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Absence of Lavrentiev Phenomenon for Functionals with (p, q)-growth

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Abstract

The aim of this thesis is to provide the absence of Lavrentiev phenomenon for functionals of the following type

$$\mathcal{F}(u) := \int_\Omega f(x, Du(x))\ dx,$$

where $\Omega \subset \mathbb{R}^n$ and $x \mapsto \frac{\partial f}{\partial z}(x,z)$ is α-Hölder continuous. Moreover, the density f is convex and satisfies the (p,q)-growth condition

$$|z|^p \leqslant f(x,z) \leqslant L(1+|z|^q),$$

with

$$1 < p < q < p + \frac{p\alpha}{n}. \tag{1}$$

For the model density represented by the double phase functional

$$f(x,z) := |z|^p + a(x)|z|^q,$$

we can do better, we can replace the relation (1) with

$$1 < p < q < p + \varkappa,$$

where $\varkappa \in (0, +\infty)$, provided

$$a(x) \leqslant C\left[a(y) + |x-y|^{\varkappa}\right].$$

Contents

Chapter 1

Introduction

We start considering two metric spaces X and Y such that

$$Y \subset X \quad \text{and} \quad \overline{Y} = X$$

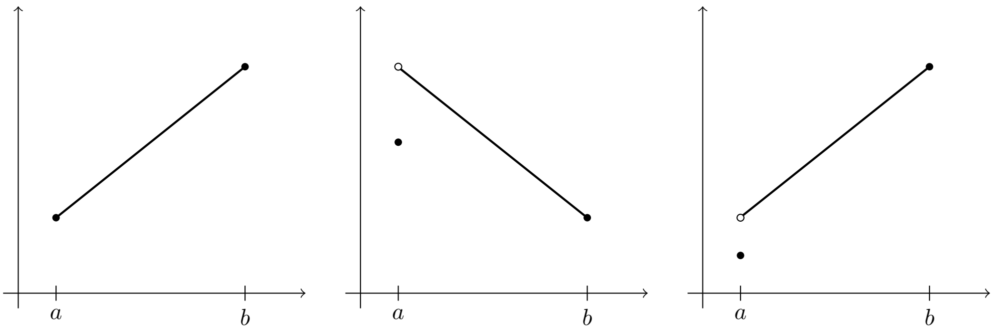
and a function $\mathcal{F} : X \to [0, +\infty]$. We obviously have that

$$\inf_X \mathcal{F} \leqslant \inf_Y \mathcal{F}$$

and if $\mathcal{F}$ is continuous the equality occurs. But if we replace the hypothesis of continuity with the lower semicontinuity then it could happen that

$$\inf_X \mathcal{F} < \inf_Y \mathcal{F}. \tag{1.1}$$

Let us look at the following figure.



For $0 < a < b$ we take

$$X = [a, b] \quad \text{and} \quad Y = (a, b).$$

In the first figure we have a continuous function. The following ones, instead, show two different lower semicontinuous functions. Then we can see how the lower semicontinuity of the last one implies that the infimum on $[a, b]$ is strictly less than the one on (a, b).

So the function has two different infimum values when considered on two spaces, one contained and dense in the other. In this thesis we aim to show the non occurence of this phenomenon for some weak lower semicontinuous functional $\mathcal{F} : X \to [0, +\infty]$ of the Calculus of Variations, where the spaces X and Y are functional spaces.

Let us introduce this phenomenon from the point of view of relaxation.

We consider a first countable topological space X and a functional $\mathcal{F} : X \to [0, +\infty]$. The sequentially lower semicontinuous (s.l.s.c.) envelope of $\mathcal{F}$ is defined as

$$\bar{\mathcal{F}}_X := \sup \{\mathcal{G} : X \to [0, +\infty] : \mathcal{G} \text{ s.l.s.c., } \mathcal{G} \leqslant \mathcal{F} \text{ on } X\}. \tag{1.2}$$

Analogously if Y is a dense subspace of X the s.l.s.c. envelope of $\mathcal{F}$ with respect to Y is

$$\bar{\mathcal{F}}_Y := \sup \{\mathcal{G} : X \to [0, +\infty] : \mathcal{G} \text{ s.l.s.c., } \mathcal{G} \leqslant \mathcal{F} \text{ on } Y\}. \tag{1.3}$$

We obviously have that

$$\bar{\mathcal{F}}_X(u) \leqslant \bar{\mathcal{F}}_Y(u) \quad \text{ for any } u \in X \tag{1.4}$$

and the strict inequality may occur.

Buttazzo and Mizel in [31] introduced the notion of Lavrentiev term, namely: for every $u \in X$

$$\mathcal{L}(u) := \begin{cases} \bar{\mathcal{F}}_Y(u) - \bar{\mathcal{F}}_X(u) & \text{if } \bar{\mathcal{F}}_X(u) < +\infty \\ \\ 0 & \text{if } \bar{\mathcal{F}}_X(u) = +\infty. \end{cases} \tag{1.5}$$

Moreover they also say that there is a Lavrentiev gap at u whenever $\mathcal{L}(u) > 0$. The functional $\bar{\mathcal{F}}_Y$, called relaxed functional, is an extension by lower semicontinuity of $\mathcal{F}$ to all of X. If $\mathcal{F}$ is lower semicontinuous, by definition we have that for all $u \in Y$

$$\mathcal{F}(u) = \bar{\mathcal{F}}_Y(u)$$

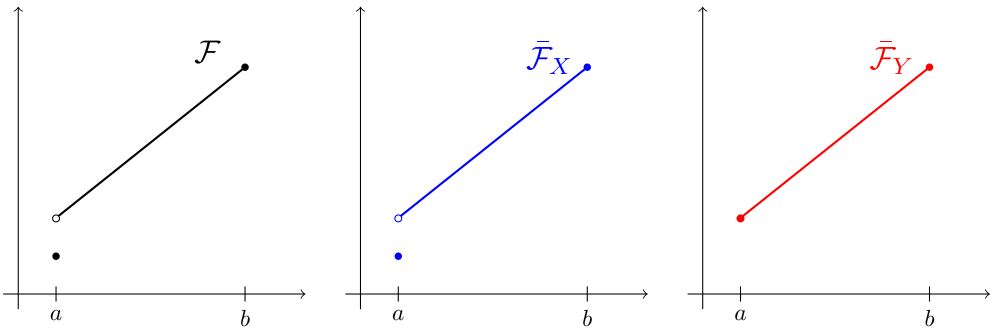
but it could happen that for some $u \in X \setminus Y$

$$\mathcal{F}(u) < \bar{\mathcal{F}}_Y(u),$$

in this case

$$\mathcal{L}(u) > 0.$$

In the following figure we can see that if $\mathcal{F}$ is lower semicontinuous then by definition $\bar{\mathcal{F}}_X$ must coincide with $\mathcal{F}$ on all of X. As far as $\bar{\mathcal{F}}_Y$ is concerned, we observe that since it is the greatest lower semicontinuous functionals that coincide with $\mathcal{F}$ on Y then $\bar{\mathcal{F}}_Y(a) > \mathcal{F}(a)$.



Adapting the view point of relaxation we have the following relaxation equality, see [29]

$$\inf\{\mathcal{F}(u) : u \in Y\} = \inf\{\bar{\mathcal{F}}_Y(u) : u \in X\}. \tag{1.6}$$

We are interested in showing that the functional $\mathcal{L}$ is equal to zero when computed on the minimizer $u \in X$ of the functional $\mathcal{F}$, in this case we get

$$\bar{\mathcal{F}}_Y(u) = \mathcal{F}(u).$$

Bearing in mind (1.6), we can conclude that $\mathcal{L}(u) = 0$ means that

$$\inf_{u \in X} \mathcal{F}(u) = \inf_{u \in Y} \mathcal{F}(u). \tag{1.7}$$

If we come back to the beginning of this introduction, we said that our aim is to show that: $\mathcal{F}$ has infimum on the given space X that is equal to the infimum on the dense subspace Y. This is nothing but (1.7). We say that the functional $\mathcal{F}$ does not present the Lavrentiev phenomenon if (1.7) holds.

In this thesis we want to prove the absence of the Lavrentiev phenomenon for some integral functionals $\mathcal{F}$ and for some functional spaces X and Y, $Y \subset X$ and $\overline{Y} = X$.

We shall mainly consider integral functionals of this form

$$\mathcal{F}(u) := \int_\Omega f(x, Du(x))\, dx,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $u : \Omega \to \mathbb{R}^N$, $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$, $n \geqslant 2$ and $N \geqslant 1$. In the framework of the Calculus of Variations the first example of Lavrentiev phenomenon is due to Lavrentiev [86]. Manià [90] simplified the example, he consided the functional

$$\mathcal{F}(u) = \int_0^1 (u^3 - x)^2 |u'|^6\, dx,$$

the spaces

$$X = \{u \in W^{1,1}([0,1], \mathbb{R}) :\ u(0) = 0,\ u(1) = 1\}$$
$$Y = \{u \in W^{1,\infty}([0,1], \mathbb{R}) :\ u(0) = 0,\ u(1) = 1\},$$

and he proved that

$$\inf_X \mathcal{F} = \mathcal{F}(x^{\frac{1}{3}}) = 0 < 7^2 3^5 2^{-18} 5^{-5} \leqslant \inf_Y \mathcal{F}.$$

As far as the one dimensional case is concerned, Ball and Mizel [10] added a coercive term to the Manià functional. Zhikov [107] treated a functional depending only on the variables x and $Du(x)$, in the case where $x \in \Omega \subset \mathbb{R}^2$. This example has been suitably generalized to the n-dimensional scalar case in [67]. It is worth to mention that, still in the vectorial case, there are examples exhibiting the phenomenon for functionals depending on (u, Du), see [5], and also depending just on Du, see [76]. We refer to the paper by Belloni and Buttazzo [18] (1995) for a more extensive list of references; as far as more recent examples are concerned, we just mention [8, 38, 50, 52, 69, 72, 101].

Another important direction of research is devoted to identify assumptions on the lagrangian f that imply the absence of Lavrentiev phenomenon. Lavrentiev himself took into account this problem proving that this phenomenon does not appear in one-dimensional problems for an integrand of the form $f(x, z)$, see [86]. Alberti and Serra Cassano [6] proved the non occurrence of the gap for any autonomous functional. An analogous result has been recently obtained for the multidimensional scalar case in [25] and [26]; see also [65] for the vectorial case. Further results in this framework can be found in [20, 98, 99]. The problem of identifying classes of functionals such that $\mathcal{L}(u)$ is equal to zero for every function $u \in X$ is less studied. For some results in this direction see, for example, [1, 18, 25, 26, 51, 92, 100].

The corresponding question for two and higher dimensional problems remained open for a very long time but now some results are avaiable, see [18, 30].

Let us now deal with the model density

$$f(x, z) = |z|^p + a(x)|z|^q, \tag{1.8}$$

where $1 < p < q < +\infty$ and the coefficient $a(x)$ is $C^{0,\alpha}(\Omega)$ and non-negative, $\alpha \in (0, 1]$. Then, according to Marcellini's terminology [94], the following (p, q)-growth is satisfied

$$|z|^p \leqslant f(x, z) \leqslant L(1 + |z|^q), \quad L \in [1, +\infty). \tag{1.9}$$

The main feature of the integrand f is the change of its growth according to the values assumed by the coefficient $a(x)$. Indeed, at points where $\{a(x) = 0\}$, f behaves like $|z|^p$ and we say that we are in the p-phase; on the other hand, at points where $\{a(x) > 0\}$ then f behaves like $|z|^q$, for large $|z|$, and we say that we are in the q-phase. Summarizing, we are dealing with a double phase functional. Model density (1.8) appeared for the first time in the Zhikov's pioneer paper [107] in the context of the study of Lavrentiev phenomenon. We remark that he considered the case: $n = 2$, $N = 1, \alpha = 1$ and $1 \leqslant p \leqslant 2$, $q > 3$. In [67] the authors showed that when the following condition is violated

$$\frac{q}{p} < \frac{n + \alpha}{n}, \tag{1.10}$$

precisely when

$$p < n < n + \alpha < q, \tag{1.11}$$

it is possible to find an example presenting the Lavrentiev phenomenon. This example is an extension to $n \geqslant 2$ of the Zhikov's one. Let us mention the recent paper [8] where p and q need not to verify $p < n < q$. We can also say that when p and q are as in (1.11) there exists a coefficient function $a(x) \in C^{0,\alpha}$ and a local minimizer u of the functional

$$\mathcal{P}(u) := \int_\Omega (|Du(x)|^p + a(x)|Du(x)|^q)\ dx$$

such that the set of its discontinuity points has a Hausdorff dimension arbitrarily close to $n - p$, see [71]. In other words, minimizers can be almost as bad as any other $W^{1,p}$-function.
The bound in (1.10) reflects in a sharp way the interaction between the exponents p, q of the growth condition and the regularity of the coefficient $a(x)$ that dictates the phase transition. This in turn relates to the kind of non-uniform ellipticity of the Euler-Lagrange equation:

$$-\text{div}A(x, Du) = 0 \tag{1.12}$$

where

$$A(x, z) = |z|^{p-2}z + \frac{q}{p}a(x)|z|^{q-2}z.$$

In fact when we evaluate (1.12) on the solution u then the non-uniform ellipticity is measured by the potential blow-up of the ratio

$$\frac{\text{highest eigenvalue of } \partial_z A(x, Du)}{\text{lowest eigenvalue of } \partial_z A(x, Du)} \approx 1 + a(x)|Du|^{q-p}. \tag{1.13}$$

Around the phase transition $\{a(x) = 0\}$ the ratio in (1.13) exhibits a potential blow-up, with respect to the gradient, of rate $q - p$; to compensate $a(x)$ is required to be suitably small. This means that since we are closed to $\{a(x) = 0\}$ then α must be large enough as required in (1.10).

Now let us ask the following question: assume that u makes finite the energy $\mathcal{P}$, then left hand side in (1.9) says that $Du \in L^p$; assume further that u minimizes the energy: does such a minimality condition boost the integrability up to $Du \in L^q$? A first account about regularity of minimizers was contained in the survey [102]. After some time the study of minimizers was further developed in [12, 42, 43]; starting from that, this case attracted a lot of interest: see [56, 57, 96, 103]. In [67] such a higher integrability result has been obtained for general densities $f(x, z)$ provided the two exponents p, q are close as prescribed in (1.10). Precisely, they assumed that: $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ has the growth condition (1.9) and, for some $\alpha \in (0, 1]$ and $\mu \in [0, 1]$, satisfies

$$L^{-1}(\mu^2 + |z_1|^2 + |z_2|^2)^{(p-2)/2}|z_1 - z_2|^2 \leqslant \left\langle \frac{\partial f}{\partial z}(x, z_1) - \frac{\partial f}{\partial z}(x, z_2); z_1 - z_2 \right\rangle \tag{1.14}$$

and
$$\left|\frac{\partial f}{\partial z}(x,z)-\frac{\partial f}{\partial z}(y,z)\right| \leqslant L|x-y|^{\alpha}(1+|z|^{q-1}), \tag{1.15}$$
where p and q are such that
$$1<p<q<p\left(\frac{n+\alpha}{n}\right). \tag{1.16}$$

As a last assumption, the Lavrentiev term (1.5) is assumed to vanish on the minimizer. Vanishing gap (1.5) has been checked in [67] for model density (1.8) as well as for other special cases, while in [68, 83] and [84] it has been satisfied in the case when the minimum point of $y \mapsto f(y,z)$ on small balls $\overline{B(x,\varepsilon)}$ is independent of z.

