UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DIPARTIMENTO DI INGEGNERIA E SCIENZE DELL'INFORMAZIONE E MATEMATICA

Dottorato di Ricerca in Matematica e Modelli

XXXVI ciclo

Titolo della tesi

Theoretical and numerical aspects of projected solution for quasiequilibrium problems

SSD SECS-S/06

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a.a. 2022/2023

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Abstract

Most of the results concerning the existence of quasiequilibrium problems require the constraint map to be a self-map. However, in some applications, the constraint map may not be a self-map. Aussel, Sultana, and Vetrivel [4] introduced the concept of projected solution for quasivariational inequalities and generalized Nash equilibria to address this issue. Cotrina and Zúñiga [27] later adapted this concept to quasiequilibrium problems.

This thesis examines an electricity market model that leads to a generalized equilibrium problem in which a constraint map cannot be a self-map. The thesis then presents an existence result for projected solutions [20] in the finite dimensional setting. This is achieved without any monotonicity assumptions and without requiring the compactness of the feasible set. Additionally, we establish the existence of projected solutions for quasivariational inequalities, quasioptimization problems, and generalized Nash equilibrium problems. Finally, we illustrate an iterative procedure for discovering a projected solution. The proposed method is founded on the concept of a gap function and a reformulation of the quasiequilibrium problem as a global optimization problem.

Introduction

Since the 1990s, the electricity market in many countries has undergone a significant restructuring process toward deregulation and liberalization. Although the details of the restructuring process and the regulatory framework may differ from country to country, in most cases, the general organization follows the same principles. One of the fundamental features of the deregulation process is the creation of a wholesale energy market, where all electricity buying and selling transactions take place. Each producer offers its production to the wholesale electricity market, where the clearing price of energy is determined by a sealed-bid auction. Producers face a complex problem when deciding which bids to submit. There is uncertainty about the demand that will arise, and there is also uncertainty about how other market participants behave. Therefore, the main question for a producer company is how to develop a bidding strategy that maximizes its profit.

In these markets, there is a central authority that manages the auctions and the electricity system in which it is located and guarantees that all users of the electricity system are treated in a fair and transparent manner. This authority has different names depending on the country and the market structure; in this thesis, we refer to it as the Independent System Operator, or ISO for short. The ISO balances the flow of electricity on the grid at all times, matching supply and demand. Since electricity cannot be stored on a large scale, wholesale electricity markets need to balance production and consumption at all times. When the ISO predicts that there will be a discrepancy between planned electricity production and demand over a period of time, it will ask generators to submit bids for the adjustment of electricity production. This discrepancy usually results either from changes in electricity demand due to weather forecast updates or from transmission due to technical incidents or network congestion. In this thesis, we do not consider possible adjustments in demand.

These characteristics distinguish the electricity market from traditional financial and other commodity markets and pose significant challenges for the design of electricity market auctions.

The hierarchical relationship between all the users of the electricity market has forced one to focus on the concept of bilevel games (or leader-follower games), which identify problems in which some variables of the problem at the upper level (the leader's problem) are constrained to be solutions to the problem at the lower level (the follower's problem). In particular, in electricity markets, producers are seen as leaders, while the ISO is seen as the common follower. In this way, the rules governing the interactions between agents are well defined. According to auctions theory, it has been possible to predict the behaviour of bids in different auction formats, which has helped auction designers choose efficient formats and avoid disastrous formats. There are two typical mechanisms: pay-as-bid and pay-as-clear. In a pay-as-bid auction, the ISO pays each producer according to the price they bid, whereas in a pay-as-clear auction, all accepted bids are paid at a single price, which is the equilibrium price. Bids above the equilibrium price, i.e., too high, are not accepted. The debate between proponents of the two formats has a long history, and the issue is still largely unsettled, although the vast majority of electricity markets use the pay-as-clear format.

However, due to high electricity prices in 2021 and 2022, the European Agency for the Cooperation of Energy Regulators is considering alternative pricing models to replace current pay-as-clear auctions and provide reliable and affordable electricity. The main aim is to decouple the price of electricity from marginal technologies, namely, gas and coal. Pay-as-bid auctions are one of the best alternatives.

In this thesis, we analyze a particular pay-as-bid model in which producers propose a bid function to the ISO rather than a price vector. The bid functions closest to the real situation are piecewise linear, convex and increasing functions, but since they are nonsmooth, they introduce more complications from a computational point of view and, above all, do not guarantee the uniqueness of the solution to the problem (ISO). For this reason, increasing quadratic bid functions, which guarantee the uniqueness of the solution of the problem (ISO) and are easier to handle computationally but deviate more from the real situation, are considered in the literature. To overcome these drawbacks, Aussel et al. in [4] introduced the concept of a projected solution, which allows the use of increasing quadratic functions for computations and, in order not to deviate too much from the real situation, to project them onto a set of piecewise linear, convex and increasing functions.

This new type of solution can be used in any situation where a classical solution cannot be achieved due to the structure of the constraint map. Indeed, requiring that the constraint map K is a self-map, i.e., that it maps the set of strategies C into itself, is the way to obtain the existence of classical solutions to a generalized equilibrium problem. However, the condition $K(C) \subseteq C$ is rather strong and is not satisfied by some applications (such as the electricity market model analyzed).

The notion of projected solution was introduced by Aussel et al. in [4] for a generalized Nash equilibrium problem and quasivariational inequality. In the same paper, the results for the projected solutions of these kinds of problems are given. Therefore, the concept of the projected solution introduced for generalized Nash games was adapted to quasiequilibrium problems in [27].

Subsequently, this concept has attracted great attention and has been developed from different perspectives. Existence results for the projected solution of quasiequilibrium problems have been given both in a finite dimensional setting [20, 27] and when the strategy space is a subset of a Banach space [15, 22]. Recently, an iterative procedure to find projected solutions of a quasiequilibrium problem defined in a normed space has been proposed [8].

In this thesis, we will focus on the study of the projected solutions for quasiequilibrium problems from different perspectives. In particular, we present the results published in [20] and then focus on the algorithmic aspect. Regarding the results, we have improved the results in the literature by not requiring any monotonicity assumptions and by not assuming the compactness of the feasible set.

From an algorithmic point of view, our work followed two directions. First, we obtained an algorithm to find a projected solution for a quasiequilibrium problem in which the constraint map K can be described by constraint functions. The main idea is to reformulate the quasiequilibrium problem as an optimization problem using an appropriate gap function and develop a descent algorithm. Second, we would like to find an algorithm that computes the projected solution of the electricity market model from which we started. This last work has not yet been completed, and what has been done thus far is presented in the Appendix.

The algorithmic results were obtained in collaboration with Giancarlo Bigi and Riccardo Cambini from the University of Pisa. I thank them for their collaboration.

Chapter 1

A motivating example: an electricity market model

We will begin this thesis by presenting a more detailed reworking of a particular pay-as-bid electricity market model. It is from this model that, in [4], the idea for the new solution concept, the projected solution, originates. As described in the introduction, when dealing with electricity markets we have to consider competition between different firms and different types of agents. This leads us to use bilevel equilibrium problems. More precisely, in this model, there are $N+1$ agents: N producers of electricity (leaders) and the ISO (follower). Each producer i proposes to the ISO the price bid function Ψ_i, and the ISO aims to determine the quantity x_i that the producer i must supply. By denoting γ_i the real cost function of producer i, D the electricity demand, that is supposed to be known, and Q_i the production capacity of producer i, the model can be written as the following N maximization problems

$$(P_i)\begin{cases} \max_{\Psi_i}[\Psi_i(x_i) - \gamma_i(x_i)] \\ \Psi_i \in C_i \subseteq L^2([0, Q_i]) \\ x = (x_1, \dots, x_N) \text{ solves } (ISO) \end{cases}$$

where (ISO) is the following problem of the ISO

$$(ISO)\begin{cases} \min_x \sum_\iota \Psi_\iota(x_\iota) \\ x_i \in [0, Q_i], \qquad \forall i = 1, \dots, N \\ \sum_\iota x_\iota = D \end{cases}$$

We analyze in detail the problem of a single producer, omitting the index i.

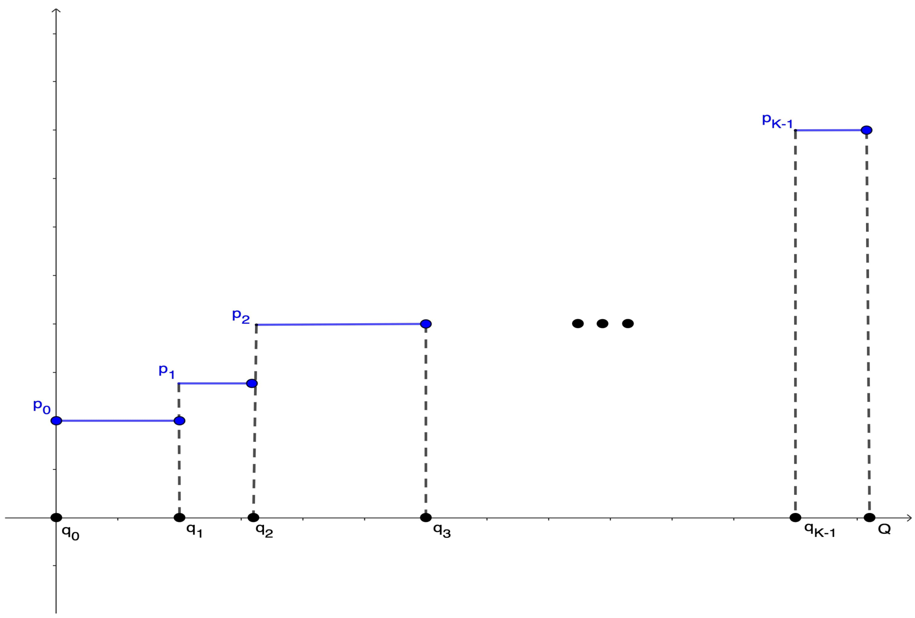

Figure 1.1: Cumulative unit price function

First, we define a cumulative unit price function $\psi : [0, Q] \to \mathbb{R}$ (Figure 1.1), representing the unit price at which the producer is willing to sell its energy. It is clear that this price is very much linked to the technology used by the producer. In fact, there are technologies that have much lower production costs because the raw material from which they produce electricity is available in unlimited and free quantities (think, for example, of wind and water power or solar radiation) and other technologies that have much higher production costs (think, for example, of gas or coal). For this reason, the domain of this function $[0, Q]$ is divided into K intervals fixed. These intervals are defined by the points q_j which verify the following condition

$$0 = q_0 < q_1 < q_2 < \cdots < q_K = Q \tag{1.1}$$

where each interval $[q_{j-1}, q_j]$ represents the amount of energy that can be produced by the j-th technology. The unit price function ψ is thus defined as

$$\psi(x) = \begin{cases} p_0 & \text{if } x = q_0 = 0 \\ p_{j-1} & \text{if } x \in (q_{j-1}, q_j] \quad \forall j = 1, \dots, K \end{cases} \tag{1.2}$$

where p_0 is the producer's minimum price and p_j is the unit price associated with the energy quantity $(q_j, q_{j+1}]$. Moreover, the vector $(p_0, p_1, \dots, p_{K-1})$ verify the condition

$$0 < \underline{p} \leq p_0 \leq p_1 \leq p_2 \leq \cdots \leq p_{K-1} \leq \overline{p} \tag{1.3}$$

where $\underline{p}$ and $\overline{p}$ are fixed and bound the range over which prices can be varied.

Therefore, the bid function Ψ, which represents the total price of the energy produced, is defined as the integral of ψ and in the producer's problem (P), the feasible region over which the bid function Ψ varies has the form

$$C^L = \left\{ \Psi : \Psi(x) = \int_0^x \psi(s)ds + p_0 \text{ and } \psi \text{ verify } (1.1), (1.2), (1.3) \right\}.$$

Then, assuming $\Psi(0) = p_0$, the bid function Ψ is defined as

$$\Psi(x) = \begin{cases} p_0 x + p_0 & \text{if } x \in [q_0, q_1] \\ p_1 x + p_0 + (p_0 - p_1) q_1 & \text{if } x \in [q_1, q_2] \\ p_2 x + p_0 + (p_0 - p_1) q_1 + (p_1 - p_2) q_2 & \text{if } x \in [q_2, q_3] \\ \quad \vdots & \\ p_{K-1} x + p_0 + \displaystyle\sum_{\nu=1}^{K-1} (p_{\nu-1} - p_\nu) q_\nu & \text{if } x \in [q_{K-1}, q_K] \end{cases}$$

or, equivalently,

$$\Psi(x) = p_j x + p_0 + \sum_{\nu=0}^{j} (p_{\nu-1} - p_\nu) q_\nu, \quad \forall x \in [q_j, q_{j+1}]$$

with $p_{-1} = 0$ and $j = 0, \dots, K-1$.

However, a mathematical model based on the bid functions Ψ of C^L would generate many computational difficulties due to the nonsmoothness of these functions. Indeed, in this case, the reformulation of these difficult problems requires modern tools of nonsmooth and variational analysis. Furthermore, as the following example shows, the uniqueness of the solution to the ISO problem is not guaranteed. In this case, for any given decision at the upper level, the problem at the lower level may have more than one solution. This creates ambiguity in the calculation of the value of the upper level objective function. This ambiguity makes it difficult for the leader to predict which point the follower will choose.

Example 1.0.1. We consider $N = 2$, $K = 2$ for all producer and the bilevel problem defined as follows

$$(P_1)\begin{cases} \max\limits_{\Psi_1}[\Psi_1(x_1) - \gamma_1(x_1)] \\ \\ \Psi_1 \in C_1^L \subseteq L^2([0, Q_1]) \\ x = (x_1, x_2) \text{ solves } (ISO) \end{cases} \qquad (P_2)\begin{cases} \max\limits_{\Psi_2}[\Psi_2(x_2) - \gamma_2(x_2)] \\ \\ \Psi_2 \in C_2^L \subseteq L^2([0, Q_2]) \\ x = (x_1, x_2) \text{ solves } (ISO) \end{cases}$$

$$(ISO)\begin{cases} \min\limits_{x}[\Psi_1(x_1) + \Psi_2(x_2)] \\ \\ x_1 \in [0, Q_1] \\ x_2 \in [0, Q_2] \\ x_1 + x_2 = 3/2 \end{cases}$$

In addition, for player 1, we consider the domain $[0, Q_1]$ divided into two intervals by the points

$$\begin{aligned} q_{1,0} &= 0 \\ q_{1,1} &= 2/3 \\ q_{1,2} &= Q_1 = 1 \end{aligned}$$

and the cumulative bid function defined as

$$\psi_1(x_1) = \begin{cases} 1/2 & \text{if } x_1 \in [0, 2/3] \\ 1 & \text{if } x_1 \in (2/3, 1] \end{cases}$$

So, calculating the integral, the price bid function is

$$\Psi_1(x_1) = \begin{cases} 1/2x_1 + 1/2 & \text{if } x_1 \in [0, 2/3] \\ x_1 + 1/6 & \text{if } x_1 \in [2/3, 1] \end{cases}$$

While, for the player 2, we consider the domain $[0, Q_2]$ divided into two intervals by the points

$$\begin{aligned} q_{2,0} &= 0 \\ q_{2,1} &= 1/3 \\ q_{2,2} &= Q_2 = 1 \end{aligned}$$

and the cumulative bid function defined as

$$\psi_2(x_2) = \begin{cases} 2/3 & \text{if } x_2 \in [0, 1/3] \\ 1 & \text{if } x_2 \in (1/3, 1] \end{cases}$$

So, the price bid function is

$$\Psi_2(x_2) = \begin{cases} 2/3x_2 + 2/3 & \text{if } x_2 \in [0, 1/3] \\ x_2 + 5/9 & \text{if } x_2 \in [1/3, 1] \end{cases}$$

We analyze the problem of the follower.

Since $x_1 + x_2 = 3/2$, we have that $x_i \in [1/2, 1]$ for all $i = 1, 2$ and so the function $\Psi_1(x_1) + \Psi_2(x_2)$ is defined as

$$\Psi_1(x_1) + \Psi_2(x_2) = \begin{cases} 1/2x_1 + x_2 + 19/18 & \text{if } (x_1, x_2) \in [1/2, 2/3] \times \big[1/2, 1\big] \\ x_1 + x_2 + 13/18 & \text{if } (x_1, x_2) \in \big[2/3, 1\big] \times \big[1/2, 1\big] \end{cases}$$

Then, the problem (ISO) does not have a unique solution. Indeed, all the points $\big(x_1, 3/2 - x_1\big)$ with $x_1 \in \big[2/3, 1\big]$ are solutions.

To overcome this problem, in the literature, each bid function $\Psi_i \in C_i^L$ is approximated by an increasing quadratic bid function in C_i^Q. Each C_i^Q is a closed and convex subset of

$$\mathcal{Q} = \{f : [0, Q_i] \to \mathbb{R} : f(x) = ax^2 + bx + c \text{ with } a > 0, b \geq 0 \text{ and } c \in \mathbb{R}\}.$$

In this way, however, there is no longer connection with the bid function Ψ_i. For instance, the coefficient c represents the producer's minimum price and is defined on the whole space $\mathbb{R}$, but this minimum value must be at least equal to $p_{i,0}$.

For this reason, the authors in [4] solve the problem of the producer i by replacing the feasible region C_i^L with the following subset of $L^2([0, Q_i], \mathbb{R})$

$$K_i(\Psi_i) = \{y_i \in C_i^Q : y_i(0) \geq \Psi_i(0)\}$$

and introduce a new concept of solution: the projected solution.

In particular, they establish that a projected solution to the problem is formed by N pairs $(\Psi_i, y_i) \in C_i^L \times C_i^Q$ which satisfy the following conditions:

1. the vector of bid functions $\Psi = (\Psi_1, \ldots, \Psi_N)$ is, between all possible vectors of bid functions of $C^L = \prod_{i=1}^N C_i^L$, the best approximation in the sense of L^2-norm of the vector $y = (y_1, \ldots, y_N)$, in other words Ψ is a projection of y on C^L;

2. for each producer $i = 1, \ldots, N$, y_i solves the following optimization problem

$$(P_i) \quad \begin{cases} \max[y_i(x_i) - \gamma_i(x_i)] \\ y_i \in K_i(\Psi_i) \\ x = (x_1, \ldots, x_N) \text{ solves } (ISO) \end{cases}$$

where

$$(ISO) \begin{cases} \min\limits_x [y_1(x_1) + \ldots + y_N(x_N)] \\ x_i \in [0, Q_i], \qquad \forall i = 1, \ldots, N \\ x_1 + \ldots + x_N = D \end{cases}$$

We observe that the objective function of the problem (ISO), under the positivity condition on the coefficients a_i, is strongly convex and continuous, and the constraint set of this problem is convex and closed. Thus the problem (ISO) admits a unique solution.

Chapter 2

Mathematical preliminaries

2.1 Set-valued maps: basic notions and continuity results

In this section, we collect the basic notions of set-valued maps, that is a function whose values are sets, which will be used throughout the thesis. The concepts given can be seen in [1, 13] and the references therein.

Definition 2.1.1. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be nonempty sets. A set-valued map φ from X to Y assigns to each $x \in X$ a subset $\varphi(x)$ of Y. We write $\varphi : X \rightrightarrows Y$ to distinguish a set-valued map from a function from X to Y.

Naturally, whenever the set-valued map is single-valued, that means $\varphi(x)$ is a singleton for any x, it reduces to a function.

The domain of a set-valued map φ is

$$\operatorname{dom} \varphi = \{x \in X : \varphi(x) \neq \emptyset\}$$

its graph is

$$\operatorname{gph} \varphi = \{(x, y) \in X \times Y : y \in \varphi(x)\}$$

and is said to be closed-valued, or to have closed values, if $\varphi(x)$ is a closed subset of Y for each x. The terms "open-valued," "compact-valued," "convex-valued," etc., are defined similarly.

For the closure of a set-valued map there is a local definition.

Definition 2.1.2. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be nonempty sets. A set-valued map $\varphi : X \rightrightarrows Y$ is said to be closed at $x \in X$ if for all $\{x_k\} \subseteq X$ such that $x_k \to x$ and $y_k \in \varphi(x_k)$ such that $y_k \to y$, then $y \in \varphi(x)$.

We say that φ is closed on X, or closed, if it is closed at every $x \in X$.

It is also possible to give a characterization of the global closure of a set-valued map in terms of its graph. Indeed, φ is closed if, and only if, its graph is a closed subset of $X \times Y$. Moreover, φ is said open if, and only if, its graph is an open subset of $X \times Y$. In general, a closed set-valued map is always closed-valued, but the converse is false.

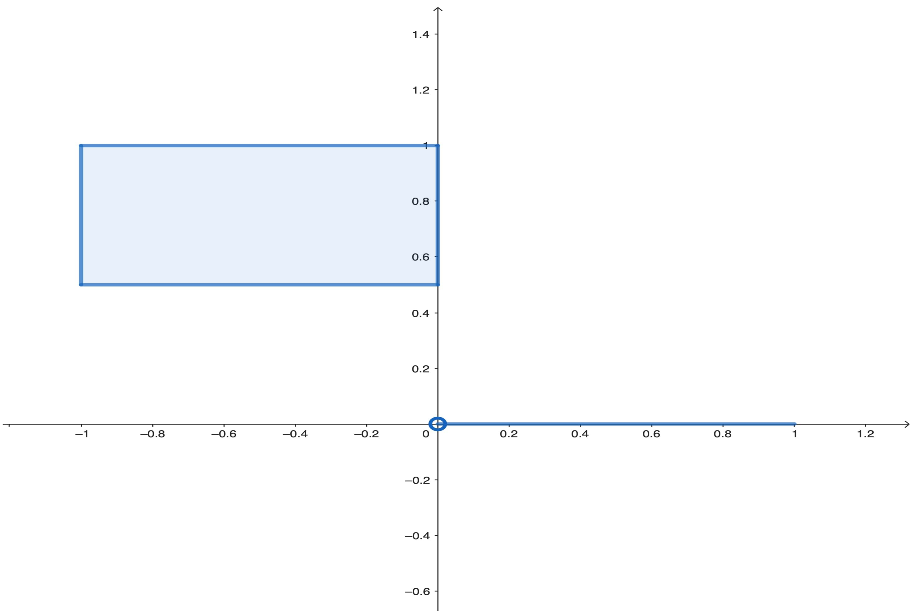

Figure 2.1: Graph of φ

For instance, the set-valued map $\varphi : [-1, 1] \rightrightarrows [0, 1]$ (Figure 2.1) defined as

$$\varphi(x) = \begin{cases} [1/2, 1] & \text{if } x \leq 0 \\ \{0\} & \text{if } x > 0 \end{cases}$$

is closed-valued but its graph is not closed. In order to introduce the continuity concept of a set-valued map, we start by reminding that a function $f : X \to Y$ is continuous at $\bar{x} \in X$ if, and only if,

$$\forall U_{f(\bar{x})} \quad \exists V_{\bar{x}} \text{ s.t. } f(x) \in U_{f(\bar{x})} \quad \forall x \in V_{\bar{x}}.$$

For the set-valued map, there are two equivalent expressions for the condition "$f(x) \in U_{f(\bar{x})}$"

$$\text{either } \{f(x)\} \subseteq U_{f(\bar{x})} \text{ or } \{f(x)\} \cap U_{f(\bar{x})} \neq \emptyset.$$

This entails two definitions of continuity for a set-valued map: upper and lower semicontinuity.

Definition 2.1.3. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be nonempty sets. A set-valued map $\varphi : X \rightrightarrows Y$ is said

- lower semicontinuous at $\bar{x}$ if for each open set Ω such that $\varphi(\bar{x}) \cap \Omega \neq \emptyset$ there exists a neighborhood $U_{\bar{x}}$ such that $\varphi(x) \cap \Omega \neq \emptyset$ for every $x \in U_{\bar{x}}$;
- upper semicontinuous at $\bar{x}$ if for each open set Ω such that $\varphi(\bar{x}) \subseteq \Omega$ there exists a neighborhood $U_{\bar{x}}$ such that $\varphi(x) \subseteq \Omega$ for every $x \in U_{\bar{x}}$;
- continuous at $\bar{x}$ if it is both upper and lower semicontinuous at $\bar{x}$.

We observe that if φ is lower semicontinuous, then it has open domain.

In a similar way to continuity, since the inverse image of a subset $A \subseteq Y$ under a function f is the set

$$f^{-1}(A) = \{x \in X : f(x) \in A\}$$

for a set-valued map there are two equivalent expressions:

$$\varphi^u(A) = \{x \in X : \varphi(x) \subseteq A\}$$

that is the upper inverse of A, and

$$\varphi^\ell(A) = \{x \in X : \varphi(x) \cap A \neq \emptyset\}$$

that is the lower inverse of A. Furthermore, for any $y \in Y$ it is possible to define a natural inverse, that is the set-valued map $\varphi^{-1} : Y \rightrightarrows X$ defined as

$$\varphi^{-1}(y) = \{x \in X : y \in \varphi(x)\} = \varphi^\ell(\{y\}).$$

The set $\varphi^{-1}(y)$ is also called the lower section of φ at y.

