Classical and Quantum Dynamics of Gyroscopic Systems and Dark Energy

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Abstract

Gyroscopic systems in classical and quantum field theory are characterized by the presence of at least two scalar degrees of freedom and by terms that mix fields and their time derivatives in the quadratic Lagrangian. In Minkowski spacetime, they naturally appear in the presence of a coupling among fields with time-dependent vacuum expectation values and fields with space-dependent vacuum expectation values, breaking spontaneously Lorentz symmetry; this is the case for a supersolid. In a cosmological background a gyroscopic system can also arise from the time dependence of non-diagonal kinetic and mass matrices. We study the classical and quantum dynamics computing the correlation functions on the vacuum state that minimizes the energy. Two regions of stability in parameter space are found: in one region, dubbed *normal*, the Hamiltonian is positive defined, while in the second region, dubbed *anomalous*, it has no definite sign. Interestingly, in the anomalous region the 2-point correlation function exhibits a resonant behaviour in a certain region of parameter space. We show that as dynamical a dark energy (with an exact equation of state $w = -1$) arises naturally as a gyroscopic system.

1 Introduction

The study of quadratic Lagrangian systems is the starting point for the analysis of classical/quantum solvable systems. Gyroscopic systems are a particular class of Lagrangians/Hamiltonians, characterized by rather surprising features even at linear level dynamics. The above systems [1] (defined on a Minkowski background) are characterized by N-degrees of freedom with $N > 1$ and their dynamics is described by the following Lagrangian and corresponding equations of motion

$$\mathcal{L} = \frac{1}{2}\dot{\varphi}^t\,\mathcal{K}\,\dot{\varphi} + \varphi^t\,\mathcal{D}\,\dot{\varphi} - \frac{1}{2}\,\varphi^t\,\mathcal{M}\,\varphi \quad \to \quad \mathcal{K}\,\ddot{\varphi} - 2\,\mathcal{D}\,\dot{\varphi} + \mathcal{M}\,\varphi = 0\,, \qquad \varphi^t = (\varphi_1, \cdots, \varphi_N)\,; \quad (1.1)$$

where $\mathcal{K}$, $\mathcal{D}$ and $\mathcal{M}$ are $N \times N$ constant matrices with the following properties: $\mathcal{K}$ is symmetric and positive defined, $\mathcal{D}$ is antisymmetric while $\mathcal{M}$ is just symmetric.
The defining property is the presence of non-dissipative velocity-dependent forces, described, in the classical mechanics picture, by the antisymmetric matrix $\mathcal{D} \neq 0$. Gyroscopic systems can be found in many area of physics and engineering[1], for a fascinating review see [2]. Such systems have a number of unusual features.

- Stability is realized in a peculiar way. Beyond a normal stability region where the mass matrix $\mathcal{M}$ is positive defined (we call such a parameter space region, the *normal stability region*), it exists a stability region also for negative defined mass matrix $\mathcal{M}$ and the matrix $\mathcal{D}$ results larger that a critical value $\mathcal{D}_c$ ($\mathcal{D} \geq \mathcal{D}_c$); we call such a parameter space region, the *gyroscopic or anomalous region*.
- Despite of the fact the Hamiltonian is time-independent and then system is conservative, time reversal symmetry is violated by the presence of single time derivative operator $\varphi\,\dot{\varphi}$ proportional to the $\mathcal{D}$ matrix [3] in the Lagrangian.
- On a Minkowski background, as a consequence of the spontaneous breaking of Lorentz invariance down to the rotational group, the elementary excitations can be interpreted as phonon-like modes of a supersolid; namely a solid coupled with a superfluid. Symmetry arguments imply the term $\mathcal{D}$ in the quadratic Lagrangian exists only when scalar fields with non-trivial vacuum configurations are coupled together.
- The quantisation of the system with a Fock representation of the canonical commutation relation is feasible only after a suitable diagonalisation of the Hamiltonian which can be written as a set of decoupled harmonic oscillators. In the Lagrangian approach such a decoupling cannot be achieved. As a general result, while in the *normal stability region* the Hamiltonian is positive defined, in the *gyroscopic (or anomalous) region* the Hamiltonian results negative defined showing an intriguing connection of a gyroscopic system with the Pais-Uhlenbeck oscillator. Interestingly, the long standing problem of the stability for interaction Pais-Uhlenbeck oscillator was recently reconsidered [4] and the resonant behavior of the 2-point correlation that we found in the anomalous region plays an important role.

On a generic time-dependent background the matrices $\mathcal{K}$, $\mathcal{D}$ and $\mathcal{M}$ are naturally time-dependent and the definition of a gyroscopic system is ambiguous due to the possibility of performing time- dependent field transformations. Focusing on the case of two scalar degrees of freedom, which represent the minimal field content for a gyroscopic system[2], we show that it possible by a suitable set of time-dependent Lagrangian field transformation to bring the system in a *canonical* form where the kinetic matrix $\mathcal{K}$ is the identity, the mass matrix is diagonal and $\mathcal{D}_c = d_c\,\epsilon_{ab}$ where $\epsilon_{ab} = -\epsilon_{ba},\ a, b = 1, 2$ and $\epsilon_{12} = 1$. In its canonical form, a system is unambiguously gyroscopic if $d_c \neq 0$ and it is characterized by three time dependent parameters: d_c and the diagonal entries of the mass matrix. In this framework we study when $d_c \neq 0$. Generally speaking, gyroscopic systems can manifest when some of the fields acquire a non-trivial background spacetime dependent "vacuum", a behavior present

[1]Dissipation can be described by adding to $\mathcal{D}$ a symmetric part.
[2]The result be generalised to the case a generic even number scalar fields.

in many multi-field systems, see for example the effective description of media [5, 6, 7, 8], massive gravity [9, 10, 11] single [12] and multi-field inflation [13], solid and supersolid inflation [14, 15, 16] and holography [17, 18].

2 Scalar Fields and Vacuum Configurations

We are interested in the study of systems characterized by a set of N scalar fields $\{\Phi^A,\ A=1,\cdots,N\}$ such that some of them acquire a non-trivial spacetime dependent vacuum expectation value (vev) that describes the background configuration of the system. Therefore the fields are split in a background configuration ϕ^A plus a fluctuation φ^A

$$\Phi^A = \phi^A + \varphi^A\,. \tag{2.1}$$

The fields $\{\varphi^A\}$ are associated to the classical/quantum small fluctuations. For instance, in the effective description of fluid dynamics, the background configuration of the fluid is described by ϕ^A while the phonon excitations are described by φ^A. It is useful to distinguish the fields according to the nature of their vev; namely

- fields with zero vev $\phi^A = 0$ and fluctuations $\varphi^A \equiv Z^A$;
- fields with a time-dependent vev, for example $\phi^A = c^A\,t$, and fluctuations $\varphi^A \equiv T^A$;
- fields with $\vec{x}$-dependent vev $\phi^A = c^A_j\,x^j$ and fluctuations $\varphi^A \equiv S^A$.

Our goal is to study the dynamics of φ^A. We shall consider the case where the underlying symmetry group of the background spacetime is partial broken by scalar fields configuration ϕ^A in such a way that spatial translations and rotations stay unbroken. Barring accidental cancellations, translational invariance requires that spatial derivatives $\partial_i\phi^A$ must be constant. In the case of fields with $\vec{x}$-dependent vev it implies that $\phi^A \equiv \phi^i_n = x^i$. In order to automatically implement such a constraint, we always require a shift symmetry for the fields with an $\vec{x}$-spatial vev, namely

$$\text{Spatial vev fields}: \quad \Phi^i_n \to \Phi^i_n + \text{constant} \qquad \text{if} \quad \phi^i_n = x^i\,, \qquad i=1,2,3 \tag{2.2}$$

where now n denote the number of different fields. In other words, being interested in systems where spatial rotations are always unbroken, in 3+1 dimensions the minimal number of $\vec{x}$-spatial vev fields is three. A triplet of scalar fields $\{\Phi^a,\ a=1,2,3\}$ is transforming as the fundamental representation of an internal $SO(3)_I$ symmetry group of the system. The vev induces the following spontaneous breaking pattern

$$SO(3)\times SO(3)_I \to SO(3)_D\,. \tag{2.3}$$

In general the total number of fields with an $\vec{x}$-spatial vev consists of n triplet of $SO(3)_I$. Then for each fluctuation we can use the Helmholtz decomposition

$$\varphi^A \equiv \varphi^i_n = S^i_n \equiv \frac{\partial_i}{\sqrt{\vec{\nabla}^2}}\,S_n + V^i_n, \qquad \partial_i V^i_n = 0, \qquad i=1,2,3 \tag{2.4}$$

to extract the scalar S_n and the vector V^i_n components. Being interested in the scalar sector, only S_n will be relevant for us.

For the fields with a time dependent vev, a shift symmetry is not strictly necessary, in particular when also the background is breaking time diff as in FRW. On the contrary when we work in a Minkowski spacetime and as soon as we require an EFT with thermodynamical properties [7], also the temporal vev fields have to be shift symmetric.

The dynamics of the fluctuations can be found by studying the structure of all operators (with up to two derivatives) consistent with rotations. At the quadratic level we have the following classification scheme:

- Operators with no derivatives:

$$\mathcal{O}^{AB}_0 = \varphi^A\,\varphi^B\,. \tag{2.5}$$

- Operators with one derivative:

$$\mathcal{O}_x^{AB} = \partial_i\varphi^A\,\varphi^B\,, \qquad \mathcal{O}_t^{AB} = \varphi^A\,\dot\varphi^B\,. \tag{2.6}$$

- Operators with two derivatives:

$$\mathcal{O}_{xx}^{AB} = \partial_i\varphi^A\,\partial_j\varphi^B\,, \qquad \mathcal{O}_{xt}^{AB} = \partial_i\varphi^A\,\dot\varphi^B\,, \qquad \mathcal{O}_{tt}^{AB} = \dot\varphi^A\dot\varphi^B\,. \tag{2.7}$$

To produce a rotational invariant quadratic Lagrangian $\mathcal{L}$, the indices in $\{\mathcal{O}_n\}$ should be saturated by using the only two available invariant tensors δ_{ij} and ϵ_{ijk}. The result is

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_1 - \mathcal{L}_0\,; \tag{2.8}$$

where

$$\mathcal{L}_2 = \mathcal{K}\cdot\mathcal{O}_{tt}\,; \tag{2.9}$$

$$\mathcal{L}_1 = \mathcal{D}^{(t)}\cdot\mathcal{O}_t + \mathcal{D}^{(tx)}\cdot\mathcal{O}_{xt}\,; \tag{2.10}$$

$$\mathcal{L}_0 = \mathcal{M}^{(0)}\cdot\mathcal{O}_0 + \mathcal{M}^{(x)}\cdot\mathcal{O}_x + \mathcal{M}^{(xx)}\cdot\mathcal{O}_{xx}\,; \tag{2.11}$$

and $\cdot$ stands for rotational invariant contractions of the relevant indices. The term $\mathcal{L}_2$ has two time derivatives and represents the kinetic term while $\mathcal{L}_0$ with no time derivatives is a mass term. Beside a rather standard kinetic matrix $\mathcal{K}_{AB}$ and a mass matrix $\mathcal{M}_{AB}$, the peculiar term is the one linear in the time derivative of the fluctuations and proportional to $\mathcal{D}_{AB}$; the presence of such a term is the defining property of a gyroscopic system. Notice that the kinetic matrix $\mathcal{K}$ and the mass matrix $\mathcal{M}$ are symmetric by construction, while one can take $\mathcal{D} = -\mathcal{D}^t$ by adding/subtracting a total derivative term. Indeed, by splitting $\mathcal{D}$ as a symmetric $\mathcal{D}^{(S)}$ and an antisymmetric $\mathcal{D}^{(A)}$ part, the former can be cast into a mass term up to a total derivative

$$\mathcal{D}^{(S)}_{AB}\,\dot\varphi_A\,\varphi_B = \frac{1}{2}\mathcal{D}^{(S)}_{AB}\,\frac{d}{dt}\left(\varphi_A\varphi_B\right) = \frac{1}{2}\frac{d}{dt}\left(\mathcal{D}^{(S)}_{AB}\,\varphi_A\,\varphi_B\right) - \frac{1}{2}\dot{\mathcal{D}}^{(S)}_{AB}\,\varphi_A\,\varphi_B\,. \tag{2.12}$$

When the structure of the Lagrangian is further restricted by imposing a shift symmetry on *all* scalar fields, namely

$$\Phi^A \to \Phi^A + \text{constant}, \tag{2.13}$$

then all operators with a single or zero derivatives (temporal or spatial) are forbidden and the structure of the shift symmetric Lagrangian reduces to

$$\mathcal{L}^{\text{shift}} = \mathcal{K}\cdot\mathcal{O}_{tt} + \mathcal{D}^{(tx)}\cdot\mathcal{O}_{tx} - \mathcal{M}^{(xx)}\cdot\mathcal{O}_{xx}\,. \tag{2.14}$$

Notice that even if (2.13) is imposed, the presence of $\mathcal{D}^{(xt)}$ breaks time reversal symmetry. In table 1 we show the structure of the quadratic Lagrangian depending on the type of vevs and on the internal shift symmetries imposed. In the general case the Lagrangian $\mathcal{L}$ contains all the operators. In the Lorentz invariant (LI) case, when the very same symmetries of Minkowski space are imposed, the system cannot be gyroscopic.