In the present thesis we aim to show the absence of Lavrentiev phenomenon in three different contexts. First of all, we deal with the functional
$$\mathcal{F}(u)=\int_{\Omega} f(x,Du(x))\,dx, \tag{1.17}$$
where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $u:\Omega \to \mathbb{R}^N$, $f:\Omega\times\mathbb{R}^{N\times n}\to\mathbb{R}$, $n \geqslant 2$ and $N \geqslant 1$. The density f satisfies the (p,q)-growth assumption (1.9), where p and q are as in (1.16), and shares the hypothesis (1.14) and (1.15) of the work [67]. But we no longer assume the vanishing of the Lavrentiev term (1.5) on the minimizers, as in [67]. On the contrary, we require that f can be approximated from below by a sequence f_k of convex functions, sharing the same hypothesis of f with the peculiarity that (1.15) holds also with p instead of q, with a constant c_k in place of L. We point out that c_k might blow up when $k\to+\infty$. Using such an approximating sequence f_k, we show the existence of suitable $W^{1,q}_{\mathrm{loc}}$ functions that do not increase the energy: more precisely for every function $u_* \in W^{1,p}(B_R,\mathbb{R}^N)$ with finite energy on B_R there exists
$$\tilde{u}_* \in (u_*+W^{1,p}_0(B_R,\mathbb{R}^N)) \cap W^{1,q}_{\mathrm{loc}}(B_R,\mathbb{R}^N))$$
with $\mathcal{F}(\tilde{u}_*) \leqslant \mathcal{F}(u_*)$, see Theorem 3.2.1. This shows the equality (1.7) with
$$X = u_0 + W^{1,p}_0(B_R,\mathbb{R}^N) \quad \text{and} \quad Y = X \cap W^{1,q}_{\mathrm{loc}}(B_R,\mathbb{R}^N),$$
where $u_0 \in W^{1,p}(B_R,\mathbb{R}^N)$ is a suitable boundary datum. The choice of Y is due to the fact that from the (p,q)-growth condition (1.9) we are interested in $W^{1,q}$-regularity so the space $W^{1,p}(B_R,\mathbb{R}^N) \cap W^{1,q}_{\mathrm{loc}}(B_R,\mathbb{R}^N)$ is the natural one. We specify that the case that attracts our interest is $p<q$, in fact for $p=q$ we have $\mathcal{L}(u)=0$ and there are many papers dealing with regularity issues for functionals with p-polynomial growth
$$|z|^p \leqslant f(x,z) \leqslant L(1+|z|^p), \quad p>1,$$
see [85, 88, 89] and the survey [102].

In the same context we prove in Theorem 3.2.5 that $\mathcal{L}(u_*)$ is zero *for any* $u_* \in$

$W^{1,p}(B_R,\mathbb{R}^N)$ with finite energy. We remark that since f is convex with respect to z, standard weak lower semicontinuity results give $\bar{\mathcal{F}}_X = \mathcal{F}$, see [74, Chapter 4]. The first step of the proof consists in considering a sequence $\{v_k\}_{k\in\mathbb{N}} \subset Y$ converging to $u_* \in X$ with respect to the strong topology of X. Then we introduce the perturbed functional

$$\mathcal{G}_k(u) := \int_{B_R} \left[f(x, Du(x)) + \frac{1}{k}(1 + k^2|Du(x) - Dv_k(x)|^2)^{\frac{p}{2}} \right] dx$$

and we apply Remark 3.2.4 to $\mathcal{G}_k$, so that $\mathcal{G}_k$ admits a minimizer u_k that belongs to Y. The proof is concluded showing that the sequence u_k converges strongly to u_* and approximates u_* in energy, i.e.:

$$\mathcal{F}(u_k) \to \mathcal{F}(u_*).$$

Let us make one last observation in this regard: under the (p,q)-growth condition (1.9), with $p < q$, the precise meaning of the integral $\mathcal{F}$ is not ambiguous if $u \in W^{1,q}_{\mathrm{loc}}$. On the contrary a priori it is not uniquely defined if $u \in W^{1,p} \setminus W^{1,q}_{\mathrm{loc}}$. In fact, a further definition being, for every $u \in W^{1,p}$

$$\bar{\mathcal{F}}(u) := \inf_{u_k} \left\{ \liminf_{k\to+\infty} \mathcal{F}(u_k) : u_k \in W^{1,p} \cap W^{1,q}_{\mathrm{loc}} \ \ \forall k \in \mathbb{N} \ \text{ and } u_k \rightharpoonup u \text{ in } W^{1,p} \right\},$$

where $\bar{\mathcal{F}}$ is exactly the relaxed functional of $\mathcal{F}$.

The second context we analyze is the following. We consider the Dirichlet problem in $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(A_i^\beta(x, Du(x))) = b^\beta(x) & \text{in } \ \Omega, \quad \beta = 1, \dots, N \\ u(x) = \tilde{u}(x) & \text{on } \ \partial\Omega, \end{cases} \tag{1.18}$$

where $A_i^\beta : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ are Carathédory functions, $b \in L^{p/(p-1)}(\Omega,\mathbb{R}^N)$ and $\tilde{u} \in W^{1,p\frac{q-1}{p-1}}(\Omega,\mathbb{R}^N)$. We set

$$A_i^\beta(x,z) = \frac{\partial f}{\partial z_i^\beta}(x,z)$$

and we assume that f satisfies (1.9), (1.14) and (1.15), with p and q in the following relation

$$2 \leqslant p \leqslant q < p\left(\frac{n+\alpha}{n}\right).$$

We aim to show that there exists a local minimizer u of the following non-autonomous energy integral

$$\mathcal{F}_b(u,\Omega) = \int_\Omega \left[f\left(x, Du(x)\right) + \langle b(x), u(x)\rangle \right] dx \tag{1.19}$$

for which the Lavrentiev phenomenon between the spaces $W^{1,p}$ and $W^{1,q}$ does not occur, more precisely

$$\inf\left\{ \mathcal{F}_b\left(v\right) : v \in u + W_0^{1,p}(\Omega',\mathbb{R}^N) \right\} = \inf\left\{ \mathcal{F}_b\left(v\right) : v \in u + W_0^{1,q}(\Omega',\mathbb{R}^N) \right\}, \tag{1.20}$$

for every $\Omega' \Subset \Omega$, see Theorem 4.1.2.

The idea to achieve this result is to apply [46, Theorem 1.1] in order to get a solution u of the Dirichlet problem (1.18) with a degree of integrability greater than q, i.e.

$$u \in (\tilde{u} + W_0^{1,p}(\Omega, \mathbb{R}^N)) \cap W_{\text{loc}}^{1,s}(\Omega, \mathbb{R}^N),$$

for all $q \leqslant s < p\left(\frac{n}{n-\alpha}\right)$. This in turn means that u is a weak solution to the following Euler system

$$\int_\Omega \sum_{\beta=1}^N \sum_{i=1}^n A_i^\beta(x, Du(x)) D_i \varphi^\beta(x)\ dx + \int_\Omega \sum_{\beta=1}^N b^\beta(x) \varphi^\beta(x)\ dx = 0,$$

for all $\varphi \in W^{1,q}(\Omega,\ \mathbb{R}^N)$ with $\text{supp}\varphi \Subset \Omega$. With the following further restriction on the exponents p and q

$$q < \frac{np - \alpha}{n - \alpha},$$

we get that u satisfies the Euler system for all test function $\varphi \in W^{1,p}(\Omega,\ \mathbb{R}^N)$ with $\text{supp}\varphi \Subset \Omega$. This will lead to the absence of the Lavrentiev phenomenon (1.20). We explore also the scalar case $N = 1$, see Theorem 4.2.2.

The last framework of this thesis is focused on the double phase functional

$$\mathcal{P}(u) = \int_\Omega (|Du(x)|^p + a(x)|Du(x)|^q)\ dx, \tag{1.21}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $n \geqslant 2$, $1 \leqslant p < q < \infty$ and weight $a : \Omega \to [0, +\infty)$ is bounded. Here we consider the space of smooth functions with compact support $C_c^\infty(\Omega, \mathbb{R})$ and the energy space

$$W(\Omega, \mathbb{R}) := \left\{ u \in W_0^{1,1}(\Omega, \mathbb{R}) : \int_\Omega M(x, |Du(x)|)\, dx < +\infty \right\} \tag{1.22}$$

endowed with a Luxemburg-type norm, where

$$M(x, t) = t^p + a(x) t^q.$$

Note that we have the inclusion $C_c^\infty \subset W$, and in turn

$$\inf_{u \in u_0 + W} \mathcal{F}(u) \leqslant \inf_{u \in u_0 + C_c^\infty} \mathcal{F}(u)\,.$$

We know that the above inequality might be strict, i.e.,

$$\inf_{u \in u_0 + W} \mathcal{F}(u) < \inf_{u \in u_0 + C_c^\infty} \mathcal{F}(u)\,, \tag{1.23}$$

which means that the Lavrentiev phenomenon between spaces C_c^∞ and W occurs. We mentioned before that the regularity of the possibly vanishing weight a dictates how far apart can be powers p and q to exclude (1.23). In particular, it is known that if

$$a \in C^{0,\alpha}(\Omega), \ \alpha \in (0,1], \ \text{ and } \ p < q \leqslant p + \alpha \max\left\{1, \frac{p}{n}\right\} \tag{1.24}$$

there is no Lavrentiev phenomenon, see [23, 28]. We wonder if is it possible to define a new class $\mathcal{Z}^\varkappa$, with $\varkappa \in (0,\infty)$, such that if $a \in \mathcal{Z}^\varkappa$ and p, q are in the relation

$$p < q \leqslant p + \varkappa$$

there is no Lavrentiev Phenomenon for the double phase functional between W and C_c^∞. The answer is positive. The weight a belongs to $\mathcal{Z}^\varkappa(\Omega)$, for $\varkappa \in (0,\infty)$, if there exists a positive constant C such that

$$a(x) \leqslant C\left(a(y) + |x-y|^\varkappa\right) \tag{1.25}$$

for all $x, y \in \Omega$. This means that no continuity or smoothness of a is required. Looking at (1.25) we can say that the key property of $a \in \mathcal{Z}^\varkappa$ is the decaying in the transition region, that needs to be at least like a power function with an exponent $\varkappa$, for $\varkappa \in (0,\infty)$. In other words, this approach proves the absence of Lavrentiev phenomenon extending the range between the exponents. In details, we show that the density of C_c^∞ in the space W, that is proved in Theorem 5.1.1, implies that $\mathcal{L}(u)$ is identically zero, see Theorem 5.1.3 and Remark 5.1.4.

1.1 Organization of the thesis

We begin displaying the notation and collecting some preliminaries in Chapter 2. Then, in Chapter 3, we present the results on the Lavrentiev gap, for the general functional (1.17), collected in the papers [58] and [60]. Based on the result of [59], in Chapter 4 we show the equality (1.20). We conclude analyzing the double phase functional (1.21) in the context of the Sobolev-type space (1.22); this result is contained in [22].

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Chapter 2

Preliminaries and notation

In this chapter we give some preliminary results specifying the notation we adopt. We divide these notions into two sections. In the first part we expose those useful for guaranteeing the absence of the Lavrentiev phenomenon in the setting of (1.17) and (1.19). While further on we provide information on the notation and basic tools used to prove the absence of Lavrentiev for the double phase functional (1.21) in the framework of Sobolev-type spaces.

2.1 General setting preliminaries

We denote by Ω an open, bounded subset of $\mathbb{R}^n$, $n \geqslant 2$, and we set

$$B_R \equiv B_R(x_0) \coloneqq \{x \in \mathbb{R}^n : |x - x_0| < R\},$$

where, unless differently specified, all the balls considered will have the same center. With c we denote a constant not necessarily the same in any two occurrences, the relevant dependence being emphasized. We specify that we deal with functional of type (1.17) where the density f is of Carathéodory type, it is convex in z and it satisfies a (p, q)-growth condition. We adopt the usual definition of local minimizer.

Definition 2.1.1. A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ if and only if $x \mapsto f(x, Du(x)) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_{\mathrm{supp}\ \varphi} f(x, Du(x))\ dx \leqslant \int_{\mathrm{supp}\ \varphi} f(x, Du(x) + D\varphi(x))\ dx,$$

for any $\varphi \in W^{1,1}(\Omega, \mathbb{R}^N)$ with supp $\varphi \Subset \Omega$.

2.1.1 Some useful lemmas

In what follows we will give several lemmas, used in the proof of Theorem 3.1.1, concerning functions belonging to fractional Sobolev spaces, more details can be found in [3]. Let us define the difference operator

$$\tau_{s,h} G(x) \coloneqq G(x + he_s) - G(x)$$

where $G : \mathbb{R}^n \to \mathbb{R}^k$, $h \in \mathbb{R}$, e_s is the unit vector in the x_s direction and $1 \leqslant s \leqslant n$. The following two lemmas are basic results about difference operator and weak derivatives.

Lemma 2.1.2. *If* $0 < \rho < R$, $|h| < R - \rho$, $1 \leqslant t < \infty$, $s \in \{1, \dots, n\}$ *and* $G, D_s G \in L^t(B_R)$, *then*

$$\int_{B_\rho} |\tau_{s,h} G(x)|^t \, dx \leqslant |h|^t \int_{B_R} |D_s G(x)|^t \, dx.$$

Lemma 2.1.3. *Let* $G \in L^2(B_R)$ *be such that*

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h} G(x)|^2 \, dx \leqslant M^2 |h|^2$$

for every $|h| < R - \rho$. *Then* $G \in W^{1,2}(B_\rho)$ *and*

$$||D_s G||_{L^2(B_\rho)} \leqslant M, \quad \forall s = 1, \dots, n.$$

For the following lemma we focus our attention on fractional Sobolev embedding theorem. We specify that this is a version localized on balls with an explicit dependence of the constant upon the radii ρ and R obtained by a suitable use of a cut-off function betweeen B_ρ and B_R. For the general version see [3].

Lemma 2.1.4. *Let* $G : \mathbb{R}^n \to \mathbb{R}^k$, $G \in L^2(B_R)$, $0 < R \leqslant 1$ *be such that*

$$\sum_{s=1}^{n} \int_{B_R} |\tau_{s,h} G(x)|^2 \eta^2(x) \, dx \leqslant M^2 |h|^{2d}$$

for some $\rho \in (0, R)$, $d \in (0,1)$, $M > 0$, $\eta : \mathbb{R}^n \to R$ *with* $\eta \in C_0^1(B_{\frac{\rho+R}{2}})$, $0 \leqslant \eta \leqslant 1$ *in* $\mathbb{R}^n$, $|D\eta| \leqslant \frac{4}{R-\rho}$ *in* $\mathbb{R}^n$, $\eta_{|B_\rho} = 1$ *and for all* h *with* $|h| \leqslant \frac{R-\rho}{4}$. *Then* $G \in W^{b,2}(B_\rho, \mathbb{R}^k) \cap L^{\frac{2n}{n-2b}}(B_\rho, \mathbb{R}^k)$ *for every* $b \in (0, d)$ *and*

$$||G||_{L^{\frac{2n}{n-2b}}(B_\rho)} \leqslant \frac{c}{(R-\rho)^{2b+2d+2}} (M + ||G||_{L^2(B_R)})$$

where $c \equiv c(n, k, b, d)$.

The following result is a consequence of Lemma 2.1 and 2.2 of [2].

Lemma 2.1.5. *For every* $p > 1$ *and* $G : B_R \to \mathbb{R}^k$ *we have*

$$|\tau_{s,h}((\mu^2 + |G(x)|^2)^{\frac{p-2}{4}} G(x))|^2 \leqslant c(k,p)(\mu^2 + |G(x)|^2 + |G(x + he_s)|^2)^{\frac{p-2}{2}} |\tau_{s,h} G(x)|^2$$

for all $x \in B_\rho$, $s = 1, \dots, n$, $|h| < R - \rho$, *with the constant* $c \equiv c(k,p)$ *independent of* μ, $0 \leqslant \mu \leqslant 1$.

For the proof of the next lemma see [74, Chapter 6, Section 3].

Lemma 2.1.6. *Let $h : [\rho, R_0] \to \mathbb{R}$ be a non-negative bounded function and $0 < \theta < 1$, $A \geqslant 0$, $\beta > 0$. Assume that*

$$h(r) \leqslant \frac{A}{(d-r)^\beta} + \theta h(d)$$

for $\rho \leqslant r < d \leqslant R_0$. Then

$$h(\rho) \leqslant \frac{cA}{(R_0 - \rho)^\beta}$$

where $c \equiv c(\theta, \beta) > 0$.