The characterization of semicontinuity by upper and lower inverse is shown in the following lemma.

Lemma 2.1.1. *Let $\varphi : X \rightrightarrows Y$ be a set-valued map. Then, the following statements are equivalent:*

1. *φ is upper (lower) semicontinuous;*
2. *$\varphi^u(V)$ is open (closed) for each open (closed) subset V of Y;*
3. *$\varphi^\ell(F)$ is closed (open) for each closed (open) subset F of Y.*

It is immediate that for a single-valued map the concepts of upper and lower semicontinuity coincide, and in this case, it is continuous as a function. While, as we see in the following two examples, for a set-valued map these two concepts are different.

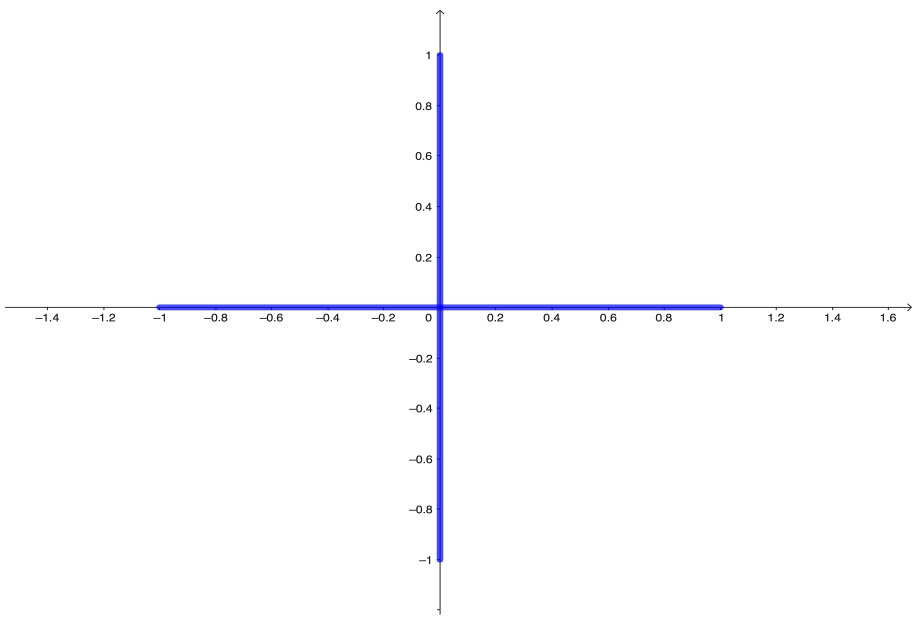

Figure 2.2: Graph of φ

Example 2.1.1. The set-valued map $\varphi : [-1, 1] \rightrightarrows [-1, 1]$ (Figure 2.2) defined as

$$\varphi(x) = \begin{cases} [-1,1] & \text{if } x = 0 \\ \{0\} & \text{if } x \neq 0 \end{cases}$$

is upper semicontinuous but not lower semicontinuous. Indeed, for all closed subset $C \subseteq [-1,1]$ we have that

$$\varphi^{\ell}(C) = \begin{cases} [-1,1] & \text{if } 0 \in C \\ \{0\} & \text{if } 0 \notin C \end{cases}$$

Then, φ is upper semicontinuous by Lemma 2.1.1. While, if we consider $C = [-1/2, 1/2]$, its upper inverse is $\varphi^u(C) = [-1,0) \cup (0,1]$ and so φ is not lower semicontinuous by Lemma 2.1.1 again.

Example 2.1.2. The set-valued map $\varphi : [-1,1] \rightrightarrows [-1,1]$ (Figure 2.3) defined as

$$\varphi(x) = \begin{cases} [-1,0] & \text{if } x < 0 \\ \{0\} & \text{if } x = 0 \\ [0,1] & \text{if } x > 0 \end{cases}$$

is lower semicontinuous but not upper semicontinuous. Indeed, for all closed subset

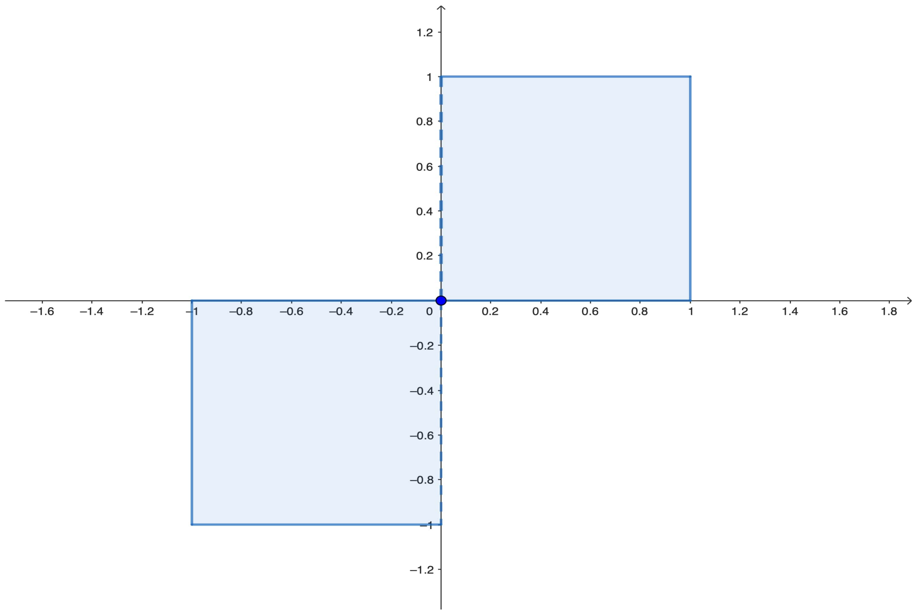

Figure 2.3: Graph of φ

$C \subseteq [-1,1]$ we have that

$$\varphi^u(C) = \begin{cases} \{0\} & \text{if } 0 \in C \\ \emptyset & \text{if } 0 \notin C \end{cases}$$

Then, φ is lower semicontinuous by Lemma 2.1.1. While, if we consider $C = [-1/2, -1/3]$, its lower inverse is $\varphi^\ell(C) = [-1, 0)$ and so φ is not upper semicontinuous by Lemma 2.1.1 again.

Because sequences are often used to describe the continuity of functions, it is also useful to describe semicontinuity in terms of sequences. In particular, we have that

- if φ has compact values, then φ is upper semicontinuous at x if, and only if, for every sequence $x_k \to x$ and $y_k \in \varphi(x_k)$ there exists a subsequence $\{y_{k_j}\} \subseteq \{y_k\}$ such that $y_{k_j} \to y \in \varphi(x)$;
- φ is lower semicontinuous at x if, and only if, for every sequence $x_k \to x$ and $y \in \varphi(x)$ there exists a sequence $y_k \in \varphi(x_k)$ such that $y_k \to y$.

We analyze the relationship between the closedness and the upper semicontinuity of a set-valued map. For closed-valued maps whose image is contained in a compact set, the two definitions coincide, but in general, a set-valued map can be closed without being upper semicontinuous, and vice versa. For instance, the set-valued map $\varphi : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as

$$\varphi(x) = \begin{cases} \{1/x\} & \text{if } x \neq 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

is closed but not upper semicontinuous. While, if we consider the set-valued map $\varphi : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as $\varphi(x) = (0, 1)$, it is upper semicontinuous but not closed.

The following theorem formalize this relationship.

Theorem 2.1.1 (Closed graph)**.** *Let $\varphi : X \rightrightarrows Y$ be a set-valued map. If Y is compact, then φ is closed if, and only if, it is upper semicontinuous and closed-valued.*

Furthermore, regarding the open graph of a set-valued map, we have the following implications:

$$\text{open graph} \quad \Rightarrow \quad \text{open lower sections} \quad \Rightarrow \quad \text{lower semicontinuity.}$$

2.2 Operations on set-valued maps

In this section, we examine the preservation of semicontinuity under various operations on set-valued maps, which we will use later on.

First of all, given a nonempty subset $C \subseteq \mathbb{R}^n$, we denote by $\operatorname{cl} C$ the closure of C and by $\operatorname{co} C$ the convex hull of C. So, we define the closure and the convex hull of a set-valued map.

Definition 2.2.1. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be nonempty sets and $\varphi : X \rightrightarrows Y$ be a set-valued map. The closure and the convex hull of φ are respectively defined as

- $(\operatorname{cl} \varphi)(x) = \operatorname{cl}(\varphi(x))$,
- $(\operatorname{co} \varphi)(x) = \operatorname{co}(\varphi(x))$.

The following two propositions show the properties of these operations.

Proposition 2.2.1. *Let $\varphi : X \rightrightarrows Y$ be a set-valued map and $x \in X$.*

1. $\operatorname{cl} \varphi$ *is lower semicontinuous at x if, and only if, φ is lower semicontinuous at x;*
2. *if φ is upper semicontinuous at x, then* $\operatorname{cl} \varphi$ *is also upper semicontinuous at x.*

Proposition 2.2.2. *Let $\varphi : X \rightrightarrows Y$ be a set-valued map with the set Y convex and $x \in X$.*

1. *if φ is compact-valued and upper semicontinuous at x, then* $\operatorname{co} \varphi$ *is upper semicontinuous at x;*
2. *if φ is lower semicontinuous at x, then* $\operatorname{co} \varphi$ *is lower semicontinuous at x;*
3. *if φ has open graph, then* $\operatorname{co} \varphi$ *has open graph.*

In general, even if φ is compact-valued and closed, then $\operatorname{co} \varphi$ may still fail to be closed. Indeed, for instance, if we consider the set-valued map $\varphi : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as

$$\varphi(x) \begin{cases} \{0, 1/x\} & \text{if } x \neq 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

it is closed with compact values but, the set-valued map $\operatorname{co}\varphi$ (Figure 2.4) defined

$$\operatorname{co}\varphi(x)\begin{cases}[0,1/x] & \text{if } x\neq 0\\ \{0\} & \text{if } x=0\end{cases}$$

is not closed.

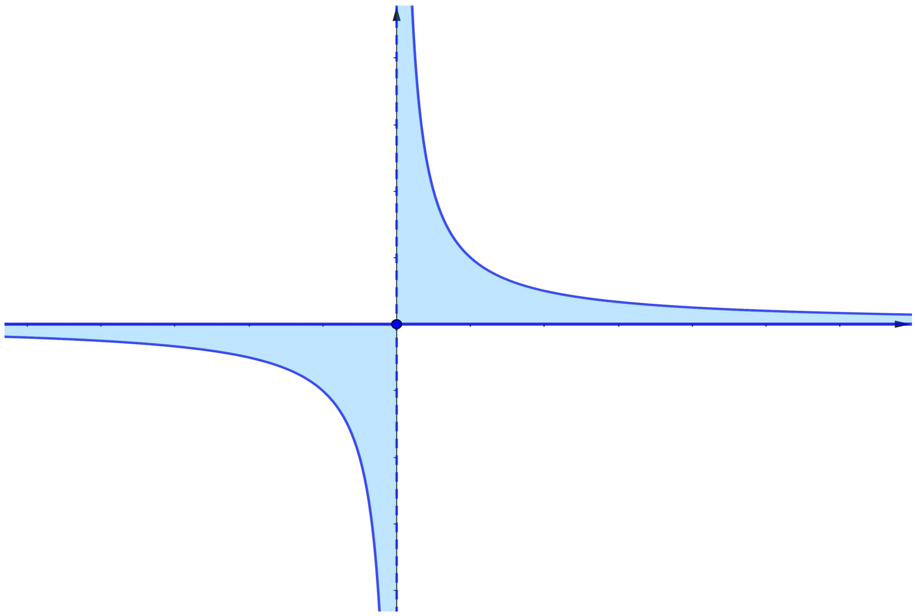

Figure 2.4: Graph of $\operatorname{co}\varphi$

Definition 2.2.2. Let $X\subseteq\mathbb{R}^n$, $Y\subseteq\mathbb{R}^m$ and $Z\subseteq\mathbb{R}^p$ be a nonempty sets and $\varphi : X \rightrightarrows Y$ and $\psi : Y \rightrightarrows Z$ be a set-valued maps. The composition of φ and ψ is the set-valued map $\psi\circ\varphi : X \rightrightarrows Z$ defined as

$$(\psi\circ\varphi)(x)=\bigcup_{y\in\varphi(x)}\psi(y).$$

The properties of the composition are shown in the following proposition.

Proposition 2.2.3. *Let $\varphi : X \rightrightarrows Y$ and $\psi : Y \rightrightarrows Z$ be set-valued maps.*

1. *If φ and ψ are upper semicontinuous, then $\psi\circ\varphi$ is upper semicontinuous;*
2. *if φ and ψ are lower semicontinuous, then $\psi\circ\varphi$ is lower semicontinuous.*

We point out that the composition of closed set-valued maps need not be closed. For instance, we consider the set-valued map $\varphi : [0, +\infty) \rightrightarrows [0, +\infty)$ defined as

$$\varphi(x) = \begin{cases} \{0, 1/x\} & \text{if } x > 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

and a set-valued map $\psi : [0, +\infty) \rightrightarrows [0, +\infty)$ defined as $\psi(y) = \{y/(1 + y)\}$. These

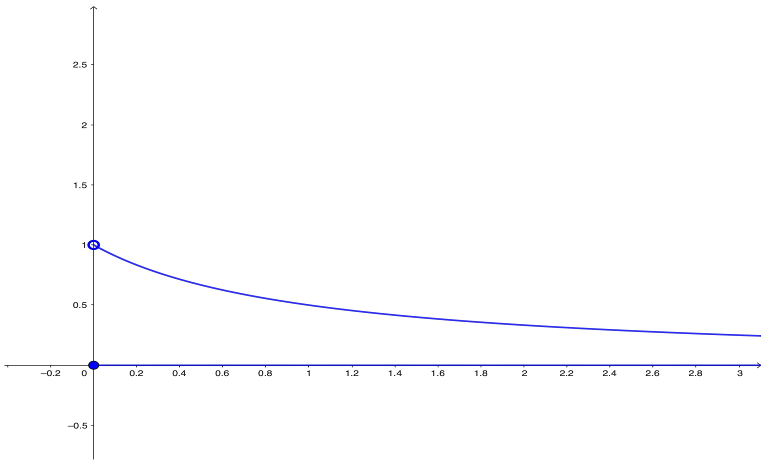

Figure 2.5: Graph of $\psi \circ \varphi$

set-valued maps are closed but, the composition $\psi \circ \varphi$ (Figure 2.5) defined as

$$(\psi \circ \varphi)(x) = \begin{cases} \{0, 1/(x + 1)\} & \text{if } x > 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

is not closed.

Definition 2.2.3. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be nonempty sets and $\varphi, \psi : X \rightrightarrows Y$ be set-valued maps. The union and the intersection of φ and ψ are respectively defined as

- $(\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x)$;
- $(\varphi \cap \psi)(x) = \varphi(x) \cap \psi(x)$.

We observe that the graph of the union (intersection) is the union (intersection) of the graphs.

The properties of the union and intersection are demonstrated by the following propositions.

Proposition 2.2.4. *Let $\varphi, \psi : X \rightrightarrows Y$ be set-valued maps. If ψ and φ are lower (upper) semicontinuous, then $\varphi \cup \psi$ is lower (upper) semicontinuous.*

Proposition 2.2.5. *Let $\varphi, \psi : X \rightrightarrows Y$ be set-valued maps.*

1. *If ψ and φ are closed, then $\varphi \cap \psi$ is closed;*
2. *if there exists $x \in X$ such that $\varphi(x) \cap \psi(x) \neq \emptyset$, ψ and φ are upper semicontinuous at x and closed-valued, then $\varphi \cap \psi$ is upper semicontinuous at x;*
3. *if there exists $x \in X$ such that $\varphi(x) \cap \psi(x) \neq \emptyset$, ψ is lower semicontinuous at x and φ has open graph, then the $\varphi \cap \psi$ is lower semicontinuous at x.*

We point out that the intersection of lower semicontinuous set-valued maps need not be lower semicontinuous. For instance, if we consider the set-valued maps $\varphi, \psi : [0, 1] \rightrightarrows [0, 1]$ defined as

$$\varphi(x) = \{x\} \quad \text{and} \quad \psi(x) = [1/2, 1]$$

they are lower semicontinuous but $\varphi \cap \psi$ defined as

$$\varphi(x) \cap \psi(x) = \begin{cases} \emptyset & \text{if } x < 1/2 \\ \{x\} & \text{if } x \geq 1/2 \end{cases}$$

is not lower semicontinuous since its domain is closed and then it is not open.

Definition 2.2.4. Let $X \subseteq \mathbb{R}^n$ and $Y_i \subseteq \mathbb{R}^{m_i}$ for all $i = 1, \ldots, k$ be nonempty sets and $\{\varphi_i\}_{i=1}^k$ be a family of the set-valued maps $\varphi_i : X \rightrightarrows Y_i$. Let $Y = \prod_{i=1}^k Y_i$, the product of the family is the set-valued map $\prod_{i=1}^k \varphi_i : X \rightrightarrows Y$ defined as

$$\left(\prod_{i=1}^k \varphi_i\right)(x) = \prod_{i=1}^k \varphi_i(x).$$

The following proposition shows the properties of the product.

Proposition 2.2.6. *The product of the family* $\{\varphi_i\}_{i=1}^k$ *satisfies the following properties:*

1. *if each* φ_i *is upper semicontinuous and compact-valued, then the product is upper semicontinuous and compact-valued;*

2. *the product is lower semicontinuous if, and only if, each* φ_i *is lower semicontinuous;*

3. *if each* φ_i *has open (closed) graph, then the product has open (closed) graph.*

It is also possible to define the product of set-valued maps when they have different domains.

Definition 2.2.5. Let $X_i \subseteq \mathbb{R}^{n_i}$ and $Y_i \subseteq \mathbb{R}^{m_i}$ for all $i = 1, \dots, k$ be nonempty sets and $\{\varphi_i\}_{i=1}^k$ be a family of the set-valued maps $\varphi_i : X_i \rightrightarrows Y_i$. Let $Y = \prod_{i=1}^k Y_i$ and $X = \prod_{i=1}^k X_i$, the product of the family is the set-valued map $\prod_{i=1}^k \varphi_i : X \rightrightarrows Y$ defined as

$$\left(\prod_{i=1}^k \varphi_i\right)(x_1, \dots, x_k) = \prod_{i=1}^k \varphi_i(x_i).$$

Also in this case, the product requires the same assumptions for semicontinuity.

Proposition 2.2.7. *The product of the family* $\{\varphi_i\}_{i=1}^k$ *satisfies the following properties:*

1. *if each* φ_i *is upper semicontinuous and compact-valued, then the product is upper semicontinuous and compact-valued;*

2. *the product is lower semicontinuous if, and only if, each* φ_i *is lower semicontinuous.*

The following property concludes this section.

Proposition 2.2.8. *Let* $X \subseteq \mathbb{R}^n$ *and* $Y \subseteq \mathbb{R}^m$ *be nonempty sets,* $C \subseteq X$ *be a subset,* $\varphi_1 : X \rightrightarrows Y$ *and* $\varphi_2 : C \rightrightarrows Y$ *be lower semicontinuous set-valued maps.*

If C is closed and φ_2 verify the condition $\varphi_2(x) \subseteq \varphi_1(x)$ for all $x \in C$, then the set-valued map $\Phi : X \rightrightarrows Y$ defined as

$$\Phi(x) = \begin{cases} \varphi_1(x) & \text{if } x \notin C \\ \varphi_2(x) & \text{if } x \in C \end{cases}$$

is lower semicontinuous.

2.3 Fixed point and the Berge maximum theorem

The first part of this section deals with continuous selection theorems, i.e. theorems that assert the existence of a continuous function whose graph is contained in the graph of a set-valued map. There are several continuous selection results that play a crucial role in various fields of study, including differential inclusion, optimal control, and mathematical economics. In the context of differential inclusion, these results help to characterize solutions to differential equations with multiple possible outcomes. Moreover, in mathematical economics, they are employed to establish the existence of equilibrium solutions in economic models.

Definition 2.3.1. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be nonempty sets and $\varphi : X \rightrightarrows Y$ be a set-valued map. The function $f : X \to Y$ is said to be a selection of φ if $f(x) \in \varphi(x)$ for each $x \in X$.

We say that f is a continuous selection if f is a selection and it is continuous. The Axiom of Choice guarantees that set-valued maps with non-empty domain always admit selections, but they may not have any other useful properties. Michael proved a series of results on the existence of continuous selections in [41]. The theorems say that a nonempty set-valued map admits a continuous selection under suitable assumptions of semicontinuity, closure, and convexity. The most famous continuous selection theorem is the following result.

Theorem 2.3.1 (Michael)**.** *Let $X \subseteq \mathbb{R}^n$ be a nonempty set and $\varphi : X \rightrightarrows \mathbb{R}^m$ be a lower semicontinuous set-valued map with nonempty and convex values. Then, there exists a continuous selection of φ.*

Under suitable assumptions on the set-valued maps, operations between set-valued maps admits a continuous selection. Specifically, in this thesis we will use the following property of the intersection.

Proposition 2.3.1. *Let $\varphi : X \rightrightarrows \mathbb{R}^m$ be a lower semicontinuous set-valued map with closed and convex values, and $\phi : X \rightrightarrows \mathbb{R}^m$ be a set-valued map with convex values and open graph. If $\varphi(x) \cap \phi(x) \neq \emptyset$ for all $x \in X$, then there exists a continuous selection of $\varphi \cap \phi$.*

As for a function, it is possible to consider the concept of fixed point for a set-valued map.

Definition 2.3.2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a function and $\varphi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map. A point $x \in \mathbb{R}^n$ is said

- fixed point of a function f if it satisfies $x = f(x)$;
- fixed point of a set-valued map φ if it satisfies $x \in \varphi(x)$.

Brouwer's fixed point theorem, one of the most famous fixed point theorems, occupies an important place in various fields of study. This result is one of the key theorems characterizing the topology of Euclidean spaces and this gives it a place among the fundamental theorems. This theorem has far-reaching applications beyond topology, extending its influence to other branches of mathematics. For instance, it plays a crucial role in proving results related to differential equations, shedding light on the behaviour of solutions in various contexts. Moreover, Brouwer's theorem finds practical applications in fields like game theory and economics, where it helps establish important concepts and principles.

Theorem 2.3.2 (Brouwer)**.** *Let $C \subseteq \mathbb{R}^n$ be a nonempty, compact and convex set and $f : C \to C$ be a function. If f is continuous, then it has a fixed point.*

A generalization of the Brouwer theorem for set-valued maps is the Kakutani fixed point theorem, developed by Shizuo Kakutani in [39]. This theorem extends Brouwer's results via the selection theorem and has found widespread application in game theory and economics.

Theorem 2.3.3 (Kakutani)**.** *Let $C \subseteq \mathbb{R}^n$ be a nonempty, compact and convex set and $\varphi : C \rightrightarrows C$ be a set-valued map. Suppose that*

(i) φ is closed;

(ii) φ has nonempty compact and convex values.

Then, φ has a fixed point.

Subsequently, there are proved also fixed point theorems in [14] for lower semicontinuous set-valued map. The main result is the following.

Theorem 2.3.4. *Let $C \subseteq \mathbb{R}^n$ be a nonempty, compact and convex set and $\varphi : C \rightrightarrows C$ be a set-valued map. Suppose that*

(i) φ is lower semicontinuous;

(ii) φ has nonempty and convex values.

Then, φ has a fixed point.

The set of fixed points of a set-valued map has the following property, which we will use later.

Lemma 2.3.1. *Let $\varphi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map. If φ has closed graph, then the set of fixed points of φ is closed.*

At the end of this section we will look at the Berge maximum theorem. As a first step, we recall the concept of semicontinuity for an extended real function.

Definition 2.3.3. Let $X \subseteq \mathbb{R}^n$ be nonempty set. A function $f : X \to [-\infty, +\infty]$ is said

- lower semicontinuous if for each $a \in \mathbb{R}$ the set $\{x \in X : f(x) \leq a\}$ is closed in X, or equivalently the set $\{x \in X : f(x) > a\}$ is open in X;
- upper semicontinuous if for each $a \in \mathbb{R}$ the set $\{x \in X : f(x) \geq a\}$ is closed in X, or equivalently the set $\{x \in X : f(x) < a\}$ is open in X;

- continuous if it is both upper and lower semicontinuous.

Clearly, a function f is lower semicontinuous if, and only if, $-f$ is upper semicontinuous, and vice versa.