The special cases with only t-dependent vev fields T^A (case $\mathcal{L}_T$), $\vec{x}$-dependent vev fields S_n (case $\mathcal{L}_S$) or both (case $\mathcal{L}_{TS}$) is presented. For the rest of the paper we will assume the presence of two scalar degrees of freedom ($N = 2$) so that all matrices will be 2×2. In Fourier space where

$$\varphi(t,\boldsymbol{x}) = \int \frac{d^3k}{(2\,\pi)^{3/2}}\;\varphi_{\boldsymbol{k}}(t)\;e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}}\,, \qquad\qquad \varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \tag{2.15}$$

the reality of the fields $\varphi(t,\boldsymbol{x}) = \varphi(t,\boldsymbol{x})^*$ imposes that $\varphi_{\boldsymbol{k}}(t) = \varphi_{-\boldsymbol{k}}(t)^*$ or $\varphi_{-\boldsymbol{k}}(t) = \varphi_{\boldsymbol{k}}(t)^*$. The Lagrangian (where only the time derivatives of the fields appear) takes the form (see 8 for more details)

$$\mathcal{L} = \frac{1}{2}\,\dot\varphi^\dagger_{\boldsymbol{k}}\,\mathcal{K}\,\dot\varphi_{\boldsymbol{k}} + \varphi^\dagger_{\boldsymbol{k}}\,\mathcal{D}\,\dot\varphi_{\boldsymbol{k}} - \frac{1}{2}\,\varphi^\dagger_{\boldsymbol{k}}\,\mathcal{M}\,\varphi_{\boldsymbol{k}}\,; \tag{2.16}$$

Table 1: *Structure of the quadratic Lagrangian. The suffix shift indicates the presence of the complete shift symmetry (2.13). $T(S)$ stands for field with only time-dependent (space-dependent) vev while TS underline that fields with both time-dependent and space-dependent vev are present. In the Lorentz invariant (LI) case the very same symmetries of Minkowski space are imposed.*

	$\mathcal{K}$	$\mathcal{D}^{(t)}$	$\mathcal{D}^{(tx)}$	$\mathcal{M}^{(0)}$	$\mathcal{M}^{(x)}$	$\mathcal{M}^{(xx)}$
$\mathcal{L}$	✓	✓	✓	✓	✓	✓
$\mathcal{L}^{\text{shift}}$	✓		✓			✓
LI	✓					✓
$\mathcal{L}_T$	✓	✓		✓		✓
$\mathcal{L}_T^{\text{shift}}$	✓					✓
$\mathcal{L}_S$	✓			✓		✓
$\mathcal{L}_S^{\text{shift}}$	✓					✓
$\mathcal{L}_{TS}$	✓		✓		✓	✓
$\mathcal{L}_{TS}^{\text{shift}}$	✓		✓			✓

where the $\mathcal{K}$, $\mathcal{D}$ and $\mathcal{M}$ matrices absorbed the $\boldsymbol{k}$ dependence of the spatial derivative of the fields. Further, the structure of the $\mathcal{D}$ matrix is fixed to be

$$\mathcal{D} = d\,\mathcal{J}, \qquad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \tag{2.17}$$

and d is a real parameter (where the integration by parts (2.12) has been used to have $\mathcal{D}$ antisymmetric). Few comments are in order. Spatial derivatives ∂_x in Fourier space are replaced by $i\,k$ and the "mass" matrix $\mathcal{M}$, kinetic mixing term $\mathcal{D}$ and the kinetic matrix $\mathcal{K}$ are both time and k-dependent. In the following we will study the k dependence of (2.16) in detail. To end up this chapter we see that a gyroscopic system with $\mathcal{D} \neq 0$ can be generated essentially

- when $\mathcal{D}^{(t)} \neq 0$: in this case we need interactions between T^A fields that are not protected by shift symmetry (typically present in FRW backgrounds and not in Minkowski space);
- when $\mathcal{D}^{(tx)} \neq 0$: in this case we need interactions between T^A and S_n fields and are present also for shift symmetry Lagrangians (i.e. in Minkowski background and for the description of the Goldstone modes of ideal thermodynamical system).

This definition of gyroscopic systems is then valid both in Minkowski space and in FRW backgrounds.

3 Canonical form of a Gyroscopic system

Taking into account the presence of non-trivial vacuum configuration, one finds that the most general structure for the quadratic Lagrangian is of the form (2.16). We stress again that to get the form (2.16) we have used only integration by parts without modifying the original equations of motion. Consider now a general linear Lagrangian field transformation $\varphi \to F(t)\,\varphi$, with F an invertible time-dependent matrix. Obviously, a time-dependent transformation will lead again to a Lagrangian of the form (2.16) but with different matrices $\tilde{\mathcal{K}}$, $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{M}}$; in particular such a transformation can induce an effective $\tilde{\mathcal{D}}$ that characterises a gyroscopic system, even if it was zero in the original field variables. Indeed, the effect of the above transformation on the equations of motion is the following

$$\varphi = F(t)\,Q \quad \Rightarrow \quad (\mathcal{K}\,F)\,\ddot{Q} - 2\,\underbrace{(\mathcal{D}\,F - \mathcal{K}\,\dot{F})}_{\tilde{\mathcal{D}}}\,\dot{Q} + (\mathcal{M}\,F + \mathcal{K}\,\ddot{F} + \mathcal{D}\,\dot{F})\,Q = 0\,. \tag{3.1}$$

When $\dot{F} \neq 0$ an effective $\tilde{\mathcal{D}}$ generically arises. As shown in appendix A, in order to remove such ambiguity one can always make a suitable linear Lagrangian field redefinition (time-dependent in general) to put the matrices entering (2.16) in the following *canonical form*

$$\mathcal{K} \to \boldsymbol{I}, \qquad \mathcal{D} \to D = d_c\, \mathcal{J}, \qquad \mathcal{M} \to M = \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}; \tag{3.2}$$

where $\boldsymbol{I}$ is the identity matrix. The argument can be generalised to the case of $N \geq 2$ degrees of freedom. Once the matrices in (2.16) are in their canonical form (3.2), we define a system "gyroscopic" if $D \neq 0$. A non-vanishing $D \neq 0$ can originate from the "canonisation" of (2.16); in particular a non-trivial d_c is generated by a time-dependent diagonalisation of the original kinetic and mass matrices $\mathcal{K}$, $\mathcal{M}$ with time-dependent rotation angles $\theta_{\mathcal{K}}$ and $\theta_{\mathcal{M}}$ respectively (see appendix A for details); namely

$$d_c = \frac{d}{\det(\mathcal{K})^{1/2}} - \frac{\mathrm{Tr}(\mathcal{K})}{\det(\mathcal{K})^{1/2}}\, \dot{\theta}_{\mathcal{K}} - 2\, \dot{\theta}_{\mathcal{M}}^2\, . \tag{3.3}$$

Thus, a system is gyroscopic when at least one of the following conditions are satisfied:

- the original $\mathcal{D} \neq 0$;
- a non-trivial time dependence of the kinetic matrix such that $\dot{\theta}_{\mathcal{K}} \neq 0$;
- a non-trivial time dependence of the mass matrix such that $\dot{\theta}_{\mathcal{M}} \neq 0$.

In a cosmological background, a typical situation where $\{\dot{\theta}_i\}$ are zero is when the matrices $\mathcal{K}$ or $\mathcal{M}$ are [3]: diagonal, there is an overall time dependence, the diagonal entries are equal, the difference between the two diagonal elements is proportional to the off-diagonal one. The cases where the matrices are time-independent (and thus $\dot{\theta}_i = 0$) include Minkowski space background and, more in general, a regime where momenta are much higher than the inverse of the curvature scale. This is the case for the definition of the Bunch-Davies vacuum in a de Sitter background (see chapter 9). Unless explicitly stated we will consider a gyroscopic system in the canonical form (3.2). One might think that after all the system (2.16) is rather simple. However, as pointed out humorously by Coleman [19], quantum field theory is based on different variations of the harmonic oscillator. The first evidence that (2.16) is less trivial than it looks is that, as shown in appendix B, there is no Lagrangian field redefinition to set $\mathcal{D} = 0$ and at the same time having a diagonal mass matrix; indeed, in a Minkowski background, setting $\mathcal{D} = 0$ through a further time dependent linear transformation of the canonical fields generates a mass term with periodic time dependence (Floquet system). While to find classical solutions is not a problem, the quantisation is not straightforward.

4 Stability of a time-independent Gyroscopic System

In this section we will focus on the simplest case where the canonical D and M matrices are *time independent*. As we already discussed, this will be the case when shift symmetry is imposed for all the fields and the metric is time-independent. The equations of motion are the following

$$\ddot{\varphi} - 2\, D\, \dot{\varphi} + M\, \varphi = 0\, . \tag{4.1}$$

Solutions of (4.1) are of the form $\varphi = e^{-i\,\omega\, t}\, v$ where v is suitable vector and ω satisfies the following algebraic equation given in terms of the linear operator $L(\omega)$

$$\det L\,(\omega) \equiv \det\left(-\omega^2\, \mathbf{I} + 2\, i\, \omega\, D + M\right) = 0 \quad \Rightarrow \quad \omega^4 - \omega^2\left(4\, d^2 + m_1^2 + m_2^2\right) + m_1^2\, m_2^2 = 0\, ; \tag{4.2}$$

thus

$$\omega_{1,2}^2 = \frac{1}{2}\left(4\, d^2 + m_1^2 + m_2^2 \pm \sqrt{\left(m_1^2 + m_2^2 + 4\, d^2\right)^2 - 4\, m_1^2\, m_2^2}\right). \tag{4.3}$$

[3] The mixing angle θ_λ for a generic 2×2 matrix λ, is given by $\tan(2\,\theta_\lambda) = \frac{2\,\lambda_{12}}{\lambda_{22} - \lambda_{11}}$ with its time derivative given by $\dot{\theta}_\lambda = \frac{(\lambda_{22} - \lambda_{11})\, \dot{\lambda}_{12} - \lambda_{12}\, (\dot{\lambda}_{22} - \dot{\lambda}_{11})}{4\, \lambda_{12}^2 + (\lambda_{22} - \lambda_{11})^2}$. It is easy to identify when $\dot{\theta}_\lambda = 0$.

Some simple general properties of ω_i can obtained by taking the transpose and then the complex conjugate of $L(\omega)$ and taking into account that $D^t = -D$. In particular we get $L(-\omega) = L(\omega)$ and $L(-\omega^*) = L(\omega)^*$. Thus, if ω is a solution, also ω^*, $-\omega$ and $-\omega^*$ are solutions. The system is stable only when ω is purely real. The region of stability can be described by using the $m_{1,2}$, d or equivalently $\omega_{1,2}$, d as independent parameters and it is given by

$$(a) \qquad m_{1,2}^2 \geq 0,\ \ d^2 > 0 \qquad \Longleftrightarrow \qquad 0 \leq d^2 \leq \frac{(\omega_1 - \omega_2)^2}{4}\,; \tag{4.4}$$

$$(b) \qquad m_{1,2}^2 \leq 0,\ \ d^2 \geq \frac{\left(\sqrt{-m_1^2} + \sqrt{-m_2^2}\,\right)^2}{4}, \quad \Longleftrightarrow \quad d^2 > \frac{(\omega_1 + \omega_2)^2}{4}\ . \tag{4.5}$$

One can also rescale ω_i by d by defining $\hat{\omega}_i \equiv \omega_i/d$, then the two stability regions corresponding to $\hat{\omega}_1 \geq \hat{\omega}_2 \geq 0$ can be rewritten as

$$(a) \quad \hat{\omega}_1 - \hat{\omega}_2 \geq 2 \tag{4.6}$$

$$(b) \quad \hat{\omega}_1 + \hat{\omega}_2 \leq 2\,. \tag{4.7}$$

Our notion of stability corresponds to what it is called marginal stability in [1]. The region of stability in parameter space is plotted in Figure 1. The intermediate range $\frac{(\omega_1-\omega_2)^2}{4} < d^2 < \frac{(\omega_1+\omega_2)^2}{4}$ is forbidden by stability. We note [4] that stability can be also achieved when the rather peculiar limit

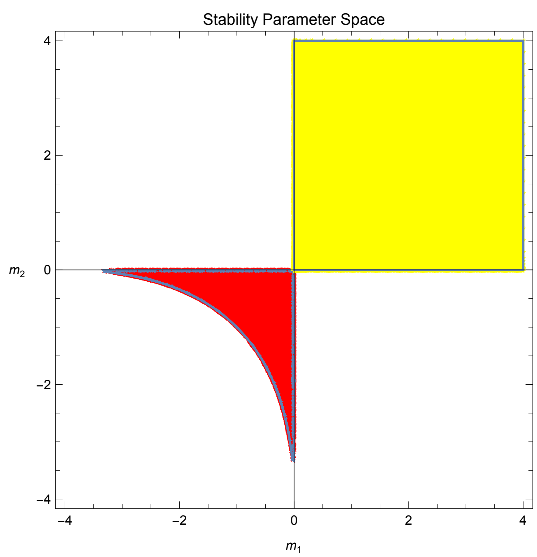

Figure 1: *For $d = 1$, in yellow the parameter space corresponding to (4.4), in red the region corresponding to (4.5).*

$\mathcal{K} \to 0$ in (2.16) is taken; namely the standard kinetic term which gives rise in the canonical form to $\ddot{\phi}$ is sub-leading when compared to the gyroscopic one. In such a limit, from (4.1) one find that $\omega^2 = m_1^2 m_2^2/(4d^2) > 0$. Of course by taking the limit $\mathcal{K} \to 0$ the number of degrees of freedom is changed and the equations of motion are not anymore second order differential equations; see [20] for an application to inflation.

It is interesting to interpret the stability conditions in Hamiltonian terms. The conjugate momenta are

$$\pi = \dot{\varphi} - D\,\varphi\,, \quad \pi = \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}\,. \tag{4.8}$$

[4]We thank the anonymous referee for pointing out such a limit.

Working in Fourier space, the Hamiltonian H can be written as

$$H = \int d^3k\, H_k \;=\; \int d^3k \left[\frac{1}{2}\left[\pi^\dagger_{\boldsymbol{k}}\,\pi_{\boldsymbol{k}} + \varphi^\dagger_{\boldsymbol{k}}\,(M - D^2)\,\varphi_{\boldsymbol{k}}\right] + \pi^\dagger_{\boldsymbol{k}}\,D\,\varphi_{\boldsymbol{k}}\right] = \tag{4.9}$$
$$\int d^3k\,\frac{1}{2}\left[(\pi_{\boldsymbol{k}} + D\,\varphi_{\boldsymbol{k}})^\dagger(\pi_{\boldsymbol{k}} + D\,\varphi_{\boldsymbol{k}}) + \varphi^\dagger_{\boldsymbol{k}}\,M\,\varphi_{\boldsymbol{k}}\right].$$

The system is stable in two disconnected regions in parameter space.
The first one, that we call *normal*, is given by (4.4) and the Hamiltonian H_k is positive defined as one can infer from (4.10). More surprising is the existence of a second region (4.5) (that we call *anomalous region*) where H_k is not positive defined, but the system is still stable. Thus the positivity of H_k is only a sufficient but not necessary condition for stability. Notice that the stability in the anomalous region is possible only if $D \neq 0$. We have focused on the case of two degrees of freedom, but more in general one can show that stability in the anomalous region can be achieved if and only if the number of degrees of freedom is even and by taking D non-singular and sufficiently large [21]; indeed the generalization of (4.3) leads to

$$4\,D^t\,D - \left(\sqrt{-M} + \sqrt{D^t\,M\,D^{-1}}\right)^2 > 0\,. \tag{4.10}$$

Another surprising property is that even if D and M are time-independent, taking $\varphi(t)$ as a solution of (4.1), then $\tilde{\varphi}(t) = \mathbb{T}\,\varphi(-t)$ is not a solution for any choice of a constant $2{\times}2$ matrix $\mathbb{T}$ unless $D = 0$. In other words in a gyroscopic system time reversal symmetry is broken even if the energy is conserved. The reason behind the lack of time reversal symmetry is the form of (4.8) that generates an Hamiltonian that is not an even function of the conjugate momenta π. For a recent critical discussion of time reversal symmetry see [3].
It is important to say some words about the effect of dissipation on a gyroscopic system where a lot of literature is present (see [2]). According to the Thomson-Tait-Chetayev theorem, for a gyroscopic system defined in the region 4.5 with a "large" D, stability is destroyed by the introduction of an arbitrarily small dissipative force.

