{n-2}.$$

From the induction hypothesis or from the base case, $D_{n-2} \leq \mathcal{D}(t)$, for each $t \in [-\bar{\tau}, n\bar{\tau} - 2\bar{\tau}]$ So, in particular, $D_{n-2} \leq \mathcal{D}(n\bar{\tau} - 2\bar{\tau})$. Therefore, combining this with (2.2.6) and with the fact that $\mathcal{D}$ is nonincreasing, we have that

$$D_{n+1} \leq \frac{\mathcal{D}(n\bar{\tau} + \bar{\tau})}{\mathcal{D}(n\bar{\tau} - 2\bar{\tau})} D_{n-2} \leq \frac{\mathcal{D}(n\bar{\tau} + \bar{\tau})}{\mathcal{D}(n\bar{\tau} - 2\bar{\tau})} \mathcal{D}(n\bar{\tau} - 2\bar{\tau}) = \mathcal{D}(n\bar{\tau} + \bar{\tau}) \leq \mathcal{D}(t),$$

for all $t \in (n\bar{\tau}, n\bar{\tau} + \bar{\tau}]$, which proves (2.2.5).
Now, notice that, for almost all time

$$\frac{d}{dt} \max_{s \in [-\bar{\tau}, t]} d_X(s) \leq \left| \frac{d}{dt} d_X(t) \right|,$$

since $\max_{s \in [-\bar{\tau}, t]} d_X(s)$ is constant or increases like $d_X(t)$. Moreover, for almost all time

$$\left| \frac{d}{dt} d_X(t) \right| \leq d_V(t).$$

To see this, let $i, j = 1, \ldots, N$ be such that $d_X(t) = |x_i(t) - xj(t)|$. Obviously, if $\left| \frac{d}{dt} d_X(t) \right| = 0$, then

$$\left| \frac{d}{dt} d_X(t) \right| = 0 \leq d_V(t).$$

So we can assume $\left| \frac{d}{dt} d_X(t) \right| > 0$. Notice that

$$\begin{aligned} \frac{d}{dt}(d_X(t))^2 = \frac{d}{dt}|x_i(t) - x_j(t)|^2 &= 2|x_i(t) - x_j(t)| \frac{d}{dt}|x_i(t) - x_j(t)| \\ &= 2|x_i(t) - x_j(t)| \frac{d}{dt} d_X(t), \end{aligned}$$

with $|x_i(t) - x_j(t)| > 0$, since otherwise $d_X(\cdot)$ wouldn't be differentiable at t. Also,

$$\frac{d}{dt}(d_X(t))^2 = 2\langle v_i(t) - v_j(t), x_i(t) - x_j(t) \rangle,$$

so that

$$|x_i(t) - x_j(t)| \frac{d}{dt} d_X(t) = \langle v_i(t) - v_j(t), x_i(t) - x_j(t) \rangle.$$

Thus,

$$|x_i(t) - x_j(t)| \left| \frac{d}{dt} d_X(t) \right| = |\langle v_i(t) - v_j(t), x_i(t) - x_j(t) \rangle| \leq |v_i(t) - v_j(t)||x_i(t) - x_j(t)|,$$

from which, dividing by $|x_i(t) - x_j(t)|$, we get

$$\left| \frac{d}{dt} d_X(t) \right| \leq |v_i(t) - v_j(t)| \leq d_V(t).$$

Therefore, for almost all time

$$\frac{d}{dt}\max_{s\in[-\bar{\tau},t]} d_X(s) \leq \left|\frac{d}{dt}d_X(t)\right| \leq d_V(t). \tag{2.2.7}$$

Next, let $\mathcal{L}:[-\bar{\tau},\infty)\to[0,\infty)$ be the function given by

$$\mathcal{L}(t) := \mathcal{D}(t) + \frac{e^{-\tilde{K}\bar{\tau}}}{3}\int_0^{2\bar{\tau}R_V^0+4M_X^0+\max_{s\in[-\bar{\tau},t]} d_X(s)} \min\left\{e^{-\tilde{K}\bar{\tau}}\min_{\sigma\in[0,r]}\tilde{\psi}(\sigma), \frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}}\right\} dr,$$

for all $t \geq -\bar{\tau}$. By definition, $\mathcal{L}$ is continuous. In addition, for each $n\geq 2$ and for $a.e.\, t\in(n\bar{\tau}, n\bar{\tau}+\bar{\tau})$, we have that

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &= \frac{d}{dt}\mathcal{D}(t) + \frac{e^{-\tilde{K}\bar{\tau}}}{3}\min\left\{e^{-\tilde{K}\bar{\tau}}\tilde{\psi}_t, \frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}}\right\}\frac{d}{dt}\max_{s\in[-\bar{\tau},t]} d_X(s)\\ &= \frac{d}{dt}\mathcal{D}(t) + \frac{e^{-\tilde{K}\bar{\tau}}}{3}\tilde{\phi}(t)\frac{d}{dt}\max_{s\in[-\bar{\tau},t]} d_X(s),\end{aligned}$$

and from (2.2.7) we get

$$\frac{d}{dt}\mathcal{L}(t) \leq \frac{d}{dt}\mathcal{D}(t) + \frac{e^{-\tilde{K}\bar{\tau}}}{3}\tilde{\phi}(t)d_V(t).$$

Now, for $a.e.\, t\in(n\bar{\tau}, n\bar{\tau}+\bar{\tau})$, with $n\geq 2$, we compute

$$\frac{d}{dt}\mathcal{D}(t) = -\frac{1}{3}\mathcal{D}(n\bar{\tau})\left(1-e^{-\tilde{K}\bar{\tau}}\int_{n\bar{\tau}}^t\tilde{\phi}(s)ds\right)^{-\frac{2}{3}} e^{-\tilde{K}\bar{\tau}}\phi(t).$$

Thus, for each $n\geq 2$ and for $a.e.\, t\in(n\bar{\tau}, n\bar{\tau}+\bar{\tau})$,

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &\leq \frac{e^{-\tilde{K}\bar{\tau}}}{3}\tilde{\phi}(t)\left(d_V(t) - \frac{\mathcal{D}(n\bar{\tau})}{\left(1-e^{-\tilde{K}\bar{\tau}}\int_{n\bar{\tau}}^t\tilde{\phi}(s)ds\right)^{\frac{2}{3}}}\right)\\ &\leq \frac{e^{-\tilde{K}\bar{\tau}}}{3}\tilde{\phi}(t)(d_V(t)-\mathcal{D}(n\bar{\tau})).\end{aligned}$$

Lastly, we can note that $d_V(t)\leq\mathcal{D}(n\bar{\tau})$, since $d_V(t)\leq D_{n+1}$ and $D_{n+1}\leq\mathcal{D}(n\bar{\tau})$ from inequality (2.2.5). Then, we get

$$\frac{d}{dt}\mathcal{L}(t) \leq 0, \tag{2.2.8}$$

for $a.e.\, t\in(n\bar{\tau}, n\bar{\tau}+\bar{\tau})$ and for each $n\geq 2$. Integrating (2.2.8) over $(2\bar{\tau}, t)$ for $t>2\bar{\tau}$ it comes that

$$\mathcal{L}(t) \leq \mathcal{L}(2\bar{\tau}). \tag{2.2.9}$$

Therefore, from (2.2.9), it holds

$$\frac{e^{-\tilde{K}\bar{\tau}}}{3}\int_0^{2\bar{\tau}R_V^0+4M_X^0+\max_{s\in[-\bar{\tau},t]}d_X(s)}\min\left\{e^{-\tilde{K}\bar{\tau}}\min_{\sigma\in[0,r]}\tilde{\psi}(\sigma),\frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}}\right\}dr\leq\mathcal{L}(2\bar{\tau}),\tag{2.2.10}$$

for all $t\geq 2\bar{\tau}$. Letting $t\to\infty$ in (2.2.10), we finally get

$$\frac{e^{-\tilde{K}\bar{\tau}}}{3}\int_0^{2\bar{\tau}R_V^0+4M_X^0+\sup_{s\in[-\bar{\tau},\infty)}d_X(s)}\min\left\{e^{-\tilde{K}\bar{\tau}}\min_{\sigma\in[0,r]}\tilde{\psi}(\sigma),\frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}}\right\}dr\leq\mathcal{L}(2\bar{\tau}).\tag{2.2.11}$$

Finally, since the function $\tilde{\psi}$ satisfies property (2.0.1), from (2.2.11), we can conclude that there exists a positive constant d^* such that

$$2\bar{\tau}R_V^0+4M_X^0+\sup_{s\in[-\bar{\tau},\infty)}d_X(s)\leq d^*.\tag{2.2.12}$$

Indeed, assume by contradiction that

$$2\bar{\tau}R_V^0+4M_X^0+\sup_{s\in[-\bar{\tau},\infty)}d_X(s)=+\infty.\tag{2.2.13}$$

Then, equation (2.2.11) reads as

$$\int_0^{+\infty}\min\left\{e^{-\tilde{K}\bar{\tau}}\min_{\sigma\in[0,r]}\tilde{\psi}(\sigma),\frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}}\right\}dr\leq\mathcal{L}(2\bar{\tau})\tag{2.2.14}$$

Now, two different situations can occur.
Case I) Assume that, for all $r\in[0,+\infty)$,

$$\frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}}\leq e^{-\tilde{K}\bar{\tau}}\min_{\sigma\in[0,r]}\tilde{\psi}(\sigma).$$

Thus,

$$\int_0^{+\infty}\min\left\{e^{-\tilde{K}\bar{\tau}}\min_{\sigma\in[0,r]}\tilde{\psi}(\sigma),\frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}}\right\}dr=\int_0^{+\infty}\frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}}dr=+\infty,$$

which is in contradiction with (2.2.14).
Case II) Assume that there exists $r_1\in[0,+\infty)$ such that

$$e^{-\tilde{K}\bar{\tau}}\min_{\sigma\in[0,r_1]}\tilde{\psi}(\sigma)<\frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}}.$$

Note that, for all $r\geq r_1$, it holds that

$$\min_{\sigma\in[0,r]}\tilde{\psi}(\sigma)\leq\min_{\sigma\in[0,r_1]}\tilde{\psi}(\sigma),$$

from which

$$e^{-\tilde{K}\bar{\tau}} \min_{\sigma\in[0,r]} \tilde{\psi}(\sigma) < \frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}}, \quad \forall r \geq r_1.$$

Thus, using (2.0.1) we can write

$$\int_0^{+\infty} \min\left\{e^{-\tilde{K}\bar{\tau}} \min_{\sigma\in[0,r]} \tilde{\psi}(\sigma), \frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}}\right\} dr \geq \int_{r_1}^{+\infty} \min\left\{e^{-\tilde{K}\bar{\tau}} \min_{\sigma\in[0,r]} \tilde{\psi}(\sigma), \frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}}\right\} dr$$
$$= e^{-\tilde{K}\bar{\tau}} \int_{r_1}^{+\infty} \min_{\sigma\in[0,r]} \tilde{\psi}(\sigma) dr = +\infty.$$

Hence, also in this case we get a contradiction.
As a consequence, in all the two possible situations we get a contradiction and we deduce the existence of a positive constant d^* for which inequality (2.2.12) is fulfilled.
Finally, we define

$$\phi^* := \min\left\{e^{-\tilde{K}\bar{\tau}}\psi_*, \frac{e^{-2\tilde{K}\bar{\tau}}}{\bar{\tau}}\right\},$$

where

$$\psi_* = \min_{r\in[0,d^*]} \tilde{\psi}(r).$$

Note that $\phi^* > 0$, being $\tilde{\psi}$ a positive function. Also, from (2.2.12), it comes that

$$\psi_* \leq \min\left\{\tilde{\psi}(r) : r \in \left[0, 2\bar{\tau}R_V^0 + 4M_X^0 + \max_{s\in[-\bar{\tau},t]} d_X(s)\right]\right\} = \tilde{\psi}_t,$$

for all $t \geq -\bar{\tau}$. Thus, we get

$$\phi^* \leq \tilde{\phi}(t), \quad \forall t \geq -\bar{\tau}.$$

This implies that, for each $n \geq 2$

$$\left(1 - e^{-\tilde{K}\bar{\tau}} \int_{n\bar{\tau}}^{n\bar{\tau}+\bar{\tau}} \tilde{\phi}(s) ds\right)^{\frac{1}{3}} \leq \left(1 - e^{-\tilde{K}\bar{\tau}} \int_{n\bar{\tau}}^{n\bar{\tau}+\bar{\tau}} \phi^* ds\right)^{\frac{1}{3}}$$
$$= \left(1 - e^{-\tilde{K}\bar{\tau}}\phi^*\bar{\tau}\right)^{\frac{1}{3}}, \tag{2.2.15}$$

with $\left(1 - e^{-\tilde{K}\bar{\tau}}\phi^*\bar{\tau}\right)^{\frac{1}{3}} < 1$.
Next, we set

$$C = \frac{1}{3\bar{\tau}} \ln\left(\frac{1}{1 - e^{-\tilde{K}\bar{\tau}}\phi^*\bar{\tau}}\right) > 0.$$

Notice that C is a constant independent of N. Moreover, we have

$$\left(1 - e^{-\tilde{K}\bar{\tau}}\phi^*\bar{\tau}\right)^{\frac{1}{3}} = e^{-C\bar{\tau}},$$

so that (2.2.15) becomes

$$\left(1 - e^{-\tilde{K}\bar{\tau}} \int_{n\bar{\tau}}^{n\bar{\tau}+\bar{\tau}} \tilde{\phi}(s)ds\right)^{\frac{1}{3}} \leq e^{-C\bar{\tau}}, \quad \forall n \geq 2. \tag{2.2.16}$$

Now we claim that, for each $n \geq 2$, it holds

$$\mathcal{D}(n\bar{\tau}) \leq D_0 e^{-C(n-2)\bar{\tau}}. \tag{2.2.17}$$

Indeed, by induction, if $n = 2$ then trivially $\mathcal{D}(2\bar{\tau}) = D_0$ and the claim holds. So suppose (2.2.17) holds true for $n \geq 2$ and prove it for $n + 1$. From the induction hypothesis and by recalling of (2.2.16), we can write

$$\begin{aligned} \mathcal{D}(n\bar{\tau} + \bar{\tau}) &= \mathcal{D}(n\bar{\tau}) \left(1 - e^{-\tilde{K}\bar{\tau}} \int_{n\bar{\tau}}^{n\bar{\tau}+\bar{\tau}} \tilde{\phi}(s)ds\right)^{\frac{1}{3}} \\ &\leq D_0 e^{-C(n-2)\bar{\tau}} e^{-C\bar{\tau}} = D_0 e^{-C(n+1-2)\bar{\tau}}. \end{aligned}$$

Hence, from (2.2.5) and (2.2.17) it follows that, for each $t > 2\bar{\tau}$, if $t \in (n\bar{\tau}, n\bar{\tau} + \bar{\tau})$, for some $n \geq 2$,

$$d_V(t) \leq D_{n+1} \leq \mathcal{D}(n\bar{\tau} + \bar{\tau}) \leq D_0 e^{-C(n+1-2)\bar{\tau}} \leq D_0 e^{-C(t-2\bar{\tau})}.$$

Thus, combining this with the fact that, for all $[-\bar{\tau}, 2\bar{\tau}]$,

$$d_V(t) \leq D_0 \leq D_0 e^{-C(t-2\bar{\tau})},$$

we can conclude that estimate (2.0.3) holds too. □

Chapter 3

First and second-order Cucker-Smale models with non-universal interaction, time delay and communication failures

In this chapter, we will investigate the asymptotic behavior of solutions to the first and second-order Cucker-Smale model (0.1.8) and (0.1.26). The aim of the analysis we will carry out is to find conditions ensuring the asymptotic consensus for both models (0.1.8) and (0.1.26), although the agents involved in the opinion formation or flocking process could not communicate with all the other components of the system and could suspend the interaction also with the agents to whom they are linked. As already pointed out in the introduction, to deal with the non-universal interaction, we will consider a network topology over the structure of the model. Moreover, consensus estimate will be established for the two aforementioned systems under a Persistence Excitation Condition. The results we will present in this chapter are contained in [39].

3.1 The first-order model

We start dealing with the first-order model. The consensus result we will prove for system (0.1.8) is the following.

Theorem 3.1.1. *Assume that the digraph $\mathcal{G}$ is strongly connected. Let $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a positive, bounded, continuous function. Assume that the weight functions $\alpha_{ij} : [0,+\infty) \to [0,1]$ are $\mathcal{L}^1$-measurable and satisfy* **(PE)**. *Moreover, suppose that the time delay functions $\tau_{ij} : [0,+\infty) \to [0,+\infty)$ are continuous and satisfy* (0.1.9). *Let $x_i^0 : [-\bar{\tau}, 0] \to \mathbb{R}^d$ be a continuous function, for any $i = 1, \dots, N$. Then, every solution $\{x_i\}_{i=1,\dots,N}$ to* (0.1.8) *with the initial conditions* (0.1.16) *satisfies the following exponential decay estimate*

$$d(t) \leq \left(\max_{i,j=1,\dots,N} \max_{r,s\in[-\bar{\tau},0]} |x_i(r) - x_j(s)| \right) e^{-C(t-\gamma(T+\bar{\tau})-\bar{\tau})}, \quad \forall t \geq 0, \tag{3.1.1}$$

where $\gamma > 0$ is the depth of the digraph, T is the positive constant in (0.1.12) *and C is a suitable positive constant.*

3.1.1 Preliminary lemmas

Let $\{x_i\}_{i=1,\dots,N}$ be solution to (0.1.8) under the initial conditions (0.1.16). We assume that the hypotheses of Theorem 3.1.1 are satisfied. We present some auxiliary lemmas.

Definition 3.1.1. Given a vector $v \in \mathbb{R}^d$, for all $n \in \mathbb{N}_0$ we define

$$I_n := [n(\gamma(T+\bar{\tau})+\bar{\tau})-\bar{\tau}, n(\gamma(T+\bar{\tau})+\bar{\tau})]$$

$$m_n^v := \min_{i=1,\dots,N} \min_{s\in I_n} \langle x_i(s), v\rangle,$$

$$M_n^v := \max_{j=1,\dots,N} \max_{s\in I_n} \langle x_j(s), v\rangle.$$

Also, we define, for all $n \in \mathbb{N}_0$,

$$\tilde{m}_n^v := \min_{i=1,\dots,N} \langle x_i(n(\gamma(T+\bar{\tau})+\bar{\tau})), v\rangle,$$

$$\tilde{M}_n^v := \max_{j=1,\dots,N} \langle x_j(n(\gamma(T+\bar{\tau})+\bar{\tau})), v\rangle.$$

Lemma 3.1.2. *For each vector $v \in \mathbb{R}^d$ and for all $n \in \mathbb{N}_0$, we have that*

$$m_n^v \le \langle x_i(t), v\rangle \le M_n^v, \tag{3.1.2}$$

for all $t \ge n(\gamma(T+\bar{\tau})+\bar{\tau})-\bar{\tau}$ and for any $i = 1,\dots,N$.

Proof. The proof follows using analogous arguments to the ones employed in Lemma 1.1.1. However, in this case, with respect to Lemma 1.1.1 the weight functions α_{ij} and the terms χ_{ij} appear in the problem's formulation. Nevertheless, one can still obtain an estimate like (1.1.5) by using the fact that both $\chi_{ij}, \alpha_{ij}, \le 1$. □

Now, we define the following quantities.

Definition 3.1.2. For all $n \in \mathbb{N}$, we define

$$D_n := \max_{i,j=1,\dots,N} \max_{r,s\in I_n} |x_i(r) - x_j(s)|.$$

Let us note that, for $n = 0$,

$$D_0 := \max_{i,j=1,\dots,N} \max_{r,s\in I_0} |x_i(r) - x_j(s)| = \max_{i,j=1,\dots,N} \max_{r,s\in[-\bar{\tau},0]} |x_i(r) - x_j(s)|.$$

So, the exponential decay estimate in (3.1.1) can be written as

$$d(t) \le e^{-C(t-\gamma(T+\bar{\tau})-\bar{\tau})} D_0, \quad \forall t \ge 0.$$

As in Chapter 1, from the previous Lemma, the following estimates can be derived.

Lemma 3.1.3. *For each $n \in \mathbb{N}_0$, we have that*

$$|x_i(s) - x_j(t)| \le D_n, \tag{3.1.3}$$

for all $s, t \ge n(\gamma(T+\bar{\tau})+\bar{\tau})-\bar{\tau}$ and for any $i, j = 1,\dots,N$.

Remark 3.1.4. Note that (3.1.3) yields

$$d(t) \leq D_n, \quad \forall t \geq n(\gamma(T+\bar{\tau})+\bar{\tau})-\bar{\tau}. \tag{3.1.4}$$

Moreover, from (3.1.3) it comes that

$$D_{n+1} \leq D_n, \quad \forall n \in \mathbb{N}_0. \tag{3.1.5}$$

Also, the agents' opinions are bounded by a constant that depends on the initial data and, as a consequence, the communication rates are bounded from below.

Lemma 3.1.5. *For every $i=1,\dots,N$, we have that*

$$|x_i(t)| \leq C_0, \quad \forall t \geq -\bar{\tau}, \tag{3.1.6}$$

where

$$C_0 := \max_{i=1,\dots,N} \max_{s\in[-\bar{\tau},0]} |x_i(s)|. \tag{3.1.7}$$

In particular,

$$\psi(x_i(t), x_j(t-\tau_{ij}(t))) \geq \psi_0, \quad \forall t \geq 0, \forall i,j=1,\dots,N, \tag{3.1.8}$$

where

$$\psi_0 := \min_{|y|,|z|\leq C_0} \psi(y,z). \tag{3.1.9}$$

3.1.2 Consensus estimate

In order to prove the consensus result, we need the following crucial proposition, inspired by a previous argument in [64].

Proposition 3.1.6. *For all $v \in \mathbb{R}^d$, it holds*

$$m_0^v + \Gamma(\tilde{M}_0^v - m_0^v) \leq \langle x_i(t), v\rangle \leq M_0^v - \Gamma(M_0^v - \tilde{m}_0^v), \tag{3.1.10}$$

for all $t \in I_1$ and for all $i=1,\dots,N$, where Γ is the positive constant defined as follows

$$\Gamma := e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\bar{\tau})+\bar{\tau})} \left(\frac{\psi_0\tilde{\alpha}}{N-1}\right)^{\gamma}. \tag{3.1.11}$$

Remark 3.1.7. Let us note that, from **(PE)**, $\Gamma \in (0,1)$ since $\tilde{\alpha}\psi_0 \leq \tilde{\alpha}K \leq 1$.

Proof. Fix $v \in \mathbb{R}^d$. Let $L = 1,\dots,N$ be such that $\langle x_L(0), v\rangle = \tilde{m}_0^v$. Note that from (3.1.2), $M_0^v \geq \tilde{m}_0^v$. Then, for a.e. $t \in [0, \gamma(T+\bar{\tau})+\bar{\tau}]$, using (3.1.2) we have

$$\begin{aligned}
\frac{d}{dt}\langle x_L(t), v\rangle &= \sum_{j:j\neq L} \chi_{Lj}\alpha_{Lj}(t)b_{Lj}(t)(\langle x_j(t-\tau_{Lj}(t)), v\rangle - \langle x_L(t), v\rangle) \\
&\leq \sum_{j:j\neq L} \chi_{Lj}\alpha_{Lj}(t)b_{Lj}(t)(M_0^v - \langle x_L(t), v\rangle) \\
&\leq \frac{K}{N-1}\sum_{j:j\neq L}(M_0^v - \langle x_L(t), v\rangle) = K(M_0^v - \langle x_L(t), v\rangle).
\end{aligned}$$

Thus, the Gronwall's inequality yields

$$\begin{aligned}\langle x_L(t), v\rangle &\le e^{-Kt}\langle x_L(0), v\rangle + M_0^v(1-e^{-Kt})\\ &= e^{-Kt}\tilde{m}_0^v + M_0^v(1-e^{-Kt})\\ &= M_0^v - e^{-Kt}(M_0^v - \tilde{m}_0^v)\\ &\le M_0^v - e^{-K(\gamma(T+\bar{\tau})+\bar{\tau})}(M_0^v - \tilde{m}_0^v).\end{aligned}$$

Hence,

$$\langle x_L(t), v\rangle \le M_0^v - e^{-K(\gamma(T+\bar{\tau})+\bar{\tau})}(M_0^v - \tilde{m}_0^v), \quad \forall t \in [0, \gamma(T+\bar{\tau})+\bar{\tau}]. \tag{3.1.12}$$

Now, let $i_1 = 1, \dots, N \setminus \{L\}$ be such that $\chi_{i_1 L} = 1$. Such an index i_1 exists since the digraph is strongly connected. Then, for a.e. $t \in [\bar{\tau}, \gamma(T+\bar{\tau})+\bar{\tau}]$, from (3.1.12) we get

$$\begin{aligned}\frac{d}{dt}\langle x_{i_1}(t), v\rangle = &\sum_{j\ne i_1, L}\chi_{i_1 j}\alpha_{i_1 j}(t)b_{i_1 j}(t)(\langle x_j(t-\tau_{i_1 j}(t)), v\rangle - \langle x_{i_1}(t), v\rangle)\\ &+\alpha_{i_1 L}(t)b_{i_1 L}(t)(\langle x_L(t-\tau_{i_1 L}(t)), v\rangle - \langle x_{i_1}(t), v\rangle)\\ \le &\sum_{j\ne i_1, L}\alpha_{i_1 j}(t)\chi_{i_1 j}b_{i_1 j}(t)(M_0^v - \langle x_{i_1}(t), v\rangle)\\ &+\alpha_{i_1 L}(t)b_{i_1 L}(t)\left(M_0^v - e^{-K(\gamma(T+\bar{\tau})+\bar{\tau})}(M_0^v - \tilde{m}_0^v) - \langle x_{i_1}(t), v\rangle\right)\\ = &(M_0^v - \langle x_{i_1}(t), v\rangle)\sum_{j\ne i_1, L}\chi_{i_1 j}\alpha_{i_1 j}(t)b_{i_1 j}(t)\\ &+\alpha_{i_1 L}(t)b_{i_1 L}(t)\left(M_0^v - e^{-K(\gamma(T+\bar{\tau})+\bar{\tau})}(M_0^v - \tilde{m}_0^v) - \langle x_{i_1}(t), v\rangle\right).\end{aligned}$$

Note that

$$\begin{aligned}\sum_{j\ne i_1, L}\chi_{i_1 j}\alpha_{i_1 j}(t)b_{i_1 j}(t) &= \sum_{j\ne i_1}\chi_{i_1 j}\alpha_{i_1 j}(t)b_{i_1 j}(t) - \alpha_{i_1 L}(t)b_{i_1 L}(t)\\ &\le \frac{K}{N-1}\sum_{j\ne i_1}\chi_{i_1 j} - \alpha_{i_1 L}(t)b_{i_1 L}(t) = \frac{KN_{i_1}}{N-1} - \alpha_{i_1 L}(t)b_{i_1 L}(t).\end{aligned}$$

Thus, it comes that

$$\begin{aligned}\frac{d}{dt}\langle x_{i_1}(t), v\rangle \le &\frac{KN_{i_1}}{N-1}(M_0^v - \langle x_{i_1}(t), v\rangle) - \alpha_{i_1 L}(t)b_{i_1 L}(t)(M_0^v - \langle x_{i_1}(t), v\rangle)\\ &+\alpha_{i_1 L}(t)b_{i_1 L}(t)\left(M_0^v - e^{-K(\gamma(T+\bar{\tau})+\bar{\tau})}(M_0^v - \tilde{m}_0^v) - \langle x_{i_1}(t), v\rangle\right)\\ = &\frac{KN_{i_1}}{N-1}(M_0^v - \langle x_{i_1}(t), v\rangle) - e^{-K(\gamma(T+\bar{\tau})+\bar{\tau})}(M_0^v - \tilde{m}_0^v)\alpha_{i_1 L}(t)b_{i_1 L}(t)\\ \le &\frac{KN_{i_1}}{N-1}(M_0^v - \langle x_{i_1}(t), v\rangle) - e^{-K(\gamma(T+\bar{\tau})+\bar{\tau})}(M_0^v - \tilde{m}_0^v)\alpha_{i_1 L}(t)\frac{\psi_0}{N-1}\\ = &\frac{KN_{i_1}}{N-1}M_0^v - e^{-K(\gamma(T+\bar{\tau})+\bar{\tau})}(M_0^v - \tilde{m}_0^v)\alpha_{i_1 L}(t)\frac{\psi_0}{N-1} - \frac{KN_{i_1}}{N-1}\langle x_{i_1}(t), v\rangle).\end{aligned}$$

Hence, the Gronwall's estimate yields

$$
\begin{aligned}
\langle x_{i_1}(t), v\rangle \leq\ & e^{-\frac{KN_{i_1}}{N-1}(t-\bar\tau)}\langle x_{i_1}(\bar\tau), v\rangle) + M_0^v(1-e^{-\frac{KN_{i_1}}{N-1}(t-\bar\tau)}) \\
& -e^{-K(\gamma(T+\bar\tau)+\bar\tau)}(M_0^v-\tilde m_0^v)\frac{\psi_0}{N-1}\int_{\bar\tau}^t \alpha_{i_1L}(s)e^{-\frac{KN_{i_1}}{N-1}(t-s)}ds \\
\leq\ & e^{-\frac{KN_{i_1}}{N-1}(t-\bar\tau)}M_0^v + M_0^v(1-e^{-\frac{KN_{i_1}}{N-1}(t-\bar\tau)}) \\
& -e^{-K(\gamma(T+\bar\tau)+\bar\tau)}(M_0^v-\tilde m_0^v)e^{-K\gamma(T+\bar\tau)}\frac{\psi_0}{N-1}\int_{\bar\tau}^t \alpha_{i_1L}(s)ds \\
=\ & M_0^v - e^{-K(2\gamma(T+\bar\tau)+\bar\tau)}(M_0^v-\tilde m_0^v)\frac{\psi_0}{N-1}\int_{\bar\tau}^t \alpha_{i_1L}(s)ds,
\end{aligned}
$$

for all $t\in[\bar\tau, \gamma(T+\bar\tau)+\bar\tau]$. In particular, for $t\in[T+\bar\tau, \gamma(T+\bar\tau)+\bar\tau]$, we find

$$
\langle x_{i_1}(t), v\rangle \leq M_0^v - e^{-K(2\gamma(T+\bar\tau)+\bar\tau)}(M_0^v-\tilde m_0^v)\frac{\psi_0}{N-1}\tilde\alpha, \tag{3.1.13}
$$

where here we have used the fact that, from (0.1.12),

$$
\int_\tau^t \alpha_{i_1L}(s)ds \geq \int_{\bar\tau}^{T+\bar\tau}\alpha_{i_1L}(s)ds \geq \tilde\alpha.
$$

Let us note that, if $\gamma=1$, estimate (3.1.13) holds for each agent. If $\gamma>1$, let us consider an index $i_2\in\{1,\dots,N\}\setminus\{i_1\}$ such that $\chi_{i_2i_1}=1$. Then, for a.e. $t\in[T+2\bar\tau, \gamma(T+\bar\tau)+\bar\tau]$, from (3.1.13) it comes that

$$
\begin{aligned}
\frac{d}{dt}\langle x_{i_2}(t), v\rangle =\ & \sum_{j\neq i_1,i_2}\chi_{i_2j}\alpha_{i_2j}(t)b_{i_2j}(t)(\langle x_j(t-\tau_{i_2j}(t)), v\rangle - \langle x_{i_2}(t), v\rangle) \\
& +\alpha_{i_2i_1}(t)b_{i_2i_1}(t)(\langle x_{i_1}(t-\tau_{i_2i_1}(t)), v\rangle - \langle x_{i_2}(t), v\rangle) \\
\leq\ & (M_0^v - \langle x_{i_2}(t), v\rangle)\sum_{j\neq i_1,i_1}\chi_{i_2j}\alpha_{i_2j}(t)b_{i_2j}(t) \\
& +\alpha_{i_2i_1}(t)b_{i_2i_1}(t)\left(M_0^v - e^{-K(2\gamma(T+\bar\tau)+\bar\tau)}(M_0^v-\tilde m_0^v)\frac{\psi_0}{N-1}\tilde\alpha - \langle x_{i_2}(t), v\rangle\right).
\end{aligned}
$$

Thus, arguing as above,

$$\begin{aligned}\frac{d}{dt}\langle x_{i_2}(t),v\rangle &\le \frac{KN_{i_2}}{N-1}(M_0^v-\langle x_{i_2}(t),v\rangle)-\alpha_{i_2i_1}(t)b_{i_2i_1}(t)(M_0^v-\langle x_{i_2}(t),v\rangle)\\ &\quad+\alpha_{i_2i_1}(t)b_{i_2i_1}(t)\left(M_0^v-e^{-K(2\gamma(T+\bar\tau)+\bar\tau)}(M_0^v-\tilde m_0^v)\frac{\psi_0}{N-1}\tilde\alpha-\langle x_{i_2}(t),v\rangle\right)\\ &=\frac{KN_{i_2}}{N-1}(M_0^v-\langle x_{i_1}(t),v\rangle)-\alpha_{i_2i_1}(t)b_{i_2i_1}(t)e^{-K(2\gamma(T+\bar\tau)+\bar\tau)}(M_0^v-\tilde m_0^v)\frac{\psi_0}{N-1}\tilde\alpha\\ &\le\frac{KN_{i_2}}{N-1}(M_0^v-\langle x_{i_1}(t),v\rangle)-\alpha_{i_2i_1}(t)e^{-K(2\gamma(T+\bar\tau)+\bar\tau)}(M_0^v-\tilde m_0^v)\left(\frac{\psi_0}{N-1}\right)^2\tilde\alpha\\ &=\frac{KN_{i_2}}{N-1}M_0^v-\alpha_{i_2i_1}(t)e^{-K(2\gamma(T+\bar\tau)+\bar\tau)}(M_0^v-\tilde m_0^v)\left(\frac{\psi_0}{N-1}\right)^2\tilde\alpha-\frac{KN_{i_2}}{N-1}\langle x_{i_2}(t),v\rangle).\end{aligned}$$

Again, using Gronwall's estimate it comes that

$$\begin{aligned}\langle x_{i_2}(t),v\rangle &\le e^{-\frac{KN_{i_2}}{N-1}(t-T-2\bar\tau)}\langle x_{i_2}(T+2\bar\tau),v\rangle)+M_0^v(1-e^{-\frac{KN_{i_2}}{N-1}(t-T-2\bar\tau)})\\ &\quad-e^{-K(2\gamma(T+\bar\tau)+\bar\tau)}(M_0^v-\tilde m_0^v)\left(\frac{\psi_0}{N-1}\right)^2\tilde\alpha\int_{T+2\bar\tau}^t\alpha_{i_2i_1}(s)e^{-\frac{KN_{i_2}}{N-1}(t-s)}ds\\ &\le M_0^v-e^{-K(3\gamma(T+\bar\tau)-T)}(M_0^v-\tilde m_0^v)\left(\frac{\psi_0}{N-1}\right)^2\tilde\alpha\int_{T+2\bar\tau}^t\alpha_{i_2i_1}(s)ds,\end{aligned}$$

for all $t\in[T+2\bar\tau,\gamma(T+\bar\tau)+\bar\tau]$. In particular, for $t\in[2T+2\bar\tau,\gamma(T+\bar\tau)+\bar\tau]$, the condition (0.1.12) yields

$$\langle x_{i_2}(t),v\rangle\le M_0^v-e^{-K(3\gamma(T+\bar\tau)-T)}(M_0^v-\tilde m_0^v)\left(\frac{\psi_0}{N-1}\right)^2\tilde\alpha^2. \tag{3.1.14}$$

Finally, iterating the above procedure along the path $i_0,i_1,\dots,i_r$, $r\le\gamma$, that starts from $i_0=L$ we find the following upper bound

$$\langle x_{i_k}(t),v\rangle\le M_0^v-e^{-K((k+1)\gamma(T+\bar\tau)-\left(\sum_{l=0}^{k-1}l\right)(T+\bar\tau)+\bar\tau)}(M_0^v-\tilde m_0^v)\left(\frac{\psi_0\tilde\alpha}{N-1}\right)^k, \tag{3.1.15}$$

for all $1\le k\le r$ and for all $t\in[k(T+\bar\tau),\gamma(T+\bar\tau)+\bar\tau]$. In particular, if the path has length γ, for $k=\gamma$, since $\sum_{l=0}^{\gamma-1}l=\frac{\gamma(\gamma-1)}{2}$, inequality (3.1.15) reads as

$$\langle x_{i_\gamma}(t),v\rangle\le M_0^v-e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\bar\tau)+\bar\tau)}(M_0^v-\tilde m_0^v)\left(\frac{\psi_0\tilde\alpha}{N-1}\right)^\gamma, \tag{3.1.16}$$

for all $t\in[\gamma(T+\bar\tau),\gamma(T+\bar\tau)+\bar\tau]$.
Let us note that (3.1.16) holds for every agent in the path starting from $i_0=L$ for $t\in[\gamma(T+\bar\tau),\gamma(T+\bar\tau)+\bar\tau]$. Then, from the arbitrariness of the path and since the digraph is strongly connected, (3.1.16) holds for all the agents.

Now, let $R = 1, \dots, N$ be such that $\tilde{M}_0^v = \langle x_R(0), v\rangle$. Then, arguing as before, we get

$$\langle x_R(t), v\rangle \geq m_0^v(1 + e^{-K(\gamma(T+\bar{\tau})+\bar{\tau})}(\tilde{M}_0^v - m_0^v)), \quad \forall t \in [0, \gamma(T+\bar{\tau})+\bar{\tau}]. \tag{3.1.17}$$

Employing the same arguments used above, we can conclude that

$$\langle x_i(t), v\rangle \geq m_0^v + e^{-K(\frac{1}{2}(\gamma^2+3\gamma)(T+\bar{\tau})+\bar{\tau})}(\tilde{M}_0^v - m_0^v)\left(\frac{\psi_0\tilde{\alpha}}{N-1}\right)^\gamma,$$

for all $t \in [\gamma(T+\bar{\tau}), \gamma(T+\bar{\tau})+\bar{\tau}]$ and for all $i = 1, \dots, N$. Finally, we can deduce that estimate (3.1.10) holds. □

The following proposition generalizes the previous one in successive time intervals. Its proof is analogous to the previous one, so we omit it.

Proposition 3.1.8. *Let $v \in \mathbb{R}^d$. For any $n \in \mathbb{N}_0$, it holds*

$$m_n^v + \Gamma(\tilde{M}_n^v - m_n^v) \leq \langle x_i(t), v\rangle \leq M_n^v - \Gamma(M_n^v - \tilde{m}_n^v), \tag{3.1.18}$$

for all $t \in I_{n+1}$ and for all $i = 1, \dots, N$, where Γ is the positive constant in (3.1.11).

Now, we are able the consensus Theorem 3.1.1.