We are now ready to see the Berge maximum theorem which is primarily used in mathematical economics and optimal control. The maximum theorem, which is due to Berge [7], states that the set of solutions to a well-behaved constrained maximization problem is upper semicontinuous in its parameters and that the value function is continuous.

Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be nonempty sets, $\varphi : X \rightrightarrows Y$ be a set-valued map and $f : \operatorname{gph} \varphi \to \mathbb{R}$ be a function. Define the value function $m : X \to [-\infty, +\infty]$ as

$$m(x) = \sup_{z \in \varphi(x)} f(x, z)$$

where, as usual, $\sup \emptyset = -\infty$, and the set-valued map $M : X \rightrightarrows Y$ of maximizers as

$$M(x) = \{y \in \varphi(x) : f(x, y) = m(x)\}.$$

The Berge maximum theorem is a consequence of the following two lemmas.

Lemma 2.3.2. *If φ and f are lower semicontinuous, then the function m is lower semicontinuous.*

Lemma 2.3.3. *If φ is upper semicontinuous with nonempty compact values and f is upper semicontinuous, then the function m is upper semicontinuous.*

Theorem 2.3.5 (Berge)**.** *If φ is continuous with nonempty compact values and f is continuous, then:*

1. *the value function m is continuous;*

2. *the argmax set-valued map M is upper semicontinuous with nonempty and compact values.*

One of the limitations of applications of this famous result is that the sets of feasible actions are assumed to be compact. This assumption was weakened by

Hogan in [38] and subsequently by Bonnas and Shapiro in [12]. In this section we will focus exclusively on Hogan's results, as they will be used in the following chapters.

Hogan weakens the assumptions of compactness by means of the following local concept of uniformly compact.

Definition 2.3.4. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be nonempty sets and $x \in X$. A set-valued map $\varphi : X \rightrightarrows Y$ is said to be uniformly compact near x if there is a neighborhood U_x of x such that the closure of the set $\bigcup_{z \in U_x} \varphi(z)$ is compact.

The paper also provides a characterization for the concept of uniformly compact.

Theorem 2.3.6. *Let φ be lower semicontinuous at x with closed and convex values on a neighborhood of $x \in X$, and f be continuous and quasiconcave with respect to its second variable. Then, $M(x)$ is nonempty and compact if, and only if, M is uniformly compact near x and nonempty valued on a neighborhood of x.*

The main results of Hogan are the following continuity results.

Theorem 2.3.7. *Let φ be lower semicontinuous, closed at $x \in X$ and uniformly compact near x, and f be continuous on $\{x\} \times \varphi(x)$. Then, m is continuous on x.*

Theorem 2.3.8. *Let φ be lower semicontinuous and closed at $x \in X$, f be continuous on $\{x\} \times \varphi(x)$ and M be uniformly compact near x with nonempty values and $M(x)$ a singleton. Then, M is lower semicontinuous and closed at x.*

Note that, if M is a single-valued map in a neighborhood of x the theorem states that M is continuous at x.

2.4 Equilibrium and quasiequilibrium problems

In various fields such as physics, chemistry, biology, engineering, and economics, the term "equilibrium" is used to describe conditions or states of a system in which all competing influences are balanced. Mathematical models, including optimization, variational inequalities, multi-objective optimization, and noncooperative games, are used to express this concept. Despite their diversity, these models share a common underlying structure that allows for a unified formulation: the famous minimax inequality of Ky Fan [35]. This is a very general mathematical model that includes formats from different disciplines. Blum and Oettli [11] considered this problem as a general equilibrium model, which is why it is now referred to as an "equilibrium problem" in the literature. In recent years, researchers have extensively studied this general problem, as highlighted in [10] and the references therein.

Definition 2.4.1. Let $C \subseteq \mathbb{R}^n$ be a nonempty set and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a function; the equilibrium problem is the following:

$$EP(f, C) \qquad \text{find } \bar{x} \in C \text{ s.t. } f(\bar{x}, u) \geq 0, \quad \forall u \in C.$$

The history of existence theorems for equilibrium problems can be traced back to 1972 when Ky Fan in [35] proposed a famous minimax result in a real Hausdorff topological vector space.

First of all, we see that the set of solutions of the EP(f, C) can be viewed as the intersection of a family of sets. Indeed, let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map defined as

$$T(u) = \{x \in C : f(x, u) \geq 0\}$$

then

$$\bar{x} \in C \text{ solves } EP(f, C) \quad \text{iff} \quad \bar{x} \in \bigcap_{u \in C} T(u). \tag{2.1}$$

Geometrical tools, such as the Knaster-Kuratowski-Mazurkiewicz lemma, are useful in proving the existence of solutions.

Lemma 2.4.1 (KKM)**.** *Let I be a finite set of indices, $\{x_i\}_{i\in I} \subseteq \mathbb{R}^n$ and $\{C_i\}_{i\in I}$ be a family of closed subsets of $\mathbb{R}^n$. If the inclusion*

$$\mathrm{co}(\{x_i\}_{i\in J}) \subseteq \bigcup_{i\in J} C_i$$

holds for any $J \subseteq I$, then the intersection

$$\mathrm{co}(\{x_i\}_{i\in I}) \cap \bigcap_{i\in I} C_i$$

is nonempty and compact.

A set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to satisfy the KKM property on a given $C \subseteq \mathbb{R}^n$ if the inclusion

$$\text{co}(\{x_i\}_{i\in I}) \subseteq \bigcup_{i\in I} G(x_i)$$

holds for any finite collection $\{x_i\}_{i\in I} \subseteq C$.

To prove that the intersection of the images of such a map is nonempty, Ky Fan introduced an equivalent version of Lemma 2.4.1. This version requires the compactness of at least one image, and he used it to prove his existence result.

Lemma 2.4.2 (Fan-KKM). *Let $C \subseteq \mathbb{R}^n$ be a nonempty set and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map with closed values. If G satisfies the KKM property on C and there exists $\bar{x} \in C$ such that $G(\bar{x})$ is compact, then the intersection $\bigcap_{x\in C} G(x)$ is nonempty and compact.*

Now, we see an adjusted version in a finite dimensional setting of the result of Ky Fan [35].

Theorem 2.4.1 (Ky Fan minimax inequality). *Let C be a nonempty, compact and convex set. Then, the EP(f, C) has at least one solution if the following properties hold:*

(i) $f(x, \cdot)$ is quasiconvex on C for all $x \in C$;

(ii) $f(\cdot, u)$ is upper semicontinuous for all $u \in C$;

(iii) $f(x, x) = 0$ for all $x \in C$.

Proof. According to characterization (2.1), a solution of EP(f, C) exists if, and only if,

$$\bigcap_{u\in C} T(u) \neq \emptyset.$$

Thanks to *(ii)* and since C is compact, $T(u)$ is compact for all $u \in C$. Additionally, T verifies the KKM property on C. Indeed, for a finite collection $\{x_i\}_{i\in I} \subseteq C$, it can be observed that any $x \in \text{co}\{x_i\}_{i\in I}$ satisfies

$$0 = f(x, x) \leq \max\{f(x, x_i) : i \in I\}$$

where the inequality descend from *(i)*. This ensures the existence of $j \in I$ such that $f(x, x_j) \geq 0$. So,

$$x \in T(x_j) \subseteq \bigcup_{i \in I} T(x_i)$$

and, thanks to Lemma 2.4.2, we have

$$\bigcap_{u \in C} T(u) \neq \emptyset.$$

□

The assumptions of Theorem 2.4.1 can be weakened also in two other different directions: either by weakening the continuity assumption of $f(\cdot, u)$ but adding some monotonicity condition, or by replacing the boundedness of the feasible set by a suitable coercivity condition (see [10] and the references therein).

The quasiequilibrium problem is a generalization of an equilibrium problem where the constraint set depends on the considered point. This setting was first studied in the context of the impulse control problem [5] and has subsequently been used by several authors to describe problems that arise in different fields, such as equilibrium problems in mechanics, economics, and network equilibrium problems. This format has gained popularity in recent years because theoretical results developed for one model can often be extended to others through the common language provided by this format.

Definition 2.4.2. Let $K : C \rightrightarrows \mathbb{R}^n$ be a set-valued map; the quasiequilibrium problem is the following:

$$QEP(f, K) \qquad \text{find } \bar{x} \in K(\bar{x}) \text{ s.t. } f(\bar{x}, u) \geq 0, \quad \forall u \in K(\bar{x}).$$

Clearly, an equilibrium problem is a quasiequilibrium problem where $K(x) = C$ for all $x \in C$.

Unlike the equilibrium problem, for which there is an extensive literature on results concerning the existence of solutions, the study of the quasiequilibrium problem is still at the beginning, and the first work in this area is [43]. Subsequently, the problem of the existence of solutions has been developed in several papers (see for example [2, 3, 19, 21, 28] and the references therein).

In this section we will see a result obtained in [21], which is a generalization for quasiequilibrium problem of Ky Fan's result. The technique used in [21] is similar to that originally introduced in [29]. The main difference being that the result in [21] requires the upper semicontinuity of f separately in its two variables and not globally: in this way, the theorem essentially collapses to the result of Ky Fan when the quasiequilibrium problem turns out to be a equilibrium problem.

Theorem 2.4.2. *Let C be a nonempty, compact and convex set, $K : C \rightrightarrows \mathbb{R}^n$ be a set-valued map with $K(C) \subseteq C$ and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a function. Then, the QEP(f, K) admits a solution if the following properties hold:*

(i) *K is lower semicontinuous with nonempty and convex values;*

(ii) $\operatorname{fix} K$ *is closed;*

(iii) *f is upper semicontinuous;*

(iv) *$f(x, \cdot)$ is quasiconvex for all $x \in C$.*

Remark 2.4.1. Using the set-valued map $F : C \rightrightarrows C$ defined as

$$F(x) = \{u \in C : f(x, u) < 0\} \tag{2.2}$$

already considered in the literature (see for example [18, 29]), it is possible to slightly relax the continuity condition on f in the previous theorem. In fact, the assumptions *(iii)* and *(iv)* can be replaced by

(iii-a) F is lower semicontinuous for all $x \in \operatorname{fix} K$ and has convex values for all $x \in \operatorname{fix} K$;

(iv-a) $F \cap K$ is lower semicontinuous for all $x \in \operatorname{bd}_C(\operatorname{fix} K)$, where $\operatorname{bd}_C(\operatorname{fix} K)$ indicates the boundary of $\operatorname{fix} K$ in C.

Since it would be desirable to find more tractable conditions on f, disjoint from those assumed on K, in the same paper [21] the authors provide sufficient conditions for the assumption *(iv-a)* in two different ways. In the first approach, they add the condition that C is a polytope, i.e. the convex hull of a finite set. In the second approach, they use the concept of affine hull of a set C, denoted by $\operatorname{aff} C$, i.e. the smallest affine set containing C, or equivalently, the intersection of all affine sets containing C.

Finally, we show that apparently different mathematical equilibrium models can be viewed as special cases of the Ky Fan inequalities by choosing appropriate values of f.

2.4.1 Variational and quasivariational inequalities

The topic of variational inequalities originates from the calculus of variations, which deals with the minimization of infinite dimensional functionals. The systematic study of this subject began in the early 1960s with Stampacchia, who used variational inequalities as an analytical tool to study free boundary problems defined by nonlinear partial differential operators arising from problems in elasticity and mechanics. Variational inequalities, along with other equilibrium problems, have been a traditional topic in mathematical physics. They have also been studied in optimization theory for applications in transportation planning, regional science, socio-economic analysis, energy modelling, and game theory.

Definition 2.4.3. Let $C \subseteq \mathbb{R}^n$ be a nonempty set and $T : C \rightrightarrows \mathbb{R}^n$ be a set-valued map; a variational inequality is the following:

$$VI(T, C) \qquad \text{find } \bar{x} \in C \text{ s.t. } \exists \bar{x}^* \in T(\bar{x}) \text{ with } \langle \bar{x}^*, x - \bar{x} \rangle \geq 0,\ \forall x \in C.$$

Every solution of the VI(T, C) is a solution of the EP(f, C) with the function f defined as

$$f(x, u) = \sup_{x^* \in T(x)} \langle x^*, u - x \rangle.$$

The reverse is true only if T has compact values.

The quasivariational inequality is an extension of a variational inequality where the constrained set is modified depending on the point considered. This type of variational inequality is commonly used to model problems in a variety of fields, including transport, telecommunications, and economics.

Definition 2.4.4. Let $K : C \rightrightarrows \mathbb{R}^n$ be a set valued map; a quasivariational inequality is the following:

$$QVI(T,K) \qquad \text{find } \bar{x} \in K(\bar{x}) \text{ s.t. } \exists \bar{x}^* \in T(\bar{x}) \text{ with } \langle \bar{x}^*, x - \bar{x} \rangle \geq 0,\ \forall x \in K(\bar{x}).$$

In a similar way to a variational inequality, a quasivariational inequality can be seen as a special case of a quasiequilibrium problem.

Bensoussan and Lyons [6] introduced the concept of quasi-variational inequality for the first time in the context of impulse control, using the single-valued map T. The case where T is a set-valued map was first considered by Chan and Pang [23].

The main existence result for quasivariational inequalities, presented in [51], assumes T is upper semicontinuous and K lower semicontinuous. Many efforts have been made to obtain existence results with weaker continuity assumptions, essentially considering general monotonicity assumptions on the set-valued map T. For a comprehensive account of such developments in the finite dimensional setting, see [33].

2.4.2 Nash and generalized Nash equilibrium problems

A strategic game is a model of interactive decision making that helps in analysing situations where two or more players make decisions that affect each other's welfare. The most important solution concept of a strategic game is the well-known Nash equilibrium [45].

The Nash equilibrium problem is a noncooperative game in which each player's objective function depends on the other players' strategies. We assume that there is a finite set of players $M = \{1, \dots, m\}$ and each player i has a set of possible strategies $C_i \subseteq \mathbb{R}^{n_i}$. The term strategy can be understood in various ways, mainly the amount of production, consumption, buying, etc. To give an example, it can be electricity (in the energy market), water (in the eco-park), goods (in the exchange) or materials (in the producing). We denote by

$$x = (x_1, \dots, x_m) \in \prod_{i \in M} C_i = C \subseteq \mathbb{R}^N = \prod_{i \in M} \mathbb{R}^{n_i}$$

the vector formed by all decision variables and by $x_{-i} \in C_{-i} = \prod_{j \neq i} C_j$ the strategy vector of all players different from player i. Each player i has an objective function $\theta_i : \mathbb{R}^N \to \mathbb{R}$ that depends on all players' strategies.

Nash introduced his equilibrium concept in [44, 45] and it is based on the following paradigm: a vector is a Nash equilibrium if none of the players has an advantage to deviate from this vector unilaterally. This concept lies at the heart of all oligopolistic competition models, among which the Nash-Cournot production/distribution problem is an important instance [47].

Definition 2.4.5. The Nash equilibrium problem consists in finding a vector $\bar{x} \in C$ such that, for each $i \in M$, one has

$$NEP(\theta_i, C_i) \qquad \theta_i(\bar{x}_i, \bar{x}_{-i}) \leq \theta_i(x_i, \bar{x}_{-i}), \quad \forall x_i \in C_i.$$

As shown in [11], Nash equilibrium problem is equivalent to EP(f, C) when f is the so-called Nikaido-Isoda function [46] defined as

$$f(x, u) = \sum_{i \in M} [\theta_i(u_i, x_{-i}) - \theta_i(x_i, x_{-i})].$$

Indeed, if $\bar{x}$ is a solution of NEP(θ_i, C_i), $\bar{x} \in C$ and all the terms of the Nikaido-Isoda function are nonnegative for any $u \in C$. Hence, $\bar{x}$ solves EP(f, C). Conversely, let $\bar{x}$ be a solution of the EP(f, C) with f the Nikaido-Isoda function. By contradiction, assume that exists an index $\underline{i} \in M$ and a strategy $u_{\underline{i}} \in C_{\underline{i}}$ such that

$$\theta_{\underline{i}}(\bar{x}_{\underline{i}}, \bar{x}_{-\underline{i}}) > \theta_{\underline{i}}(u_{\underline{i}}, \bar{x}_{-\underline{i}}).$$

Since $f(\bar{x}, u) \geq 0$ for all $u \in C$, choosing $u_j = \bar{x}_j$ for all $j \neq \underline{i}$ leads to the contradiction

$$\begin{aligned} f(\bar{x}, u) &= \sum_{i \in M} [\theta_i(u_i, \bar{x}_{-i}) - \theta_i(\bar{x}_i, \bar{x}_{-i})] \\ &= \theta_{\underline{i}}(u_{\underline{i}}, \bar{x}_{-\underline{i}}) - \theta_{\underline{i}}(\bar{x}_{\underline{i}}, \bar{x}_{-\underline{i}}) < 0 \end{aligned}$$

and hence $\bar{x}$ is a solution for the NEP(θ_i, C_i).

The generalized Nash equilibrium problem was introduced by Debreu in [30] as a generalization of the Nash equilibrium problem.

Definition 2.4.6. Let $K_i : C_{-i} \rightrightarrows \mathbb{R}^{n_i}$ be a strategy set-valued map; the generalized Nash equilibrium problem consists in finding $\bar{x} \in C$ such that, for each $i \in M$, one has

$$GNEP(\theta_i, K_i) \qquad \bar{x}_i \in K_i(\bar{x}_{-i}) \text{ and } \theta_i(\bar{x}_i, \bar{x}_{-i}) \leq \theta_i(x_i, \bar{x}_{-i}), \quad \forall x_i \in K_i(\bar{x}_{-i}).$$

The generalized Nash equilibrium problem considers the strategy set of each player to depend on the decision variables of all other players. This model is more realistic as it allows for constraints that depend on other players, such as the limited availability of goods on the market. For this reason, research in this area has gained significant attention over the years. It is an interdisciplinary field that encompasses, for example, economics, computer science, engineering, mathematics, and operations research. We refer the reader to [32] for a detailed overview of the historical development of the generalized Nash equilibrium problem as well as the literature review, solution theory, algorithms and its many other applications.

Similarly to NEP(θ_i, C_i) also GNEP(θ_i, K_i) can be seen as a special case of QEP(f, K) taking the Nikaido-Isoda function and the set-valued map K defined as

$$K(x) = \prod_{i \in M} K_i(x_{-i}). \tag{2.3}$$

2.4.3 Potential game and quasioptimization problems

In game theory, a game is said to be potential game if the incentive of all players to change their strategy can be expressed using a single global function called potential function. Potential functions were first introduced by Rosenthal in [49] for strategic games, and later studied by Monder and Shapley in [42] for Nash equilibrium problems. The case of potential functions for generalized Nash equilibrium problems was first considered by Facchinei, Piccialli and Sciandrone in [34]. These functions are useful tool for analysing the equilibrium properties of games, allowing optimization theory to be applied to the study of Nash equilibria. In this thesis, we will see only the weighted potential games. The results given in this section can be seen in [50] and the references therein.

We consider a generalized Nash equilibrium problem with a finite set of players $M = \{1, \ldots, m\}$. This game is said weighted potential game if there exists a function $P : \mathbb{R}^N \to \mathbb{R}$ and m scalars $\alpha_i > 0$ such that for all $i \in M$ and $x_{-i} \in \mathbb{R}^{n-i}$ we have

$$P(x_i', x_{-i}) - P(x_i'', x_{-i}) = \alpha_i \left(\theta_i(x_i', x_{-i}) - \theta_i(x_i'', x_{-i})\right), \quad \forall x_i', x_i'' \in \mathbb{R}^{n_i} \tag{2.4}$$

where the function P is called weighted potential function.

A potential game can be characterized as follows.

Proposition 2.4.1. *A GNEP(θ_i, K_i) is a weighted potential game if, and only if, there exist $m + 1$ functions $P : \mathbb{R}^N \to \mathbb{R}$ and $P_i : \mathbb{R}^{n-i} \to \mathbb{R}$, and m scalars $\alpha_i > 0$ such that for all $i \in M$ we have*

$$\alpha_i \theta_i(x_i, x_{-i}) = P(x_i, x_{-i}) + P_i(x_{-i}), \quad \forall x \in \mathbb{R}^N. \tag{2.5}$$

Proof. If GNEP(θ_i, K_i) is a weighted potential game, there exists a function $P : \mathbb{R}^N \to \mathbb{R}$ and m scalars $\alpha_i > 0$ such that for all $i \in M$ and $x_{-i} \in \mathbb{R}^{n-i}$ we have

$$\alpha_i \theta_i(x_i', x_{-i}) = P(x_i', x_{-i}) - P(x_i'', x_{-i}) + \alpha_i \theta_i(x_i'', x_{-i}), \quad \forall x_i', x_i'' \in \mathbb{R}^{n_i}. \tag{2.6}$$

If we fix x_{-i} and x_i', we see that the term $\alpha_i \theta_i(x_i'', x_{-i}) - P(x_i'', x_{-i})$ does not depend on x_i'', otherwise equality (2.6) would not hold. Then,

$$P_i(x_{-i}) = \alpha_i \theta_i(x_i'', x_{-i}) - P(x_i'', x_{-i})$$

and (2.5) hold.

Vice versa, if (2.5) is true, the function P is a weighted potential function for the GNEP(θ_i, K_i). Indeed, for all $i \in M$ and $x_{-i} \in \mathbb{R}^{n-i}$ we have

$$\begin{aligned} P(x_i', x_{-i}) - P(x_i'', x_{-i}) &= \alpha_i \theta_i(x_i', x_{-i}) - P_i(x_{-i}) - (\alpha_i \theta_i(x_i'', x_{-i}) - P_i(x_{-i})) \\ &= \alpha_i \theta_i(x_i', x_{-i}) - \alpha_i \theta_i(x_i'', x_{-i}) \end{aligned}$$

for all $x_i', x_i'' \in \mathbb{R}^{n_i}$. □

If a game has a potential function, this implies favourable conditions for the existence and tractability of Nash equilibria.

Theorem 2.4.3. *Let GNEP(θ_i, K_i) be a weighted potential game with P a weighted potential function and K the set-valued map defined in (2.3). A point $\bar{x} \in C$ is a solution of the GNEP(θ_i, K_i) if $\bar{x}$ solves the problem*

$$\bar{x} \in K(\bar{x}) \quad \textit{and} \quad P(\bar{x}) = \min_{x \in K(\bar{x})} P(x). \tag{2.7}$$

Proof. Since $\bar{x}$ is a minimum point of the function P in the set $K(\bar{x})$, for all $i \in M$ we have

$$P(\bar{x}_i, \bar{x}_{-i}) - P(x_i, x_{-i}) \leq 0 \quad \forall (x_i, x_{-i}) \in K_i(\bar{x}_i) \times \prod_{j \neq i} K_j(\bar{x}_{-j}).$$

Thus, choosing $x_{-i} = \bar{x}_{-i} \in \prod_{j \neq i} K_j(\bar{x}_{-j})$ and using (2.4) we obtain there exist m scalars $\alpha_i > 0$ such that for all $i \in M$

$$\alpha_i \left(\theta_i(\bar{x}_i, \bar{x}_{-i}) - \theta_i(x_i, \bar{x}_{-i}) \right) \leq 0, \quad \forall x_i \in K_i(\bar{x}_{-i})$$

which means that $\bar{x}$ is a solution of GNEP(θ_i, K_i). □

The following example shows that being the minimum point of the potential function is a sufficient, but not necessary, condition to be the solution of the GNEP(θ_i, K_i).

Example 2.4.1. Let $m = 2$. We denote by $x \in C_1 = [-5, 5]$ the decision variables of player 1 and by $y \in C_2 = [-10, 10]$ the decision variables of player 2. The objective functions $\theta_i : \mathbb{R}^2 \to \mathbb{R}$ are defined as

$$\theta_1(x, y) = 2x^2 - 6xy$$

$$\theta_2(x, y) = 5y^2 - 15xy$$

and the strategy set-valued maps $K_i : C_{-i} \rightrightarrows \mathbb{R}$ are defined as

$$K_1(y) = [y - 1, 10]$$

$$K_2(x) = [-5, x + 1].$$

This is a weighted potential game. Indeed, the function $P : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$P(x, y) = x^2 + y^2 - 3xy$$

is a weighted potential function with $\alpha_1 = 1/2$ and $\alpha_2 = 1/5$.

Clearly, the vector $(0,0)$ is a solution of the GNEP(θ_i, K_i) but it is simple to see that it is not a minimum point of P. Indeed, if we consider $x = y \in [-1,1] \setminus \{0\}$ we have that

$$P(x,x) = -x^2 < 0 = P(0,0).$$

The problem (2.7) is called quasioptimization problem. This term was introduced by Facchinei and Kanzow in [32] to emphasise that this is not a standard optimization problem because the feasible set depends on the considered point.

A formal definition of this type of problem concludes this section.