5 Symplectic Classical Dynamics

As discussed in the previous section the impossibility to get rid of D at the Lagrangian level makes the Hamiltonian formalism the ideal tool both to study the classical dynamics and to quantize the system. Following the approach described in [22, 23], it is convenient to introduce a compact notation to denote a generic point in the phase space described by a 4-dimensional vector $z(t, \vec{x})$ or equivalently its Fourier transform $z_{\boldsymbol{k}}(t)$ defined by

$$z_{\boldsymbol{k}} = \begin{pmatrix}\varphi_{\boldsymbol{k}}\\ \pi_{\boldsymbol{k}}\end{pmatrix}, \qquad \varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \quad \pi = \begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix}. \tag{5.1}$$

The Hamiltonian can be written as (for quadratic systems see also [24], [25], [26])

$$H = \int d^3k\,\frac{1}{2}\,z^t_{-\boldsymbol{k}}\,\mathcal{H}_k\,z_{\boldsymbol{k}} = \int d^3k\,\frac{1}{2}\,z^\dagger_{\boldsymbol{k}}\,\mathcal{H}_k\,z_{\boldsymbol{k}} \tag{5.2}$$

The Hamiltonian density matrix in Fourier space can be read off from (2.16) and (4.10) in the canonical form as

$$\mathcal{H}_k = \begin{pmatrix} M - D^2 & D^t\\ D & \boldsymbol{I}\end{pmatrix} = \begin{pmatrix} m_1^2 + d^2 & 0 & 0 & -d\\ 0 & m_2^2 + d^2 & d & 0\\ 0 & d & 1 & 0\\ -d & 0 & 0 & 1\end{pmatrix} \tag{5.3}$$

with $k = |\boldsymbol{k}|$. The Poisson brackets among the basic variables can be written as

$$\{z_m(t,\boldsymbol{x}),\, z_n(t,\boldsymbol{y})\} = \Omega_{mn}\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \quad \Rightarrow \quad \{z_{\boldsymbol{k},m}(t),\, z^*_{\boldsymbol{q},n}(t)\} = \Omega_{mn}\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{q})\,, \tag{5.4}$$

where Ω is the following 4×4 antisymmetric matrix that encodes the symplectic structure

$$\Omega = \begin{pmatrix} 0 & \boldsymbol{I} \\ -\boldsymbol{I} & 0 \end{pmatrix} . \tag{5.5}$$

The Hamilton equations can be written in terms of the Poisson brackets as a set of linear first order differential equations

$$\dot{z}_{\boldsymbol{k}}(t) = \{z_{\boldsymbol{k}}(t),\ H\} = \Omega\,\mathcal{H}_k\, z_{\boldsymbol{k}}(t) \tag{5.6}$$

that are equivalent to (4.1). From now on for notation simplicity we will omit the suffix $\boldsymbol{k}$ in $z_{\boldsymbol{k}}$. We exploit the freedom in the choice of canonical variables to find a symplectic transformation[5] that diagonalises the Hamiltonian $\mathcal{H}_k$

$$S^t\,\mathcal{H}_k\,S = \Lambda_{\mathcal{H}} \tag{5.7}$$

where $\Lambda_{\mathcal{H}}$ is a diagonal matrix and the symplectic matrix S, namely

$$S^t\,\Omega\,S = \Omega\,. \tag{5.8}$$

The symplectic decomposition (5.7) is different from a similarity transformation used in the standard diagonalization procedure. Once S is found, time evolution is rather simple in the new basis $\tilde{z}$ defined by $z = S\,\tilde{z}$ where the system can be interpreted as a collection of decoupled harmonic oscillators and then quantization becomes standard. To find S we consider the following ansatz

$$S = \begin{pmatrix} \boldsymbol{I} & B \\ C & J \end{pmatrix}, \qquad J = \begin{pmatrix} j_{11} & 0 \\ 0 & j_{22} \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}. \tag{5.9}$$

Imposing that S is symplectic produces a 2-parameter family of symplectic matrices; the parameters b and c can be fixed by imposing that $\Lambda_{\mathcal{H}}$ is diagonal and one gets

$$\Lambda_{\mathcal{H}} = \begin{pmatrix} \frac{2\,\omega_1^2\,(\omega_1^2-\omega_2^2)}{T_{11}} & 0 & 0 & 0 \\ 0 & \frac{2\,\omega_2^2\,(\omega_1^2-\omega_2^2)}{T_{22}} & 0 & 0 \\ 0 & 0 & \frac{T_{11}}{2\,(\omega_1^2-\omega_2^2)} & 0 \\ 0 & 0 & 0 & \frac{T_{22}}{2\,(\omega_1^2-\omega_2^2)} \end{pmatrix}; \tag{5.10}$$

where[6]

$$\begin{aligned} T_{11} &= 4\,d^2 + m_1^2 - m_2^2 + \omega_1^2 - \omega_2^2\,, \quad T_{22} = -4\,d^2 + m_1^2 - m_2^2 + \omega_1^2 - \omega_2^2\,, \\ T_{33} &= 4\,d^2 - m_1^2 + m_2^2 + \omega_1^2 - \omega_2^2\,, \quad T_{44} = -4\,d^2 - m_1^2 + m_2^2 + \omega_1^2 - \omega_2^2\,; \end{aligned} \tag{5.11}$$

and

$$j_{11} = j_{22} = 1 + b\,c\,, \qquad c = \frac{m_1^2 - m_2^2 - \omega_1^2 + \omega_2^2}{4\,d}\,, \qquad b = \frac{2\,d}{\omega_1^2 - \omega_2^2}\,. \tag{5.12}$$

Notice also that

$$T_{11}\,T_{33} = 16\,d^2\,\omega_1^2 \to \Lambda_{\mathcal{H}_{11}}\,\Lambda_{\mathcal{H}_{33}} = \frac{\omega_1^2}{4}\,, \qquad T_{22}\,T_{44} = 16\,d^2\,\omega_2^2 \to \Lambda_{\mathcal{H}_{22}}\,\Lambda_{\mathcal{H}_{44}} = \frac{\omega_2^2}{4}\,. \tag{5.13}$$

The system is classically stable when $\omega_{1,2} \in \mathbb{R}$ and $\omega_1^2 \geq \omega_2^2 \geq 0$; however as discussed in the previous section stability does not implies that the Hamiltonian is positive definite. Indeed

$$\begin{aligned} &T_{11} \geq 0 \text{ and } T_{33} \geq 0 \ \text{ for } \ 0 \leq d^2 \leq \frac{(\omega_1-\omega_2)^2}{4} \text{ and } d^2 \geq \frac{(\omega_1+\omega_2)^2}{4}\,; \\ &T_{22} \geq 0 \text{ and } T_{44} \geq 0 \ \text{ for } \ 0 \leq d^2 \leq \frac{(\omega_1-\omega_2)^2}{4}\,. \end{aligned} \tag{5.14}$$

[5]According to the Williamson theorem, given a positive definite and symmetric Hamiltonian H, it always exists a symplectic transformation that diagonalises H; see appendix C and [27] for a recent discussion.

[6]Note that $m_1^2 - m_2^2 = \sqrt{(4\,d^2 - \omega_1^2 - \omega_2^2)^2 - 4\,\omega_1^2\,\omega_2^2}$ and $\omega_1^2 - \omega_2^2 = \sqrt{(4\,d^2 - m_1^2 - m_2^2)^2 - 4\,m_1^2\,m_2^2}$, that allow to express all the quantities as functions of the independent parameters d, $\omega_{1,2}$ or d, $m_{1,2}$.

For $0 \leq d^2 \leq \frac{(\omega_1-\omega_2)^2}{4}$ the diagonal Hamiltonian is positive defined and corresponds to the normal region (4.4) of stability.
In the anomalous region (4.5), $d^2 \geq \frac{(\omega_1+\omega_2)^2}{4}$ and the system is still stable, but now $T_{11,}, T_{33} > 0$ while $T_{22,}, T_{44} < 0$ and the Hamiltonian can be written as the sum of one standard harmonic oscillator plus a second ghost-like harmonic oscillator.
This can be shown explicitly by exploiting the fact that we can still perform a further canonical transformation to reduce the oscillators Hamiltonian to the standard form

$$\tilde{z} = N_\pm \, z_c \,, \qquad N_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} n_\pm & 0 \\ 0 & n_\pm^{-1} \end{pmatrix} \; ; \tag{5.15}$$

exploiting (5.13), the 2×2 submatrix is taken as

$$n_\pm = \begin{pmatrix} \sqrt{\frac{T_{11}}{2\,\omega_1\,(\omega_1^2-\omega_2^2)}} & 0 \\ 0 & \sqrt{\frac{\pm T_{22}}{2\,\omega_2\,(\omega_1^2-\omega_2^2)}} \end{pmatrix} \; ; \tag{5.16}$$

the notation $\pm$ refers to the case where T_{22} is positive or negative according to (5.14). The integrand that defines the Hamiltonian in (5.2) reads

$$\Lambda_c^{(\pm)} = N_\pm^t \, \Lambda_{\mathcal{H}} \, N_\pm = \begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \pm\,\omega_2 & 0 & 0 \\ 0 & 0 & \omega_1 & 0 \\ 0 & 0 & 0 & \pm\,\omega_2 \end{pmatrix} \,, \qquad H_k^{(\pm)} = \frac{1}{2}\, z_c^\dagger \, \Lambda_c^{(\pm)} \, z_c \,. \tag{5.17}$$

The explicit expressions for the Hamiltonian is then given by

$$H_k^{(+)} = \frac{1}{2} \sum_{i=1,2} \omega_i \, \left(\pi_{c_i}^2 + \varphi_{c_i}^2\right) , \qquad H_k^{(-)} = \frac{\omega_1}{2} \, \left(\pi_{c_1}^2 + \varphi_{c_1}^2\right) - \frac{\omega_2}{2} \, \left(\pi_{c_2}^2 + \varphi_{c_2}^2\right) . \tag{5.18}$$

The quantization of the system in the anomalous region that corresponds to $H_c^{(-)}$ has a ghost character. The complete canonical transformation that relates the original variables z and z_c is given by

$$z = S\, \tilde{z} = S\, N_\pm \, z_c \,, \qquad \Lambda_c^{(\pm)} = N_\pm^t \, S^t \, \mathcal{H} \, S \, N_\pm \,. \tag{5.19}$$

The time evolution of z_c is very simple

$$\begin{aligned} z_c(t) &= e^{\Omega\, \Lambda_c^{(\pm)}\, t} \, z_c(0) \equiv G_c^{(\pm)}(t)\, z_c(0) \,, \\ G_c^{(\pm)}(t) &= \begin{pmatrix} \cos(t\,\omega_1) & 0 & \sin(t\,\omega_1) & 0 \\ 0 & \cos(t\,\omega_2) & 0 & \pm\sin(t\,\omega_2) \\ -\sin(t\,\omega_1) & 0 & \cos(t\,\omega_1) & 0 \\ 0 & \mp\sin(t\,\omega_2) & 0 & \cos(t\,\omega_2) \end{pmatrix} . \end{aligned} \tag{5.20}$$

The matrix $G_c^{(\pm)}$ is also symplectic. From $G_c^{(\pm)}$ the evolution of the original variables $z(t)$ can be also found by using (5.19):

$$z(t) = G(t)\, z(0) \,, \qquad G(t) = S\, N_\pm \, G_c^{(\pm)}(t)\, N_\pm^{-1}\, S^{-1} \,. \tag{5.21}$$

6 Quantization

One of the problem in the quantization of classical field theory is that, contrary to the case of system with a finite number of degrees of freedom, the procedure is not unique [28]; indeed, given two representations of the canonical commutation relations, in general it is not guaranteed that they are unitary equivalent. The most widely used quantization scheme is based on the Fock space construction according with when a suitable set of creation and annihilation operators are defined, physical states are built by acting with them on the vacuum state. While in flat spacetime Poincare' symmetry allows a natural selection of the vacuum state, in general this is not the case and different set of creation and annihilation operators can be constructed related by a Bogolyubov transformation. A well known example is the study of quantum field in a non-trivial gravitational background [29, 30, 31, 32]. Typically the first step is to write the non-interacting Hamiltonian of the system as a set of decoupled harmonic oscillators; given a quadratic Hamiltonian, one can introduce creation and annihilation operators starting from a set of canonical variables $z_{\boldsymbol{k}}$ and the classical Poisson brackets by promoting them to quantum operators ($z_{\boldsymbol{k}} \to \hat{z}_{\boldsymbol{k}}$) which satisfy the equal time canonical commutation relations (CCR) [33] (see [22, 23] for notations and method)

$$[\hat{z}(t,\boldsymbol{x})_m,\, \hat{z}(t,\boldsymbol{y})_n] = i\,\Omega_{mn}\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\,. \tag{6.1}$$

The easiest way to construct a Fock representation of the CCR is to start from the diagonal form of the Hamiltonian (5.17) in terms of canonical variables z_c. In the Heisenberg picture we define

$$b_{\boldsymbol{k}_j}(t) = \frac{1}{\sqrt{2}}\left(\hat{\varphi}_{\boldsymbol{k}\,c_j} + i\,\hat{\pi}_{\boldsymbol{k}\,c_j}\right)\,, \qquad b^\dagger_{\boldsymbol{k}_j}(t) = \frac{1}{\sqrt{2}}\left(\hat{\varphi}_{\boldsymbol{k}\,c_j} - i\,\hat{\pi}_{\boldsymbol{k}\,c_j}\right) \quad j=1,2\,, \tag{6.2}$$

which leads to

$$\left[b_{\boldsymbol{k}_m}(t),\, b^\dagger_{\boldsymbol{q}_n}(t)\right] = \delta_{mn}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{q})\,. \tag{6.3}$$