To prove in Theorem 3.2.5 that the Lavrentiev term is identically zero we use the following classical result.

Lemma 2.1.7. *Let us set*

$$X = W^{1,p}(B_R, \mathbb{R}^N) \quad \textit{and} \quad Y = W^{1,p}(B_R, \mathbb{R}^N) \cap W^{1,q}_{loc}(B_R, \mathbb{R}^N).$$

Let $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$ be sequentially lower semicontinuous with

$$\int_{B_R} |Du(x)|^p \, dx \leqslant \mathcal{F}(u). \tag{2.1}$$

Let $u \in X$ be such that $\mathcal{F}(u) < +\infty$. Then

$$\mathcal{L}(u) = 0$$

if and only if there exists a sequence

$$\{u_k\}_{k \in \mathbb{N}} \subset Y$$

such that $u_k \rightharpoonup u$ weakly in X and

$$\mathcal{F}(u_k) \to \mathcal{F}(u).$$

Proof. Let us start assuming that for every $u \in X$ there exists a sequence $\{\tilde{u}_k\}_{k \in \mathbb{N}} \in Y$ such that $\tilde{u}_k \rightharpoonup u$ weakly in X and $\mathcal{F}(\tilde{u}_k) \to \mathcal{F}(u)$. We distinguish two cases.

- If $\mathcal{F}(u) = +\infty$, since $\mathcal{F}$ is s.l.s.c. we have

$$\bar{\mathcal{F}}_X(u) = \mathcal{F}(u) = +\infty \quad \Longrightarrow \quad \mathcal{L}(u) = 0.$$

- If $\mathcal{F}(u) < +\infty$, using the sequentially lower semicontinuity of $\mathcal{F}$ and the definition (1) we get

$$\begin{aligned} \mathcal{F}(u) = \bar{\mathcal{F}}_X(u) \leqslant \bar{\mathcal{F}}_Y(u) \\ &= \inf_{u_k} \left\{ \liminf_{k \to +\infty} \mathcal{F}(u_k) : u_k \in Y \quad \forall k \in \mathbb{N} \text{ and } u_k \rightharpoonup u \text{ in } X \right\} \\ &\leqslant \liminf_{k \to +\infty} \mathcal{F}(\tilde{u}_k) = \mathcal{F}(u), \end{aligned}$$

  where the last inequality is due to the particular sequence $\{\tilde{u}_k\}_{k \in \mathbb{N}}$ for which we have $\mathcal{F}(\tilde{u}_k) \to \mathcal{F}(u)$; so $\mathcal{L}(u) = 0$.

Now let us suppose $\mathcal{L}(u) = 0$.

- If $\mathcal{F}(u) = +\infty$, we have
$$\bar{\mathcal{F}}_Y(u) = \bar{\mathcal{F}}_X(u) = +\infty.$$
So, for all $\{u_k\}_{k\in\mathbb{N}} \subset Y$ such that $u_k \rightharpoonup u$ in X it holds
$$\liminf_{k\to+\infty} \mathcal{F}(u_k) = +\infty.$$
Then, for every $M > 0$ there exists k_M such that for all $k \geqslant k_M$ we have
$$M \leqslant \inf_{s\geqslant k} \mathcal{F}(u_s) \leqslant \mathcal{F}(u_k).$$
Therefore,
$$\mathcal{F}(u_k) \to \mathcal{F}(u).$$

- If $\mathcal{F}(u) < +\infty$, then
$$\begin{aligned}\mathcal{F}(u) = \bar{\mathcal{F}}_X(u) = \bar{\mathcal{F}}_Y(u)&\\ = \inf_{u_k}\left\{\liminf_{k\to+\infty} \mathcal{F}(u_k) : u_k \in Y \quad \forall k \in \mathbb{N} \text{ and } u_k \rightharpoonup u \text{ in } X\right\},\end{aligned}$$
and so for all $r > 0$ there exists $u_{r,k} \xrightharpoonup[k\to+\infty]{X} u$ such that
$$\bar{\mathcal{F}}_Y(u) \leqslant \liminf_{k\to+\infty} \mathcal{F}(u_{r,k}) \leqslant \bar{\mathcal{F}}_Y(u) + \frac{1}{r}.$$
Consequently, there exists a subsequence $\tilde{u}_{r,k} := u_{r,s_k}$ such that $\tilde{u}_{r,k} \xrightharpoonup[k\to+\infty]{X} u$ and
$$\liminf_{k\to+\infty} \mathcal{F}(u_{r,k}) = \lim_{k\to+\infty} \mathcal{F}(\tilde{u}_{r,k}).$$
Now, let p' be the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$, and let $\{g_h\}_{h\in\mathbb{N}}$ be a sequence dense in $L^{p'}(B_R, \mathbb{R})$. For $r = 1$ we have
$$\bar{\mathcal{F}}_Y(u) \leqslant \lim_{k\to+\infty} \mathcal{F}(\tilde{u}_{1,k}) \leqslant \bar{\mathcal{F}}_Y(u) + \frac{1}{1},$$
$$\tilde{u}_{1,k} \xrightharpoonup[k\to+\infty]{X} u, \quad \tilde{u}_{1,k} \xrightarrow[k\to+\infty]{L^p} u$$
and
$$\left|\int_{B_R} g_1(D_i\tilde{u}^j_{1,k} - D_iu^j)\, dx\right| \xrightarrow[k\to\infty]{} 0,$$
for all $i = 1, \dots, n$ and $j = 1, \dots, N$. Then there exists $k_1 \geqslant 1$ such that
$$\bar{\mathcal{F}}_Y(u) - \frac{1}{1} \leqslant \mathcal{F}(\tilde{u}_{1,k_1}) \leqslant \bar{\mathcal{F}}_Y(u) + \frac{1}{1} + \frac{1}{1},$$

$$||\tilde{u}_{1,k_1} - u||_{L^p} \leqslant \frac{1}{1}$$

and

$$\left| \int_{B_R} g_1(D_i \tilde{u}^j_{1,k_1} - D_i u^j)\ dx \right| \leqslant \frac{1}{1},$$

for all $i = 1, \ldots, n$ and $j = 1, \ldots, N$. For $r = 2$ we have

$$\bar{\mathcal{F}}_Y(u) \leqslant \lim_{k \to +\infty} \mathcal{F}(\tilde{u}_{2,k}) \leqslant \bar{\mathcal{F}}_Y(u) + \frac{1}{2},$$

$$\tilde{u}_{2,k} \xrightarrow[k \to +\infty]{X} u, \quad \tilde{u}_{2,k} \xrightarrow[k \to +\infty]{L^p} u$$

and

$$\left| \int_{B_R} g_1(D_i \tilde{u}^j_{2,k} - D_i u^j)\ dx \right| \xrightarrow[k \to +\infty]{} 0, \quad \left| \int_{B_R} g_2(D_i \tilde{u}^j_{2,k} - D_i u^j)\ dx \right| \xrightarrow[k \to +\infty]{} 0,$$

for all $i = 1, \ldots, n$ and $j = 1, \ldots, N$. Then, there exists $k_2 > k_1$ such that

$$\bar{\mathcal{F}}_Y(u) - \frac{1}{2} \leqslant \mathcal{F}(\tilde{u}_{2,k_2}) \leqslant \bar{\mathcal{F}}_Y(u) + \frac{1}{2} + \frac{1}{2},$$

$$||\tilde{u}_{2,k_2} - u||_{L^p} \leqslant \frac{1}{2}$$

and

$$\left| \int_{B_R} g_1(D_i \tilde{u}^j_{2,k_2} - D_i u^j)\ dx \right| + \left| \int_{B_R} g_2(D_i \tilde{u}^j_{2,k_2} - D_i u^j)\ dx \right| \leqslant \frac{1}{2},$$

for all $i = 1, \ldots, n$ and $j = 1, \ldots, N$. Hence, iterating the process, we have $1 \leqslant k_1 < k_2 < \cdots < k_h$ such that

$$\bar{\mathcal{F}}_Y(u) - \frac{1}{h} \leqslant \mathcal{F}(\tilde{u}_{h,k_h}) \leqslant \bar{\mathcal{F}}_Y(u) + \frac{1}{h} + \frac{1}{h},$$

$$||\tilde{u}_{h,k_h} - u||_{L^p} \leqslant \frac{1}{h}$$

and

$$\left| \int_{B_R} g_s(D_i \tilde{u}^j_{h,k_h} - D_i u^j)\ dx \right| \leqslant \frac{1}{h},$$

for all $s = 1, \ldots, h$, $i = 1, \ldots, n$ and $j = 1, \ldots, N$. Let us set

$$w_h \coloneqq \tilde{u}_{h,k_h}.$$

We observe that, by (2.1)

$$\int_{B_R} |Dw_h(x)|^p\ dx \leqslant \mathcal{F}(w_h) \leqslant \bar{\mathcal{F}}_Y(u) + 2,$$

hence

$$||Dw_h||_{L^p(B_R)} \leqslant (\bar{\mathcal{F}}_Y(u) + 2)^{\frac{1}{p}}.$$

Moreover, for every $g \in L^{p'}(B_R, \mathbb{R})$ and for every $s \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that

$$||g - g_k||_{L^{p'}(B_R)} \leqslant \frac{1}{s}.$$

Therefore, for all $h \geqslant k$, for all $i = 1, \dots, n$ and $j = 1, \dots, N$, we have

$$\begin{aligned}\left|\int_{B_R} g(D_i w_h^j - D_i u^j)\,dx\right| &= \left|\int_{B_R} [g_k + (g - g_k)](D_i w_h^j - D_i u^j)\,dx\right| \\ &= \left|\int_{B_R} g_k(D_i w_h^j - D_i u^j)\,dx\right| + \left|\int_{B_R} (g - g_k)(D_i w_h^j - D_i u^j)\,dx\right| \\ &\leqslant \frac{1}{h} + ||g - g_k||_{L^{p'}(B_R)} ||Dw_h - Du||_{L^p(B_R)} \\ &\leqslant \frac{1}{h} + \frac{1}{s}\left[(\bar{\mathcal{F}}_Y(u) + 2)^{\frac{1}{p}} + ||Du||_{L^p(B_R)}\right] \\ &\leqslant \frac{1}{s}\left[1 + (\bar{\mathcal{F}}_Y(u) + 2)^{\frac{1}{p}} + ||Du||_{L^p(B_R)}\right],\end{aligned}$$

where the last inequality is in force if $h \geqslant s$. Recalling that $\bar{\mathcal{F}}_Y(u) < +\infty$, we just got that for every $g \in L^{p'}(B_R, \mathbb{R})$ it holds

$$\int_{B_R} g(D_i w_h^j - D_i u^j)\,dx \xrightarrow[h \to +\infty]{} 0,$$

for all $i = 1, \dots, n$ and $j = 1, \dots, N$. Then, summarizing, $\{w_h\}_{h \in \mathbb{N}} \subset Y$ and

$$\tilde{w}_h \xrightarrow[h \to +\infty]{X} u, \quad \mathcal{F}(w_h) \longrightarrow \mathcal{F}(u).$$

This ends the proof.

□

Next two lemmas are used in the proof of Theorem 4.1.1. The first one is a generalized version of Lemma 2.1.6.

Lemma 2.1.8. *Let $Z(t)$ be a bounded non-negative function in the interval $[\rho, R]$. Assume that for $\rho \leqslant r < d \leqslant R$*

$$Z(r) \leqslant [A(d - r)^{-\alpha} + B(d - r)^{-\beta} + C] + \theta Z(d),$$

with $A, B, C \geqslant 0$, $\alpha, \beta > 0$ and $0 \leqslant \theta < 1$. Then

$$Z(\rho) \leqslant c(\alpha, \theta)[A(R - \rho)^{-\alpha} + B(R - \rho)^{-\beta} + C].$$

Lemma 2.1.9. *Assume that (4.11) and (4.12) hold. Let $A_{\varepsilon,i}^{\beta}$ be defined as in (4.32), see Chapter 5. Then there exists $c > 0$ such that for all $\varepsilon \in (0, 1)$, for all $x \in \Omega$ and for all $z, \tilde{z} \in \mathbb{R}^{N \times n}$*

$$|z|^p \leqslant c\left(\sum_{\beta=1}^{N}\sum_{i=1}^{n} A_{\varepsilon,i}^{\beta}(x, z)(z_i^\beta - \tilde{z}_i^\beta) + (1 + |\tilde{z}|)^{\frac{p(q-1)}{p-1}}\right). \tag{2.2}$$

Proof. By (4.11)

$$\begin{aligned}|z|^p &\leqslant 2^{p-1}(|z-\tilde z|^{p-2}|z-\tilde z|^2+|\tilde z|^p)\\ &\leqslant c\left((|z|^2+|\tilde z|^2)^{\frac{p-2}{2}}|z-\tilde z|^2+|\tilde z|^p\right)\\ &\leqslant \tilde c\left(\sum_{\beta,i}(A^\beta_{\varepsilon,i}(x,z)-A^\beta_{\varepsilon,i}(x,\tilde z))(z^\beta_i-\tilde z^\beta_i)+|\tilde z|^p\right),\quad \text{for all } z,\tilde z\in\mathbb{R}^{N\times n},\end{aligned}$$

for some constant $\tilde c>0$. By (4.12) and the Young inequality

$$\tilde c\left|\sum_{\beta,i}A^\beta_{\varepsilon,i}(x,\tilde z)(z^\beta_i-\tilde z^\beta_i)\right|\leqslant c(M+1)(1+|\tilde z|)^{q-1}|z-\tilde z|\leqslant \frac12|z|^p+c(1+|\tilde z|)^{\frac{p(q-1)}{p-1}}.$$

Thus (2.2) follows. □

Lastly, we state and prove respectively [93, Lemma 2.1] and [75, Lemma 2.1].

Lemma 2.1.10 (Lemma 2.1 in [93])**.** *Let $\Omega\subset\mathbb{R}^n$ and $f:\Omega\times\mathbb{R}^{N\times n}\to\mathbb{R}$, $f=f(x,z)$, be a convex function with respect to z such that*

$$|f(x,z)|\leqslant L(1+|z|^q),\quad \textit{for all } z\in\mathbb{R}^{N\times n},\tag{2.3}$$

where $q\geqslant 1$ and L is a positive constant. Then

$$\left|\frac{\partial f}{\partial z}(x,z)\right|\leqslant c(1+|z|^{q-1}),\quad \textit{for all } z\in\mathbb{R}^{N\times n},$$

where $c\equiv c(n,N,q,L)$ is a positive constant.