Proof of Theorem 3.1.1. Let $\{x_i\}_{i=1,\dots,N}$ be solution to (0.1.8) under the initial conditions (0.1.16). Fix $v \in \mathbb{R}^d$. Let us define the quantities

$$\mathcal{D}_n^v := M_n^v - m_n^v, \quad \forall n \in \mathbb{N}_0,$$

where M_n^v, m_n^v are the constants introduced in Definition 3.1.1. Note that, for all $n \in \mathbb{N}_0$, we have $\mathcal{D}_n^v \geq 0$, being $M_n^v \geq m_n^v$.
Let $\Gamma \in (0, 1)$ be the constant in (3.1.11). We claim that

$$\mathcal{D}_{n+1}^v \leq (1-\Gamma)\mathcal{D}_n^v, \quad \forall n \in \mathbb{N}_0. \tag{3.1.19}$$

Indeed, fix $n \in \mathbb{N}_0$. Let $i, j = 1, \dots, N$ and $s, t \in I_{n+1}$ be such that $\langle x_i(s), v\rangle = M_{n+1}^v$ and $\langle x_j(t), v\rangle = m_{n+1}^v$. Then, applying Lemma 3.1.8, we can write

$$\begin{aligned}\mathcal{D}_{n+1}^v &= M_{n+1}^v - m_{n+1}^v = \langle x_i(s), v\rangle - \langle x_j(t), v\rangle \\ &\leq M_n^v - m_n^v - \Gamma(M_n^v - \tilde{m}_n^v) - \Gamma(\tilde{M}_n^v - m_n^v).\end{aligned} \tag{3.1.20}$$

Now, we distinguish four cases.
Case I) Assume that $M_n^v - \tilde{m}_n^v = 0$ and $\tilde{M}_n^v - m_n^v = 0$. Then, since from (3.1.2)

$$m_n^v \leq \tilde{m}_n^v \leq \tilde{M}_n^v = m_n^v,$$

we get

$$m_n^v = \tilde{m}_n^v = M_n^v.$$

As a consequence, (3.1.20) becomes

$$\mathcal{D}^v_{n+1} \le 0 = (1-\Gamma)\mathcal{D}^v_n.$$

Case II) Assume that $M^v_n - \tilde{m}^v_n = 0$ and $\tilde{M}^v_n - m^v_n > 0$. Then, since from (3.1.2)

$$\tilde{m}^v_n \le \tilde{M}^v_n \le M^v_n = \tilde{m}^v_n,$$

we can write

$$\tilde{M}^v_n = M^v_n.$$

As a consequence, (3.1.20) becomes

$$\mathcal{D}^v_{n+1} \le M^v_n - m^v_n - \Gamma\tilde{M}^v_n + \Gamma m^v_n = (1-\Gamma)(M^v_n - m^v_n) = (1-\Gamma)\mathcal{D}^v_n.$$

Case III) Assume that $M^v_n - \tilde{m}^v_n > 0$ and $\tilde{M}^v_n - m^v_n = 0$. Then, from (3.1.2) we have

$$m^v_n \le \tilde{m}^v_n \le \tilde{M}^v_n = m^v_n,$$

from which

$$\tilde{m}^v_n = m^v_n.$$

As a consequence, (3.1.20) becomes

$$\mathcal{D}^v_{n+1} \le M^v_n - m^v_n - \Gamma M^v_n + \Gamma\tilde{m}^v_n = (1-\Gamma)(M^v_n - m^v_n) = (1-\Gamma)\mathcal{D}^v_n.$$

Case IV) Assume that $M^v_n - \tilde{m}^v_n > 0$ and $\tilde{M}^v_n - m^v_n > 0$. In this case, using the fact that $\tilde{M}^v_n \ge \tilde{m}^v_n$, from (3.1.20) we get

$$\mathcal{D}^v_{n+1} \le (1-\Gamma)(M^v_n - m^v_n) - \Gamma\tilde{M}^v_n + \Gamma\tilde{m}^v_n \le (1-\Gamma)(M^v_n - m^v_n) = (1-\Gamma)\mathcal{D}^v_n.$$

Hence, (3.1.19) is fulfilled.
As a consequence, since the positive constant Γ in (3.1.19) does not depend on the choice of the vector v, we find the following estimate:

$$D_{n+1} \le (1-\Gamma)D_n, \quad \forall n \in \mathbb{N}_0. \tag{3.1.21}$$

To see this, fix $n \in \mathbb{N}$. Let $i, j = 1, \dots, N$ and $s, t \in I_{n+1}$ be such that

$$D_{n+1} = |x_i(s) - x_j(t)|.$$

Let us define the unit vector

$$v = \frac{x_i(s) - x_j(t)}{|x_i(s) - x_j(t)|}.$$

Then, using (3.1.2) and (3.1.19),

$$\begin{aligned}
D_{n+1} &= \langle x_i(s) - x_j(t), v\rangle = \langle x_i(s), v\rangle - \langle x_j(t), v\rangle \\
&\le M^v_{n+1} - m^v_{n+1} = \mathcal{D}^v_{n+1} \\
&\le (1-\Gamma)\mathcal{D}^v_n = (1-\Gamma)(M^v_n - m^v_n) \\
&\le (1-\Gamma)\max_{k,l=1,\dots,N}\max_{r,w\in I_n}|x_k(r) - x_l(w)| = (1-\Gamma)D_n.
\end{aligned}$$

Thus, (3.1.21) holds true.
Now, from (3.1.21) it comes that

$$D_n \leq (1-\Gamma)^n D_0, \quad \forall n \in \mathbb{N}_0. \tag{3.1.22}$$

Let us note that (3.1.22) can be rewritten as

$$D_n \leq e^{-nC(\gamma(T+\bar{\tau})+\bar{\tau})} D_0, \quad \forall n \in \mathbb{N}_0, \tag{3.1.23}$$

where

$$C = \frac{1}{\gamma(T+\bar{\tau})+\bar{\tau}} \ln\left(\frac{1}{1-\Gamma}\right).$$

Now, let $t \geq 0$. Thus, $t \in [n(\gamma(T+\bar{\tau})+\bar{\tau}), (n+1)(\gamma(T+\bar{\tau})+\bar{\tau})]$, for some $n \in \mathbb{N}_0$. Then, using (3.1.4) and (3.1.23), it comes that

$$d(t) \leq D_n \leq e^{-nC(\gamma(T+\bar{\tau})+\bar{\tau})} D_0 \leq e^{-C(t-\gamma(T+\bar{\tau})-\bar{\tau})} D_0,$$

which concludes our proof. □

3.2 The second-order model

Now, we focus on the second-order model (0.1.26). We will prove the following flocking result.

Theorem 3.2.1. *Assume that the digraph $\mathcal{G}$ is strongly connected. Let $\tilde{\psi} : \mathbb{R} \to \mathbb{R}$ be a positive, bounded, continuous function that satisfies*

$$\int_0^{+\infty} \left(\min_{r\in[0,t]} \tilde{\psi}(r)\right)^\gamma dt = +\infty, \tag{3.2.1}$$

where γ is the depth of the digraph. Assume that the weight functions $\alpha_{ij} : [0,+\infty) \to [0,1]$ are $\mathcal{L}^1$-measurable and satisfy **(PE)**. *Moreover, suppose that the time delay functions $\tau_{ij} : [0,+\infty) \to [0,+\infty)$ are continuous and satisfy* (0.1.9). *Let $x_i^0, v_i^0 : [-\bar{\tau}, 0] \to \mathbb{R}^d$ be continuous functions, for any $i = 1,\dots,N$. Then, for every solution $\{(x_i,v_i)\}_{i=1,\dots,N}$ to* (0.1.26) *with the initial conditions* (0.1.28)*, there exists a positive constant d^* such that*

$$\sup_{t\geq -\bar{\tau}} d_X(t) \leq d^*, \tag{3.2.2}$$

and there exists a positive constant μ for which the following exponential decay estimate holds

$$d_V(t) \leq \left(\max_{i,j=1,\dots,N} \max_{r,s\in[-\bar{\tau},0]} |v_i(r) - v_j(s)|\right) e^{-\mu(t-\gamma(T+\bar{\tau})-\bar{\tau})}, \quad \forall t \geq 0. \tag{3.2.3}$$

Remark 3.2.2. Let us note that, if the function $\tilde{\psi}$ is nonincreasing and the interaction is universal, i.e. $\gamma = 1$, then the condition (3.2.1) reduces to

$$\int_0^{+\infty} \tilde{\psi}(t)dt = +\infty,$$

which is the classical assumption to obtain the unconditional flocking (see e.g. [101]). Since here we deal with an influence function not necessarily monotonic and the interaction is not universal, we require the stronger assumption (3.2.1) (cf. [40] for the case of universal interaction).

3.2.1 Preliminary lemmas

Let $\{x_i, v_i\}_{i=1,\dots,N}$ be solution to (0.1.26) under the initial conditions (0.1.28). We assume that the hypotheses of Theorem 3.2.1 are satisfied. The following lemmas hold. We omit their proofs since they can be proved using the same arguments employed in Chapter 1 and in the previous section.

Definition 3.2.1. Given a vector $v \in \mathbb{R}^d$, for all $n \in \mathbb{N}_0$ we define

$$r_n^v := \min_{j=1,\dots,N} \min_{s \in I_n} \langle v_j(s), v \rangle,$$
$$R_n^v := \max_{j=1,\dots,N} \max_{s \in I_n} \langle v_j(s), v \rangle,$$

where, as in the previous section,

$$I_n = [n(\gamma(T+\bar{\tau})+\bar{\tau}) - \bar{\tau}, n(\gamma(T+\bar{\tau})+\bar{\tau})].$$

Also, we define, for all $n \in \mathbb{N}_0$,

$$\tilde{r}_n^v := \min_{j=1,\dots,N} \langle v_j(n(\gamma(T+\bar{\tau})+\bar{\tau})), v \rangle,$$

$$\tilde{R}_n^v := \max_{j=1,\dots,N} \langle v_j(n(\gamma(T+\bar{\tau})+\bar{\tau})), v \rangle.$$

Lemma 3.2.3. *For each vector $v \in \mathbb{R}^d$ and for any $n \in \mathbb{N}_0$, we have that*

$$r_n^v \le \langle v_i(t), v \rangle \le R_n^v, \tag{3.2.4}$$

for all $t \ge n(\gamma(T+\bar{\tau})+\bar{\tau}) - \bar{\tau}$ and for any $i = 1, \dots, N$.

Definition 3.2.2. For all $n \in \mathbb{N}_0$, we define

$$F_n := \max_{i,j=1,\dots,N} \max_{r,s \in I_n} |v_i(r) - v_j(s)|.$$

Remark 3.2.4. Let us note that

$$F_0 := \max_{i,j=1,\dots,N} \max_{r,s \in I_0} |v_i(r) - v_j(s)| = \max_{i,j=1,\dots,N} \max_{r,s \in [-\bar{\tau},0]} |v_i(r) - v_j(s)|.$$

Then, the exponential decay estimate in (3.2.3) can be written as

$$d_V(t) \le e^{-\mu(t-\gamma(T+\bar{\tau})-\bar{\tau})} F_0, \quad \forall t \ge 0.$$

Lemma 3.2.5. *For each $n \in \mathbb{N}_0$, we have that*

$$|v_i(s) - v_j(t)| \leq F_n, \tag{3.2.5}$$

for all $s, t \geq n(\gamma(T + \bar{\tau}) + \bar{\tau}) - \bar{\tau}$ and for any $i, j = 1, \ldots, N$.

Remark 3.2.6. Let us note that (3.2.5) yields

$$d_V(t) \leq F_n, \quad \forall t \geq n(\gamma(T + \bar{\tau}) + \bar{\tau}) - \bar{\tau}. \tag{3.2.6}$$

Furthermore, from (3.2.5) it follows that

$$F_{n+1} \leq F_n, \quad \forall n \in \mathbb{N}_0. \tag{3.2.7}$$

Also, we can find a bound on the velocities $|v_i(t)|$, which is uniform with respect to t and $i = 1, \ldots, N$, and that depends on the initial velocities.

Lemma 3.2.7. *For every $i = 1, \ldots, N$, we have that*

$$|v_i(t)| \leq C_0^V, \quad \forall t \geq -\bar{\tau}, \tag{3.2.8}$$

where

$$C_0^V := \max_{i=1,\ldots,N} \max_{s \in [-\bar{\tau}, 0]} |v_i(s)|. \tag{3.2.9}$$

Now, we provide the following result in which an estimate for the position diameters is established. Since this result can be proved with analogous arguments to the ones employed in Lemma 2.1.4 of Chapter 2, we omit its proof.

Lemma 3.2.8. *For every $i, j = 1, \ldots, N$, we get*

$$|x_i(t) - x_j(t - \tau_{ij}(t))| \leq \bar{\tau} C_0^V + M_0^X + d_X(t), \quad \forall t \geq 0, \tag{3.2.10}$$

where C_0^V is the positive constants in (3.2.9) and

$$M_0^X := \max_{i=1,\ldots,N} \max_{s,t \in [-\bar{\tau}, 0]} |x_i(s) - x_i(t)|. \tag{3.2.11}$$

3.2.2 Flocking estimate

To prove the flocking result we need, as before, a crucial proposition. First of all, we give the following definition.

Definition 3.2.3. We define

$$\tilde{\phi}(t) := \min \left\{ \psi(r) : r \in \left[0, \bar{\tau} C_0^V + M_X^0 + \max_{s \in [-\bar{\tau}, t]} d_X(s) \right] \right\},$$

for all $t \geq -\bar{\tau}$.

Remark 3.2.9. Let us note that from (3.2.10)

$$\tilde{\psi}(|x_i(t) - x_j(t - \tau_{ij}(t))|) \geq \tilde{\phi}(t), \quad \forall t \geq 0, \forall i, j = 1, \dots, N.$$

from which

$$a_{ij}(t) \geq \frac{1}{N-1}\tilde{\phi}(t), \quad \forall t \geq 0, \forall i, j = 1, \dots, N. \tag{3.2.12}$$

Proposition 3.2.10. *For all $v \in \mathbb{R}^d$, it holds*

$$r_0^v + \Gamma_1(\tilde{R}_0^v - r_0^v) \leq \langle v_i(t), v\rangle \leq R_0^v - \Gamma_1(R_0^v - \tilde{r}_0^v), \tag{3.2.13}$$

for all $t \in I_1$ and for all $i = 1, \dots, N$, where Γ_1 is the positive constant defined as follows

$$\Gamma_1 := e^{-\tilde{K}(\frac{1}{2}(\gamma^2+3\gamma)(T+\bar{\tau})+\bar{\tau})} \left(\frac{\tilde{\phi}(\gamma(T+\bar{\tau})+\bar{\tau})\tilde{\alpha}}{N-1}\right)^{\gamma}. \tag{3.2.14}$$

Remark 3.2.11. Let us note that, from **(PE)**, $\Gamma_1 \in (0,1)$ since $\tilde{\alpha}\tilde{K} \leq 1$.

Proof. Fix $v \in \mathbb{R}^d$. Let $L = 1, \dots, N$ be such that $\langle v_L(0), v\rangle = \tilde{r}_0^v$. Note that from (3.2.4), $R_0^v \geq \tilde{r}_0^v$. Then, for a.e. $t \in [0, \gamma(T+\bar{\tau})+\bar{\tau}]$, from (3.2.4)

$$\begin{aligned}
\frac{d}{dt}\langle v_L(t), v\rangle &= \sum_{j:j\neq L} \chi_{Lj}\alpha_{Lj}(t)a_{Lj}(t)(\langle v_j(t-\tau_{Lj}(t)), v\rangle - \langle v_L(t), v\rangle) \\
&\leq \sum_{j:j\neq L} \chi_{Lj}\alpha_{Lj}(t)a_{Lj}(t)(R_0^v - \langle v_L(t), v\rangle) \\
&\leq \frac{K}{N-1}\sum_{j:j\neq L}(R_0^v - \langle v_L(t), v\rangle) = \tilde{K}(R_0^v - \langle v_L(t), v\rangle).
\end{aligned}$$

Thus, the Gronwall's inequality yields

$$\begin{aligned}
\langle v_L(t), v\rangle &\leq e^{-\tilde{K}t}\langle v_L(0), v\rangle + R_0^v(1 - e^{-\tilde{K}t}) \\
&= R_0^v - e^{-\tilde{K}t}(R_0^v - \tilde{r}_0^v) \\
&\leq R_0^v - e^{-\tilde{K}(\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v - \tilde{r}_0^v).
\end{aligned}$$

Therefore, we have

$$\langle v_L(t), v\rangle \leq R_0^v - e^{-\tilde{K}(\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v - \tilde{r}_0^v), \quad \forall t \in [0, \gamma(T+\bar{\tau})+\bar{\tau}]. \tag{3.2.15}$$

Now, let $i_1 = 1, \dots, N \setminus \{L\}$ be such that $\chi_{i_1 L} = 1$. Such an index i_1 exists since the digraph is strongly connected. Then, for a.e. $t \in [\bar{\tau}, \gamma(T+\bar{\tau})+\bar{\tau}]$, from (3.2.15) we get

$$
\begin{aligned}
\frac{d}{dt}\langle v_{i_1}(t), v\rangle = & \sum_{j\neq i_1, L} \chi_{i_1 j}\alpha_{i_1 j}(t) a_{i_1 j}(t)(\langle v_j(t-\tau_{i_1 j}(t)), v\rangle - \langle v_{i_1}(t), v\rangle) \\
& +\alpha_{i_1 L}(t) a_{i_1 L}(t)(\langle v_L(t-\tau_{i_1 L}(t)), v\rangle - \langle v_{i_1}(t), v\rangle) \\
\leq & \sum_{j\neq i_1, L} \alpha_{i_1 j}(t)\chi_{i_1 j} a_{i_1 j}(t)(R_0^v - \langle v_{i_1}(t), v\rangle) \\
& +\alpha_{i_1 L}(t) a_{i_1 L}(t)\left(R_0^v - e^{\tilde{K}(\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v - \tilde{r}_0^v) - \langle v_{i_1}(t), v\rangle\right) \\
= & (R_0^v - \langle v_{i_1}(t), v\rangle)\sum_{j\neq i_1, L} \chi_{i_1 j}\alpha_{i_1 j}(t) a_{i_1 j}(t) \\
& +\alpha_{i_1 L}(t) a_{i_1 L}(t)\left(R_0^v - e^{-\tilde{K}(\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v - \tilde{r}_0^v) - \langle v_{i_1}(t), v\rangle\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sum_{j\neq i_1, L} \chi_{i_1 j}\alpha_{i_1 j}(t) a_{i_1 j}(t) &= \sum_{j\neq i_1} \chi_{i_1 j}\alpha_{i_1 j}(t) a_{i_1 j}(t) - \alpha_{i_1 L}(t) a_{i_1 L}(t) \\
&\leq \frac{\tilde{K}}{N-1}\sum_{j\neq i_1, L} \chi_{i_1 j} - \alpha_{i_1 L}(t) a_{i_1 L}(t) = \frac{\tilde{K} N_{i_1}}{N-1} - \alpha_{i_1 L}(t) a_{i_1 L}(t).
\end{aligned}
$$

Thus, from (3.2.12) it comes that

$$
\begin{aligned}
\frac{d}{dt}\langle v_{i_1}(t), v\rangle \leq & \frac{\tilde{K} N_{i_1}}{N-1}(R_0^v - \langle v_{i_1}(t), v\rangle) - \alpha_{i_1 L}(t) a_{i_1 L}(t)(R_0^v - \langle v_{i_1}(t), v\rangle) \\
& +\alpha_{i_1 L}(t) a_{i_1 L}(t)\left(R_0^v - e^{-\tilde{K}(\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v - \tilde{r}_0^v) - \langle v_{i_1}(t), v\rangle\right) \\
\leq & \frac{\tilde{K} N_{i_1}}{N-1}(R_0^v - \langle v_{i_1}(t), v\rangle) - \alpha_{i_1 L}(t)\frac{\tilde{\phi}(t)}{N-1} e^{-\tilde{K}(\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v - \tilde{r}_0^v) \\
= & \frac{\tilde{K} N_{i_1}}{N-1} R_0^v - \alpha_{i_1 L}(t)\frac{\tilde{\phi}(t)}{N-1} e^{-\tilde{K}(\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v - \tilde{r}_0^v) - \frac{\tilde{K} N_{i_1}}{N-1}\langle v_{i_1}(t), v\rangle).
\end{aligned}
$$

Hence, the Gronwall's estimate yields

$$\begin{aligned}
\langle v_{i_1}(t), v\rangle &\le e^{-\frac{\tilde{K}N_{i_1}}{N-1}(t-\bar{\tau})}\langle v_{i_1}(\bar{\tau}), v\rangle) + R_0^v(1-e^{-\frac{\tilde{K}N_{i_1}}{N-1}(t-\bar{\tau})}) \\
&\quad -e^{-\tilde{K}(\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v-\tilde{r}_0^v)\frac{1}{N-1}\int_{\bar{\tau}}^t \tilde{\phi}(s)\alpha_{i_1L}(s)e^{-\frac{\tilde{K}N_{i_1}}{N-1}(t-s)}ds \\
&\le e^{-\frac{\tilde{K}N_{i_1}}{N-1}(t-\bar{\tau})}R_0^v + R_0^v(1-e^{-\frac{\tilde{K}N_{i_1}}{N-1}(t-\bar{\tau})}) \\
&\quad -e^{-\tilde{K}(\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v-\tilde{r}_0^v)e^{-\tilde{K}\gamma(T+\bar{\tau})}\frac{1}{N-1}\int_{\bar{\tau}}^t \tilde{\phi}(s)\alpha_{i_1L}(s)ds \\
&= R_0^v - e^{-\tilde{K}(2\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v-\tilde{r}_0^v)\frac{1}{N-1}\int_{\bar{\tau}}^t \tilde{\phi}(s)\alpha_{i_1L}(s)ds,
\end{aligned}$$

for all $t\in[\bar{\tau},\gamma(T+\bar{\tau})+\bar{\tau}]$. Note that, since $\tilde{\phi}$ is a nonincreasing function,

$$\tilde{\phi}(t)\ge\tilde{\phi}(\gamma(T+\bar{\tau})+\bar{\tau}),\quad \forall t\in[0,\gamma(T+\bar{\tau})+\bar{\tau}]. \tag{3.2.16}$$

Then, we can write

$$\langle v_{i_1}(t), v\rangle \le R_0^v - e^{-\tilde{K}(2\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v-\tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\bar{\tau})+\bar{\tau})}{N-1}\int_{\bar{\tau}}^t \alpha_{i_1L}(s)ds,$$

for all $t\in[\bar{\tau},\gamma(T+\bar{\tau})+\bar{\tau}]$. In particular, for $t\in[T+\bar{\tau},\gamma(T+\bar{\tau})+\bar{\tau}]$, we find

$$\langle v_{i_1}(t), v\rangle \le R_0^v - e^{-\tilde{K}(2\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v-\tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\bar{\tau})+\bar{\tau})}{N-1}\tilde{\alpha}, \tag{3.2.17}$$

where here we have used the fact that (0.1.12) implies the following inequality

$$\int_{\bar{\tau}}^t \alpha_{i_1L}(s)ds \ge \int_{\bar{\tau}}^{T+\bar{\tau}} \alpha_{i_1L}(s)ds \ge \tilde{\alpha}.$$

Now, if $\gamma=1$, (3.2.17) holds true for each agent. On the other hand, if $\gamma>1$, let us consider an index $i_2\in\{1,\dots,N\}\setminus\{i_1\}$ such that $\chi_{i_2i_1}=1$. Then, for a.e. $t\in[T+2\bar{\tau},\gamma(T+\bar{\tau})+\bar{\tau}]$, from (3.2.17) it comes that

$$\begin{aligned}
\frac{d}{dt}\langle v_{i_2}(t), v\rangle &= \sum_{j\ne i_1,i_2}\chi_{i_2j}\alpha_{i_2j}(t)a_{i_2j}(t)(\langle v_j(t-\tau_{i_2j}(t)), v\rangle - \langle v_{i_2}(t), v\rangle) \\
&\quad +\alpha_{i_2i_1}(t)a_{i_2i_1}(t)(\langle v_{i_1}(t-\tau_{i_2i_1}(t)), v\rangle - \langle v_{i_2}(t), v\rangle) \\
&\le (R_0^v - \langle v_{i_2}(t), v\rangle)\sum_{j\ne i_1,i_2}\chi_{i_2j}\alpha_{i_2j}(t)a_{i_2j}(t) \\
&\quad +\alpha_{i_2i_1}(t)a_{i_2i_1}(t)\left(R_0^v - e^{-\tilde{K}(2\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v-\tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\bar{\tau})+\bar{\tau})}{N-1}\tilde{\alpha} - \langle v_{i_2}(t), v\rangle\right).
\end{aligned}$$

Hence, arguing as above we obtain

$$\frac{d}{dt}\langle v_{i_2}(t), v\rangle \le \frac{\tilde{K}N_{i_2}}{N-1}(R_0^v - \langle x_{i_2}(t), v\rangle) - \alpha_{i_2i_1}(t)a_{i_2i_1}(t)(R_0^v - \langle v_{i_2}(t), v\rangle)$$

$$+\alpha_{i_2i_1}(t)a_{i_2i_1}(t)\left(R_0^v - e^{-\tilde{K}(2\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\bar{\tau})+\bar{\tau})}{N-1}\tilde{\alpha} - \langle x_{i_2}(t), v\rangle\right)$$

$$\le \frac{\tilde{K}N_{i_2}}{N-1}(R_0^v - \langle v_{i_2}(t), v\rangle) - \alpha_{i_2i_1}(t)e^{-\tilde{K}(2\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\bar{\tau})+\bar{\tau})}{(N-1)^2}\tilde{\phi}(t)\tilde{\alpha}.$$

Again, using Gronwall's estimate it comes that

$$\langle v_{i_2}(t), v\rangle \le e^{-\frac{\tilde{K}N_{i_2}}{N-1}(t-T-2\bar{\tau})}\langle v_{i_2}(T+2\bar{\tau}), v\rangle) + R_0^v(1 - e^{-\frac{\tilde{K}N_{i_2}}{N-1}(t-T-2\bar{\tau})})$$

$$-e^{-\tilde{K}(2\gamma(T+\bar{\tau})+\bar{\tau})}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\bar{\tau})+\bar{\tau})}{(N-1)^2}\tilde{\alpha}\int_{T+2\bar{\tau}}^t \tilde{\phi}(s)\alpha_{i_2i_1}(s)e^{-\frac{\tilde{K}N_{i_2}}{N-1}(t-s)}ds$$

$$\le R_0^v - e^{-\tilde{K}(3\gamma(T+\bar{\tau})-T)}(R_0^v - \tilde{r}_0^v)\frac{\tilde{\phi}(\gamma(T+\bar{\tau})+\bar{\tau})}{(N-1)^2}\tilde{\alpha}\int_{T+2\bar{\tau}}^t \tilde{\phi}(s)\alpha_{i_2i_1}(s)ds,$$

for all $t \in [T+2\bar{\tau}, \gamma(T+\bar{\tau})+\bar{\tau}]$. In particular, for $t \in [2T+2\bar{\tau}, \gamma(T+\bar{\tau})+\bar{\tau}]$, the condition (0.1.12) and the inequality (3.2.16) imply that

$$\langle v_{i_2}(t), v\rangle \le R_0^v - e^{-\tilde{K}(3\gamma(T+\bar{\tau})-T)}(R_0^v - \tilde{r}_0^v)\left(\frac{\tilde{\phi}(\gamma(T+\bar{\tau})+\bar{\tau})}{N-1}\right)^2\tilde{\alpha}^2. \tag{3.2.18}$$

Finally, iterating the above procedure along the path $i_0, i_1, \ldots, i_r$, with $r \le \gamma$, starting from $i_0 = L$ we find the following upper bound

$$\langle v_{i_k}(t), v\rangle \le R_0^v - e^{-\tilde{K}((k+1)\gamma(T+\bar{\tau})-(T+\bar{\tau})\left(\sum_{l=0}^{k-1} l\right)+\bar{\tau})}(R_0^v - \tilde{r}_0^v)\left(\frac{\tilde{\phi}(\gamma(T+\bar{\tau})+\bar{\tau})\tilde{\alpha}}{N-1}\right)^k, \tag{3.2.19}$$

for all $1 \le k \le r$ and for all $t \in [k(T+\bar{\tau}), \gamma(T+\bar{\tau})+\bar{\tau}]$. In particular, if the path has length γ, for $k = \gamma$, since $\sum_{l=0}^{\gamma-1} l = \frac{\gamma(\gamma-1)}{2}$, inequality (3.2.19) reads as

$$\langle v_{i_\gamma}(t), v\rangle \le R_0^v - e^{-\tilde{K}(\frac{1}{2}(\gamma^2+3\gamma)(T+\bar{\tau})+\bar{\tau})}(R_0^v - \tilde{r}_0^v)\left(\frac{\tilde{\phi}(\gamma(T+\bar{\tau})+\bar{\tau})\tilde{\alpha}}{N-1}\right)^\gamma, \tag{3.2.20}$$

for all $t \in [\gamma(T+\bar{\tau}), \gamma(T+\bar{\tau})+\bar{\tau}]$. Arguing as in Proposition 3.1.6, we can say that (3.2.20) holds for every $i = 1, \ldots, N$.
Now, let $R = 1, \ldots, N$ be such that $\tilde{R}_0^v = \langle v_R(0), v\rangle$. Then, arguing as before, we get

$$\langle v_R(t), v\rangle \ge r_0^v + e^{-\tilde{K}(\gamma(T+\bar{\tau})+\bar{\tau})}(\tilde{R}_0^v - r_0^v), \quad \forall t \in [0, \gamma(T+\bar{\tau})+\bar{\tau}]. \tag{3.2.21}$$

Employing the same arguments used above, we can conclude that

$$\langle v_i(t), v\rangle \geq r_0^v + e^{-\tilde{K}(\frac{1}{2}(\gamma^2+3\gamma)(T+\bar{\tau})+\bar{\tau})}(\tilde{R}_0^v - r_0^v))\left(\frac{\tilde{\phi}(\gamma(T+\bar{\tau})+\bar{\tau})\tilde{\alpha}}{N-1}\right)^{\gamma},$$

for all $t \in [\gamma(T+\bar{\tau}), \gamma(T+\bar{\tau})+\bar{\tau}]$ and for all $i = 1, \dots, N$. Finally, we can deduce that estimate (3.2.13) holds. □

The following proposition extends the previous one in successive time intervals. We omit its proof since it is analogous to the previous one.

Proposition 3.2.12. *Let $v \in \mathbb{R}^d$. For any $n \in \mathbb{N}$, it holds*

$$r_n^v + \Gamma_{n+1}(\tilde{R}_n^v - r_n^v) \leq \langle v_i(t), v\rangle \leq R_n^v - \Gamma_{n+1}(R_n^v - \tilde{r}_n^v), \tag{3.2.22}$$

for all $t \in I_{n+1}$ and for all $i = 1, \dots, N$, where Γ_{n+1} is the positive constant defined as

$$\Gamma_{n+1} := e^{-\tilde{K}(\frac{1}{2}(\gamma^2+3\gamma)(T+\bar{\tau})+\bar{\tau})}\left(\frac{\tilde{\phi}((n+1)(\gamma(T+\bar{\tau})+\bar{\tau}))\tilde{\alpha}}{N-1}\right)^{\gamma}. \tag{3.2.23}$$

Remark 3.2.13. Let us note that from (3.2.22) it comes that

$$R_{n+1}^v - r_{n+1}^v \leq (1-\Gamma_{n+1})(R_n^v - r_n^v), \quad \forall n \in \mathbb{N}_0. \tag{3.2.24}$$

where $\Gamma_{n+1} \in (0,1)$ is the constant in (3.2.23).
Indeed, given $n \in \mathbb{N}_0$, let $i, j = 1, \dots, N$ and $s, t \in I_{n+1}$ be such that $\langle v_i(s), v\rangle = R_{n+1}^v$ and $\langle v_j(t), v\rangle = r_{n+1}^v$. Then, applying Lemma 3.2.12, we can write

$$\begin{aligned} R_{n+1}^v - r_{n+1}^v &= \langle v_i(s), v\rangle - \langle v_j(t), v\rangle \\ &\leq R_n^v - r_n^v - \Gamma_{n+1}(R_n^v - \tilde{r}_n^v) - \Gamma_{n+1}(\tilde{R}_n^v - r_n^v). \end{aligned} \tag{3.2.25}$$

Then, arguing as in the proof of Theorem 3.1.1, we get that estimate (3.2.24) holds true. Also, setting $C^* := e^{-\tilde{K}(\frac{1}{2}(\gamma^2+3\gamma)(T+\bar{\tau})+\bar{\tau})}\left(\frac{\tilde{\alpha}}{N-1}\right)^{\gamma}$, it holds that

$$\Gamma_{n+1} = C^*(\tilde{\phi}((n+1)(\gamma(T+\bar{\tau})+\bar{\tau})))^{\gamma}, \quad \forall n \in \mathbb{N}_0. \tag{3.2.26}$$

As a consequence, (3.2.24) can be written as

$$R_{n+1}^v - r_{n+1}^v \leq (1 - C^*(\tilde{\phi}((n+1)(\gamma(T+\bar{\tau})+\bar{\tau})))^{\gamma})(R_n^v - r_n^v), \quad \forall n \in \mathbb{N}_0. \tag{3.2.27}$$

In particular, from (3.2.24) and (3.2.27), arguing as in Theorem 3.1.1, it comes that

$$F_{n+1} \leq (1-\Gamma_{n+1})F_n, \quad \forall n \in \mathbb{N}_0, \tag{3.2.28}$$

or, equivalently,

$$F_{n+1} \leq (1 - C^*(\tilde{\phi}((n+1)(\gamma(T+\bar{\tau})+\bar{\tau})))^{\gamma})F_n, \quad \forall n \in \mathbb{N}_0, \tag{3.2.29}$$

Now, we are able to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Let $\{(x_i, v_i)\}_{i=1,\dots,N}$ be solution to (0.1.26) under the initial conditions (0.1.28). Let us define

$$\tilde{\Gamma}_{n+1} = \frac{\Gamma_{n+1}}{\gamma(T+\bar{\tau})+\bar{\tau}}, \quad \forall n \in \mathbb{N}_0.$$

Let us introduce the function $\mathcal{E} : [-\bar{\tau}, +\infty) \to [0, +\infty)$,

$$\mathcal{E}(t) := \begin{cases} F_0, & t \in [-\bar{\tau}, \gamma(T+\bar{\tau})+\bar{\tau}], \\ \mathcal{E}(n(\gamma(T+\bar{\tau})+\bar{\tau}))\left(1-\tilde{\Gamma}_{n+1}(t-n(\gamma(T+\bar{\tau})+\bar{\tau}))\right), & \\ & t \in (n(\gamma(T+\bar{\tau})+\bar{\tau}), (n+1)(\gamma(T+\bar{\tau})+\bar{\tau})],\ n \geq 1. \end{cases}$$

By definition, $\mathcal{E}$ is continuous, positive and nonincreasing. Moreover, we claim that

$$F_n \leq \mathcal{E}(t), \quad \forall t \in [-\bar{\tau}, n(\gamma(T+\bar{\tau})+\bar{\tau})],\ \forall n \in \mathbb{N}_0. \tag{3.2.30}$$

We prove this by induction. For $n = 1$, from (3.2.7) we can immediately say that

$$F_1 \leq F_0 = \mathcal{E}(t), \quad \forall t \in [-\bar{\tau}, \gamma(T+\bar{\tau})+\bar{\tau}].$$

Now, assume that (3.2.30) holds for some $n \geq 1$. We have to show that (3.2.30) is true also for $n+1$. From the induction hypothesis and by using again (3.2.7), we have that

$$F_{n+1} \leq F_n \leq \mathcal{E}(t),$$

for all $t \in [-\bar{\tau}, n(\gamma(T+\bar{\tau})+\bar{\tau})]$. It lasts to prove that $F_{n+1} \leq \mathcal{E}(t)$, for all $t \in (n(\gamma(T+\bar{\tau})+\bar{\tau}), (n+1)(\gamma(T+\bar{\tau})+\bar{\tau})]$. From (3.2.28), it comes that

$$\begin{aligned} \mathcal{E}(t) &\geq \mathcal{E}((n+1)(\gamma(T+\bar{\tau})+\bar{\tau})) = \mathcal{E}(n(\gamma(T+\bar{\tau})+\bar{\tau}))(1-\tilde{\Gamma}_{n+1}(\gamma(T+\bar{\tau})+\bar{\tau})) \\ &= (1-\Gamma_{n+1})F_n \geq F_{n+1}, \end{aligned}$$

for all $t \in (n(\gamma(T+\bar{\tau})+\bar{\tau}), (n+1)(\gamma(T+\bar{\tau})+\bar{\tau})]$, where in the above inequalities we have used the fact that $\mathcal{E}$ is nonincreasing. Hence, (3.2.30) is proven.
Now, for almost all time (see Chapter 2 for further details)

$$\frac{d}{dt} \max_{s \in [-\bar{\tau}, t]} d_X(s) \leq \left| \frac{d}{dt} d_X(t) \right| \leq d_V(t). \tag{3.2.31}$$

Next, let us define the function $\mathcal{W} : [-\bar{\tau}, +\infty) \to [0, +\infty)$,

$$\mathcal{W}(t) := (\gamma(T+\bar{\tau})+\bar{\tau})\mathcal{E}(t) + C^* \int_0^{\bar{\tau} C_0^V + M_0^X + \max_{s \in [-\bar{\tau}, t+\gamma(T+\bar{\tau})+\bar{\tau}]} d_X(s)} \left(\min_{\sigma \in [0, r]} \tilde{\psi}(\sigma) \right)^{\gamma} dr,$$

for all $t \geq -\bar{\tau}$. By construction, $\mathcal{W}$ is continuous. Also, for each $n \geq 1$ and for a.e. $t \in (n(\gamma(T+\bar{\tau})+\bar{\tau}), (n+1)(\gamma(T+\bar{\tau})+\bar{\tau})$, from (3.2.6), (3.2.30) and (3.2.31) it follows that

$$\begin{aligned}\frac{d}{dt}\mathcal{W}(t) &= (\gamma(T+\bar{\tau})+\bar{\tau})\frac{d}{dt}\mathcal{E}(t) + C^*(\tilde{\phi}(t+\gamma(T+\bar{\tau})+\bar{\tau}))^\gamma \frac{d}{dt}\max_{s\in[-\bar{\tau},t+(\gamma(T+\bar{\tau})+\bar{\tau})]} d_X(s) \\ &\leq -\mathcal{E}(n\gamma(T+\bar{\tau})+\bar{\tau})C^*(\tilde{\phi}((n+1)(\gamma(T+\bar{\tau})+\bar{\tau})))^\gamma \\ &\qquad + C^*(\tilde{\phi}(t+\gamma(T+\bar{\tau})+\bar{\tau}))^\gamma d_V(t+(\gamma(T+\bar{\tau})+\bar{\tau})) \\ &\leq C^* F_n(-(\tilde{\phi}((n+1)(\gamma(T+\bar{\tau})+\bar{\tau})))^\gamma + (\tilde{\phi}((n+1)(\gamma(T+\bar{\tau})+\bar{\tau}))^\gamma)) = 0.\end{aligned}$$

Then,

$$\frac{d}{dt}\mathcal{W}(t) \leq 0, \quad \text{a.e. } t > \gamma(T+\bar{\tau})+\bar{\tau}, \tag{3.2.32}$$

which implies

$$\mathcal{W}(t) \leq \mathcal{W}(\gamma(T+\bar{\tau})+\bar{\tau}), \quad \forall t \geq \gamma(T+\bar{\tau})+\bar{\tau}. \tag{3.2.33}$$

Now, by definition of $\mathcal{W}$, being $\mathcal{E}$ a nonnegative function, we have

$$C^* \int_0^{\bar{\tau}C_0^V + M_0^X + \max_{s\in[-\bar{\tau},t+\gamma(T+\bar{\tau})+\bar{\tau}]} d_X(s)} \left(\min_{\sigma\in[0,r]} \tilde{\psi}(\sigma)\right)^\gamma dr \leq \mathcal{W}(\gamma(T+\bar{\tau})+\bar{\tau}),$$

for all $t \geq \gamma(T+\bar{\tau})+\bar{\tau}$. Letting $t \to \infty$ in the above inequality, we can conclude that

$$C^* \int_0^{\bar{\tau}C_0^V + M_0^X + \sup_{s\in[-\bar{\tau},+\infty)} d_X(s)} \left(\min_{\sigma\in[0,r]} \tilde{\psi}(\sigma)\right)^\gamma dr \leq \mathcal{W}(\gamma(T+\bar{\tau})+\bar{\tau})). \tag{3.2.34}$$

Finally, since the function $\tilde{\psi}$ satisfies property (3.2.1), from (3.2.34), we can conclude that there exists a positive constant d^* such that

$$\bar{\tau}C_0^V + M_0^X + \sup_{s\in[-\bar{\tau},+\infty)} d_X(s) \leq d^*. \tag{3.2.35}$$

Now, let us define

$$\hat{\phi} := \min_{r\in[0,d^*]} \tilde{\psi}(r).$$

Note that $\tilde{\phi}^* > 0$. Also, (3.2.35) yields

$$\hat{\phi} \leq \tilde{\phi}(t), \quad \forall t \geq -\bar{\tau}. \tag{3.2.36}$$

Then, from (3.2.29) and (3.2.36) we have

$$F_{n+1} \leq (1 - C^*\hat{\phi}^\gamma)F_n, \quad \forall n \in \mathbb{N}_0. \tag{3.2.37}$$

Thus, thanks to an induction argument, we can write

$$F_n \leq (1 - C^*\hat{\phi}^\gamma)^n F_0, \quad \forall n \in \mathbb{N}_0.$$

Note that the above inequality can be rewritten as

$$F_n \leq e^{-n\mu(\gamma(T+\bar{\tau})+\bar{\tau})} F_0, \quad \forall n \in \mathbb{N}_0, \tag{3.2.38}$$

where

$$\mu = \frac{1}{\gamma(T+\bar{\tau})+\bar{\tau}} \ln\left(\frac{1}{1-C^*\hat{\phi}^\gamma}\right).$$

Finally, let $t \geq 0$. Then, $t \in [n(\gamma(T+\bar{\tau})+\bar{\tau}), (n+1)(\gamma(T+\bar{\tau})+\bar{\tau})]$, for some $n \in \mathbb{N}_0$. Then, using (3.2.6) and (3.2.38)

$$d_V(t) \leq F_n \leq e^{-n\mu(\gamma(T+\bar{\tau})+\bar{\tau})} F_0 \leq e^{-\mu(t-\gamma(T+\bar{\tau})-\bar{\tau})} F_0,$$

which concludes our proof. □

Chapter 4

The Hegselmann-Krause model with time delay and communication case: the all-to-all interaction case

In this chapter, we deal again with the Hegselmann-Krause model (0.1.8). In particular, we consider the case in which $\chi_{ij} = 1$, for all $i, j = 1, \dots, N$, namely the case of all-to-all interaction. Note that, in this case, $\gamma = 1$. Moreover, we suppose that $\tau_{ij} = \tau(t)$, for a.e. $t \geq 0$ and for all $i, j = 1, \dots, N$, where $\tau(\cdot)$ is a suitable time delay function that satisfies (0.1.9), i.e.

$$0 \leq \tau(t) \leq \bar{\tau}, \quad \forall t \geq 0. \tag{4.0.1}$$

Also, we assume that $\alpha_{ij}(t) = \alpha(t)$, for a.e. $t \geq 0$ and for all $i, j = 1, \dots, N$, where $\alpha : [0, +\infty) \to [0, 1]$ is a suitable weight function that satisfies the Persistence Excitation Condition **(PE)**, that now reads as

(PE) there exist two positive constants T and $\tilde{\alpha}$ such that

$$\int_t^{t+T} \alpha(s) ds \geq \tilde{\alpha}, \quad \forall t \geq 0. \tag{4.0.2}$$

Without loss of generality, we can assume that $\tilde{\alpha} K \leq 1$ and that $T \geq \bar{\tau}\tau$.