In an equivalent and more compact matrix notation

$$B_{\boldsymbol{k}}(t) = U\,\hat{z}_{\boldsymbol{k}_c}(t)\,; \tag{6.4}$$

where

$$B_{\boldsymbol{k}}(t) = \left(b_{\boldsymbol{k}_1}(t),\, b_{\boldsymbol{k}_2}(t), b^\dagger_{-\boldsymbol{k}_1}(t),\, b^\dagger_{-\boldsymbol{k}_2}(t)\right)^t\,, \qquad U = \frac{1}{\sqrt{2}}\begin{pmatrix} \boldsymbol{I} & i\,\boldsymbol{I} \\ \boldsymbol{I} & -i\,\boldsymbol{I}\end{pmatrix}\,; \tag{6.5}$$

with the corresponding inverse relation

$$\hat{z}_{\boldsymbol{k}_c}(t) = U^\dagger\; B_{\boldsymbol{k}}(t)\,. \tag{6.6}$$

It is easy to show that

$$b_{\boldsymbol{k}_j}(t) = e^{-i\,\omega_j\,t}\,b_{\boldsymbol{k}_j}\,, \qquad\qquad j=1,2\,. \tag{6.7}$$

The vacuum state relative to the b operator is defined as

$$b_{\boldsymbol{k}_j}(t_0)\,|0_b\rangle = 0\,, \qquad \langle 0_b|0_b\rangle = 1\;.) \tag{6.8}$$

Actually, given the time evolution (6.7), setting $t_0 = 0$, (6.8) holds for any t. The Hamiltonian in terms of these creation and annihilation operators has the standard form for two independent harmonic oscillators

$$H_k^{(+)} = \sum_{i=1}^{2}\frac{\omega_i}{2}\left(b_{\boldsymbol{k}_i}\,b^\dagger_{\boldsymbol{k}_i} + b^\dagger_{\boldsymbol{k}_i}\,b_{\boldsymbol{k}_i}\right) \tag{6.9}$$

in the normal region and

$$H_k^{(-)} = \frac{\omega_1}{2}\left(b_{\boldsymbol{k}_1}\,b^\dagger_{\boldsymbol{k}_1} + b^\dagger_{\boldsymbol{k}_1}\,b_{\boldsymbol{k}_1}\right) - \frac{\omega_2}{2}\left(b_{\boldsymbol{k}_2}\,b^\dagger_{\boldsymbol{k}_2} + b^\dagger_{\boldsymbol{k}_2}\,b_{\boldsymbol{k}_2}\right) \tag{6.10}$$

in the anomalous region of stability. The correlation function for canonical fields can be easily obtained [22, 23] as

$$\begin{aligned}\langle 0_b|\hat{z}_{\boldsymbol{k}\,c_m}(t)\,\hat{z}^\dagger_{\boldsymbol{q}\,c_n}(t)|0_b\rangle &= U^\dagger_{mr}\,\langle 0_b|B_{\boldsymbol{k}_r}(t)\,B_{\boldsymbol{q}_s}(t)|0_b\rangle\,U_{sn} \\ &= \delta^{(3)}(\boldsymbol{k}-\boldsymbol{q})\;U^\dagger_{mr}\;\left(\delta_{r\,1}\,\delta_{s\,3} + \delta_{r\,2}\,\delta_{s\,4}\right)\;U_{sn} \equiv \delta^{(3)}(\boldsymbol{k}-\boldsymbol{q})\;\Sigma_{mn} \\ \Sigma = \frac{1}{2}\left(\boldsymbol{I}+i\,\Omega\right) &= \begin{pmatrix} \frac{1}{2} & 0 & \frac{i}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{i}{2} & 0 & \frac{1}{2}\end{pmatrix}\end{aligned} \tag{6.11}$$

The same correlation function for the original fields can be also computed by using (5.19), namely $\hat{z}_{\boldsymbol{k}}(t) = S\, N_\pm\, \hat{z}_{\boldsymbol{k}\,c}(t)$

$$
\begin{aligned}
&\langle 0_b | \hat{z}_{\boldsymbol{k}_m}(t)\, \hat{z}^\dagger_{\boldsymbol{q}_n}(t) | 0_b \rangle = \delta^{(3)}(\boldsymbol{k} - \boldsymbol{q})\, Z_{mn}\,; \\
&Z = S\, N_\pm\, \Sigma\, N_\pm^t\, S^t \;\equiv \mathcal{Z} + \frac{i}{2}\,\Omega;
\end{aligned}
\tag{6.12}
$$

with Z hermitian matrix with entries:

$$
\mathcal{Z} = \frac{1}{(\omega_1^2 - \omega_2^2)} \begin{pmatrix} \frac{4\,d^2\,\omega_2}{\pm T_{22}} + \frac{T_{11}}{4\,\omega_1} & 0 & 0 & \mathcal{Z}_{14} \\ 0 & \frac{4\,d^2\,\omega_1}{T_{11}} + \frac{\pm T_{22}}{4\,\omega_2} & \mathcal{Z}_{23} & 0 \\ 0 & \mathcal{Z}_{23} & \mathcal{Z}_{33} & 0 \\ \mathcal{Z}_{14} & 0 & 0 & \mathcal{Z}_{44} \end{pmatrix}\,;
\tag{6.13}
$$

where

$$
\begin{aligned}
\mathcal{Z}_{14} &= \frac{T_{11}\left(m_1^2 - m_2^2 - \omega_1^2 + \omega_2^2\right)}{16\,d\,\omega_1} + \frac{d\,\omega_2\left(m_1^2 - m_2^2 + \omega_1^2 - \omega_2^2\right)}{\pm T_{22}}\,; \\
\mathcal{Z}_{23} &= \frac{d\,\omega_1\left(m_1^2 - m_2^2 + \omega_1^2 - \omega_2^2\right)}{T_{11}} + \frac{\pm T_{22}\left(m_1^2 - m_2^2 - \omega_1^2 + \omega_2^2\right)}{16\,d\,\omega_2}\,; \\
\mathcal{Z}_{33} &= \frac{1}{64}\left[\frac{\pm T_{22}\left(m_1^2 - m_2^2 - \omega_1^2 + \omega_2^2\right)^2}{d^2\,\omega_2} + \frac{16\,\omega_1\,\left(m_1^2 - m_2^2 + \omega_1^2 - \omega_2^2\right)^2}{T_{11}}\right]\,; \\
\mathcal{Z}_{44} &= \frac{1}{64}\left(\frac{T_{11}\left(m_1^2 - m_2^2 - \omega_1^2 + \omega_2^2\right)^2}{d^2\,\omega_1} + \frac{16\,\omega_2\left(m_1^2 - m_2^2 + \omega_1^2 - \omega_2^2\right)^2}{\pm T_{22}}\right)\,;
\end{aligned}
\tag{6.14}
$$

While both Σ and Z are symplectic matrices with the same symplectic eigenvalues, the canonical transformation (5.16) is singular in the limit $\omega_2 \to \omega_1$ and it is interesting to see what happens to the correlation. From (4.3), $\omega_2 = \omega_1$ is possible when

$$
\begin{cases} d = d_{\mathrm{cr}} = \frac{\left(\sqrt{-m_1^2} + \sqrt{-m_2^2}\,\right)}{2} & \text{anomalous region } m_{1,2}^2 < 0 \\ d = 0,\; m_2^2 = m_1^2 > 0 & \text{normal region } m_{1,2}^2 > 0 \end{cases}\,.
\tag{6.15}
$$

In the anomalous region when $d = d_{\mathrm{cr}}$, we have that $\omega_1^2 = \omega_2^2 = \bar{\omega}^2 \equiv \left(m_1^2\, m_2^2\right)^{1/2}$, then setting $d = d_{\mathrm{cr}} + \frac{\epsilon^2}{2}$ it gives $\omega_1^2 = \bar{\omega}^2 + 2\,\bar{\omega}\,d_{\mathrm{cr}}\,\epsilon$, $\omega_2^2 = \bar{\omega}^2 - 2\,\bar{\omega}\,d_{\mathrm{cr}}\,\epsilon$, where ϵ is a dimensionless small parameter measuring the distance of d from d_{cr}. The non-trivial part $\mathcal{Z}$ of the correlation function (6.12) has the following behavior at the leading order in ϵ

$$
\mathcal{Z} = \begin{pmatrix} \frac{1}{2\,\epsilon\,\sqrt{-m_1^2}} & 0 & 0 & \frac{\sqrt{-m_2^2} - \sqrt{-m_1^2}}{4\,\epsilon\,\sqrt{-m_1^2}} \\ 0 & \frac{1}{2\,\epsilon\,\sqrt{-m_2^2}} & \frac{\sqrt{-m_1^2} + \sqrt{-m_2^2}}{4\,\epsilon\,\sqrt{-m_2^2}} & 0 \\ 0 & \frac{\sqrt{-m_1^2} + \sqrt{-m_2^2}}{4\,\epsilon\,\sqrt{-m_2^2}} & \frac{\left(m_1^2 - m_2^2\right)^2}{8\,\epsilon\,\sqrt{-m_2^2}\left(\sqrt{-m_1^2} + \sqrt{-m_2^2}\right)^2} & 0 \\ \frac{\sqrt{-m_2^2} - \sqrt{-m_1^2}}{4\,\epsilon\,\sqrt{-m_1^2}} & 0 & 0 & \frac{\left(m_1^2 - m_2^2\right)^2}{8\,\epsilon\,\sqrt{-m_1^2}\left(\sqrt{-m_1^2} + \sqrt{-m_2^2}\right)^2} \end{pmatrix}.
\tag{6.16}
$$

Thus $\mathcal{Z}$ shows a resonant singular behavior when $\omega_1 \approx \omega_2$ in the anomalous region of stability, which is peculiar also from a classical point view: even a very small coupling can trigger a runaway behavior of classical solutions that far from $\omega_1 \approx \omega_2$ are well behaved at least when interactions are not too big [4]. On the other hand when $d \to 0$ in the normal region of stability no resonant behavior is present; indeed we get

$$\mathcal{Z} = \begin{pmatrix} \frac{1}{2\,m_1} & 0 & 0 & 0 \\ 0 & \frac{1}{2\,m_1} & 0 & 0 \\ 0 & 0 & \frac{m_1}{2} & 0 \\ 0 & 0 & 0 & \frac{m_1}{2} \end{pmatrix}. \tag{6.17}$$

The entries Z_{11} and Z_{22} are particularly important, since they represent the autocorrelations (power spectra) of the original fields φ_1 and φ_2, whose behavior in both the stability regions is shown in figure 2.

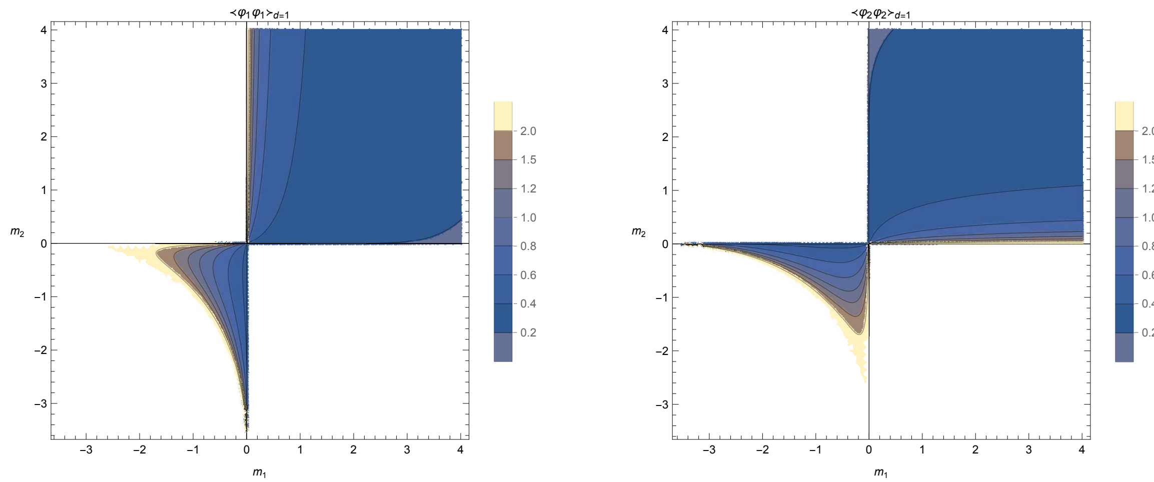

Figure 2: *Contour plot of* $\mathcal{Z}_{11}$ *and* $\mathcal{Z}_{22}$ *for* $d = 1$.