Proof. The proof is obvious if $\frac{\partial f}{\partial z}(x,z)=0$. Let us suppose $\frac{\partial f}{\partial z}(x,z)\neq 0$. Since $z\mapsto f(x,z)$ is convex we have

$$f(x,\tilde z)\geqslant f(x,z)+\frac{\partial f}{\partial z}(x,z)(z-\tilde z),\tag{2.4}$$

for all $z,\tilde z\in\mathbb{R}^{N\times n}$. Now, let us choose $\tilde z:=z+h$, where

$$h:=(1+|z|^q)^{\frac1q}\frac{\frac{\partial f}{\partial z}(x,z)}{\left|\frac{\partial f}{\partial z}(x,z)\right|}.$$

Then, by (2.4),

$$\begin{aligned}(1+|z|^q)^{\frac1q}\left|\frac{\partial f}{\partial z}(x,z)\right| &= \frac{\partial f}{\partial z}(x,z)h\\ &\leqslant f(x,z+h)-f(x,z)\\ &\leqslant |f(x,z+h)|+|f(x,z)|\\ &\leqslant L(1+|z+h|^q)+L(1+|z|^q)\\ &\leqslant (2^q+1)2L(1+|z|^q).\end{aligned}$$

Consequently,

$$\begin{aligned}\left|\frac{\partial f}{\partial z}(x,z)\right| &\leqslant (2^q+1)2L(1+|z|^q)^{1-\frac{1}{q}}\\ &\leqslant 2^{3+q-\frac{1}{q}}L(1+|z|^{q-1}).\end{aligned}$$

□

Let us now define the function $f_\mu(z) := (\mu^2+|z|^2)^{\frac{p}{2}}$, where $p>1$ and $\mu\in[0,1]$. We set

$$V(z) := \frac{1}{p}\frac{\partial f_\mu}{\partial z}(z) = (\mu^2+|z|^2)^{\frac{p-2}{2}}z \tag{2.5}$$

and

$$W(t) := (\mu^2+t^2)^{\frac{p-2}{2}}t.$$

We observe that $|V(z)| = W(|z|)$. Now,

$$W'(t) = (\mu^2+t^2)^{\frac{p-4}{2}}[\mu^2+(p-1)t^2] \tag{2.6}$$

is positive for $t>0$, so W is increasing and since $W(0)=0$ and $\lim_{t\to+\infty}W(t)=\infty$, it follows that W maps $[0,+\infty)$ bijectively onto $[0,+\infty)$. This in turn implies that $V(z) = W(|z|)\frac{z}{|z|}$ is also bijective. We state and prove the following lemma that we will use in Chapter 3, section 3.4. We point out that the case that attracts our interest is $1<p<2$.

Lemma 2.1.11 (Lemma 2.1 in [75]). *There exist positive constants c_1 and c_2 depending only on $p>1$ such that, for all $z_1,z_2\in\mathbb{R}^{N\times n}$ with $z_1\neq z_2$, we have*

$$c_1 \leqslant \frac{|V(z_1)-V(z_2)|}{(\mu^2+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_1-z_2|} \leqslant c_2. \tag{2.7}$$

Proof. We just consider the case $1<p<2$, the proof of the lemma for $p>2$ is given in [73]. We start observing that (2.6) provides the estimate

$$(p-1)(\mu^2+t^2)^{\frac{p-2}{2}} \leqslant W'(t) \leqslant (\mu^2+t^2)^{\frac{p-2}{2}}. \tag{2.8}$$

Let us assume $0<s<t$ where $s=|z_1|$ and $t=|z_2|$. Considered as a function of $\theta := s^{-1}t^{-1}\langle z_1,z_2\rangle\in[-1,1]$, the square of the expression in (2.7)

$$\frac{W(s)^2+W(t)^2-2W(s)W(t)\theta}{(\mu^2+s^2+t^2)^{p-2}(s^2+t^2-2st\theta)}$$

attains its extremal values at the end points $\theta=-1$ and $\theta=1$. Therefore it is enough to estimate

$$\frac{W(t)\pm W(s)}{(\mu^2+s^2+t^2)^{\frac{p-2}{2}}(t\pm s)}.$$

Since $\mu^2 + s^2 + t^2 \leqslant \mu^2 + (s+t)^2$, we have

$$1 = \frac{(\mu^2 + s^2 + t^2)^{\frac{p-2}{2}} s + (\mu^2 + s^2 + t^2)^{\frac{p-2}{2}} t}{(\mu^2 + s^2 + t^2)^{\frac{p-2}{2}} (s+t)} \leqslant \frac{W(s) + W(t)}{W(s+t)} \leqslant 2.$$

Applying the Mean Value Theorem to the function W, we obtain

$$\frac{W(t) - W(s)}{(\mu^2 + s^2 + t^2)^{\frac{p-2}{2}} (t-s)} = \frac{W'(r)}{(\mu^2 + s^2 + t^2)^{\frac{p-2}{2}}} \geqslant \frac{(p-1)(\mu^2 + r^2)^{\frac{p-2}{2}}}{(\mu^2 + s^2 + t^2)^{\frac{p-2}{2}}} \geqslant p - 1,$$

for some r with $s < r < t$. We have also used (2.8) and $\mu^2 + r^2 < \mu^2 + s^2 + t^2$. If $3s \leqslant t$ then $t - s \geqslant \frac{1}{2}(t+s)$. Therefore

$$\frac{W(t) - W(s)}{(\mu^2 + s^2 + t^2)^{\frac{p-2}{2}} (t-s)} \leqslant \frac{W(t)}{(\mu^2 + (s+t)^2)^{\frac{p-2}{2}} \frac{1}{2}(t+s)} = \frac{2W(t)}{W(s+t)} \leqslant 2.$$

If $t < 3s$ then $s < r < t$ implies that $t < 3r$, so we have $\mu^2 + s^2 + t^2 < 10(\mu^2 + r^2)$. This gives

$$\frac{W(t) - W(s)}{(\mu^2 + s^2 + t^2)^{\frac{p-2}{2}} (t-s)} = \frac{W'(r)}{(\mu^2 + s^2 + t^2)^{\frac{p-2}{2}}} \leqslant \frac{(\mu^2 + r^2)^{\frac{p-2}{2}}}{(\mu^2 + s^2 + t^2)^{\frac{p-2}{2}}} \leqslant 10^{\frac{1}{2}|p-2|}.$$

Then (2.7) holds with $c_1 := p - 1$ and $c_2 := \max\left\{2, 10^{\frac{1}{2}|p-2|}\right\} \leqslant \sqrt{10}$. $\square$

It follows that, by the definition (2.5) of V and by (2.7), we have

$$p(p-1) \leqslant \frac{\left|\frac{\partial f_\mu}{\partial z}(z_1) - \frac{\partial f_\mu}{\partial z}(z_2)\right|}{(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} |z_1 - z_2|} \leqslant p\sqrt{10}.$$

With this lemma we conclude the preliminary part concerning the general setting.

2.2 Sobolev-type space preliminaries

Given a set $\Omega \subseteq \mathbb{R}^n$, $\gamma \in (0,1]$, and a function $f : \Omega \to \mathbb{R}$, we denote by

$$[f]_{0,\gamma} := \sup_{x,y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\gamma}, \tag{2.9}$$

the Hölder seminorm of f. We say that two real functions f, g are comparable, if there exists a constant $c > 0$ such that

$$f \leqslant g \leqslant cf.$$

We moreover say that the function $f : [0, +\infty) \to [0, +\infty)$ satisfies Δ_2-condition if there exists a constant $c > 0$ such that

$$f(2t) \leqslant cf(t),$$

for any t. We denote such situation by $f \in \Delta_2$. Let us introduce some basic facts concerning spaces of Musielak–Orlicz type [35, 36, 77]. With the function $M : \Omega \times [0, +\infty) \to \mathbb{R}$, given by

$$M(x,t) = t^p + a(x)t^q \qquad \text{for } \; p, q > 1\,, \quad 0 \leqslant a \in L^\infty\,,$$

we can define the corresponding Musielak–Orlicz space

$$L_M(\Omega) = \left\{ \xi : \Omega \to \mathbb{R}^n \text{ measurable and such that } \int_\Omega M(x, |\xi(x)|)\, dx < \infty \right\}$$

equipped with the Luxemburg norm

$$||\xi||_{L_M(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega M\left(x, \frac{|\xi(x)|}{\lambda}\right) dx \leqslant 1 \right\}.$$

Related Sobolev space $W(\Omega) = \left\{ u \in W_0^{1,1}(\Omega, \mathbb{R}) : \int_\Omega M(x, |Du(x)|)\, dx < \infty \right\}$ is considered with the norm

$$||u||_{W(\Omega)} := ||u||_{L^1(\Omega)} + ||Du||_{L_M(\Omega)}.$$

We say that, a sequence $\{\xi_k\}_{k\in\mathbb{N}}$ converges to ξ modularly in $L_M(\Omega)$ if

$$\int_\Omega M(x, |\xi_k - \xi|)\, dx \xrightarrow{k\to\infty} 0\,, \tag{2.10}$$

and we denote it by $\xi_k \xrightarrow[k\to\infty]{M} \xi$. We mention the Generalized Vitali Convergence Theorem from the [36, Theorem 3.4.4], stating that

$$\begin{aligned} \xi_k \to \xi \text{ modularly} \iff & \text{the family } \{M(x, |\xi_k(x)|)\}_{k\in\mathbb{N}} \text{ is uniformly integrable} \\ & \text{and } \{\xi_k\}_{k\in\mathbb{N}} \text{ converges in measure to } \xi. \end{aligned} \tag{2.11}$$

By the choice of M, it is equivalent to say that the sequence $\{u_k\}_{k\in\mathbb{N}} \subset W(\Omega, \mathbb{R})$ converges to $u \in W(\Omega, \mathbb{R})$ in the strong topology of $W(\Omega, \mathbb{R})$ and that

$$u_k \xrightarrow[k\to\infty]{L^1} u \text{ in } L^1(\Omega) \quad \text{and} \quad Du_k \xrightarrow[k\to\infty]{M} Du \text{ modularly.} \tag{2.12}$$

In order to prove that every $u \in W$ can be approximated in W-norm by a sequence in C_c^∞ we can just consider $u \in W \cap L^\infty$. Indeed, by the following lemma we have that $W \cap L^\infty$ is dense in W.

Lemma 2.2.1. *The space $W(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$ is dense in $W(\Omega, \mathbb{R})$.*

Proof. Let $u \in W(\Omega, \mathbb{R})$. Consider truncation of u defined as

$$T_k(u) := \begin{cases} u & \text{if } |u| \leqslant k \\ k\dfrac{u}{|u|} & \text{if } |u| > k. \end{cases}$$

Clearly, $T_k(u) \in L^\infty(\Omega, \mathbb{R})$. Moreover, chain rule for Sobolev maps implies that $DT_k(u) = Du\mathbb{1}_{|u|\leqslant k}$ so that $DT_k(u) \to Du$ almost everywhere (a.e.) as $k \to \infty$. As $M(x, 0) = 0$ we have that

$$0 \leqslant M(x, |DT_k(u)|) = M(x, |Du|)\mathbb{1}_{|u|\leqslant k} \leqslant M(x, |Du|),$$

so that the sequence $\{M(x, |DT_k(u)|)\}_{k\in\mathbb{N}}$ is uniformly integrable. Then by (2.11) and (2.12) we conclude the proof. $\square$

Now we introduce the approximation method by convolution with shrinking. This method is of use in many papers concerning the absence of the Lavrentiev phenomenon and density of smooth functions in Musielak–Orlicz–Sobolev spaces, see [4, 21, 23, 28]. Let us fix $n, m \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^n$ be a bounded star-shaped domain with respect to a ball $B(x_0, R)$. For $\delta > 0$ define $\kappa_\delta = 1 - \frac{\delta}{R}$. Moreover, let ρ_δ be a standard regularizing kernel on $\mathbb{R}^n$, that is $\rho_\delta(x) = \rho(x/\delta)/\delta^n$, where $\rho \in C^\infty(\mathbb{R}^n)$, $\operatorname{supp}\rho \Subset B(0, 1)$ and $\int_{\mathbb{R}^n} \rho(x)\, dx = 1$, $\rho(x) = \rho(-x)$, such that $0 \leqslant \rho \leqslant 1$. Then for any measurable function $v : \mathbb{R}^n \to \mathbb{R}^m$, we define the function $S_\delta v : \mathbb{R}^n \to \mathbb{R}^m$ by

$$S_\delta v(x) := \int_\Omega \rho_\delta(x - y)v\left(x_0 + \frac{y - x_0}{\kappa_\delta}\right) dy = \int_{B_\delta(0)} \rho_\delta(y)v\left(x_0 + \frac{x - y - x_0}{\kappa_\delta}\right) dy\,. \tag{2.13}$$

By direct computations, one can show that $S_\delta v$ has a compact support in Ω for $\delta \in (0, R/4)$. Moreover, we observe that for $v \in W(\Omega, \mathbb{R})$ it holds that

$$DS_\delta v = \tfrac{1}{\kappa_\delta} S_\delta(Dv)\,. \tag{2.14}$$

We introduce other useful properties of this approximation in the next two lemmas.

Lemma 2.2.2 (Lemma 3.1 in [23])**.** *If $v \in L^1(\Omega, \mathbb{R})$, then $S_\delta v$ converges to v in $L^1(\Omega)$, and so in measure, as $\delta \to 0$.*

Proof. Without loss of generality, we assume that Ω is star-shaped with respect to a ball $B(0, R)$. For the general case we can change variables moving the center of the ball to the origin and applying the result for this case, then we reverse the change of variables. We start observing that

$$\begin{aligned} ||S_\delta v - v||_{L^1} &= \int_{\mathbb{R}^n} \left| \int_{B_\delta(0)} \rho_\delta(y)v\left(\frac{x - y}{k_\delta}\right) dy - v(x)\right| dx \\ &\leqslant \int_{\mathbb{R}^n} \int_{B_\delta(0)} \rho_\delta(y) \left|v\left(\frac{x - y}{k_\delta}\right) - v(x)\right| dy\, dx. \end{aligned}$$

Now, for some ball $B = B(0,r)$, $r > 0$, taking a function $g \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$, we have

$$\begin{aligned}
||S_\delta g - g||_{L^1} &\leqslant \int_B \int_{B_\delta(0)} \rho_\delta(y) \left| g\left(\frac{x-y}{k_\delta}\right) - g(x) \right| dy\, dx \\
&\leqslant ||Dg||_{L^\infty} \int_B \int_{B_\delta(0)} \rho_\delta(y) \left(|x| \left(\frac{1}{k_\delta} - 1\right) + \frac{|y|}{k_\delta} \right) dy\, dx \\
&\leqslant ||Dg||_{L^\infty} \int_B \int_{B_\delta(0)} \rho_\delta(y) \left(r \left(\frac{1}{k_\delta} - 1\right) + \frac{\delta}{k_\delta} \right) dy\, dx \\
&\leqslant ||Dg||_{L^\infty} V_n^r \left(r \left(\frac{1}{k_\delta} - 1\right) + \frac{\delta}{k_\delta} \right) \xrightarrow[\delta \to 0]{} 0.
\end{aligned}$$

where V_n^r is the volume of the n-ball $B(0,r)$. We fix any $\varepsilon > 0$ and take $g \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ such that $||v - g||_{L^1} < \varepsilon$. By Young inequality, it holds that

$$\begin{aligned}
||S_\delta v - v||_{L^1} &\leqslant ||S_\delta v - S_\delta g||_{L^1} + ||S_\delta g - g||_{L^1} + ||g - v||_{L^1} \\
&\leqslant k_\delta^n \varepsilon + ||S_\delta g - g||_{L^1} + \varepsilon \xrightarrow[\delta \to 0]{} 2\varepsilon.
\end{aligned}$$

By taking $\varepsilon \to 0$, we obtain $||S_\delta v - v||_{L^1} \xrightarrow[\delta \to 0]{} 0$. Since convergence in L^1 implies convergence in measure we just proved that $\lim_{\delta \to 0} S_\delta v = v$. $\square$

Lemma 2.2.3 (Lemma 3.3 in [23]). *Let $v \in W_0^{1,1}(\Omega, \mathbb{R})$, where Ω is a star-shaped domain with respect to a ball $B(x_0, R)$. It holds that*

- *if $v \in L^\infty(\Omega, \mathbb{R})$, then*
$$||DS_\delta(v)||_{L^\infty} \leqslant \delta^{-1} ||v||_{L^\infty} ||D\rho||_{L^1}; \tag{2.15}$$
- *if $v \in C^{0,\gamma}(\Omega, \mathbb{R})$, $\gamma \in (0,1]$, then*
$$||DS_\delta(v)||_{L^\infty} \leqslant \frac{\delta^{\gamma-1}}{\kappa_\delta^\gamma} [v]_{0,\gamma} ||D\rho||_{L^1}. \tag{2.16}$$