In this situation, the results in Chapter 3 for the first-order model can be improved. Indeed, we have seen in the proof of Theorem 3.1.1 that the constant C in the exponential decay estimate depends on the number of agents N. Although the result 3.1.1 is very general, the dependence of the number of agents in the decay estimate satisfied by the solution's diameter is not so good, especially when the number of agents becomes too large.

So, in this chapter we will show that, in the case of universal interaction, the C constant in the exponential decay estimate (3.1.1) can be chosen independent of N, whenever the time delay and the weight functions are not pair-dependent. The results in this chapter are the analogous in [42].

The consensus result we will prove now is the following.

Theorem 4.0.1. *Assume $\chi_{ij} = 1$, for all $i, j = 1, \ldots, N$. Let $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a positive, bounded, continuous function. Assume that the weight function $\alpha : [0, +\infty) \to [0, 1]$ is $\mathcal{L}^1$-measurable and satisfies* (4.0.2) *and that the time delay function $\tau : [0, +\infty) \to [0, +\infty)$ is continuous and satisfies* (4.0.1). *Let $x_i^0 : [-\bar{\tau}, 0] \to \mathbb{R}^d$ be a continuous function, for any $i = 1, \ldots, N$. Then, every solution $\{x_i\}_{i=1,\ldots,N}$ to* (0.1.8) *with the initial conditions* (0.1.16) *satisfies the exponential decay estimate*

$$d(t) \leq \left(\max_{i,j=1,\ldots,N} \max_{r,s \in [-\bar{\tau},0]} |x_i(r) - x_j(s)| \right) e^{-C(t-3T+\bar{\tau})}, \quad \forall t \geq 0, \tag{4.0.3}$$

where T is the positive constant in (0.1.12) *and C is a suitable positive constant, independent of N.*

4.1 Proof of the consensus estimate

Let $\{x_i\}_{i=1,\ldots,N}$ be solution to (0.1.8) under the initial conditions (0.1.16). We assume that the hypotheses of Theorem 4.0.1 are satisfied. Let us first give the following definitions.

Definition 4.1.1. Given a vector $v \in \mathbb{R}^d$, for all $n \in \mathbb{N}_0$, we define

$$m_n^v := \min_{i=1,\ldots,N} \min_{s \in [nT-\bar{\tau}, nT]} \langle x_i(s), v \rangle,$$

$$M_n^v := \max_{j=1,\ldots,N} \max_{s \in [nT-\bar{\tau}, nT]} \langle x_j(s), v \rangle.$$

Definition 4.1.2. For all $n \in \mathbb{N}_0$, we define

$$D_n := \max_{i,j=1,\ldots,N} \max_{r,s \in [nT-\bar{\tau}, nT]} |x_i(r) - x_j(s)|.$$

Let us note that

$$D_0 = \max_{i,j=1,\ldots,N} \max_{r,s \in [-\bar{\tau}, 0]} |x_i(r) - x_j(s)|.$$

So, the exponential decay estimate (4.0.3) can be rewritten as

$$d(t) \leq D_0 e^{-C(t-3T+\bar{\tau})}, \quad \forall t \geq 0.$$

In this chapter, we won't prove the asymptotic consensus with the method provided in Chapter 3, namely we won't refine the estimate in Lemma 3.1.2 to obtain an estimate like (3.1.18).

We will rather use a similar approach to the one employed in Chapter 1. In particular, we need the following fundamental results. Since they are analogous to the correspondent results in Chapter 1 and Chapter 3, we omit their proofs.

Lemma 4.1.1. *For each vector $v \in \mathbb{R}^d$ and for all $n \in \mathbb{N}_0$, we have that*

$$m_n^v \leq \langle x_i(t), v \rangle \leq M_n^v, \tag{4.1.1}$$

for all $t \geq nT - \bar{\tau}$ and for any $i = 1, \ldots, N$.

Lemma 4.1.2. *For each $n \in \mathbb{N}_0$, we have that*

$$|x_i(s) - x_j(t)| \leq D_n, \tag{4.1.2}$$

for all $s, t \geq nT - \bar{\tau}$ and for any $i, j = 1, \dots, N$. In particular,

$$d(t) \leq D_n, \quad \forall t \geq nT - \bar{\tau}. \tag{4.1.3}$$

Remark 4.1.3. Let us note that from (4.1.2) it comes that

$$D_{n+1} \leq D_n, \quad \forall n \in \mathbb{N}_0. \tag{4.1.4}$$

Lemma 4.1.4. *For every $i = 1, \dots, N$, we have that*

$$|x_i(t)| \leq C_0 := \max_{i=1,\dots,N} \max_{s \in [-\bar{\tau}, 0]} |x_i(s)|, \quad \forall t \geq -\bar{\tau}. \tag{4.1.5}$$

In particular,

$$\psi(x_i(t), x_j(t - \tau_{ij}(t))) \geq \psi_0 := \min_{|y|,|z| \leq C_0} \psi(y, z), \quad \forall t \geq 0, \forall i, j = 1, \dots, N. \tag{4.1.6}$$

Lemma 4.1.5. *For all $i, j = 1, \dots, N$, unit vector $v \in \mathbb{R}^d$ and $n \in \mathbb{N}_0$, we have*

$$\langle x_i(t) - x_j(t), v \rangle \leq e^{-K(t-\bar{t})} \langle x_i(\bar{t}) - x_j(\bar{t}), v \rangle + (1 - e^{-K(t-\bar{t})}) D_n, \tag{4.1.7}$$

for all $t \geq \bar{t} \geq nT$.
Moreover, for all $n \in \mathbb{N}_0$, we get

$$D_{n+1} \leq e^{-KT} d(nT) + (1 - e^{-KT}) D_n. \tag{4.1.8}$$

Now, we prove Theorem 4.0.1.

Proof of Theorem 4.0.1. Let $\{x_i\}_{i=1,\dots,N}$ be solution to (0.1.8) under the initial conditions (0.1.16). We first claim that there exist a positive constant $C^* \in (0, 1)$, independent of $N \in \mathbb{N}$, such that

$$d(nT) \leq C^* D_{n-2}, \quad \forall n \geq 2. \tag{4.1.9}$$

Indeed, let $n \geq 2$. Note that inequality (4.1.9) is trivially satisfied if $d(nT) = 0$. So, we can assume $d(nT) > 0$. Let $i, j = 1, \dots, N$ be such that $d(nT) = |x_i(nT) - x_j(nT)|$. We define the unit vector

$$v = \frac{x_i(n(T) - x_j(nT)}{|x_i(nT) - x_j(nT)|}.$$

Then,

$$d(nT) = \langle x_i(nT) - x_j(nT), v \rangle.$$

Now, we distinguish two different situations.

Case I. Assume that there exists $\bar{t} \in [(n-1)T - \bar{\tau}, nT]$ such that

$$\langle x_i(\bar{t}) - x_j(\bar{t}), v \rangle < 0.$$

Note that, being $T \geq \bar{\tau}$, it holds $nT \geq \bar{t} \geq (n-1)T - \bar{\tau} \geq (n-2)T$. Then, we can apply (4.1.7) and we get

$$\begin{aligned} d(nT) &\leq e^{-K(nT-\bar{t})}\langle x_i(\bar{t}) - x_j(\bar{t}), v\rangle + (1 - e^{-K(nT-\bar{t})})D_{n-2} \\ &\leq (1 - e^{-K(nT-\bar{t})})D_{n-2} \\ &\leq (1 - e^{-K(T+\bar{\tau})})D_{n-2}. \end{aligned} \tag{4.1.10}$$

Case II. Assume it rather holds

$$\langle x_i(t) - x_j(t), v\rangle \geq 0, \quad \forall t \in [(n-1)T - \bar{\tau}, nT]. \tag{4.1.11}$$

Then, arguing as in Lemma 1.2.1 we get

$$\frac{d}{dt}\langle x_i(t) - x_j(t), v\rangle \leq (K - \psi_0\alpha(t))\,(M^v_{n-1} - m^v_{n-1}) - K\langle x_i(t) - x_j(t), v\rangle,$$

for a.e. $t \in [(n-1)T, nT]$. Hence, Gronwall's inequality yields

$$\begin{aligned} \langle x_i(t) - x_j(t), v\rangle \leq{}& e^{-K(t-(n-1)T)}\langle x_i((n-1)T) - x_j((n-1)T), v\rangle \\ &+(M^v_{n-1} - m^v_{n-1}) \int_{(n-1)T}^{t} (K - \psi_0\alpha(s))\, e^{-K(t-s)} ds, \end{aligned}$$

for all $t \in [(n-1)T, nT]$. In particular, for $t = nT$, it comes that

$$\begin{aligned} d(nT) &\leq e^{-KT}\langle x_i((n-1)T) - x_j((n-1)T), v\rangle \\ &\qquad +(M^v_{n-1} - m^v_{n-1}) \int_{(n-1)T}^{nT} (K - \psi_0\alpha(s))\, e^{-K(nT-s)} ds \\ &\leq \left(e^{-KT} + K \int_{(n-1)T}^{nT} e^{-K(nT-s)} ds \right. \\ &\qquad \left. -\psi_0 \int_{(n-1)T}^{nT} \alpha(s) e^{-K(nT-s)} ds \right) D_{n-1} \\ &\leq \left(1 - \psi_0 e^{-KT} \int_{(n-1)T}^{nT} \alpha(s) ds \right) D_{n-1}. \end{aligned}$$

Then, since from the Persistence Excitation Condition (4.0.2) we have that

$$\int_{(n-1)T}^{nT} \alpha(s) ds \geq \tilde{\alpha},$$

we get

$$d(nT) \leq \left(1 - \psi_0 e^{-KT}\tilde{\alpha}\right) D_{n-1} \leq \left(1 - \psi_0 e^{-K((T+2\bar{\tau}))}\tilde{\alpha}\right) D_{n-2}.$$

Now, we set

$$C^* := \max\left\{1 - e^{-K(T+\bar{\tau})}, 1 - \psi_0 e^{-KT}\tilde{\alpha}\right\}. \tag{4.1.12}$$

Then, recalling of (4.1.10), we deduce $C^* \in (0,1)$ is the constant for which (4.1.9) holds. Finally, let us define

$$C = \frac{1}{3T} \ln \left(\frac{1}{1 - e^{-KT}(1 - C^*)} \right). \tag{4.1.13}$$

Then, arguing as in Theorem 1.0.1, we can conclude that C, that does not depend on N, is the positive constant for which the exponential decay estimate (4.0.3) is fulfilled.

□

4.2 The continuum model

Now, the continuum model associated with the particle system (0.1.8) is given by

$$\begin{aligned} &\partial_t \mu_t + \operatorname{div}\left(F[\mu_{t-\tau(t)}]\mu_t\right) = 0, \quad t > 0, \\ &\mu_s = g_s, \quad x \in \mathbb{R}^d, \quad s \in [-\bar{\tau}, 0], \end{aligned} \tag{4.2.14}$$

where the velocity field F is defined as

$$F[\mu_{t-\tau(t)}](x) = \int_{\mathbb{R}^d} \alpha(t)\psi(x,y)(y-x)\, d\mu_{t-\tau(t)}(y), \tag{4.2.15}$$

and $g_s \in \mathcal{C}([-\bar{\tau}, 0]; \mathcal{M}(\mathbb{R}^d))$.

As in Chapter 1, we assume that the potential $\psi(\cdot,\cdot)$ in (4.2.15) is Lipschitz continuous.

Definition 4.2.1. Let $T > 0$. We say that $\mu_t \in \mathcal{C}([0,T); \mathcal{M}(\mathbb{R}^d))$ is a measure-valued solution to (4.2.14) on the time interval $[0,T)$ if for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times [0,T))$ we have:

$$\int_0^T \int_{\mathbb{R}^d} \left(\partial_t \varphi + F[\mu_{t-\tau(t)}](x) \cdot \nabla_x \varphi \right) d\mu_t(x) dt + \int_{\mathbb{R}^d} \varphi(x,0) dg_0(x) = 0. \tag{4.2.16}$$

Since the consensus result for the particle model (0.1.8) holds without any upper bounds on the time delay τ, one can deduce the following consensus theorem for the PDE model (4.2.14) without requiring a smallness assumption on the time delay τ. We omit the proof since, once we have the result for the particle system (0.1.8) with estimates independent of the number of agents, the consensus estimate for the continuum model is obtained with arguments analogous to the ones used in [37] and [88]. On the other hand, we formulate the theorem since the ones stated in [37, 88] require an upper bound on the time delay size inherited from the result for the particle system. Now, the more general result for the ODE system (0.1.8) allows us to extend the applicability of the convergence result for the continuum model (4.2.14).

Theorem 4.2.1. *Let $\mu_t \in \mathcal{C}([0,T]; \mathcal{P}_1(\mathbb{R}^d))$ be a measure-valued solution to* (1.3.1) *with compactly supported initial datum $g_s \in \mathcal{C}([-\bar{\tau}, 0]; \mathcal{P}_1(\mathbb{R}^d))$ and let F as in* (1.3.2). *Then, there exists a constant $C > 0$ such that*

$$d_X(\mu_t) \leq \left(\max_{s \in [-\bar{\tau}, 0]} d_X(g_s) \right) e^{-Ct}, \quad \forall t \geq 0.$$

Chapter 5

Opinion formation and flocking models with attractive-repulsive interaction

In this chapter, we analyze first and second-order Cucker-Smale models with attractive-repulsive interaction. We will find conditions ensuring the asymptotic consensus for both models (0.1.17) and (0.1.29), despite the agents repeal each other in the intervals of negative interaction, i.e. in which $\alpha(t) = 1$. Compensating the behaviour of the solutions to the considered models in the *bad* intervals, i.e. the intervals in which the agents repeal each other, with the *good* behavior in the intervals in which the influence among the agents is positive, we establish the convergence to consensus for the Hegselmann-Krause model with attractive-repulsive interaction (0.1.17) and the exhibition of asymptotic flocking for the Cucker-Smale model with attractive-repulsive interaction (0.1.29) under quite general assumptions. The results contained in this chapter are taken from [43].

5.1 The Hegselmann-Krause model

In this Section, we deal with the first-order model (0.1.17). For solutions to (0.1.17), the following consensus result holds.

Theorem 5.1.1. *Let $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a positive, bounded, continuous function. Assume that the sequence $\{t_n\}_n$ of definition* (0.1.19) *satisfies* (0.1.20). *Assume also that the following conditions hold:*

$$\sum_{p=0}^{\infty} \ln \left(\frac{e^{K(t_{2p+2}-t_{2p+1})}}{2 - e^{K(t_{2p+2}-t_{2p+1})}} \right) < +\infty, \tag{5.1.1}$$

$$\sum_{p=0}^{\infty} \ln \left(\max \left\{ 1 - e^{-K(t_{2p+1}-t_{2p})}, 1 - \frac{\psi_0}{K}(1 - e^{-K(t_{2p+1}-t_{2p})}) \right\} \right) = -\infty, \tag{5.1.2}$$

where

$$\psi_0 := \min_{|y|,|z| \le M^0} \psi(y, z), \tag{5.1.3}$$

being
$$M^0 = e^{K\sum_{p=0}^{\infty}(t_{2p+2}-t_{2p+1})}\max_{i=1,\dots,N}|x_i(0)|. \tag{5.1.4}$$

Then, every solution $\{x_i\}_{i=1,\dots,N}$ *to* (0.1.17) *with the initial conditions* (0.1.18) *converges to consensus.*

Remark 5.1.2. Let us note that condition (5.1.1) implies that

$$\sum_{p=0}^{+\infty}(t_{2p+2}-t_{2p+1}) < +\infty, \tag{5.1.5}$$

so that the quantity M^0 in (5.1.4) is finite and it makes sense to consider the minimum given by (5.1.3). Indeed, being $e^{K(t_{2p+2}-t_{2p+1})} > 1$, it turns out that

$$2 - e^{K(t_{2p+2}-t_{2p+1})} < 1.$$

Then, using (0.1.20), from the above inequality it comes that

$$1 < \frac{1}{2 - e^{K(t_{2p+2}-t_{2p+1})}},$$

from which

$$e^{K(t_{2p+2}-t_{2p+1})} < \frac{e^{K(t_{2p+2}-t_{2p+1})}}{2 - e^{K(t_{2p+2}-t_{2p+1})}}.$$

Hence,

$$K\sum_{p=0}^{\infty}(t_{2p+2}-t_{2p+1}) \le \sum_{p=0}^{\infty}\ln\left(\frac{e^{K(t_{2p+2}-t_{2p+1})}}{2 - e^{K(t_{2p+2}-t_{2p+1})}}\right).$$

So, (5.1.1) implies (5.1.5). As a consequence, from (5.1.5) we deduce that $t_{2p+2}-t_{2p+1} \to 0$, as $p \to \infty$.

Remark 5.1.3. Assume that

$$t_{2n+1} - t_{2n} > \frac{1}{K}\ln\left(1 + \frac{K}{\psi_0}\right), \quad \forall n \in \mathbb{N}_0. \tag{5.1.6}$$

Note that $\ln\left(1 + \frac{K}{\psi_0}\right) > \ln 2$, so that

$$t_{2p+2} - t_{2p+1} < t_{2q+1} - t_{2q}, \quad \forall p, q \in \mathbb{N}_0.$$

Then, in this situation the condition (5.1.2) can be simplified. Indeed, for all $p \in \mathbb{N}_0$, from (5.1.6) we have that

$$K(t_{2p+1} - t_{2p}) > \ln\left(1 + \frac{K}{\psi_0}\right),$$

which implies

$$e^{-K(t_{2p+1}-t_{2p})} < \frac{1}{1 + \frac{K}{\psi_0}}.$$

Then,

$$\left(1+\frac{\psi_0}{K}\right)e^{-K(t_{2p+1}-t_{2p})}<\frac{\psi_0}{K},$$

and this gives

$$\frac{\psi_0}{K}\left(e^{-K(t_{2p+1}-t_{2p})}-1\right)<-e^{-K(t_{2p+1}-t_{2p})}.$$

Thus,

$$1-\frac{\psi_0}{K}\left(1-e^{-K(t_{2p+1}-t_{2p})}\right)<1-e^{-K(t_{2p+1}-t_{2p})},$$

and

$$\max\left\{1-e^{-K(t_{2p+1}-t_{2p})},1-\frac{\psi_0}{K}(1-e^{-K(t_{2p+1}-t_{2p})})\right\}=1-e^{-K(t_{2p+1}-t_{2p})}.$$

So, (5.1.2) becomes

$$\sum_{p=0}^{+\infty}\ln\left(1-e^{-K(t_{2p+1}-t_{2p})}\right)=-\infty. \tag{5.1.7}$$

However, the above condition (5.1.7) is automatically satisfied. Indeed, we can assume that

$$t_{2n+2}-t_{2n+1}\leq T,\quad \forall n\in\mathbb{N}_0,$$

for some $T>0$, eventually splitting the intervals of positive interaction into subintervals of length at most T. Then,

$$\sum_{p=0}^{\infty}\ln\left(1-e^{-K(t_{2p+1}-t_{2p})}\right)\leq\sum_{p=0}^{\infty}\ln\left(1-e^{-KT}\right)=-\infty,$$

from which (5.1.7) is fulfilled. Thus, (5.1.6) implies (5.1.2).

5.1.1 Preliminary estimates

Let $\{x_i\}_{i=1,\dots,N}$ be solution to (0.1.17) under the initial conditions (0.1.18). In this section, we present some preliminary lemmas. We first give some results that are related to the behavior of the solution $\{x_i\}_{i=1,\dots,N}$ in the intervals of positive interaction.

Lemma 5.1.4. *For each $v\in\mathbb{R}^d$ and $n\in\mathbb{N}_0$, we have that*

$$\min_{j=1,\dots,N}\langle x_j(t_{2n}),v\rangle\leq\langle x_i(t),v\rangle\leq\max_{j=1,\dots,N}\langle x_j(t_{2n}),v\rangle, \tag{5.1.8}$$

for all $t\in[t_{2n},t_{2n+1}]$ and $i=1,\dots,N$.

Proof. The proof follows using similar arguments to the ones employed in Lemma 1.1.1 of Chapter 1. □

As in Chapter 1, from the above Lemma one can deduce the following estimates.

Lemma 5.1.5. *For each* $n \in \mathbb{N}_0$ *and* $i, j = 1, \dots, N$*, we get*

$$|x_i(s) - x_j(t)| \leq d(t_{2n}), \quad \forall s, t \in [t_{2n}, t_{2n+1}]. \tag{5.1.9}$$

Remark 5.1.6. Let us note that from (1.1.8), in particular, it follows that

$$d(t_{2n+1}) \leq d(t_{2n}), \quad \forall n \in \mathbb{N}_0. \tag{5.1.10}$$

Now, we deal with the intervals in which the agents repeal each other.

Lemma 5.1.7. *For each* $v \in \mathbb{R}^d$ *and* $n \in \mathbb{N}_0$*, we have that*

$$\min_{j=1,\dots,N} \langle x_j(t_{2n+2}), v \rangle \leq \langle x_i(t), v \rangle \leq \max_{j=1,\dots,N} \langle x_j(t_{2n+2}), v \rangle, \tag{5.1.11}$$

for all $t \in [t_{2n+1}, t_{2n+2}]$ *and* $i = 1, \dots, N$*.*

Proof. The proof follows using similar arguments to the ones employed in Lemma 1.1.1 of Chapter 1. □

As in Chapter 1, from the previous lemmas one can prove the following estimates.

Lemma 5.1.8. *For each* $n \in \mathbb{N}_0$ *and* $i, j = 1, \dots, N$*, we get*

$$|x_i(s) - x_j(t)| \leq d(t_{2n+2}), \quad \forall s, t \in [t_{2n+1}, t_{2n+2}]. \tag{5.1.12}$$

Remark 5.1.9. Let us note that from (5.1.12), in particular, it follows that

$$d(t_{2n+2}) \geq d(t_{2n+1}), \quad \forall n \in \mathbb{N}_0. \tag{5.1.13}$$

Also, in the intervals in which the particles attract each other, the solutions of the system under consideration have a bound that is uniform with respect to $i = 1, \dots, N$, but that depends on the maximum value assumed by the opinions of the agents at the left end point of the good interval. To this aim, let us define

$$M_n^0 := \max_{i=1,\dots,N} |x_i(t_n)|, \quad \forall n \in \mathbb{N}_0. \tag{5.1.14}$$

Let us note that, in particular, for $n = 0$

$$M_0^0 := \max_{i=1,\dots,N} |x_i(0)|,$$

which is the same constant that appears in (5.1.4).

Lemma 5.1.10. *For every* $i = 1, \dots, N$*, we have that*

$$|x_i(t)| \leq M_{2n}^0, \quad \forall t \in [t_{2n}, t_{2n+1}], \tag{5.1.15}$$

where M_{2n}^0 *is the positive constant defined as in* (5.1.14)*.*

Proposition 5.1.11. *For all $i, j = 1, \dots, N$, unit vector $v \in \mathbb{R}^d$ and $n \in \mathbb{N}_0$ we have that*

$$\langle x_i(t) - x_j(t), v\rangle \le e^{-K(t-\bar{t})}\langle x_i(\bar{t}) - x_j(\bar{t}), v\rangle + (1 - e^{-K(t-\bar{t})})d(t_{2n}), \tag{5.1.16}$$

for all $t_{2n+1} \ge t \ge \bar{t} \ge t_{2n}$.

Now, we find a bound from below for the influence function ψ. As in the previous chapters, this will be crucial to prove the asymptotic consensus. However, in this case, the bound from below on the influence function requires finer arguments with respect to the analysis carried out in the previous chapters. Indeed, in the previous chapters the fact that the agents' opinions had a bound, uniform with respect to t and i, allowed us to immediately deduce the existence of a bound from below on the influence function. Here, the bound on the agents' opinions depends on the values assumed by the agents opinions at some points (see estimate (5.1.15)) and holds only on the intervals of positive interaction.

Proposition 5.1.12. *Assume* (5.1.5)*. Then, for all $t \ge 0$, we have that*

$$\psi(x_i(t), x_j(t)) \ge \psi_0, \quad \forall i, j = 1, \dots, N, \tag{5.1.17}$$

where ψ_0 is the positive constant in (5.1.3)*.*

Remark 5.1.13. Let us note that the previous result holds in particular under assumption (5.1.1), which implies (5.1.5) as already pointed out.

Proof of Proposition 5.1.12. From (5.1.15), it follows that

$$\max_{i=1,\dots,N} |x_i(t)| \le M^0, \quad \forall t \ge 0. \tag{5.1.18}$$

To see this, fix $t \ge 0$. Then, there exists $n \in \mathbb{N}_0$ such that $t \in [t_{2n}, t_{2n+2}]$. Thus, if $t \in [t_{2n}, t_{2n+1}]$, from (5.1.15) we have that

$$|x_i(t)| \le M^0_{2n} = \max_{i=1,\dots,N} |x_i(t_{2n})|, \quad \forall i = 1, \dots, N. \tag{5.1.19}$$

On the other hand, assume that $t \in (t_{2n+1}, t_{2n+2})$. Given $i = 1, \dots, N$, if $|x_i(t)| > 0$, we define the unit vector

$$v = \frac{x_i(t)}{|x_i(t)|}.$$

Then,

$$|x_i(t)| = \langle x_i(t), v\rangle.$$

Now, for all $s \in [t_{2n+1}, t)$, it holds that

$$\begin{aligned}
\frac{d}{ds}\langle x_i(s), v\rangle &= -\frac{1}{N-1}\sum_{j:j\neq i} \psi(x_i(s), x_j(s))(\langle x_j(s), v\rangle - \langle x_i(s), v\rangle) \\
&= \frac{1}{N-1}\sum_{j:j\neq i} \psi(x_i(s), x_j(s))(\langle x_i(s), v\rangle - \langle x_j(s), v\rangle).
\end{aligned}$$

Thus, denoted with

$$m_{t_{2n+2}} = \min_{l=1,\dots,N} \langle x_l(t_{2n+2}), v\rangle,$$

the first inequality in (5.1.11) implies that

$$\langle x_l(s), v\rangle \geq m_{t_{2n+2}}, \quad \forall s \in [t_{2n+1}, t],\ \forall l = 1, \dots, N.$$

As a consequence, we get

$$\begin{aligned}\frac{d}{ds}\langle x_i(s), v\rangle &\leq \frac{1}{N-1}\sum_{j:j\neq i}\psi(x_i(s), x_j(s))(\langle x_i(s), v\rangle - m_{t_{2n+2}})\\ &\leq K(\langle x_i(s), v\rangle - m_{t_{2n+2}}).\end{aligned}$$

So, the Gronwall's inequality yields

$$\begin{aligned}\langle x_i(s), v\rangle &\leq e^{K(s-t_{2n+1})}\langle x_i(t_{2n+1}), v\rangle - Km_{t_{2n+2}}\int_{t_{2n+1}}^{s} e^{K(s-r)}dr\\ &= e^{K(s-t_{2n+1})}\langle x_i(t_{2n+1}), v\rangle + m_{t_{2n+2}}(1 - e^{K(s-t_{2n+1})}),\end{aligned}$$

for all $s \in [t_{2n+1}, t]$. In particular, for $s = t$ it comes that

$$\begin{aligned}\langle x_i(t), v\rangle &\leq e^{K(t-t_{2n+1})}\langle x_i(t_{2n+1}), v\rangle + m_{t_{2n+2}}(1 - e^{K(t-t_{2n+1})})\\ &= e^{K(t-t_{2n+1})}(\langle x_i(t_{2n+1}), v\rangle - m_{t_{2n+2}}) + m_{t_{2n+2}}\\ &\leq e^{K(t_{2n+2}-t_{2n+1})}(\langle x_i(t_{2n+1}), v\rangle - m_{t_{2n+2}}) + m_{t_{2n+2}}\\ &= e^{K(t_{2n+2}-t_{2n+1})}\langle x_i(t_{2n+1}), v\rangle + m_{t_{2n+2}}(1 - e^{K(t_{2n+2}-t_{2n+1})})\\ &\leq e^{K(t_{2n+2}-t_{2n+1})}\langle x_i(t_{2n+1}), v\rangle\\ &\leq e^{K(t_{2n+2}-t_{2n+1})}|x_i(t_{2n+1})|.\end{aligned}$$

Thus, using (5.1.15) we get

$$|x_i(t)| = \langle x_i(t), v\rangle \leq e^{K(t_{2n+2}-t_{2n+1})}M^0_{2n}.$$

Of course, the above inequality is satisfied also if $|x_i(t)| = 0$. Thus, combining this last inequality with (5.1.19), being $e^{K(t_{2n+2}-t_{2n+1})} > 1$, we can conclude that

$$|x_i(t)| \leq e^{K(t_{2n+2}-t_{2n+1})}M^0_{2n}, \quad \forall n \in \mathbb{N}_0,\ t \in [t_{2n}, t_{2n+2}],\ i = 1, \dots, N. \tag{5.1.20}$$

Now, let us note that, using an induction argument, from (5.1.20) it follows that

$$M^0_{2n+2} = \max_{i=1,\dots,N}|x_i(t_{2n+2})| \leq M^0_0\prod_{p=0}^{n} e^{K(t_{2p+2}-t_{2p+1})}, \quad \forall n \geq 0.$$

As a consequence, for all $t \geq 0$, it holds

$$|x_i(t)| \leq M_0^0 \prod_{p=0}^{n} e^{K(t_{2p+2}-t_{2p+1})} = e^{K\sum_{p=0}^{n}(t_{2p+2}-t_{2p+1})} M_0^0.$$

Then, for all $t \geq 0$,

$$|x_i(t)| \leq M_0^0 e^{K\sum_{p=0}^{\infty}(t_{2p+2}-t_{2p+1})},$$

which proves (5.1.18).
Finally, from (5.1.18), we deduce that

$$\psi(x_i(t), x_i(t)) \geq \psi_0,$$

for all $t \geq 0$. □

5.1.2 Asymptotic consensus

Now, before moving to the proof of Theorem 5.1.1, we provide some estimates on the sequence of diameters $\{d(t_n)\}_n$. First of all, thanks to the presence of a uniform bound from below on the influence function ψ, the following fundamental result holds in the intervals of positive interaction.

Proposition 5.1.14. *Assume* (5.1.5). *Then, for all $n \in \mathbb{N}_0$, there exists a constant $C_{2n} \in (0, 1)$, independent of $N \in \mathbb{N}_0$, such that*

$$d(t_{2n+1}) \leq C_{2n} d(t_{2n}). \tag{5.1.21}$$

Proof. Let $n \in \mathbb{N}_0$. Trivially, if $d(t_{2n+1}) = 0$, then of course inequality (5.1.21) holds for any positive constant. So, suppose $d(t_{2n+1}) > 0$. Let $i, j = 1, \dots, N$ be such that $d(t_{2n+1}) = |x_i(t_{2n+1}) - x_j(t_{2n+1})|$. We set

$$v = \frac{x_i(t_{2n+1}) - x_j(t_{2n+1})}{|x_i(t_{2n+1}) - x_j(t_{2n+1})|}.$$

Then, v is a unit vector for which we can write

$$d(t_{2n+1}) = \langle x_i(t_{2n+1}) - x_j(t_{2n+1}), v\rangle.$$

Let us define

$$M_{t_{2n}} = \max_{l=1,\dots,N} x_l(t_{2n}), v\rangle,$$

$$m_{t_{2n}} = \min_{l=1,\dots,N} \langle x_l(t_{2n}), v\rangle.$$

Then $M_{t_{2n}} - m_{t_{2n}} \leq d(t_{2n})$.
Now, we distinguish two different situations.