From the fact that S and $N_\pm$ are symplectic they have unit determinant, we get

$$\mathrm{Det}\,(Z) = \mathrm{Det}\,(\Sigma) \Rightarrow \mathrm{Det}\,(\mathcal{Z}) = \frac{1}{16}\,. \tag{6.18}$$

As shown in [23] the part $\mathcal{Z}$ of the correlation matrix Z is relevant to study decoherence once one of two modes is traced out. In particular, the so called *purity* γ is a measure of the entanglement between the two dof.

$$\gamma^2 \equiv \left[4\,(\mathcal{Z}_{11}\;\mathcal{Z}_{33} - \mathcal{Z}_{13}^2)\right]^{-1}\,. \tag{6.19}$$

In particular for a pure state we have $\gamma = 1$ while for mixed states we have $0 \leq \gamma \leq 1$. The limit $\gamma \to 0$ corresponds to the maximally decoherence case [23]. For our gyroscopic system in particular we get in the two regions of stability the following results

$$\gamma_\pm^2 = \frac{\hat\omega_1\,(\hat\omega_1 \pm \hat\omega_2)^{\,2}\hat\omega_2}{4\,[(\hat\omega_1 \pm \hat\omega_2)^{\,2} - 4]\,(\hat\omega_1\hat\omega_2 \pm 1)}\,, \tag{6.20}$$

that we show in figure 3. We give also the following limits on the boundaries of the parameter space

$$\lim_{d\to 0}\gamma_+^2 = 1, \qquad \lim_{d\to\infty}\gamma_+^2 = \frac{4\,\sqrt{m_1\,m_2}}{(\sqrt{m_1} + \sqrt{m_2})^2} \tag{6.21}$$

$$\lim_{d\to d_c}\gamma_-^2 = 0, \qquad \lim_{d\to\infty}\gamma_-^2 = \frac{4\,\sqrt{m_1\,m_2}}{(\sqrt{-m_1} + \sqrt{-m_2})^2} \tag{6.22}$$

$$\lim_{m_1\to m_2}\gamma_\pm^2 = 1 \tag{6.23}$$

As discussed at the beginning of the section the Fock representation is by non means unique. Indeed given a set a canonical variable one can define an alternative set of creation/annihilation

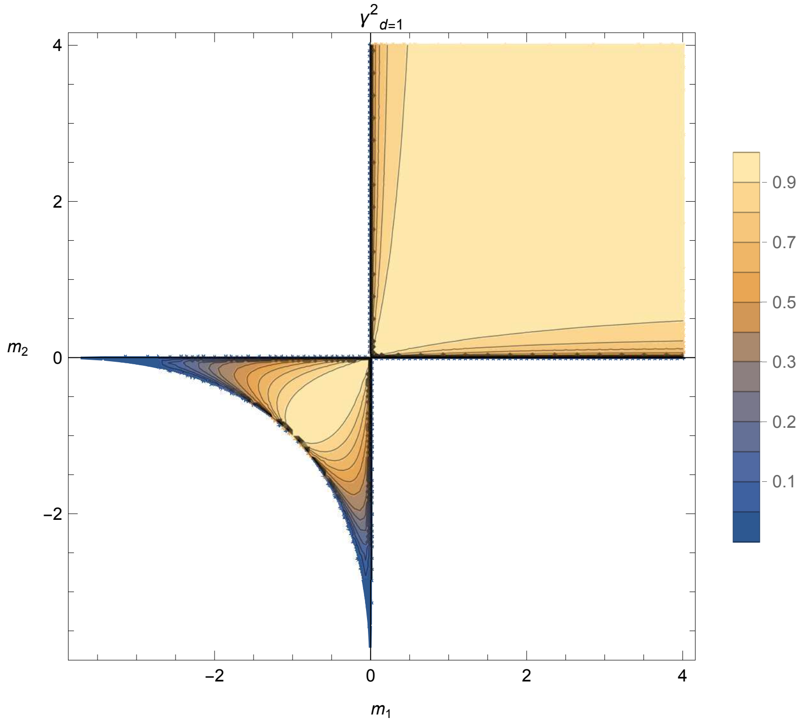

Figure 3: *Contour plot of the square for the purity γ^2 once we fixed $d = 1$.*

operators by applying the above construction for $\hat{z}_c$ to $\hat{z}$. Similarly to (6.2) we can define a new set of creation/annihilation operators a and $a^\dagger$ such that

$$a_{\boldsymbol{k}_j}(t) = \frac{1}{\sqrt{2}\, y_j}\left(y_j^2\, \hat{\varphi}_{\boldsymbol{k}j} + i\, \hat{\pi}_{\boldsymbol{k}j}\right), \qquad a^\dagger_{\boldsymbol{k}_j}(t) = \frac{1}{\sqrt{2}\, y_j}\left(y_j^2\, \hat{\varphi}_{\boldsymbol{k}j} - i\, \hat{\pi}_{\boldsymbol{k}j}\right) \quad j = 1, 2\,, \tag{6.24}$$

or, in matrix representation

$$A_{\boldsymbol{k}}(t) = U\, Y\, \hat{z}_{\boldsymbol{k}}(t)\,; \tag{6.25}$$

where

$$A_{\boldsymbol{k}}(t) = \Big(a_{\boldsymbol{k}_1}(t),\, a_{\boldsymbol{k}_2}(t), a^\dagger_{-\boldsymbol{k}_1}(t),\, a^\dagger_{-\boldsymbol{k}_2}(t)\Big)^t\,, \qquad Y = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}\,, \qquad y = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}\,, \tag{6.26}$$

together with the obvious inverse relations. The important physical difference is that we do not have a unique option in the choice of the y matrix; the choice of $N_\pm$ in the definition of $\hat{z}_c$ (and that of B) was instrumental to get the Hamiltonian in the standard form (6.9) for an harmonic oscillator. As before, the a-type operators also define the associated vacuum state by

$$a_{\boldsymbol{k}_j}(t_0)|0_a\rangle = 0\,, \qquad \langle 0_a|0_a\rangle = 1\,. \tag{6.27}$$

The Hamiltonian in the new basis is the following

$$H_k = \frac{1}{2}\, A^\dagger_{\boldsymbol{k}}(t)\, \mathcal{H}_A\, A_{\boldsymbol{k}}(t)\,, \qquad \mathcal{H}_A = U\, Y^{-1t}\, \mathcal{H}_k\, Y^{-1}\, U^\dagger\,. \tag{6.28}$$

As before $A_{\boldsymbol{k}}(t)$ are in the Heisenberg picture and

$$A_{\boldsymbol{k}}(t) = \mathcal{G}_A(t)\, A_{\boldsymbol{k}}(0) \equiv \mathcal{G}_A(t)\, A_{\boldsymbol{k}}\,, \qquad \mathcal{G}_A(t) = U\, Y\, G(t)\, Y^{-1}\, U^\dagger\,, \tag{6.29}$$

and $G(t)$ gives the time evolution of $z_{\boldsymbol{k}}$ according to (5.21). By using (6.29), (5.21), and that $G_c^\dagger\,\Lambda_c^{(\pm)}\,G_c = \Lambda_c^{(\pm)}$, the Hamiltonian can be written in terms of $A_{\boldsymbol{k}}$ instead of $A_{\boldsymbol{k}}(t)$

$$H_k = \frac{1}{2}\,A_{\boldsymbol{k}}^\dagger\,\mathcal{H}_A\,A_{\boldsymbol{k}}\,, \tag{6.30}$$

which shows that H_k is time-independent. In general, H_k can be written in terms of ten real parameters which can be interpreted as squeezing and rotation parameters [22, 23] or equivalently as Bogolyubov coefficients. Explicitly we get

$$\mathcal{H}_A = \begin{pmatrix} P & Q \\ Q^\dagger & P \end{pmatrix} \quad \text{where } P = P^\dagger = \begin{pmatrix} F_1 & F_{12}\,e^{i\,\phi} \\ F_{12}\,e^{-i\,\phi} & F_2 \end{pmatrix}\,, \quad Q = \begin{pmatrix} R_1\,e^{i\,\Theta_1} & R_{12}\,e^{i\,\xi} \\ R_{12}\,e^{i\,\xi} & R_2\,e^{i\,\Theta_2} \end{pmatrix}\,; \tag{6.31}$$

with

$$F_1 = \frac{d^2+m_1^2+y_1^4}{2\,y_1^2}\,, \qquad F_2 = \frac{d^2+m_2^2+y_2^4}{2\,y_2^2}\,, \qquad F_{12} = \frac{d\left(y_1^2+y_2^2\right)}{4\,y_1\,y_2}\,, \quad \phi = \frac{\pi}{2} \tag{6.32}$$

$$\begin{aligned} &R_1 = \frac{d^2+m_1^2-y_1^4}{2\,y_1^2}\,, \qquad R_2 = \frac{d^2+m_2^2-y_2^4}{2\,y_2^2}\,, \qquad R_{12} = \frac{d\left(y_1^2-y_2^2\right)}{2\,y_1\,y_2}\,, \\ &\Theta_1 = \Theta_2 = 0\,,\ \xi = \frac{\pi}{2}\,. \end{aligned} \tag{6.33}$$

Thus

$$H_k \;=\; \sum_{i=1}^{2}\Big[F_i\,\left(a_{\boldsymbol{k}_i}^\dagger\,a_{\boldsymbol{k}_i} + a_{-\boldsymbol{k}_i}^\dagger\,a_{-\boldsymbol{k}_i} + \text{h.c.}\right) + R_i\,\left(e^{i\,\Theta_i}\,a_{\boldsymbol{k}_i}^\dagger\,a_{-\boldsymbol{k}_i}^\dagger + \text{h.c.}\right)\Big] + \tag{6.34}$$

$$F_{12}\,e^{i\,\theta}\,\left(a_{\boldsymbol{k}_1}^\dagger\,a_{\boldsymbol{k}_2} + \ a_{-\boldsymbol{k}_1}^\dagger\,a_{-\boldsymbol{k}_2}\right) + \text{h.c.} + R_{12}\,e^{i\,\xi}\,\left(a_{\boldsymbol{k}_1}^\dagger\,a_{-\boldsymbol{k}_2}^\dagger + a_{\boldsymbol{k}_2}^\dagger\,a_{-\boldsymbol{k}_1}^\dagger\right) + \text{h.c.}\,. \tag{6.35}$$

The physical interpretation of the various terms is the following [22, 23]

- *Harmonic*: $F_{i=1,2}$ non-standard normalization of the number operator;
- *Parametric*: $R_{i=1,2}$ gives rise to particle creation;
- *Transferring*: F_{12} transfers particles from one sector to the other;
- *Entangling*: R_{12} represents cross-sector particle creation.

By a suitable choice of $y_{1,2}$, one can set $R_{12}=0$ and $R_1=0$ or $R_2=0$.

Let us discuss the relation between the quantization performed by using the above explicit covariant symplectic formalism and the more traditional one that makes an ansatz for quantum fields according to which they can be written, in Fourier space, as linear combination of creation and annihilation operators [22]; for instance in the case of a single scalar field

$$\hat{\varphi}_{\boldsymbol{k}}(t) = \phi_k(t)\,b_{\boldsymbol{k}} + \phi_k(t)^*\,b_{\boldsymbol{k}}^\dagger\,, \tag{6.36}$$

where $\phi_k(t)$ is a solution of the linear equation of motion in Fourier space. The requirement that the field $\hat{\varphi}(t,\boldsymbol{x})$ together with its conjugate momentum $\hat{\pi}(t,\boldsymbol{x})$ satisfy the equal time canonical commutation rules gives a condition on the Wronskian of the solutions $\phi_k(t)$ and $\phi_k(t)^*$ of the equation of motion of the field. As matter of fact such condition is equivalent to the symplectic character of the transformations relating the field variables, the matrices S and N in our case. The symplectic treatment is particular useful when the conjugate momenta are not simply proportional to the time derivative of the fields; this is the case in a gyroscopic system, see eq. (4.8). To write the quantum field in the form (6.36) we can use (6.4) and (5.19) to get

$$\begin{aligned} &\hat{z}_{\boldsymbol{k}}(t) = S\,N_\pm\,U^\dagger\,B_{\boldsymbol{k}}(t)\,, \qquad B_{\boldsymbol{k}}(t) = U\,\hat{z}_{\boldsymbol{k}\,c}(t) = U\,G_{\pm c}(t)\,\hat{z}_{\boldsymbol{k}\,c}(0) = U\,G_{\pm c}(t)\,U^\dagger\,B_{\boldsymbol{k}}(0) \\ &\quad\Rightarrow \qquad \hat{z}_{\boldsymbol{k}}(t) = S\,N_\pm\,G_{\pm c}(t)\,U^\dagger\,B_{\boldsymbol{k}}(0) \equiv \mathbb{E}(t)\,B_{\boldsymbol{k}}(0)\,. \end{aligned} \tag{6.37}$$

The time-dependent 4×4 symplectic matrix $\mathbb{E}(t)$ is determined by U, the canonical transformations S and $N_\pm$ and the symplectic time evolution matrix $G_{\pm c}(t)$ and can be written in terms of the suitable "classical modes" that can be read out from $\mathbb{E}(t)$; in particular the expression for the quantum fields $\hat{\varphi}_{1,2}$ is the following

$$
\begin{gathered}
\begin{pmatrix} \hat{\varphi}_{\boldsymbol{k}_1}(t) \\ \hat{\varphi}_{\boldsymbol{k}_2}(t) \end{pmatrix} = L(t) \begin{pmatrix} b_1(\boldsymbol{k}) \\ b_2(\boldsymbol{k}) \end{pmatrix} + L^*(t) \begin{pmatrix} b_1(\boldsymbol{k})^\dagger \\ b_2(\boldsymbol{k})^\dagger \end{pmatrix} , \\
L(t) = \begin{pmatrix} \mathbb{E}_{11} & \mathbb{E}_{12} \\ \mathbb{E}_{21} & \mathbb{E}_{22} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{T_{11}}\, e^{-i\,\omega_1\,t}}{2\sqrt{\omega_1}\,\sqrt{\omega_1^2-\omega_2^2}} & -\frac{2\,i\,d\,\sqrt{\omega_2}\,e^{\mp\,i\,\omega_2\,t}}{\sqrt{\pm T_{22}}\,\sqrt{\omega_1^2-\omega_2^2}} \\ -\frac{2\,i\,d\,\sqrt{\omega_1}\,e^{-i\,\omega_1\,t}}{\sqrt{T_{11}}\,\sqrt{\omega_1^2-\omega_2^2}} & \frac{\sqrt{\pm T_{22}}\,e^{\mp\,i\,\omega_2\,t}}{2\sqrt{\omega_2}\,\sqrt{\omega_1^2-\omega_2^2}} \end{pmatrix} .
\end{gathered}
\tag{6.38}
$$

Clearly the matrix L and L^* are just submatrices of $\mathbb{E}$. One can check that the quantum fields satisfy the equations of motion (4.1). Of course the same representation can be used for the quantum fields expressed in terms of the alternative set of creation and annihilation operators (6.2) and their modes

$$
\hat{z}_{\boldsymbol{k}}(t) = \tilde{\mathbb{E}}_{\boldsymbol{k}}(t)\, A_{\boldsymbol{k}}(0)\,; \tag{6.39}
$$

by using (6.29) and (6.25), the symplectic matrix $\tilde{\mathbb{E}}_{\boldsymbol{k}}(t)$ is given by

$$
\begin{aligned}
\hat{z}_{\boldsymbol{k}}(t) &= Y^{-1}\, U^\dagger\, A_{\boldsymbol{k}}(t) = Y^{-1}\, U^\dagger\, \mathcal{G}_A(t)\, A_{\boldsymbol{k}}(0) \\
&\Rightarrow \tilde{\mathbb{E}}_{\boldsymbol{k}}(t) = Y^{-1}\, U^\dagger\, \mathcal{G}_A(t)\,.
\end{aligned}
\tag{6.40}
$$

The classical modes in $\tilde{\mathbb{E}}_{\boldsymbol{k}}(t)$ are solutions of the classical equations of motion, namely $\dot{\tilde{\mathbb{E}}}_{\boldsymbol{k}} = \Omega\,\mathcal{H}\,\tilde{\mathbb{E}}_{\boldsymbol{k}}$. It is interesting to note that the initial conditions at $t=0$ can be related to the freedom in the choice of the matrix Y, see [22]. The initial conditions on $\mathcal{G}_A(t)$ is related to the initial conditions for the classical modes

$$
\mathcal{G}_A(0) = \boldsymbol{I} \quad \Rightarrow \quad \tilde{\mathbb{E}}_{\boldsymbol{k}}(0) = Y^{-1}\, U^\dagger \,= \begin{pmatrix} \frac{y_1}{\sqrt{2}} & 0 & \frac{y_1}{\sqrt{2}} & 0 \\ 0 & \frac{y_2}{\sqrt{2}} & 0 & \frac{y_2}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}y_1} & 0 & \frac{i}{\sqrt{2}y_1} & 0 \\ 0 & -\frac{i}{\sqrt{2}y_2} & 0 & \frac{i}{\sqrt{2}y_2} \end{pmatrix} \tag{6.41}
$$

The relation among the classical modes $\mathbb{E}_{\boldsymbol{k}}$ and $\tilde{\mathbb{E}}_{\boldsymbol{k}}$ and the corresponding sets of creation and annihilation operators is a Bogolyubov transformation [29, 32, 31, 22].