Proof. As in Lemma 2.2.2 without loss of generality we assume that $x_0 = 0$. We start proving (2.15). Note that $DS_\delta(v) = v\left(\frac{x}{k_\delta}\right) * (D\rho_\delta)$, therefore by Hölder inequality we obtain

$$\begin{aligned}
||DS_\delta v||_{L^\infty} &\leqslant ||v||_{L^\infty} \int_{\mathbb{R}^n} |D\rho_\delta(x)|\, dx \\
&= ||v||_{L^\infty} \int_{\mathbb{R}^n} \delta^{-n-1} \left| D\rho\left(\frac{x}{\delta}\right) \right| dx \\
&= \delta^{-1} ||v||_{L^\infty} ||D\rho||_{L^1},
\end{aligned}$$

which is (2.15). To prove (2.16) we first observe that $DS_\delta(v) = \frac{1}{k_\delta} S_\delta(Dv)$. Let us fix any $x \in \Omega$ and denote $\tilde{v}(y) := v(y) - v(x/k_\delta)$. We have

$$DS_\delta(v) = \frac{1}{k_\delta} S_\delta(Dv) = \frac{1}{k_\delta} S_\delta(D\tilde{v}) = DS_\delta(\tilde{v}).$$

Therefore, it holds that

$$\begin{aligned}|DS_\delta(v)(x)| &\leqslant \int_\Omega \left|\tilde{v}\left(\frac{y}{k_\delta}\right)\right| |(D\rho_\delta)(y-x)|\, dy \\ &= \int_{B(x,\delta)} \left|\tilde{v}\left(\frac{y}{k_\delta}\right)\right| |(D\rho_\delta)(y-x)|\, dy \\ &\leqslant \left(\frac{\delta}{k_\delta}\right)^\gamma [v]_{0,\gamma} ||D\rho_\delta||_{L^1}, \end{aligned} \tag{2.17}$$

where we used the fact that

$$\left|\tilde{v}\left(\frac{y}{k_\delta}\right)\right| = \left|v\left(\frac{y}{k_\delta}\right) - v\left(\frac{x}{k_\delta}\right)\right| \leqslant \left(\frac{|x-y|}{k_\delta}\right)^\gamma [v]_{0,\gamma}, \quad \text{for } y \in B(x,\delta).$$

To end the proof, we observe that $||D\rho_\delta||_{L^1} = \delta^{-1}||D\rho||_{L^1}$ and hence by (2.17) we obtain (2.16). $\square$

Chapter 3

Non occurrence of Lavrentiev gap for a class of functionals with non-standard growth

In this chapter we prove the absence of the Lavrentiev phenomenon for the following non-autonomous functional

$$\mathcal{F}(u) := \int_\Omega f(x, Du(x))\ dx, \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $u : \Omega \to \mathbb{R}^N$, $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$, $n \geqslant 2$ and $N \geqslant 1$. The density $f(x, z)$ satisfies a (p, q)-growth condition with respect to z and can be approximated, from below, by means of a suitable sequence of functions f_k. We consider $B_R \Subset \Omega$ and the spaces

$$X = u_0 + W_0^{1,p}(B_R, \mathbb{R}^N) \quad \text{and} \quad Y = X \cap W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N),$$

where $u_0 \in W^{1,p}(B_R, \mathbb{R}^N)$ is a suitable boundary value. We also prove that the lower semicontinuous envelope $\bar{\mathcal{F}}_Y$ coincides with $\mathcal{F}$ or, in other words, that the Lavrentiev term is equal to zero for any admissible function $u \in W^{1,p}(B_R, \mathbb{R}^N)$. We perform the approximations by means of functions preserving the values on the boundary of B_R.

Let us compare the present approach with the one in [67]. Both of them approximate the original density $f(x, z)$. In [67] such an approximation is performed from above by means of $f_\sigma(x, z) = f(x, z) + \sigma|z|^q$; in this way $f_\sigma(x, z)$ has the same growth q from below and from above; after minimizing the corresponding integral energy

$$\mathcal{F}_\sigma(u) = \int f_\sigma(x, Du(x))\, dx,$$

the authors find good estimates for minimizer u_σ; the difficult task is to show that $\mathcal{F}_\sigma(u_\sigma) \to \mathcal{F}(u)$: for this reason in [67] it is assumed that $\mathcal{L}(u) = 0$. On the other hand, in the present paper, we perform the approximation from below by means of f_k. After minimizing the integral energy

$$\mathcal{F}_k(u) := \int f_k(x, Du(x))\,dx, \tag{3.2}$$

we find good estimates for minimizer u_k; now it is easy to show that $\mathcal{F}_k(u_k) \to \mathcal{F}(u)$. At this point the question arises: when is such an approximating sequence f_k available? We give a first answer in Theorem 3.4.1 in the case of $f(x, z) = \tilde{f}(x, |z|)$, for some $\tilde{f} : \Omega \times [0, +\infty) \to \mathbb{R}$.

Let us mention that approximation from below has been used in [37, 44, 45, 62]. Specifically in [37, 44, 45] and [62] the approximation is made with C^2 functions. On the contrary, in our proof we need not C^2 approximation: our f_k is only C^1; we need p growth from below and from above, with $f_k \leqslant f_{k+1}$. Approximation from above is easier when f does not depend on x: $f(x, z) = f(z)$; see [66].

Let us give a last remark. The strict inequality $q < p\left(\frac{n+\alpha}{n}\right)$ in our assumption (1.16) is used in the proof when checking (3.8). As far as the bordeline case $q = p\left(\frac{n+\alpha}{n}\right)$ is concerned, we mention [12] where the authors are able to deal with densities f satisfing

$$c_1(|z|^p + a(x)|z|^q) \leqslant f(x, u, z) \leqslant c_2(|z|^p + a(x)|z|^q)$$

for suitable positive constants c_1, c_2; see also [55].

The present chapter is divided into five section. In the first one we get an a priori estimate for the minimizer of the approximating functional (3.2). Then we pass to prove the absence of the Lavrentiev Phenomenon for (3.1). This result is then properly used to prove that $\mathcal{L}(u) = 0$ for every $u \in W^{1,p}$ by means of a perturbation of the functional (3.1). In Section 3.3 we apply the penalizing method to prove that the gap is identically zero for other functionals: in particular those previously considered in [70] and [37]. In the first case a vectorial problem set in Morrey spaces is considered. In the second one we are concerned with a multidimensional scalar problem whose lagrangian is of sum type. In Section 3.4 we explain how to get the approximating functions f_k in the case of a radial lagrangian. We conclude with an example of density for which we do not have the Lavrentiev gap.

3.1 A priori estimate

The theorem presented in this section allows us to obtain an a priori estimate for the minimizer $u_k \in W^{1,p}(B_R, \mathbb{R}^N)$ of the functional

$$\mathcal{F}_k(u) = \int_{B_R} f_k(x, Du(x))\,dx.$$

For simplicity of notation and because the result is applicable to any density satisfing the hypothesis of the theorem (that is to say, not necessarily for approximating functions) we will not write the index k. The a priori estimate is obtained by using fractional differentiability and it is inspired by Theorem 4 in [67].

Theorem 3.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\alpha \in (0,1]$ and p, q be such that*

$$1 < p < q < p\left(\frac{n+\alpha}{n}\right).$$

Let $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Carathéodory function. Assume that there exist $L \in [1, +\infty)$, $M \in (0, +\infty)$ and $\mu \in [0,1]$ such that

$$z \mapsto f(x,z) \in C^1(\mathbb{R}^{N\times n}), \quad \text{for all } x \in \Omega, \tag{$\tilde{\mathrm{H}}1$}$$

$$\tilde{L}^{-1}|z|^p \leqslant f(x,z) \leqslant \tilde{L}(1+|z|^q), \tag{$\tilde{\mathrm{H}}2$}$$

$$\tilde{L}^{-1}(\mu^2+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_1-z_2|^2 \leqslant \left\langle \frac{\partial f}{\partial z}(x,z_1) - \frac{\partial f}{\partial z}(x,z_2), z_1 - z_2 \right\rangle, \tag{$\tilde{\mathrm{H}}3$}$$

$$\left|\frac{\partial f}{\partial z}(x,z) - \frac{\partial f}{\partial z}(y,z)\right| \leqslant \tilde{L}|x-y|^\alpha(1+|z|^{q-1}), \tag{$\tilde{\mathrm{H}}4$}$$

$$\left|\frac{\partial f}{\partial z}(x,z)\right| \leqslant M(1+|z|^{p-1}), \tag{$\tilde{\mathrm{H}}5$}$$

$$\left|\frac{\partial f}{\partial z}(x,z) - \frac{\partial f}{\partial z}(y,z)\right| \leqslant M|x-y|^\alpha(1+|z|^{p-1}). \tag{$\tilde{\mathrm{H}}6$}$$

Assume that $u \in W^{1,p}(B_R, \mathbb{R}^N)$ is a minimizer of

$$\mathcal{F}(u) = \int_{B_R} f(x, Du(x))\,dx$$

with a suitable boundary datum, where $B_R \Subset \Omega, R \in (0,1]$.

Then, for all $t \in \left[p, \frac{np}{n-\alpha}\right)$ and all $\rho \in (0,R)$ there exist two constants

$$c \equiv c(n, N, q, p, \tilde{L}, R, \rho, t, \alpha), \quad \beta \equiv \beta(n, q, p, t, \alpha),$$

belonging to $(0, +\infty)$, such that

$$\int_{B_\rho} |Du(x)|^t\,dx \leqslant c\left(\int_{B_R} [f(x, Du(x)) + 1]\,dx\right)^\beta. \tag{3.3}$$

Remark 3.1.2. It is important to note that the constant c and the exponent β in (3.3) do not depend on the constant M appearing in ($\tilde{\mathrm{H}}5$) and ($\tilde{\mathrm{H}}6$), this will be used later.

Proof of Theorem 3.1.1. First of all we observe that the hypothesis ($\tilde{\mathrm{H}}2$) and ($\tilde{\mathrm{H}}3$) imply that there exists a constant $\tilde{c} \equiv \tilde{c}(n, N, q, \tilde{L})$ such that

$$\left|\frac{\partial f}{\partial z}(x,z)\right| \leqslant \tilde{c}(1+|z|^{q-1}), \tag{$\tilde{\mathrm{H}}7$}$$

for the proof of this inequality see Lemma 2.1.10.

We divide the proof into three steps.

<u>Step 1</u>: weak Euler-Lagrange equation for the minimizer $u \in W^{1,p}(B_R, \mathbb{R}^N)$

Note first that $z \mapsto f(x,z) \in C^1(\mathbb{R}^{N\times n}), \forall x \in \Omega$, then

$$\begin{aligned}
|f(x,z)| &= \left| f(x,0) + \int_0^1 \frac{\partial}{\partial t}\left(f(x,tz)\right)\, dt \right| \\
&\leqslant |f(x,0)| + \int_0^1 \left|\frac{\partial f}{\partial z}(x,tz)\right| |z|\, dt \\
&\leqslant |f(x,0)| + \int_0^1 M(1+|tz|^{p-1})|z|\, dt \\
&\leqslant |f(x,0)| + M|z| + M|z|^p.
\end{aligned}$$

The above inequality, together with hypothesis ($\tilde{\mathrm{H}}5$), leads to the following weak form of the Euler-Lagrange equation for $u \in W^{1,p}(B_R, \mathbb{R}^N)$

$$\int_{B_R} \sum_{i=1}^{N}\sum_{j=1}^{n} \left[\frac{\partial f}{\partial z_j^i}(x, Du(x)) \frac{\partial \varphi^i}{\partial x_j}(x)\right] dx = 0, \quad \forall \varphi \in W_0^{1,p}(B_R, \mathbb{R}^N). \tag{3.4}$$

<u>Step 2</u>: the minimizer $u \in W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N)$

We pick $0 < \rho \leqslant r < d \leqslant R \leqslant 1$ and a cut-off function $\eta \in C_0^\infty(B_{\frac{d+r}{2}})$ such that $0 \leqslant \eta \leqslant 1$ and

$$\eta_{|B_r} \equiv 1, \quad |D\eta| \leqslant \frac{4}{d-r}.$$

We choose $s \in \{1, \dots, n\}$ and $h \in \mathbb{R}$:

$$0 < |h| \leqslant \frac{d-r}{4}.$$

Now we substitute $\varphi := \tau_{s,-h}(\eta^2 \tau_{s,h} u)$ into (3.4) and deduce

$$\underbrace{\int_{B_R} \eta^2 \tau_{s,h}\left(\frac{\partial f}{\partial z}(x, Du)\right) \tau_{s,h} Du\, dx}_{\mathrm{I}}$$
$$= \underbrace{- \int_{B_R} \tau_{s,h}\left(\frac{\partial f}{\partial z}(x, Du)\right) 2\eta D\eta \otimes \tau_{s,h} u\, dx}_{\mathrm{II}}\, .$$

We point out that such a φ is admissible because of the $p-$growth condition of f itself and $u \in W^{1,p}(B_R, \mathbb{R}^N)$.

Now we study I:

$$\begin{aligned}
&\int_{B_R} \eta^2 \tau_{s,h}\left(\frac{\partial f}{\partial z}(x, Du)\right) \tau_{s,h} Du\, dx \\
&\quad = \int_{B_R} \eta^2 \left[\frac{\partial f}{\partial z}(x + he_s, Du(x + he_s)) - \frac{\partial f}{\partial z}(x, Du(x))\right] \tau_{s,h} Du\, dx \\
&\quad = \int_{B_R} \eta^2 \left[\frac{\partial f}{\partial z}(x + he_s, Du(x + he_s)) - \frac{\partial f}{\partial z}(x, Du(x + he_s))\right] \tau_{s,h} Du\, dx \\
&\qquad + \int_{B_R} \eta^2 \left[\frac{\partial f}{\partial z}(x, Du(x + he_s)) - \frac{\partial f}{\partial z}(x, Du(x))\right] \tau_{s,h} Du\, dx.
\end{aligned}$$

In the following we estimate the terms above. We start with

$$\begin{aligned}
&\int_{B_R} \eta^2 \left[\frac{\partial f}{\partial z}(x, Du(x + he_s)) - \frac{\partial f}{\partial z}(x, Du(x))\right] \tau_{s,h} Du\, dx \\
&\overset{(\tilde{\mathrm{H}}3)}{\geqslant} \int_{B_R} \eta^2 [\tilde{L}^{-1} (\mu^2 + |Du(x + he_s)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} |\tau_{s,h} Du|^2]\, dx \\
&\geqslant \int_{B_R} c(N, p, \tilde{L}) \eta^2 |\tau_{s,h}[(\mu^2 + |Du(x)|^2)^{\frac{p-2}{4}} Du(x)]|^2\, dx,
\end{aligned} \tag{E1}$$

where for the last inequality we have used Lemma 2.1.5. Now let us move on with

$$\begin{aligned}
&\left|\int_{B_R} \eta^2 \left[\frac{\partial f}{\partial z}(x + he_s, Du(x + he_s)) - \frac{\partial f}{\partial z}(x, Du(x + he_s))\right] \tau_{s,h} Du\, dx\right| \\
&\overset{(\tilde{\mathrm{H}}6)}{\leqslant} M \left\{\int_{B_R} \eta^2 [1 + |Du(x + he_s)|^{p-1}] |\tau_{s,h} Du(x)|\, dx\right\} |h|^\alpha \\
&\leqslant c(n, N, p, M) \left\{\int_{B_{\frac{d+r}{2}}} [1 + |Du(x)|^p + |Du(x + he_s)|^p]\, dx\right\} |h|^\alpha \\
&\leqslant c(n, N, p, M) \left\{\int_{B_d} (1 + |Du(x)|^p)\, dx\right\} |h|^\alpha.
\end{aligned} \tag{E2}$$