Case I. Assume that there exists $\bar{t} \in [t_{2n}, t_{2n+1})$ such that

$$\langle x_i(\bar{t}) - x_j(\bar{t}), v\rangle < 0.$$

Then, from (5.1.16) with $t_{2n+1} \geq \bar{t} \geq t_{2n}$, we have

$$\begin{aligned} d(t_{2n+1}) &\leq e^{-K(t_{2n+1}-\bar{t})}\langle x_i(\bar{t}) - x_j(\bar{t}), v\rangle + (1 - e^{-K(t_{2n+1}-\bar{t})})d(t_{2n}) \\ &\leq (1 - e^{-K(t_{2n+1}-\bar{t})})d(t_{2n}) \\ &\leq (1 - e^{-K(t_{2n+1}-t_{2n})})d(t_{2n}). \end{aligned} \tag{5.1.22}$$

Case II. Assume it rather holds

$$\langle x_i(t) - x_j(t), v\rangle \geq 0, \quad \forall t \in [t_{2n}, t_{2n+1}). \tag{5.1.23}$$

Then, for every $t \in [t_{2n}, t_{2n+1})$, we have that

$$\begin{aligned} \frac{d}{dt}\langle x_i(t) - x_j(t), v\rangle &= \frac{1}{N-1}\sum_{l:l\neq i}\psi(x_i(t), x_l(t))\langle x_l(t) - x_i(t), v\rangle \\ &\quad -\frac{1}{N-1}\sum_{l:l\neq j}\psi(x_i(t), x_l(t))\langle x_l(t) - x_j(t), v\rangle \\ &= \frac{1}{N-1}\sum_{l:l\neq i}\psi(x_i(t), x_l(t))(\langle x_l(t), v\rangle - M_{t_{2n}} + M_{t_{2n}} - \langle x_i(t), v\rangle) \\ &\quad +\frac{1}{N-1}\sum_{l:l\neq j}\psi(x_i(t), x_l(t))(\langle x_j(t), v\rangle - m_{t_{2n}} + m_{t_{2n}} - \langle x_l(t), v\rangle) \\ &:= S_1 + S_2. \end{aligned}$$

Now, being $t \in [t_{2n}, t_{2n+1})$, from (5.1.8) we have that

$$m_{t_{2n}} \leq \langle x_k(t), v\rangle \leq M_{t_{2n}}, \quad \forall k = 1, \ldots, N. \tag{5.1.24}$$

Therefore, using (5.1.24), we get

$$\begin{aligned} S_1 &= \frac{1}{N-1}\sum_{l:l\neq i}\psi(x_i(t), x_l(t-))(\langle x_l(t), v\rangle - M_{t_{2n}}) \\ &\quad +\frac{1}{N-1}\sum_{l:l\neq i}\psi(x_i(t), x_l(t))(M_{t_{2n}} - \langle x_i(t), v\rangle) \\ &\leq \frac{1}{N-1}\psi_0\sum_{l:l\neq i}(\langle x_l(t), v\rangle - M_{t_{2n}}) + K(M_{t_{2n}} - \langle x_i(t), v\rangle), \end{aligned}$$

and

$$\begin{aligned} S_2 &= \frac{1}{N-1}\sum_{l:l\neq j}\psi(x_i(t), x_l(t))(\langle x_j(t), v\rangle - m_{t_{2n}}) \\ &\quad +\frac{1}{N-1}\sum_{l:l\neq j}\psi(x_i(t), x_l(t))(m_{t_{2n}} - \langle x_l(t), v\rangle) \\ &\leq K(\langle x_j(t), v\rangle - m_{t_{2n}}) + \frac{1}{N-1}\psi_0\sum_{l:l\neq j}(m_{t_{2n}} - \langle x_l(t), v\rangle). \end{aligned}$$

Combining this last fact with (5.1.24) it comes that

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t)-x_j(t),v\rangle &\le K(M_{t_{2n}}-m_{t_{2n}}-\langle x_i(t)-x_j(t),v\rangle)\\
&\quad+\frac{1}{N-1}\psi_0\sum_{l:l\neq i,j}(\langle x_l(t),v\rangle-M_{t_{2n}}+m_{t_{2n}}-\langle x_l(t),v\rangle)\\
&\quad+\frac{1}{N-1}\psi_0(\langle x_j(t),v\rangle-M_{t_{2n}}+m_{t_{2n}}-\langle x_i(t),v\rangle)\\
&=K(M_{t_{2n}}-m_{t_{2n}})-K\langle x_i(t)-x_j(t),v\rangle+\frac{N-2}{N-1}\psi_0(-M_{t_{2n}}+m_{t_{2n}})\\
&\quad+\frac{1}{N-1}\psi_0(\langle x_j(t),v\rangle-M_{t_{2n}}+m_{t_{2n}}-\langle x_i(t_{2n}),v\rangle).
\end{aligned}$$

Now, from (5.1.23) we get

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t)-x_j(t),v\rangle &\le K(M_{t_{2n}}-m_{t_{2n}})-K\langle x_i(t)-x_j(t),v\rangle\\
&\quad+\frac{N-2}{N-1}\psi_0(-M_{t_{2n}}+m_{t_{2n}})+\frac{1}{N-1}\psi_0(-M_{t_{2n}}+m_{t_{2n}})\\
&\quad-\frac{1}{N-1}\psi_0\langle x_i(t)-x_j(t),v\rangle\\
&\le K(M_{t_{2n}}-m_{t_{2n}})-K\langle x_i(t)-x_j(t),v\rangle+\psi_0(-M_{t_{2n}}+m_{t_{2n}})\\
&=(K-\psi_0)(M_{t_{2n}}-m_{t_{2n}})-K\langle x_i(t)-x_j(t),v\rangle.
\end{aligned}$$

Hence, from Gronwall's inequality it comes that

$$\begin{aligned}
\langle x_i(t)-x_j(t),v\rangle &\le e^{-K(t-t_{2n})}\langle x_i(t_{2n})-x_j(t_{2n}),v\rangle\\
&\quad+(M_{t_{2n}}-m_{t_{2n}})\int_{t_{2n}}^{t}(K-\psi_0)\,e^{-K(t-s)}ds,
\end{aligned}$$

for all $t\in[t_{2n},t_{2n+1})$. In particular, for $t=t_{2n+1}$, it comes that

$$\begin{aligned}
d(t_{2n+1}) &\le e^{-K(t_{2n+1}-t_{2n})}\langle x_i(t_{2n})-x_j(t_{2n}),v\rangle+(M_{t_{2n}}-m_{t_{2n}})\int_{t_{2n}}^{t_{2n+1}}(K-\psi_0)e^{-K(t_{2n+1}-s)}ds\\
&\le e^{-K(t_{2n+1}-t_{2n})}|x_i(t_{2n})-x_j(t_{2n})|+(M_{t_{2n}}-m_{t_{2n}})\int_{t_{2n}}^{t_{2n+1}}(K-\psi_0)e^{-K(t_{2n+1}-s)}ds\\
&\le\left(e^{-K(t_{2n+1}-t_{2n})}+K\int_{t_{2n}}^{t_{2n+1}}e^{-K(t_{2n+1}-s)}ds-\psi_0\int_{t_{2n}}^{t_{2n+1}}e^{-K(t_{2n+1}-s)}ds\right)d(t_{2n})\\
&=\left(e^{-K(t_{2n+1}-t_{2n})}+1-e^{-K(t_{2n+1}-t_{2n})}-\frac{\psi_0}{K}(1-e^{-K(t_{2n+1}-t_{2n})})\right)d(t_{2n})\\
&=\left(1-\frac{\psi_0}{K}(1-e^{-K(t_{2n+1}-t_{2n})})\right)d(t_{2n}).
\end{aligned}$$

So, if we set

$$C_{2n} := \max\left\{1 - e^{-K(t_{2n+1}-t_{2n})}, 1 - \frac{\psi_0}{K}(1 - e^{-K(t_{2n+1}-t_{2n})})\right\}, \tag{5.1.25}$$

$C_{2n} \in (0,1)$ is the constant for which (5.1.21) holds. □

In the bad intervals, the previous estimate is not valid, being the diameter of the solution evaluated at t_{2n+2} larger than the diameter evaluated at t_{2n+1}. However, the growth of the diameter in the intervals of negative interaction can be in some way controlled, as the following lemma shows.

Proposition 5.1.15. *Assume* (0.1.20). *Then, for all* $n \in \mathbb{N}_0$, *we have that*

$$d(t_{2n+2}) \leq \frac{e^{K(t_{2n+2}-t_{2n+1})}}{2 - e^{K(t_{2n+2}-t_{2n+1})}} d(t_{2n+1}). \tag{5.1.26}$$

Proof. Let $n \in \mathbb{N}_0$. Let all $i, j = 1, \dots, N$ be such that

$$d(t_{2n+2}) = |x_i(t_{2n+2}) - x_j(t_{2n+2})|.$$

If $d(t_{2n+2}) = 0$, then from (5.1.13) also $d(t_{2n+1}) = 0$ and of course inequality (5.1.26) is fulfilled. So, we can assume that $d(t_{2n+2}) > 0$. In this case, let $v \in \mathbb{R}^d$ be the so defined unit vector

$$v = \frac{x_i(t_{2n+2}) - x_j(t_{2n+2})}{|x_i(t_{2n+2}) - x_j(t_{2n+2})|}.$$

Then,

$$d(t_{2n+2}) = \langle x_i(t_{2n+2}) - x_j(t_{2n+2}), v\rangle.$$

Moreover, we set

$$M_{t_{2n+2}} = \max_{k=1,\dots,N} \langle x_k(t_{2n+2}), v\rangle,$$
$$m_{t_{2n+2}} = \min_{k=1,\dots,N} \langle x_k(t_{2n+2}), v\rangle.$$

Thus, from (5.1.11), for all $t \in [t_{2n+1}, t_{2n+2}]$, it holds

$$m_{t_{2n+2}} \leq \langle x_k(t), v\rangle \leq M_{t_{2n+2}}, \quad \forall k = 1, \dots, N. \tag{5.1.27}$$

Now, for all $t \in [t_{2n+1}, t_{2n+2})$, using the first inequality in (5.1.27) and the fact that $\alpha(t) = -1$, we have that

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t), v\rangle &= -\frac{1}{N-1}\sum_{l:l\neq i} \psi(x_i(t), x_l(t))(\langle x_l(t), v\rangle - \langle x_i(t), v\rangle) \\
&\leq -\frac{1}{N-1}\sum_{l:l\neq i} \psi(x_i(t), x_l(t))(m_{t_{2n+2}} - \langle x_i(t), v\rangle) \\
&= \frac{1}{N-1}\sum_{l:l\neq i} \psi(x_i(t), x_l(t))(\langle x_i(t), v\rangle) - m_{t_{2n+2}}) \\
&\leq K(\langle x_i(t), v\rangle) - m_{t_{2n+2}}).
\end{aligned}$$

Therefore, the Gronwall's inequality yields

$$\begin{aligned}\langle x_i(t), v\rangle &\le e^{K(t-t_{2n+1})}\langle x_i(t_{2n+1}), v\rangle - Km_{t_{2n+2}}\int_{t_{2n+1}}^{t} e^{K(t-s)}ds \\ &= e^{K(t-t_{2n+1})}\langle x_i(t_{2n+1}), v\rangle + m_{t_{2n+2}}(1-e^{K(t-t_{2n+1})}).\end{aligned} \tag{5.1.28}$$

On the other hand, using the second inequality in (5.1.27), we get

$$\begin{aligned}\frac{d}{dt}\langle x_i(t), v\rangle &= -\frac{1}{N-1}\sum_{l:l\neq j}\psi(x_j(t), x_l(t))(\langle x_l(t), v\rangle - \langle x_j(t), v\rangle) \\ &\ge -\frac{1}{N-1}\sum_{l:l\neq j}\psi(x_j(t), x_l(t))(M_{t_{2n+2}} - \langle x_j(t), v\rangle) \\ &= \frac{1}{N-1}\sum_{l:l\neq j}\psi(x_j(t), x_l(t))(\langle x_j(t), v\rangle) - M_{t_{2n+2}}) \\ &\ge K(\langle x_j(t), v\rangle) - M_{t_{2n+2}}).\end{aligned}$$

Hence, using the Gronwall's inequality, we can write

$$\begin{aligned}\langle x_j(t), v\rangle &\ge e^{K(t-t_{2n+1})}\langle x_j(t_{2n+1}), v\rangle - KM_{t_{2n+2}}\int_{t_{2n+1}}^{t} e^{K(t-s)}ds \\ &= e^{K(t-t_{2n+1})}\langle x_j(t_{2n+1}), v\rangle + M_{t_{2n+2}}(1-e^{K(t-t_{2n+1})}).\end{aligned} \tag{5.1.29}$$

Thus, combining (5.1.28) and (5.1.29), we can conclude that, for all $t \in [t_{2n+1}, t_{2n+2}]$, it holds

$$\begin{aligned}\langle x_i(t) - x_j(t), v\rangle &= \langle x_i(t), v\rangle - \langle x_j(t), v\rangle \\ &\le e^{K(t-t_{2n+1})}\langle x_i(t_{2n+1}), v\rangle + m_{t_{2n+2}}(1-e^{K(t-t_{2n+1})}) \\ &\quad -e^{K(t-t_{2n+1})}\langle x_j(t_{2n+1}), v\rangle - M_{t_{2n+2}}(1-e^{K(t-t_{2n+1})}) \\ &\le e^{K(t-t_{2n+1})}\langle x_i(t_{2n+1}) - x_j(t_{2n+1}), v\rangle + (m_{t_{2n+2}} - M_{t_{2n+2}})(1-e^{K(t-t_{2n+1})}) \\ &= e^{K(t-t_{2n+1})}\langle x_i(t_{2n+1}) - x_j(t_{2n+1}), v\rangle + (M_{t_{2n+2}} - m_{t_{2n+2}})(e^{K(t-t_{2n+1})}-1) \\ &\le e^{K(t-t_{2n+1})}d(t_{2n+1}) + d(t_{2n+2})(e^{K(t-t_{2n+1})}-1).\end{aligned}$$

For $t = t_{2n+2}$, we obtain

$$d(t_{2n+2}) = \langle x_i(t_{2n+2}) - x_j(t_{2n+2}), v\rangle \le e^{K(t_{2n+2}-t_{2n+1})}d(t_{2n+1}) + d(t_{2n+2})(e^{K(t_{2n+2}-t_{2n+1})}-1),$$

from which

$$d(t_{2n+2})(2-e^{K(t_{2n+2}-t_{2n+1})}) \le e^{K(t_{2n+2}-t_{2n+1})}d(t_{2n+1}).$$

Thus, since $2-e^{K(t_{2n+2}-t_{2n+1})} > 0$ from (0.1.20), we have that

$$d(t_{2n+2}) \le \frac{e^{K(t_{2n+2}-t_{2n+1})}}{2-e^{K(t_{2n+2}-t_{2n+1})}}d(t_{2n+1}),$$

i.e. (5.1.26) is proven. □

Proof of Theorem 5.1.1. Let $\{x_i\}_{i=1,\dots,N}$ be solution to (0.1.17), (0.1.18). Then, for all $n \in \mathbb{N}_0$, using (5.1.21), (5.1.25) and (5.1.26)we have that

$$\begin{aligned} d(t_{2n+2}) &\leq \frac{e^{K(t_{2n+2}-t_{2n+1})}}{2 - e^{K(t_{2n+2}-t_{2n+1})}} d(t_{2n+1}) \\ &\leq \frac{e^{K(t_{2n+2}-t_{2n+1})}}{2 - e^{K(t_{2n+2}-t_{2n+1})}} \max\left\{1 - e^{-K(t_{2n+1}-t_{2n})}, 1 - \frac{\psi_0}{K}(1 - e^{-K(t_{2n+1}-t_{2n})})\right\} d(t_{2n}). \end{aligned}$$

Thus, using an induction argument, we get

$$\begin{aligned} d(t_{2n+2}) &\leq \prod_{p=0}^{n} \left(\frac{e^{K(t_{2p+2}-t_{2p+1})}}{2 - e^{K(t_{2p+2}-t_{2p+1})}} \max\left\{1 - e^{-K(t_{2p+1}-t_{2p})}, 1 - \frac{\psi_0}{K}(1 - e^{-K(t_{2p+1}-t_{2p})})\right\} \right) d(0) \\ &= e^{\sum_{p=0}^{\infty} \ln\left(\frac{e^{K(t_{2p+2}-t_{2p+1})}}{2 - e^{K(t_{2p+2}-t_{2p+1})}} \max\left\{1 - e^{-K(t_{2p+1}-t_{2p})}, 1 - \frac{\psi_0}{K}(1 - e^{-K(t_{2p+1}-t_{2p})})\right\}\right)} d(0) \\ &= e^{\sum_{p=0}^{\infty} \left[\ln\left(\frac{e^{K(t_{2p+2}-t_{2p+1})}}{2 - e^{K(t_{2p+2}-t_{2p+1})}} \right) + \ln\left(\max\left\{1 - e^{-K(t_{2p+1}-t_{2p})}, 1 - \frac{\psi_0}{K}(1 - e^{-K(t_{2p+1}-t_{2p})})\right\}\right) \right]} d(0). \end{aligned}$$

Now, $\sum_{p=0}^{\infty} \ln\left(\frac{e^{K(t_{2p+2}-t_{2p+1})}}{2 - e^{K(t_{2p+2}-t_{2p+1})}} \right) < +\infty$ from (5.1.1). Then, the solution $\{x_i\}_{i=1,\dots,N}$ converges to consensus since the following condition is satisfied from (5.1.2):

$$\sum_{p=0}^{\infty} \ln\left(\max\left\{1 - e^{-K(t_{2p+1}-t_{2p})}, 1 - \frac{\psi_0}{K}(1 - e^{-K(t_{2p+1}-t_{2p})})\right\}\right) = -\infty.$$

□

5.1.3 Exponential consensus

We conclude this Section with another consensus result for the Hegselmann-Krause model (0.1.17). Namely, under a stronger condition than (5.1.2) on the sequence $\{t_n\}_n$, we are able to prove that the consensus is achieved exponentially fast.

Theorem 5.1.16. *Let $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a positive, bounded, continuous function. Assume that the sequence $\{t_n\}_n$ of definition* (0.1.19) *satisfies* (0.1.20). *Assume* (5.1.5) *and that the following condition holds:*

$$\sup_{n\in\mathbb{N}} \left(\frac{e^{K(t_{2n+2}-t_{2n+1})}}{2 - e^{K(t_{2n+2}-t_{2n+1})}} \max\left\{1 - e^{-K(t_{2n+1}-t_{2n})}, 1 - \frac{\psi_0}{K}(1 - e^{-K(t_{2n+1}-t_{2n})})\right\} \right) = c < 1. \tag{5.1.30}$$

Then, every solution $\{x_i\}_{i=1,\dots,N}$ to (0.1.17) *with the initial conditions* (0.1.18) *satisfies the following exponential decay estimate*

$$d(t) \leq e^{-\gamma\left(t - \frac{\ln 2}{K} - T\right)} d(0), \quad \forall t \geq 0, \tag{5.1.31}$$

for two suitable positive constants γ and T, independent of N.

Remark 5.1.17. The assumption (5.1.30) implies (5.1.2). Indeed, if (5.1.30) holds,

$$\sum_{p=0}^{\infty} \ln\left(\frac{e^{K(t_{2p+2}-t_{2p+1})}}{2-e^{K(t_{2p+2}-t_{2p+1})}} \max\left\{1-e^{-K(t_{2p+1}-t_{2p})}, 1-\frac{\psi_0}{K}(1-e^{-K(t_{2p+1}-t_{2p})})\right\}\right)$$
$$\leq \sum_{p=0}^{\infty} \ln c = -\infty,$$

i.e.

$$\sum_{p=0}^{\infty} \ln\left(\frac{e^{K(t_{2p+2}-t_{2p+1})}}{2-e^{K(t_{2p+2}-t_{2p+1})}} \max\left\{1-e^{-K(t_{2p+1}-t_{2p})}, 1-\frac{\psi_0}{K}(1-e^{-K(t_{2p+1}-t_{2p})})\right\}\right) = -\infty. \tag{5.1.32}$$

Therefore, being $\frac{e^{K(t_{2p+2}-t_{2p+1})}}{2-e^{K(t_{2p+2}-t_{2p+1})}} > 1$, it comes that

$$\sum_{p=0}^{\infty} \ln\left(\max\left\{1-e^{-K(t_{2p+1}-t_{2p})}, 1-\frac{\psi_0}{K}(1-e^{-K(t_{2p+1}-t_{2p})})\right\}\right)$$
$$\leq \sum_{p=0}^{\infty} \ln\left(\frac{e^{K(t_{2p+2}-t_{2p+1})}}{2-e^{K(t_{2p+2}-t_{2p+1})}} \max\left\{1-e^{-K(t_{2p+1}-t_{2p})}, 1-\frac{\psi_0}{K}(1-e^{-K(t_{2p+1}-t_{2p})})\right\}\right).$$

Then, the condition (5.1.2) is satisfied.

Proof of Theorem 5.1.16. Let $\{x_i\}_{i=1,\dots,N}$ be solution to (0.1.17), (0.1.18). Then, since (5.1.5) holds, for all $n \in \mathbb{N}_0$,

$$d(t_{2n+2}) \leq \frac{e^{K(t_{2n+2}-t_{2n+1})}}{2-e^{K(t_{2n+2}-t_{2n+1})}} \max\left\{1-e^{-K(t_{2n+1}-t_{2n})}, 1-\frac{\psi_0}{K}(1-e^{-K(t_{2n+1}-t_{2n})})\right\} d(t_{2n}).$$

Therefore, using (5.1.30), we get

$$d(t_{2n+2}) \leq c d(t_{2n}), \tag{5.1.33}$$

with $c \in (0,1)$. As a consequence, using an induction argument,

$$d(t_{2n}) \leq c^n d(0), \quad \forall n \in \mathbb{N}_0. \tag{5.1.34}$$

Now, let $t \geq 0$. Then, there exists $n \in \mathbb{N}_0$ such that $t \in [t_{2n}, t_{2n+2})$. Thus, if $t \in [t_{2n}, t_{2n+1}]$, using (5.1.9),

$$d(t) \leq d(t_{2n}).$$

On the other hand, if $t \in (t_{2n+1}, t_{2n+2})$, from (5.1.12) and (5.1.33) we get

$$d(t) \leq d(t_{2n+2}) \leq c d(t_{2n}).$$

Therefore, being $c < 1$, in both cases

$$d(t) \leq d(t_{2n}).$$

So, using (5.1.34), we can write

$$d(t) \leq c^n d(0) = e^{-n \ln\left(\frac{1}{c}\right)} d(0).$$

At this point, we distinguish two different situations.
Case I) Assume that

$$T := \sup_{n \in \mathbb{N}_0} (t_{2n+1} - t_{2n}) < +\infty.$$

So setting

$$\gamma := \ln\left(\frac{1}{c}\right) \frac{1}{\frac{\ln 2}{K} + T},$$

it comes that

$$d(t) \leq e^{-n\gamma\left(\frac{\ln 2}{K} + T\right)}$$

Now, using (0.1.20), it holds that $t_{2n+2} \leq (n+1)\left(\frac{\ln 2}{K} + T\right)$. Thus, being $t \leq t_{2n+2}$, we can conclude that

$$d(t) \leq e^{-\gamma\left(t - \frac{\ln 2}{K} - T\right)} d(0),$$

which proves (5.1.31).
Case II) Assume that

$$\sup_{n \in \mathbb{N}} (t_{2n+1} - t_{2n}) = +\infty.$$

We pick $\tilde{T} > 0$. Without loss of generality, eventually splitting the intervals in which the weight function $\alpha = 1$ in subintervals of length at most $\tilde{T}$, we can assume that

$$t_{2n+1} - t_{2n} \leq \tilde{T}, \quad \forall n \in \mathbb{N}_0. \tag{5.1.35}$$

Then, setting

$$\gamma := \ln\left(\frac{1}{c}\right) \frac{1}{\frac{\ln 2}{K} + \tilde{T}}.$$

Then, reasoning as in the previous case, we get that

$$d(t) \leq e^{-\gamma\left(t - \frac{\ln 2}{K} - \tilde{T}\right)} d(0),$$

which proves (5.1.31). □

5.2 The Cucker-Smale model

In this Section, we deal with the second-order model (0.1.29). The flocking result we will prove is the following.

Theorem 5.2.1. *Let $\tilde{\psi} : \mathbb{R} \to \mathbb{R}$ be a positive, bounded, continuous function satisfying*

$$\int_0^\infty \min_{r \in [0,x]} \tilde{\psi}(r) dx = +\infty. \tag{5.2.1}$$

Assume that the sequence $\{t_n\}_n$ *of definition* (0.1.19) *satisfies* (0.1.31) *and*

(5.2.2)

Moreover, assume that the following condition hold:

$$\sum_{p=0}^{\infty} \ln \left(\frac{e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}{2 - e^{\tilde{K}(t_{2p+2}-t_{2p+1})}} \right) < +\infty, \tag{5.2.3}$$

$$\sum_{p=0}^{\infty} \ln \left(\max \left\{ 1 - e^{-\tilde{K}(t_{2p+1}-t_{2p})}, 1 - \frac{\tilde{\psi}_0}{\tilde{K}}(1 - e^{-\tilde{K}(t_{2p+1}-t_{2p})}) \right\} \right) = -\infty, \tag{5.2.4}$$

where

$$\tilde{\psi}_0 := \min_{|y| \leq M^0} \tilde{\psi}(y), \tag{5.2.5}$$

being

$$M^0 := e^{\sum_{p=0}^{\infty} \ln \left(\frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2-e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} \right)} d_V(0). \tag{5.2.6}$$

Then, every solution $\{x_i, v_i\}_{i=1,\dots,N}$ *to* (0.1.29) *with the initial conditions* (0.1.30) *exhibits asymptotic flocking.*

Remark 5.2.2. Let us note that (5.2.3) implies (5.1.5). This follows from the same arguments used in Remark 5.1.2. In particular, from (5.1.5) we have $t_{2p+2} - t_{2p+1} \to 0$, as $p \to +\infty$.

5.2.1 Preliminary estimates

Let $\{(x_i, v_i)\}_{i=1,\dots,N}$ be solution to (0.1.29) under the initial conditions (0.1.30). We present some auxiliary lemmas that will be needed for the proof of Theorem 5.2.1. We omit some proofs of these preliminary results, since they are analogous to the proofs of correspondent results in Chapter 1 and in the previous section. In the intervals of positive interaction, we have the following estimates.

Lemma 5.2.3. *For each* $v \in \mathbb{R}^d$ *and* $n \in \mathbb{N}_0$, *we have that*

$$\min_{j=1,\dots,N} \langle v_j(t_{2n}), v \rangle \leq \langle v_i(t), v \rangle \leq \max_{j=1,\dots,N} \langle v_j(t_{2n}), v \rangle, \tag{5.2.7}$$

for all $t \in [t_{2n}, t_{2n+1}]$ *and* $i = 1, \dots, N$.

Lemma 5.2.4. *For each* $n \in \mathbb{N}_0$ *and* $i, j = 1, \dots, N$, *we get*

$$|v_i(s) - v_j(t)| \leq d_V(t_{2n}), \quad \forall s, t \in [t_{2n}, t_{2n+1}]. \tag{5.2.8}$$

Remark 5.2.5. Let us note that from (5.2.8), in particular, it follows that

$$d_V(t_{2n+1}) \leq d_V(t_{2n}), \quad \forall n \in \mathbb{N}_0. \tag{5.2.9}$$

In the intervals on negative interaction, we have rather the following estimates on the velocity diameters.

Lemma 5.2.6. *For each* $v \in \mathbb{R}^d$ *and* $n \in \mathbb{N}_0$, *we have that*

$$\min_{j=1,\dots,N} \langle v_j(t_{2n+2}), v\rangle \le \langle v_i(t), v\rangle \le \max_{j=1,\dots,N} \langle v_j(t_{2n+2}), v\rangle, \tag{5.2.10}$$

for all $t \in [t_{2n+1}, t_{2n+2}]$ *and* $i = 1, \dots, N$.

Lemma 5.2.7. *For each* $n \in \mathbb{N}_0$ *and* $i, j = 1, \dots, N$, *we get*

$$|v_i(s) - v_j(t)| \le d_V(t_{2n+2}), \quad \forall s, t \in [t_{2n+1}, t_{2n+2}]. \tag{5.2.11}$$

Remark 5.2.8. Let us note that from (5.2.11), in particular, it follows that

$$d_V(t_{2n+2}) \ge d_V(t_{2n+1}), \quad \forall n \in \mathbb{N}_0. \tag{5.2.12}$$

Now, in the intervals of positive interaction, the agents' velocities are bounded uniformly with respect to $i = 1, \dots, N$ by a positive constant that depends on the maximum value assumed by velocities at the left end point of good intervals. To this aim, let us define

$$\tilde{M}_n^0 := \max_{i=1,\dots,N} |v_i(t_n)|, \quad \forall n \in \mathbb{N}_0. \tag{5.2.13}$$

Lemma 5.2.9. *For every* $i = 1, \dots, N$, *we have that*

$$|v_i(t)| \le \tilde{M}_{2n}^0, \quad \forall t \in [t_{2n}, t_{2n+1}], \tag{5.2.14}$$

where $\tilde{M}_{2n}^0$ *is the positive constant in* (5.2.13).

Now, we prove that the agents' velocities are uniformly bounded by a positive constant that depends on the initial data, as we did in the previous section. Indeed, estimate (5.2.14) provide us a bound on the agents' velocities which is uniform with respect to $i = 1, \dots, N$ but that is not uniform with respect to t, since the constant $\tilde{M}_{2n}^0$ depends on the sequence $\{t_n\}$. To find a uniform bound on the velocities we have to employ finer arguments, that are the analogous we used to prove estimate (5.1.18) for the first-order model. The following result holds.

Lemma 5.2.10. *Assume* (5.2.3). *Then, for all* $t \ge 0$, *we have that*

$$\max_{i=1,\dots,N} |v_i(t)| \le \tilde{M}^0, \tag{5.2.15}$$

where

$$\tilde{M}^0 := e^{\tilde{K} \sum_{p=0}^{\infty} (t_{2p+2} - t_{2p+1})} \tilde{M}_0^0 \tag{5.2.16}$$

Remark 5.2.11. Let us note that, in the case of the second-order model (0.1.29), estimate (5.2.15) does not allow us to deduce the existence of a lower bound on the influence function $\tilde{\psi}$, as we did in the previous Section. This is due to the fact that now the influence function depends on the distance between the agents' positions. So, to get a bound from below on the influence function $\tilde{\psi}$ we will have to prove that the position diameters are bounded. The boundedness of the position diameters will follow from the introduction of a suitable Lyapunov functional.

Also, the following fundamental result holds.

Proposition 5.2.12. *For all $i, j = 1, \dots, N$, unit vector $v \in \mathbb{R}^d$ and $n \in \mathbb{N}_0$ we have that*

$$\langle v_i(t) - v_j(t), v\rangle \le e^{-\tilde{K}(t-\bar{t})}\langle v_i(\bar{t}) - v_j(\bar{t}), v\rangle + (1 - e^{-\tilde{K}(t-\bar{t})})d_V(t_{2n}), \tag{5.2.17}$$

for all $t_{2n+1} > t \ge \bar{t} \ge t_{2n}$.

Now, from (5.2.12) the velocity diameter is nondecreasing in the intervals of negative interaction. This prevents the decay of the velocity diameters and, as a consequence, the asymptotic flocking. However, we can control the growth of the velocity diameters, as we did in the previous section for the first-order model.

Proposition 5.2.13. *Assume* (0.1.31)*. Then, for all $n \in \mathbb{N}_0$, we have that*

$$d_V(t_{2n+2}) \le \frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} d_V(t_{2n+1}). \tag{5.2.18}$$

Finally, we prove the following crucial result, that provides a bound on the velocity diameters.

Proposition 5.2.14. *Assume* (5.2.3)*. Then, for all $t \ge 0$, it holds that*

$$d_V(t) \le \bar{M}^0, \tag{5.2.19}$$

where

$$\bar{M}^0 := e^{\sum_{p=0}^{\infty} \ln\left(\frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}\right)} d_V(0). \tag{5.2.20}$$

Proof. For all $n \in \mathbb{N}_0$, from (5.2.9) and (5.2.18) we have that

$$d_V(t_{2n+2}) \le \frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} d_V(t_{2n+1}) \le \frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} d_V(t_{2n}).$$

Then,

$$\begin{aligned} d_V(t_{2n}) &\le \prod_{p=0}^{n} \left(\frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}\right) d_V(0) = e^{\sum_{p=0}^{n} \ln\left(\frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}\right)} d_V(0) \\ &\le e^{\sum_{p=0}^{\infty} \ln\left(\frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}\right)} d_V(0). \end{aligned}$$

We have so proved that

$$d_V(t_{2n}) \le \bar{M}^0, \quad \forall n \in \mathbb{N}_0. \tag{5.2.21}$$

As a consequence, for all $t \ge 0$, since $t \in [t_{2n}, t_{2n+2})$, for some $n \in \mathbb{N}_0$, using (5.1.9) and (5.1.12) we can conclude that (5.2.19) holds true. □

5.2.2 Asymptotic flocking

Before moving to the proof of Theorem 5.2.1, we provide an estimate on the velocity diameters in the intervals of positive interaction, as we did in the previous section.

Now, we pick $T > \frac{\ln 2}{\tilde{K}}$. We can assume, eventually splitting the intervals of positive interaction into subintervals of length at most T, that

$$t_{2n+1} - t_{2n} \leq T, \quad \forall n \in \mathbb{N}_0. \tag{5.2.22}$$

As a consequence, from (0.1.31) and (5.2.22), being $T > \frac{\ln 2}{\tilde{K}}$, we can write

$$t_{n+1} - t_n \leq T, \quad \forall n \in \mathbb{N}_0. \tag{5.2.23}$$

Definition 5.2.1. For all $t \geq 0$, we define

$$\tilde{\psi}_t := \min\left\{\tilde{\psi}(r) : r \in \left[0, \max_{s\in[0,t]} d_X(s)\right]\right\}, \tag{5.2.24}$$

and

$$\phi(t) := \min\left\{e^{-\tilde{K}T}\tilde{\psi}_t, \frac{e^{-\tilde{K}T}}{T}\right\}. \tag{5.2.25}$$

Remark 5.2.15. Let us note that, for all $t \geq 0$ and $i, j = 1, \ldots, N$,

$$|x_i(t) - x_j(t)| \leq \max_{s\in[0,t]} d_X(s).$$

As a consequence, it holds that

$$\tilde{\psi}(|x_i(t) - x_j(t)|) \geq \tilde{\psi}_t > 0, \quad \forall t \geq 0,\ i, j = 1, \ldots, N. \tag{5.2.26}$$

Proposition 5.2.16. *For all $n \in \mathbb{N}_0$,*

$$d_V(t_{2n+1}) \leq \left(1 - \int_{t_{2n}}^{t_{2n+1}} \phi(s)ds\right) d_V(t_{2n}). \tag{5.2.27}$$

Remark 5.2.17. Let us note that

$$\int_{t_{2n}}^{t_{2n+1}} \phi(s)ds \in (0, 1), \quad \forall n \in \mathbb{N}_0,$$

since from (5.2.25), we have that $\phi(t) < 1/T$ and from (5.2.22) it holds $t_{2n+1} - t_{2n} \leq T$. Therefore,

$$1 - \int_{t_{2n}}^{t_{2n+1}} \phi(s)ds \in (0, 1).$$

Proof of Proposition 5.2.16. Let $n \in \mathbb{N}_0$. Trivially, if $d_V(t_{2n+1}) = 0$, then of course inequality (5.2.27) holds. So, suppose $d_V(t_{2n+1}) > 0$. Let $i, j = 1, \ldots, N$ be such that $d_V(t_{2n+1}) = |v_i(t_{2n+1}) - v_j(t_{2n+1})|$. We set

$$v = \frac{v_i(t_{2n+1}) - v_j(t_{2n+1})}{|v_i(t_{2n+1}) - v_j(t_{2n+1})|}.$$

Then, v is a unit vector for which we can write

$$d_V(t_{2n+1}) = \langle v_i(t_{2n+1}) - v_j(t_{2n+1}), v\rangle.$$

Let us define

$$M_{t_{2n}} = \max_{l=1,\dots,N} v_l(t_{2n}), v\rangle,$$

$$m_{t_{2n}} = \min_{l=1,\dots,N} \langle v_l(t_{2n}), v\rangle.$$

Then $M_{t_{2n}} - m_{t_{2n}} \leq d_V(t_{2n})$.
Now, we distinguish two different situations.

Case I. Assume that there exists $\bar{t} \in [t_{2n}, t_{2n+1}]$ such that

$$\langle v_i(\bar{t}) - v_j(\bar{t}), v\rangle < 0.$$

Then, from (5.2.17) with $t_{2n+1} \geq \bar{t} \geq t_{2n}$, we have

$$\begin{aligned} d_V(t_{2n+1}) &\leq e^{-\tilde{K}(t_{2n+1}-\bar{t})}\langle v_i(\bar{t}) - v_j(\bar{t}), v\rangle + (1 - e^{-\tilde{K}(t_{2n+1}-\bar{t})})d_V(t_{2n}) \\ &\leq (1 - e^{-\tilde{K}(t_{2n+1}-\bar{t})})d_V(t_{2n}) \\ &\leq (1 - e^{-\tilde{K}T})d_V(t_{2n}) \\ &\leq \left(1 - \int_{t_{2n}}^{t_{2n+1}} \phi(s)ds\right) d_V(t_{2n}). \end{aligned} \tag{5.2.28}$$

Case II. Assume it rather holds

$$\langle v_i(t) - v_j(t), v\rangle \geq 0, \quad \forall t \in [t_{2n}, t_{2n+1}]. \tag{5.2.29}$$

Then, for every $t \in [t_{2n}, t_{2n+1}]$, we have that

$$\begin{aligned} \frac{d}{dt}\langle v_i(t) - v_j(t), v\rangle &= \frac{1}{N-1}\sum_{l:l\neq i} \tilde{\psi}(|x_i(t) - x_j(t)|)\langle v_l(t) - v_i(t), v\rangle \\ &\quad -\frac{1}{N-1}\sum_{l:l\neq j} \tilde{\psi}(|x_i(t) - x_j(t)|)\langle v_l(t) - v_j(t), v\rangle \\ &= \frac{1}{N-1}\sum_{l:l\neq i} \tilde{\psi}(|x_i(t) - x_j(t)|)(\langle v_l(t), v\rangle - M_{t_{2n}} + M_{t_{2n}} - \langle v_i(t), v\rangle) \\ &\quad +\frac{1}{N-1}\sum_{l:l\neq j} \tilde{\psi}(|x_i(t) - x_j(t)|)(\langle v_j(t), v\rangle - m_{t_{2n}} + m_{t_{2n}} - \langle v_l(t), v\rangle) \\ &:= S_1 + S_2. \end{aligned}$$

Now, being $t \in [t_{2n}, t_{2n+1}]$, from (5.2.7) we have that

$$m_{t_{2n}} \leq \langle v_k(t), v\rangle \leq M_{t_{2n}}, \quad \forall k = 1, \dots, N. \tag{5.2.30}$$

Therefore, we get

$$\begin{aligned}
S_1 &= \frac{1}{N-1}\sum_{l:l\neq i}\tilde{\psi}(|x_i(t)-x_j(t)|)(\langle v_l(t),v\rangle - M_{t_{2n}}) \\
&\quad + \frac{1}{N-1}\sum_{l:l\neq i}\tilde{\psi}(|x_i(t)-x_j(t)|)(M_{t_{2n}} - \langle v_i(t),v\rangle) \\
&\leq \frac{1}{N-1}\tilde{\psi}_t\sum_{l:l\neq i}(\langle v_l(t),v\rangle - M_{t_{2n}}) + \tilde{K}(M_{t_{2n}} - \langle v_i(t),v\rangle),
\end{aligned}$$

and

$$\begin{aligned}
S_2 &= \frac{1}{N-1}\sum_{l:l\neq j}\tilde{\psi}(|x_i(t)-x_j(t)|)(\langle v_j(t),v\rangle - m_{t_{2n}}) \\
&\quad + \frac{1}{N-1}\sum_{l:l\neq j}\tilde{\psi}(|x_i(t)-x_j(t)|)(m_{t_{2n}} - \langle v_l(t),v\rangle) \\
&\leq \tilde{K}(\langle v_j(t),v\rangle - m_{t_{2n}}) + \frac{1}{N-1}\tilde{\psi}_t\sum_{l:l\neq j}(m_{t_{2n}} - \langle v_l(t),v\rangle).
\end{aligned}$$

Combining this last fact with (5.2.30) it comes that

$$\begin{aligned}
\frac{d}{dt}\langle v_i(t)-v_j(t),v\rangle &\leq \tilde{K}(M_{t_{2n}} - m_{t_{2n}} - \langle v_i(t)-v_j(t),v\rangle) \\
&\quad + \frac{1}{N-1}\tilde{\psi}_t\sum_{l:l\neq i,j}(\langle v_l(t),v\rangle - M_{t_{2n}} + m_{t_{2n}} - \langle v_l(t),v\rangle) \\
&\quad + \frac{1}{N-1}\tilde{\psi}_t(\langle v_j(t),v\rangle - M_{t_{2n}} + m_{t_{2n}} - \langle v_i(t),v\rangle) \\
&= \tilde{K}(M_{t_{2n}} - m_{t_{2n}}) - \tilde{K}\langle v_i(t)-v_j(t),v\rangle + \frac{N-2}{N-1}\tilde{\psi}_t(-M_{t_{2n}} + m_{t_{2n}}) \\
&\quad + \frac{1}{N-1}\tilde{\psi}_t(\langle v_j(t),v\rangle - M_{t_{2n}} + m_{t_{2n}} - \langle v_i(t_{2n}),v\rangle).
\end{aligned}$$

Now, from (5.2.29) we get

$$\begin{aligned}
\frac{d}{dt}\langle v_i(t)-v_j(t),v\rangle &\leq \tilde{K}(M_{t_{2n}} - m_{t_{2n}}) - \tilde{K}\langle v_i(t)-v_j(t),v\rangle \\
&\quad + \frac{N-2}{N-1}\tilde{\psi}_t(-M_{t_{2n}} + m_{t_{2n}}) + \frac{1}{N-1}\tilde{\psi}_t(-M_{t_{2n}} + m_{t_{2n}}) \\
&\quad - \frac{1}{N-1}\tilde{\psi}_t\langle v_i(t)-v_j(t),v\rangle \\
&\leq \tilde{K}(M_{t_{2n}} - m_{t_{2n}}) - \tilde{K}\langle v_i(t)-v_j(t),v\rangle + \tilde{\psi}_t(-M_{t_{2n}} + m_{t_{2n}}) \\
&= \left(\tilde{K} - \tilde{\psi}_t\right)(M_{t_{2n}} - m_{t_{2n}}) - \tilde{K}\langle v_i(t)-v_j(t),v\rangle.
\end{aligned}$$