7 Gyroscopic Systems and the Pais-Uhlenbeck Oscillator

It is intriguing to relate the anomalous region of stability of a gyroscopic system with the Pais-Uhlenbeck oscillator [34]. The Pais-Uhlenbeck higher derivative Lagrangian can be written as

$$
L_{\rm PU} = \frac{1}{2}\left[\ddot{\varphi}^2 - \left(\omega_1^2+\omega_2^2\right)\dot{\varphi}^2 + \omega_1^2\,\omega_2^2\,\varphi^2\right] . \tag{7.1}
$$

that gives the following fourth order equation of motion

$$
\varphi^{(4)} + (\omega_1^2+\omega_2^2)\,\ddot{\varphi} + \omega_1^2\,\omega_2^2\,\varphi = 0\,. \tag{7.2}
$$

The solution is of the form $\varphi \sim \exp(i\,\omega\,t)$ and ω satisfies exactly (4.2). Though a gyroscopic system has at least 2 degrees of freedom, it is easy to see that (7.2) is equivalent to a system of second order coupled equations [35]

$$
\ddot{\varphi}_1 + \mu_1\,\varphi_1 - \rho_1\,\varphi_2 = 0, \qquad\qquad \ddot{\varphi}_2 + \mu_2\,\varphi_2 - \rho_2\,\varphi_1 = 0 \tag{7.3}
$$

where the real constants μ_i and ρ_i are constrained by

$$\mu_1 + \mu_2 = \omega_1^2 + \omega_2^2 \,, \qquad \mu_1\, \mu_2 - \rho_1\, \rho_2 = \omega_1^2\, \omega_2^2 \,. \tag{7.4}$$

Exploiting the freedom in the choice of μ_i and ρ_i it is easy to realize that the Pais-Uhlenbeck oscillator admits different classically equivalent Lagrangian formulations [35, 36]. Depending on the choice of ρ_i, the two second order equations can be derived by two different Lagrangians of the form

$$L_{\mathrm{PU}_{a/b}} = \frac{1}{2}\left[\dot{\varphi_1}^2 \pm \dot{\varphi_2}^2 - \frac{1}{2}\,\left(\mu_1\, \varphi_1^2 \pm \mu_2\, \varphi_2^2 - 2\, \rho_1\, \varphi_1\, \varphi_2\right)\right] \qquad \text{for } \rho_1 = \pm \rho_2 \,; \tag{7.5}$$

After a Lagrangian field redefinition, L_{PU_a} leads to an Hamiltonian that is positive defined[7], while L_{PU_b} leads to an Hamiltonian that is not positive defined. Our representation of the Pais-Uhlenbeck oscillator is rather different, as it is evident form the equations of motion (4.1)

$$\ddot{\varphi}_1 + m_1^2\, \varphi_1 - 2\, d\, \dot{\varphi}_2 = 0, \qquad \ddot{\varphi}_2 + m_2^2\, \varphi_2 - 2\, d\, \dot{\varphi}_1 = 0 \,; \tag{7.6}$$

nevertheless they are still equivalent to (7.2), moreover the Hamiltonian H is positive definite in the region of normal stability (4.4) or indefinite in the anomalous region of stability (4.5). It is interesting to realize that the equivalence at the level of equations of motion of the PU oscillator and L_{PU_a}, L_{PU_b} in general is altered when interactions are introduced [37]. For instance, if one introduces an interaction potential of the form $\lambda\, \varphi_1^2\, \varphi_2^2$, only the Lagrangian L_{PU_b} generates equations of motion that are equivalent to a PU oscillator with the same interaction. This is not the case for L_{PU_a} and for our gyroscopic system. However, by introducing a non-dynamical field, one modify the Lagrangian to extend the equivalence also to other cases [4].

8 Examples of Gyroscopic Systems

In this section we give a number of specific examples of gyroscopic system considered on a Friedmann-Robertson-Walker (FRW) cosmological background. When perturbations of the metric are considered, departing the homogeneous FRW metric the gyroscopic nature of such a system is not altered but the treatment is more involved; see [38, 16, 15] for the discussion in the context of inflation and the computation of primordial non-Gaussianity. On general grounds in $1+3$ dimensions we can define out of N scalar fields the following set of composite operators shift symmetric but in general not invariant under internal $SO(3)$

$$C^{AB} = g^{\mu\nu}\, \partial_\mu \Phi^A\, \partial_\nu \Phi^B \,, \qquad A, B = 1, 2, \cdots, N \,. \tag{8.1}$$

Depending of what kind of vev the fields develop one can distinguish the following cases giving a sketch of the operators involved.

- All fields have time dependent vevs and C^{AB} is a singlet under internal $SO(3)$ for any choice of A, B and we have $N(N+1)/2$ operators.
- All fields have space-dependent vev and to be consistent with the unbroken $SO(3)_d$ diagonal group as discussed in section 2 the fields must be arranged in n triplets of $SO(3)_d$, namely $\Phi^A \to \Phi^a_i$ with $a = 1, 2, 3$ and $i = 1, 2, \cdots, n = N/3$. The basic combination of fields is

  $$B^{ab}_{ij} = g^{\mu\nu}\, \partial_\mu \Phi^a_i\, \partial_\nu \Phi^b_j \,, \qquad a, b = 1, 2, 3 \quad i, j = 1, 2, \cdots, N/3 \tag{8.2}$$

  from which one can form a number of $SO(3)_d$ invariant operators

  $$X^{(S)}_{i_1,\ldots,i_n} = \mathrm{Tr}[B_{i_1 i_2} \ldots B_{i_{n-1} i_n}] \,. \tag{8.3}$$

[7] This is possible when $\omega_1^2 \neq \omega_2^2$.

- Finally, the most involved case is when the fields develop both space and time dependent vevs. For simplicity let suppose that there is a single field Φ^0 that has a time dependent vev and the $N-1 = 3\,n$ remaining fields arranged as triplets: Φ^a_i with vev $\phi^a_i = x^a$. This time the basic building block conveniently organized according to the $SO(3)_d$ transformations are, besides (8.3),

$$X_{i_1 j_2,\dots,i_n} = \mathrm{Tr}\left(Z_{i_1 j_2}\cdots Z_{i_{n-1} i_n}\right)\,, \tag{8.4}$$

where

$$Z^{ab}_{ij} = (g^{\mu\nu}\,\partial_\mu\Phi^0\,\partial_\nu\Phi^a_i)\,(g^{\alpha\beta}\,\partial_\alpha\Phi^0\,\partial_\beta\Phi^b_j)\,. \tag{8.5}$$

The above operators showed here does not exhaust the list of possible single derivative $SO(3)_d$ invariant operators. Indeed, many others can be built out of $u^\mu_{ijk} = \epsilon^{\mu\nu\rho\sigma}\,\epsilon_{abc}\,\partial_\nu\Phi^a_i\,\partial_\rho\Phi^b_j\partial_\sigma\Phi^c_k$. Given the complexity of the most general case and to illustrate the general picture described in section 2 we consider the case of two scalar fields Φ_1 and Φ_2 in a $1+1$ dimensional FRW background[8]. The generalization to the case of 1+3 dimensions is not very difficult. In 1+1 dimensions a single scalar field can develop a spatially dependent vev still preserving homogeneity and spatial translational invariance while to do same in $1+3$ dimensions in a rotational invariant way we need at least three dof. In 1+1 we have just three basic operators.

Taking the metric

$$g_{\mu\nu} = a^2\,\eta_{\mu\nu}\,, \tag{8.6}$$

the most general action is of the form

$$S = \int d^2x\,\sqrt{-g}\,U(\partial\Phi,\,\Phi)\,. \tag{8.7}$$

The number of operators is limited to

$$X_1 = g^{\mu\nu}\,\partial_\mu\Phi^1\,\partial_\nu\Phi^1 \qquad X_2 = g^{\mu\nu}\,\partial_\mu\Phi^2\,\partial_\nu\Phi^2\,, \qquad X_3 = g^{\mu\nu}\,\partial_\mu\Phi^1\,\partial_\nu\Phi^2\,. \tag{8.8}$$

8.1 Time dependent vevs

Consider first the case where both scalars have a time-dependent vev, namely

$$\Phi^1 = \phi_1(t) + T_1\,, \qquad\qquad \Phi^2 = \phi_2(t) + T_2\,. \tag{8.9}$$

The Lagrangian U has the form $U = U(X_1, X_2, X_3, \Phi_1, \Phi_2)$. The kinetic matrix $\mathcal{K}$ has the following matrix elements

$$\begin{aligned}
&\mathcal{K} = \frac{1}{a^2}\begin{pmatrix}\mathcal{K}_{11} & \mathcal{K}_{12}\\ \mathcal{K}_{12} & \mathcal{K}_{22}\end{pmatrix}\,;\\
&\mathcal{K}_{11} = -2\,a^2\,U_{X_1} + 4\,\dot\phi_1^2\,U_{X_1^2} + \dot\phi_2\left[4\,\dot\phi_1\,U_{X_1X_3} + \dot\phi_2\,U_{X_3^2}\right]\,;\\
&\mathcal{K}_{12} = -a^2\,U_{X_3} + 2\,\dot\phi_1^2\,U_{X_1X_3} + \dot\phi_2\,\left[\dot\phi_1\,\left(4\,U_{X_1X_2} + U_{X_3^2}\right) + 2\,\dot\phi_2\,U_{X_2X_3}\right]\,;\\
&\mathcal{K}_{22} = -2\,a^2\,U_{X_2} + \dot\phi_1^2\,U_{X_3^2} + 4\,\dot\phi_2\,\left[\dot\phi_1\,U_{X_2X_3} + \dot\phi_2\,U_{X_2^2}\right]\,.
\end{aligned} \tag{8.10}$$

$$\mathcal{D} = \frac{1}{2}\left[\dot\phi_1\,\left(U_{\Phi_1X_3} - 2\,U_{\Phi_2X_1}\right) + \dot\phi_2\,\left(2\,U_{\Phi_1X_2} - U_{\Phi_2X_3}\right)\right]\,\mathcal{J}\,. \tag{8.11}$$

$$\mathcal{M} = -\begin{pmatrix} 2\,U_{X_1} & U_{X_3}\\ U_{X_3} & 2\,U_{X_2}\end{pmatrix}\,k^2 + \begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{12} & \alpha_{22}\end{pmatrix}\,; \tag{8.12}$$

the explicit expressions for α_{ij} are omitted for sake of brevity and will be not relevant for the discussion. The main features are the following

[8]For simplicity we take the spatially flat case.

- The kinetic matrix is k-independent and the off diagonal elements are related to the presence of the operators X_3 and $U_{X_1 X_2} \neq 0$.
- In the presence of shift symmetry: $U_{\Phi_i} = 0$ and then automatically $\mathcal{D} = 0$.
- The mass matrix is k-dependent.

As a physical example in 1+3 dimensions, we can consider two scalar fields fluids such that $\mathcal{K}$ is diagonal (no kinetic mixing) and with constant sound speeds;

$$U(X_1, X_2, \Phi_1, \Phi_2) = A_1\, X_1^{\frac{1+c_{s_1}^2}{2\,c_{s_1}^2}} + A_2\, X_2^{\frac{1+c_{s_2}^2}{2\,c_{s_2}^2}} + V(\Phi_1, \Phi_2)\,; \tag{8.13}$$

which at the background level gives $\dot{\phi}_{1,2} = a^{1-c_{s_{1,2}}^2}$

$$\mathcal{K} = \begin{pmatrix} \frac{(1+c_{s_1}^2)a^{1+c_{s_1}^2}}{c_{s_1}^2} & 0 \\ 0 & \frac{(1+c_{s_2}^2)a^{1+c_{s_2}^2}}{c_{s_2}^2} \end{pmatrix}, \qquad \mathcal{D} = 0\,, \qquad \mathcal{M} = \begin{pmatrix} \frac{(1+c_{s_1}^2)a^{1+c_{s_1}^2}}{c_{s_1}^2}\, k^2 & a^4\, m_{12}^2 \\ a^4\, m_{12}^2 & \frac{(1+c_{s_2}^2)a^{1+c_{s_2}^2}}{c_{s_2}^2}\, k^2 \end{pmatrix}. \tag{8.14}$$

The canonical fields can be introduced to get rid of $\mathcal{K}$ by $\varphi = \mathcal{K}^{-1/2}\varphi_c$; as a result the new quadratic Lagrangian has $\mathcal{K} = \mathbb{I}$, still $\mathcal{D} = 0$ and

$$M_c = \begin{pmatrix} c_{s_1}^2\, k^2 - \frac{1}{4}(1+c_{s_1}^2)^2\mathcal{H}^2 - \frac{1}{2}(1+c_{s_1}^2)\mathcal{H}' & \frac{a^{2-c_{s_1}^2-c_{s_2}^2}\, c_{s_1}\, c_{s_2}\, m_{12}^2}{(1+c_{s_1}^2)^{1/2}(1+c_{s_2}^2)^{1/2}} \\ \frac{a^{2-c_{s_1}^2-c_{s_2}^2}\, c_{s_1}\, c_{s_2}\, m_{12}^2}{(1+c_{s_1}^2)^{1/2}(1+c_{s_2}^2)^{1/2}} & c_{s_2}^2\, k^2 - \frac{1}{4}(1+c_{s_2}^2)^2\mathcal{H}^2 - \frac{1}{2}(1+c_{s_2}^2)\mathcal{H}' \end{pmatrix}. \tag{8.15}$$

The mass matrix can be diagonalized by an orthogonal transformation with a time dependent mixing angle θ which will inevitably lead to a gyroscopic system with

$$D = \begin{pmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{pmatrix}; \qquad \tan(2\,\theta) = \frac{8\, c_{s_1}\, c_{s_2} a^{3-c_{s_1}/2-c_{s_2}/2}}{(c_{s_1}^2 - c_{s_2}^2)(1+c_{s_1}^2)^{1/2}(1+c_{s_2}^2)^{1/2}\left[2\,\mathcal{H}' + (2+c_{s_1}^2+c_{s_2}^2)\mathcal{H}^2\right]}. \tag{8.16}$$

Even the case of two non-canonical scalar fields with a mass mixing in a FRW background is a gyroscopic system in disguise [13].