Definition 2.4.7. Let $C \subseteq \mathbb{R}^n$ be a nonempty set, $h : \mathbb{R}^n \to \mathbb{R}$ be an objective function and $K : C \rightrightarrows \mathbb{R}^n$ be a set-valued map; the quasioptimization problem is the following:

$$QOP(h,K) \qquad \text{find } \bar{x} \in C \text{ s.t. } \bar{x} \in K(\bar{x}) \text{ and } h(\bar{x}) \leq h(u), \quad \forall u \in K(\bar{x}).$$

Even QOP(h,K) can be seen as a special case of QEP(f,K) taking

$$f(x,u) = h(u) - h(x).$$

Chapter 3

Existence results for projected solution

This chapter aims to extensively collect and discuss the results of [20]. Precisely, we study the existence results of projected solutions for quasiequilibrium problems and from this, we deduce the existence of projected solutions for quasivariational inequalities, quasioptimization problems and generalized Nash equilibrium problems.

As shown in the previous chapter, there are only a few results regarding the existence of quasiequilibrium problems. In most of these results, the constraint map K is assumed to be a self-map, that is, it maps C to itself. However, in some applications (such as the motivating example in Chapter 1), the constraint map may not be a self-map, i.e. $K(C) \nsubseteq C$, and the existence of a fixed point of K may not be verified. For this reason, the authors in [4] introduced the concept of projected solution for quasivariational inequalities and generalized Nash equilibrium problems. This solution is applicable even when the constraint map is not a self-map. Later, the authors in [27] adapted this concept to quasiequilibrium problems.

First of all, let us recall some properties of the metric projection that we are going to use in the following chapters. For further details, interested readers can refer to the book [31] and the references therein.

Definition 3.0.1. Let $C \subseteq \mathbb{R}^n$ be a nonempty set and $y \in \mathbb{R}^n$. The metric projection of y onto C is the set (possibly empty)

$$P_C(y) = \{x \in C : \|y - x\| \leq \|y - u\|, \ \forall u \in C\}$$

where $\|\cdot\|$ is the euclidean norm. The points of $P_C(y)$ are called best approximations of y in C.

If C is closed, then the set $P_C(y)$ is nonempty and if C is also convex there is a unique best approximation, which will be denoted by $x = p_C(y)$. Moreover, in this case, the metric projection map $p_C : \mathbb{R}^n \to C$ is a continuous and nonexpansive function, i.e.,

$$\|p_C(y_1) - p_C(y_2)\| \leq \|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathbb{R}^n.$$

Finally, we give the variational characterization of the best approximation from a convex set. If C is closed and convex, Kolmogorov's characterization says that whenever $x \in C$ we have

$$x = p_C(y) \quad \Leftrightarrow \quad \langle y - x, u - x\rangle \leq 0, \quad \forall u \in C.$$

This characterization can be equivalently written $y \in x + N_C(x)$ where $N_C(x)$ is the normal cone of C at x, i.e.

$$N_C(x) = \{x^* \in \mathbb{R}^n : \langle x^*, u - x\rangle \leq 0, \ \forall u \in C\}.$$

3.1 Quasiequilibrium problems

In this section we introduce the concept of projected solution for a quasiequilibrium problems, first studied in [27], and we present the existence results of this solution.

Definition 3.1.1. Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set, $K : C \rightrightarrows \mathbb{R}^n$ be a set-valued map and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a function. A pair $(\bar{x}, \bar{y}) \in C \times \mathbb{R}^n$ is said to be a projected solution of the QEP(f, K) if, and only if,

- $\bar{x} = p_C(\bar{y})$;
- $\bar{y}$ solves EP$(f, K(\bar{x}))$, i.e.,
  $\bar{y} \in K(\bar{x})$ s.t. $f(\bar{y}, u) \geq 0$, for all $u \in K(\bar{x})$.

Notice that, if $\bar{y} \in C$ then $\bar{x} = \bar{y}$ is the classical solution of the quasiequilibrium problem. Therefore, the concept of projected solution corresponds to the classical solution when $K(C) \subseteq C$.

When K is not a self-map, two situations are possible: either $K(C) \cap C = \emptyset$ or $K(C) \cap C \neq \emptyset$. The following examples show the possible relationships between classical and projected solutions in these situations.

Example 3.1.1. Consider the nonempty, closed and convex set $C = [0, +\infty) \subset \mathbb{R}$ and the set-valued map $K : C \rightrightarrows \mathbb{R}$ defined as

$$K(x) = [-x - 2, -1].$$

Since $K(C) = (-\infty, -1]$, this is the case $K(C) \cap C = \emptyset$. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function defined as

$$f(x, u) = -x + u.$$

The classical solution does not exist, but it is simple to see that the pair $(0, -2)$ is the unique projected solution of the QEP(f, K).

Example 3.1.2. Consider the nonempty, closed and convex set $C = [0, +\infty) \subset \mathbb{R}$ and the set-valued map $K : C \rightrightarrows \mathbb{R}$ defined as

$$K(x) = [-x - 1, 1].$$

Notice that $K(C) = (-\infty, 1]$, $K(C) \cap C \neq \emptyset$ and fix $K = [0, 1]$. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function defined as before

$$f(x, u) = -x + u.$$

Clearly, each fixed point of K is not a classical solution since

$$f(x, -x - 1) = -2x - 1 < 0$$

but it is simple to see that the pair $(0, -1)$ is the unique projected solution of the QEP(f, K). It is interesting to notice that $x = 0$ is a fixed point.

Example 3.1.3. Consider the nonempty, closed and convex set $C = [-2, 2] \subset \mathbb{R}$ and the set-valued map $K : C \rightrightarrows \mathbb{R}$ defined as

$$K(x) = \left[x - 1, \frac{4x - 1}{3}\right].$$

Notice that $K(C) = [-3, 7/3]$, $C \subseteq K(C)$ and $\operatorname{fix} K = [1, 2]$. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function defined as

$$f(x, u) = x^2 - u^2.$$

The QEP(f, K) admits both a classical solution $x = 1$ and the projected solutions $(-2, -3)$ and $(2, 7/3)$ in addition to $(1, 1)$. Notice that $x = 2$ is a fixed point.

There are only a few results regarding the existence of projected solutions for quasiequilibrium problems in finite dimensional space. The first existence result is in [27], where the authors assume suitable assumptions on auxiliary set-valued maps. In particular, they consider a nonempty, compact and convex set $C \subseteq \mathbb{R}^n$, the set-valued map $Q : C \times \mathbb{R}^n \rightrightarrows C \times \mathbb{R}^n$ defined as

$$Q(x, y) = p_C(y) \times K(x)$$

the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined as

$$F(y) = \{u \in \mathbb{R}^n : f(y, u) < 0\}$$

and the set-valued map $R : C \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined as

$$R(x, y) = F(y) \cap K(x).$$

The following theorem is the main result in [27].

Theorem 3.1.1. *Let C be a nonempty, compact and convex set. Then, the QEP(f, K) admits a projected solution if the following properties hold:*

(i) Q is lower semicontinuous with nonempty convex values;

(ii) $Q(C \times \mathbb{R}^n)$ is bounded;

(iii) $\operatorname{fix} Q$ *is closed;*

(iv) *R is lower semicontinuous with convex values on* $\operatorname{fix} Q$*;*

(v) $f(y, y) \geq 0$ *for all* $y \in M$*, where*

$$M = \{y \in K(C) : \text{ there exists } x \in C \text{ such that } (x, y) \in \operatorname{fix} Q\}.$$

We analyze in detail the assumptions of Theorem 3.1.1 about the set-valued map Q. Specifically, we can rephrase them in relation to the set-valued map K.

Remark 3.1.1.

- Thanks to Proposition 2.2.7 and the continuity of the function p_C, we can concluded that the assumption *(i)* on Q is equivalent to requiring K to be lower semicontinuous with convex values.

- The set $Q(C \times \mathbb{R}^n)$ can be written as

$$Q(C \times \mathbb{R}^n) = p_C(\mathbb{R}^n) \times K(C) = C \times K(C).$$

  Then it is bounded if, and only if, $K(C)$ and C are bounded. Anyway, if we assume that C is closed, assumption *(ii)* of Theorem 3.1.1 stresses the fact that C must be compact.

- Finally, the set $\operatorname{fix} Q$ may be equivalently rewritten as

$$\operatorname{fix} Q = \{(x, y) \in C \times \mathbb{R}^n : y \in \operatorname{fix}(K \circ p_C) \text{ and } x = p_C(y)\}.$$

  For this reason, since p_C is continuous, the closedness of $\operatorname{fix} Q$ is equivalent to the closedness of $\operatorname{fix}(K \circ p_C)$.

In the same paper, the authors obtain the following result with suitable assumptions for the set-valued map K and the function f.

Corollary 3.1.1. *Let C be a nonempty, compact and convex set, and assume that $K(C)$ is a compact subset of $\mathbb{R}^n$. Then, the QEP(f, K) admits a projected solution if the following properties hold:*

(i) *K is closed and lower semicontinuous with convex values;*

(ii) f *is continuous and quasiconvex with respect to its second argument;*

(iii) f *vanishes on the diagonal of* $\mathbb{R}^n \times \mathbb{R}^n$.

Unlike the previous results where the compactness of the feasible region C is required, we obtain an existence result of projected solutions by assuming the closedness of C only, and without requiring any monotonicity assumptions on the function f.

Theorem 3.1.2. *Let* C *be a nonempty, closed and convex set, and assume that* $K(C)$ *is bounded. Then, the QEP*(f, K) *admits a projected solution if the following properties hold:*

(i) K *is continuous with nonempty, closed and convex values;*

(ii) $f(y, y) \geq 0$ *for all* $y \in K(C)$*;*

(iii) f *is upper semicontinuous on* $K(C) \times \mathbb{R}^n$*;*

(iv) $f(y, \cdot)$ *is quasiconvex on* $\mathbb{R}^n$ *for all* $y \in K(C)$.

Proof. Define $\widetilde{C} = \operatorname{cl}\operatorname{co} K(C)$ which is compact.

We consider the set-valued map $\widetilde{K} : \widetilde{C} \rightrightarrows \widetilde{C}$ defined as

$$\widetilde{K}(y) = K(p_C(y))$$

and the set-valued map $F : \operatorname{fix} \widetilde{K} \rightrightarrows \mathbb{R}^n$ defined as

$$F(y) = \{u \in \mathbb{R}^n : f(y, u) < 0\}.$$

The set-valued map $\widetilde{K}$ is continuous since composition of two continuous set-valued maps (Proposition 2.2.3). Moreover, it has nonempty, closed and convex values from *(i)*. The Kakutani fixed point theorem guarantees that $\operatorname{fix} \widetilde{K}$ is nonempty. Moreover, from the Closed graph theorem, the map $\widetilde{K}$ has closed graph and hence $\operatorname{fix} \widetilde{K}$ is closed thanks to Lemma 2.3.1. Clearly, $\operatorname{fix} \widetilde{K} \subseteq K(C)$ and hence F has convex values from *(iv)*. Since $(y, u) \in \operatorname{gph} F$ is equivalent to affirm that $f(y, u) < 0$, the fact that F has open graph descends from *(iii)*.

By contradiction assume that $F(y) \cap \widetilde{K}(y) \neq \emptyset$ for all $y \in \operatorname{fix} \widetilde{K}$. From Proposition 2.3.1, $F \cap \widetilde{K}$ has a continuous selection $g : \operatorname{fix} \widetilde{K} \to \widetilde{C}$. The set-valued map $\Phi : \widetilde{C} \rightrightarrows \widetilde{C}$ defined as

$$\Phi(y) = \begin{cases} \widetilde{K}(y) & \text{if } y \notin \operatorname{fix} \widetilde{K} \\ \{g(y)\} & \text{if } y \in \operatorname{fix} \widetilde{K} \end{cases}$$

is lower semicontinuous thanks to Proposition 2.2.8 and there exists a continuous selection $\varphi : \widetilde{C} \to \widetilde{C}$ of Φ by Michael selection theorem. Then φ extends g out from $\operatorname{fix} \widetilde{K}$. The Brouwer fixed point theorem affirms that φ has a fixed point $y \in \widetilde{C}$. Clearly, $y \in \operatorname{fix} \widetilde{K}$ and this implies that y is a fixed point of g, i.e., $y = g(y) \in F(y)$. Hence $f(y, y) < 0$ which contradicts *(ii)*.

Therefore, there exists $\bar{y} \in \operatorname{fix} \widetilde{K}$, i.e.

$$\bar{y} \in \widetilde{K}(\bar{y}) = K(p_C(\bar{y}))$$

such that $F(\bar{y}) \cap \widetilde{K}(\bar{y}) = \emptyset$, i.e.

$$f(\bar{y}, u) \geq 0, \quad \forall u \in \widetilde{K}(\bar{y})$$

which means that $(p_C(\bar{y}), \bar{y})$ is a projected solution. □

In the following, we will analyze the assumptions of Theorem 3.1.2 in more detail. Specifically, our focus will be on their use in the proof.

Remark 3.1.2. The fact that $f(y, y) \geq 0$ for all $y \in K(C)$ is used only to contradict that $F(y) \cap \widetilde{K}(y) \neq \emptyset$ for all $y \in \operatorname{fix} \widetilde{K}$. For this reason, assumption *(ii)* may be relaxed by

(ii-a) $f(y, y) \geq 0$ for all $y \in \operatorname{fix} \widetilde{K}$.

Remark 3.1.3. The upper semicontinuity of K and the closedness of its values are used only to prove that $\operatorname{fix} \widetilde{K}$ is closed. We notice that $\operatorname{fix} \widetilde{K}$ is not empty, even without the hypothesis that K is closed (Proposition 2.3.4). For this reason, assumption *(i)* may be substituted by

(i-a) K is lower semicontinuous with nonempty, convex values;

(i-b) $\operatorname{fix}\widetilde{K}$ is closed.

The next example shows that assumptions *(i-a)* and *(i-b)* together are strictly weaker than *(i)*.

Example 3.1.4. Consider the nonempty, closed and convex set $C = [0, +\infty) \times [0, +\infty) \subset \mathbb{R}^2$, and the set-valued map $K : C \rightrightarrows \mathbb{R}^2$ defined as

$$K(x) = \begin{cases} \{(u_1, u_2) \in \mathbb{R}^2 : u_1 + u_2 = -1,\ u_1 \le 0,\ u_2 \le 0\} & \text{if } x = (0,0) \\ \{(u_1, u_2) \in \mathbb{R}^2 : u_1 + u_2 > -1,\ u_1 < 0,\ u_2 < 0\} & \text{if } x \ne (0,0) \end{cases}$$

Consequently, the set $\widetilde{C} \subseteq \mathbb{R}^2$ is defined as

$$\widetilde{C} = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 + u_2 \ge -1,\ u_1 \le 0,\ u_2 \le 0\}$$

and the set-valued $\widetilde{K} : \widetilde{C} \rightrightarrows \widetilde{C}$ is defined as

$$\widetilde{K}(y_1, y_2) = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 + u_2 = -1,\ u_1 \le 0,\ u_2 \le 0\}$$

Clearly, K has not closed values and it is not upper semicontinuous at $(0, 0)$. Moreover, we observe that $K(C) \cap C = \emptyset$, the set

$$\operatorname{fix}\widetilde{K} = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 = -1,\ y_1 \le 0,\ y_2 \le 0\}$$

is closed and K is lower semicontinuous with convex values. Now consider the function $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(y, u) = u_1 - y_1.$$

This function satisfies all the assumptions of Theorem 3.1.2 hence, the existence of a projected solution is guaranteed by Remark 3.1.3 and Theorem 3.1.2. It is an easy calculation to show that $(\bar{x}, \bar{y}) = ((0, 0), (-1, 0))$ is the unique projected solution of the quasiequilibrium problem.

Remark 3.1.4. The upper semicontinuity of f and the quasiconvexity of $f(y, \cdot)$ are used only to show that $F \cap \widetilde{K}$ admits a continuous selection in $\operatorname{fix}\widetilde{K}$. For this reason, just ask that $F \cap \widetilde{K}$ verifies the assumption of the Michael selection theorem. Then the assumptions *(iii)* and *(iv)* may be changed in *(iii-a)* $F \cap \widetilde{K}$ is lower semicontinuous with convex values on $\operatorname{fix} \widetilde{K}$.

The following result can be drawn from the previous remarks.

Theorem 3.1.3. *Let C be a nonempty, closed and convex set, and assume that $K(C)$ is bounded. Then, the QEP(f, K) admits a projected solution if the following properties hold:*

(i-a) K is lower semicontinuous with nonempty convex values;

(i-b) $\operatorname{fix} \widetilde{K}$ is closed;

(ii-a) $f(y, y) \geq 0$ for all $y \in \operatorname{fix} \widetilde{K}$;

(iii-a) $F \cap \widetilde{K}$ is lower semicontinuous with convex values on $\operatorname{fix} \widetilde{K}$.

Thanks to Remark 3.1.1 and Remark 3.1.2, the assumptions of Theorem 3.1.1 coincide with those of Theorem 3.1.3, except for the compactness of C, which is not required in our result. This fact allows us to consider a large class of quasiequilibrium problems as reported in the following example.

Example 3.1.5. Let $C = [0, +\infty) \subset \mathbb{R}$ be a nonempty, closed and convex set, $K : C \rightrightarrows \mathbb{R}$ be a set-valued map defined as

$$K(x) = \left[-\frac{1}{x+1}, \frac{1}{x+1} \right]$$

and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function defined as

$$f(y, u) = -y + u.$$

Since C is not compact then Theorem 3.1.1 and Corollary 3.1.1 cannot be applied. Instead, all the assumptions of Theorem 3.1.2 are fulfilled: the constraint map K is continuous, with nonempty, closed, and convex values, $K(C) = [-1, 1]$ is compact, f is continuous, quasiconvex, and $f(y, y) = 0$ for all $y \in \mathbb{R}$. Therefore a projected solution exists and it is easy to see that $(0, -1)$ is the projected solution.

Furthermore, if we consider suitable assumptions on the auxiliary set-valued map $T : C \rightrightarrows \mathbb{R}^n$ defined as

$$T(x) = K(x) \cap (x + N_C(x))$$

it is possible to characterize the closedness of $\operatorname{fix} \widetilde{K}$ in a simple way.

In general, there is no relationship between the closedness of a set-valued map and the closedness of its range. For instance the map $\Phi : [1, +\infty) \rightrightarrows \mathbb{R}$ defined as

$$\Phi(x) = \left[\frac{1}{x+1}, \frac{1}{x}\right]$$

has closed graph, but $\Phi([1, +\infty)) = (0, 1]$. Vice versa the map $\Phi : [1, +\infty) \rightrightarrows \mathbb{R}$ defined as

$$\Phi(x) = [x, x + 1)$$

has not closed values, but $\Phi([1, +\infty)) = [1, +\infty)$. Instead, the map T enjoys this property.

Proposition 3.1.1. *Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set, $K : C \rightrightarrows \mathbb{R}^n$ and $T : C \rightrightarrows \mathbb{R}^n$ be set-valued map with T defined as*

$$T(x) = K(x) \cap (x + N_C(x)).$$

Then, $T(C)$ is closed if, and only if, T is closed.

Proof. Assume that T is closed and consider $\{y_k\} \subseteq T(C)$ such that $y_k \to y \in \mathbb{R}^n$. By assumption, for each $k \in \mathbb{N}$ there exists $x_k \in C$ such that $y_k \in T(x_k)$ and this is equivalent to affirm

$$\begin{cases} y_k \in K(x_k) \\ x_k = p_C(y_k) \end{cases}$$

Since p_C is continuous, then

$$x_k = p_C(y_k) \to p_C(y) = x.$$

Therefore, $(x_k, y_k) \in \operatorname{gph} T$ and $(x_k, y_k) \to (x, y)$, hence $y \in T(x) \subseteq T(C)$ because T is closed.

For the converse, we consider $\{(x_k, y_k)\} \subseteq \operatorname{gph} T$ such that $(x_k, y_k) \to (x, y) \in C \times \mathbb{R}^n$. Since $y_k \in T(x_k)$ we have that

$$\begin{cases} y_k \in \ K(x_k) \\ x_k = p_C(y_k) \end{cases}$$

Since $(x_k, y_k) \to (x, y)$ and p_C is continuous, then $x = p_C(y)$. Furthermore, $y_k \in T(x_k) \subseteq T(C)$ and $T(C)$ is closed, then $y \in T(C)$, i.e., there exists $z \in C$ such that $y \in T(z)$. This implies that $z = p_C(y)$. The uniqueness of the best approximation guarantees that $x = z$ and $y \in T(x)$. □

The set fix $\widetilde{K}$ may be characterized by means of T, indeed $y \in T(x)$ is equivalent to affirm that $y \in K(p_C(y))$ and $x = p_C(y)$. Thanks to this reformulation and Proposition 3.1.1, assumption *(i-b)* may be replaced by

(i-b′) T is closed,

or, equivalently, by

(i-b″) $T(C)$ is closed.

In other words, we have the following equivalence

$$\operatorname{fix}(K \circ p_C) \text{ closed} \quad \Leftrightarrow \quad T(C) \text{ closed} \quad \Leftrightarrow \quad T \text{ closed.}$$

Moreover, since the set-valued map $x + N_C(x)$ is closed, the closedness of K is sufficient to ensure the closedness of $\operatorname{fix}(K \circ p_C)$.

Let us now consider three different problems where we obtain existence results for projected solutions applying Theorem 3.1.2: quasivariational inequalities, generalized Nash equilibrium problems and quasioptimization problems.

3.2 Quasivariational inequalities

The concept of projected solution for the quasivariational inequalities was first investigated in [4].

Definition 3.2.1. Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set, $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $K : C \rightrightarrows \mathbb{R}^n$ be two set-valued maps. A pair $(\bar{x}, \bar{y}) \in C \times \mathbb{R}^n$ is said to be a projected solution of QVI(Φ, K) if, and only if,

- $\bar{x} = p_C(\bar{y})$;
- $\bar{y}$ solves VI$(\Phi, K(\bar{x}))$, i.e.,

  $\bar{y} \in K(\bar{x})$ s.t. $\exists \bar{y}^* \in \Phi(\bar{y})$ with $\langle \bar{y}^*, y - \bar{y} \rangle \geq 0$, for all $y \in K(\bar{x})$.

The existence of a projected solution for quasivariational inequalities was first proven in [4]. This was done assuming that the set-valued map K has convex values and nonempty interior and the operator Φ is pseudomonotone. Subsequently, an existence result for this type of problem was achieved in [27], where the authors avoid these two restrictive assumptions but require (adapting opportunely the notations) the compactness of C and $K(C)$. Later, we establish the following existence result avoiding the two restrictive assumptions in [4] and requiring only the closedness of C and the boundedness of $K(C)$.

Theorem 3.2.1. *Let C be a nonempty, closed and convex set, and assume that $K(C)$ is bounded. Then, $QVI(\Phi, K)$ admits a projected solution if the following properties hold:*

(i) K is lower semicontinuous with nonempty and convex values;

(ii) fix$(K \circ p_C)$ *is closed;*

(iii) Φ is upper semicontinuous on $K(C)$ with nonempty, compact and convex values.

Proof. The result descends from Theorem 3.1.2 and Remark 3.1.3 taking

$$f(y, u) = \max_{y^* \in \Phi(y)} \langle y^*, u - y \rangle.$$

First, we check that f verifies the assumptions of Theorem 3.1.2. Clearly $f(y, y) = 0$. Fixed $y \in K(C)$, the set

$$\begin{aligned} \{u \in \mathbb{R}^n : f(y, u) < a\} &= \left\{ u \in \mathbb{R}^n : \max_{y^* \in \Phi(y)} \langle y^*, u - y \rangle < a \right\} \\ &= \bigcap_{y^* \in \Phi(y)} \{u \in \mathbb{R}^n : \langle y^*, u - y \rangle < a\} \end{aligned}$$

is convex for all $a \in \mathbb{R}$ being an intersection of subspaces. The function f is upper semicontinuity on $K(C)\times\mathbb{R}^n$ from *(iii)* and Lemma 2.3.3. Then, thanks to Theorem 3.1.2 with Remark 3.1.3, there exists $(\bar{x}, \bar{y}) \in C \times \mathbb{R}^n$ with $\bar{y} \in K(\bar{x})$ and $\bar{x} = p_C(\bar{y})$ such that

$$\max_{\bar{y}^* \in \Phi(\bar{y})} \langle \bar{y}^*, u - \bar{y} \rangle \geq 0, \quad \forall u \in K(\bar{x})$$

which is equivalent to affirming that

$$\inf_{u \in K(\bar{x})} \max_{y^* \in \Phi(\bar{y})} \langle y^*, u - \bar{y} \rangle \geq 0.$$

Thanks to Sion's minimax theorem we have that

$$\max_{\bar{y}^* \in \Phi(\bar{y})} \inf_{u \in K(\bar{x})} \langle \bar{y}^*, u - \bar{y} \rangle \geq 0$$

which means that $(\bar{x}, \bar{y})$ is a projected solution of the QVI(Φ, K). □

Theorem 3.2.1 allows us to consider a large class of quasivariational inequalities that do not satisfy the assumptions of the results in [4, 27], as illustrated in the following examples.