Hence, from Gronwall's inequality it comes that

$$\langle v_i(t) - v_j(t), v\rangle \le e^{-\tilde{K}(t-t_{2n})}\langle v_i(t_{2n}) - v_j(t_{2n}), v\rangle$$
$$+(M_{t_{2n}} - m_{t_{2n}})\int_{t_{2n}}^{t}\left(\tilde{K} - \tilde{\psi}_s\right)e^{-\tilde{K}(t-s)}ds,$$

for all $t \in [t_{2n}, t_{2n+1}]$. In particular, for $t = t_{2n+1}$, from (5.2.8) it comes that

$$\begin{aligned}
d_V(t_{2n+1}) &\le e^{-\tilde{K}(t_{2n+1}-t_{2n})}\langle v_i(t_{2n}) - v_j(t_{2n}), v\rangle + (M_{t_{2n}} - m_{t_{2n}})\int_{t_{2n}}^{t_{2n+1}}(\tilde{K} - \tilde{\psi}_s)e^{-\tilde{K}(t_{2n+1}-s)}ds \\
&\le e^{-\tilde{K}(t_{2n+1}-t_{2n})}|v_i(t_{2n}) - v_j(t_{2n})| + (M_{t_{2n}} - m_{t_{2n}})\int_{t_{2n}}^{t_{2n+1}}(\tilde{K} - \tilde{\psi}_s)e^{-\tilde{K}(t_{2n+1}-s)}ds \\
&\le \left(e^{-\tilde{K}(t_{2n+1}-t_{2n})} + \tilde{K}\int_{t_{2n}}^{t_{2n+1}}e^{-\tilde{K}(t_{2n+1}-s)}ds - \int_{t_{2n}}^{t_{2n+1}}\tilde{\psi}_s e^{-\tilde{K}(t_{2n+1}-s)}ds\right)d_V(t_{2n}) \\
&= \left(e^{-\tilde{K}(t_{2n+1}-t_{2n})} + 1 - e^{-\tilde{K}(t_{2n+1}-t_{2n})} - \int_{t_{2n}}^{t_{2n+1}}\tilde{\psi}_s e^{-\tilde{K}(t_{2n+1}-s)}ds\right)d_V(t_{2n}) \\
&= \left(1 - \int_{t_{2n}}^{t_{2n+1}}\tilde{\psi}_s e^{-\tilde{K}(t_{2n+1}-s)}ds\right)d_V(t_{2n}) \\
&\le \left(1 - e^{-\tilde{K}T}\int_{t_{2n}}^{t_{2n+1}}\tilde{\psi}_s ds\right)d_V(t_{2n}) \\
&\le \left(1 - \int_{t_{2n}}^{t_{2n+1}}\phi(s)ds\right)d_V(t_{2n}).
\end{aligned}$$

So, taking into account (5.2.28), we can conclude that (5.2.27) holds. □

Proof of Theorem 5.2.1. Let $\{(x_i, v_i)\}_{i=1,\dots,N}$ be solution to (0.1.29) under the initial conditions (0.1.30). We define the function $\mathcal{D} : [0,\infty) \to [0,\infty)$,

$$\mathcal{D}(t) := \begin{cases} d_V(0), & t = 0, \\ \left(1 - \int_{t_{2n}}^{t}\phi(s)ds\right)d_V(t_{2n}), & t \in (t_{2n}, t_{2n+1}],\ n \in \mathbb{N}_0, \\ \left(1 - \int_{t_{2n+1}}^{t}\phi(s)ds\right)d_V(t_{2n+2}), & t \in (t_{2n+1}, t_{2n+2}],\ n \in \mathbb{N}_0. \end{cases}$$

By construction, $\mathcal{D}$ is piecewise continuous. Indeed, $\mathcal{D}$ is continuous everywhere except at points t_n, $n \in \mathbb{N}_0$. Moreover, for all $n \in \mathbb{N}$, $n \ge 1$,

$$\lim_{t\to t_{2n}^+}\mathcal{D}(t) = d_V(t_{2n}) \ge \left(1 - \int_{t_{2n-1}}^{t_{2n}}\phi(s)ds\right)d_V(t_{2n}) = \mathcal{D}(t_{2n}), \tag{5.2.31}$$

$$\lim_{t\to t_{2n+1}^+} \mathcal{D}(t) = d_V(t_{2n+2}) \le \frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2-e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} d_V(t_{2n+1})$$
$$\le \frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2-e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} \left(1-\int_{t_{2n}}^{t_{2n+1}} \phi(s)ds\right) d_V(t_{2n}) \tag{5.2.32}$$
$$= \frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2-e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} \mathcal{D}(t_{2n+1}).$$

Also, $\mathcal{D}$ is nonincreasing in all intervals of the form $(t_n, t_{n+1}]$, $n \in \mathbb{N}_0$.
Now, notice that, for almost all times t,

$$\frac{d}{dt} \max_{s\in[0,t]} d_X(s) \le \left| \frac{d}{dt} d_X(t) \right|,$$

since $\max_{s\in[0,t]} d_X(s)$ is constant or increases like $d_X(t)$. Moreover, for almost all times

$$\left| \frac{d}{dt} d_X(t) \right| \le d_V(t).$$

Therefore, for almost all times

$$\frac{d}{dt} \max_{s\in[0,t]} d_X(s) \le \left| \frac{d}{dt} d_X(t) \right| \le d_V(t). \tag{5.2.33}$$

Next, we define the function $\mathcal{L} : [0,\infty) \to [0,\infty)$ as follows:

$$\mathcal{L}(t) := \mathcal{D}(t) + \int_0^{\max_{s\in[0,t]} d_X(s)} \min\left\{ e^{-\tilde{K}T} \min_{\sigma\in[0,r]} \tilde{\psi}(\sigma), \frac{e^{-\tilde{K}T}}{T} \right\} dr,$$

for all $t \ge 0$. By definition, $\mathcal{L}$ is piecewise continuous, i.e. $\mathcal{L}$ is continuous everywhere except at points t_n, $n \in \mathbb{N}_0$.
In addition, for each $n \in \mathbb{N}$ and for all $t \in (t_{2n}, t_{2n+1})$, we have that

$$\frac{d}{dt}\mathcal{L}(t) = \frac{d}{dt}\mathcal{D}(t) + \min\left\{ e^{-\tilde{K}T} \tilde{\psi}_t, \frac{e^{-\tilde{K}T}}{T} \right\} \frac{d}{dt} \max_{s\in[0,t]} d_X(s)$$
$$= \frac{d}{dt}\mathcal{D}(t) + \phi(t) \frac{d}{dt} \max_{s\in[0,t]} d_X(s),$$

and from (5.2.33) we get

$$\frac{d}{dt}\mathcal{L}(t) \le \frac{d}{dt}\mathcal{D}(t) + \phi(t) d_V(t)$$
$$= -\phi(t) d_V(t_{2n}) + \phi(t) d_V(t)$$
$$= \phi(t)(d_V(t) - d_V(t_{2n})).$$

Thus, since $d_V(t) \leq d_V(t_{2n})$ from (5.2.8), we can deduce that

$$\frac{d}{dt}\mathcal{L}(t) \leq 0, \quad \forall t \in (t_{2n}, t_{2n+1}).$$

As a consequence, for all $t_{2n} < s < t \leq t_{2n+1}$, it comes that

$$\mathcal{L}(t) \leq \mathcal{L}(s).$$

Letting $s \to t_{2n}^+$, we get

$$\mathcal{L}(t) \leq \lim_{s\to t_{2n}^+} \mathcal{L}(s) = \lim_{s\to t_{2n}^+} \mathcal{D}(s) + \int_0^{\max_{s\in[0,t_{2n}]} d_X(s)} \min\left\{ e^{-\tilde{K}T} \min_{\sigma\in[0,r]} \tilde{\psi}(\sigma), \frac{e^{-\tilde{K}T}}{T} \right\} dr,$$

for all $t \in (t_{2n}, t_{2n+1}]$. Thus, using (0.1.31), (5.2.8) and (5.2.31), we can write

$$\begin{aligned}
\mathcal{L}(t) &\leq d_V(t_{2n}) + \int_0^{\max_{s\in[0,t_{2n}]} d_X(s)} \min\left\{ e^{-\tilde{K}T} \min_{\sigma\in[0,r]} \tilde{\psi}(\sigma), \frac{e^{-\tilde{K}T}}{T} \right\} dr \\
&= \frac{1}{\left(1 - \int_{t_{2n-1}}^{t_{2n}} \phi(s)ds\right)} \mathcal{D}(t_{2n}) + \int_0^{\max_{s\in[0,t_{2n}]} d_X(s)} \min\left\{ e^{-\tilde{K}T} \min_{\sigma\in[0,r]} \tilde{\psi}(\sigma), \frac{e^{-\tilde{K}T}}{T} \right\} dr \\
&\leq \frac{1}{\left(1 - \int_{t_{2n-1}}^{t_{2n}} \phi(s)ds\right)} \left[\mathcal{D}(t_{2n}) + \int_0^{\max_{s\in[0,t_{2n}]} d_X(s)} \min\left\{ e^{-\tilde{K}T} \min_{\sigma\in[0,r]} \tilde{\psi}(\sigma), \frac{e^{-\tilde{K}T}}{T} \right\} dr \right] \\
&\leq \frac{1}{\left(1 - \int_{t_{2n-1}}^{t_{2n}} \phi(s)ds\right)} \mathcal{L}(t_{2n}),
\end{aligned}$$

for all $t \in (t_{2n}, t_{2n+1}]$. So,

$$\mathcal{L}(t) \leq \frac{1}{\left(1 - \int_{t_{2n-1}}^{t_{2n}} \phi(s)ds\right)} \mathcal{L}(t_{2n}), \quad \forall t \in [t_{2n}, t_{2n+1}]. \tag{5.2.34}$$

On the other hand, for all $t \in (t_{2n+1}, t_{2n+2})$, we have that

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) &= \frac{d}{dt}\mathcal{D}(t) + \min\left\{ e^{-\tilde{K}T}\tilde{\psi}_t, \frac{e^{-\tilde{K}T}}{T} \right\} \frac{d}{dt} \max_{s\in[0,t]} d_X(s) \\
&= \frac{d}{dt}\mathcal{D}(t) + \phi(t) \frac{d}{dt} \max_{s\in[0,t]} d_X(s),
\end{aligned}$$

and from (5.2.33) we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) &\leq \frac{d}{dt}\mathcal{D}(t) + \phi(t) d_V(t) \\
&= -\phi(t) d_V(t_{2n+2}) + \phi(t) d_V(t) \\
&= \phi(t)(d_V(t) - d_V(t_{2n+2})).
\end{aligned}$$

Thus, since $d_V(t) \leq d_V(t_{2n+2})$ from (5.2.11), we can deduce that

$$\frac{d}{dt}\mathcal{L}(t) \leq 0, \quad \forall t \in (t_{2n+1}, t_{2n+2}).$$

As a consequence, for all $t_{2n+1} < s < t \leq t_{2n+2}$, it comes that

$$\mathcal{L}(t) \leq \mathcal{L}(s).$$

Letting $s \to t_{2n+1}^+$, we get

$$\mathcal{L}(t) \leq \lim_{s \to t_{2n+1}^+} \mathcal{L}(s) = \lim_{s \to t_{2n+1}^+} D(s) + \int_0^{\max_{s\in[0,t_{2n+1}]} d_X(s)} \min\left\{ e^{-\tilde{K}T} \min_{\sigma\in[0,r]} \tilde{\psi}(\sigma), \frac{e^{-\tilde{K}T}}{T} \right\} dr,$$

for all $t \in (t_{2n+1}, t_{2n+2}]$. Thus, using (5.2.32), we can write

$$\begin{aligned} \mathcal{L}(t) &\leq \frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} \mathcal{D}(t_{2n+1}) + \int_0^{\max_{s\in[0,t_{2n+1}]} d_X(s)} \min\left\{ e^{-\tilde{K}T} \min_{\sigma\in[0,r]} \tilde{\psi}(\sigma), \frac{e^{-\tilde{K}T}}{T} \right\} dr \\ &\leq \frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} \mathcal{L}(t_{2n+1}), \end{aligned}$$

for all $t \in (t_{2n+1}, t_{2n+2}]$. Therefore,

$$\mathcal{L}(t) \leq \frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} \mathcal{L}(t_{2n+1}), \quad \forall t \in [t_{2n+1}, t_{2n+2}]. \tag{5.2.35}$$

Now, combining (5.2.34) and (5.2.35), it turns out that

$$\mathcal{L}(t_{2n+2}) \leq \frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} \frac{1}{\left(1 - \int_{t_{2n-1}}^{t_{2n}} \phi(s)ds\right)} \mathcal{L}(t_{2n}), \quad \forall n \in \mathbb{N}. \tag{5.2.36}$$

Thus, thanks to an induction argument, from (5.2.36) it follows that

$$\mathcal{L}(t_{2n+2}) \leq \prod_{p=1}^{n} \left(\frac{e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}{2 - e^{\tilde{K}(t_{2p+2}-t_{2p+1})}} \frac{1}{\left(1 - \int_{t_{2p-1}}^{t_{2p}} \phi(s)ds\right)} \right) \mathcal{L}(t_2), \tag{5.2.37}$$

for all $n \in \mathbb{N}$.
Now, let $t \geq t_4$. Then, there exists $n \geq 2$ such that $t \in [t_{2n}, t_{2n+2}]$. As a consequence, if $t \in [t_{2n}, t_{2n+1}]$, from (5.2.34) and (5.2.37) with $n-1 \geq 1$, we get

$$\begin{aligned} \mathcal{L}(t) &\leq \frac{1}{1 - \int_{t_{2n-1}}^{t_{2n}} \phi(s)ds} \mathcal{L}(t_{2n}) \leq \frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} \frac{1}{1 - \int_{t_{2n-1}}^{t_{2n}} \phi(s)ds} \mathcal{L}(t_{2n}) \\ &\leq \frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} \frac{1}{1 - \int_{t_{2n-1}}^{t_{2n}} \phi(s)ds} \prod_{p=1}^{n-1} \left(\frac{e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}{2 - e^{\tilde{K}(t_{2p+2}-t_{2p+1})}} \frac{1}{\left(1 - \int_{t_{2p-1}}^{t_{2p}} \phi(s)ds\right)} \right) \mathcal{L}(t_2) \\ &= \prod_{p=1}^{n} \left(\frac{e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}{2 - e^{\tilde{K}(t_{2p+2}-t_{2p+1})}} \frac{1}{\left(1 - \int_{t_{2p-1}}^{t_{2p}} \phi(s)ds\right)} \right) \mathcal{L}(t_2). \end{aligned}$$

On the other hand, if $t \in [t_{2n+1}, t_{2n+2}]$, from (5.2.34), (5.2.35) and (5.2.37) it comes that

$$\mathcal{L}(t) \leq \frac{e^{\tilde{K}(t_{2n+2}-t2n+1)}}{2-e^{\tilde{K}(t_{2n+2}-t2n+1)}}\mathcal{L}(t_{2n+1}) \leq \frac{e^{\tilde{K}(t_{2n+2}-t2n+1)}}{2-e^{\tilde{K}(t_{2n+2}-t2n+1)}}\frac{1}{1-\int_{t_{2n-1}}^{t_{2n}}\phi(s)ds}\mathcal{L}(t_{2n})$$
$$\leq \prod_{p=1}^{n}\left(\frac{e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}{2-e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}\frac{1}{\left(1-\int_{t_{2p-1}}^{t_{2p}}\phi(s)ds\right)}\right)\mathcal{L}(t_2).$$

Thus, for all $t \geq t_4$,

$$\begin{aligned}\mathcal{L}(t) &\leq \prod_{p=1}^{n}\left(\frac{e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}{2-e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}\frac{1}{\left(1-\int_{t_{2p-1}}^{t_{2p}}\phi(s)ds\right)}\right)\mathcal{L}(t_2)\\ &= e^{\sum_{p=1}^{n}\left[\ln\left(\frac{e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}{2-e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}\right)+\ln\left(\frac{1}{1-\int_{t_{2p-1}}^{t_{2p}}\phi(s)ds}\right)\right]}\mathcal{L}(t_2).\end{aligned} \tag{5.2.38}$$

Now, from (5.2.3), $\sum_{p=1}^{+\infty}\ln\left(\frac{e^{\tilde{K}(t_{2n+2}-t_{2p+1})}}{2-e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}\right) < +\infty$. Also, $\sum_{p=1}^{+\infty}\ln\left(\frac{1}{1-\int_{t_{2p-1}}^{t_{2p}}\phi(s)ds}\right) < +\infty$. Indeed, from (5.2.23) it turns out that

$$\int_{t_{2p-1}}^{t_{2p}}\phi(s)ds \leq \frac{e^{-\tilde{K}T}}{T}(t_{2p}-t_{2p-1}), \quad \forall p \geq 1.$$

Then, since from (5.2.3) we have that $t_{2p}-t_{2p-1} \to 0$, as $p \to \infty$ (see Remark 5.2.2), we can write

$$\ln\left(\frac{1}{1-\int_{t_{2p-1}}^{t_{2p}}\phi(s)ds}\right) \leq \ln\left(\frac{1}{1-\frac{e^{-\tilde{K}T}}{T}(t_{2p}-t_{2p-1})}\right)$$
$$= -\ln\left(1-\frac{e^{-\tilde{K}T}}{T}(t_{2p}-t_{2p-1})\right) \sim \frac{e^{-\tilde{K}T}}{T}(t_{2p}-t_{2p-1}).$$

As a consequence, from (5.1.5) it holds $\sum_{p=0}^{+\infty}\ln\left(\frac{1}{1-\int_{t_{2p-1}}^{t_{2p}}\phi(s)ds}\right) < +\infty$.

So, setting

$$C := e^{\sum_{p=0}^{\infty}\left[\ln\left(\frac{e^{\tilde{K}(t_{2n+2}-t_{2p+1})}}{2-e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}\right)+\ln\left(\frac{1}{1-\int_{t_{2p-1}}^{t_{2p}}\phi(s)ds}\right)\right]}\mathcal{L}(t_2),$$

taking into account of (5.2.38), we can conclude that

$$\mathcal{L}(t) \leq C, \quad \forall t \geq t_4. \tag{5.2.39}$$

So, for all $t \geq t_4$, by definition of $\mathcal{L}$,

$$\int_0^{\max_{s\in[0,t]} d_X(s)} \min\left\{e^{-\tilde{K}T}\min_{\sigma\in[0,r]}\tilde{\psi}(\sigma), \frac{e^{-\tilde{K}T}}{T}\right\}dr \leq \mathcal{L}(t) \leq C.$$

Letting $t \to \infty$ in the above inequality, we finally get

$$\int_0^{\sup_{s\in[0,\infty)} d_X(s)} \min\left\{e^{-\tilde{K}T} \min_{\sigma\in[0,r]} \tilde{\psi}(\sigma), \frac{e^{-\tilde{K}T}}{T}\right\} dr \leq C. \tag{5.2.40}$$

Then, since the function $\tilde{\psi}$ satisfies (5.2.1), from (5.2.40), there exists a positive constant d^* such that

$$\sup_{s\in[0,\infty)} d_X(s) \leq d^*. \tag{5.2.41}$$

Now, we define

$$\phi^* := \min\left\{e^{-\tilde{K}T}\psi_*, \frac{e^{-\tilde{K}T}}{T}\right\},$$

where

$$\psi_* = \min_{r\in[0,d^*]} \tilde{\psi}(r).$$

Note that $\phi^* > 0$, being $\tilde{\psi}$ a positive function. Also, from (5.2.41), it comes that

$$\psi_* \leq \min\left\{\tilde{\psi}(r) : r \in \left[0, \max_{s\in[0,t]} d_X(s)\right]\right\} = \tilde{\psi}_t,$$

for all $t \geq 0$. Thus, we get

$$\phi^* \leq \phi(t), \quad \forall t \geq 0.$$

As a consequence, for all $n \in \mathbb{N}_0$,

$$\int_{t_{2n}}^{t_{2n+1}} \phi(s)ds \geq \phi^*(t_{2n+1} - t_{2n}),$$

from which

$$1 - \int_{t_{2n}}^{t_{2n+1}} \phi(s)ds \leq 1 - \phi^*(t_{2n+1} - t_{2n}).$$

So, recalling of inequality (5.2.27), we can write

$$d_V(t_{2n+1}) \leq (1 - \phi^*(t_{2n+1} - t_{2n}))\, d_V(t_{2n}), \quad \forall n \in \mathbb{N}_0. \tag{5.2.42}$$

Now, using (5.2.18) and (5.2.42), we have that

$$\begin{aligned} d_V(t_{2n+2}) &\leq \frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} d_V(t_{2n+1}) \leq \frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} \left(1 - \int_{t_{2n}}^{t_{2n+1}} \phi(s)ds\right) d_V(t_{2n}) \\ &\leq \frac{e^{\tilde{K}(t_{2n+2}-t_{2n+1})}}{2 - e^{\tilde{K}(t_{2n+2}-t_{2n+1})}} (1 - \phi^*(t_{2n+1} - t_{2n}))\, d_V(t_{2n}). \end{aligned}$$

Thus, using an induction argument, we get

$$d_V(t_{2n+2}) \leq \prod_{p=0}^{n} \left(\frac{e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}{2 - e^{\tilde{K}(t_{2p+2}-t_{2p+1})}} \left(1 - \phi^*(t_{2p+1} - t_{2p})\right) \right) d_V(0)$$

$$= e^{\sum_{p=0}^{\infty} \ln\left(\frac{e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}{2-e^{\tilde{K}(t_{2p+2}-t_{2p+1})}} (1-\phi^*(t_{2p+1}-t_{2p}))\right)} d_V(0)$$

$$= e^{\sum_{p=0}^{\infty} \left[\ln\left(\frac{e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}{2-e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}\right) + \ln(1-\phi^*(t_{2p+1}-t_{2p}))\right]} d_V(0).$$

Now, $\sum_{p=0}^{\infty} \ln\left(\frac{e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}{2-e^{\tilde{K}(t_{2p+2}-t_{2p+1})}}\right) < +\infty$ from (5.2.3). Then, the solution $\{x_i, v_i\}_{i=1,\ldots,N}$ exhibits asymptotic flocking if the following condition is satisfied:

$$\sum_{p=0}^{\infty} \ln\left((1 - \phi^*(t_{2p+1} - t_{2p}))\right) = -\infty. \tag{5.2.43}$$

However, the above condition is guaranteed since, from (5.2.2),

$$1 - \phi^*(t_{2p+1} - t_{2p}) \leq 1 - \frac{\phi^*}{\tilde{K}} \in (0, 1), \quad \forall p \in \mathbb{N}_0.$$

Thus,

$$\sum_{p=0}^{\infty} \ln\left((1 - \phi^*(t_{2p+1} - t_{2p}))\right) \leq \sum_{p=0}^{\infty} \ln\left(1 - \frac{\phi^*}{\tilde{K}}\right) = -\infty,$$

from which (5.2.43) is fulfilled. □

Chapter 6

Linear evolution equations with time-dependent time delay

In this chapter, we will study well-posedness and exponential stability for the abstract model (0.2.32). Also, we will extend the results that hold for the linear model (0.2.32) to a nonlinear model with a Lipschitz perturbation. Applications to the wave equation and to an elasticity system will be also provided. All the results contained in this chapter are taken from [44].

6.1 Well-posedness

In this section, we prove a well-posedness result for the abstract model (0.2.32). Since we are dealing with time-dependent time delays, so the time delay is not necessarily constant, we cannot employ the step-by-step procedure that is usually used for delay equations. Indeed, as we will see in the proof of the well-posedness result, the standard step-by-step procedure can be used only in the case in which the time delay function is bounded from below by a positive constant. In the general case, namely in the case in which the time delay function is a generic continuous function that satisfies (0.2.33), we have to argue differently and we can prove well-posedness through a fixed point approach.

Theorem 6.1.1. *Let $f : [-\bar{\tau}, 0] \to H$ be a continuous function. Then, the problem* (0.2.32) *has a unique (weak) solution given by Duhamel's formula*

$$U(t) = S(t)U_0 + \int_0^t S(t-s)\, k(s)BU(s-\tau(s))\ ds, \tag{6.1.1}$$

for all $t \geq 0$.

Proof. Let $f \in C([-\bar{\tau}, 0]; H)$. We give two different proofs, the first one only valid under the additional assumption that the time delay is bounded from below by a positive constant.
Case 1. Assume that

$$\tau(t) \geq \tau_0 > 0, \quad \forall t \geq 0, \tag{6.1.2}$$

for a suitable positive constant τ_0. We can argue step-by-step, as in the proof of Proposition 2.1 of [72], by restricting ourselves each time to time intervals of length τ_0.
First we consider $t \in [0, \tau_0]$. Then, from (6.1.2), $t - \tau(t) \in [-\bar{\tau}, 0]$. So, setting $F(t) = k(t)BU(t - \tau(t))$, $t \in [0, \tau_0]$, we have that $F(t) = k(t)Bf(t - \tau(t))$, $t \in [0, \tau_0]$. Then, problem (0.2.32) can be rewritten, in the interval $[0, \tau_0]$, as a standard inhomogeneous evolution problem:

$$\begin{aligned} &U'(t) = AU(t) + F(t) \quad \text{in } (0, \tau_0), \\ &U(0) = U_0. \end{aligned} \tag{6.1.3}$$

Since $k \in \mathcal{L}^1_{loc}([-\bar{\tau}, +\infty); \mathbb{R})$, B is a bounded linear operator and $f \in C([-\bar{\tau}, 0]; H)$, we have that $F \in \mathcal{L}^1((0, \tau_0); H)$. Therefore, applying [92, Corollary 2.2] there exists a unique solution $U \in C([0, \tau_0]; H)$ of (6.1.3) satisfying the Duhamel's formula

$$U(t) = S(t)U_0 + \int_0^t S(t-s)F(s)ds, \quad t \in [0, \tau_0].$$

Therefore,

$$U(t) = S(t)U_0 + \int_0^t S(t-s)k(s)BU(s - \tau(s))ds, \quad t \in [0, \tau_0].$$

Next, we consider the time interval $[\tau_0, 2\tau_0]$ and also define $F(t) = k(t)BU(t - \tau(t))$, for $t \in [\tau_0, 2\tau_0]$. Note that, if $t \in [\tau_0, 2\tau_0]$, then $t - \tau(t) \in [-\bar{\tau}, \tau_0]$ and so $U(t - \tau(t))$ is known from the first step. Then $F_{|_{[\tau_0, 2\tau_0]}}$ is a known function and it belongs to $\mathcal{L}^1((\tau_0, 2\tau_0); H)$. So we can rewrite our model (0.2.32) in the time interval $[\tau_0, 2\tau_0]$ as the inhomogeneous evolution problem

$$\begin{aligned} &U'(t) = AU(t) + F(t) \quad \text{for } \ t \in (\tau_0, 2\tau_0), \\ &U(\tau) = U(\tau_0^-). \end{aligned} \tag{6.1.4}$$

Then, by the standard theory of abstract Cauchy problems, we have a unique continuous solution $U : [\tau_0, 2\tau_0] \to H$ satisfying

$$U(t) = S(t - \tau_0)U(\tau_0^-) + \int_{\tau_0}^t S(t-s)F(s)ds, \quad t \in [\tau_0, 2\tau_0],$$

and so

$$U(t) = S(t - \tau_0)U(\tau_0^-) + \int_{\tau_0}^t S(t-s)k(s)BU(s - \tau(s))ds, \quad t \in [\tau_0, 2\tau_0].$$

Putting together the partial solutions obtained in the first and second steps we have a unique continuous solution $U : [0, 2\tau_0] \to \mathbb{R}$ satisfying the Duhamel's formula

$$U(t) = S(t)U_0 + \int_0^t S(t-s)k(s)BU(s - \tau(s))ds, \quad t \in [0, 2\tau_0].$$

Iterating this procedure we can find a unique solution $U \in C([0, +\infty); H)$ satisfying the representation formula (6.1.1).

Case 2. Let $\tau(\cdot)$ be a continuous function satisfying (0.2.33). In this case, since assumption (6.1.2) does not necessarily hold, we cannot use the step-by-step procedure above, as we did in Case 1. We have rather to employ a different method, based on the use of the Banach's fixed point Theorem.
Now, since $k \in \mathcal{L}^1_{loc}([-\bar{\tau}, +\infty); \mathbb{R})$, there exists $T > 0$ such that

$$\|k\|_{\mathcal{L}^1([0,T];\mathbb{R})} = \int_0^T |k(s)|ds < \frac{1}{M\|B\|}. \tag{6.1.5}$$

We define the set

$$C_f([-\bar{\tau}, T]; H) := \{U \in C([-\bar{\tau}, T]; H) : U(s) = f(s), \forall s \in [-\bar{\tau}, 0]\}.$$

Let us note that $C_f([-\bar{\tau}, T]; H) \neq \emptyset$, since it suffices to take

$$U(t) = \begin{cases} U_0, & t \in (0, T], \\ f(t), & t \in [-\bar{\tau}, 0], \end{cases}$$

to have that $U \in C_f([-\bar{\tau}, T]; H)$.
It is immediate to see that $C_f([-\bar{\tau}, T]; H)$ with the norm

$$\|U\|_{C([-\bar{\tau},T];H)} = \max_{r \in [-\bar{\tau},T]} \|U(r)\|, \quad \forall U \in C([-\bar{\tau}, T]; H),$$

is a Banach space.
Now, we define the map $\Gamma : C_f([-\bar{\tau}, T]; H) \to C_f([-\bar{\tau}, T]; H)$ given by

$$\Gamma U(t) = \begin{cases} S(t)U_0 + \int_0^t S(t-s)k(s)BU(s-\tau(s))\, ds, & t \in [0, T], \\ f(t), & t \in [-\bar{\tau}, 0). \end{cases}$$

We claim that Γ is well-defined. Indeed, let $U \in C_f([-\bar{\tau}, T]; H)$. Then, from the semigroup theory, $t \mapsto S(t)U_0$ is continuous. Also, since $U(\cdot)$ is continuous in $[-\bar{\tau}, T]$, $\tau(\cdot)$ is a continuous function and B is a bounded linear operator from H into itself, $[0, T] \ni t \mapsto BU(t-\tau(t))$ is continuous. Moreover, $k \in \mathcal{L}^1([0, T]; \mathbb{R})$. So $k(\cdot)BU(\cdot-\tau(\cdot)) \in \mathcal{L}^1([0, T]; H)$. Hence, the map $t \mapsto \int_0^t S(t-s)k(s)BU(s-\tau(s))ds$ is continuous in $[0, T]$. Thus, $\Gamma U \in C([0, T]; H)$. Finally, since $\Gamma U = f$ in $[\bar{\tau}, 0]$ with $f \in C([-\bar{\tau}, 0]; H)$ and $f(0) = U_0$, $\Gamma U \in C_f([\bar{\tau}, T]; H)$ and Γ is well-defined.
Now, let $U, V \in C_f([\bar{\tau}, T]; H)$. For all $t \in [-\bar{\tau}, 0]$,

$$\|\Gamma U(t) - \Gamma V(t)\| = 0.$$

On the other hand, for all $t \in (0, T]$,

$$\begin{aligned} \|\Gamma U(t) - \Gamma V(t)\| &\leq \int_0^t |k(s)| \|S(t-s)\|_{\mathcal{L}(H)} \|BU(s-\tau(s)) - BV(s-\tau(s))\| ds \\ &\leq M\|B\| \int_0^t |k(s)| \|U(s-\tau(s)) - V(s-\tau(s))\| ds \\ &\leq M\|B\| \|U - V\|_{C([-\bar{\tau},T];H)} \int_0^T |k(s)| ds \\ &= MB\|k\|_{\mathcal{L}^1([0,T];\mathbb{R})} \|U - V\|_{C([-\bar{\tau},T];H)}. \end{aligned}$$

Thus,

$$\|\Gamma U(t) - \Gamma V(t)\| \le MB\|k\|_{\mathcal{L}^1([0,T];\mathbb{R})}\|U - V\|_{C([-\bar{\tau},T];H)}, \quad \forall t \in [-\bar{\tau}, T],$$

from which

$$\|\Gamma U - \Gamma V\|_{C([-\bar{\tau},T];H)} \le M\|B\|\|k\|_{\mathcal{L}^1([0,T];\mathbb{R})}\|U - V\|_{C([-\bar{\tau},T];H)}.$$

Now, from (6.1.5) we have that $M\|B\|\|k\|_{\mathcal{L}^1([0,T];\mathbb{R})} < 1$. Hence, Γ is a contraction. Then, from the Banach's Theorem, Γ has a unique fixed point $U \in C_f([-\bar{\tau}, T]; H)$, i.e. (0.2.32) has a unique solution $U \in C([0,T]; H)$ given by the Duhamel's formula

$$U(t) = S(t)U_0 + \int_0^t S(t-s)k(s)BU(s - \tau(s))\, ds, \quad \forall t \in [0, T].$$

Now, let us note that the solution U is bounded. Indeed, for all $t \in [0, T]$,

$$\begin{aligned}\|U(t)\| &\le M\|U_0\| + M\|B\| \int_0^t |k(s)|\|U(s-\tau(s))\|ds \\ &\le M\|U_0\| + M\|B\|\|k\|_{\mathcal{L}^1([0,T];\mathbb{R})} \max_{r\in[-\bar{\tau},0]}\|f(r)\| + M\|B\| \int_0^t |k(s)| \max_{r\in[0,s]}\|U(r)\|ds.\end{aligned}$$

Then,

$$\begin{aligned}\max_{r\in[0,t]}\|U(r)\| &\le M\left(\|U_0\| + \|B\|\|k\|_{\mathcal{L}^1([0,T];\mathbb{R})} \max_{r\in[-\bar{\tau},0]}\|f(r)\|\right) \\ &\quad + M\|B\| \int_0^t |k(s)| \max_{r\in[0,s]}\|U(r)\|ds.\end{aligned}$$

Hence, the Gronwall's estimate yields

$$\max_{r\in[0,t]}\|U(r)\| \le M\left(\|U_0\| + \|B\|\|k\|_{\mathcal{L}^1([0,T];\mathbb{R})} \max_{r\in[-\bar{\tau},0]}\|f(r)\|\right) e^{M\|B\|\int_0^t |k(s)|ds},$$

from which, taking into account of (6.1.5),

$$\|U(t)\| \le e\left(M\|U_0\| + \max_{r\in[-\bar{\tau},0]}\|f(r)\|\right), \quad \forall t \in [0, T].$$

Thus, the solution U is bounded and we can extend it up to some maximal interval $[0, \delta)$, $\delta > 0$. We claim that $\delta = +\infty$. Indeed, assume by contradiction that $\delta < +\infty$. Then, being U bounded, we can consider the following problem

$$\begin{aligned}&V'(t) = AV(t) + k(t)BV(t - \tau(t)), \quad t \in (0, \infty), \\ &V(t) = U(t) \quad t \in [\delta - \bar{\tau}, \delta), \\ &V(\delta) = U(\delta^-).\end{aligned} \tag{6.1.6}$$

Arguing as before, there exists $T' > 0$ such that

$$\|k\|_{\mathcal{L}^1([\delta,T'],\mathbb{R})} = \int_\delta^{T'} |k(s)|ds < \frac{1}{M\|B\|}. \tag{6.1.7}$$

We then set

$$C_U([\delta-\bar\tau,T'];H)=\{V\in C([\delta-\bar\tau,T'];H): V(s)=U(s),\forall s\in[\delta-\bar\tau,\delta), V(\delta)=U(\delta^-)\},$$

which is a nonempty closed subset of $C([\delta-\bar\tau,T'];H)$.
Next, we define the map $\Gamma: C_U([-\bar\tau,T];H)\to C_U([-\bar\tau,T];H)$ given by

$$\Gamma V(t)=\begin{cases} S(t-\delta)U(\delta^-)+\int_\delta^t S(t-s)k(s)BV(s-\tau(s)), & t\in[\delta,T'],\\ U(t), & t\in[\delta-\bar\tau,\delta).\end{cases}$$

We have that Γ is well-defined and, using the same arguments employed at the beginning of Case 2, (6.1.7) implies that Γ is a contraction. So, Γ has a unique fixed point, i.e. (6.1.6) has a unique continuous V solution given by the Duhamel formula

$$V(t)=S(t-\delta)U(\delta^-)+\int_\delta^t S(t-s)k(s)BV(s-\tau(s)),\quad t\in[\delta,T'].$$

Thus, putting together the solutions U, V, we get the existence of a unique continuous solution to (0.2.32) that satisfies the Duhamel's formula (6.1.1) and that is defined in $[0,\delta')$, with $\delta'>\delta$. This contradicts the maximality of δ. Hence, $\delta=+\infty$, i.e. (0.2.32) has a unique global solution $U\in C([0,+\infty);H)$ given by (6.1.1). □

6.2 Exponential stability

Now, we establish a stability result for the system (0.2.32). Namely, under an appropriate relation between the problem's parameters we prove that the system (0.2.32) is exponentially stable. In particular, we assume that

$$M\|B\|e^{\omega\bar\tau}\int_0^t |k(s)|ds\le\gamma+\omega' t,\quad \forall t\ge 0, \tag{6.2.1}$$

for suitable constants $\gamma\ge 0$ and $\omega'\in[0,\omega)$.

Theorem 6.2.1. *Assume* (6.2.1). *Then, for every $f\in C([-\bar\tau,0];H)$ the solution $U\in C([0,+\infty);H)$ to* (0.2.32) *with the initial datum f satisfies the exponential decay estimate*

$$||U(t)||\le Me^{\gamma}\left(\|U_0\|+e^{\omega\bar\tau}K\|B\|\max_{s\in[-\bar\tau,0]}\{\|e^{\omega s}f(s)\|\}\right)e^{-(\omega-\omega')t}, \tag{6.2.2}$$

for any $t\ge 0$.