8.2 Space-dependent vevs

In this case it is convenient to define as in four dimensions $\Phi^i = x^i + \frac{\partial_x}{\sqrt{\vec{\nabla}^2}}\, S_i$ $(i = 1, 2)$ and, as discussed in section 2, we need shift symmetry; thus the generic Lagrangian has the form $U(X_1, X_2, X_3)$ and for the matrices $\mathcal{K}$, $\mathcal{D}$ and $\mathcal{M}$ we get

$$\mathcal{K} = -\begin{pmatrix} 2\, U_{X_1} & U_{X_3} \\ -U_{X_3} & 2\, U_{X_2} \end{pmatrix} \qquad \mathcal{D} = 0\,; \tag{8.17}$$

$$\begin{aligned} \mathcal{M} &= -k^2 \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{12} & \mathcal{M}_{22} \end{pmatrix}; \\ \mathcal{M}_{11} &= \frac{4\left(U_{X_1^2} + U_{X_1 X_3}\right) + U_{X_3^2}}{a^2} + 2\, U_{X_1}\,; \\ \mathcal{M}_{12} &= \frac{4\, U_{X_1 X_2} + 2\, \left(U_{X_1 X_3} + U_{X_2 X_{12}}\right) + U_{X_3^2}}{a^2} + U_{X_3}\,; \\ \mathcal{M}_{22} &= \frac{4\left(U_{X_2^2} + U_{X_2 X_3}\right) + U_{X_3^2}}{a^2} + 2\, U_{X_2}\,. \end{aligned} \tag{8.18}$$

In this case

- the off diagonal elements are induced by the presence of the operator X_3;
- we have always $\mathcal{D} = 0$;
- the mass matrix is quadratic in k and it is not diagonal when the operator X_3 is present and $U_{X_1 X_2} \neq 0$.

An explicit example in Minkowski spacetime can be found in [39] where the effective field theory for the interactions between acoustic and gapped phonons was studied. In a FRW set up, we see that the non diagonally kinetic and mass matrices will induce an effective D matrix once the Lagrangian will be rewritten in the canonical form (3.3).

8.3 Mixed vevs

Finally, in the mixed vevs case we have

$$\Phi^1 = \phi(t) + T\,, \qquad \Phi^2 = x + \frac{\partial_x}{\sqrt{\vec{\nabla}^2}}\, S\,. \tag{8.19}$$

Now the $SO(3)$ invariant operators are X_1 and X_2 of (8.8) and

$$\tilde{X}_3 = X_3^2\,. \tag{8.20}$$

as can be by deduced by (8.4,8.5) and the Lagrangian is then of the form $U(X_1, X_2, \tilde{X}_3, \Phi_1)$. Omitting for simplicity the kinetic matrix, we have

$$\mathcal{D} = \frac{k\,\left(2\,U_{X_1 X_2} - U_{\tilde{X}_3}\right)\,\dot{\phi}}{a^2}\,\mathcal{J}\,; \tag{8.21}$$

$$\mathcal{M} = -\begin{pmatrix} 2\,k^2\,\left(a^2 U_{X_1} + U_{\tilde{X}_3}\right) & k\,\beta_{12} \\ k\,\beta_{12} & 2\,k^2\,\left(U_{X_2} + 2\frac{U_{X_2^2}}{a^2}\right) + k^2\,\gamma \end{pmatrix} + \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix}\,; \tag{8.22}$$

once again the expression of α_{ij}, β_{12} and γ are not relevant for the discussion. The main features are

- the kinetic matrix is always diagonal;
- The system is genuinely gyroscopic being $\mathcal{D} \neq 0$ when $U_{\tilde{X}_3} \neq 0$ and $U_{X_1 X_2} \neq 0$ with an overall k dependence.

The present case is the most interesting one and will be further studied in 1+3 dim in the following.

9 Bunch Davies vacuum

The Bunch-Davies (BD) vacuum is the vacuum of election to set the initial conditions for cosmological perturbations during a de Sitter or quasi de Sitter period. A simple and physical way to define the BD vacuum is to invoke the equivalence principle according with at very small scales gravity does not influence local physics. In this context, by choosing conformal time as in (8.6), we impose that at early time, namely when $a \to 0$, the gyroscopic system behaves as in a Minkowski space, namely the Lagrangian in such limit is time independent. By taking $a = t^{2/(1+3\,w)}$, the required time-independent Lagrangian at early time is obtained when the matrices entering the gyroscopic system are of the form

$$\mathcal{K} = \begin{pmatrix} \bar{\kappa}_1\, a^{\xi_1} & 0 \\ 0 & \bar{\kappa}_2\, a^{\xi_2} \end{pmatrix}\,, \quad d = \bar{d}\, a^{\varsigma + \frac{\xi_1+\xi_2}{2}}\,, \quad \mathcal{M}_{ij} = \bar{m}_{ij} + \hat{m}_{ij}\, a^{\eta_{ij}}\,; \quad \dot{\theta}_k = 0\,;$$

all quantities with a bar are constant in time and

$$\eta_{11} = \xi_1 \leq 0, \ \ \eta_{22} = \xi_2 \leq 0, \ \ \eta_{12} \geq \frac{\xi_1 + \xi_2}{2}\,, \ \ w < -\frac{1}{3}\,, \ \ \varsigma \leq 0\,. \tag{9.1}$$

By using the procedure of appendix A to reach the canonical form (3.2) for the matrices of a gyroscopic system, we get in the limit $a \to 0$

$$
\begin{aligned}
&\mathcal{L}^{(BD)} = \frac{1}{2}\,\dot\varphi^t\,\dot\varphi + d_c\,\varphi^t\,\mathcal{J}\,\dot\varphi - \frac{1}{2}\,\varphi^t\,M^{(BD)}\,\varphi\,;\\
&d_c = \begin{cases} 0 & \varsigma \neq 0 \\ \frac{\bar d}{\sqrt{\bar\kappa_1\,\bar\kappa_2}} & \varsigma = 0 \end{cases}\,;
\end{aligned}
\tag{9.2}
$$

the constant mass matrix $M^{(BD)}$ is given by

$$
M^{(BD)} = \begin{pmatrix} \frac{\hat m^2_{1,1}}{\bar\kappa_1} & \frac{\hat m^2_{1,2}}{\sqrt{\bar\kappa_1}\sqrt{\bar\kappa_2}}\;\delta^{(\xi_1+\xi_2)/2}_{\eta_{12}} \\ \frac{\hat m^2_{1,2}}{\sqrt{\bar\kappa_1}\sqrt{\bar\kappa_2}}\;\delta^{(\xi_1+\xi_2)/2}_{\eta_{12}} & \frac{\hat m^2_{2,2}}{\bar\kappa_2} \end{pmatrix} + \delta^0_{\xi_i}\begin{pmatrix} \frac{\bar m^2_{1,1}}{\bar\kappa_1} & \frac{\bar m^2_{1,2}}{\sqrt{\bar\kappa_1}\sqrt{\bar\kappa_2}} \\ \frac{\bar m^2_{1,2}}{\sqrt{\bar\kappa_1}\sqrt{\bar\kappa_2}} & \frac{\bar m^2_{2,2}}{\bar\kappa_2} \end{pmatrix}\,. \tag{9.3}
$$

Notice that the d is different from zero only when $\varsigma = 0$. In a rotational invariant theory, the dependence of spatial momentum of the mass terms is of the form

$$
\hat m^2_{i,i} = c^2_{s_i}\;k^2\;\bar\kappa_i\,; \tag{9.4}
$$

and $c^2_{s_i}$ play the role of non-trivial sound speeds. At small scales for which k is very large, $\mathcal{D}$ is relevant only if $\bar d \propto k$. This is the case when fields which develop both space and time dependent vev. An example is supersolid inflation [16, 15] where $w = -1$, $\xi_1 = -\xi_2 = 4$, $\eta_{12} = 1$, $\varsigma = 0$. On the contrary, when only fields which develop time-dependent vev are present, the matrix $\mathcal{D}$ is not important in the limit of large k used in the selection of the BD vacuum [13].

10 Dynamical Dark Energy as a Gyroscopic System

One of the open questions in modern cosmology is the nature of dark energy that is driving the present expansion of our universe. The simplest option is to add a non-dynamical cosmological constant to the Einstein equations. Alternatively one may try to device a dynamical model for dark energy associated with a medium of some sort with pressure p, energy density ρ and an equation of state $p \approx -\rho$. A perfect fluid with a single scalar degree of freedom[9] does not work: the energy momentum conservation forces ρ to be a constant and one gets back to a cosmological constant. To move on one needs to go beyond a perfect fluid and/or add degrees of freedom. As matter of fact, four scalar fields $\{\Phi^A,\ A = 0,1,2,3\}$, three $\{\Phi^a,\ a = 1,2,3\}$ with an $\vec x$-dependent vev and Φ^0 with a time-dependent vev can be used to describe the most general non-dissipative self-gravitating medium at the leading derivative expansion and the analysis of section (2) applies. In particular, a genuine $\mathcal{D}$ term is present and we are dealing with a gyroscopic system. The action is

$$
S = M^2_{pl}\int d^4x\,\sqrt{-g}\,U(b, y, \chi, \tau_Y, \tau_Z)\,; \tag{10.1}
$$

where

$$
\begin{aligned}
&b = (\mathrm{Det}[B^{ab}])^{1/2}\,, \qquad y = u^\mu\partial_\mu\Phi^0\,, \qquad \chi = (-g^{\mu\nu}\,\partial_\mu\Phi^0\partial_\nu\Phi^0)^{1/2}\,,\\
&B^{ab} = g^{\mu\nu}\,\partial_\mu\Phi^a\,\partial_\nu\Phi^b\,, \qquad \tau_Y = \frac{\mathrm{Tr}(B^2)}{\mathrm{Tr}(B)^2}\,, \qquad \tau_Z = \frac{\mathrm{Tr}(B^3)}{\mathrm{Tr}(B)^3} \qquad a,b = 1,2,3
\end{aligned}
\tag{10.2}
$$

and

$$
u^\mu = -\frac{\epsilon^{\mu\nu\alpha\beta}}{6\,b\,\sqrt{-g}}\epsilon_{abc}\,\partial_\nu\Phi^a\,\partial_\alpha\Phi^b\,\partial_\beta\Phi^c\,, \qquad u^2 = -1\,. \tag{10.3}
$$

[9]Two additional degrees of freedom corresponding to a single transverse vector are present, however, thank to the conservation of the vorticity, their dynamics is trivial.

The energy-momentum tensor (EMT) has the form

$$T_{\mu\nu} = (U - b\,U_b)g_{\mu\nu} + (y\,U_y - b\,U_b)\,u_\mu\,u_\nu + \chi\,U_\chi\,v_\mu\,v_\nu + Q^{(Y)}_{\mu\nu}\,U_{\tau_Y} + Q^{(Z)}_{\mu\nu}\,U_{\tau_Z}\,; \tag{10.4}$$

with

$$v_\mu = \chi^{-1}\,\partial_\mu \Phi^0\,. \tag{10.5}$$

In flat space or on a spatially flat FRW spacetime[10] $\bar{u}_\mu = \bar{v}_\mu$, $Q^{(Z)}_{\mu\nu} = Q^{(Y)}_{\mu\nu} = 0$ and the EMT is the one of a perfect fluid with

$$\bar{\rho} = -U + \bar{\chi}\,U_\chi + \bar{y}\,U_y\,, \qquad \bar{p} = U - \bar{b}\,U_b\,. \tag{10.6}$$

Depending on che choice of U, different equation of state for the medium can be considered. In [40] there were studied models, dubbed Λ-media, featuring an exact equation of state $p + \rho = 0$ (i.e. $w = p/\rho = -1$), valid not only at the background level in a FRW metric but also at the non-perturbative one; this is the case by taking

$$U(b, y, \chi, \tau_Y, \tau_Z) \equiv b^{1+w}\,U_w(b^{-w}\,\chi,\; b^{-w}\,y,\; \tau_Y,\; \tau_Z)\,. \tag{10.7}$$

To study stability, away from any possible Jeans instability, it is sufficient to consider the limit of very large spatial momentum k and forget about the expansion of the universe and metric perturbations[11]. The scalar fields fluctuate according with

$$\Phi^0 = \phi(t) + \pi_0\,, \qquad \Phi^a = x^a + \pi^a\,. \tag{10.8}$$

The π^a excitations are decomposed according to $\pi^a = \pi^a_\perp + \partial_a \pi_L$. The transverse part $\pi^a_\perp$ with $\partial_a \pi^a_\perp = 0$ describes vector modes that are not considered here while π_L represents phonon modes. As discussed in section 2, the structure of the vevs of the scalar fields is such that the quadratic Lagrangian derived from (10.1) has precisely the form (2.16) and reads

$$\mathcal{L}_{ph} = \frac{(\bar{p} + \bar{\rho} + M_1)}{2}\dot{S}^2 + M_0\,\dot{T}^2 + \frac{(M_1 - 2\,M_4)\;k}{2}\;\left(S\,\dot{T} - T\,\dot{S}\right) + (M_3 - M_2)\;k^2\,S^2 + \frac{M_1}{2}\,k^2\,T^2\,, \tag{10.9}$$

where we have set $S = k\,\pi_L$, $T = \pi_0$ and

$$\begin{aligned} &M_0 = \frac{1}{2}\left(U_{\chi\chi} + 2\,U_{y\chi} + U_{yy}\right)\,, \quad M_1 = -U_\chi\,, \quad M_3 = \frac{1}{2}U_{bb}\,, \quad M_2 = \frac{U_{\tau_Y} + U_{\tau_Z}}{27}\,, \\ &M_4 = U_{b\chi} + U_{by} - \frac{1}{2}U_\chi - U_y\,. \end{aligned} \tag{10.10}$$

By defining the dimensionless associated parameters c_i through $M_i = \bar{\rho}\,c_i$ we have that a Λ-medium with $w = -1$ gives the constraints[12]

$$w = -1, \quad c_0 = c_4, \quad c_2 = 3\,(c_3 - c_4)\,. \tag{10.11}$$

The positivity of the kinetic terms imposes

$$c_1 > 0, \quad c_0 > 0\,. \tag{10.12}$$

In the *normal region* of stability (4.4) the positivity of the mass matrix requires that

$$(c_3 - c_2) \leq 0, \quad c_1 \leq 0 \tag{10.13}$$

[10]The bar denotes background quantities and in Minkowski spacetime $\bar{b} = \bar{\chi} = \bar{y} = 1$.