Example 3.2.1. Let $C = [-2, 0] \times [0, 2] \subset \mathbb{R}^2$ be a nonempty, closed and convex set, $K : C \rightrightarrows \mathbb{R}^2$ be a set-valued map defined as

$$K(x_1, x_2) = \{(u_1, u_2) \in \mathbb{R}^2 : -4 - x_2 \leq u_1 \leq 0, -1 \leq u_2 \leq 1\}$$

and $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a set-valued map defined as

$$\Phi(y_1, y_2) = (y_1^2, 1 + y_2^2).$$

Clearly, fix $K = [-2, 0] \times [0, 1]$ but the QVI(Φ, K) has no classic solution. Instead, all the assumptions of Theorem 3.2.1 are satisfied and the existence of a projected solution is guaranteed. In particular, it is easy to see that there are two projected solutions: $((0, 0), (0, -1))$ and $((-2, 0), (-4, -1))$. Notice that Φ is not pseudomonotone, indeed $\langle \Phi(0, 0), (-2, 0) - (0, 0) \rangle = 0$ and $\langle \Phi(-2, 0), (-2, 0) - (0, 0) \rangle = -8$. Therefore, the result in [4] cannot be applied.

Example 3.2.2. Let $C = [0, +\infty)$ be a nonempty, closed and convex set, $K : C \rightrightarrows \mathbb{R}$ be a set-valued map defined as

$$K(x) = [-4 - x, -1]$$

and $\Phi : \mathbb{R} \to \mathbb{R}$ be a set-valued map defined as

$$\Phi(y) = \{y^2\}.$$

Clearly, QVI(Φ, K) has no classic solution since $\operatorname{fix} K = \emptyset$. However, the existence of a projected solution is guaranteed since all the assumptions of Theorem 3.2.1 are satisfied. In particular, there is only one projected solution of QVI(Φ, K): $(0, -4)$. Notice that C is not compact and then, the result in [27] cannot be applied.

3.3 Generalized Nash equilibrium problems

The concept of projected solution for generalized Nash equilibrium problem has been introduced in [4] where the existence of such equilibrium was investigated.

We consider a finite set of players $M = \{1, \dots, m\}$. For each player i, let $C_i \subseteq \mathbb{R}^{n_i}$ be nonempty, closed and convex set, $C = \prod_{i \in M} C_i \subseteq \mathbb{R}^N = \prod_{i \in M} \mathbb{R}^{n_i}$ be a set, $\theta_i : \mathbb{R}^N \to \mathbb{R}$ be an objective function and $K_i : C_{-i} \rightrightarrows \mathbb{R}^{n_i}$ be a strategy set-valued map.

Definition 3.3.1. A pair $(\bar{x}, \bar{y}) \in C \times \mathbb{R}^N$ is said to be a projected solution of the GNEP(θ_i, K_i) if, and only if,

- $\bar{x} = p_C(\bar{y})$;
- $\bar{y}$ solves NEP($\theta_i, K_i(\bar{x}_{-i})$), i.e., for each $i \in M$

  $\bar{y}_i \in K_i(\bar{x}_{-i})$ s.t. $\theta_i(\bar{y}_i, \bar{y}_{-i}) \leq \theta_i(y_i, \bar{y}_{-i})$, for all $y_i \in K_i(\bar{x}_{-i})$.

In order to apply Theorem 3.1.2, we need to introduce "cumulative" constraint set-valued maps. First, we denote by $\overline{K}_i$ the set-valued map $\overline{K}_i : C \rightrightarrows \mathbb{R}^{n_i}$ defined as $\overline{K}_i(x) = K_i(x_{-i})$, hence we consider the set-valued map $\overline{K} : C \rightrightarrows \mathbb{R}^N$ defined as

$$\overline{K}(x) = \overline{K}_1(x) \times \cdots \times \overline{K}_m(x).$$

Finding a projected solution for a generalized Nash equilibrium problem amounts to finding a projected solution for the quasiequilibrium problem associated to the Nikaido-Isoda function [46]

$$f(y,u) = \sum_{i=1}^{m} [\theta_i(u_i, y_{-i}) - \theta_i(y_i, y_{-i})]$$

with $\overline{K}(x) = \prod_{i \in M} \overline{K}_i(x)$ defined as above. Indeed, if $(\bar{x}, \bar{y})$ is a projected solution for GNEP(θ_i, K_i), then $\bar{x} = p_C(\bar{y})$ and $\bar{y}_i \in K_i(\bar{x}_{-i})$ for all $i \in M$. Therefore $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_m) \in \prod_{i \in M} \overline{K}_i(\bar{x}) = \overline{K}(\bar{x})$. Furthermore, all the terms of the Nikaido-Isoda function are nonnegative for any $y_i \in K_i(\bar{x}_{-i})$ and then for any $y \in \overline{K}(\bar{x})$, hence $(\bar{x}, \bar{y})$ is a projected solution for the QEP$(f, \overline{K})$.

Conversely, if $(\bar{x}, \bar{y})$ is a projected solution for the QEP$(f, \overline{K})$, then $\bar{x} = p_C(\bar{y})$ and $\bar{y} \in \overline{K}(\bar{x})$. By contradiction, assume that exists an index $i \in M$ and a strategy $u_i \in \overline{K}_i(\bar{x})$ such that $\theta_i(u_i, \bar{y}_{-i}) < \theta_i(\bar{y}_i, \bar{y}_{-i})$. Since $f(\bar{y}, u) \geq 0$ for all $u \in \overline{K}(\bar{x})$, choosing $u_j = \bar{y}_j$ for all $j \neq i$ leads to the contradiction

$$f(\bar{y}, u) = \theta_i(u_i, \bar{y}_{-i}) - \theta_i(\bar{y}_i, \bar{y}_{-i}) < 0$$

and hence $(\bar{x}, \bar{y})$ is a projected solution for GNEP(θ_i, K_i).

An existence result for projected Nash equilibria was proved first in [4] assuming that the set-valued maps K_i are either single-valued or have convex values and nonempty interior, and the convexity of the functions θ_i. Subsequently, an existence result has been achieved in [27] where they avoid these last restrictive assumptions but require the compactness of C. Later, we establish the following result that is analogous to [27] but we require only the closedness of C and not the compactness.

Theorem 3.3.1. *For each $i \in M$, let C_i be a nonempty, closed and convex set, and assume that $K_i(C_{-i})$ is bounded. Then, the GNEP(θ_i, K_i) admits a projected solution if the following properties hold:*

(i) K_i are lower semicontinuous with nonempty, convex values for all $i \in M$;

(ii) fix$(\overline{K} \circ p_C)$ *is closed;*

(iii) θ_i are continuous and $\theta_i(\cdot, y_{-i})$ are convex on $\mathbb{R}^N$ for all $i \in M$.

Proof. We have already shown that finding a projected solution for a generalized Nash equilibrium problem is equivalent to finding a projected solution for the quasiequilibrium problem associated to the Nikaido-Isoda function and $\overline{K}$. So, it is sufficient to check that all the conditions of Theorem 3.1.2 with Remark 3.1.3 are fulfilled. The set $C = \prod_{i \in M} C_i$ is nonempty, closed and convex since the product of nonempty, closed and convex sets. Analogously, the set $\overline{K}(C) = \prod_{i \in M} \overline{K}_i(C)$ is bounded. The set-valued map $\overline{K}$ is lower semicontinuous, and it has nonempty convex values from *(i)*. Furthermore, f is upper semicontinuous on $\overline{K}(C) \times \mathbb{R}^N$ from *(iii)*. Moreover,

$$f(y, y) = \sum_{i=1}^{m} [\theta_i(y_i, y_{-i}) - \theta_i(y_i, y_{-i})] = 0.$$

The quasiconvexity of $f(y, \cdot)$ descends to the fact that the Nikaido-Isoda function is convex since it is the sum of convex functions. Then, thanks to Theorem 3.1.2 with Remark 3.1.3, there exists a projected solution. □

As shown in [16], using a different proof, it is possible to relax the assumptions on the functions θ_i. Specifically, if we do not rely on the equivalence with the quasiequilibrium problem via the Nikaido-Isoda function, we can require only quasiconvexity with respect to its player's variable of the functions θ_i, rather than convexity.

Theorem 3.3.2. *For each $i \in M$ let C_i be a nonempty, closed and convex set, and assume that $K_i(C_{-i})$ is bounded. Then, the GNEP(θ_i, K_i) admits a projected solution if the following properties hold:*

(i) K_i are continuous with nonempty, closed and convex values for all $i \in M$;

(ii) θ_i are continuous and $\theta_i(\cdot, y_{-i})$ are quasiconvex on $\mathbb{R}^N$ for all $i \in M$.

Proof. Define $\widetilde{C}_i = \operatorname{cl} \operatorname{co} K_i(C_{-i})$ and $\widetilde{C} = \prod_{i=1}^{m} \widetilde{C}_i$ which are compact and convex. Consider the set-valued map $L_i : C \times \widetilde{C} \rightrightarrows \widetilde{C}_i$ defined as

$$L_i(x, y) = \{u_i \in K_i(x_{-i}) : \theta_i(u_i, y_{-i}) \leq \theta_i(w_i, y_{-i}) \quad \forall w_i \in K_i(x_{-i})\}$$

the set-valued map $L = \prod_{i=1}^{m} L_i : C \times \widetilde{C} \rightrightarrows \widetilde{C}$ defined as

$$L(x, y) = L_1(x, y) \times \cdots \times L_m(x, y)$$

and the set-valued map $\widehat{L} : \widetilde{C} \rightrightarrows \widetilde{C}$ defined as

$$\widehat{L}(y) = L(p_C(y), y).$$

To apply the Berge Theorem, we need to introduce the set-valued map $\widehat{K}_i : C \times \widetilde{C} \rightrightarrows \widetilde{C}_i$ defined as

$$\widehat{K}_i(x, y) = K_i(x_{-i})$$

and the set-valued map $\widehat{\theta}_i : C \times \widetilde{C} \times \widetilde{C}_i \to \mathbb{R}$ defined as

$$\widehat{\theta}_i(x, y, v_i) = \theta_i(v_i, y_{-i}).$$

Clearly $\widehat{\theta}_i$ is continuous from *(ii)* and $\widehat{K}_i$ is continuous with nonempty and compact values from *(i)*. Then, thanks to Berge's Theorem, L_i are upper semicontinuous with nonempty and compact values. Hence, L is upper semicontinuous with nonempty and compact values (Proposition 2.2.6) and so $\widehat{L}$ is upper semicontinuous (Proposition 2.2.3) with nonempty and compact values. Moreover, $\widehat{L}$ has convex values because θ_i is quasiconvex concerning its player's variable and K_i has convex values. Then the Kakutani fixed point theorem guarantees that there exists a fixed point y of $\widehat{L}$. Therefore, $y \in L(p_C(y), y)$ which means that the pair $(p_C(y), y)$ is a projected solution for GNEP(θ_i, K_i). □

3.4 Quasioptimization problems

The concept of projected solution for the quasioptimization problems was first investigated in [4].

Definition 3.4.1. Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set, $h : \mathbb{R}^n \to \mathbb{R}$ be an objective function and $K : C \rightrightarrows \mathbb{R}^n$ be a set-valued map. A pair $(\bar{x}, \bar{y}) \in C \times \mathbb{R}^n$ is said to be a projected solution of the QOP(h, K) if, and only if,

- $\bar{x} = p_C(\bar{y})$;

- $\bar{y} \in K(\bar{x})$ s.t. $h(\bar{y}) \leq h(u)$, for all $u \in K(\bar{x})$.

The existence result of projected solutions for the quasioptimization problem was proved first in [4] using the quasivariational inequality with the normal operator N_h^a associated to the adjusted level sets of h. Subsequently, using a common approach (see for instance [27]) we establish the following existence result as a direct consequence of Theorem 3.1.2 and Remark 3.1.3.

Theorem 3.4.1. *Let C be a nonempty, closed and convex set, and assume that $K(C)$ is bounded. Then, the QOP(h, K) admits a projected solution if the following properties hold:*

(i) K is lower semicontinuous with nonempty, convex values;

(ii) $\operatorname{fix}(K \circ p_C)$ is closed;

(iii) h continuous and quasiconvex on $\mathbb{R}^n$.

Proof. It is sufficient to apply Theorem 3.1.2 with Remark 3.1.3 to the auxiliary function

$$f(y, u) = h(u) - h(y).$$

First, we check that f verifies the assumptions of Theorem 3.1.2. Clearly $f(y, y) = 0$ and the set

$$\{u \in \mathbb{R}^n : f(y, u) < a\} = \{u \in \mathbb{R}^n : h(u) < h(y) + a\}$$

is convex for all $y \in K(C)$ and $a \in \mathbb{R}$. Moreover, f is continuous. Then, all the assumptions of Theorem 3.1.2 with the Remark 3.1.3 are satisfied and then there exist $\bar{x} \in C$ and $\bar{y} \in K(\bar{x})$ with $\bar{x} = p_C(\bar{y})$ such that

$$h(u) - h(\bar{y}) = f(\bar{y}, u) \geq 0, \quad \forall u \in K(\bar{x})$$

which means that $(\bar{x}, \bar{y})$ is a projected solution of QOP(h, K). □

Clearly, as observed in Remark 3.1.4, the assumption *(iii)* on continuity and quasiconvexity of h may be replaced by the assumption *(iii-a)*. This is equivalent to affirm that the set-valued map $H : \operatorname{fix} \widetilde{K} \rightrightarrows \mathbb{R}^n$ defined as

$$H(y) = \{u \in \widetilde{K}(y) : h(y) > h(u)\}$$

is lower semicontinuous and convex values. Thanks to this fact, Theorem 3.4.1 can be compared with the analogous in [27]. The only difference is that our result requires only the closedness of C, while in [27] it requires the compactness of C.

Chapter 4

A descent method for projected solution

The unique algorithm for finding a projected solution of a quasiequilibrium problem has been proposed in [8] in a normed space setting. The authors prove that the limit points of the sequence generated by the iterative procedure are projected solutions if the sequence is asymptotically regular. Therefore, the algorithm is useful from a theoretical standpoint to demonstrate the existence of a projected solution to a quasiequilibrium problem under appropriate assumptions. However, the algorithm has not yet been implemented in a numerical context. In fact, the major drawback of the algorithm from a numerical standpoint is that an equilibrium problem must be solved at each step. This chapter presents a more efficient approach for finding a projected solution of quasiequilibrium problems. The main idea is to reformulate the quasiequilibrium problem as an optimization problem through a suitable gap function and develop a descent algorithm, assuming that the set-valued map K can be described by constraining functions.

The next section will review the main notions and results needed for the rest of the chapter.

4.1 Some concepts on nonsmooth analysis

The Clarke subdifferential calculus is the most well-known and frequently used aspect of nonsmooth analysis, which deals with differential analysis in the absence of differentiability. Although nonsmooth analysis has classical roots, it has only recently experienced significant growth. One reason for this development is the recognition that nondifferentiable phenomena are more widespread and play a more significant role than previously believed. In particular, in recent years, nonsmooth analysis has become increasingly important in various fields, including functional analysis, optimization, differential equations (as in the theory of viscosity solutions), control theory, and, increasingly, in analysis generally (critical point theory, inequalities, fixed point theory, variational methods, and others). The concepts given in this section can be seen in [24, 25] and the references therein.

Definition 4.1.1. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function near $x \in \mathbb{R}^n$.

- The Clarke directional derivative of ψ at x in the direction $v \in \mathbb{R}^n$ is defined as
$$\psi^\circ(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{\psi(y + tv) - \psi(y)}{t}.$$
- The Clarke subdifferential of ψ at x is the subset of $\mathbb{R}^n$ given by
$$\partial\psi(x) = \{\xi \in \mathbb{R}^n : \psi^\circ(x; v) \geq \langle \xi, v \rangle \text{ for all } v \in \mathbb{R}^n\}$$
whose elements are said to be generalized subgradients.

Note that the definition of Clarke directional derivative does not require the existence of any limit, as it involves only an upper limit and concerns only the behavior of ψ near x. This definition differs from the traditional definition of the directional derivative, as the base point y of the difference quotient varies. Additionally, the concept of Clarke subdifferential reduces to the derivative if ψ is continuously differentiable.

Proposition 4.1.1. *Let ψ be a Lipschitz function near $x \in \mathbb{R}^n$ of rank L, then the Clarke directional derivative and the Clarke subdifferential have the following basic properties:*

- *the function $v \to \psi^\circ(x; v)$ is finite, positively homogeneous, subadditive and satisfies*
$$|\psi^\circ(x; v)| \leq L\|v\|;$$

- *ψ° is upper semicontinuous as a function of (x, v) and it is Lipschitz of rank L as a function of v;*

- *$\partial\psi(x)$ is a nonempty, convex and compact subset of $\mathbb{R}^n$;*

- *$\partial\psi$ is upper semicontinuous at x;*

- *for every $v \in \mathbb{R}^n$ we have*
$$\psi^\circ(x; v) = \max\{\langle \xi, v\rangle : \xi \in \partial\psi(x)\}.$$

The mean value theorem is a crucial result in real analysis. It is used to prove statements about a function on an interval based on local hypotheses about derivatives at points within the interval. The following result extends the mean value theorem.

Theorem 4.1.1 (Lebourg). *Let $x, y \in \mathbb{R}^n$. If ψ is Lipschitz on an open set containing the line segment $[x, y]$, then there exists a point $u \in (x, y)$ such that*

$$\psi(y) - \psi(x) \in \langle \partial\psi(u), y - x\rangle = \{\langle \xi, y - x\rangle : \xi \in \partial\psi(u)\}.$$

It turns out that the differential concept most naturally linked to the theory of this section is that of strict differentiability. This concept is a modified version of the typical notion of differentiability. In particular, it is more restrictive as it allows both points used in the difference quotient to "move".

Definition 4.1.2. The function ψ is said to be strictly differentiable at $x \in \mathbb{R}^n$ if there exists an element $\xi \in \mathbb{R}^n$ such that for each $v \in \mathbb{R}^n$ we have

$$\lim_{\substack{y \to x \\ t \downarrow 0}} \frac{\psi(y + tv) - \psi(y)}{t} = \langle \xi, v\rangle$$

and provided the convergence is uniform for v in compact sets.

A characterization of this concept says that a function ψ is strictly differentiable at x if, and only if, ψ is Lipschitz near x and $\partial\psi(x)$ is a singleton.

The following proposition is a consequence of this characterization.

Proposition 4.1.2. *If ψ is continuously differentiable at $x \in \mathbb{R}^n$, then ψ is strictly differentiable at x and hence Lipschitz near x.*

Rademacher's Theorem states that each Lipschitz function on an open subset of $\mathbb{R}^n$ is differentiable almost everywhere on that subset. Thanks to this, the Clarke subdifferential may be characterized by

$$\partial\psi(x) = \text{co}\left\{\xi = \lim_{k\to\infty} \nabla\psi(x_k) : x_k \to x,\ x_k \notin \Omega_\psi\right\}$$

where Ω_ψ is the set of points in $\mathbb{R}^n$ at which ψ fails to be differentiable.

Let us now consider a vector-valued function $\Psi : \mathbb{R}^n \to \mathbb{R}^m$ such that each component is a Lipschitz function near $x \in \mathbb{R}^n$. As before, Rademacher's Theorem asserts that Ψ is differentiable on any neighborhood of x in which Ψ is Lipschitz and, thanks to the previous characterization of Clarke subdifferential, the generalized Jacobian is defined as follows.

Definition 4.1.3. Let $\Psi : \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function such that each component is a Lipschitz function near $x \in \mathbb{R}^n$. The generalized Jacobian of Ψ at x is the set

$$\partial\Psi(x) = \text{co}\left\{A = \lim_{k\to\infty} \nabla\Psi(x_k) : x_k \to x,\ x_k \notin \Omega_\Psi\right\}$$

where $\nabla\Psi$ is the Jacobian matrix.

The generalized Jacobian has the following properties.

Proposition 4.1.3. *Let $\Psi : \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function such that each component is a Lipschitz function near $x \in \mathbb{R}^n$, then its generalized Jacobian have the following basic properties:*

- *$\partial\Psi(x)$ is a nonempty, convex and compact subset of $\mathbb{R}^{m\times n}$;*
- *$\partial\Psi$ is closed at x;*

- *$\partial\Psi$ is upper semicontinuous at x.*

Theorem 4.1.2. *Let $\Psi : \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function, $\gamma : \mathbb{R}^m \to \mathbb{R}$ and $\psi : \mathbb{R}^n \to \mathbb{R}$ be functions with ψ defined as $\psi = \gamma \circ \Psi$. If Ψ is Lipschitz near $x \in \mathbb{R}^n$ and γ is Lipschitz near $\Psi(x) = y$, then ψ is Lipschitz near x and one has*

$$\partial\psi(x) \subseteq \mathrm{co}(\partial\gamma(y)\partial\Psi(x)).$$

If in addition, γ is strictly differentiable at $\Psi(x)$, then

$$\partial\psi(x) = \nabla\gamma(y)\partial\Psi(x).$$

We conclude this section by recalling the concept of subdifferential of convex function. The subdifferential arises in convex analysis, which is the study of convex functions, often in relation to convex optimization. In particular, it is the generalization of the gradient for convex functions that are not necessarily differentiable.

Definition 4.1.4. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a convex function. The subdifferential of ψ at $x \in \mathbb{R}^n$ is the convex set

$$\partial\psi(x) = \{x^* \in \mathbb{R}^n : \psi(y) \geq \psi(x) + \langle x^*, y - x\rangle \quad \forall y \in \mathbb{R}^n\}.$$

The concept of Clarke subdifferential reduces to the subdifferential when ψ is convex.

Now, we consider a function $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that is convex with respect to its second variable. The following theorem is a direct consequence of a result in [48] and it provides the upper semicontinuity of the set-valued map which assigns to each pair (x, y) the subdifferential $\partial\varphi(x, \cdot)(y)$.

Theorem 4.1.3. *Let $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $\psi(x, \cdot)$ is convex for each x. Fixed $(\bar{x}, \bar{y})$ and $\varepsilon > 0$ there exists $r > 0$ such that*

$$\partial\psi(x, \cdot)(y) \subseteq \partial\psi(\bar{x}, \cdot)(\bar{y}) + B(0, \varepsilon)$$

provided that $x \in B(\bar{x}, r)$ and $y \in B(\bar{y}, r)$.

4.2 A gap function for quasiequilibrium problems

Gap functions were originally developed for variational inequalities [26, 36, 52] because they provide an equivalent differentiable optimization formulation. They were later extended to equilibrium problems in [40]. However, reformulating quasiequilibrium problems as optimization problems presents some difficulties that are not encountered in the case of equilibrium problems. The gap function may not be differentiable, even if both the equilibrium and constraining functions are. Additionally, monotonicity assumptions are necessary for both the equilibrium and constraining functions.

In [9], the authors present a numerical method for solving quasiequilibrium problems when the constraint map K is described by differentiable constraint functions. The algorithm presented in this chapter is a modification of the algorithm proposed in [9]. The presence of the projection function, however, makes the problem more challenging to handle. Nevertheless, we are able to prove the convergence of our method by taking advantage of the nonexpansiveness of the projection and utilizing some properties of Clarke's subdifferential.

Let $C \subseteq \mathbb{R}^n$ be a nonempty set and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a function. Throughout the remainder of this chapter, we consider the functions $g_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with $i = 1, \ldots, m$ and the set-valued map $K : C \rightrightarrows \mathbb{R}^n$ explicitly described as

$$K(x) = \{u \in \mathbb{R}^n : g_i(x, u) \leq 0, \ i = 1, \ldots, m\}. \tag{4.1}$$

In [38] the author obtains the following result on the semicontinuity of this kind of set-valued map we will use later.

Theorem 4.2.1. *Let K be a set-valued map defined as in (4.1) and $x \in C$. If each g_i is continuous on $\{x\} \times K(x)$, convex with respect to its second variable and there exists $u \in \mathbb{R}^n$ such that $g_i(x, u) < 0$ for all i, then K is lower semicontinuous at x.*

From now on, we will work under the following assumptions on the quasiequilibrium problem.