Proof. Let $f\in C([-\bar\tau,0];H)$. Let $U\in C([0,+\infty);H)$ be the solution to (0.2.32) with the initial condition f. From Duhamel's Formula, we have that

$$||U(t)||\le Me^{-\omega t}||U_0||+Me^{-\omega t}\int_0^t e^{\omega s}|k(s)|\cdot||BU(s-\tau(s))||ds,\quad \forall t\ge 0.$$

Then, for all $t \geq \bar{\tau}$, we deduce that

$$\begin{aligned} ||U(t)|| \leq Me^{-\omega t}||U_0|| + Me^{-\omega t}\int_0^{\bar{\tau}} e^{\omega s}|k(s)|\cdot||BU(s-\tau(s))||ds \\ +Me^{-\omega t}\int_{\bar{\tau}}^{t} e^{\omega s}|k(s)|\cdot||BU(s-\tau(s))||ds \\ \leq Me^{-\omega t}||U_0|| + M||B||e^{-\omega t}e^{\omega\bar{\tau}}\int_0^{\bar{\tau}} e^{\omega(s-\tau(s))}|k(s)|\cdot||U(s-\tau(s))||ds \\ +M||B||e^{-\omega t}e^{\omega\bar{\tau}}\int_{\bar{\tau}}^{t} e^{\omega(s-\tau(s))}|k(s)|\cdot||U(s-\tau(s))||ds. \end{aligned} \tag{6.2.3}$$

Now, observe that

$$\begin{aligned} &\int_0^{\bar{\tau}} e^{\omega(s-\tau(s))}|k(s)|\cdot\|U(s-\tau(s))\|ds \\ &\leq \int_0^{\bar{\tau}} |k(s)|\left(\max_{r\in[-\bar{\tau},0]}\{e^{\omega r}\|f(r)\|\} + \max_{r\in[0,s]}\{e^{\omega r}\|U(r)\|\}\right)ds \\ &\leq K\max_{r\in[-\bar{\tau},0]}\{e^{\omega r}\|f(r)\|\} + \int_0^{\bar{\tau}}|k(s)|\max_{r\in[0,s]}\{e^{\omega r}\|U(r)\|\}\,ds. \end{aligned} \tag{6.2.4}$$

Then, using (6.2.4) in (6.2.3), we deduce

$$\begin{aligned} ||U(t)|| \leq Me^{-\omega t}\left(||U_0|| + e^{\omega\bar{\tau}}K\|B\|\max_{r\in[-\bar{\tau},0]}\{e^{\omega r}\|f(r)\|\}\right) \\ +M\|B\|e^{-\omega t}e^{\omega\bar{\tau}}\int_0^t |k(s)|\max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||\}\,ds, \end{aligned} \tag{6.2.5}$$

for all $t \geq \bar{\tau}$. On the other hand, for all $t \in [0,\bar{\tau}]$, it holds that

$$||U(t)|| \leq Me^{-\omega t}||U_0|| + M\|B\|e^{-\omega t}e^{\omega\bar{\tau}}\int_0^t e^{\omega(s-\tau(s))}|k(s)|\cdot||U(s-\tau(s))||ds.$$

Then, arguing as before,

$$\begin{aligned} &\int_0^t e^{\omega(s-\tau(s))}|k(s)|\cdot\|U(s-\tau(s))\|ds \\ &\leq \int_0^t |k(s)|\left(\max_{r\in[-\bar{\tau},0]}\{e^{\omega r}\|f(r)\|\} + \max_{r\in[0,s]}\{e^{\omega r}\|U(r)\|\}\right)ds \\ &\leq K\max_{r\in[-\bar{\tau},0]}\{e^{\omega r}\|f(r)\|\} + \int_0^t|k(s)|\max_{r\in[0,s]}\{e^{\omega r}\|U(r)\|\}\,ds \end{aligned}$$

for all $t \in [0, \bar{\tau}]$. Hence, (6.2.5) holds also for $t \in [0, \bar{\tau}]$. So, we deduce

$$e^{\omega t}||U(t)|| \le M \left(||U_0|| + e^{\omega\bar{\tau}} K \|B\| \max_{s\in[-\bar{\tau},0]} \{\|e^{\omega s} f(s)\|\} \right)$$
$$+M\|B\|e^{\omega\bar{\tau}} \int_0^t |k(s)| \max_{r\in[s-\bar{\tau},s]\cap[0,s]} \{e^{\omega r}||U(r)||\} \, ds, \quad \forall t \ge 0.$$

Now, we note that it also holds

$$\max_{s\in[t-\bar{\tau},t]\cap[0,t]} \{e^{\omega s}||U(s)||\} \le M \left(||U_0|| + e^{\omega\bar{\tau}} K \|B\| \max_{s\in[-\bar{\tau},0]} \{\|e^{\omega s} f(s)\|\} \right)$$
$$+M\|B\|e^{\omega\bar{\tau}} \int_0^t |k(s)| \max_{r\in[s-\bar{\tau},s]\cap[0,s]} \{e^{\omega r}||U(r)||\} \, ds, \quad \forall t \ge 0.$$

Hence, if we denote

$$\tilde{u}(t) := \max_{s\in[t-\bar{\tau},t]\cap[0,t]} \{e^{\omega s}||U(s)||\},$$

Gronwall's estimate implies

$$\tilde{u}(t) \le \tilde{M} e^{M\|B\|e^{\omega\bar{\tau}} \int_0^t |k(s)|ds}, \quad \forall t \ge 0,$$

where

$$\tilde{M} := M \left(\|U_0\| + e^{\omega\bar{\tau}} K \|B\| \max_{s\in[-\bar{\tau},0]} \{\|e^{\omega s} f(s)\|\} \right).$$

Then,

$$e^{\omega t}\|U(t)\| \le \tilde{M} e^{M\|B\|e^{\omega\bar{\tau}} \int_0^t |k(s)|ds}, \quad \forall t \ge 0.$$

Finally, assumption (6.2.1), yields

$$\|U(t)\| \le \tilde{M} e^{M\|B\|e^{\omega\bar{\tau}} \int_0^t |k(s)|ds} e^{-\omega t}$$
$$\le \tilde{M} e^{\gamma+\omega' t} e^{-\omega t} = \tilde{M} e^{\gamma} e^{-(\omega-\omega')t},$$

for all $t \ge 0$, which proves the exponential decay estimate (6.2.2). □

6.3 A nonlinear model

As an easy generalization of the previous results, we can prove well-posedness and exponential stability for the following nonlinear model

$$\begin{aligned} &U'(t) = AU(t) + k(t)BU(t - \tau(t)) + G(U(t)), \quad t \in (0, \infty), \\ &U(t) = f(t) \quad t \in [-\bar{\tau}, 0], \end{aligned} \tag{6.3.1}$$

where A,B, $k(\cdot), \tau(\cdot)$ are as before, and we denote $U_0 := f(0)$. Moreover, $G : H \to H$ is Lipschitz continuous, namely there exists $L > 0$ such that

$$\|G(U_1) - G(U_2)\| \le L\|U_1 - U_2\|, \quad \forall\, U_1, U_2 \in H, \tag{6.3.2}$$

and we assume that $G(0) = 0$.

Analogously to before, one can first give a well-posedness result. See [84] for the proof in the case of constant time delay.

Theorem 6.3.1. *Let $f : [-\bar{\tau}, 0] \to H$ be a continuous function. Then, the problem* (6.3.1) *has a unique (weak) solution given by Duhamel's formula*

$$U(t) = S(t)U_0 + \int_0^t S(t-s)[G(U(s)) + k(s)BU(s - \tau(s))]\, ds, \tag{6.3.3}$$

for all $t \geq 0$.

Proof. Let $f \in C([-\bar{\tau}, 0], H)$. As before, we can give two different proofs.
Case 1 Assume that (6.1.2) holds tue. We can argue step-by-step, as before, by restricting ourselves each time to time intervals of length τ_0.
First we consider $t \in [0, \tau_0]$. Then, from (6.1.2), $t - \tau(t) \in [-\bar{\tau}, 0]$. So, setting $F(t) = k(t)BU(t - \tau(t))$, $t \in [0, \tau_0]$, we have that $F(t) = k(t)Bf(t - \tau(t))$, $t \in [0, \tau_0]$. Then, problem (6.3.1) can be rewritten, in the interval $[0, \tau_0]$, as a standard inhomogeneous evolution problem:

$$\begin{aligned} &U'(t) = AU(t) + G(U(t)) + F(t) \quad \text{in } (0, \tau_0), \\ &U(0) = U_0. \end{aligned} \tag{6.3.4}$$

Since $k \in \mathcal{L}^1_{loc}([-\bar{\tau}, +\infty); \mathbb{R})$ and $f \in C([-\bar{\tau}, 0]; H)$, then we have that $F \in \mathcal{L}^1((0, \tau_0); H)$. Therefore, applying the standard theory for nonlinear evolution equations (see e.g. [92]), there exists a unique solution $U \in C([0, \tau_0]; H)$ of (6.3.4) satisfying the Duhamel's formula

$$U(t) = S(t)U_0 + \int_0^t S(t-s)[G(U(s)) + F(s)]ds, \quad t \in [0, \tau_0].$$

Therefore,

$$U(t) = S(t)U_0 + \int_0^t S(t-s)[G(U(s)) + k(s)BU(s - \tau(s))]ds, \quad t \in [0, \tau_0].$$

Next, we consider the time interval $[\tau_0, 2\tau_0]$ and also define $F(t) = k(t)BU(t - \tau(t))$, for $t \in [\tau_0, 2\tau_0]$. Note that, if $t \in [\tau_0, 2\tau_0]$, then $t - \tau(t) \in [-\bar{\tau}, \tau_0]$ and so $U(t - \tau(t))$ is known from the first step. Then $F_{|_{[\tau_0, 2\tau_0]}}$ is a known function and it belongs to $\mathcal{L}^1((\tau_0, 2\tau_0); H)$. So we can rewrite our model (6.3.1) in the time interval $[\tau_0, 2\tau_0]$ as the inhomogeneous evolution problem

$$\begin{aligned} &U'(t) = AU(t) + G(U(t)) + F(t) \quad \text{for } \ t \in (\tau_0, 2\tau_0), \\ &U(\tau) = U(\tau_0^-). \end{aligned} \tag{6.3.5}$$

Then, we have a unique continuous solution $U : [\tau_0, 2\tau_0] \to H$ satisfying

$$U(t) = S(t - \tau_0)U(\tau_0^-) + \int_{\tau_0}^t S(t-s)[G(U(s)) + F(s)]ds, \quad t \in [\tau_0, 2\tau_0],$$

and so

$$U(t) = S(t-\tau_0)U(\tau_0^-) + \int_{\tau_0}^t S(t-s)[G(U(s)) + k(s)BU(s-\tau(s))]ds, \quad t \in [\tau_0, 2\tau_0].$$

Putting together the partial solutions obtained in the first and second steps we have a unique continuous solution $U : [0, 2\tau_0) \to \mathbb{R}$ satisfying the Duhamel's formula

$$U(t) = S(t)U_0 + \int_0^t S(t-s)[G(U(s)) + k(s)BU(s-\tau(s))]ds, \quad t \in [0, 2\tau_0].$$

Iterating this procedure we can find a unique solution $U \in C([0, +\infty); H)$ satisfying the representation formula (6.3.3).
Case 2 Let $\tau(\cdot)$ be a continuous function satisfying (0.2.33).
Now, since $k \in \mathcal{L}^1_{loc}([-\bar{\tau}, +\infty); \mathbb{R})$, there exists $T > 0$ sufficiently small such that

$$LT + \|B\|\|k\|_{\mathcal{L}^1([0,T];\mathbb{R})} = LT + \|B\| \int_0^T |k(s)|ds < \frac{1}{M}, \tag{6.3.6}$$

where L is the Lipschitz constant in (6.3.2). We define the set

$$\tilde{C}_f([-\bar{\tau}, T]; H) := \{U \in C([-\bar{\tau}, T]; H) : U(s) = f(s), \forall s \in [-\bar{\tau}, 0]\}.$$

Let us note that $\tilde{C}_f([-\bar{\tau}, T]; H)$ is a nonempty and closed subset of $C([-\bar{\tau}, T]; H)$. Hence, $(\tilde{C}_f([-\bar{\tau}, T]; H), \|\cdot\|_{C([-\bar{\tau},T];H)})$ is a Banach space.
Next, we define the map $\tilde{\Gamma} : \tilde{C}_f([-\bar{\tau}, T]; H) \to \tilde{C}_f([-\bar{\tau}, T]; H)$ given by

$$\tilde{\Gamma}U(t) = \begin{cases} S(t)U_0 + \int_0^t S(t-s)[G(U(s)) + k(s)BU(s-\tau(s))]\, ds, & t \in [0, T], \\ f(t), & t \in [-\bar{\tau}, 0). \end{cases}$$

Let us note that $\tilde{\Gamma}$ is well-defined.
Moreover, $\tilde{\Gamma}$ is a contraction. Indeed, let $U, V \in \tilde{C}_f([\bar{\tau}, T]; H)$. Then, for all $t \in [-\bar{\tau}, 0]$,

$$\|\tilde{\Gamma}U(t) - \tilde{\Gamma}V(t)\| = 0.$$

On the other hand, for all $t \in (0, T]$, since G is Lipschitz continuous we get

$$\begin{aligned} \|\tilde{\Gamma}U(t) - \tilde{\Gamma}V(t)\| &\leq \int_0^t \|S(t-s)\|_{\mathcal{L}(H)}\|G(U(s)) - G(V(s))\|ds \\ &\quad + \int_0^t |k(s)|\|S(t-s)\|_{\mathcal{L}(H)}\|BU(s-\tau(s)) - BV(s-\tau(s))\|ds \\ &\leq M(LT + \|B\|\|k\|_{\mathcal{L}^1([0,T];\mathbb{R})})\|U - V\|_{C([-\bar{\tau},T];H)}. \end{aligned}$$

Thus,

$$\|\tilde{\Gamma}U - \tilde{\Gamma}V\|_{C([-\bar{\tau},T];H)} \leq M(LT + \|B\|\|k\|_{\mathcal{L}^1([0,T];\mathbb{R})})\|U - V\|_{C([-\bar{\tau},T];H)}.$$

As a consequence, from (6.3.6) $\tilde{\Gamma}$ is a contraction. Thus, from the Banach's Theorem, $\tilde{\Gamma}$ has a unique fixed point $U \in \tilde{C}_f([-\bar{\tau}, T]; H)$, i.e. (0.2.32) has a unique solution $U \in C([0,T]; H)$ given by the Duhamel formula

$$U(t) = S(t)U_0 + \int_0^t S(t-s)[G(U(s)) + k(s)BU(s-\tau(s))]\, ds, \quad \forall t \in [0,T].$$

Now, note that the solution U is bounded. So, arguing as in **Case 2** of Theorem 6.1.1, we can conclude that (6.3.1) has a unique continuous global solution that satisfies the Duhamel's formula (6.3.3). □

As in the previous section, under an appropriate relation between the problem's parameters, also the system (6.3.1) is exponentially stable.

Theorem 6.3.2. *Assume* (6.2.1) *and* $L < \frac{\omega - \omega'}{M}$. *Then, for every* $f \in C([-\bar{\tau}, 0]; H)$, *the solution* $U \in C([0, +\infty); H)$ *to* (6.3.1) *with the initial datum* f *satisfies the exponential decay estimate*

$$||U(t)|| \le Me^{\gamma} \left(\|U_0\| + e^{\omega\bar{\tau}} K \|B\| \max_{s \in [-\bar{\tau}, 0]} \{\|e^{\omega s} f(s)\|\} \right) e^{-(\omega - \omega' - ML)t}, \tag{6.3.7}$$

for any $t \ge 0$.

Proof. Let $f \in C([-\bar{\tau}, 0]; H)$ and let U be the unique global solution to (6.3.1) with initial datum f. Then, from Duhamel's formula (6.3.3), we have that

$$\begin{aligned} ||U(t)|| \le Me^{-\omega t}||U_0|| + Me^{-\omega t} \int_0^t e^{\omega s}||G(U(s))||ds \\ + M\|B\|e^{-\omega t} \int_0^t e^{\omega s}|k(s)| \cdot ||U(s - \tau(s))||)ds, \end{aligned}$$

for all $t \ge 0$. Now, using the same arguments employed in Theorem 6.2.1, we get

$$\begin{aligned} \int_0^t e^{\omega s}|k(s)| \cdot ||U(s-\tau(s))||\, ds \le e^{\omega\bar{\tau}} \int_0^{\bar{\tau}} e^{\omega(s-\tau(s))}|k(s)| \cdot ||U(s-\tau(s))||\, ds \\ + e^{\omega\bar{\tau}} \int_{\bar{\tau}}^t e^{\omega(s-\tau(s))}|k(s)| \cdot ||U(s-\tau(s))||\, ds \\ \le e^{\omega\bar{\tau}} K \max_{r \in [-\bar{\tau}, 0]} \{||e^{\omega r} f(r)||\} + e^{\omega\bar{\tau}} \int_0^t |k(s)| \cdot \max_{r \in [s-\bar{\tau}, s] \cap [0,s]} \{e^{\omega r}||U(r)||\}ds, \end{aligned} \tag{6.3.8}$$

for all $t \ge \bar{\tau}$. Also, for all $t \in [0, \bar{\tau}]$,

$$\begin{aligned} \int_0^t e^{\omega s}|k(s)| \cdot ||U(s-\tau(s))||\, ds \le e^{\omega\bar{\tau}} \int_0^t e^{\omega(s-\tau(s))}|k(s)| \cdot ||U(s-\tau(s))||\, ds \\ \le e^{\omega\bar{\tau}} K \max_{r \in [-\bar{\tau}, 0]} \{||e^{\omega r} f(r)||\} + e^{\omega\bar{\tau}} \int_0^t |k(s)| \cdot \max_{r \in [s-\bar{\tau}, s] \cap [0,s]} \{e^{\omega r}||U(r)||\}ds, \end{aligned}$$

i.e. (6.3.8) holds true for all $t \geq 0$. Hence, from (6.3.8) we get

$$||U(t)|| \leq Me^{-\omega t}\left(||U_0|| + e^{\omega\bar{\tau}}K\|B\| \max_{r\in[-\bar{\tau},0]}\{||e^{\omega r}f(r)||\}\right)$$

$$+Me^{-\omega t}e^{\omega\bar{\tau}}||B||\int_0^t |k(s)|\cdot \max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||\}ds + Me^{-\omega t}\int_0^t e^{\omega s}||G(U(s))||ds.$$

Now, let us note that, being $G(0)=0$ and being G Lipschitz continuous of constant L,

$$\int_0^t e^{\omega s}||G(U(s))||ds \leq L\int_0^t e^{\omega s}||U(s)|| \leq L\int_0^t \max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||\}ds.$$

As a consequence, we can write

$$||U(t)|| \leq Me^{-\omega t}(||U_0|| + e^{\omega\bar{\tau}}K||B|| \max_{r\in[-\bar{\tau},0]}\{||e^{\omega r}f(r)||\})$$

$$+Me^{-\omega t}\int_0^t (e^{\omega\bar{\tau}}||B|||k(s)| + L)\cdot \max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||\}ds.$$

Then, we have that

$$e^{\omega t}||U(t)|| \leq M\left(||U_0|| + e^{\omega\bar{\tau}}K||B|| \max_{r\in[-\bar{\tau},0]}\{\|e^{\omega r}f(r)\|\}\right)$$

$$+M\int_0^t (e^{\omega\bar{\tau}}||B|||k(s)| + L)\cdot \max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||\}\, ds, \quad \forall t \geq 0.$$

Now, we note that

$$\max_{r\in[t-\bar{\tau},t]\cap[0,t]}\{e^{\omega r}||U(r)||\} \leq M\left(||U_0|| + e^{\omega\bar{\tau}}K||B|| \max_{s\in[-\bar{\tau},0]}\{\|e^{\omega s}f(s)\|\}\right)$$

$$+M\int_0^t (e^{\omega\bar{\tau}}||B|||k(s)| + L)\cdot \max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||\}\, ds, \qquad \forall t \geq 0.$$

Hence, if we denote with

$$\tilde{u}(t) := \max_{r\in[t-\bar{\tau},t]\cap[0,t]}\{e^{\omega r}||U(r)||\},$$

Gronwall's estimate yields

$$\tilde{u}(t) \leq \tilde{M}e^{M\|B\|e^{\omega\bar{\tau}}\int_0^t |k(s)|ds + MLt}, \quad \forall t \geq 0$$

where

$$\tilde{M} := M\left(\|U_0\| + e^{\omega\bar{\tau}}K||B|| \max_{r\in[-\bar{\tau},0]}\{\|e^{\omega r}f(r)\|\}\right).$$

Then,

$$e^{\omega t}\|U(t)\| \leq \tilde{M}e^{M\|B\|e^{\omega\bar{\tau}}\int_0^t |k(s)|ds + MLt}.$$

Finally, by the assumption (6.2.1) and the assumption on the Lipschitz constant L, we get the exponential decay estimate (6.3.7). □

6.4 Examples

We conclude this chapter by providing applications of the results established in the previous sections.

As concrete examples, we will consider the wave equation with localized frictional damping and delay feedback and an elasticity system with analogous feedback laws.

6.4.1 The damped wave equation

Let Ω be an open bounded subset of $\mathbb{R}^d$, with boundary $\partial\Omega$ of class C^2, and let $\mathcal{O} \subset \Omega$ be an open subset which satisfies the geometrical control property in [15]. For instance, $\mathcal{O} \subset \Omega$ can be a neighborhood of the whole boundary $\partial\Omega$ or, denoting by m the standard multiplier $m(x) = x - x_0$, $x_0 \in \mathbb{R}^d$, as in [75], $\mathcal{O}$ can be the intersection of Ω with an open neighborhood of the set

$$\Gamma_0 = \{\, x \in \Gamma \ : \ m(x) \cdot \nu(x) > 0 \,\}.$$

Moreover, let $\tilde{\mathcal{O}} \subset \Omega$ be another open subset. Denoting by $\chi_{\mathcal{O}}$ and $\chi_{\tilde{\mathcal{O}}}$ the characteristic functions of the sets $\mathcal{O}$ and $\tilde{\mathcal{O}}$ respectively, we consider the following wave equation

$$\begin{aligned} & u_{tt}(x,t) - \Delta u(x,t) + a\chi_{\mathcal{O}}(x) u_t(x,t) \\ & \qquad\qquad + k(t)\chi_{\tilde{\mathcal{O}}}(x) u_t(x, t-\tau(t)) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\ & u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,+\infty), \\ & u(x,s) = u_0(x,s), \quad u_t(x,s) = u_1(x,s), \quad (x,s) \in \Omega \times [-\bar{\tau}, 0], \end{aligned} \tag{6.4.1}$$

where a is a positive constant, $\tau(t)$ is the time delay function satisfying $0 \le \tau(t) \le \bar{\tau}$, and the delayed damping coefficient $k(\cdot) : [-\bar{\tau}, +\infty) \to (0, +\infty)$ is a $\mathcal{L}^1_{loc}([-\bar{\tau}, +\infty))$ function satisfying (0.2.35). Denoting $v(t) = u_t(t)$ and $U(t) = (u(t), v(t))^T$, for any $t \ge 0$,, we can rewrite system (6.4.1) in the abstract form (0.2.32), with $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$,

$$A = \begin{pmatrix} 0 & Id \\ \Delta & -a\chi_{\mathcal{O}} \end{pmatrix}$$

and

$$B \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -\chi_{\tilde{\mathcal{O}}} v \end{pmatrix}, \quad \forall\, t \ge 0.$$

We know that A generates an exponentially stable C_0-semigroup $\{S(t)\}_{t\ge 0}$ (see e.g. [70]), namely there exist $\omega, M > 0$ such that

$$||S(t)||_{\mathcal{L}(\mathcal{H})} \le M e^{-\omega t}, \quad \forall\, t \ge 0.$$

Hence, under the assumption (6.2.1), the stability estimate of Theorem 6.2.1 holds for such a model. Then, we can deduce an exponential decay estimate for the energy functional

$$E(t) := \frac{1}{2}\int_\Omega |u_t(x,t)|^2 dx + \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 dx + \frac{1}{2}\int_{t-\bar{\tau}}^t \int_{\tilde{\mathcal{O}}} |k(s)| \cdot |u_t(x,s)|^2 dx ds.$$

Theorem 6.4.1. *Assume* (6.2.1). *Then, for all initial data* $(u_0, u_1) \in C([-\bar{\tau}, 0]; H_0^1(\Omega) \times L^2(\Omega))$, *the solution to* (6.4.1) *satisfies the energy decay estimate*

$$E(t) \leq C_* e^{-\beta t}, \quad t \geq 0, \tag{6.4.2}$$

where C_* *is a constant depending on the initial data and* $\beta > 0$.

Proof. From the energy's definition,

$$\begin{aligned} E(t) &= \frac{1}{2}\|U(t)\|^2 + \frac{1}{2}\int_{t-\bar{\tau}}^{t}\int_{\tilde{\mathcal{O}}} |k(s)| \cdot |u_t(x,s)|^2 dx ds \\ &\leq \frac{1}{2}\|U(t)\|^2 + \frac{1}{2}\int_{t-\bar{\tau}}^{t} |k(s)| \|U(s)\|^2 ds. \end{aligned} \tag{6.4.3}$$

Then, from Theorem 6.2.1,

$$\|U(t)\| \leq C_0 e^{-(\omega-\omega')t}, \quad \forall t \geq 0,$$

for a suitable constant C_0 depending on the initial data. So, we can estimate

$$\int_{t-\bar{\tau}}^{t} |k(s)| \|U(s)\|^2 ds \leq C_0 K e^{(\omega-\omega')\bar{\tau}}\, e^{-(\omega-\omega')t}, \quad \forall t \geq 0.$$

By using the last two inequalities in (6.4.3), we obtain the exponential decay estimate (6.4.2). □

Remark 6.4.2. As another example, we could consider the damped plate equation (see e.g. [82] for the model details). The analysis is analogous to the wave case above. Then, under suitable assumptions, the exponential stability result holds for that model.

6.4.2 A damped elasticity system

Let $\Omega \subset \mathbb{R}^d$, and let $\tilde{\mathcal{O}}, \mathcal{O} \subset \Omega$ be as in the previous example. We consider the following elastodynamic system

$$\begin{aligned} & u_{tt}(x,t) - \mu\Delta u(x,t) - (\lambda+\mu)\nabla \operatorname{div} u + a\chi_{\mathcal{O}}(x) u_t(x,t) \\ & \qquad\qquad + k(t)\chi_{\tilde{\mathcal{O}}}(x) u_t(x, t-\tau(t)) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\ & u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,+\infty), \\ & u(x,s) = u_0(x,s), \quad u_t(x,s) = u_1(x,s), \quad (x,s) \in \Omega \times [-\bar{\tau}, 0], \end{aligned} \tag{6.4.4}$$

where a is a positive constant, $\tau(t)$ is the time delay function satisfying $0 \leq \tau(t) \leq \bar{\tau}$, and the delayed damping coefficient $k(\cdot) : [-\bar{\tau}, +\infty) \to (0, +\infty)$ is a $\mathcal{L}^1_{loc}([-\bar{\tau}, +\infty))$ function satisfying (0.2.35). Note that, in this case, the function u is vector-valued and takes values in $\mathbb{R}^d$ while λ and μ are positive constants usually called Lamé coefficients.

Denoting $v(t) = u_t(t)$ and $U(t) = (u(t), v(t))^T$, for any $t \geq 0$,, we can rewrite system (6.4.4) in the abstract form (0.2.32), with $\mathcal{H} = H_0^1(\Omega)^d \times L^2(\Omega)^d$,

$$A = \begin{pmatrix} 0 & Id \\ \mu\Delta + (\lambda+\mu)\nabla \mathrm{div} & -a\chi_{\mathcal{O}} \end{pmatrix}$$

and

$$B\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -\chi_{\tilde{\mathcal{O}}} v \end{pmatrix}, \quad \forall\, t \geq 0.$$

We know that A generates an exponentially stable C_0-semigroup $\{S(t)\}_{t\geq 0}$ (see e.g. [52]), namely there exist $\omega, M > 0$ such that

$$||S(t)||_{\mathcal{L}(\mathcal{H})} \leq Me^{-\omega t}, \quad \forall\, t \geq 0.$$

Hence, under the assumption (6.2.1), the stability estimate of Theorem 6.2.1 holds for such a model. Therefore, we can deduce an exponential decay estimate for the energy functional

$$\mathcal{E}(t) := \frac{1}{2}\int_\Omega |u_t(x,t)|^2 dx + \frac{1}{2}\int_\Omega \Big[\mu \sum_{i,j=1}^n \Big(\frac{\partial}{\partial x_i} u_j(x,t)\Big)^2 + (\lambda+\mu)|dvu|\Big] dx$$
$$+\frac{1}{2}\int_{t-\bar{\tau}}^t \int_{\tilde{\mathcal{O}}} |k(s)| \cdot |u_t(x,s)|^2 dx ds.$$

Theorem 6.4.3. *Assume* (6.2.1)*. Then, for all initial data* $(u_0, u_1) \in C([-\bar{\tau}, 0]; H_0^1(\Omega)^d \times L^2(\Omega)^d)$*, the solution to* (6.4.4) *satisfies the energy decay estimate*

$$\mathcal{E}(t) \leq \bar{C}e^{-\beta^* t}, \quad t \geq 0, \tag{6.4.5}$$

where $\bar{C}$ *is a constant depending on the initial data and* $\beta^* > 0$.

Proof. The proof comes analogously to the one of Theorem 6.4.1. □

Chapter 7

Semilinear evolution equations with memory and time-dependent time delay feedback

In this chapter, we will establish well-posedness and exponential stability for solutions to system (0.2.36) corresponding to sufficiently small initial data. All the results in this chapter are contained in [45].

Now, we start our analysis by writing (0.2.36) in abstract form. First of all, in the spirit of [48], we define an auxiliary function and we give an equivalent formulation of our model (0.2.36). We introduce the energy of the considered model, that takes into account the memory damping and of the time-dependent time delay feedback. Due to the presence of time-variable time delays in the feedback law, we define another auxiliary energy functional, which is instead not needed in [89] since, there, the constant time delay case is considered. Then, introducing suitable spaces, we write our model in abstract form.

As in Dafermos [48], we define the function

$$\eta^t(s) := u(t) - u(t-s), \quad s, t \in (0, +\infty), \tag{7.0.1}$$

so that we can rewrite (0.2.36) in the following way:

$$\begin{aligned} &u_{tt}(t) + (1-\tilde{\beta})Au(t) + \int_0^{+\infty} \beta(s)A\eta^t(s)ds + k(t)BB^*u_t(t-\tau(t)) \\ &\qquad\qquad\qquad\qquad\qquad\qquad = \nabla\psi(u(t)),\ t \in (0,+\infty), \\ &\eta_t^t(s) = -\eta_s^t(s) + u_t(t), \quad t, s \in (0, +\infty), \\ &u(0) = u_0(0), \\ &u_t(t) = g(t), \quad t \in [-\overline{\tau}, 0], \\ &\eta^0(s) = \eta_0(s) = u_0(0) - u_0(-s) \quad s \in (0, +\infty). \end{aligned} \tag{7.0.2}$$

Let us define the energy of the model (0.2.36) (equivalently (7.0.2)) as

$$\begin{aligned} E(t) := E(u(t)) = \frac{1}{2}||u_t(t)||_H^2 + \frac{1-\tilde{\beta}}{2}||A^{\frac{1}{2}}u(t)||_H^2 - \psi(u) \\ +\frac{1}{2}\int_0^{+\infty}\beta(s)||A^{\frac{1}{2}}\eta^t(s)||_H^2 ds + \frac{1}{2}\int_{t-\bar{\tau}}^t |k(s)|\cdot||B^*u_t(s)||_H^2 ds. \end{aligned} \tag{7.0.3}$$

Note that, apart from the last term, this is the natural energy for nonlinear wave-type equations with memory (cf. e.g. [6]). The additional term

$$\frac{1}{2}\int_{t-\bar{\tau}}^t |k(s)|\cdot\|B^*u_t(s)\|_H^2\, ds$$

is crucial in order to deal with the delay feedback in the case of time-varying time delay (cf. [72, 89] for similar terms).

Moreover, let us define the functional

$$\mathcal{E}(t) := \max\left\{\frac{1}{2}\max_{s\in[-\bar{\tau},0]}\|g(s)\|_H^2,\ \max_{s\in[0,t]}E(s)\right\}.$$

In particular, for $t=0$,

$$\mathcal{E}(0) := \max\left\{\frac{1}{2}\max_{s\in[-\bar{\tau},0]}\|g(s)\|_H^2,\ E(0)\right\}.$$

In order to write (7.2.23) as an abstract first-order equation, we introduce the following Hilbert spaces. Let $L^2_\beta((0,+\infty);D(A^{\frac{1}{2}}))$ be the Hilbert space of the $D(A^{\frac{1}{2}})-$valued functions in $(0,+\infty)$ endowed with the scalar product

$$\langle\varphi,\psi\rangle_{L^2_\beta((0,+\infty);D(A^{\frac{1}{2}}))} = \int_0^\infty \beta(s)\langle A^{\frac{1}{2}}\varphi, A^{\frac{1}{2}}\psi\rangle_H ds$$

and denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} = D(A^{\frac{1}{2}})\times H\times L^2_\beta((0,+\infty);D(A^{\frac{1}{2}})),$$

equipped with the inner product

$$\left\langle\begin{pmatrix}u\\v\\w\end{pmatrix},\begin{pmatrix}\tilde{u}\\\tilde{v}\\\tilde{w}\end{pmatrix}\right\rangle_{\mathcal{H}} := (1-\tilde{\beta})\langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}\tilde{u}\rangle_H + \langle v,\tilde{v}\rangle_H + \int_0^\infty \beta(s)\langle A^{\frac{1}{2}}w, A^{\frac{1}{2}}\tilde{w}\rangle_H ds. \tag{7.0.4}$$

Setting $U=(u,u_t,\eta^t)$, we can restate (0.2.36) in the abstract form

$$\begin{aligned} &U'(t) = \mathcal{A}U(t) - k(t)\mathcal{B}U(t-\tau(t)) + F(U(t)),\\ &U(s) = \tilde{g}(s), \quad s\in[-\bar{\tau},0], \end{aligned} \tag{7.0.5}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ -(1-\tilde{\beta})Au - \int_0^{+\infty}\beta(s)Aw(s)ds \\ -w_s + v \end{pmatrix}$$

with domain

$$\begin{aligned} D(\mathcal{A}) = \{&(u,v,w) \in D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}}) \times L^2_\beta((0,+\infty); D(A^{\frac{1}{2}})) : \\ &(1-\tilde{\beta})u + \int_0^{+\infty}\beta(s)w(s)ds \in D(A), \quad w_s \in L^2_\beta((0,+\infty); D(A^{\frac{1}{2}}))\}, \end{aligned} \tag{7.0.6}$$

in the Hilbert space $\mathcal{H}$, and the operator $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{B}\begin{pmatrix} u \\ v \\ w \end{pmatrix} := \begin{pmatrix} 0 \\ BB^*v \\ 0 \end{pmatrix}.$$

Note that, by (0.2.38), it turns out that $\|\mathcal{B}\|_{\mathcal{L}(\mathcal{H})} = b^2$. Moreover, $\tilde{g}(s) = (u_0(0), g(s), \eta_0)$ for $s \in [-\bar{\tau}, 0]$, and we denote $U_0 := \tilde{g}(0) = (u_0(0), u_1, \eta_0)$. Also, $F(U) := (0, \nabla\psi(u), 0)^T$.

Now, from (H2) and (H3) we deduce that the function F satisfies:

(F1) $F(0) = 0$;

(F2) for each $r > 0$ there exists a constant $L(r) > 0$ such that

$$||F(U) - F(V)||_{\mathcal{H}} \le L(r)||U - V||_{\mathcal{H}} \tag{7.0.7}$$

whenever $||U||_{\mathcal{H}} \le r$ and $||V||_{\mathcal{H}} \le r$.

It is well-known (see e.g. [57]) that the operator $\mathcal{A}$ in the problem's formulation (7.0.5), corresponding to the linear undelayed part of the model, generates an exponentially stable semigroup $\{S(t)\}_{t\ge 0}$, namely there exist two constants $M, \omega > 0$ such that

$$||S(t)||_{\mathcal{L}(\mathcal{H})} \le Me^{-\omega t}, \quad \forall t \ge 0. \tag{7.0.8}$$

Our stability results will be obtained under an assumption on the coefficient $k(t)$ of the delay feedback. More precisely, we assume (cf. [72]) that there exist two constants $\omega' \in [0, \omega)$ and $\gamma \in \mathbb{R}$ such that

$$b^2 Me^{\omega\bar{\tau}}\int_0^t |k(s)|ds \le \gamma + \omega' t, \quad \text{for all } t \ge 0. \tag{7.0.9}$$

Note that (7.0.9) includes, as particular cases, k integrable or k in L^∞ with $\|k\|_\infty$ sufficiently small.

7.1 Local well-posedness and preliminary estimates

In this section, we present some preliminary results that will be crucial to prove global well-posedness and exponential stability for (0.2.36).

First of all, the following local well-posedness result holds. As we pointed out in Chapter 6, since we are dealing with time-dependent time delays, we cannot argue employing the classical step-by-step argument that is usually used for delay equations. To establish local well-posedness, we have rather to use a fixed point approach.

Theorem 7.1.1. *Let us consider the system* (7.0.5) *with initial datum* $\tilde{g} \in C([-\bar{\tau}, 0]; \mathcal{H})$. *Then, there exists a unique local solution* $U(\cdot)$ *defined on a time interval* $[0, \delta)$.

Proof. Let $\tilde{g} \in C([-\bar{\tau}, 0]; \mathcal{H})$. We set

$$C := \max\left\{2M \max_{s\in[-\bar{\tau},0]} \|\tilde{g}(s)\|_{\mathcal{H}},\ \max_{s\in[-\bar{\tau},0]} \|\tilde{g}(s)\|_{\mathcal{H}}\right\}.$$

Let $\xi > 0$ be a sufficiently small time such that

$$\xi L(C) + b^2 \|k\|_{\mathcal{L}^1([0,\xi];\mathbb{R})} < \frac{1}{4M}, \tag{7.1.1}$$

with $L(C)$ as in (7.0.7). Let us denote

$$C_{\tilde{g}}([-\bar{\tau}, \xi]; \mathcal{H}) := \{U \in C([-\bar{\tau}, \xi]; \mathcal{H}) : U(s) = \tilde{g}(s),\ \forall s \in [-\bar{\tau}, 0]\}.$$

Note that $C_{\tilde{g}}([-\bar{\tau}, \xi]; \mathcal{H})$ is a nonempty and closed subset of $C([-\bar{\tau}, \xi]; \mathcal{H})$. As a consequence, $(C_{\tilde{g}}([-\bar{\tau}, \xi]; \mathcal{H}), \|\cdot\|_{C([-\bar{\tau},\xi];\mathcal{H})})$ is a Banach space. Moreover, let us denote

$$C^C_{\tilde{g}}([-\bar{\tau}, \xi]; \mathcal{H}) := \{U \in C_{\tilde{g}}([-\bar{\tau}, \xi]; \mathcal{H}) : \|U(t)\|_{\mathcal{H}} \le C,\ \forall t \in [-\bar{\tau}, \xi]\}.$$

Let us note that $C^C_{\tilde{g}}([-\bar{\tau}, \xi]; \mathcal{H})$ is nonempty since it suffices to take

$$U(s) = \begin{cases} U_0, & s \in [0, \xi], \\ \tilde{g}(s), & s \in [-\bar{\tau}, 0), \end{cases}$$

to have that $U \in C_{\tilde{g}}([-\bar{\tau}, \xi]; \mathcal{H})$ and $\|U(t)\|_{\mathcal{H}} \le \|\tilde{g}\|_{C([-\bar{\tau},\xi];\mathcal{H})} \le C$, for all $t \in [-\bar{\tau}, \xi]$. So, U belongs to $C^C_{\tilde{g}}([-\bar{\tau}, \xi]; \mathcal{H})$. Also, it is easy to see that $C^C_{\tilde{g}}([-\bar{\tau}, \xi]; \mathcal{H})$ is closed in $C([-\bar{\tau}, \xi]; \mathcal{H})$. Hence, $(C^C_{\tilde{g}}([-\bar{\tau}, \xi]; \mathcal{H}), \|\cdot\|_{C([-\bar{\tau},\xi];\mathcal{H})})$ is a Banach space too.
Next, we define the map $\Gamma : C^C_{\tilde{g}}([-\bar{\tau}, \xi]; \mathcal{H}) \to C^C_{\tilde{g}}([-\bar{\tau}, \xi]; \mathcal{H})$ given by

$$\Gamma U(t) = \begin{cases} S(t)U_0 + \int_0^t S(t-s)[F(U(s)) + k(s)\mathcal{B}U(s-\tau(s))]\, ds, & t \in (0, \xi], \\ \tilde{g}(t), & t \in [-\bar{\tau}, 0]. \end{cases}$$

We claim that Γ is well-defined. Indeed, let $U \in C^C_{\tilde{g}}([-\bar{\tau}, \xi]; H)$. Then, from the semigroup theory, $t \mapsto S(t)U_0$ is continuous. Also, since $U(\cdot)$ is continuous in $[-\bar{\tau}, \xi]$, $\tau(\cdot)$ is a continuous function and $\mathcal{B}$ is a bounded linear operator from H into itself, $[0, \xi] \ni t \mapsto$

$\mathcal{B}U(t-\tau(t))$ is continuous. Moreover, $k \in \mathcal{L}^1([0,\xi];\mathbb{R})$. So $k(\cdot)\mathcal{B}U(\cdot-\tau(\cdot)) \in \mathcal{L}^1([0,\xi];\mathcal{H})$. Also, since $U(\cdot)$ is continuous in $[-\bar{\tau},\xi]$ and $F(\cdot)$ is locally Lipschitz continuous in $\mathcal{H}$ from (F_2), the map $t \to F(U(t))$ is continuous in $[-\bar{\tau},\xi]$. So, $F((U(\cdot))) \in \mathcal{L}^1([0,\xi];\mathcal{H})$. As a consequence, the map $t \mapsto \int_0^t S(t-s)[F(U(s)) + k(s)\mathcal{B}U(s-\tau(s))]ds$ is continuous in $[0,\xi]$. Thus, $\Gamma U \in C([0,\xi];H)$. Furthermore, $\Gamma U = \tilde{g}$ in $[\bar{\tau},0]$. Finally, for all $t \in [-\bar{\tau},0]$,

$$\|\Gamma U(t)\|_{\mathcal{H}} = \|\tilde{g}(t)\|_{\mathcal{H}} \leq C.$$

On the other hand, for all $t \in (0,\xi]$, from (F_2) with $F(0) = 0$ and $\|U\|_{C([-\bar{\tau},\xi];\mathcal{H})} \leq C$, we can write

$$\begin{aligned}
\|\Gamma U(t)\|_{\mathcal{H}} &\leq Me^{-\omega t}\|U_0\|_{\mathcal{H}} + M\int_0^t e^{-\omega(t-s)}(\|F(U(s))\|_{\mathcal{H}} + b^2|k(s)|\|U(s-\tau(s))\|_{\mathcal{H}})ds \\
&\leq M\|U_0\|_{\mathcal{H}} + M\int_0^t (L(C) + b^2|k(s)|)(\|U(s)\|_{\mathcal{H}} + \|U(s-\tau(s))\|_{\mathcal{H}})ds \\
&\leq M\|U_0\|_{\mathcal{H}} + 2MC(\xi L(C) + b^2\|k\|_{\mathcal{L}^1([0,\xi];\mathbb{R})}).
\end{aligned}$$

Thus, using (7.1.1), by definition of C we get

$$\|\Gamma U(t)\|_{\mathcal{H}} \leq \frac{C}{2} + 2MC(\xi L(C) + b^2\|k\|_{\mathcal{L}^1([0,\xi];\mathbb{R})}) \leq \frac{C}{2} + 2MC\frac{1}{4M} = C.$$

Thus,

$$\|\Gamma U(t)\|_{\mathcal{H}} \leq C, \quad \forall t \in [-\bar{\tau},\xi].$$

So, we can conclude that Γ is well defined.