[11]As shown in [41] one gets the very same result by the full analysis of quadratic perturbation in perturbed FRW universe.

[12]This can be seen as the consequence of a Lifshitz scaling symmetry [14].

which is clearly incompatible with (10.12); thus there is no room for stability in the normal region. Stability in the *anomalous region* defined by (4.5) requires that

$$(c_3 - c_2) = 3\,c_4 - 2\,c_3 \geq 0, \quad c_1 \geq 0\,, \tag{10.14}$$

$$0 < c_1 \leq c_4 - \sqrt{c_4\,(3\,c_4 - 2\,c_3)}\,, \tag{10.15}$$

and no inconsistency is present and one easily check that U of the form (10.7) does the job. A point worth to be stressed is that stability with $w = -1$ requires $c_2 \neq 0$ which signals the presence of a solid component in the medium associated to the operators τ_Y and τ_Z; incidentally the very same operators turn on an anisotropic stress part in the EMT (10.4) and by the gravitational Higgs also generate a mass for the graviton. Thus there is no fluid-superfluid medium that is stable with $w = -1$. At the same time the actual realization of stability also requires that $c_4 \neq 0$ together with $c_1 \neq 0$ which means stability also needs for the presence of a superfluid component related to the operator χ. The bottom line is that one can model a dynamical dark energy with $w = -1$ that is stable at the quadratic level, the price to be paid is that the Hamiltonian is not positive definite and is connected with the Pais-Uhlenbeck oscillator. The results in [4] indicates that even in the presence of non-linearities the existence of unavoidable pathologies are far from being automatic.

11 Conclusions

We analyzed the classical and quantum dynamics of quadratic non-dissipative gyroscopic system that are characterized by a Lagrangian with a term of the form $\varphi\,\dot{\varphi}$ that is non-trivial when at least two degrees of freedom are present. In Minkowski spacetime such a term naturally appears when one consider a set of coupled scalar fields in which some fields acquire a space-dependent vev and others a time-dependent one, spontaneously breaking the Lorentz group down to the rotation group $SO(3)$. The minimal number of scalar degrees of freedom for a gyroscopic system is two and they can be interpreted as the Goldstone modes for the spontaneous breaking of temporal and spatial translations but also as the phonon-like excitations of a supersolid, a medium which has a superfluid and a solid component. We studied the classical and quantum dynamics by using symplectic techniques. The system is classically stable in two different regions in parameter space. In what we call *normal* region, the Hamiltonian is positive defined while in the *anomalous* region the Hamiltonian is not positive defined; indeed, after a suitable canonical transformation can be written as the sum of a standard harmonic oscillator and a ghost-like oscillator. As a result, a gyroscopic system in the anomalous region of stability is related to the physics of the Pais-Uhlenbeck oscillator. In the anomalous region of stability a resonant behavior in the 2-point correlation function can take place and it is intriguing that it is behind the slow instability found in [4] when the modes of the standard and the ghost modes are coupled. The very same resonant behavior is behind the maximization of the entanglement when the ghost mode is traced over.

On a time dependent background as FRW (we always retain rotational invariance), the definition of gyroscopic systems becomes ambiguous due to the possibility of performing a time-dependent field redefinition, however it is possible to identify a set of field redefinitions such that a generic Lagrangian with two scalar fields can be brought in the canonical form (3.2) for which the kinetic matrix is the identity, the mass matrix is diagonal and there is a gyroscopic term that mixes the fields with their time derivative through the antisymmetric matrix D. The presence in the canonical form of the Lagrangian of a non zero matrix D is taken as the definition of a gyroscopic system. It turns out that a non-trivial D can be induced by a time-dependent non-diagonal kinetic and/or mass matrix in the original Lagrangian which can be important for the existence of the Bunch-Davies vacuum. As a result, in a time-dependent background, two coupled fluids/superfluids and two coupled solids can also be gyroscopic, see (8.13) and (8.2). Finally, we have shown that dynamical dark energy can be described as a gyroscopic system in the anomalous region of stability.

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A Canonical Form

A generic symmetric time-dependent matrix $\mathcal{T}$ can be diagonalized by an orthogonal transformation

$$\mathcal{R}^t\,\mathcal{T}\,\mathcal{R} = \mathcal{T}_d = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \qquad \mathcal{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{A.1}$$

The kinetic matrix $\mathcal{K}$ in (2.16) is positive definite with eigenvalues $\kappa_1 > \kappa_2 > 0$ and $\mathcal{K}_d = \mathrm{Diag}(\kappa_1,\,\kappa_2)$. By using (A.1), after the following field redefinition

$$\varphi = A_{\mathcal{K}}\,\varphi', \qquad A_{\mathcal{K}} = \mathcal{R}_{\mathcal{K}}\,\mathcal{K}_d^{-1/2}, \tag{A.2}$$

one gets the Lagrangian

$$\mathcal{L}' = \frac{1}{2}\,\dot\varphi'\,\mathcal{K}'\,\dot\varphi' + \varphi'\,\mathcal{D}'\,\dot\varphi' - \frac{1}{2}\,\varphi'\,\mathcal{M}'\,\varphi'\,; \tag{A.3}$$

where

$$K' = \boldsymbol{I}, \tag{A.4}$$

$$\mathcal{D}' = A^t\,D\,A - \dot\theta_{\mathcal{K}}\,\frac{\mathrm{Tr}(\mathcal{K})}{\mathrm{Det}(\mathcal{K})^{1/2}}\,\mathcal{J} \;\equiv\; d'\,\mathcal{J}, \qquad d' = \frac{d - \dot\theta_{\mathcal{K}}\,\mathrm{Tr}(\mathcal{K})}{\det(\mathcal{K})^{1/2}}\,; \tag{A.5}$$

$$\mathcal{M}' = A^t\,\mathcal{M}\,A - \dot A^t\,\mathcal{K}\,\dot A + \frac{1}{2}\frac{d}{dt}\left(\dot A^t\,\mathcal{K}\,A + A^t\,\mathcal{K}\,\dot A\right). \tag{A.6}$$

One can also diagonalize the mass with $\mathcal{M}' = \mathrm{Diag}(\tilde m_1^2,\,\tilde m_2^2)$ to arrive to the canonical form given in (2.16) by a final field redefinition

$$\varphi' = \mathcal{R}_{\mathcal{M}}\,\varphi'' = A_{\mathcal{M}}^{-1}\,\varphi\,. \tag{A.7}$$

The structure of the Lagrangian in the canonical form for a gyroscopic system is given by

$$\mathcal{L}'' = \frac{1}{2}\,\dot\varphi''\,\dot\varphi'' + \varphi''\,D\,\dot\varphi'' - \frac{1}{2}\,\varphi''\,M\,\varphi''\,; \tag{A.8}$$

where $\varphi'' = \mathcal{R}_{\mathcal{M}}^{-1}\,\mathcal{K}_d^{1/2}\,\mathcal{R}_{\mathcal{K}}^{-1}\,\varphi$ and

$$D = \mathcal{D}' - \dot\theta_{\mathcal{K}}\,\mathcal{J} = d_c\,\mathcal{J}, \qquad d_c = \frac{d}{\mathrm{Det}(\mathcal{K})^{1/2}} - \dot\theta_{\mathcal{K}}\,\frac{\mathrm{Tr}(\mathcal{K})}{\mathrm{Det}(\mathcal{K})^{1/2}} - 2\,\dot\theta_{\mathcal{M}}^2, \tag{A.9}$$

$$M = \mathcal{M}' - 2\,\dot\theta_{\mathcal{M}}^2\,\boldsymbol{I} = \begin{pmatrix} \tilde m_1^2 - 2\,\dot\theta_{\mathcal{M}}^2 & 0 \\ 0 & \tilde m_2^2 - 2\,\dot\theta_{\mathcal{M}}^2 \end{pmatrix} \equiv \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}. \tag{A.10}$$

B Lagrangian Transformation

When the matrices are in the canonical form (3.2) but time-dependent, the equations of motion derived from (2.16) are the following

$$\ddot\varphi - 2\,D\dot\varphi + \left(M + \dot D\right)\varphi = 0\,. \tag{B.1}$$

Let us now show that is not possible by a Lagrangian field redefinition to set $D = 0$. Taking $\varphi = L\,\tilde\varphi$, we get the following new form for the matrices in (2.16)

$$\begin{aligned} &\boldsymbol{I} \to K_n = L^t\,K\,L\,, \qquad D \to D_n = \dot L^t\,L + L^t\,D\,L\,, \\ &M \to M_n = L^t\,M\,L - \dot L^t\,\dot L - 2\,L^t D\,\dot L\,. \end{aligned} \tag{B.2}$$

To find $D_n = 0$, L has to satisfies

$$\tilde{D} = L^t D L + \frac{1}{2}\left(\dot{L}^t L - L^t \dot{L}\right) = 0 \quad \Rightarrow \quad 2\,D + L\dot{L}^t - \dot{L}L^t = 0\,. \tag{B.3}$$

Clearly there is no solution if $\dot{L} = 0$. If L is orthogonal then the solution is

$$\dot{L} = D\,L\,. \tag{B.4}$$

However, by a time-dependent field redefinition of the form

$$L = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{pmatrix}\,, \qquad \dot{\theta} = d\,, \tag{B.5}$$

the resulting Hamiltonian takes the form

$$H_n = \frac{1}{2}\,P^t P + \frac{1}{2}\,\tilde{\varphi}^t M_n(t)\tilde{\varphi} \tag{B.6}$$

and it is inevitably time-dependent. The time dependence is rather special and it is of the Floquet type, namely

$$\tilde{H}(t) = H_n(t+T)\,, \qquad T = \frac{2\pi}{d}\,. \tag{B.7}$$

Such time dependent Hamiltonian is often found in condensed matter physics. In the canonical form M is diagonal and thus we get

$$M_n = \begin{pmatrix} d^2 + m_1^2\cos(d\,t)^2 + m_2^2\sin(d\,t)^2 & \left(m_1^2 - m_2^2\right)\sin(d\,t)\,\cos(d\,t) \\ \left(m_1^2 - m_2^2\right)\sin(d\,t)\,\cos(d\,t) & d^2 + m_2^2\cos(d\,t)^2 + m_1^2\sin(d\,t)^2 \end{pmatrix}\,. \tag{B.8}$$

C Hamiltonian Diagonalization

The Hamiltonian diagonalization requires a symplectic decomposition (congruence transformation). According to the Williamson theorem [27], if $\mathcal{H}$ is a *real symmetric positive* matrix of order $2\,n$, there exists a symplectic matrix S such that C.3 holds and given $\Lambda_{\mathcal{H}} = \text{diag}\,(\lambda_j = i\,\omega_j)$ we have $\det\left(\Omega\,\mathcal{H} \pm i\,\omega_j\,\mathbb{I}_{4\times4}\right) = 0$ and the matrix S admits the decomposition

$$S = \sqrt{\Lambda_{\mathcal{H}}}\,\mathbb{O}\,\sqrt{\mathcal{H}^{-1}}\,; \tag{C.1}$$

where $\mathbb{O}$ is an orthogonal matrix satisfying

$$\mathbb{O}\,\sqrt{\mathcal{H}}\,\Omega\,\sqrt{\mathcal{H}}\,\mathbb{O}^t = \Lambda_{\mathcal{H}}\,\Omega\,. \tag{C.2}$$

All the eigenvalues of the matrix $\Omega\,\mathcal{H}$ are purely imaginary and

$$S^t \cdot \Omega \cdot \mathcal{H} \cdot S = \text{diag}\,(i\,\omega_1,\, -i\,\omega_1,\, i\,\omega_2,\, -i\,\omega_2)\,. \tag{C.3}$$

Classical Stability requires that the fundamental frequencies $\Omega_i \in \mathbb{C}$ and $\lambda_i^* = -\lambda_i$. Once the Hamiltonian is diagonal, the Hamilton equations (5.6) are simple to solve. Imposing the initial conditions $z_{\boldsymbol{k}}(t=0) = z_0$, we can formally write the solution of the above system as

$$z_{\vec{k}}(t) = G_k(t,\, t_0) \cdot z_0\,; \tag{C.4}$$

where $G_k(t,\, t_0)$ satisfies

$$\partial_t G_k(t) = \Omega \cdot \mathcal{H}_k \cdot G_k(t,\, t_0), \qquad G_k(t_0,\, t_0) = \mathbb{I}_{4\times4}\,. \tag{C.5}$$

The solution is

$$G_k(t,\, t_0) = T_\tau\, e^{\int_0^t d\tau\; \Omega\cdot\mathcal{H}_k(\tau)}\,. \tag{C.6}$$

When $\mathcal{H}_k$ is time independent the T ordering disappears and we have

$$G_k(t) = e^{\Omega\cdot\mathcal{H}_k\, t}\,. \tag{C.7}$$

The time evolution matrix G is symplectic:

$$\{z,\, z^\dagger\} = G\cdot\{z_0,\, z_0^\dagger\}\cdot G^\dagger \quad\Rightarrow\quad \Omega = G\cdot\Omega\cdot G^\dagger = G\cdot\Omega\cdot G^t,\,; \tag{C.8}$$

with a real G.

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