We pass to the one we have labelled II

$$
\begin{aligned}
&\left|\int_{B_R}\tau_{s,h}\left(\frac{\partial f}{\partial z}(x,Du)\right)2\eta D\eta\otimes\tau_{s,h}u\,dx\right| \\
&\overset{(\tilde{H}5)}{\leqslant} c(n,N,M)\int_{B_R}\eta|D\eta|[1+|Du(x)|^{p-1}+|Du(x+he_s)|^{p-1}]|\tau_{s,h}u(x)|\,dx \\
&\leqslant \frac{c(n,N,M)}{d-r}\left\{\int_{B_{\frac{d+r}{2}}}[1+|Du(x)|^p+|Du(x+he_s)|^p]\,dx\right\}^{\frac{p-1}{p}} \\
&\quad\times\left\{\int_{B_{\frac{d+r}{2}}}|\tau_{s,h}u(x)|^p\,dx\right\}^{\frac{1}{p}} \qquad \text{(Hölder inequality)} \\
&\leqslant \frac{c(n,N,p,M)}{d-r}\left\{\int_{B_d}(1+|Du(x)|^p)\,dx\right\}|h|^\alpha, \qquad \text{(E3)}
\end{aligned}
$$

where for the last inequality we have used Lemma 2.1.2. Finally we sum up on $s\in\{1,\cdots,n\}$ and obtain

$$
\begin{aligned}
&\int_{B_R}\sum_{s=1}^n|\tau_{s,h}[(\mu^2+|Du(x)|^2)^{\frac{p-2}{4}}Du(x)]|^2\eta^2\,dx \\
&\qquad\leqslant\frac{c(n,N,p,\tilde{L},M)}{d-r}\left\{\int_{B_d}(1+|Du(x)|^p)\,dx\right\}|h|^\alpha.
\end{aligned}
$$

In view of this result we can apply Lemma 2.1.4 in order to have

$$
(\mu^2+|Du(x)|^2)^{\frac{p-2}{4}}Du(x)\in L^{\frac{2n}{n-2\theta}}(B_r),\quad \forall\theta\in\left(0,\frac{\alpha}{2}\right)
$$

and also we can perform the following estimate

$$
\begin{aligned}
&||(\mu^2+|Du(x)|^2)^{\frac{p-2}{4}}Du(x)||_{L^{\frac{2n}{n-2\theta}}(B_r)} \\
&\leqslant\frac{c}{(d-r)^{2\theta+\alpha+2}}\left[\int_{B_d}(1+|Du|^p)\,dx+\frac{c}{d-r}\int_{B_d}(1+|Du|^p)\,dx\right]^{\frac{1}{2}} \\
&\leqslant\frac{c}{(d-r)^{2\theta+\alpha+3}}\left(\int_{B_d}(1+|Du|^p)\,dx\right)^{\frac{1}{2}},
\end{aligned}
$$

notably

$$
\int_{B_r}(1+|Du|)^{q\delta}\,dx\leqslant\frac{c}{(d-r)^\sigma}\left(\int_{B_d}(1+|Du|)^p\,dx\right)^{\frac{n}{n-2\theta}} \tag{3.5}
$$

for some $c\equiv c(n,N,p,q,\tilde{L},M,\theta,\alpha)$, where

$$
\sigma:=(2\theta+\alpha+3)\frac{2n}{n-2\theta},\quad \delta:=\frac{p}{q}\frac{n}{n-2\theta},
$$

we enforce $\delta > 1$, that is

$$\theta > \frac{n}{2}\frac{q-p}{q}. \tag{3.6}$$

Let us note that

$$\frac{n}{2}\frac{q-p}{q} < \frac{\alpha}{2}$$

because it can be written as

$$\frac{q}{p} < \frac{n}{n-\alpha},$$

which is true for the hypothesis on p and q, therefore we pick $\theta \in \left(\frac{n}{2}\frac{q-p}{q}, \frac{\alpha}{2}\right)$.

Hence we can conclude that $u \in W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N)$.

Step 3: the minimizer u satisfies

$$\int_{B_\rho} |Du(x)|^t \, dx \leqslant c \left(\int_{B_R} [f(x, Du(x)) + 1] \, dx \right)^\beta$$

for t, c, β as in the statement of the theorem, in particular the constant c and the exponent β do not depend on M

In the following we are going to do again the passages of the Step 2 with some differences. Firstly we remember that now $u \in W^{1,p}(B_R, \mathbb{R}^N) \cap W^{1,q}(B_{\tilde{R}}, \mathbb{R}^N)$, where $0 < \rho \leqslant r < d \leqslant \tilde{R} < R \leqslant 1$ and $\tilde{R} = \frac{\rho+R}{2}$. We consider the ball $B_{\tilde{R}}$ in place of B_R and the weak Euler-Lagrange equation for u which is still valid for any $\varphi \in W^{1,q}_0(B_{\tilde{R}}, \mathbb{R}^N)$. After defining the cut-off function and the test function like before, we can make the same calculations of the last step up to the estimate (E1), taking care of putting $B_{\tilde{R}}$ instead of B_R.

In the estimates (E2) and (E3) we use hypothesis ($\tilde{\mathrm{H}}4$) and ($\tilde{\mathrm{H}}7$) instead of ($\tilde{\mathrm{H}}6$) and ($\tilde{\mathrm{H}}5$) respectively, that is

$$\begin{aligned}
&\left| \int_{B_{\tilde{R}}} \eta^2 \left[\frac{\partial f}{\partial z}(x + he_s, Du(x + he_s)) - \frac{\partial f}{\partial z}(x, Du(x + he_s)) \right] \tau_{s,h} Du \, dx \right| \\
&\overset{(\tilde{\mathrm{H}}4)}{\leqslant} \tilde{L} \left\{ \int_{B_{\tilde{R}}} \eta^2 [1 + |Du(x + he_s)|^{q-1}] |\tau_{s,h} Du(x)| \, dx \right\} |h|^\alpha \\
&\leqslant c(n, N, q, \tilde{L}) \left\{ \int_{B_{\frac{d+r}{2}}} [1 + |Du(x)|^q + |Du(x + he_s)|^q] \, dx \right\} |h|^\alpha \\
&\leqslant c(n, N, q, \tilde{L}) \left\{ \int_{B_d} (1 + |Du(x)|^q) \, dx \right\} |h|^\alpha.
\end{aligned}$$

Moreover,

$$
\begin{aligned}
&\left|\int_{B_{\tilde{R}}} \tau_{s,h}\left(\frac{\partial f}{\partial z}(x,Du)\right) 2\eta D\eta \otimes \tau_{s,h} u \, dx\right| \\
&\overset{(\tilde{H}7)}{\leqslant} c(n,N,q,\tilde{L}) \int_{B_{\tilde{R}}} \eta |D\eta| [1 + |Du(x)|^{q-1} + |Du(x+he_s)|^{q-1}] |\tau_{s,h} u(x)| \, dx \\
&\leqslant \frac{c(n,N,q,\tilde{L})}{d-r} \left\{ \int_{B_{\frac{d+r}{2}}} [1 + |Du(x)|^q + |Du(x+he_s)|^q] \, dx \right\}^{\frac{q-1}{q}} \\
&\qquad \times \left\{ \int_{B_{\frac{d+r}{2}}} |\tau_{s,h} u(x)|^q \, dx \right\}^{\frac{1}{q}} \qquad \text{(Hölder inequality)} \\
&\leqslant \frac{c(n,N,q,\tilde{L})}{d-r} \left\{ \int_{B_d} (1 + |Du(x)|^q) \, dx \right\} |h|^\alpha,
\end{aligned}
$$

where for the last inequality we have used Lemma 2.1.2. As in the previous step we sum up on $s \in \{1, \cdots, n\}$ in order to have

$$
\begin{aligned}
&\int_{B_{\tilde{R}}} \sum_{s=1}^{n} |\tau_{s,h}[(\mu^2 + |Du(x)|^2)^{\frac{p-2}{4}} Du(x)]|^2 \eta^2 \, dx \\
&\qquad \leqslant \frac{c}{d-r} \left\{ \int_{B_d} (1 + |Du(x)|^q) \, dx \right\} |h|^\alpha
\end{aligned}
$$

for some $c \equiv c(n,N,p,q,\tilde{L})$.

Applying Lemma 2.1.4 we get

$$
\begin{aligned}
&||(\mu^2 + |Du(x)|^2)^{\frac{p-2}{4}} Du(x)||_{L^{\frac{2n}{n-2\theta}}(B_r)} \\
&\leqslant \frac{c}{(d-r)^{2\theta+\alpha+2}} \left[\int_{B_d} (1 + |Du|^p) \, dx + \frac{c}{d-r} \int_{B_d} (1 + |Du|^q) \, dx \right]^{\frac{1}{2}} \\
&\leqslant \frac{c}{(d-r)^{2\theta+\alpha+3}} \left(\int_{B_d} (1 + |Du|^q) \, dx \right)^{\frac{1}{2}}
\end{aligned}
$$

and

$$
\int_{B_r} (1 + |Du|)^{q\delta} \, dx \leqslant \frac{c}{(d-r)^\sigma} \left(\int_{B_d} (1 + |Du|)^q \, dx \right)^{\frac{n}{n-2\theta}}
$$

for some $c \equiv c(n,N,p,q,\tilde{L},\theta,\alpha)$ and σ, δ like above.

We now perform the following estimates with a $\gamma > \delta$ that we will define later

$$\begin{aligned}\int_{B_r} (1+|Du|)^{q\delta}\,dx &\leqslant \frac{c}{(d-r)^\sigma}\left(\int_{B_d} (1+|Du|)^{\frac{q\delta}{\gamma}}(1+|Du|)^{q\left(1-\frac{\delta}{\gamma}\right)}\,dx\right)^{\frac{q\delta}{p}}\\ &\leqslant \frac{c}{(d-r)^\sigma}\left(\int_{B_d}(1+|Du|)^{q\delta}\,dx\right)^{\frac{q\delta}{p\gamma}}\\ &\quad\times\left(\int_{B_d}(1+|Du|)^{q\frac{\gamma-\delta}{\gamma-1}}\,dx\right)^{\frac{q\delta}{p}\frac{\gamma-1}{\gamma}},\end{aligned}$$

where for the last inequality we have used the Hölder inequality.

With the choices

$$\varepsilon := \frac{q}{p}\frac{\delta}{\gamma}, \quad \nu := \frac{\gamma-\delta}{\gamma-1}, \quad \lambda := \gamma - 1,$$

we can write the last inequality like

$$\int_{B_r}(1+|Du|)^{q\delta}\,dx \leqslant \left(\int_{B_d}(1+|Du|)^{q\delta}\,dx\right)^{\varepsilon}\frac{c}{(d-r)^\sigma}\left(\int_{B_d}(1+|Du|)^{q\nu}\,dx\right)^{\lambda\varepsilon}. \tag{3.7}$$

Finally, we want to choose γ in such a way that

$$\varepsilon < 1, \quad q\nu \leqslant p, \quad \nu < \delta,$$

which is nothing more than

$$\delta\frac{q}{p} < \gamma, \quad \gamma \leqslant \frac{q\delta - p}{q-p},$$

i.e.

$$\delta\frac{q}{p} < \frac{q\delta-p}{q-p},$$

which we can write as

$$\theta > \frac{n}{2}\frac{q-p}{p}.$$

Now we point out that

$$\frac{n}{2}\frac{q-p}{p} < \frac{\alpha}{2} \tag{3.8}$$

since this inequality is equivalent to

$$\frac{q}{p} < \frac{n+\alpha}{n},$$

that is the constraint on p and q. Bearing in mind the conditions (3.6) we take $\theta \in \left(\max\left\{\frac{n}{2}\frac{q-p}{q}, \frac{n}{2}\frac{q-p}{p}\right\}, \frac{\alpha}{2}\right) = \left(\frac{n}{2}\frac{q-p}{p}, \frac{\alpha}{2}\right)$.

Now we can apply Young's inequality to (3.7), and thus getting

$$\begin{aligned}\int_{B_r}(1+|Du|)^{q\delta}\,dx \leqslant \frac{c}{(d-r)^{\frac{\sigma}{1-\varepsilon}}}\left(\int_{B_d}(1+|Du|)^p\,dx\right)^{\frac{\lambda\varepsilon}{1-\varepsilon}} \\ +\frac{1}{2}\int_{B_d}(1+|Du|)^{q\delta}\,dx. \end{aligned} \tag{3.9}$$

If the last integral was computed on the same set of the one on the left-hand side we would have finished; unfortunately, the sets of integration are different and we have to use Lemma 2.1.6 with the following setting

$$R_0 \coloneqq \tilde{R}-\omega, \quad 0<\omega<\tilde{R}-\rho$$

$$A \coloneqq \left(\int_{B_{\tilde{R}}}(1+|Du|)^p\,dx\right)^{\frac{\lambda\varepsilon}{1-\varepsilon}}, \quad h(r)\coloneqq \int_{B_r}(1+|Du|)^{q\delta}\,dx.$$

So, with the choice $\rho \leqslant r < d \leqslant \tilde{R}-\omega$ the inequality (3.9) turns into

$$\int_{B_\rho}(1+|Du|)^{\frac{pn}{n-2\theta}}\,dx \leqslant \frac{c}{(\tilde{R}-\omega-\rho)^{\frac{\sigma}{1-\varepsilon}}}\left(\int_{B_{\tilde{R}}}(1+|Du|)^p\,dx\right)^{\frac{\lambda\varepsilon}{1-\varepsilon}}$$

for some $c \equiv c(n,N,p,q,\tilde{L},\theta,\alpha)$. At this point we let $\omega \to 0$ and then we apply ($\tilde{\mathrm{H}}2$) and replace $\tilde{R}=\frac{R+\rho}{2}$. Finally, let us pick θ such that $t<\frac{pn}{n-2\theta}$, that is,

$$\theta > \frac{n}{2}\left(1-\frac{p}{t}\right).$$

We specify that

$$\frac{n}{2}\left(1-\frac{p}{t}\right)<\frac{\alpha}{2}$$

corresponds to

$$t<\frac{pn}{n-\alpha}.$$

Hence, taking $\theta \in \left(\max\left\{\frac{n}{2}\frac{q-p}{p}, \frac{n}{2}\left(1-\frac{p}{t}\right)\right\}, \frac{\alpha}{2}\right)$ we get the expected estimate. □

3.2 Absence of the Lavrentiev gap

In this section we prove that $\mathcal{L}(u)=0$ for every $u \in W^{1,p}$. Next theorem is the one that guarantees us the absence of the Lavrentiev phenomenon for functional (3.1). As mentioned in the introduction the density has a (p,q)-growth condition and satisfies (1.14) and (1.15). The peculiarity of this density is that it can be approximated from below with functions f_k satisfying the hypothesis of Theorem 3.1.1. The assumption (1.16) always remains in force; to emphasize the importance of this condition we refer to the work [67] where a counterexample is shown in which a local minimizer $u \in W^{1,p}_{\mathrm{loc}}$ does not belong to $W^{1,q}_{\mathrm{loc}}$ if we have (1.11). From now on we use the following notation

$$\mathcal{F}(u, A) := \int_A f(x, Du(x))\, dx.$$

Theorem 3.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\alpha \in (0,1]$ and p, q be such that*

$$1 < p < q < p\left(\frac{n+\alpha}{n}\right).$$

Let $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Carathéodory function. Assume that there exist $L \in [1, +\infty)$ and $\mu \in [0,1]$ such that

$$z \mapsto f(x, z) \in C^1(\mathbb{R}^{N\times n}), \quad \textit{for all } x \in \Omega, \tag{H1}$$

$$L^{-1}|z|^p \leqslant f(x, z) \leqslant L(1 + |z|^q), \tag{H2}$$

$$L^{-1}(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_1 - z_2|^2 \leqslant \left\langle \frac{\partial f}{\partial z}(x, z_1) - \frac{\partial f}{\partial z}(x, z_2), z_1 - z_2 \right\rangle, \tag{H3}$$

$$\left|\frac{\partial f}{\partial z}(x, z) - \frac{\partial f}{\partial z}(y, z)\right| \leqslant L|x - y|^\alpha (1 + |z|^{q-1}). \tag{H4}$$

Assume also that that there exist a sequence of Carathéodory function $f_k : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$, a sequence $c(k) \in (0, +\infty)$ and $\tilde{L} \in (0, +\infty)$ such that

$$z \mapsto f_k(x, z) \in C^1(\mathbb{R}^{N\times n}), \quad \textit{for all } x \in \Omega, \tag{AP1}$$

$$\tilde{L}^{-1}|z|^p \leqslant f_k(x, z) \leqslant \tilde{L}(1 + |z|^q), \tag{AP2}$$

$$\tilde{L}^{-1}(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_1 - z_2|^2 \leqslant \left\langle \frac{\partial f_k}{\partial z}(x, z_1) - \frac{\partial f_k}{\partial z}(x, z_2), z_1 - z_2 \right\rangle, \tag{AP3}$$

$$\left|\frac{\partial f_k}{\partial z}(x, z) - \frac{\partial f_k}{\partial z}(y, z)\right| \leqslant \tilde{L}|x - y|^\alpha (1 + |z|^{q-1}), \tag{AP4}$$

$$\left|\frac{\partial f_k}{\partial z}(x, z)\right| \leqslant c(k)(1 + |z|^{p-1}), \tag{AP5}$$

$$\left|\frac{\partial f_k}{\partial z}(x, z) - \frac{\partial f_k}{\partial z}(y, z)\right| \leqslant c(k)|x - y|^\alpha (1 + |z|^{p-1}), \tag{AP6}$$

$$f_k(x,z) \leqslant f(x,z), \quad \textit{for all } (x,z) \in \Omega \times \mathbb{R}^{N\times n}, \tag{AP7}$$

$$\lim_{k\to+\infty} f_k(x,z) = f(x,z), \quad \textit{for all } (x,z) \in \Omega \times \mathbb{R}^{N\times n}, \tag{AP8}$$

$$f_k(x,z) \leqslant f_{k+1}(x,z), \quad \textit{for all } (x,z) \in \Omega \times \mathbb{R}^{N\times n}. \tag{AP9}$$

Let $u_* \in W^{1,p}(B_R,\mathbb{R}^N)$ *be a function such that*

$$\mathcal{F}(u_*,B_R) < +\infty$$

where $B_R \Subset \Omega, R \in (0,1]$.