Next, we claim that Γ is a contraction. Indeed, let $U, V \in C^C_{\tilde{g}}([-\bar{\tau},\xi];H)$. Then, for all $t \in [-\bar{\tau},0]$,

$$\|\Gamma U(t) - \Gamma V(t)\|_{\mathcal{H}} = 0.$$

On the other hand, for all $t \in (0,\xi]$, since $\|U\|_{C([-\bar{\tau},\xi];\mathcal{H})}, \|V\|_{C([-\bar{\tau},\xi];\mathcal{H})} \leq C$, from (F_2) it follows that

$$\begin{aligned}
\|\Gamma U(t) - \Gamma V(t)\|_{\mathcal{H}} &\leq \int_0^t \|S(t-s)\|_{\mathcal{L}(\mathcal{H})}\|F(U(s)) - F(V(s))\|_{\mathcal{H}}ds \\
&\quad + \int_0^t \|S(t-s)\|_{\mathcal{L}(\mathcal{H})}|k(s)|\|\mathcal{B}U(s-\tau(s)) - \mathcal{B}V(s-\tau(s))\|_{\mathcal{H}}ds \\
&\leq M(L(C)\xi + b^2\|k\|_{\mathcal{L}^1([0,\xi];\mathbb{R})})\|U-V\|_{C([-\bar{\tau},\xi];\mathcal{H})}.
\end{aligned}$$

Thus,

$$\|\Gamma U - \Gamma V\|_{C([-\bar{\tau},\xi];\mathcal{H})} \leq M(L(C)\xi + b^2\|k\|_{\mathcal{L}^1([0,\xi];\mathbb{R})})\|U-V\|_{C([-\bar{\tau},\xi];\mathcal{H})}.$$

As a consequence, since from (7.1.1) $M(L(C)\xi + b^2\|k\|_{\mathcal{L}^1([0,T];\mathbb{R})}) < \frac{1}{4} < 1$, the map Γ is a contraction. Thus, from the Banach's Theorem, Γ has a unique fixed point $U \in C^C_{\tilde{g}}([-\bar{\tau},\xi];\mathcal{H})$. So, the fixed point $U \in C^C_{\tilde{g}}([-\bar{\tau},\xi];\mathcal{H})$ is a local solution to (7.2.23) that can be extended to some maximal interval $[0,\delta)$ since $\|U\|_{C([0,\xi];\mathcal{H})} \leq C$.

Now, we prove that the fixed point U is the unique local mild solution to (7.0.5). Indeed, assume that (7.0.5) has another local mild solution V defined in a time interval $[0, \delta')$. Let $t_0 > 0$ be such that both U and V are defined in the time interval $[0, t_0]$. We denote with $c := \max\{\|U\|_{C([-\bar{\tau}, t_0]; \mathcal{H})}, \|V\|_{C([-\bar{\tau}, t_0]; \mathcal{H})}\}$. Then, for every $t \in [0, t_0]$, we have that

$$\begin{aligned}\|U(t) - V(t)\|_{\mathcal{H}} \le M \int_0^t b^2 |k(s)| \|U(s - \tau(s)) - V(s - \tau(s))\|_{\mathcal{H}} ds \\ + ML(c) \int_0^t \|U(s) - V(s)\|_{\mathcal{H}} ds \\ \le M \int_0^t (b^2 |k(s)| + L(c)) \max_{r \in [s - \bar{\tau}, s]} \|U(r) - V(r)\|_{\mathcal{H}} ds,\end{aligned}$$

from which

$$\max_{r \in [t - \bar{\tau}, t]} \|U(r) - V(r)\|_{\mathcal{H}} \le M \int_0^t (b^2 |k(s)| + L(c)) \max_{r \in [s - \bar{\tau}, s]} \|U(r) - V(r)\|_{\mathcal{H}} ds.$$

Thus, the Gronwall's estimate yields

$$\max_{r \in [t - \bar{\tau}, t]} \|U(r) - V(r)\|_{\mathcal{H}} \le 0,$$

and

$$\|U(t) - V(t)\|_{\mathcal{H}} = 0, \quad \forall t \in [0, t_0].$$

So, U and V coincide on every closed interval $[0, t_0]$ in which they both exist. Then, $\delta = \delta'$ and U is the unique local mild solution to (7.2.23). □

Remark 7.1.2. Assume that the time delay function $\tau(\cdot)$ is bounded from below by a positive constant, namely

$$\tau(t) \ge \tau_0, \quad \forall t \ge 0, \tag{7.1.2}$$

for some $\tau_0 > 0$. In this case, Theorem 7.1.1 can be proved in a simpler way. Indeed, in $[0, \tau_0]$, we can rewrite the abstract system (7.0.5) as an undelayed problem:

$$\begin{aligned} U'(t) &= \mathcal{A}U(t) - k(t)\mathcal{B}\tilde{g}(t - \tau(t)) + F(U(t)), \quad t \in (0, \tau_0), \\ U(0) &= U_0. \end{aligned}$$

Then, we can apply the classical theory of nonlinear semigroups (see e.g. [92, 91]) obtaining the existence of a unique solution on a set $[0, \delta)$, with $\delta \le \tau_0$.

Now, we present some preliminary estimates.

Lemma 7.1.3. *Let $u : [0, T) \to \mathbb{R}$ be a solution of* (0.2.36). *Assume that*

$$E(t) \ge \frac{1}{4} ||u_t(t)||_H^2, \quad \forall t \ge 0. \tag{7.1.3}$$

Then,

$$E(t) \le \bar{C}(t) \mathcal{E}(0), \quad \forall t \ge 0, \tag{7.1.4}$$

where

$$\bar{C}(t) = e^{3b^2 \int_0^t |k(s)| ds}. \tag{7.1.5}$$

Proof. Differentiating the energy, we obtain

$$\frac{dE(t)}{dt} = \langle u_t(t), u_{tt}(t)\rangle_H + (1-\tilde{\beta})\langle A^{\frac{1}{2}}u(t), A^{\frac{1}{2}}u_t(t)\rangle_H - \langle \nabla\psi(u(t)), u_t(t)\rangle_H$$
$$+\frac{1}{2}|k(t)|\cdot||B^*u_t(t)||_H^2 - \frac{1}{2}|k(t-\bar{\tau})|\cdot||B^*u_t(t-\bar{\tau})||_H^2$$
$$+\int_0^{+\infty}\beta(s)\langle A^{\frac{1}{2}}\eta^t(s), A^{\frac{1}{2}}\eta_t^t(s)\rangle_H ds.$$

Then, since from (7.0.2) it holds that

$$u_{tt}(t) = \nabla\psi(u(t)) - (1-\tilde{\beta})Au(t) - \int_0^{+\infty}\beta(s)A\eta^t(s)ds - k(t)BB^*u_t(t-\tau(t)),$$

we get

$$\frac{dE(t)}{dt} = \langle u_t(t), \nabla\psi(u(t))\rangle_H - (1-\tilde{\beta})\langle u_t(t), Au(t)\rangle_H - \int_0^{+\infty}\beta(s)\langle u_t(t), A\eta^t(s)\rangle_H ds$$
$$-k(t)\langle u_t(t), BB^*u_t(t-\tau(t))\rangle_H + (1-\tilde{\beta})\langle A^{\frac{1}{2}}u(t), A^{\frac{1}{2}}u_t(t)\rangle_H - \langle\nabla\psi(u(t)), u_t(t)\rangle_H$$
$$+\frac{1}{2}|k(t)|\cdot||B^*u_t(t)||_H^2 - \frac{1}{2}|k(t-\bar{\tau})|\cdot||B^*u_t(t-\bar{\tau})||_H^2$$
$$+\int_0^{+\infty}\beta(s)\langle A^{\frac{1}{2}}\eta^t(s), A^{\frac{1}{2}}\eta_t^t(s)\rangle_H ds.$$

Let us note that, being A a self-adjoint positive operator, also $A^{\frac{1}{2}}$ is self-adjoint. This together with the second inequality in (7.0.2), i.e. $\eta_t^t = -\eta_s^t + u_t$, yields

$$\frac{dE(t)}{dt} = -\int_0^{+\infty}\beta(s)\langle u_t(t), A\eta^t(s)\rangle_H ds - k(t)\langle u_t(t), BB^*u_t(t-\tau(t))\rangle_H$$
$$+\frac{1}{2}|k(t)|\cdot||B^*u_t(t)||_H^2 - \frac{1}{2}|k(t-\bar{\tau})|\cdot||B^*u_t(t-\bar{\tau})||_H^2$$
$$+\int_0^{+\infty}\beta(s)\langle A\eta^t(s), \eta_t^t(s)\rangle_H ds$$
$$= -k(t)\langle u_t(t), BB^*u_t(t-\tau(t))\rangle_H + \frac{1}{2}|k(t)|\cdot||B^*u_t(t)||_H^2$$
$$-\frac{1}{2}|k(t-\bar{\tau})|\cdot||B^*u_t(t-\bar{\tau})||_H^2 - \int_0^{+\infty}\beta(s)\langle A\eta^t(s), \eta_s^t(s)\rangle_H ds.$$

Now, we claim that

$$\int_0^{+\infty}\beta(s)\langle\eta_s^t, A\eta^t(s)\rangle_H ds \geq 0. \tag{7.1.6}$$

Indeed, since $A^{\frac{1}{2}}$ is self-adjoint, we can write

$$\frac{1}{2}\frac{d}{ds}||A^{\frac{1}{2}}\eta^t(s)||_H^2 = \langle A^{\frac{1}{2}}\eta_s^t, A^{\frac{1}{2}}\eta^t(s)\rangle_H = \langle\eta_s^t, A\eta^t(s)\rangle_H.$$

Thus, since $\eta^t(0) = 0$ and $\beta(t)||A^{\frac{1}{2}}\eta^t(s)||_H^2 \to 0$, as $t \to +\infty$, (see [56] for details) it comes that

$$\int_0^{+\infty} \beta(s)\langle \eta_s^t, A\eta^t(s)\rangle_H ds = -\frac{1}{2}\int_0^{+\infty} \beta'(s)||A^{\frac{1}{2}}\eta^t(s)||_H^2 ds.$$

Finally, using again (iv) on the memory kernel $\beta(\cdot)$, we can say that

$$\begin{aligned}\int_0^{+\infty} \beta(s)\langle \eta_s^t, A\eta^t(s)\rangle_H ds &= -\frac{1}{2}\int_0^{+\infty} \beta'(s)||A^{\frac{1}{2}}\eta^t(s)||_H^2 ds \\ &\geq \frac{\delta}{2}\int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta^t(s)||_H^2 ds \geq 0,\end{aligned},$$

which proves (7.1.6).
Next, from (7.1.6), we can estimate the derivative of the energy in the following way:

$$\begin{aligned}\frac{dE(t)}{dt} &\leq -k(t)\langle u_t(t), BB^*u_t(t-\tau(t))\rangle_H + \frac{1}{2}|k(t)|\cdot||B^*u_t(t)||_H^2 - \frac{1}{2}|k(t-\bar{\tau})|\cdot||B^*u_t(t-\bar{\tau})||_H^2 \\ &\leq -k(t)\langle u_t(t), BB^*u_t(t-\tau(t))\rangle_H + \frac{1}{2}|k(t)|\cdot||B^*u_t(t)||_H^2.\end{aligned}$$

Therefore, using the definition of adjoint and Young inequality, we get

$$\begin{aligned}\frac{dE(t)}{dt} &\leq -k(t)\langle B^*u_t(t), B^*u_t(t-\tau(t))\rangle_H + \frac{1}{2}|k(t)|\cdot||B^*u_t(t)||_H^2 \\ &\leq \frac{1}{2}|k(t)|\cdot||B^*u_t(t)||_H^2 + \frac{1}{2}|k(t)|\cdot||B^*u_t(t-\tau(t))||_H^2 + \frac{1}{2}|k(t)|\cdot||B^*u_t(t)||_H^2 \\ &\leq \frac{3}{2}|k(t)|\max_{s\in[t-\bar{\tau},t]}||B^*u_t(s)||_H^2.\end{aligned}$$

Now, let us note that, from (7.1.3), for $t \geq \bar{\tau}$ it holds that

$$\max_{s\in[t-\bar{\tau},t]}\{||B^*u_t(s)||_H^2\} \leq \max_{s\in[0,t]}\{||B^*u_t(s)||_H^2\} \leq b^2\max_{s\in[0,t]}\{||u_t(s)||_H^2\} \leq 2b^2\max_{s\in[0,t]}E(s) \leq 2b^2\mathcal{E}(t).$$

On the other hand, if $t \in [0,\bar{\tau})$, using again (7.1.3), or

$$\max_{s\in[t-\bar{\tau},t]}\{||B^*u_t(s)||_H^2\} = \max_{s\in[0,t]}\{||B^*u_t(s)||_H^2\} \leq 2b^2\mathcal{E}(t),$$

or

$$\max_{s\in[t-\bar{\tau},t]}\{||B^*u_t(s)||_H^2\} = \max_{s\in[-\bar{\tau},0]}\{||B^*u_t(s)||_H^2\} \leq b^2\max_{s\in[-\bar{\tau},0]}\{||g(s)||_H^2\} \leq 2b^2\mathcal{E}(t).$$

Therefore,

$$\max_{s\in[t-\bar{\tau},t]}\{||B^*u_t(s)||_H^2\} \leq 2b^2\mathcal{E}(t), \quad \forall t \geq 0,$$

from which

$$\frac{dE(t)}{dt} \leq 3b^2|k(t)|\mathcal{E}(t), \quad \forall t \geq 0.$$

As a consequence, since $\mathcal{E}(t)$ is constant or increases like $E(t)$, it turns out that

$$\frac{d\mathcal{E}(t)}{dt} \leq 3b^2|k(t)|\mathcal{E}(t), \quad \forall t \geq 0.$$

Then, the Gronwall's inequality yields

$$\mathcal{E}(t) \leq e^{3b^2 \int_0^t |k(s)|ds}\mathcal{E}(0).$$

By definition of $\mathcal{E}(t)$, we finally get

$$E(t) \leq \mathcal{E}(t) \leq e^{3b^2 \int_0^t |k(s)|ds}\mathcal{E}(0),$$

from which

$$E(t) \leq e^{3b^2 \int_0^t |k(s)|ds}\mathcal{E}(0),$$

that ends the proof. □

The following lemma allows us to find a bound from below on the energy of the model, provided that a smallness condition is satisfied by the initial data.

Lemma 7.1.4. *Let $U(t) = (u(t), u_t(t), \eta^t)$ be a non-zero solution to (7.0.5) defined on an interval $[0, \delta)$, and let $T > \delta$. Let h be the strictly increasing function appearing in (0.2.40).*

1. *If $h(||A^{\frac{1}{2}}u_0(0)||_H) < \frac{1-\tilde{\beta}}{2}$, then $E(0) > 0$.*
2. *Assume that $h(||A^{\frac{1}{2}}u_0(0)||_H) < \frac{1-\tilde{\beta}}{2}$ and that*

$$h\left(\frac{2}{(1-\tilde{\beta})^{\frac{1}{2}}}C^{\frac{1}{2}}\mathcal{E}^{\frac{1}{2}}(0)\right) < \frac{1-\tilde{\beta}}{2}, \tag{7.1.7}$$

for some positive constant $C \geq \bar{C}(T)$, with $\bar{C}(\cdot)$ defined in (7.1.5). Then

$$\begin{aligned} E(t) > \frac{1}{4}||u_t(t)||_H^2 + \frac{1-\tilde{\beta}}{4}||A^{\frac{1}{2}}u(t)||_H^2 + \frac{1}{4}\int_{t-\bar{\tau}}^t |k(s)| \cdot ||B^*u_t(s)||_H^2 ds \\ + \frac{1}{4}\int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta^t(s)||_H^2 ds, \end{aligned} \tag{7.1.8}$$

for all $t \in [0, \delta)$. In particular,

$$E(t) > \frac{1}{4}\|U(t)\|_{\mathcal{H}}^2, \quad \forall t \in [0, \delta). \tag{7.1.9}$$

Remark 7.1.5. Let us note that (7.1.7) implies that

$$h\left(\frac{2}{(1-\tilde{\beta})^{\frac{1}{2}}}\bar{C}^{\frac{1}{2}}(T)\mathcal{E}^{\frac{1}{2}}(0)\right) < \frac{1-\tilde{\beta}}{2}, \tag{7.1.10}$$

being the positive constant C in (7.1.7) bigger or equal than $\bar{C}(T)$ and being the function h strictly increasing.

Proof. From the assumption (H3) on the function ψ, we can write

$$\begin{aligned}|\psi(u)| &\leq \int_0^1 |\langle \nabla\psi(su), u\rangle_H| ds \\ &\leq ||A^{\frac{1}{2}}u||_H^2 \int_0^1 h(s||A^{\frac{1}{2}}u||_H) s ds \\ &\leq h(||A^{\frac{1}{2}}u||_H)||A^{\frac{1}{2}}u||_H^2 \int_0^1 s ds = \frac{1}{2} h(||A^{\frac{1}{2}}u||_H)||A^{\frac{1}{2}}u||_H^2,\end{aligned} \tag{7.1.11}$$

where we used the fact that h is a strictly increasing function and the fact that $||u||_{D(A^{\frac{1}{2}})} = (1-\tilde{\beta})||A^{\frac{1}{2}}u||_H$ with $\tilde{\beta} < 1$.
Now, being U a non-zero solution to (7.0.5), the initial datum $\tilde{g}$ satisfies $\mathcal{B}\tilde{g} \neq 0$. Indeed, if the initial datum $\tilde{g}$ is such that $\mathcal{B}\tilde{g} \equiv 0$, then the unique solution to (7.0.5) is $U \equiv 0$. As a consequence $B^*g \neq 0$ since, otherwise, being B a linear operator, we would have $0 = BB^*g = \mathcal{B}\tilde{g}$. Hence, from the assumption $h(\|A^{\frac{1}{2}}u_0(0)\|_H) < \frac{1-\tilde{\beta}}{2}$ and from (7.1.11), we have that

$$\begin{aligned}E(0) &= \frac{1}{2}||u_1||_H^2 + \frac{1-\tilde{\beta}}{2}||A^{\frac{1}{2}}u_0(0)||_H^2 - \psi(u_0(0)) + \frac{1}{2}\int_{-\bar{\tau}}^0 |k(s)|\cdot||B^*g(s)||_H^2 ds \\ &\quad + \frac{1}{2}\int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta_0(s)||_H^2 ds \\ &\geq \frac{1}{2}||u_1||_H^2 + \frac{1-\tilde{\beta}}{2}||A^{\frac{1}{2}}u_0(0)||_H^2 - \frac{1}{2}h(||A^{\frac{1}{2}}u_0(0)||_H)||A^{\frac{1}{2}}u_0(0)||_H^2 \\ &\quad + \frac{1}{2}\int_{-\bar{\tau}}^0 |k(s)|\cdot||B^*g(s)||_H^2 ds + \frac{1}{2}\int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta_0(s)||_H^2 ds \\ &\geq \frac{1}{2}||u_1||_H^2 + \frac{1-\tilde{\beta}}{4}||A^{\frac{1}{2}}u_0(0)||_H^2 + \frac{1}{2}\int_{-\bar{\tau}}^0 |k(s)|\cdot||B^*g(s)||_H^2 ds \\ &\quad + \frac{1}{2}\int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta_0(s)||_H^2 ds > 0.\end{aligned}$$

So, the claim 1 is proven.
In order to prove the second statement, we argue by contradiction. Let us denote

$$r := \sup\{s \in [0,\delta) : \text{(7.1.8) holds, } \forall t \in [0,s)\}.$$

We suppose by contradiction that $r < \delta$. Then, by continuity, it holds

$$\begin{aligned}E(r) &= \frac{1}{4}||u_t(r)||_H^2 + \frac{1-\tilde{\beta}}{4}||A^{\frac{1}{2}}u(r)||_H^2 + \frac{1}{4}\int_{r-\bar{\tau}}^r |k(s)|\cdot||B^*u_t(s)||_H^2 ds \\ &\quad + \frac{1}{4}\int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta^r(s)||_H^2 ds.\end{aligned} \tag{7.1.12}$$

In particular, (7.1.12) implies that

$$\frac{1-\tilde{\beta}}{4}\|A^{\frac{1}{2}}u(r)\|_H^2 \leq E(r).$$

Also, by definition of r, for all $t \in [0, r]$,

$$\begin{aligned} E(t) \geq & \frac{1}{4}||u_t(t)||_H^2 + \frac{1-\tilde{\beta}}{4}||A^{\frac{1}{2}}u(t)||_H^2 \\ & + \frac{1}{4}\int_{t-\bar{\tau}}^{t} |k(s)| \cdot ||B^* u_t(s)||_H^2 ds + \frac{1}{4}\int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta^t(s)||_H^2 ds \\ \geq & \frac{1}{4}||u_t(t)||_H^2. \end{aligned}$$

Thus, the assumption (7.1.3) of Lemma 7.1.3 is satisfied and we can write

$$E(t) \leq \bar{C}(t)\mathcal{E}(0), \quad \forall t \in [0, r],$$

from which, being $\bar{C}(t) \leq \bar{C}(T)$, it comes that

$$E(t) \leq \bar{C}(T)\mathcal{E}(0), \quad \forall t \in [0, r].$$

In particular, for $t = r$,

$$E(r) \leq \bar{C}(T)\mathcal{E}(0).$$

As a consequence,

$$\frac{1-\tilde{\beta}}{4}||A^{\frac{1}{2}}u(r)||_H^2 \leq E(r) \leq \bar{C}(T)\mathcal{E}(0).$$

Thus, since h is strictly increasing, from (7.1.7) (which implies (7.1.10)) we have that

$$h(||A^{\frac{1}{2}}u(r)||_H) \leq h\left(\frac{2}{(1-\tilde{\beta})^{\frac{1}{2}}}\bar{C}^{\frac{1}{2}}(T)\mathcal{E}^{\frac{1}{2}}(0)\right) < \frac{1-\tilde{\beta}}{2}. \qquad (7.1.13)$$

Finally, using (7.1.11) and (7.1.13) we can conclude that

$$\begin{aligned} E(r) = & \frac{1}{2}||u_t(r)||_H^2 + \frac{1-\tilde{\beta}}{2}||A^{\frac{1}{2}}u(r)||_H^2 - \psi(u(r)) + \frac{1}{2}\int_{r-\bar{\tau}}^{r} |k(s)| \cdot ||B^* u_t(s)||_H^2 ds \\ & + \frac{1}{2}\int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta^r(s)||_H^2 ds \\ > & \frac{1}{4}||u_t(r)||_H^2 + \frac{1-\tilde{\beta}}{4}||A^{\frac{1}{2}}u(r)||_H^2 + \frac{1}{4}\int_{r-\bar{\tau}}^{r} |k(s)| \cdot ||B^* u_t(s)||_H^2 ds \\ & + \frac{1}{4}\int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta^r(s)||_H^2 ds. \end{aligned}$$

This contradicts the maximality of r. So, $r = \delta$ and the proof is completed. □

7.2 Global well-posedness and stability

Now, we prove our main results. First, we give a stability result for the abstract model (7.0.5) under suitable assumptions. More precisely, we prove an exponential stability estimate for solutions to (7.0.5) corresponding to *small* initial data.

Theorem 7.2.1. *Assume* (7.0.9). *Let* $\tilde{g} \in C([-\bar{\tau}, 0]; \mathcal{H})$ *and let* U *be a solution to* (7.0.5) *with the initial datum* $\tilde{g}$*, defined in a time interval* $[0, T]$*,* $T > 0$*, that satisfies*

$$\|U(t)\|_{\mathcal{H}} \leq C, \quad \forall t \in [0, T], \tag{7.2.1}$$

for some $C > 0$ *such that* $L(C) < \frac{\omega - \omega'}{M}$.
Then, U *satisfies the exponential decay estimate*

$$\|U(t)\|_{\mathcal{H}} \leq M e^{\gamma} \left(||U_0||_{\mathcal{H}} + e^{\omega\bar{\tau}} K b^2 \max_{r \in [-\bar{\tau}, 0]} \{ e^{\omega r} ||\tilde{g}(r)||_{\mathcal{H}} \} \right) e^{-(\omega - \omega' - ML(C))t}, \tag{7.2.2}$$

for all $t \in [0, T]$.

Proof. Let $\tilde{g} \in C([-\bar{\tau}, 0]; \mathcal{H})$. Let U be a solution to (7.0.5) with the initial datum $\tilde{g}$. From Duhamel's formula, for all $t \in [0, T]$ we have

$$U(t) = S(t)U_0 + \int_0^t S(t - s)[-k(s)\mathcal{B}U(s - \tau(s)) + F(U(s))]ds.$$

Thus, using (7.0.8), we get

$$\begin{aligned} ||U(t)||_{\mathcal{H}} \leq\ & ||S(t)||_{\mathcal{L}(\mathcal{H})} ||U_0||_{\mathcal{H}} + \int_0^t ||S(t - s)||_{\mathcal{L}(\mathcal{H})} |k(s)| \cdot ||\mathcal{B}U(s - \tau(s))||_{\mathcal{H}} ds \\ & + \int_0^t ||S(t - s)||_{\mathcal{L}(\mathcal{H})} ||F(U(s))||_{\mathcal{H}} ds \\ \leq\ & M e^{-\omega t} ||U_0||_{\mathcal{H}} + M e^{-\omega t} \int_0^t e^{\omega s} |k(s)| \cdot ||\mathcal{B}U(s - \tau(s))||_{\mathcal{H}} ds \\ & + M e^{-\omega t} \int_0^t e^{\omega s} ||F(U(s))||_{\mathcal{H}} ds. \end{aligned}$$

Now, using the assumptions (F_1) and (F_2) on F and taking into account of (7.2.1), we can write

$$||F(U(s))||_{\mathcal{H}} = ||F(U(s)) - F(0)||_{\mathcal{H}} \leq L(C) ||U(s)||_{\mathcal{H}}.$$

This last fact together with (0.2.37) implies that

$$\begin{aligned} ||U(t)||_{\mathcal{H}} &\le Me^{-\omega t}||U_0||_{\mathcal{H}} + Me^{-\omega t}\int_0^t e^{\omega s}|k(s)|\cdot||\mathcal{B}U(s-\tau(s))||_{\mathcal{H}}ds \\ &\quad + ML(C)e^{-\omega t}\int_0^t e^{\omega s}||U(s)||_{\mathcal{H}}ds \\ &\le Me^{-\omega t}||U_0||_{\mathcal{H}} + Me^{-\omega t}e^{\omega\bar{\tau}}\int_0^t e^{\omega(s-\tau(s))}|k(s)|\cdot||\mathcal{B}U(s-\tau(s))||_{\mathcal{H}}ds \\ &\quad + ML(C)e^{-\omega t}\int_0^t e^{\omega s}||U(s)||_{\mathcal{H}}ds. \end{aligned} \tag{7.2.3}$$

Now, we assume $t \ge \bar{\tau}$. We split

$$\begin{aligned} \int_0^t e^{\omega(s-\tau(s))}|k(s)|\cdot||\mathcal{B}U(s-\tau(s))||_{\mathcal{H}}ds &= \int_0^{\bar{\tau}} e^{\omega(s-\tau(s))}|k(s)|\cdot||\mathcal{B}U(s-\tau(s))||_{\mathcal{H}}ds \\ &\quad + \int_{\bar{\tau}}^t e^{\omega(s-\tau(s))}|k(s)|\cdot||\mathcal{B}U(s-\tau(s))||_{\mathcal{H}}ds. \end{aligned} \tag{7.2.4}$$

We first estimate, using (0.2.39) with $t = \bar{\tau}$,

$$\begin{aligned} &\int_0^{\bar{\tau}} e^{\omega(s-\tau(s))}|k(s)|\cdot||\mathcal{B}U(s-\tau(s))||_{\mathcal{H}}ds \\ &\quad \le b^2\int_0^{\bar{\tau}}|k(s)|\left(\max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\} + \max_{r\in[0,s]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\}\right)ds \\ &\quad \le Kb^2\max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\} + b^2\int_0^{\bar{\tau}}|k(s)|\max_{r\in[0,s]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\}ds. \\ &\quad = Kb^2\max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\} + b^2\int_0^{\bar{\tau}}|k(s)|\max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\}ds. \end{aligned}$$

Also,

$$\begin{aligned} \int_{\bar{\tau}}^t e^{\omega(s-\tau(s))}|k(s)|\cdot||\mathcal{B}U(s-\tau(s))||_{\mathcal{H}}ds &\le b^2\int_{\bar{\tau}}^t e^{\omega(s-\tau(s))}|k(s)|\cdot||U(s-\tau(s))||_{\mathcal{H}}ds \\ &\le b^2\int_{\bar{\tau}}^t |k(s)|\max_{r\in[s-\bar{\tau},s]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\}ds \\ &= b^2\int_{\bar{\tau}}^t |k(s)|\max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\}ds. \end{aligned}$$

Therefore, (7.2.4) becomes

$$\begin{aligned} \int_0^t e^{\omega(s-\tau(s))}|k(s)|\cdot||\mathcal{B}U(s-\tau(s))||_{\mathcal{H}}ds &\le Kb^2\max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\} \\ &\quad + \int_0^t b^2|k(s)|\max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\}ds, \end{aligned} \tag{7.2.5}$$

for all $t \geq \bar{\tau}$.
On the other hand, if $t < \bar{\tau}$, using (0.2.39) it rather holds

$$\int_0^t e^{\omega(s-\tau(s))}|k(s)| \cdot ||\mathcal{B}U(s-\tau(s))||_{\mathcal{H}} ds$$

$$\leq b^2 \int_0^t |k(s)| \max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\}ds + b^2 \int_0^t |k(s)| \max_{r\in[0,s]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\}ds$$

$$= b^2 \int_0^t |k(s)| \max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\}ds + b^2 \int_0^t |k(s)| \max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\}ds$$

$$\leq b^2 \int_0^{\bar{\tau}} |k(s)| \max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\}ds + b^2 \int_0^t |k(s)| \max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\}ds$$

$$\leq Kb^2 \max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\} + b^2 \int_0^t |k(s)| \max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\}ds.$$

So, (7.2.5) holds for every $t \in [0,T]$. Putting (7.2.5) in (7.2.3), we can write

$$||U(t)||_{\mathcal{H}} \leq Me^{-\omega t}\left(||U_0||_{\mathcal{H}} + e^{\omega\bar{\tau}}Kb^2 \max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\}\right)$$

$$+Me^{-\omega t}e^{\omega\bar{\tau}}b^2 \int_0^t |k(s)| \max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\}ds + ML(C)e^{-\omega t}\int_0^t e^{\omega s}||U(s)||_{\mathcal{H}} ds,$$

$$\leq Me^{-\omega t}\left(||U_0||_{\mathcal{H}} + e^{\omega\bar{\tau}}Kb^2 \max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\}\right)$$

$$+e^{-\omega t}\int_0^t (Me^{\omega\bar{\tau}}b^2|k(s)| + ML(C)) \max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\}ds,$$

from which

$$e^{\omega t}||U(t)||_{\mathcal{H}} \leq M\left(||U_0||_{\mathcal{H}} + e^{\omega\bar{\tau}}Kb^2 \max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\}\right)$$

$$+ \int_0^t (Me^{\omega\bar{\tau}}b^2|k(s)| + ML(C)) \max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\}ds,$$

for all $t \in [0,T]$. Thus,

$$\max_{r\in[t-\bar{\tau},t]\cap[0,t]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\} \leq M\left(||U_0||_{\mathcal{H}} + e^{\omega\bar{\tau}}Kb^2 \max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\}\right)$$

$$+ \int_0^t (Me^{\omega\bar{\tau}}b^2|k(s)| + ML(C)) \max_{r\in[s-\bar{\tau},s]\cap[0,s]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\}ds.$$

Hence, denoted with

$$\tilde{U}(t) := \max_{r\in[t-\bar{\tau},t]\cap[0,t]}\{e^{\omega r}||U(r)||_{\mathcal{H}}\},$$

using Gronwall's inequality we get

$$\|\tilde{U}(t)\|_{\mathcal{H}} \leq M\left(||U_0||_{\mathcal{H}} + e^{\omega\bar{\tau}}Kb^2 \max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\}\right) e^{Mb^2 e^{\omega\bar{\tau}}\int_0^t |k(s)|ds + ML(C)t}.$$

Finally,

$$e^{\omega t}\|U(t)\|_{\mathcal{H}} \leq M\left(||U_0||_{\mathcal{H}} + e^{\omega\bar{\tau}}Kb^2 \max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\}\right) e^{Mb^2 e^{\omega\bar{\tau}}\int_0^t |k(s)|ds + ML(C)t},$$

which implies that

$$\|U(t)\|_{\mathcal{H}} \leq M\left(||U_0||_{\mathcal{H}} + e^{\omega\bar{\tau}}Kb^2 \max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\}\right) e^{Mb^2 e^{\omega\bar{\tau}}\int_0^t |k(s)|ds + ML(C)t} e^{-\omega t}.$$

From (7.0.9), we can conclude that

$$\|U(t)\|_{\mathcal{H}} \leq Me^{\gamma}\left(||U_0||_{\mathcal{H}} + e^{\omega\bar{\tau}}Kb^2 \max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\}\right) e^{-(\omega-\omega'-ML(C))t},$$

for all $t \in [0,T]$, which proves the exponential decay estimate (7.2.2). □

Now, we prove that, for sufficiently small initial data, the system (0.2.36) has a unique global solution that satisfies (7.2.1), whenever the coefficient of the delay feedback $k(t)$ satisfies (7.0.9).

Theorem 7.2.2. *Assume* (7.0.9). *There exist $\rho > 0$ and $C_\rho > 0$, with $L(C_\rho) < \frac{\omega-\omega'}{2M}$, for which, if $\tilde{g} = (u_0(0), g, \eta_0)$ is such that*

$$||u_1||_H^2 + (1-\tilde{\beta})||A^{\frac{1}{2}}u_0(0)||_H^2 + \int_{-\bar{\tau}}^0 |k(s)| \cdot ||B^* g(s)||_H^2 + \int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta_0(s)||_H^2 ds < \rho^2 \tag{7.2.6}$$

and

$$\max_{s\in[-\bar{\tau},0]} ||g(s)||_H < \rho, \tag{7.2.7}$$

then the problem (7.0.2) *with the initial datum $\tilde{g}$ has a unique global solution $U \in C([0,+\infty)),\mathcal{H})$ that satisfies*

$$\|U(t)\|_{\mathcal{H}} \leq C_\rho, \quad \forall t \geq 0. \tag{7.2.8}$$

Proof. Let us fix a time T sufficiently large, $T \geq \bar{\tau}$, such that

$$C_T := 4M^2 e^{2\gamma} \max\left\{(1+Kb^2e^{\omega\bar{\tau}}),\, e^{\omega\bar{\tau}}\right\}\left(1+e^{2\omega\bar{\tau}}K^2b^4\right)e^{-(\omega-\omega')T} < 1. \tag{7.2.9}$$

Also, we set

$$C_T^* := \sup\left\{e^{3b^2\int_{nT}^{(n+1)T}|k(s)|ds} : n \in \mathbb{N}\right\}. \tag{7.2.10}$$

The assumption (0.2.39) ensures that $C_T^* < +\infty$. Note that $C_T^* \geq \bar{C}(T)$, where $\bar{C}(T)$ is defined in (7.1.5). Now, we pick $\rho > 0$ in such a way that

$$\rho \leq \frac{(1-\tilde{\beta})^{\frac{1}{2}}}{2(C_T^*)^{\frac{1}{2}}} h^{-1}\left(\frac{1-\tilde{\beta}}{2}\right). \tag{7.2.11}$$

Let $(u_0(0), u_1, \eta_0)$ and g be such that (7.2.6) and (7.2.7) holds true. Then, the initial datum $\tilde{g}$ satisfies the following smallness condition:

$$\max_{s\in[-\bar{\tau},0]} ||\tilde{g}(s)||_{\mathcal{H}} \leq \sqrt{2}\rho. \tag{7.2.12}$$

Now, from Theorem 7.1.1 there exists a unique local solution $U(\cdot)$ to (7.0.5) with the initial datum $\tilde{g}(s)$, $s \in [-\bar{\tau}, 0]$ which is defined on a time interval $[0, \delta)$ and that satisfies the Duhamel's formula

$$U(t) = S(t)U_0 + \int_0^t S(t-s)[-k(s)\mathcal{B}U(s-\tau(s)) + F(U(s))]\, ds, \tag{7.2.13}$$

for all $t \in [0, \delta)$. Without loss of generality, we can suppose that $\delta \leq T$. Also, we can assume that U is a non-zero solution to (7.0.5), since, otherwise, (7.2.8) is trivially satisfied. Thus, using (7.2.11), the assumption (7.2.6) on the initial data, and recalling that h is a strictly increasing function, we get

$$\begin{aligned} h(||A^{\frac{1}{2}}u_0(0)||_H) &< h\left(\frac{\rho}{(1-\tilde{\beta})^{\frac{1}{2}}}\right) \leq h\left(\frac{1}{2(C_T^*)^{\frac{1}{2}}} h^{-1}\left(\frac{1-\tilde{\beta}}{2}\right)\right) \\ &\leq h\left(h^{-1}\left(\frac{1-\tilde{\beta}}{2}\right)\right) = \frac{1-\tilde{\beta}}{2}, \end{aligned}$$

were in the above inequality we used the fact that $2(C_T^*)^{\frac{1}{2}} \geq 1$. Hence, since we have $h(||A^{\frac{1}{2}}u_0(0)||_H) < \frac{1-\tilde{\beta}}{2}$, from Lemma 7.1.4 it follows that $E(0) > 0$.