Assumption A. *The set $C \subseteq \mathbb{R}^n$ is nonempty, closed and convex, the functions f and g_i verify the following:*

- *$f(y, y) = 0$ for all $y \in \mathbb{R}^n$;*
- *f and g_i are continuously differentiable;*
- *$f(y, \cdot)$ and $g_i(y, \cdot)$ are convex for all $y \in \mathbb{R}^n$.*

In order to consider the projection, we define the modified set-valued map $\widetilde{K} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as

$$\widetilde{K}(y) = \{u \in \mathbb{R}^n : x = p_C(y) \text{ and } g_i(x, u) \leq 0,\ i = 1, \ldots, m\}.$$

Notice that $\widetilde{K}(y) = K(y)$ whenever $y \in C$. Moreover, we denote by

$$D_S = \{y \in \mathbb{R}^n : \exists u \in \mathbb{R}^n \text{ s.t. } x = p_C(y) \text{ and } g_i(x, u) < 0,\ i = 1, \ldots, m\}$$

the set of all the points y such that $\widetilde{K}(y)$ satisfies the Slater condition. Since p_C and g_i are continuous, the set D_S is open.

Before using the gap function, let us examine the following parametric minimization problem. Given $\alpha > 0$, we consider the following

$$\min \left\{ f(y, u) + \alpha \|y - u\|^2/2 \ :\ u \in \widetilde{K}(y) \right\}. \tag{4.2}$$

Theorem 4.2.2. *Let the Assumption A be satisfied and $\alpha > 0$, then*

(1) for every $y \in \mathbb{R}^n$ the problem (4.2) has a unique solution $u(y)$;

(2) the function $u : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on D_S.

Proof.

(1) Given $y \in \mathbb{R}^n$, the function $f_\alpha(y, \cdot)$ defined as

$$\begin{aligned} f_\alpha(y, u) &= f(y, u) + \frac{\alpha}{2}\|y - u\|^2 - \frac{\alpha}{2}\|u\|^2 \\ &= f(y, u) + \frac{\alpha}{2}\left[\|y\|^2 - 2\langle y, u\rangle\right] \end{aligned}$$

is convex, due to the fact that it is the sum of two convex functions. Hence, picked $y^* \in \partial f_\alpha(y, \cdot)(y)$ we have

$$\begin{aligned} f_\alpha(y, u) &\geq f_\alpha(y, y) + \langle y^*, u - y\rangle \\ &= -\frac{\alpha}{2}\|y\|^2 + \langle y^*, u - y\rangle \end{aligned}$$

then, from the definition of f_α, we deduce that

$$\begin{aligned} f(y,u) + \frac{\alpha}{2}\|y-u\|^2 &\geq \frac{\alpha}{2}\left(\|u\|^2 - \|y\|^2\right) + \langle y^*, u-y\rangle \\ &\geq \frac{\alpha}{2}\left(\|u\|^2 - \|y\|^2\right) - \|y^*\| \cdot \|u-y\|. \end{aligned}$$

Let $\bar{y} \in C$ be fixed. From Theorem 4.1.3 there exists $r > 0$ such that

$$\partial f_\alpha(y,\cdot)(y) \in \partial f_\alpha(\bar{y},\cdot)(\bar{y}) + B(0,\varepsilon)$$

provided that $y \in B(\bar{y}, r)$. Therefore $\|y^*\|$ is bounded in a neighborhood of $\bar{y}$ and

$$\frac{\alpha}{2}\left(\|u\|^2 - \|y\|^2\right) - \|y^*\| \cdot \|u-y\| \to \infty$$

as $\|u\| \to +\infty$. So the function $f(y,\cdot) + \alpha\|y - \cdot\|^2/2$ is coercive. In addiction, this function is strongly convex. Moreover, $\widetilde{K}$ has closed valued since g and p_C are continuous. Then, since f is continuous, the problem (4.2) has a unique solution.

(2) Let $\{(y_k, u_k)\} \subseteq \operatorname{gph} \widetilde{K}$ be convergent to (y,u) and $x_k = p_C(y_k)$. Since p_C is continuous and C is closed and convex, there exists $x \in C$ such that

$$x_k = p_C(y_k) \to p_C(y) = x.$$

Moreover, since $g_i(x_k, u_k) \leq 0$ for each i and g_i is continuous, then the set-valued map $\widetilde{K}$ is closed. Furthermore, $\widetilde{K}$ is lower semicontinuous at any $y \in D_S$, thanks to Theorem 4.2.1, since $g_i(p_C(y),\cdot)$ are convex and satisfy Slater condition. The map u is uniformly compact near any $y \in D_S$ by Theorem 2.3.6 since f is continuous and u is single-valued. Hence u is continuous at $y \in D_S$ by Theorem 2.3.8. □

The next result shows that the fixed points of u are closely linked to the projected solution of the quasiequilibrium problem.

Proposition 4.2.1. *Let Assumption A be satisfied and $\alpha > 0$, then the pair $(p_C(\bar{y}), \bar{y})$ is a projected solution of QEP(f,K) if, and only if, $u(\bar{y}) = \bar{y}$.*

Proof. If $(p_C(\bar{y}), \bar{y})$ is a projected solution of QEP(f,K), then $\bar{y} \in \widetilde{K}(\bar{y}) = K(p_C(\bar{y}))$ and

$$0 \leq f(\bar{y}, u) \leq f(\bar{y}, u) + \alpha\|\bar{y} - u\|^2/2, \quad \forall u \in \widetilde{K}(\bar{y})$$

then $u(\bar{y}) = \bar{y}$ since $f(\bar{y}, \bar{y}) + \alpha\|\bar{y} - \bar{y}\|^2/2 = 0$.

Vice versa, if $\bar{y} = u(\bar{y})$ then $\bar{y} \in \widetilde{K}(\bar{y})$. Moreover, the problem (4.2) is a convex optimization problem, therefore $\bar{y}$ satisfies the optimality condition

$$\langle \nabla_2 f(\bar{y}, \bar{y}), u - \bar{y} \rangle \geq 0, \quad \forall u \in \widetilde{K}(\bar{y}). \tag{4.3}$$

Since $f(y, \cdot)$ is convex we have

$$\begin{aligned} f(\bar{y}, u) &\geq f(\bar{y}, \bar{y}) + \langle \nabla_2 f(\bar{y}, \bar{y}), u - \bar{y} \rangle \\ &= \langle \nabla_2 f(\bar{y}, \bar{y}), u - \bar{y} \rangle \end{aligned}$$

which, thanks to (4.3), implies that $(p_C(\bar{y}), \bar{y})$ is a projected solution of QEP(f, K). □

The following step is to demonstrate that the value function associated to the minimization problem (4.2)

$$\begin{aligned} \varphi_\alpha(y) &= -\min\left\{ f(y, u) + \alpha\|y - u\|^2/2 \ : \ u \in \widetilde{K}(y) \right\} \\ &= -f(y, u(y)) - \alpha\|y - u(y)\|^2/2 \end{aligned} \tag{4.4}$$

is a gap function. Indeed, this approach transforms the task of finding a projected solution of a quasiequilibrium problem into an optimization problem. This is achieved through the following characterizations.

Theorem 4.2.3. *Let Assumption A be satisfied and $\alpha > 0$, then*

(1) $\varphi_\alpha(y) \geq 0$ for all $y \in \mathbb{R}^n$ such that $g_i(p_C(y), y) \leq 0$, for each $i = 1, \ldots, m$;

(2) the pair $(p_C(\bar{y}), \bar{y})$ is a projected solution of QEP(f, K) if, and only if, $\varphi_\alpha(\bar{y}) = 0$ and $g_i(p_C(\bar{y}), \bar{y}) \leq 0$, for each $i = 1, \ldots, m$.

Proof.

(1) By assumptions $y \in \widetilde{K}(y)$, therefore

$$\min\left\{ f(y, u) + \alpha\|y - u\|^2/2 \ : \ u \in \widetilde{K}(y) \right\} \leq f(y, y) + \alpha\|y - y\|^2/2 \ = 0$$

that is, $\varphi_\alpha(y) \geq 0$.

(2) If $(p_C(\bar{y}), \bar{y})$ is a projected solution of QEP(f, K), then $\bar{y} \in \widetilde{K}(\bar{y})$, that is, $g_i(p_C(\bar{y}), \bar{y}) \leq 0$ for all i. Furthermore, for Proposition 4.2.1, $u(\bar{y}) = \bar{y}$ and so

$$\varphi(\bar{y}) = -f(\bar{y}, \bar{y}) - \alpha\|\bar{y} - \bar{y}\|^2/2 = 0.$$

Vice versa, $\bar{y} \in \widetilde{K}(\bar{y})$ is the optimal solution of problem (4.2) since $f(\bar{y}, \bar{y}) + \alpha\|\bar{y} - \bar{y}\|^2/2 = 0$. Hence, $u(\bar{y}) = \bar{y}$ and Proposition 4.2.1 guarantees that $(p_C(\bar{y}), \bar{y})$ is a projected solution of QEP(f, K). □

When $y \in D_S$, the set of Karush-Kuhn-Tucker multipliers $\Lambda(y) \subseteq \mathbb{R}^m_+$ becomes a nonempty and compact set. Moreover, for each $\lambda(u) \in \Lambda(y)$ the following optimality condition for the optimization problem (4.2)

$$\begin{cases} \nabla_2 f(y, u(y)) + \alpha(u(y) - y) + \sum_{i=1}^{m} \lambda_i(y) \nabla_2 g_i(p_C(y), u(y)) = 0 \\ \lambda_i(y) g_i(p_C(y), u(y)) = 0, \quad i = 1, \ldots, m \end{cases} \tag{4.5}$$

characterizes the solution $u(y)$.

The next result describes the Lipschitz structure of the gap function φ_α. Additionally, an upper estimate of the Clarke directional derivative of φ_α is provided.

Theorem 4.2.4. *Let Assumption A be satisfied, $\alpha > 0$ and $y \in D_S$ be fixed, then*

(1) φ_α is Lipschitz near y;

(2) for any $d \in \mathbb{R}^n$

$$\begin{aligned} \varphi^\circ_\alpha(y; d) \quad \leq \quad & -\Bigg[\langle \nabla_1 f(y, u(y)) + \alpha(y - u(y)), d\rangle + \\ & + \min_{\lambda \in \Lambda(y)} \min_{A \in \partial p_C(y)} \sum_{i=1}^{m} \lambda_i \langle \nabla_1 g_i(p_C(y), u(y)) A, d\rangle \Bigg]. \end{aligned}$$

Proof. For any fixed $y \in D_S$, the Lagrangian function associated to the optimization problem (4.2) is

$$L(y, u, \lambda) = f(y, u) + \alpha\|y - u\|^2/2 + \sum_{i=1}^{m} \lambda_i g_i(p_C(y), u).$$

Since $y \in D_S$ and $u(y)$ is the solution of the optimization problem (4.2), there exists $\lambda(y) \geq 0$ such that the pair $(u(y), \lambda(y))$ is a saddle point of the Lagrangian, i.e.

$$L(y, u(y), \lambda) \leq L(y, u(y), \lambda(y)) \leq L(y, u, \lambda(y)) \tag{4.6}$$

for all $u \in \mathbb{R}^n$ and $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda_i \geq 0$ for all i. Furthermore, $\lambda(y) \in \Lambda(y)$ and then for the condition (4.5) we have

$$\begin{aligned} L(y, u(y), \lambda(y)) &= f(y, u(y)) + \alpha\|y - u(y)\|^2/2 + \sum_{i=1}^{m} \lambda_i(y) g_i(p_C(y), u(y)) \\ &= f(y, u(y)) + \alpha\|y - u(y)\|^2/2 \\ &= -\varphi_\alpha(y). \end{aligned}$$

(1) Take $v, w \in D_S$. The left inequality in (4.6) with $y = w$ and $\lambda = \lambda(v)$ becomes

$$\varphi_\alpha(w) \leq -\left[f(w, u(w)) + \alpha\|w - u(w)\|^2/2 + \sum_{i=1}^{m} \lambda_i(v) g_i(p_C(w), u(w)) \right]$$

and the right inequality in (4.6) with $y = v$ and $u = u(w)$ becomes

$$-\varphi_\alpha(v) \leq f(v, u(w)) + \alpha\|v - u(w)\|^2/2 + \sum_{i=1}^{m} \lambda_i(v) g_i(p_C(v), u(w)).$$

Then, we have

$$\begin{aligned} \varphi_\alpha(w) - \varphi_\alpha(v) \leq\ & f(v, u(w)) - f(w, u(w)) + \\ & + \alpha(\|v - u(w)\|^2 - \|w - u(w)\|^2)/2 + \\ & + \sum_{i=1}^{m} \lambda_i(v)[g_i(p_C(v), u(w)) - g_i(p_C(w), u(w))]. \end{aligned} \tag{4.7}$$

By Assumption A, all the functions are locally Lipschitz, i.e. Lipschitz on every compact subset of $\mathbb{R}^n$. Now, fixed $\bar{y} \in D_S$ and $\bar{u} = u(\bar{y})$ there exists $\delta_{\bar{y}} > 0$ such that $B(\bar{y}, \delta_{\bar{y}}) \subset D_S$ and, since u is continuous (Theorem 4.2.2), there exists $\delta_{\bar{u}} > 0$ such that $u(B(\bar{y}, \delta_{\bar{y}})) \subseteq B(\bar{u}, \delta_{\bar{u}})$. Let L_f be the Lipschitz constant of f in $B(\bar{y}, \delta_{\bar{y}}) \times B(\bar{u}, \delta_{\bar{u}})$. Hence, for all $v, w \in B(\bar{y}, \delta_{\bar{y}})$

$$|f(v, u(w)) - f(w, u(w))| \leq L_f(\|v - w\| + \|u(w) - u(w)\|) = L_f\|v - w\|.$$

Moreover, for all $v, w \in B(\bar{y}, \delta_{\bar{y}})$,

$$\begin{aligned}
&\Big| \|v - u(w)\|^2 - \|w - u(w)\|^2 \Big| \\
&\quad = \Big(\|v - u(w)\| + \|w - u(w)\| \Big) \Big| \|v - u(w)\| - \|w - u(w)\| \Big| \\
&\quad \leq L\|v - w\|
\end{aligned}$$

since

$$\begin{aligned}
\|v - u(w)\| &\leq \|v - \bar{y}\| + \|\bar{y} - \bar{u}\| + \|\bar{u} - u(w)\| \\
&\leq \delta_{\bar{y}} + \|\bar{y} - \bar{u}\| + \delta_{\bar{u}} = L
\end{aligned}$$

a similar approach can be applied to $\|w - u(w)\|$, and

$$\Big| \|v - u(w)\| - \|w - u(w)\| \Big| \leq \|v - w\|.$$

Finally, since p_C is continuous, there exists $\delta_{\bar{x}} > 0$ such that $p_C(B(\bar{y}, \delta_{\bar{y}})) \subseteq B(\bar{x}, \delta_{\bar{x}})$ where $\bar{x} = p_C(\bar{y})$. Hence, for all $v, w \in B(\bar{y}, \delta_{\bar{y}})$ and for each i we have

$$\begin{aligned}
|g_i(p_C(v), u(w)) - g_i(p_C(w), u(w))| &\leq L_i\|p_C(v) - p_C(w)\| \\
&\leq L_i\|v - w\|
\end{aligned}$$

where L_i is the Lipschitz constant of g_i in $B(\bar{x}, \delta_{\bar{x}}) \times B(\bar{u}, \delta_{\bar{u}})$ and the last inequality descends from the nonexpansivity of the projection. Moreover, Lemma 2 in [37] guarantees that each multiplier is locally bounded, that is, for each i there exists L'_i such that $|\lambda_i(v)| \leq L'_i$ holds for all $v \in B(\bar{y}, \delta_{\bar{y}})$. Collecting all these inequalities we have

$$\varphi(w) - \varphi(v) \leq \left(L_f + \alpha L + \sum_{i=1}^{m} L'_i L_i \right) \|v - w\|$$

for all $v, w \in B(\bar{y}, \delta)$. Therefore φ is Lipschitz near $\bar{y}$.

(2) Fixed $d \in \mathbb{R}^n$ there is $\delta > 0$ such that $z + td \in B(y, \delta) \subset D_S$ for any $z \in B(y, \delta)$ and $t > 0$ small enough. Let $y_k \to y$, $t_k \to 0^+$ and $y_k^t = y_k + t_k d$ be such that

$$\varphi_\alpha^\circ(y; d) = \limsup_{k \to \infty} \frac{\varphi_\alpha(y_k^t) - \varphi_\alpha(y_k)}{t_k}.$$

Taking (4.7) with $w = y_k^t$ and $v = y_k$, we apply the mean value theorem to the functions $f(\cdot, u(y_k^t))$, $\| \cdot - u(y_k^t)\|^2/2$ and $g_i(p_C(\cdot), u(y_k^t))$ in the right-hand side of the inequality. Since f is continuously differentiable, for each k there exist y_k', y_k'' in the segment (y_k, y_k^t) such that

$$\begin{aligned} f\left(y_k, u(y_k^t)\right) - f(y_k^t, u(y_k^t)\big) &= -t_k \langle \nabla_1 f\left(y_k', u(y_k^t)\right), d\rangle \\ \left(\|y_k - u(y_k^t)\|^2 - \|y_k^t - u(y_k^t)\|^2\right)/2 &= -t_k \langle y_k'' - u(y_k^t), d\rangle. \end{aligned}$$

Since each constraint function $g_i(p_C(\cdot), u(y_k^t))$ is not continuously differentiable but locally Lipschitz only, we need to apply the Lebourg mean value result. Therefore for each i and k there is $y_k^i \in (y_k, y_k^t)$ such that

$$\begin{aligned} g_i(p_C(y_k), u(y_k^t)) - g_i(p_C(y_k^t), u(y_k^t)) &\in -\langle \partial g_i(p_C(\cdot), u(y_k^t))(y_k^i), y_k - y_k^t\rangle \\ &= -t_k \langle \nabla_1 g_i(p_C(y_k^i), u(y_k^t)) \partial p_C(y_k^i), d\rangle \end{aligned}$$

where the inclusion is due to Theorem 4.1.1 and the equality descends from Theorem 4.1.2. Therefore, there exists $A_k^i \in \partial p_C(y_k^i)$ such that

$$\begin{aligned} \varphi_\alpha^\circ(y; d) \leq - \limsup_{k\to\infty} \Bigg[&\langle \nabla_1 f(y_k', u(y_k^t)) + \alpha(y_k'' - u(y_k^t)), d\rangle + \\ &+ \sum_{i=1}^m \lambda_i(y_k) \langle \nabla_1 g_i(p_C(y_k^i), u(y_k^t)) A_k^i, d\rangle \Bigg]. \end{aligned}$$

Since $\nabla_1 f$, $\nabla_1 g_i$, p_C and u are continuous, then

$$\begin{aligned} u(y_k^t) &\to u(y) \\ p_C(y_k^i) &\to p_C(y) \\ y_k'' - u(y_k^t) &\to y - u(y) \\ \nabla_1 f(y_k', u(y_k^t)) &\to \nabla_1 f(y, u(y)) \\ \nabla_1 g_i(p_C(y_k^i), u(y_k^t)) &\to \nabla_1 g_i(p_C(y), u(y)). \end{aligned}$$

Moreover, since ∂p_c is an upper semicontinuous set-valued map with compact values (Proposition 4.1.3) for each i the sequence $\{A_k^i\}$ is definitively contained in a compact set and then, without loss of generality, we may assume that

$$A_k^i \to A^i \in \partial p_C(y).$$

Furthermore, Lemma 2 in [37] guarantees that the set-valued map Λ is uniformly bounded on a neighborhood of y and closed at y. Hence, without loss of generality, there exists $\lambda(y) \in \Lambda(y)$ such that $\lambda_i(y_k) \to \lambda_i(y)$ for each i. As a consequence, the inequality

$$\varphi_\alpha^\circ(y; d) \leq - \Bigg[\langle \nabla_1 f(y, u(y)) + \alpha(y - u(y)), d\rangle + \sum_{i=1}^{m} \lambda_i(y) \langle \nabla_1 g_i(p_C(y), u(y)) A^i, d\rangle\Bigg]$$

holds. □

4.3 The descent numerical method

A descent direction for φ_α can be obtained under suitable monotonicity conditions on f and constraining functions. As proposed in [40] for equilibrium problems, we require that f is strict ∇-monotone on a suitable set X, that is

$$\langle \nabla_1 f(y, u) + \nabla_2 f(y, u), u - y\rangle > 0 \tag{4.8}$$

for any $y, u \in X$.

Furthermore, as the upper estimate of the Clarke directional derivative of φ_α only involves active constraints, it is appropriate to limit the monotonicity assumptions accordingly. The following definition extends the notion of ∇-monotonicity introduced in [40] when γ is a locally Lipschitz function.

Definition 4.3.1. Let $X \subseteq \mathbb{R}^n$ be fixed. A locally Lipschitz function $\gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to satisfy the active ∂-monotone condition on X if

$$\langle \xi_1 + \xi_2, u - y\rangle \geq 0, \quad \forall \xi_i \in \partial_i \gamma(y, u) \tag{4.9}$$

for any $y, u \in X$ with $\gamma(y, u) = 0$, where $\partial_1\gamma(y, u) = \partial\gamma(\cdot, u)(y)$ and $\partial_2\gamma(y, u) = \partial\gamma(y, \cdot)(u)$.

Unfortunately, satisfying active ∂-monotonicity is generally not easy. However, the following proposition demonstrates a class of constraint functions that satisfy this condition on a specific set.

Proposition 4.3.1. *Let* $C = \prod_{i=1}^{n}[a_i, b_i] \subseteq \mathbb{R}^n$. *If* $m \leq 1$ *and* $q \in \mathbb{R}$ *then the function* $g(p_C(\cdot), \cdot)$ *where*

- $g(x, u) = u_j - mx_j - q$
- $g(x, u) = mx_j - u_j + q$

is active ∂*-monotone on the set* $X = \{x \in \mathbb{R}^n : g(p_C(x), x) \leq 0\}$.

Proof. We prove the claim only for $g(x, u) = u_j - mx_j - q$ since the second case is conceptually similar. Let $y, u \in \mathbb{R}^n$ be the solutions of the system

$$\begin{cases} u_j \leq m(p_C(u))_j + q \\ y_j \leq m(p_C(y))_j + q \\ u_j = m(p_C(y))_j + q \end{cases}$$

where the two inequalities descend from the fact that $y, u \in X$ and the equality is $g(p_C(y), u) = 0$. Hence, $u_j \geq y_j$. Moreover,

$$\frac{\partial}{\partial y_i} (p_C(y))_j = 0 \quad \forall i \neq j$$

and

$$\frac{\partial}{\partial y_j} (p_C(y))_j = \begin{cases} \{1\} & \text{if } y_j \in (a_j, b_j) \\ [0,1] & \text{if } y_j \in \{a_j, b_j\} \\ \{0\} & \text{if } y_j \notin [a_j, b_j] \end{cases}$$

Therefore, the inequality (4.9) becomes

$$\langle -m\xi_j e_j + e_j, u - y\rangle = (1 - m\xi_j)(u_j - y_j) \geq 0$$

where $\xi_j \in \frac{\partial}{\partial y_j} (p_C(y))_j$. Since $m \leq 1$, then $1 - m\xi_j \geq 0$ and the condition is satisfied on X. □

The active ∂-monotonicity of the constraint function heavily depends on the structure of C due to the presence of the projection map, as shown in the following example.

Example 4.3.1. Let $n = 2$ and $C = \{x \in \mathbb{R}^2 : \|x\|^2 \leq 1\}$. Take

$$g(x, u) = u_1 - mx_1 - q$$

as in Proposition 4.3.1. We show that, for each $m, q \in \mathbb{R}$, except the case $m < 0$ and $q = 0$, the function $g(p_C(\cdot), \cdot)$ is not active ∂-monotone on the set $X = \{x \in \mathbb{R}^2 : g(p_C(x), x) \leq 0\}$.

First, we notice that for each $y \in \mathbb{R}^2$ with $\|y\| > 1$ we have

$$(p_C(y))_1 = \frac{y_1}{\|y\|} \quad \text{and} \quad \nabla(p_C(y))_1 = \left(-\frac{y_2^2}{\|y\|^3}, \frac{y_1 y_2}{\|y\|^3}\right).$$

Now, we find two points $y, u \in \mathbb{R}^2$ with $\|y\| > 1$ that solve the system

$$\begin{cases} u_1 \leq m(p_C(u))_1 + q \\ y_1 \leq m(p_C(y))_1 + q \\ u_1 = m(p_C(y))_1 + q \end{cases}$$

and verify the inequality

$$\left(1 - \frac{my_2^2}{\|y\|^3}\right)(u_1 - y_1) + \frac{m}{\|y\|^3} y_1 y_2 (u_2 - y_2) < 0 \tag{4.10}$$

which is the contradictions of (4.9).