Then there exists $\tilde{u}_* \in (u_* + W_0^{1,p}(B_R,\mathbb{R}^N)) \cap W^{1,q}_{loc}(B_R,\mathbb{R}^N)$ *such that*

$$\mathcal{F}(\tilde{u}_*,B_R) \leqslant \mathcal{F}(u_*,B_R)$$

and

$$\int_{B_\rho} |D\tilde{u}_*(x)|^t \, dx \leqslant c \left(\int_{B_R} [f(x,Du_*(x)) + 1] \, dx \right)^\beta,$$

for all $\rho \in (0,R)$, *all* $t \in \left[p, \frac{np}{n-\alpha}\right)$ *and for some positive constants* $c(n,N,q,p,\tilde{L},R,\rho,t,\alpha)$ *and* $\beta(n,q,p,t,\alpha)$. *This shows that the Lavrentiev phenomenon for* $\mathcal{F}$ *does not occur. Precisely*

$$\inf_{u\in u_0+W_0^{1,p}(B_R,\mathbb{R}^N)} \mathcal{F}(u,B_R) = \inf_{u\in (u_0+W_0^{1,p}(B_R,\mathbb{R}^N))\cap W^{1,q}_{loc}(B_R,\mathbb{R}^N)} \mathcal{F}(u,B_R),$$

where $u_0 \in W^{1,p}(B_R,\mathbb{R}^N)$ *and* $\mathcal{F}(u_0,B_R) < +\infty$.

Proof. We divide the proof into four steps.

Step 1: a priori estimates for the minimizer of the functional

$$\mathcal{F}_k(u,B_R) = \int_{B_R} f_k(x,Du(x)) \, dx$$

Consider the above functional $\mathcal{F}_k$ and observe the following assertions:

- $f_k(x,z) \geqslant 0, \ \forall (x,z) \in \bar{B}_R \times \mathbb{R}^{N\times n}$,
- $z \mapsto f_k(x,z) \in C^1(\mathbb{R}^{N\times n}), \ \forall x \in \bar{B}_R$,
- $z \mapsto f_k(x,z)$ is convex for (AP3), $\ \forall x \in \bar{B}_R$,
- $f_k(x,z) \geqslant \tilde{L}^{-1}|z|^p, \ \forall (x,z) \in \bar{B}_R \times \mathbb{R}^{N\times n}$,
- $\mathcal{F}_k(u_*,B_R) \leqslant \mathcal{F}(u_*,B_R) < +\infty$ for (AP7).

Therefore we can apply the direct method of the Calculus of Variations: for any $k \in \mathbb{N}$ there exists $u_k \in u_* + W_0^{1,p}(B_R, \mathbb{R}^N)$ such that

$$\min_{u \in u_* + W_0^{1,p}(B_R, \mathbb{R}^N)} \mathcal{F}_k(u, B_R) = \mathcal{F}_k(u_k, B_R).$$

We are in a position to use the Theorem 3.1.1 for the functional $\mathcal{F}_k$ in order to have the following estimate

$$\begin{aligned} \int_{B_\rho} |Du_k(x)|^t \, dx &\leqslant c \left(\int_{B_R} [f_k(x, Du_k(x)) + 1] \, dx \right)^\beta \\ &\leqslant c \left(\int_{B_R} [f_k(x, Du_*(x)) + 1] \, dx \right)^\beta \\ &\overset{\text{(AP7)}}{\leqslant} c \left(\int_{B_R} [f(x, Du_*(x)) + 1] \, dx \right)^\beta \\ &< +\infty \end{aligned}$$

for all $\rho \in (0, R)$, all $t \in \left[p, \frac{np}{n-\alpha}\right)$ and for some positive constants $c(n, N, q, p, \tilde{L}, R, \rho, t, \alpha)$ and $\beta(n, q, p, t, \alpha)$. Thus,

$$\int_{B_\rho} |Du_k(x)|^t \, dx \leqslant c \left(\int_{B_R} [f(x, Du_*(x)) + 1] \, dx \right)^\beta . \tag{3.10}$$

Step 2: there exists another function u_∞ satisfying the a priori estimate with an energy lower than or equal to the one of u_*

As in (4.23) of [44], we consider the following chain of inequalities where $h \leqslant k$

$$\int_{B_R} \tilde{L}^{-1} |Du_k|^p \, dx \leqslant \mathcal{F}_h(u_k, B_R) \leqslant \mathcal{F}_k(u_k, B_R) \leqslant \mathcal{F}_k(u_*, B_R) \leqslant \mathcal{F}(u_*, B_R) < +\infty. \tag{3.11}$$

By the previous inequality, up to not relabelled subsequences, we may suppose that there exists u_∞ such that

$$\begin{aligned} u_\infty - u_* &\in W_0^{1,p}(B_R, \mathbb{R}^N) \\ Du_k &\rightharpoonup Du_\infty \ \text{ weakly in } L^p(B_R). \end{aligned} \tag{3.12}$$

Bearing in mind that the functional $\mathcal{F}_h$ is weakly lower semi-continuous we obtain

$$\mathcal{F}_h(u_\infty, B_R) \leqslant \liminf_{k \to +\infty} \mathcal{F}_h(u_k, B_R) \overset{(3.11)}{\leqslant} \mathcal{F}(u_*, B_R).$$

In turn, this fact and the monotone convergence theorem for $h \to +\infty$, i.e.

$$\mathcal{F}_h(u_\infty, B_R) \to \mathcal{F}(u_\infty, B_R)$$

imply that

$$\mathcal{F}(u_\infty, B_R) \leqslant \mathcal{F}(u_*, B_R). \tag{3.13}$$

Moreover, for (3.10), (3.12) and the weak lower semi-continuity of $w \mapsto \int_{B_\rho} |w|^t\, dx$ we can perform the following estimate for u_∞:

$$\int_{B_\rho} |Du_\infty(x)|^t\, dx \leqslant \liminf_{k\to+\infty} \int_{B_\rho} |Du_k(x)|^t\, dx \leqslant c\left(\int_{B_R} [f(x, Du_*(x)) + 1]\, dx\right)^\beta. \tag{3.14}$$

In view of this result we can assert that

$$u_\infty \in (u_* + W_0^{1,p}(B_R, \mathbb{R}^N)) \cap W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N)$$

because $t \in \left[p, \frac{np}{n-\alpha}\right)$ and

$$\frac{np}{n-\alpha} > p\left(\frac{n+\alpha}{n}\right) > q.$$

Remark 3.2.2. In order to get (3.14) we use (3.10) where the constant c is independent of k: this is obtained in Step 3 of Theorem 3.1.1.

Step 3: lemma for the existence of a regular minimizing sequence

Lemma 3.2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let p, q be such that*

$$1 < p < q < p\left(\frac{n+\alpha}{n}\right), \quad \alpha \in (0, 1].$$

Let f and f_k be as in the Theorem 3.2.1, then there exists a sequence $\{\tilde{v}_m\}_{m\in\mathbb{N}} \subset (v_0 + W_0^{1,p}(B_R, \mathbb{R}^N)) \cap W^{1,q}_{loc}(B_R, \mathbb{R}^N)$ such that

$$\mathcal{F}(\tilde{v}_m, B_R) \longrightarrow \inf_{v\in v_0+W_0^{1,p}(B_R,\mathbb{R}^N)} \mathcal{F}(v, B_R),$$

where $B_R \Subset \Omega$, $R \in (0, 1]$, $v_0 \in W^{1,p}(B_R, \mathbb{R}^N)$ and $\mathcal{F}(v_0, B_R) < +\infty$.

Proof. Let us start considering a minimizing sequence $\{v_m\}_{m\in\mathbb{N}}$ for the functional $\mathcal{F}$ on the class of function $v_0 + W_0^{1,p}(B_R, \mathbb{R}^N)$, i.e.

$$\mathcal{F}(v_m, B_R) \longrightarrow \inf_{v\in v_0+W_0^{1,p}(B_R,\mathbb{R}^N)} \mathcal{F}(v, B_R) =: \mathcal{J}. \tag{3.15}$$

Now we use the Step 1 and Step 2 of the Theorem 3.2.1 with v_m, $m \in \mathbb{N}$, in place of u_*, namely

$$\forall m \in \mathbb{N} \quad \exists \tilde{v}_m \in (v_m + W_0^{1,p}(B_R, \mathbb{R}^N)) \cap W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N) \tag{3.16}$$

but $v_m \in v_0 + W_0^{1,p}(B_R, \mathbb{R}^N)$, so

$$\tilde{v}_m \in (v_0 + W_0^{1,p}(B_R, \mathbb{R}^N)) \cap W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N), \quad \forall m \in \mathbb{N},$$

in particular by (3.14)

$$\int_{B_\rho} |D\tilde{v}_m(x)|^t \, dx \quad \leqslant \quad c\left(\int_{B_R} [f(x, Dv_m(x)) + 1] \, dx\right)^\beta \overset{(3.15)}{\leqslant} c(|B_R| + \mathcal{J} + 1)^\beta \qquad (3.17)$$

for all $\rho \in (0, R)$ and $t \in \left[p, \frac{np}{n-\alpha}\right)$.

Moreover, by (3.16) and (3.13)

$$\mathcal{J} \leqslant \mathcal{F}(\tilde{v}_m, B_R) \leqslant \mathcal{F}(v_m, B_R) \to \mathcal{J}$$

thus we can conclude that

$$\mathcal{F}(\tilde{v}_m, B_R) \to \mathcal{J}.$$

□

Step 4: conclusion about the absence of the Lavrentiev phenomenon

Let us define

$$\mathcal{J} := \inf_{u \in u_0 + W_0^{1,p}(B_R, \mathbb{R}^N)} \mathcal{F}(u, B_R)$$

$$\tilde{\mathcal{J}} := \inf_{u \in (u_0 + W_0^{1,p}(B_R, \mathbb{R}^N)) \cap W_{\mathrm{loc}}^{1,q}(B_R, \mathbb{R}^N)} \mathcal{F}(u, B_R),$$

now, by the previous lemma we get a sequence

$$\{\tilde{u}_s\}_{s \in \mathbb{N}} \subset (u_0 + W_0^{1,p}(B_R, \mathbb{R}^N)) \cap W_{\mathrm{loc}}^{1,q}(B_R, \mathbb{R}^N)$$

such that

$$\mathcal{F}(\tilde{u}_s, B_R) \to \mathcal{J}.$$

Recalling that

$$\mathcal{J} \leqslant \tilde{\mathcal{J}} \leqslant \mathcal{F}(\tilde{u}_s, B_R)$$

we can conclude that

$$\mathcal{J} = \tilde{\mathcal{J}},$$

which implies the absence of the Lavrentiev phenomenon. □

Remark 3.2.4. In order to formulate our hypothesis we do not need to assume the existence of a minimizer $u \in u_0 + W_0^{1,p}(B_R, \mathbb{R}^N)$ for $\mathcal{F}$, but we can observe that this is a straightfoward consequence of direct methods of the Calculus of Variations. We can achieve even that

$$u \in (u_0 + W_0^{1,p}(B_R, \mathbb{R}^N)) \cap W_{\mathrm{loc}}^{1,q}(B_R, \mathbb{R}^N)$$

for such a minimizer u; let us see in detail how to do it: an application of Lemma 3.2.3 gives us a sequence

$$\{\tilde{u}_m\}_{m \in \mathbb{N}} \subset (u_0 + W_0^{1,p}(B_R, \mathbb{R}^N)) \cap W_{\mathrm{loc}}^{1,q}(B_R, \mathbb{R}^N)$$

such that

$$\mathcal{F}(\tilde{u}_m, B_R) \longrightarrow \inf_{u \in u_0 + W_0^{1,p}(B_R, \mathbb{R}^N)} \mathcal{F}(u, B_R) =: \mathcal{J}$$

from which it follows that

$$\int_{B_R} L^{-1} |D\tilde{u}_m|^p \, dx \overset{\text{(H2)}}{\leqslant} \mathcal{F}(\tilde{u}_m, B_R) < \mathcal{J} + 1.$$

In turn, this fact implies that, up to not relabelled subsequences, there exists $\tilde{u}_\infty$ such that

$$\tilde{u}_\infty - u_0 \in W_0^{1,p}(B_R, \mathbb{R}^N)$$

$$D\tilde{u}_m \rightharpoonup D\tilde{u}_\infty \quad \text{weakly in } L^p(B_R),$$

this, together with (3.17) and the weak lower semi-continuity of $w \mapsto \int_{B_\rho} |w|^t \, dx$ leads to

$$\int_{B_\rho} |D\tilde{u}_\infty(x)|^t \, dx \leqslant c(|B_R| + \mathcal{J} + 1)^\beta$$

for all $\rho \in (0, R)$ and $t \in \left[p, \frac{np}{n-\alpha}\right)$, and thus

$$\tilde{u}_\infty \in (u_0 + W_0^{1,p}(B_R, \mathbb{R}^N)) \cap W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N).$$

Recalling the weak lower semi-continuity of $\mathcal{F}$ we can write

$$\mathcal{J} \leqslant \mathcal{F}(\tilde{u}_\infty, B_R) \leqslant \liminf_{m \to +\infty} \mathcal{F}(\tilde{u}_m, B_R) = \mathcal{J},$$

then the definition of $\mathcal{J}$ implies that $\tilde{u}_\infty$ is a minimizer and $\tilde{u}_\infty = u$ by the strict convexity of $\mathcal{F}$.