Also, using (7.1.11) and the fact that $h(||A^{\frac{1}{2}}u_0(0)||_H) < \frac{1-\tilde{\beta}}{2}$, we can write

$$\begin{aligned}
E(0) &= \frac{1}{2}||u_1||_H^2 + \frac{1-\tilde{\beta}}{2}||A^{\frac{1}{2}}u_0(0)||_H^2 - \psi(u_0(0)) + \frac{1}{2}\int_{-\bar{\tau}}^0 |k(s)|\cdot||B^*u_t(s)||_H^2 ds \\
&\quad + \frac{1}{2}\int_0^{+\infty}\beta(s)||A^{\frac{1}{2}}\eta_0(s)||_H^2 ds \\
&\le \frac{1}{2}||u_1||_H^2 + \frac{1-\tilde{\beta}}{2}||A^{\frac{1}{2}}u_0(0)||_H^2 + \frac{1}{2}h(||A^{\frac{1}{2}}u_0(0)||_H)||A^{\frac{1}{2}}u_0(0)||_H^2 \\
&\quad + \frac{1}{2}\int_{-\bar{\tau}}^0 |k(s)|\cdot||B^*g(s)||_H^2 ds + \frac{1}{2}\int_0^{+\infty}\beta(s)||A^{\frac{1}{2}}\eta_0(s)||_H^2 ds \\
&\le \frac{1}{2}||u_1||_H^2 + \frac{1-\tilde{\beta}}{2}||A^{\frac{1}{2}}u_0(0)||_H^2 + \frac{1-\tilde{\beta}}{4}||A^{\frac{1}{2}}u_0(0)||_H^2 \\
&\quad + \frac{1}{2}\int_{-\bar{\tau}}^0 |k(s)|\cdot||B^*g(s)||_H^2 ds + \frac{1}{2}\int_0^{+\infty}\beta(s)||A^{\frac{1}{2}}\eta_0(s)||_H^2 ds \\
&= \frac{1}{2}||u_1||_H^2 + \frac{3}{4}(1-\tilde{\beta})||A^{\frac{1}{2}}u_0(0)||_H^2 + \frac{1}{2}\int_{-\bar{\tau}}^0 |k(s)|\cdot||B^*g(s)||_H^2 ds \\
&\quad + \frac{1}{2}\int_0^{+\infty}\beta(s)||A^{\frac{1}{2}}\eta_0(s)||_H^2 ds < \rho^2.
\end{aligned}$$

The above inequality, together with (7.2.11), implies that

$$h\left(\frac{2}{(1-\tilde{\beta})^{\frac{1}{2}}}(C_T^*)^{\frac{1}{2}}E^{\frac{1}{2}}(0)\right) < h\left(\frac{2}{(1-\tilde{\beta})^{\frac{1}{2}}}(C_T^*)^{\frac{1}{2}}\rho\right) \le h\left(h^{-1}\left(\frac{1-\tilde{\beta}}{2}\right)\right) = \frac{1-\tilde{\beta}}{2}. \tag{7.2.14}$$

Furthermore, from (7.2.7),

$$h\left(\frac{2}{(1-\tilde{\beta})^{\frac{1}{2}}}(C_T^*)^{\frac{1}{2}}\cdot\frac{1}{\sqrt{2}}\max_{s\in[-\bar{\tau},0]}||g(s)||_H\right) < h\left(\frac{\sqrt{2}}{(1-\tilde{\beta})^{\frac{1}{2}}}(C_T^*)^{\frac{1}{2}}\rho\right) \le \frac{1-\tilde{\beta}}{2}. \tag{7.2.15}$$

Then, from (7.2.14), (7.2.15) and by definition of $\mathcal{E}(0)$, we have that

$$h\left(\frac{2}{(1-\tilde{\beta})^{\frac{1}{2}}}(C_T^*)^{\frac{1}{2}}\mathcal{E}^{\frac{1}{2}}(0)\right) < \frac{1-\tilde{\beta}}{2}. \tag{7.2.16}$$

Since $C_T^* \ge \bar{C}(T)$, the above inequality (7.2.16) allows us to apply Lemma 7.1.4, that ensures that (7.1.8) is satisfied for all $t \in [0,\delta)$. In particular,

$$E(t) \ge \frac{1}{4}||u_t(t)||_H^2, \quad \forall t \in [0,\delta).$$

Combining (7.1.4) and (7.1.8) with $\bar{C}(t) \leq \bar{C}(T) \leq C_T^*$, it turns out that

$$\frac{1}{4}||u_t(t)||_H^2 + \frac{1-\tilde{\beta}}{4}||A^{\frac{1}{2}}u(t)||_H^2 + \frac{1}{4}\int_{t-\bar{\tau}}^t |k(s)| \cdot ||B^* u_t(s)||_H^2 ds + \frac{1}{4}\int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta^t(s)||_H^2 ds$$
$$< E(t) \leq C_T^* \mathcal{E}(0), \tag{7.2.17}$$

for all $t \in [0, \delta)$. Thus, since the solution U is bounded from (7.2.17), we can extend it in $t = \delta$ and in the whole interval $[0, T]$. Moreover, (7.2.17) holds for all $t \in [0, T]$.
Now, using (7.2.16) and (7.2.17) with $t = T$, we can write

$$h(||A^{\frac{1}{2}}u_0(T)||_H) \leq h\left(\frac{2}{(1-\tilde{\beta})^{\frac{1}{2}}}(C_T^*)^{\frac{1}{2}}\mathcal{E}^{\frac{1}{2}}(0)\right) < \frac{1-\tilde{\beta}}{2}. \tag{7.2.18}$$

Furthermore, from the smallness assumption (7.2.6) on the initial data and from (7.2.17), it comes that

$$\frac{1}{4}||U(t)||_{\mathcal{H}}^2 \leq E(t) \leq C_T^* \rho^2,$$

where here we have used the fact that $\mathcal{E}(0) < \rho^2$. Thus,

$$||U(t)||_{\mathcal{H}} \leq C_\rho := 2(C_T^*)^{\frac{1}{2}}\rho. \tag{7.2.19}$$

Next, eventually choosing a smaller value of ρ, we can suppose that $L(C_\rho) < \frac{\omega-\omega'}{2M}$. We have so proved that there exist $\rho > 0$, $C_\rho > 0$ such that, whenever the initial data $(u_0(0), g, \eta_0)$ satisfy the smallness condition (7.2.6) and (7.2.7), then the system (7.0.5) with the initial data U_0, $\tilde{g}$, has a unique solution U defined in the time interval $[0, T]$ such that $||U(t)||_{\mathcal{H}} \leq C_\rho$.
As a consequence, from Theorem 7.2.1, being $L(C_\rho) < \frac{\omega-\omega'}{2M}$, the following estimate holds

$$\|U(t)\|_{\mathcal{H}} \leq Me^\gamma \left(||U_0||_{\mathcal{H}} + e^{\omega\bar{\tau}}Kb^2 \max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\}\right) e^{-\frac{\omega-\omega'}{2}t},$$

for all $t \in [0, T]$. Thus, using (7.2.12), we can write

$$\|U(t)\|_{\mathcal{H}} \leq Me^\gamma \left(||U_0||_{\mathcal{H}} + e^{\omega\bar{\tau}}Kb^2\sqrt{2}\rho\right) e^{-\frac{\omega-\omega'}{2}t},$$

for any $t \in [0, T]$. Then,

$$\begin{aligned}\|U(t)\|_{\mathcal{H}}^2 &\leq M^2 e^{2\gamma}\left(||U_0||_{\mathcal{H}} + e^{\omega\bar{\tau}}Kb^2\sqrt{2}\rho\right)^2 e^{-(\omega-\omega')t} \\ &\leq 2M^2 e^{2\gamma}\left(||U_0||_{\mathcal{H}}^2 + (e^{\omega\bar{\tau}}Kb^2\sqrt{2}\rho)^2\right) e^{-(\omega-\omega')t} \\ &= 2M^2 e^{2\gamma}\left(||U_0||_{\mathcal{H}}^2 + e^{2\omega\bar{\tau}}K^2 b^4 2\rho^2\right) e^{-(\omega-\omega')t},\end{aligned}$$

from which, taking into account (7.2.12) with $U_0 = \tilde{g}(0)$,

$$\|U(t)\|_{\mathcal{H}}^2 \leq 4M^2 e^{2\gamma}\rho^2\left(1 + e^{2\omega\bar{\tau}}K^2 b^4\right) e^{-(\omega-\omega')t}, \tag{7.2.20}$$

for all $t \in [0, T]$.
Also, for all $s \in [T-\bar{\tau}, T] \subseteq [0, T]$, (1.1.18) yields

$$\begin{aligned} ||u_t(s)||_H^2 \leq \|U(s)\|_{\mathcal{H}}^2 & \\ &\leq 4M^2 e^{2\gamma}\rho^2 \left(1 + e^{2\omega\bar{\tau}} K^2 b^4\right) e^{-(\omega-\omega')s} \\ &\leq 4M^2 e^{2\gamma}\rho^2 \left(1 + e^{2\omega\bar{\tau}} K^2 b^4\right) e^{-(\omega-\omega')(T-\bar{\tau})} \\ &\leq 4M^2 e^{2\gamma} e^{\omega\bar{\tau}}\rho^2 \left(1 + e^{2\omega\bar{\tau}} K^2 b^4\right) e^{-(\omega-\omega')T}. \end{aligned}$$

As a consequence, from (0.2.39)

$$\begin{aligned} \int_{T-\bar{\tau}}^{T} |k(s)| \cdot ||B^* u_t(s)||_H^2 ds &\leq 4M^2 e^{2\gamma} e^{\omega\bar{\tau}} b^2 \rho^2 \left(1 + e^{2\omega\bar{\tau}} K^2 b^4\right) e^{-(\omega-\omega')T} \int_{T-\bar{\tau}}^{T} |k(s)| ds \\ &\leq 4KM^2 e^{2\gamma} e^{\omega\bar{\tau}} b^2 \rho^2 \left(1 + e^{2\omega\bar{\tau}} K^2 b^4\right) e^{-(\omega-\omega')T}. \end{aligned}$$

This last fact together with (7.2.20) implies that

$$\begin{aligned} &\|U(T)\|_{\mathcal{H}}^2 + \int_{T-\bar{\tau}}^{T} |k(s)| \cdot ||B^* u_t(s)||_H^2 ds \\ &\quad \leq 4M^2 e^{2\gamma}\rho^2 \left(1 + e^{2\omega\bar{\tau}} K^2 b^4\right) \left(1 + K b^2 e^{\omega\bar{\tau}}\right) e^{-(\omega-\omega')T} \leq C_T \rho^2 < \rho^2. \end{aligned} \tag{7.2.21}$$

Moreover, (7.2.20) implies that

$$\max_{s\in[T-\bar{\tau},T]} ||u_t(s)||_H^2 \leq 4M^2 e^{2\gamma} e^{\omega\bar{\tau}} \rho^2 \left(1 + e^{2\omega\bar{\tau}} K^2 b^4\right) e^{-(\omega-\omega')T} \leq C_T \rho^2 < \rho^2,$$

from which

$$\max_{s\in[T-\bar{\tau},T]} ||u_t(s)||_H < \rho. \tag{7.2.22}$$

The conditions (7.2.21) and (7.2.22) allow us to apply the same arguments employed before on the interval $[T, 2T]$. Namely, we consider the initial value problem

$$\begin{aligned} &V'(t) = \mathcal{A}V(t) - k(t)\mathcal{B}V(t-\tau(t)) + F(V(t)), \quad t \in [T, 2T], \\ &V(s) = U(s), \quad s \in [T-\bar{\tau}, T], \end{aligned} \tag{7.2.23}$$

where $U(\cdot)$ is the solution to (7.0.5) in the interval $[0, T]$.
We define now the energy of the solution

$$\begin{aligned} \tilde{E}(t) := \tilde{E}(v(t)) = \frac{1}{2}||v_t(t)||_H^2 + \frac{1-\tilde{\beta}}{2}||A^{\frac{1}{2}} v(t)||_H^2 - \psi(v) & \\ + \frac{1}{2}\int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta^t(s)||_H^2 ds + \frac{1}{2}\int_{t-\bar{\tau}}^{t} |k(s)| \cdot ||B^* v_t(s)||_H^2 ds, & \end{aligned} \tag{7.2.24}$$

where $\eta^t(s) = v(t) - v(t-s)$, and the functional

$$\tilde{\mathcal{E}}(t) := \max\left\{\frac{1}{2}\max_{s\in[T-\bar{\tau},T]} \|u_t(s)\|_H^2,\ \max_{s\in[T,t]} \tilde{E}(s)\right\}. \tag{7.2.25}$$

Let us note that $\tilde{E}(T) = E(T)$.
Now, from Theorem 7.1.1, the problem (7.2.23) with the initial datum $U(s)$, $s \in [T-\bar{\tau}, T]$, has a unique local solution $V(\cdot)$ defined on a time interval $[T, T+\delta)$ given by the Duhamel's formula

$$V(t) = S(t-T)U(T) + \int_0^t S(t-T-s)[-k(s)\mathcal{B}V(s-\tau(s)) + F(V(s))]\, ds, \qquad (7.2.26)$$

for all $t \in [T, T+\delta)$. We can assume that $\delta \leq \bar{\tau}$ and that V is a non-zero solution, since, otherwise, (7.2.8) is obviously satisfied for all $t \geq T$.
First of all, the inequality (7.2.18) yields $\tilde{E}(T) > 0$.
Let us note that, if $\tilde{E}(t) \geq \frac{1}{4}\|v_t(t)\|_H^2$, for all $t \in [T, T+\delta)$, arguing as in Lemma 7.1.3,

$$\tilde{E}(t) \leq e^{3b^2 \int_T^t |k(s)| ds}\, \tilde{\mathcal{E}}(T), \quad \forall t \in [T, T+\delta). \qquad (7.2.27)$$

Also, using the same arguments employed in Lemma 7.1.4 and observing that

$$C_T^* \geq e^{3b^2 \int_T^{2T} |k(s)| ds},$$

if

$$h\left(\frac{2}{(1-\tilde{\beta})^{\frac{1}{2}}}(C_T^*)^{\frac{1}{2}}\tilde{\mathcal{E}}^{\frac{1}{2}}(T)\right) < \frac{1-\tilde{\beta}}{2}, \qquad (7.2.28)$$

then

$$\begin{aligned} \tilde{E}(t) > \frac{1}{4}||v_t(t)||_H^2 + \frac{1-\tilde{\beta}}{4}||A^{\frac{1}{2}}v(t)||_H^2 + \frac{1}{4}\int_{t-\bar{\tau}}^t |k(s)| \cdot ||B^* v_t(s)||_H^2 ds \\ + \frac{1}{4}\int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta^t(s)||_H^2 ds, \end{aligned} \qquad (7.2.29)$$

for all $t \in [T, T+\delta)$. In particular,

$$\tilde{E}(t) > \frac{1}{4}\|V(t)\|_{\mathcal{H}}^2, \quad \forall t \in [T, T+\delta). \qquad (7.2.30)$$

Now, from (7.2.21), we have $\tilde{E}(T) < \rho^2$. This together with (7.2.22) implies that

$$\tilde{\mathcal{E}}(T) < \rho^2. \qquad (7.2.31)$$

Hence, using (7.2.11) we get

$$h\left(\frac{2}{(1-\tilde{\beta})^{\frac{1}{2}}}(C_T^*)^{\frac{1}{2}}\tilde{\mathcal{E}}(T)^{\frac{1}{2}}\right) < h\left(\frac{2}{(1-\tilde{\beta})^{\frac{1}{2}}}(C_T^*)^{\frac{1}{2}}\rho\right) < \frac{1-\tilde{\beta}}{2}. \qquad (7.2.32)$$

So, (7.2.28) is satisfied. As a consequence, inequalities (7.2.29) and (7.2.30) hold for all $t \in [T, T+\delta)$. In particular, from (7.2.30) we get

$$\tilde{E}(t) \geq \frac{1}{4}\|v_t(t)\|_H^2, \quad \forall t \in [T, T+\delta).$$

Therefore, also inequality (7.2.27) is fulfilled. Combining (7.2.27) and (7.2.29), we finally get

$$\frac{1}{4}||v_t(t)||_H^2 + \frac{1-\tilde{\beta}}{4}||A^{\frac{1}{2}}v(t)||_H^2 + \frac{1}{4}\int_{t-\bar{\tau}}^{t} |k(s)| \cdot ||B^* v_t(s)||_H^2 ds + \frac{1}{4}\int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta^t(s)||_H^2 ds$$
$$< \tilde{E}(t) \le e^{3b^2 \int_T^{2T} |k(s)|ds}\tilde{\mathcal{E}}(T) \le C_T^* \tilde{\mathcal{E}}(T). \tag{7.2.33}$$

Thus, the solution V is bounded and we can extend it in $t = T + \delta$ and in the whole interval $[T, 2T]$. Moreover, (7.2.33) and (7.2.30) hold true for all $t \in [T, 2T]$. So, taking into account of (7.2.31), it holds

$$\|V(t)\|_{\mathcal{H}} \le 2(C_T^*)^{\frac{1}{2}} \tilde{\mathcal{E}}^{\frac{1}{2}}(T) \le 2(C_T^*)^{\frac{1}{2}}\rho = C_\rho. \tag{7.2.34}$$

Putting together the two partial solutions (7.2.13) and (7.2.26) to (7.0.5) obtained in the time intervals $[0, T]$ and $[T, 2T]$, respectively, we get the existence of a unique solution $U \in C([0, 2T]; \mathcal{H})$ to (7.0.5) that is defined in the time interval $[0, 2T]$ and that satisfies the Duhamel's formula (7.2.13), for all $t \in [0, 2T]$. Moreover, from (7.2.19) and (7.2.34), the solution U satisfies (7.2.1) with $C = C_\rho$. Thus, since $L(C_\rho) < \frac{\omega-\omega'}{2M}$, the exponential decay estimate (7.2.2) is satisfied by the solution U, i.e.

$$\|U(t)\|_{\mathcal{H}} \le Me^{\gamma}\left(||U_0||_{\mathcal{H}} + e^{\omega\bar{\tau}}Kb^2 \max_{r\in[-\bar{\tau},0]}\{e^{\omega r}||\tilde{g}(r)||_{\mathcal{H}}\}\right)e^{-\frac{\omega-\omega'}{2}t}, \quad \forall t \in [0, 2T].$$

Again, we deduce that (7.2.20) holds for all $t \in [0, 2T]$. Furthermore, for all $s \in [2T - \bar{\tau}, 2T]$,

$$||u_t(s)||_H^2 \le ||U(s)||_{\mathcal{H}}^2 \le 4M^2 e^{2\gamma} e^{\omega\bar{\tau}} \rho^2 \left(1 + e^{2\omega\bar{\tau}} K^2 b^4\right) e^{-(\omega-\omega')2T}.$$

As a consequence,

$$\begin{aligned}
\|U(2T)\|_{\mathcal{H}}^2 + \int_{2T-\bar{\tau}}^{2T} |k(s)| \cdot ||B^* u_t(s)||_H^2 ds &\le 4M^2 e^{2\gamma}\rho^2(1 + e^{2\omega\bar{\tau}}K^2 b^4)e^{-(\omega-\omega')2T} \\
&\quad +4M^2 e^{2\gamma} e^{\omega\bar{\tau}} K b^2 \rho^2 (1 + e^{2\omega\bar{\tau}} K^2 b^4) e^{-(\omega-\omega')2T} \\
&\le 4M^2 e^{2\gamma}\rho^2(1 + e^{2\omega\bar{\tau}}K^2 b^4)(1 + Kb^2 e^{\omega\bar{\tau}})e^{-(\omega-\omega')2T} \\
&\le 4M^2 e^{2\gamma}\rho^2(1 + e^{2\omega\bar{\tau}}K^2 b^4)(1 + Kb^2 e^{\omega\bar{\tau}})e^{-(\omega-\omega')T} \\
&\le C_T \rho^2 < \rho^2.
\end{aligned} \tag{7.2.35}$$

Also,

$$\begin{aligned}
\max_{s\in[2T-\bar{\tau},2T]} ||u_t(s)||_H^2 &\le 4M^2 e^{2\gamma} e^{\omega\bar{\tau}} \rho^2 \left(1 + e^{2\omega\bar{\tau}} K^2 b^4\right) e^{-(\omega-\omega')2T} \\
&\le 4M^2 e^{2\gamma} e^{\omega\bar{\tau}} \rho^2 \left(1 + e^{2\omega\bar{\tau}} K^2 b^4\right) e^{-(\omega-\omega')T} \le C_T\rho^2 < \rho^2,
\end{aligned}$$

from which

$$\max_{s\in[2T-\bar{\tau},2T]} ||u_t(s)||_H < \rho. \tag{7.2.36}$$

The conditions (7.2.35) and (7.2.36) allows us to apply the same arguments employed before on the interval $[2T, 3T]$, obtaining the existence of a unique solution to (7.0.5) defined in the time interval $[0, 3T]$ and that satisfies (7.2.8) for all $t \in [0, 3T]$. Iterating this procedure, we finally get the existence of a unique global solution $U \in C([0, +\infty); \mathcal{H})$ to (7.0.5) with the initial datum $\tilde{g}(s)$, $s \in [-\bar{\tau}, 0]$, that satisfies (7.2.8). □

We have then proved that, for suitably small initial data, solutions to problem (7.0.5) are globally defined and bounded by a positive constant C_ρ that satisfies $L_{C_\rho} < \frac{\omega-\omega'}{M}$ (see Theorem 7.2.2). Thus, solutions to problem (7.0.5) satisfy the exponential decay estimate (7.2.2). Therefore, we are ready to prove the energy decay for model (7.0.2).

Theorem 7.2.3. *Let us consider model* (7.0.2) *and assume* (7.0.9). *There exists* $\rho > 0$ *such that, if the following smallness conditions on the initial data are satisfied:*

$$||u_1||_H^2 + (1-\tilde{\beta})||A^{\frac{1}{2}}u_0(0)||_H^2 + \int_{-\bar{\tau}}^0 |k(s)| \cdot ||B^*g(s)||_H^2 + \int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta_0(s)||_H^2 ds < \rho^2, \tag{7.2.37}$$

and

$$\max_{s\in[-\bar{\tau},0]} ||g(s)||_H < \rho, \tag{7.2.38}$$

then (7.0.2) *has a unique solution globally defined. Moreover, the energy satisfies the exponential decay estimate*

$$E(t) \le \tilde{C}e^{-\mu t}, \tag{7.2.39}$$

where $\mu := \omega - \omega'$ *and* $\tilde{C}$ *is a constant depending on the initial data.*

Proof. Let $\rho > 0$ be the positive constant in Theorem 7.2.2. Let us consider initial data for which the smallness conditions (7.2.37) and (7.2.38) hold true. Then, from Theorem 7.2.2, the problem (7.0.5) has a unique global $U(\cdot)$ solution that satisfies $||U(t)||_{\mathcal{H}} < C_\rho$, with $L(C_\rho) < \frac{\omega-\omega'}{2M}$, and the exponential decay estimate (7.2.2). From (7.1.11) and (7.2.17), we have that

$$\begin{aligned}
E(t) &= \frac{1}{2}||u_t(t)||_H^2 + \frac{1-\tilde{\beta}}{2}||A^{\frac{1}{2}}u(t)||_H^2 - \psi(u(t)) + \frac{1}{2}\int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta^t(s)||_H^2 ds \\
&\quad + \frac{1}{2}\int_{t-\bar{\tau}}^t |k(s)| \cdot ||B^*u_t(s)||_H^2 ds \\
&\le \frac{1}{2}||u_t(t)||_H^2 + \frac{1-\tilde{\beta}}{2}||A^{\frac{1}{2}}u(t)||_H^2 + \frac{1-\tilde{\beta}}{4}||A^{\frac{1}{2}}u(t)||_H^2 + \frac{1}{2}\int_0^{+\infty} \beta(s)||A^{\frac{1}{2}}\eta^t(s)||_H^2 ds \\
&\quad + \frac{b^2}{2}\int_{t-\bar{\tau}}^t |k(s)| \cdot ||u_t(s)||_H^2 ds \\
&\le ||U(t)||_{\mathcal{H}}^2 + \frac{b^2}{2}\int_{t-\bar{\tau}}^t |k(s)| \cdot ||U(s)||_{\mathcal{H}}^2 ds
\end{aligned}$$

for any $t \geq \bar{\tau}$. Now, applying Theorem 7.2.1, there exists a positive constant $\hat{C}$ such that

$$||U(t)||_{\mathcal{H}} \leq \hat{C} e^{-\frac{\omega-\omega'}{2}t}, \quad \forall t \geq 0, \tag{7.2.40}$$

where here we have used the fact that $L(C_\rho) < \frac{\omega-\omega'}{2M}$. Thus,

$$E(t) \leq \hat{C}^2 e^{-(\omega-\omega')t} + \frac{\hat{C}^2 b^2}{2} \int_{t-\bar{\tau}}^{t} |k(s)| e^{-(\omega-\omega')s} ds \leq \hat{C}^2 e^{-(\omega-\omega')t} + \frac{\hat{C}^2 b^2}{2} K e^{\omega\bar{\tau}} e^{-(\omega-\omega')t},$$

for all $t \geq \bar{\tau}$. Setting

$$\tilde{C} := \max\left\{\hat{C}^2, \frac{\hat{C}^2 b^2}{2} K e^{\omega\bar{\tau}}\right\},$$

we can write

$$E(t) \leq \tilde{C} e^{-(\omega-\omega')t},$$

from which,

$$E(t) \leq \tilde{C} e^{-\mu t}, \quad \forall t \geq \bar{\tau},$$

where $\mu = \omega - \omega'$. Hence, (7.2.39) holds true for all $t \geq \bar{\tau}$. □

7.3 Examples

We conclude this chapter by providing two applications of the previous results. In both examples, we establish global well-posedness and exponential stability for the wave equation with memory and different source terms.

7.3.1 The wave equation with memory and source term

Let Ω be a non-empty bounded set in $\mathbb{R}^n$, with boundary Γ of class C^2, and let $\mathcal{O} \subset \Omega$ be a nonempty open subset of Ω. We assume $n \geq 3$. The lower dimension cases could be studied analogously. We consider the following wave equation:

$$\begin{aligned}
&u_{tt}(x,t) - \Delta u(x,t) + \int_0^{+\infty} \beta(s)\Delta u(x,t-s)ds + k(t)\chi_{\mathcal{O}} u_t(x,t-\tau(t)) \\
&\qquad\qquad\qquad\qquad = |u(x,t)|^\sigma u(x,t), \qquad \text{in } \Omega \times (0,+\infty), \\
&u(x,t) = 0, \qquad \text{in } \Gamma \times (0,+\infty), \\
&u(x,t) = u^0(x,t) \qquad \text{in } \Omega \times (-\infty, 0], \\
&u_t(x,0) = u_1(x), \qquad \text{in } \Omega, \\
&u_t(x,t) = g(x,t), \qquad \text{in } \Omega \times [-\bar{\tau}, 0],
\end{aligned} \tag{7.3.1}$$

where the time delay function $\tau(\cdot)$ satisfies (0.2.37), $\beta : (0,+\infty) \to (0,+\infty)$ is a locally absolutely continuous memory kernel satisfying the assumptions (i)-(iv), $\sigma > 0$ and the damping coefficient $k(\cdot)$ is a function in $L^1_{loc}([-\bar{\tau},+\infty))$ for which (0.2.39) holds. Then, system (7.3.1) falls in the form (0.2.36) with $A = -\Delta$ and $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$.

Moreover, $D(A^{\frac{1}{2}}) = H_0^1(\Omega)$.
Here, the operator $B : L^2(\Omega) \to L^2(\Omega)$ is defined as

$$Bu(x) = \chi_{\mathcal{O}} u(x) = \begin{cases} u(x), & x \in \mathcal{O}, \\ 0, & x \in \Omega \setminus \mathcal{O}, \end{cases}$$

for all $u \in L^2(\Omega)$. By definition, B is a bounded linear operator from $L^2(\Omega)$ into itself. Furthermore, it is easy to see that $B^* = B$.
Let us define η_s^t as in (7.0.1). Then, system (7.3.1) can be rewritten as follows:

$$\begin{array}{l} u_{tt}(x,t) - (1-\tilde{\beta})\Delta u(x,t) - \displaystyle\int_0^{+\infty} \beta(s)\Delta\eta^t(x,s)ds + k(t)\chi_{\mathcal{O}} u_t(x,t-\tau) \\ \qquad\qquad\qquad = |u(x,t)|^\sigma u(x,t), \quad \text{in } \Omega \times (0,+\infty), \\ \eta_t^t(x,s) = -\eta_s^t(x,s) + u_t(x,t), \qquad \text{in } \Omega \times (0,+\infty) \times (0,+\infty), \\ u(x,t) = 0, \qquad \text{in } \Gamma \times (0,+\infty), \\ \eta^t(x,s) = 0, \qquad \text{in } \Gamma \times (0,+\infty), \quad \text{for } t \geq 0, \\ u(x,0) = u_0(x) := u^0(x,0), \qquad \text{in } \Omega, \\ u_t(x,0) = u_1(x) := \dfrac{\partial u^0}{\partial t}(x,t)\Big|_{t=0}, \qquad \text{in } \Omega, \\ \eta^0(x,s) = \eta_0(x,s) := u^0(x,0) - u^0(x,-s), \qquad \text{in } \Omega \times (0,+\infty), \\ u_t(x,t) = g(x,t), \qquad \text{in } \Omega \times [-\overline{\tau}, 0]. \end{array} \tag{7.3.2}$$

In order to reformulate (7.3.2) as an abstract first order equation, we introduce the Hilbert space $L^2_\beta((0,+\infty); H_0^1(\Omega))$ endowed with the inner product

$$\langle \phi, \varphi \rangle_{L^2_\beta((0,+\infty);H_0^1(\Omega))} := \int_\Omega \left(\int_0^{+\infty} \beta(s)\nabla\phi(x,s)\nabla\varphi(x,s)ds \right) dx,$$

and the Hilbert space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2_\beta((0,+\infty); H_0^1(\Omega)),$$

equipped with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \right\rangle_{\mathcal{H}} := (1-\tilde{\beta}) \int_\Omega \nabla u \nabla \tilde{u} dx + \int_\Omega v\tilde{v} dx + \int_\Omega \int_0^{+\infty} \beta(s)\nabla w \nabla \tilde{w} ds dx.$$

We set $U = (u, u_t, \eta^t)$. Then, (7.3.2) can be rewritten in the form (7.0.5), where

$$\mathcal{A} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ (1-\tilde{\beta})\Delta u + \int_0^{+\infty} \beta(s)\Delta w(s)ds \\ -w_s + v \end{pmatrix},$$

with domain

$$\begin{aligned} D(\mathcal{A}) = \; & \{(u,v,w) \in H_0^1(\Omega) \times H_0^1(\Omega) \times L^2_\beta((0,+\infty); H_0^1(\Omega)) : \\ & (1-\tilde{\beta})u + \int_0^{+\infty} \beta(s)w(s)ds \in H^2(\Omega) \cap H_0^1(\Omega), \; w_s \in L^2_\beta((0,+\infty); H_0^1(\Omega))\}, \end{aligned}$$

$\mathcal{B}(u, v, \eta^t)^T := (0, \chi_{\mathcal{O}} v, 0)^T$, $\tilde{g} = (u_0, g, \eta^0)^T$, in $[-\bar{\tau}, 0]$, and $F(U(t)) = (0, |u(t)|^\sigma u(t), 0)^T$.
Now, we consider the functional

$$\psi(u) := \frac{1}{\sigma + 2} \int_\Omega |u(x)|^{\sigma+2} dx, \quad \forall u \in H_0^1(\Omega).$$

By Sobolev's embedding theorem, ψ is well-defined for $0 < \sigma \le \frac{4}{n-2}$. Also, the Gâteaux derivative of ψ at any point $u \in H_0^1(\Omega)$ is

$$D\psi(u)(v) = \int_\Omega |u(x)|^\sigma u(x) v(x) dx,$$

for all $v \in H_0^1(\Omega)$. Moreover, as in [6], if $0 < \sigma \le \frac{2}{n-2}$, then ψ satisfies the assumptions (H1), (H2), (H3).
Let us define the energy in this way:

$$\begin{aligned} E(t) := & \frac{1}{2} \int_\Omega |u_t(x,t)|^2 dx + \frac{1 - \tilde{\beta}}{2} \int_\Omega |\nabla u(x,t)|^2 dx - \psi(u(x,t)) \\ & + \frac{1}{2} \int_{t-\bar{\tau}}^t \int_{\mathcal{O}} |k(s)| \cdot |u_t(x,s)|^2 dx ds + \frac{1}{2} \int_0^{+\infty} \beta(s) \int_\Omega |\nabla \eta^t(x,s)|^2 dx ds. \end{aligned}$$

Then, from Theorem 7.2.3 we have that (7.3.1) is well-posed and that, for solutions corresponding to suitably small initial data, an exponential decay estimate holds provided that the condition (7.0.9) is satisfied.

7.3.2 The wave equation with memory and integral source term

Let Ω be a non-empty bounded set in $\mathbb{R}^n$, with boundary Γ of class C^2. Moreover, let $\mathcal{O} \subset \Omega$ be a nonempty open subset of Ω. We consider the following wave equation:

$$\begin{aligned} & u_{tt}(x,t) - \Delta u(x,t) + \int_0^{+\infty} \beta(s) \Delta u(x, t-s) ds + k(t) \chi_{\mathcal{O}} u_t(x, t - \tau(t)) \\ & \qquad = \left(\int_\Omega |u(x,t)|^2 \right)^{\frac{p}{2}} u(x,t), \qquad \text{in } \Omega \times (0, +\infty), \\ & u(x,t) = 0, \qquad \text{in } \Gamma \times (0, +\infty), \\ & u(x,t) = u_0(x,t) \qquad \text{in } \Omega \times (-\infty, 0], \\ & u_t(x,0) = u_1(x), \qquad \text{in } \Omega, \\ & u_t(x,t) = g(x,t), \qquad \text{in } \Omega \times [-\bar{\tau}, 0], \end{aligned} \tag{7.3.3}$$

where the time delay function $\tau(\cdot)$ satisfies (0.2.37), $\beta : (0, +\infty) \to (0, +\infty)$ is a locally absolutely continuous memory kernel such that the assumptions (i)-(iv) are fulfilled, $p \ge 1$, and the damping coefficient $k(\cdot)$ is a function in $L^1_{loc}([-\bar{\tau}, +\infty))$ for which (0.2.39) holds true. Then, system (7.3.3) falls in the form (0.2.36) with $A = -\Delta$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $D(A^{\frac{1}{2}}) = H_0^1(\Omega)$. Also, the operator B is as in the previous example.
Let us note that system (7.3.3) is analogous to system (7.3.1), with the only difference

given by the nonlinearity. So, arguing as in Example 7.3.1, (7.3.3) can be rewritten as (7.3.2) and then as an abstract first order equation.
Now, consider the functional

$$\psi(u) := \frac{1}{p+2}\left(\int_\Omega |u(x)|^2 dx\right)^{\frac{p+2}{2}} = \frac{1}{p+2}\|u\|^{p+2}_{L^2(\Omega)}, \quad \forall u \in L^2(\Omega).$$

Note that ψ is well defined. Also, the Gâteaux derivative of ψ at any point $u \in L^2(\Omega)$ is given by

$$D\psi(u)(v) = \left(\int_\Omega |u(x)|^2 dx\right)^{\frac{p}{2}} \int_\Omega u(x)v(x)dx,$$

for any $v \in L^2(\Omega)$. Then, ψ is defined in the whole $L^2(\Omega)$ and

$$\nabla\psi(u) = \left(\int_\Omega |u(x)|^2 dx\right)^{\frac{p}{2}} u(x), \quad \forall u \in L^2(\Omega),$$

is the unique vector representing $D\psi(u)$ in the Riesz isomorphism. So (H_1) is trivially satisfied. Arguing as in [78], we can find a positive constant $C > 0$ such that

$$\|\nabla\psi(u) - \nabla\psi(v)\|^2_{L^2(\Omega)} \le C(\|u\|^{2p}_{H^1_0(\Omega)} + \|v\|^{2p}_{H^1_0(\Omega)})\|u - v\|^2_{H^1_0(\Omega)}, \tag{7.3.4}$$

for all $u, v \in H^1_0(\Omega)$. Thus, for any $r > 0$ and for all $u, v \in H^1_0(\Omega)$ with $\|\nabla u\|_{L^2(\Omega)}, \|\nabla v\|_{L^2(\Omega)} \le r$, since from Poincaré inequality $\|\cdot\|_{H^1_0(\Omega)}$ and $\|\nabla(\cdot)\|_{L^2(\Omega)}$ are equivalent norms on $H^1_0(\Omega)$, from (7.3.4) we get

$$\|\nabla\psi(u) - \nabla\psi(v)\|^2_{L^2(\Omega)} \le 2r^{2p}C\|\nabla u - \nabla v\|^2_{L^2(\Omega)},$$

from which

$$\|\nabla\psi(u) - \nabla\psi(v)\|_{L^2(\Omega)} \le \sqrt{2C}r^p\|\nabla u - \nabla v\|_{L^2(\Omega)},$$

Hence, (H_2) is satisfied.
Finally, we prove that (H_3) holds true. Note that $\psi(0), \nabla\psi(0) = 0$. Also, using (7.3.4) with $v = 0$ and Poincaré inequality, for all $u \in H^1_0(\Omega)$ we can write

$$\|\nabla\psi(u)\|^2_{L^2(\Omega)} \le C\|\nabla u\|^{2p}_{L^2(\Omega)}\|\nabla u\|^2_{L^2(\Omega)},$$

which implies

$$\|\nabla\psi(u)\|_{L^2(\Omega)} \le \sqrt{C}\|\nabla u\|^p_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}$$

Thus, (H_3) is fulfilled with $h(z) = \sqrt{C}z^p$, for all $z \ge 0$, which is a continuous and strictly increasing function.
Let us define the energy as follows:

$$E(t) := \frac{1}{2}\int_\Omega |u_t(x,t)|^2 dx + \frac{1-\tilde{\beta}}{2}\int_\Omega |\nabla u(x,t)|^2 dx - \psi(u(x,t))$$
$$+\frac{1}{2}\int_{t-\tau}^t \int_{\mathcal{O}} |k(s)| \cdot |u_t(x,s)|^2 dx ds + \frac{1}{2}\int_0^{+\infty} \beta(s)\int_\Omega |\nabla\eta^t(x,s)|^2 dx ds.$$

Then, applying Theorem 7.2.3 to this model, we get well-posedness and exponential decay of the energy for solutions corresponding to suitably small initial data provided that the condition (7.0.9) is satisfied.

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