If $m > 0$ and $q \in \mathbb{R}$, consider the points

$$\begin{aligned} y &= (-t, t) \\ u &= (-t + \Delta, t + \alpha\Delta) \end{aligned}$$

with

$$t = \frac{m}{\sqrt{2}} - q + \Delta$$

and $\alpha, \Delta > 0$ two parameters. If $\alpha > 0$ and $\Delta > \max\{0, q + (1 - m)/\sqrt{2}\}$, then $\|y\| > 1$ and y, u solve the system. Moreover if $\alpha > \max\{0, 1 + 2\sqrt{2}(\Delta - q)/m\}$ the inequality (4.10) is verified.

While, if $m < 0$ and $q < 0$, take

$$\begin{aligned} y &= \left(t, \sqrt{\alpha^2 - t^2}\right) \\ u &= \left(t, -\sqrt{\alpha^2 - t^2}\right) \end{aligned}$$

with

$$t = \frac{\alpha q}{\alpha - m}.$$

Choosing $\alpha > \max\{1, m - q\}$ we have $\|y\| > 1$ and y, u solve the system and verify the inequality (4.10).

Finally, if $m < 0$ and $q > 0$, choose the points

$$\begin{aligned} y &= (t, 1) \\ u &= (t + \Delta, 1 + \alpha\Delta) \end{aligned}$$

with

$$\begin{aligned} t &= \frac{q}{1 - m} \\ \Delta &= \frac{mq}{\sqrt{q^2 + (1 - m)^2}} - \frac{mq}{1 - m} \end{aligned}$$

and $\alpha > 1/t$, then $\|y\| > 1$ and y, u solve the system. If, in addition,

$$\alpha > \frac{1}{t} - \frac{1}{\Delta}\left(\frac{1}{t^2} + 1\right) > \frac{1}{t}$$

the inequality (4.10) is satisfied.

Moreover, the case with $m = 0$ is meaningless since it coincides to K constant. While, the case $q = 0$ and $m < 0$ is the only one in which the active ∂-monotonicity on X holds. Indeed, the only points that verify the system are

$$\begin{aligned} y &= (0, y_2) \\ u &= (0, u_2) \end{aligned}$$

with $y_2, u_2 \in \mathbb{R}$, and the inequality (4.9) holds in such points.

Under the assumptions of monotonicity on f and the constraint functions, the following descent property provides the tool for designing the algorithm.

Theorem 4.3.1. *Let $y \in D_S \cap \operatorname{fix} \widetilde{K}$ and suppose*

(i) Assumption A is satisfied and $\alpha > 0$;

(ii) f *is strictly* ∇*-monotone on* $\widetilde{K}(y)$*;*

(iii) $g_i(p_C(\cdot),\cdot)$ *is active* ∂*-monotone on* $\widetilde{K}(y)$ *for any* $i = 1,\ldots,m$*.*

If the pair $(p_C(y), y)$ *is not a projected solution of QEP*(f,K)*, then* $\varphi_\alpha^\circ(y; u(y)-y) < 0$*.*

Proof. Since the pair $(p_C(y), y)$ is not a projected solution of QEP(f,K), Proposition 4.2.1 implies that $u(y) \neq y$ and Theorem 4.2.4 implies that there exist m multipliers λ_i and m elements of the generalized Jacobian $A^i \in \partial p_C(y)$ such that

$$\begin{aligned}
\varphi_\alpha^\circ(y; u(y)-y) \;&\leq\; -\langle \nabla_1 f(y,u(y)) + \alpha(y-u(y)), u(y)-y\rangle + \\
&\quad -\sum_{i=1}^{m} \lambda_i \langle \nabla_1 g_i(p_C(y),u(y))A^i, u(y)-y\rangle \\
&=\; -\langle \nabla_1 f(y,u(y)) + \nabla_2 f(y,u(y)), u(y)) - y\rangle + \\
&\quad -\sum_{i=1}^{m} \lambda_i \langle \nabla_1 g_i(p_C(y),u(y))A^i + \nabla_2 g_i(p_C(y),u(y)), u(y)-y\rangle \\
&<\; -\sum_{i=1}^{m} \lambda_i \langle \nabla_1 g_i(p_C(y),u(y))A^i + \nabla_2 g_i(p_C(y),u(y)), u(y)-y\rangle
\end{aligned}$$

where the equality is due to (4.5) and the last inequality to the strict ∇-monotonicity of f. Moreover, each addend

$$\langle \nabla_1 g_i(p_C(y),u(y))A^i + \nabla_2 g_i(p_C(y),u(y)), u(y)-y\rangle \geq 0$$

since $g_i(p_C(\cdot),\cdot)$ is active ∂-monotone and

$$\nabla_1 g_i(p_C(y),u(y))A^i \in \partial_1 g_i(p_C(\cdot),u(y))(y)$$

for Theorem 4.1.2. Therefore $\varphi_\alpha^\circ(y; u(y)-y) < 0$. □

The descent direction of Theorem 4.3.1 can be used to devise a descent algorithm in the same way as [9]. In particular, if the current iterate y_k does not verify the stop criterion, a step along the descent direction $u(y_k) - y_k$ is performed by exploiting an inexact line search.

Algorithm	
Step 0:	Choose $\beta, \gamma \in (0,1)$, $y_0 \in \operatorname{fix} \widetilde{K}$ and set $k = 0$.
Step 1:	Compute $x_k = p_C(y_k)$.
Step 2:	Compute $u(y_k) = \operatorname{argmin}\{f(y_k, u) + \alpha\|y_k - u\|^2/2 \ : \ u \in K(x_k)\}$.
Step 3:	If $d_k = u(y_k) - y_k = 0$ STOP. Otherwise, compute the smallest non-negative integer s such that $\varphi_\alpha(y_k + \gamma^s d_k) - \varphi_\alpha(y_k) \leq -\beta\gamma^{2s}\|d_k\|$.
Step 4:	Set $t_k = \gamma^s$, $y_{k+1} = y_k + t_k d_k$, $k = k+1$ and go to Step 1.

Theorem 4.3.2. *Let Assumption A be satisfied, $\alpha > 0$ and*

(i) $\operatorname{fix} \widetilde{K} \subseteq D_S$*;*

(ii) $\operatorname{fix} \widetilde{K}$ *convex;*

(iii) $\widetilde{K}(y) \subseteq \operatorname{fix} \widetilde{K}$ *for any* $y \in \operatorname{fix} \widetilde{K}$*;*

(iv) f *is strictly* ∇*-monotone on* $\operatorname{fix} \widetilde{K}$*;*

(v) $g_i(p_C(\cdot), \cdot)$ *is active* ∂*-monotone on* $\operatorname{fix} \widetilde{K}$ *for any* i*.*

Then either Algorithm stops at a projected solution of QEP(f, K) *after a finite number of iterations or produces a sequence* $\{(x_k, y_k)\}$ *such that any of its cluster points is a projected solution of QEP*(f, K)*.*

Proof. First, we see that the line search procedure is finite. By contradiction, assume that there exists some iteration k that satisfies

$$\varphi_\alpha(y_k + \gamma^s d_k) - \varphi_\alpha(y_k) > -\beta\gamma^{2s}\|d_k\|$$

for all $s \in \mathbb{N}$. For the definition of the Clarke directional derivative we have

$$\begin{aligned} \varphi_\alpha^\circ(y_k; d_k) &\geq \limsup_{s\to\infty} \frac{\varphi_\alpha(y_k + \gamma^s d_k) - \varphi_\alpha(y_k)}{\gamma^s} \\ &\geq \limsup_{s\to\infty} -\beta\gamma^s\|d_k\| = 0 \end{aligned}$$

then, thanks to Theorem 4.3.1, $(p_C(y_k), y_k)$ is a projected solution of QEP(f, K). But y_k does not satisfy the stopping criterion of Step 3 and therefore $u(y_k) \neq y_k$ which contradicts Theorem 4.2.3.

Furthermore, if the algorithm stops after a finite number k of iterations, the pair (x_k, y_k) is a projected solution of QEP(f, K) for Theorem 4.2.3.

Now, assume that the algorithm generates the sequence $\{(x_k, y_k)\}$ and let (x^*, y^*) be a cluster point. We show that $\{y_k\} \subseteq \operatorname{fix} \widetilde{K}$. We give a proof by induction on k. Assuming that $y_k \in \operatorname{fix} \widetilde{K}$ we have $u(y_k) \in \widetilde{K}(y_k) \subseteq \operatorname{fix} \widetilde{K}$ from *(i)*. Hence, $y_{k+1} = t_k u(y_k) + (1 - t_k) y_k \in \operatorname{fix} \widetilde{K}$ from *(ii)*. Moreover, the set $\operatorname{fix} \widetilde{K}$ is closed since $\widetilde{K}$ is a closed set-valued map, hence $y^* \in \operatorname{fix} \widetilde{K}$.

Without loss of generality, assume that $y_k \to y^*$ and hence

$$x_k = p_C(y_k) \to p_C(y^*) = x^*.$$

Since u is continuous (Theorem 4.2.2) we have

$$d_k = u(y_k) - y_k \to u(y^*) - y^* = d^*.$$

If $d^* = 0$, thanks to Theorem 4.2.3, (x^*, y^*) is a projected solution of QEP(f, K). By contradiction assume that $d^* \neq 0$. The sequence $\{\varphi_\alpha(y_k)\}$ is bounded from below (Theorem 4.2.3) and monotone decreasing (Step 3). Then it admits finite limit and

$$0 = \lim_{k \to \infty} [\varphi_\alpha(y_k) - \varphi_\alpha(y_{k+1})] \geq \limsup_{k \to \infty} \beta t_k^2 \|d_k\| \geq 0$$

implies that $t_k \to 0$ since $d^* \neq 0$.

Since s is the smallest non-negative integer that verifies Step 3, we have that

$$\varphi_\alpha(y_k + t_k \gamma^{-1} d_k) - \varphi_\alpha(y_k) > -\beta (t_k \gamma^{-1})^2 \|d_k\| \tag{4.11}$$

while, Theorem 4.1.1 guarantees the existence of $\theta_k \in (0, 1)$ and some $\xi_k \in \partial \varphi_\alpha(y_k + \theta_k t_k \gamma^{-1} d_k)$ such that

$$\varphi_\alpha(y_k + t_k \gamma^{-1} d_k) - \varphi_\alpha(y_k) = \langle \xi_k, t_k \gamma^{-1} d_k \rangle. \tag{4.12}$$

From (4.11) and (4.12) we obtain

$$\langle \xi_k, d_k \rangle > -\beta t_k \gamma^{-1} \|d_k\|$$

From the definition of the Clarke subdifferential, since $\xi_k \in \partial \varphi_\alpha(y_k + \theta_k t_k \gamma^{-1} d_k)$, we have

$$\langle \xi_k, d_k \rangle \leq \varphi_\alpha^\circ(y_k + \theta_k t_k \gamma^{-1} d_k; d_k)$$

which implies

$$\varphi_\alpha^\circ(y_k + \theta_k t_k \gamma^{-1} d_k; d_k) > -\beta t_k \gamma^{-1} \|d_k\|.$$

Additionally, since $y_k \to y^*$, $d_k \to d^*$ and $t_k \to 0$, follows

$$y_k + \theta_k t_k \gamma^{-1} d_k \to y^*.$$

Therefore, from the upper semicontinuity of the Clarke directional derivate φ_α° (Proposition 4.1.1), we have

$$\begin{aligned} \varphi_\alpha^\circ(y^*; d^*) &\geq \limsup_{k\to\infty} \varphi_\alpha^\circ(y_k + \theta_k t_k \gamma^{-1} d_k; d_k) \\ &\geq \limsup_{k\to\infty} \left[-\beta t_k \gamma^{-1} \|d_k\|\right] = 0. \end{aligned}$$

Then, thanks to Theorem 4.3.1 and Theorem 4.2.3, y^* is a solution of QEP$(f, \widetilde{K})$ and $d^* = 0$ which contradicts the assumption that $d^* \neq 0$. □

Remark 4.3.1. In order to verify assumptions (ii), (iii) and (v) of Theorem 4.3.2 it is sufficient to show that are verified by each single constraint map $\widetilde{K}_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined as

$$\widetilde{K}_i(y) = \{u \in \mathbb{R}^n : x = p_C(y) \text{ and } g_i(x, u) \leq 0\}.$$

Indeed, for each y we have

$$\begin{aligned} \widetilde{K}(y) &= \bigcap_{i=1}^{m} \widetilde{K}_i(y) \\ \operatorname{fix} \widetilde{K} &= \bigcap_{i=1}^{m} \operatorname{fix} \widetilde{K}_i. \end{aligned}$$

Hence, the convexity of each $\operatorname{fix} \widetilde{K}_i$ guarantees the convexity of $\operatorname{fix} \widetilde{K}$. Unfortunately, the convexity of each $\operatorname{fix} \widetilde{K}_i$ is not guaranteed unless the mapping $y \mapsto g_i(p_C(y), y)$ is quasiconvex. In fact, even if the function g_i is affine the set $\operatorname{fix} \widetilde{K}_i$ could be not convex.

Moreover, if assume that (iii) holds for each $\widetilde{K}_i$ then

$$y \in \widetilde{K}(\operatorname{fix} \widetilde{K}) = \bigcap_{j=1}^{m} \widetilde{K}_j \left(\bigcap_{i=1}^{m} \operatorname{fix} \widetilde{K}_i \right) \quad \Rightarrow \quad y \in \widetilde{K}_i(\operatorname{fix} \widetilde{K}_i) \subseteq \operatorname{fix} \widetilde{K}_i$$

for each $i = 1, \dots, m$; hence (iii) holds for $\widetilde{K}$.

Finally, if $g_i(p_C(\cdot), \cdot)$ is active ∂-monotone on $\operatorname{fix} \widetilde{K}_i$ then $g_i(p_C(\cdot), \cdot)$ is active ∂-monotone on the smaller set $\operatorname{fix} \widetilde{K}$.

4.4 Numerical test

Preliminary tests were conducted to analyze the sensitivity of the algorithm with respect to its parameters. To the best of our knowledge, this is the first algorithm to have been implemented for finding a projected solution of a quasiequilibrium problem. It has been implemented in MATLAB R2024a and the built-in functions fmincon and quadprog from the Optimization Toolbox were exploited to evaluate the gap function φ_α at steps 2 and 3, while projections have been performed through explicit formulas. The tests have been run on a MacBook Pro M1, 2020.

Precisely, we considered QEP(f, K) with the set $C = B(0, r_1) \subseteq \mathbb{R}^n$ with $r_1 \in (0, 2)$, the function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x, y) = \langle Px + Qy + u, y - x \rangle$$

where $P, Q \in \mathbb{R}^{n \times n}$ are symmetric positive definite matrices and $u \in \mathbb{R}^n$, and the set-valued map $K : C \rightrightarrows \mathbb{R}^n$ defined as

$$K(x) = B(x, r_2) = \{z \in \mathbb{R}^n \ : \ \|x - z\|^2 \leq r_2\}$$

where $r_1 + r_2 = 2$.

All the assumptions of convergence Theorem 4.3.2 are verified. Indeed, the set

$$\operatorname{fix} \widetilde{K} = B(0, r_1 + r_2) = B(0, 2)$$

is convex,

$$\widetilde{K}(y) = B(p_C(y), r_2) \subseteq \operatorname{fix} \widetilde{K}$$

for all $y \in \operatorname{fix} \widetilde{K}$, the function f is strictly ∇-monotone on $\operatorname{fix} \widetilde{K}$ and the function

$$\widetilde{g}(y, z) = \|p_C(y) - z\|^2 - r_2$$

is active ∂-monotone on $\operatorname{fix} \widetilde{K}$.

Instances have been produced relying on uniformly distributed pseudorandom numbers on the data of the function f and the size of the balls C and $K(x)$. In particular, the formulas $P = AA^T + \varepsilon I$ and $Q = AA^T + \varepsilon I$ have been exploited where the matrices A and B have entries drawn from $[-1, 1]$, $\varepsilon \in (0, 0.5]$, u has been taken in $[-1, 1]^n$ with norm at most 5, while r_1 has been taken in $[1, 1.5]$ and $r_2 = 2 - r_1$. Sizes have been set to $n = 50$ and $n = 100$. Finally, the value 10^{-3} was used as the threshold for the stopping criterion at Step 2.

First, we ran the algorithm for different choices of the parameters β and γ on a set of 100 random instances with $\alpha = 1$ and random starting points in fix $\widetilde{K}$. Results are given in Tables 4.1 and 4.2: each row reports the CPU time, the average and the minimum and maximum number of iterations and the average and maximum number of steps of the linesearch at Step 3 (the minimum has been omitted since it always turns out to be 1).

The tables show that the performance of the algorithm is much more sensitive to the choice of β and small values of β and intermediate values of γ provide a good choice. Moreover, the tests often computed projected solutions (x^*, y^*) with $y^* \notin C$ so that y^* is not a solution in the standard sense. The percentage of such "truly" projected solution was 64% in the tests with $n = 50$ and 58% with $n = 100$, and independent of the choice of the parameters.

Afterwards, similar tests have been performed to analyze the sensitivity with respect to α. According to indications of the previous tests, the values for the other parameters have been set to $\beta = 0.2$ and $\gamma = 0.5$. Tables 4.3 and 4.4 report the results of the tests over 100 instances and show that small values of α provide a good choice, though the impact on the performance is not particularly strong. In these tests the percentage of such "truly" projected solution was similar to the previous one, namely 60% for $n = 50$ and 60% for $n = 100$.

Table 4.1: $n = 50$ and $\alpha = 1$: sensitivity with respect to β and γ.

		time	iterations			linesearch steps	
β	γ	(sec)	min	avg	max	avg	max
0.2	0.3	0.3268	19	24.42	29	1.63	2.00
0.4	0.3	0.4366	26	30.98	37	1.77	2.00
0.6	0.3	0.8786	28	48.12	64	2.21	2.63
0.8	0.3	1.1430	30	59.11	83	12.34	2.72
0.2	0.5	0.2319	14	17.74	22	1.56	2.38
0.4	0.5	0.4956	17	27.49	33	2.22	2.92
0.6	0.5	0.7154	25	34.53	44	2.58	3.51
0.8	0.5	1.0316	30	43.78	56	2.90	3.64
0.2	0.7	0.2439	12	15.87	19	1.82	2.98
0.4	0.7	0.5762	16	23.58	29	2.98	4.52
0.6	0.7	0.8247	21	29.10	37	3.49	4.71
0.8	0.7	1.1960	23	35.73	45	4.13	5.58
0.2	0.9	0.4061	12	14.93	18	3.31	6.98
0.4	0.9	1.1070	15	20.96	26	6.50	10.93
0.6	0.9	1.8719	18	26.42	33	8.73	13.51
0.8	0.9	2.6303	21	31.45	39	10.31	15.18

Table 4.2: $n = 100$ and $\alpha = 1$: sensitivity with respect to β and γ.

		time	iterations			linesearch steps	
β	γ	(sec)	min	avg	max	avg	max
0.2	0.3	1.1858	19	24.38	29	1.63	2.00
0.4	0.3	1.6051	25	30.95	35	1.77	2.00
0.6	0.3	3.2414	30	48.05	63	2.20	2.60
0.8	0.3	4.1895	32	58.68	76	2.33	2.68
0.2	0.5	0.8390	15	17.67	22	1.55	2.37
0.4	0.5	1.8229	19	27.36	32	2.22	2.95
0.6	0.5	2.6459	27	34.53	42	2.58	3.55
0.8	0.5	3.8023	31	43.71	54	2.89	3.60
0.2	0.7	0.8866	13	15.88	19	1.82	2.98
0.4	0.7	2.1273	18	23.53	28	2.99	4.53
0.6	0.7	3.0560	22	29.14	35	3.49	4.72
0.8	0.7	4.4469	25	35.81	43	4.13	5.58
0.2	0.9	1.4843	13	14.95	17	3.30	6.97
0.4	0.9	4.0794	17	21.03	25	6.46	10.95
0.6	0.9	6.9246	20	26.39	31	8.70	13.48
0.8	0.9	9.7355	23	31.41	37	10.28	15.17

Table 4.3: $n = 50$, $\beta = 0.2$ and $\gamma = 0.5$: sensitivity with respect to α.

	time	iterations			linesearch steps	
α	(sec)	min	avg	max	avg	max
0.5	0.1702	12	15.02	21	1.50	2.22.
1.0	0.2311	15	18.13	23	1.56	2.35
1.5	0.2999	17	21.17	25	1.61	2.37
2.0	0.3468	20	24.22	29	1.61	2.38

Table 4.4: $n = 100$, $\beta = 0.2$ and $\gamma = 0.5$: sensitivity with respect to α.

	time	iterations			linesearch steps	
α	(sec)	min	avg	max	avg	max
0.5	0.6053	13	14.52	20	1.49	2.16
1.0	0.8446	15	17.46	22	1.54	2.32
1.5	1.1333	17	20.57	25	1.60	2.38
2.0	1.3069	20	23.78	29	1.60	2.39

Appendix A

A numerical approach for a pay-as-bid electricity market model

This thesis concludes with a work in progress. We aim to obtain an algorithm that is used to find a projected solution of the pay-as-bid electricity market model presented in Chapter 1.

The electricity market model under consideration poses three primary challenges: a bilevel structure, the presence of projection, and the need to solve N problems simultaneously. To address the bilevel structure, we propose characterizing the solution of the follower problem using the KKT conditions. In this way, the lower level "disappears", and we obtain a single level, which is a game. We employ the same procedure for the projection. Indeed, determining the projection is equivalent to solving a convex optimization problem whose unique solution can be characterized by the KKT conditions. Finally, because of the properties of the potential games, it is possible to solve a single optimization problem instead of N simultaneous minimization problems.

First, let us review the model. As shown in the first chapter, finding a projected solution of the electricity market model considered consists of finding N pairs $(\Psi_i, y_i) \in C_i^L \times C_i^Q$ that verify the following conditions:

1. the vector of bid functions $\Psi = (\Psi_1, \ldots, \Psi_N)$ is a projection of $y = (y_1, \ldots, y_N)$ on $C_1^L \times \cdots \times C_N^L$;

2. for each producer $i = 1, \ldots, N$, y_i solves the following optimization problem

$$(P_i) \quad \begin{cases} \max[y_i(x_i) - \gamma_i(x_i)] \\ y_i \in K_i(\Psi_i) \\ x = (x_1, \ldots, x_N) \text{ solves } (ISO) \end{cases}$$

where

$$(ISO) \begin{cases} \min\limits_{x}[y_1(x_1) + \ldots + y_N(x_N)] \\ x_i \in [0, Q_i], \qquad \forall i = 1, \ldots, N \\ x_1 + \ldots + x_N = D \end{cases}$$

For simplicity, the index i is omitted. We recall that for each producer, the set C^Q is a closed and convex subset of

$$\mathcal{Q} = \left\{ f : [0, Q] \to \mathbb{R} : f(x) = ax^2 + bx + c \text{ with } a > 0, b \geq 0 \text{ and } c \in \mathbb{R} \right\}$$

the set $K(\Psi) \subseteq L^2([0, Q], \mathbb{R})$ is defined as

$$K(\Psi) = \left\{ y \in C^Q : y(0) \geq \Psi(0) \right\}$$

and each bid function $\Psi \in C^L$, assuming $\Psi(0) = p_0$, is defined as

$$\Psi(x) = \begin{cases} p_0 x + p_0 & \text{if } x \in [q_0, q_1] \\ p_1 x + p_0 + (p_0 - p_1) q_1 & \text{if } x \in [q_1, q_2] \\ p_2 x + p_0 + (p_0 - p_1) q_1 + (p_1 - p_2) q_2 & \text{if } x \in [q_2, q_3] \\ \vdots & \\ p_{K-1} x + p_0 + \sum\limits_{\nu=1}^{K-1} (p_{\nu-1} - p_\nu) q_\nu & \text{if } x \in [q_{K-1}, q_K] \end{cases}$$

or, equivalently,

$$\Psi(x) = \alpha_j x + \beta_j, \quad \forall x \in [q_j, q_{j+1}]$$

where

$$\alpha_j = p_j$$

$$\beta_j = p_0 + \sum_{\nu=0}^{j} (p_{\nu-1} - p_\nu) q_\nu$$

with $p_{-1} = 0$ and $j = 0, \ldots, K-1$.

The problem can be seen in a finite dimensional setting through the use of an isometry. To achieve this, we consider the piecewise quadratic functions $f : [0, Q] \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} f_0(x) = a_0 x^2 + b_0 x + c_0 & \text{if } x \in [q_0, q_1] \\ f_1(x) = a_1 x^2 + b_1 x + c_1 & \text{if } x \in [q_1, q_2] \\ f_2(x) = a_2 x^2 + b_2 x + c_2 & \text{if } x \in [q_2, q_3] \\ \quad \vdots & \\ f_{K-1}(x) = a_{K-1} x^2 + b_{K-1} x + c_{K-1} & \text{if } x \in [q_{K-1}, q_K] \end{cases} \tag{A.1}$$

and denote by $\mathcal{P} \subseteq L^2([0, Q])$ the following set

$$\mathcal{P} = \left\{ f : [0, Q] \to \mathbb{R} : f \text{ is defined as in } (A.1) \text{ with } a, b, c \in \mathbb{R}^K \right\} \tag{A.2}$$

where $a = (a_0, \ldots, a_{K-1})$, $b = (b_0, \ldots, b_{K-1})$ and $c = (c_0, \ldots, c_{K-1})$. Clearly, C^Q and C^L are subsets of $\mathcal{P}$.

There exists an isomorphism $T : \mathcal{P} \to \mathbb{R}^{3K}$ defined as

$$T(f) = (a_0, b_0, c_0; a_1, b_1, c_1; \ldots; a_{K-1}, b_{K-1}, c_{K-1})$$

where the parameters (a_j, b_j, c_j) characterize the polynomial of the function f in the j-th interval. The isomorphism T enables us to view the elements of C^L and C^Q as vectors of $\mathbb{R}^{3K}$ in the following way

$$\begin{array}{lclcl} T(C^L) & = & \overline{C}^L & = & \{\mathbf{v} \in \mathbb{R}^{3K} : \mathbf{v} = (0, \alpha_0, \beta_0; 0, \alpha_1, \beta_1; \ldots; 0, \alpha_{K-1}, \beta_{K-1})\} \\ T(C^Q) & = & \overline{C}^Q & = & \{\mathbf{w} \in \mathbb{R}^{3K} : \mathbf{w} = (a, b, c; a, b, c; \ldots; a, b, c)\} \end{array}$$

Thanks to this, for each $\Psi \in C^L$, we have that

$$K(\Psi) = T^{-1}(\overline{K}(T(\Psi)))$$

with $\overline{K} : \overline{C}^L \rightrightarrows \overline{C}^Q$ defined as

$$\overline{K}(\mathbf{v}) = \{\mathbf{w} \in \overline{C}^Q \;:\; a > 0,\; b \geq 0,\; c \geq \beta_0\}$$

Moreover, if we consider on $\mathbb{R}^{3K}$ the distance $d^\bullet$ induced by the distance in L^2, i.e.

$$d^\bullet(T(f), T(g)) = \|f - g\|_{L^2}$$

we have that T is an isometry and, for each $y \in C^Q$, we have

$$p_{C^L}(y) = T^{-1}(p_{\overline{C}^L}(T(y)))$$

with $p_{\overline{C}^L} : \overline{C}^Q \to \overline{C}^L$ the projection of $\mathbf{w} \in \overline{C}^Q$ onto $\overline{C}^L$ defined through the distance $d^\bullet$. This enables us to view the problem in a finite dimensional setting.

In addiction, we assume that, for each producer i, the function of the real cost of production is defined as

$$\gamma(x) = Ax^2 + Bx$$

with $A > 0$ and $B \geq 0$, and the bid function $y \in C^L$ is such that:

- $a = A$, which means that the bid function y is forced to be "relatively closed" to the function of the real cost of production γ;
- $b \in \left[\underline{b}, \overline{b}\right]$ with $0 < \underline{b} < \overline{b}$;
- $c = \Psi(0) = p_0$.

The final bilevel problem, using all of these assumptions and the isometry T, is as follows

$$(P_i) \quad \begin{cases} \max\limits_{b_i, p_{i,0}} \left[(b_i - B_i)x_i + p_{i,0}\right] \\ b_i \in [\underline{b}_i, \overline{b}_i] \\ p_{i,0} \in [\underline{p}_i, \overline{p}_i] \\ x = (x_1, \ldots, x_N) \text{ solves } (ISO) \end{cases}$$

$$(ISO) \quad \begin{cases} \min\limits_{x_1,\ldots,x_N} \left[(A_1 x_1^2 + b_1 x_1 + p_{1,0}) + \ldots + (A_N x_N^2 + b_N x_N + p_{N,0})\right] \\ x_i \in [0, Q_i], \qquad \forall i = 1, \ldots, N \\ x_1 + \ldots + x_N = D \end{cases}$$

with the additional requirement $T(\Psi_i) = p_{\overline{C_i}^L}(T(y_i))$ for each i.

As mentioned in the introduction of this chapter, this problem presents three main computational challenges: bilevel structure, presence of projection, and the need to solve N problems simultaneously. Regarding the presence of the lower level, the problem (ISO) is strongly convex and its unique solution can be characterized using KKT conditions. Thus, the constraint "x solves (ISO)" in the upper level problem can be replaced with

$$\begin{cases} 2A_\nu x_\nu + b_\nu + \lambda_\nu^+ - \lambda_\nu^- + \mu = 0 & \nu = 1, \dots, N \\ \lambda_\nu^+(Q_\nu - x_\nu) = 0 & \nu = 1, \dots, N \\ \lambda_\nu^- x_\nu = 0 & \nu = 1, \dots, N \\ \lambda_\nu^\pm \geq 0 & \nu = 1, \dots, N \\ x_\nu \in [0, Q_\nu] & \nu = 1, \dots, N \\ x_1 + \ldots + x_N = D & \end{cases}$$

While, regarding the projection, the situation is slightly more complicated. Let us consider a single producer, omitting the index i. In addition to calculating $T(\Psi) = p_{\overline{C}^L}(T(y))$, we need to require that $y \in K(\Psi)$, i.e., $\Psi(0) = y(0) = p_0$. Let us focus on determining the projection, which is equivalent to solving the following convex optimization problem

$$\min \left\{ \frac{1}{2} d^\bullet(T(y), T(\Psi')) : \; T(\Psi') \in \overline{C}^L \right\}$$

where

$$\begin{aligned} d^\bullet(T(y), T(\Psi')) &= \|y - \Psi'\|_{L^2}^2 \\ &= \frac{1}{2} \sum_{\nu=0}^{K-1} \int_{q_\nu}^{q_{\nu+1}} [At^2 + (b - p'_\nu)t + p_0 - p'_0 - \sum_{\tau=0}^{\nu} (p'_{\tau-1} - p'_\tau) q_\tau]^2 dt \\ &= A(p') + B(p') \end{aligned}$$

with

$$\begin{aligned} A(p') &= \frac{1}{2} \int_{q_0}^{q_1} [At^2 + (b - p'_0)t + p_0 - p'_0]^2 dt \\ B(p') &= \frac{1}{2} \sum_{\nu=1}^{K-1} \int_{q_\nu}^{q_{\nu+1}} [At^2 + (b - p'_\nu)t + p_0 - p'_0 - \sum_{\tau=0}^{\nu} (p'_{\tau-1} - p'_\tau) q_\tau]^2 dt \end{aligned}$$

The problem's variable results $p' = (p'_0, \dots, p'_{K-1}) \in \mathbb{R}^K$ constrained to the condition

$$0 < \underline{p} \leq p'_0 \leq \cdots \leq p'_{K-1} \leq \overline{p}$$

and the projection problem becomes

$$\min \left\{ A(p') + B(p') \ : \ 0 < \underline{p} \leq p'_0 \leq \cdots \leq p'_{K-1} \leq \overline{p} \right\}$$

which can be characterized using the KKT conditions, again. To calculate the relative KKT conditions, we need to find the partial derivatives of A and B. For this purpose, we will use the Leibniz integral rule. First, we denote by

$$f_\nu(t, p') = At^2 + (b - p'_\nu)t + p_0 - p'_0 - \sum_{\tau=0}^{\nu} (p'_{\tau-1} - p'_\tau) q_\tau$$

so, the partial derivatives of the function A are

$$D_j A(p') = \begin{cases} -\displaystyle\int_{q_0}^{q_1} (1+t) f_0(t, p') dt & \text{if } j = 0 \\ 0 & \text{if } j = 1, \dots, K-1 \end{cases}$$

while those of the function B result

$$D_j B(p') = \begin{cases} -(1+q_1) \displaystyle\sum_{\nu=1}^{K-1} \int_{q_\nu}^{q_{\nu+1}} f_\nu(t, p') dt & \text{if } j = 0 \\ (q_j - q_{j+1}) \displaystyle\sum_{\nu=j+1}^{K-1} \int_{q_\nu}^{q_{\nu+1}} f_\nu(t, p') dt + \displaystyle\int_{q_j}^{q_{j+1}} (q_j - t) f_j(t, p') dt & \text{if } j = 1, \dots, K-2 \\ \displaystyle\int_{q_{K-1}}^{q_K} (q_{K-1} - t) f_{K-1}(t, p') dt & \text{if } j = K-1 \end{cases}$$

Now we can write the KKT conditions related to the minimum problem characterizing the projection, which are as follows

$$\begin{cases} D_jA(p') + D_jB(p') - \lambda'_j + \lambda'_{j+1} = 0 & j = 0, \ldots, K-1 \\ \lambda'_0(\underline{p} - p'_0) = 0 & \\ \lambda'_j(p'_{j-1} - p'_j) = 0 & j = 1, \ldots, K-1 \\ \lambda'_K(p'_{K-1} - \overline{p}) = 0 & \\ \lambda'_j \geq 0 & j = 0, \ldots, K \\ \underline{p} \leq p'_0 \leq p'_1 \leq p'_2 \leq \cdots \leq p'_{K-1} \leq \overline{p} & \end{cases}$$

If both of the KKT conditions described above are used, the lower level will "disappear" and we will get a single level game. In particular, the problem (P_i) becomes

$$\begin{cases} \max_{b_i, p_{i,0}} [(b_i - B_i)x_i + p_{i,0}] & \\ & \\ b_i \in [\underline{b_i}, \overline{b_i}] & \\ & \\ 2A_\nu x_\nu + b_\nu + \lambda_\nu^+ - \lambda_\nu^- + \mu = 0 & \nu = 1, \dots, N \\ \lambda_\nu^+ (Q_\nu - x_\nu) = 0 & \nu = 1, \dots, N \\ \lambda_\nu^- x_\nu = 0 & \nu = 1, \dots, N \\ \lambda_\nu^\pm \geq 0 & \nu = 1, \dots, N \\ x_\nu \in [0, Q_\nu] & \nu = 1, \dots, N \\ x_1 + \ldots + x_N = D & \\ & \\ D_j A(p'_i) + D_j B(p'_i) - \lambda'_{i,j} + \lambda'_{i,j+1} = 0 & j = 0, \dots, K_i - 1 \\ \lambda'_{i,0}(\underline{p}_i - p'_{i,0}) = 0 & \\ \lambda'_{i,j}(p'_{i,j-1} - p'_{i,j}) = 0 & j = 1, \dots, K_i - 1 \\ \lambda'_{i,K_i}(p'_{i,K_i - 1} - \overline{p}_i) = 0 & \\ \lambda'_{i,j} \geq 0 & j = 0, \dots, K_i \\ \underline{p}_i \leq p'_{i,0} \leq p'_{i,1} \leq p'_{i,2} \leq \dots \leq p'_{i,K_i-1} \leq \overline{p}_i & \\ p'_{i,0} = p_{i,0} & \end{cases}$$

Finally, in regards to the need to solve multiple problems simultaneously, our game turns out to be a weighted potential game. Indeed, for each parameter $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N_{++}$, the function $P : \prod_{\nu=1}^N \left[\underline{b_\nu}, \overline{b_\nu}\right] \times \left[\underline{p_\nu}, \overline{p_\nu}\right] \to \mathbb{R}$ defined as

$$P(b_1, p_{1,0}, \dots, b_N, p_{N,0}) = \sum_{\nu=1}^N \alpha_\nu [(b_\nu - B_\nu)x_\nu + p_{\nu,0}]$$

verifies the condition (2.4) and thus it is a weighted potential function. Therefore, by using the properties of potential games, it is possible to solve a single optimization problem instead of N simultaneous problems. In particular, any possible solution of the following optimization problem (P_α)

$$
\begin{cases}
\displaystyle\max_{b_1,p_{1,0},\dots,b_N,p_{N,0}} \sum_{\nu=1}^{N} \alpha_\nu[(b_\nu - B_\nu)x_\nu + p_{\nu,0}] & \\
\\
b_\nu \in [\underline{b_\nu}, \overline{b_\nu}] & \nu = 1,\dots,N \\
\\
2A_\nu x_\nu + b_\nu + \lambda_\nu^+ - \lambda_\nu^- + \mu = 0 & \nu = 1,\dots,N \\
\lambda_\nu^+(Q_\nu - x_\nu) = 0 & \nu = 1,\dots,N \\
\lambda_\nu^- x_\nu = 0 & \nu = 1,\dots,N \\
\lambda_\nu^\pm \geq 0 & \nu = 1,\dots,N \\
x_\nu \in [0, Q_\nu] & \nu = 1,\dots,N \\
x_1 + \ldots + x_N = D & \\
\\
D_j A(p'_\nu) + D_j B(p'_\nu) - \lambda'_{\nu,j} + \lambda'_{\nu,j+1} = 0 & \nu = 1,\dots,N \quad j = 0,\dots,K_\nu - 1 \\
\lambda'_{\nu,0}(\underline{p}_\nu - p'_{\nu,0}) = 0 & \nu = 1,\dots,N \\
\lambda'_{\nu,j}(p'_{\nu,j-1} - p'_{\nu,j}) = 0 & \nu = 1,\dots,N \quad j = 0,\dots,K_\nu - 1 \\
\lambda'_{\nu,K_\nu}(p'_{\nu,K_\nu-1} - \overline{p}_\nu) = 0 & \nu = 1,\dots,N \\
\lambda'_{\nu,j} \geq 0 & \nu = 1,\dots,N \quad j = 0,\dots,K_\nu - 1 \\
\underline{p}_\nu \leq p'_{\nu,0} \leq p'_{\nu,1} \leq p'_{\nu,2} \leq \cdots \leq p'_{\nu,K_\nu-1} \leq \overline{p}_\nu & \nu = 1,\dots,N \\
\\
p'_{\nu,0} = p_{\nu,0} & \nu = 1,\dots,N
\end{cases}
$$

allows us to find a projected solution of the game (Proposition 2.4.3). In particular, the functions Ψ_i are identified by the parameters $p'_{i,0},\dots,p'_{i,K_i-1}$.

Our idea is to develop an algorithm that allows us to find a projected solution of the electricity market model using the problem (P_α). Unfortunately, the complementarity conditions paired with the KKT conditions needed to manage the binding constraints make the problem difficult from a computational point of view. In this light, various approaches have been proposed in the literature to manage them (dummy binary variables, big-M, outer approximations, branch-and-bounds, and so on). Our idea is to use a transversality condition, recently introduced by Cambini and Riccardi in [17] that can be substituted for the complementarity conditions while still guaranteeing the optimality conditions based on those of KKT. The transversality condition enables efficient solving of a specific class of Max-Min problems, among which our problem is included. Indeed, as noted by the authors in [17], using the transversality condition guarantees us stable behaviour as the number of variables increases and significantly reduces the algorithm's execution time.

Bibliography

[1] C.D. Aliprantis and K.C. Border. *Infinite dimensional analysis*. Berlin: Springer-Verlag, 2006.

[2] J.P. Aubin. *Optima and equilibria*. Vol. 140. Springer-Verlag, Berlin, 1993, pp. xvi+417. DOI: 10.1007/978-3-662-02959-6.

[3] D. Aussel, J. Cotrina, and A. Iusem. “An existence result for quasi-equilibrium problems”. In: *J. Convex Anal.* 24 (2017), pp. 55–66.

[4] D. Aussel, A. Sultana, and V. Vetrivel. “On the Existence of Projected Solutions of Quasi-Variational Inequalities and Generalized Nash Equilibrium Problems”. In: *J. Optim. Theory Appl.* 170 (2016), pp. 818–837. DOI: 10.1007/s10957-016-0951-9.

[5] A. Bensoussan, M. Goursat, and J.L. Lions. “Contrôle impulsionnel et inéquations quasivariationnelles stationnaires”. In: *C. R. Acad. Sci. Paris Sér* 276 (1973), A1279–A1284.

[6] A. Bensoussan and J.L. Lions. “Contrôle impulsionnel et inéquations quasivariationnelles d’evolution”. In: *C. R. Acad. Sci. Paris Sér* 276 (1973), pp. 1333–1338.

[7] C. Berge. *Topological spaces*. Paris: English translation by E. M. Patterson of Espaces topologiques et fonctions multivoques, published by Dunod, 1963.

[8] M. Bianchi, E. Miglierina, and M. Ramazannejad. “On Projected Solutions for Quasi Equilibrium Problems with Non-self Constraint Map”. In: (2023). arXiv: 2303.08608 [math.OC].

[9] G. Bigi and M. Passacantando. “Gap functions for quasi-equilibria”. In: *J. Global Optim.* 66 (2016), pp. 791–810. DOI: 10.1007/s10898-016-0458-9.

[10] G. Bigi et al. *Nonlinear programming techniques for equilibria*. Cham: Springer, 2019.

[11] E. Blum and W. Oettli. “From optimization and variational inequalities to equilibrium problems”. In: *Math. Stud.* 63 (1993), pp. 1–23.

[12] J.F. Bonnans and A. Shapiro. *Perturbation analysis of optimization problems*. Springer Series in Operations Research. Springer-Verlag, New York, 2000, pp. xviii+601. DOI: 10.1007/978-1-4612-1394-9.

[13] K. Border. *Fixed Point Theorems with Applications to Economics and Game Theory*. Cambridge: Cambridge University Press, 1985.

[14] F. E. Browder. “The Fixed Point Theory of Multi-valued Mappings in Topological Vector Spaces”. In: *Mathematische Annalen* 177 (1968), pp. 283–301.

[15] O. Bueno and J. Cotrina. “Existence of projected solutions for generalized Nash equilibrium problems”. In: *J. Optim. Theory Appl.* 191 (2021), pp. 344–362. DOI: 10.1007/s10957-021-01941-9.

[16] C. Calderón and J. Cotrina. “Remarks on projected solutions for generalized Nash games”. In: (2023). DOI: 110.48550/arXiv.2307.13171. arXiv: 2307.13171.

[17] R. Cambini and R. Riccardi. *Optimality conditions for differentiable linearly constrained pseudoconvex programs*. DOI: 10.1007/s10203-024-00454-09.

[18] M. Castellani and M. Giuli. “An existence result for quasiequilibrium problems in separable Banach spaces”. In: *J. Math. Anal. Appl.* 425 (2015), pp. 85–95. DOI: 10.1016/j.jmaa.2014.12.022.

[19] M. Castellani and M. Giuli. “Ekeland’s principle for cyclically antimonotone equilibrium problems”. In: *Nonlinear Anal. Real World Appl.* 32 (2016), pp. 213–228. DOI: 10.1016/j.nonrwa.2016.04.011.

[20] M. Castellani, M. Giuli, and S. Latini. "Projected solutions for finite-dimensional quasiequilibrium problems". In: *Comput. Manag. Sci.* 20 (2023), p. 9. DOI: 10.1007/s10287-023-00444-4.

[21] M. Castellani, M. Giuli, and M. Pappalardo. "A Ky Fan minimax inequality for quasiequilibria on finite-dimensional spaces". In: *J. Optim. Theory Appl.* 179 (2018), pp. 53–64. DOI: 10.1007/s10957-018-1319-0.

[22] M. Castellani et al. "Projected solutions of generalized quasivariational problems in Banach spaces". In: *Nonlinear Anal. Real World Appl.* 76 (2024), Paper No. 104021, 12. DOI: 10.1016/j.nonrwa.2023.104021.

[23] D. Chan and J.S. Pang. "The generalized Quasi variational inequality problem". In: *Math. Oper. Res.* 7 (1982), pp. 211–222. DOI: 10.1287/moor.7.2.211.

[24] F. H. Clarke. *Optimization and nonsmooth analysis.* Vol. 5. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1990, pp. xii+308. DOI: 10.1137/1.9781611971309.

[25] F. H. Clarke et al. *Nonsmooth analysis and control theory.* Vol. 178. Graduate Texts in Mathematics. Springer-Verlag, New York, 1998.

[26] G. Cohen. "Auxiliary problem principle extended to variational inequalities". In: *J. Optim. Theory Appl.* 59 (1988), pp. 325–333. DOI: 10.1007/BF00938316.

[27] J. Cotrina and J. Zúñiga. "Quasi-equilibrium problems with non-self constraint map". In: *J. Global Optim.* 75 (2019), pp. 177–197. DOI: 10.1007/s10898-019-00762-5.

[28] P. Cubiotti. "Existence of Nash equilibria for generalized games without upper semicontinuity". In: *Internat. J. Game Theory* 26 (1997), pp. 267–273. DOI: 10.1007/BF01295855.

[29] P. Cubiotti. "Existence of solutions for lower semicontinuous quasi-equilibrium problems". In: *Comput. Math. Appl.* 30 (1995), pp. 11–22. DOI: 10.1016/0898-1221(95)00171-T.

[30] G. Debreu. "A social equilibrium existence theorem". In: *Proc. Nat. Acad. Sci. U.S.A.* 38 (1952), pp. 886–893. DOI: 10.1073/pnas.38.10.886.

[31] F. Deutsch. *Best approximation in inner product spaces.* New York: Springer-Verlag, 2001.

[32] F. Facchinei and C. Kanzow. "Generalized Nash equilibrium problems". In: *4OR* 5 (2007), pp. 173–210. DOI: 10.1007/s10288-007-0054-4.

[33] F. Facchinei and J. S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems.* New York: Springer, 2003.

[34] F. Facchinei, V. Piccialli, and M. Sciandrone. "Decomposition algorithms for generalized potential games". In: *Comput. Optim. Appl.* 50 (2011), pp. 237–262. DOI: 10.1007/s10589-010-9331-9.

[35] K. Fan. "A minimax inequality and applications". In: *Inequalities III.* Ed. by O. Shisha. New York: Academic Press, 1972, pp. 103–113.

[36] M. Fukushima. "Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems". In: *Math. Programming* 53 (1992), pp. 99–110. DOI: 10.1007/BF01585696.

[37] W.W. Hogan. "Directional derivatives for extremal-value functions with applications to the completely convex case". In: *Operations Res.* 21 (1973), pp. 188–209. URL: https://doi.org/10.1287/opre.21.1.188.

[38] W.W. Hogan. "Point-to-Set maps in mathematical programming". In: *SIAM Review* 15 (1973), pp. 591–603. URL: http://www.jstor.org/stable/2028579.

[39] S. Kakutani. "A Generalization of Brouwer's Fixed Point Theorem". In: *Duke Math. J.* 8 (1941), pp. 416–427.

[40] G. Mastroeni. "Gap functions for equilibrium problems". In: *J. Global Optim.* 27 (2003), pp. 411–426. DOI: 10.1023/A:1026050425030.

[41] E. Michael. "Continuous selections. I." In: *Ann. of Math.* 63 (1956), pp. 361–382. DOI: 10.2307/1969615.

[42] D. Monderer and L.S. Shapley. "Potential games". In: *Games Econom. Behav.* 14 (1996), pp. 124–143. DOI: `10.1006/game.1996.0044`.

[43] U. Mosco. "Implicit variational problems and quasi variational inequalities. Nonlinear operators and the calculus of variations". In: *Lecture Notes in Math.* Ed. by Dold A. and Eckmann B. Vol. 543. Berlin: Springer, 1976, pp. 103–113.

[44] J. Nash. "Equilibrium points in n-person games". In: *Proc. Nat. Acad. Sci. U.S.A.* 36 (1950), pp. 48–49. DOI: `10.1073/pnas.36.1.48`. URL: `https://doi.org/10.1073/pnas.36.1.48`.

[45] J. Nash. "Non-cooperative games". In: *Ann. of Math. (2)* 54 (1951), pp. 286–295. DOI: `10.2307/1969529`. URL: `https://doi.org/10.2307/1969529`.

[46] H. Nikaido and K. Isoda. "Note on non-cooperative convex games". In: *Pacific J. Math.* 5 (1955), pp. 807–815.

[47] K. Okuguchi. *Expectations and stability in oligopoly models.* Vol. 138. Lecture Notes in Economics and Mathematical Systems. Springer-Verlag, Berlin-New York, 1976.

[48] R. Tyrrell Rockafellar. *Convex analysis.* Vol. No. 28. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1970.

[49] R.W. Rosenthal. "A class of games possessing pure-strategy Nash equilibria". In: *Internat. J. Game Theory* 2 (1973), pp. 65–67. DOI: `10.1007/BF01737559`.

[50] S. Sagratella. "Algorithms for generalized potential games with mixed-integer variables". In: *Comput. Optim. Appl.* 68 (2017), pp. 689–717. DOI: `10.1007/s10589-017-9927-4`.

[51] N.X. Tan. "Quasi-variational inequality in topological linear locally convex Hausdorff spaces". In: *Math. Nachr.* 122 (1985), pp. 231–245. DOI: `10.1002/mana.19851220123`.

[52] D. L. Zhu and P. Marcotte. "An extended descent framework for variational inequalities". In: *J. Optim. Theory Appl.* 80 (1994). DOI: `10.1007/BF02192941`.