Now, we see how adding a suitable penalization to the functional (3.1) and using the previous remark to this new functional, allows us to conclude the Lavrentiev term is identically zero, when considering the spaces

$$X = W^{1,p}(B_R, \mathbb{R}^N), \quad Y = W^{1,p}(B_R, \mathbb{R}^N) \cap W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N).$$

This in turn implies that the relaxed functional

$$\bar{\mathcal{F}}(u, B_R) := \inf_{u_j} \left\{ \liminf_{j \to +\infty} \mathcal{F}(u_j) : u_j \in W^{1,p} \cap W^{1,q}_{\mathrm{loc}} \ \ \forall j \in \mathbb{N} \ \text{ and } u_j \rightharpoonup u \text{ in } W^{1,p} \right\},$$

is represented by $\mathcal{F}$ itself, namely

$$\bar{\mathcal{F}}(u) = \mathcal{F}(u), \quad \text{for all } u \in W^{1,p}(B_R, \mathbb{R}^n).$$

Here we cannot proceed in the same way as when u is the minimizer of $\mathcal{F}$ and we need to modify the functional to construct a sequence converging both strongly and in energy to the fixed function u. The theorem is the following.

Theorem 3.2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\alpha \in (0,1]$ and p, q be such that*

$$1 < p < q < p\left(\frac{n+\alpha}{n}\right).$$

Let $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Carathéodory function. Assume that there exist $L \in [1, +\infty)$ and $\mu \in [0,1]$ such that f satisfies (H1)-(H4).

Assume also that that there exist a sequence of Carathéodory function $f_k : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$, a sequence $c(k) \in (0, +\infty)$ and $\tilde{L} \in (0, +\infty)$ such that f_k satisfies (AP1)-(AP9).

Let u_* be a function in $W^{1,p}(B_R, \mathbb{R}^N)$ such that $\mathcal{F}(u_*, B_R) < +\infty$, where $B_R \Subset \Omega, R \in (0,1]$. Then

$$\mathcal{L}(u_*) = 0.$$

Proof. By Lemma 2.1.7 we need to show that there exists a sequence $\{u_\ell\}_{\ell\in\mathbb{N}} \subset W^{1,p}(B_R, \mathbb{R}^N) \cap W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N)$ such that

$$u_\ell \rightharpoonup u_* \quad \text{weakly in } W^{1,p}(B_R, \mathbb{R}^N)$$

and

$$\mathcal{F}(u_\ell, B_R) \to \mathcal{F}(u_*, B_R).$$

We divide the proof into two steps.

Step 1: we construct a suitable sequence

$$\{u_\ell\}_{\ell\in\mathbb{N}} \subset (u_* + W^{1,p}_0(B_R, \mathbb{R}^N)) \cap W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N).$$

We start observing that by Theorem 3.10 in [74] it is possible to find a $\rho > 0$ and a function $\bar{u}_* \in W^{1,p}_0(B_{R+\rho}, \mathbb{R}^N)$: $\bar{u}_* = u_*$ in B_R. Moreover, $\bar{u}_*$ can be extended to all of $\mathbb{R}^n$ by setting $\bar{u}_* = 0$ outside $B_{R+\rho}$ and such an extension belongs to $W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$; we keep calling this extension u_*. Now let $\ell \in \mathbb{N}$, we define v_ℓ, the regularized of u_*, in the following way

$$v_\ell(x) = \int_{B(x,\frac{1}{\ell})} u_*(y)\ell^n \psi(\ell(x-y))\, dy,$$

where $\psi \in C^\infty_c(B(0,1), \mathbb{R}), \psi \geqslant 0$ and $\int_{B(0,1)} \psi(x)\, dx = 1$. Then $\operatorname{supp} v_\ell$ is compact in $\mathbb{R}^n$ and $v_\ell \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^N) \subset W^{1,p}(\Omega, \mathbb{R}^N)$. Since $v_\ell \to u_*$ strongly in $W^{1,p}(B_R, \mathbb{R}^N)$, up to subsequence, still denoted v_ℓ, we have

$$||u_* - v_\ell||_{W^{1,p}(B_R)} \leqslant \frac{1}{\ell}. \tag{3.18}$$

We define

$$\mathcal{G}_\ell(u, B_R) = \int_{B_R} \left[f(x, Du(x)) + \frac{1}{\ell}(1 + \ell^2 |Du(x) - Dv_\ell(x)|^2)^{\frac{p}{2}} \right] dx,$$

we call $g_\ell(x,z) := f(x,z) + \frac{1}{\ell}(1 + \ell^2|z - Dv_\ell(x)|^2)^{\frac{p}{2}}$ and we observe that

- $g_\ell(x,z) \geqslant 0,\ \forall (x,z) \in \Omega \times \mathbb{R}^{N\times n}$;
- $z \mapsto g_\ell(x,z) \in C^1(\mathbb{R}^{N\times n}),\ \forall x \in \Omega$;
- $z \mapsto g_\ell(x,z)$ is convex $\ \forall x \in \Omega$ (sum of convex functions);
- $g_\ell(x,z) \geqslant L^{-1}|z|^p + \dfrac{1}{\ell}(1+\ell^2|z-Dv_\ell(x)|^2)^{\frac{p}{2}} > L^{-1}|z|^p,\ \forall (x,z) \in \Omega \times \mathbb{R}^{N\times n}$;
- $\mathcal{G}_\ell(u_*, B_R) = \mathcal{F}(u_*, B_R) + \dfrac{1}{\ell}\displaystyle\int_{B_R} (1+\ell^2|Du_*(x)-Dv_\ell(x)|^2)^{\frac{p}{2}}\ dx < +\infty$.

Therefore we can apply the direct method of the Calculus of Variations: for any $\ell \in \mathbb{N}$ there exists $u_\ell \in u_* + W_0^{1,p}(B_R, \mathbb{R}^N)$ such that

$$\min_{u \in u_* + W_0^{1,p}(B_R,\mathbb{R}^N)} \mathcal{G}_\ell(u, B_R) = \mathcal{G}_\ell(u_\ell, B_R).$$

Now let us observe that $g_\ell : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a Carathéodory function and let us define $h_\ell(x,z) \coloneqq \frac{1}{\ell}(1+\ell^2|z-Dv_\ell(x)|^2)^{\frac{p}{2}}$, then by the convexity of $z \mapsto h_\ell(x,z)$ we have

$$\begin{aligned}\left\langle \frac{\partial g_\ell}{\partial z}(x,z_1) - \frac{\partial g_\ell}{\partial z}(x,z_2), z_1 - z_2 \right\rangle =& \left\langle \frac{\partial f}{\partial z}(x,z_1) - \frac{\partial f}{\partial z}(x,z_2), z_1 - z_2 \right\rangle \\ &+ \left\langle \frac{\partial h_\ell}{\partial z}(x,z_1) - \frac{\partial h_\ell}{\partial z}(x,z_2), z_1 - z_2 \right\rangle \\ \geqslant& L^{-1}(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_1 - z_2|^2.\end{aligned}$$

We point out that since $v_\ell \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^N)$ there exists $M = M(\ell) \in [0, +\infty)$ such that $|Dv_\ell(x)| + |D^2 v_\ell(x)| \leqslant M$, for all $x \in \mathbb{R}^n$. Hence we have

$$\begin{aligned} g_\ell(x,z) &= f(x,z) + \frac{1}{\ell}(1+\ell^2|z-Dv_\ell(x)|^2)^{\frac{p}{2}} \\ &\leqslant f(x,z) + \frac{2^{\frac{p}{2}}}{\ell}(1+\ell^p|z-Dv_\ell(x)|^p) \\ &\leqslant L(1+|z|^q) + \frac{2^{\frac{p}{2}}}{\ell} + 2^{\frac{p}{2}}\ell^{p-1}Lf(x, z-Dv_\ell(x)) \\ &\leqslant L(1+|z|^q) + \frac{2^{\frac{p}{2}}}{\ell} + 2^{\frac{p}{2}}\ell^{p-1}L^2(1+|z-Dv_\ell(x)|^q) \\ &\leqslant L(1+|z|^q) + \frac{2^{\frac{p}{2}}}{\ell} + \tilde{c}_1(1+|z|^q + |Dv_\ell(x)|^q) \\ &\leqslant L(1+|z|^q) + \tilde{c}_2(1+|z|^q) \\ &\leqslant (L+\tilde{c}_2)(1+|z|^q), \end{aligned}$$

where $\tilde{c}_1 = \tilde{c}_1(\ell, p, q, L)$ and $\tilde{c}_2 = \tilde{c}_2(\ell, p, q, L, M)$ are suitable positive constants. Now let us compute the derivative with respect to x of $\frac{\partial h_\ell}{\partial z_i^\alpha}(x,z) = pl(1+\ell^2|z-Dv_\ell(x)|^2)^{\frac{p-2}{2}}(z_i^\alpha -$

$D_i v_\ell^\alpha(x)$):

$$\begin{aligned}\frac{\partial^2 h_\ell}{\partial x_j \partial z_i^\alpha}(x,z) = pl \Big[&\ell^2 (p-2)(1+\ell^2|z-Dv_\ell(x)|^2)^{\frac{p-4}{2}} \\ &\times (z_i^\alpha - D_i v_\ell^\alpha(x)) \sum_{\beta=1}^N \sum_{r=1}^n (z_r^\beta - D_r v_\ell^\beta(x))(-D_j D_r v_\ell^\beta(x)) \\ &+ (1+\ell^2|z-Dv_\ell(x)|^2)^{\frac{p-2}{2}} (-D_j D_i v_\ell^\alpha(x)) \Big].\end{aligned}$$

Since $x \mapsto \frac{\partial h_\ell}{\partial z}(x,z)$ is $C^1(\mathbb{R}^n)$ we can conclude that $\frac{\partial h_\ell}{\partial z}$ is α-Hölder continuous with respect to x, more precisely we find that

$$\begin{aligned}\left|\frac{\partial h_\ell}{\partial z}(x,z) - \frac{\partial h_\ell}{\partial z}(y,z)\right| &\leqslant H|x-y|^\alpha (1+|z|)^{p-2} \\ &\leqslant \tilde{H}|x-y|^\alpha (1+|z|^{q-1}),\end{aligned}$$

where H and $\tilde{H}$ are positive constants depending on $\ell, \Omega, p, q, n, N, M$. Consequentially

$$\begin{aligned}\left|\frac{\partial g_\ell}{\partial z}(x,z) - \frac{\partial g_\ell}{\partial z}(y,z)\right| &\leqslant \left|\frac{\partial f}{\partial z}(x,z) - \frac{\partial f}{\partial z}(y,z)\right| + \left|\frac{\partial h_\ell}{\partial z}(x,z) - \frac{\partial h_\ell}{\partial z}(y,z)\right| \\ &\leqslant L|x-y|^\alpha (1+|z|^{q-1}) + \tilde{H}|x-y|^\alpha (1+|z|^{q-1}) \\ &\leqslant (L+\tilde{H})|x-y|^\alpha (1+|z|^{q-1}).\end{aligned}$$

Now let us define

$$g_k^\ell(x,z) \coloneqq f_k(x,z) + \frac{1}{\ell}(1+\ell^2|z-Dv_\ell(x)|^2)^{\frac{p}{2}},$$

where $\{f_k\}_{k\in\mathbb{N}}$ is the approximating sequence defined in the statement of the Theorem. First of all we observe that g_k^ℓ is C^1 with respect to z and it satisfies (AP3) with the same constant $\tilde{L}^{-1}$. Moreover, we note that g_k^ℓ satisfies (AP2), (AP4), (AP6), to be more precise

$$\tilde{L}^{-1}|z|^p \leqslant g_k^\ell(x,z) \leqslant \tilde{L}_1(1+|z|^q),$$

$$\left|\frac{\partial g_k^\ell}{\partial z}(x,z) - \frac{\partial g_k^\ell}{\partial z}(y,z)\right| \leqslant (\tilde{L}+\tilde{H})|x-y|^\alpha (1+|z|^{q-1}),$$

$$\left|\frac{\partial g_k^\ell}{\partial z}(x,z) - \frac{\partial g_k^\ell}{\partial z}(y,z)\right| \leqslant (c(k)+2^{p-1}H)|x-y|^\alpha (1+|z|^{p-1}),$$

where $\tilde{L}_1$ depends on $\tilde{L}, \ell, p, q, M$. As far as (AP5) is concerned we observe that

$$\begin{aligned}\left|\frac{\partial g_k^\ell}{\partial z}(x,z)\right| &\leqslant \left|\frac{\partial f_k}{\partial z}(x,z)\right| + \left|\frac{\partial h}{\partial z}(x,z)\right| \\ &\leqslant c(k)(1+|z|^{p-1}) + pl(1+\ell^2|z-Dv_\ell(x)|^2)^{\frac{p-2}{2}}|z-Dv_\ell(x)| \\ &\leqslant c(k)(1+|z|^{p-1}) + pl(1+\ell^2|z-Dv_\ell|^2)^{\frac{p-2}{2}}(1+\ell^2|z-Dv_\ell|^2)^{\frac{1}{2}} \\ &\leqslant c(k)(1+|z|^{p-1}) + \bar{c}_1(1+|z|^{p-1}+|Dv_\ell(x)|^{p-1}) \\ &\leqslant c(k)(1+|z|^{p-1}) + \bar{c}_2(1+|z|^{p-1}) \\ &\leqslant (c(k)+\bar{c}_2)(1+|z|^{p-1}),\end{aligned}$$

with $\bar{c}_1 = \bar{c}_1(p,\ell)$ and $\bar{c}_2 = \bar{c}_2(p,\ell,M)$ suitable positive constants. In the end, it is straightforward to check that g_k^ℓ satisfies (AP7), (AP8) and (AP9), more precisely

$$g_k^\ell(x,z) \leqslant g_\ell(x,z);$$

$$\lim_{k\to+\infty} g_k^\ell(x,z) = g_\ell(x,z);$$

$$g_k^\ell(x,z) \leqslant g_\ell^{k+1}(x,z).$$

Observing that $\mathcal{G}_\ell(u_*) < +\infty$ we are at last in position to apply Remark 3.2.4 to the functional $\mathcal{G}_\ell$, where $\{g_k^\ell\}_{k\in\mathbb{N}}$ plays the role of the approximating sequence. So we get that $u_\ell \in W^{1,q}_{\mathrm{loc}}(B_R, \mathbb{R}^N)$.

<u>Step 2</u>: we show that $u_\ell \to u_*$ strongly in $W^{1,p}(B_R,\mathbb{R}^N)$ and $\mathcal{F}(u_\ell,B_R) \to \mathcal{F}(u_*,B_R)$

Let us consider the following chain of inequalities

$$\begin{aligned}\mathcal{F}(u_\ell,B_R) \leqslant \mathcal{G}_\ell(u_\ell,B_R) \leqslant \mathcal{G}_\ell(u_*,B_R) &\leqslant \mathcal{F}(u_*,B_R) + \frac{2^{\frac{p}{2}}\mathrm{meas}(B_R)}{\ell} \\ &\quad + 2^{\frac{p}{2}}\ell^{p-1}||Du_* - Dv_\ell||^p_{L^p(B_R)} \\ &\overset{(3.18)}{\leqslant} \mathcal{F}(u_*,B_R) + \frac{2^{\frac{p}{2}}\mathrm{meas}(B_R)}{\ell} + \frac{2^{\frac{p}{2}}}{\ell}.\end{aligned} \tag{3.19}$$

Therefore

$$\limsup_{\ell\to+\infty}\mathcal{F}(u_\ell,B_R) \leqslant \limsup_{\ell\to+\infty}\left(\mathcal{F}(u_*,B_R) + \frac{2^{\frac{p}{2}}\mathrm{meas}(B_R)}{\ell} + \frac{2^{\frac{p}{2}}}{\ell}\right) = \mathcal{F}(u_*,B_R). \tag{3.20}$$

Bearing in mind the positivity of $\mathcal{F}$ we observe that

$$\begin{aligned}\ell^{p-1}\int_{B_R}|Du_\ell(x)-Dv_\ell(x)|^p\,dx &\leqslant \int_{B_R}\frac{1}{\ell}(1+\ell^2|Du_\ell(x)-Dv_\ell(x)|^2)^{\frac{p}{2}} \\ &= \mathcal{G}_\ell(u_\ell,B_R) - \mathcal{F}(u_\ell,B_R) \\ &\overset{(3.19)}{\leqslant} \mathcal{F}(u_*,B_R) + \frac{2^{\frac{p}{2}}\mathrm{meas}(B_R)}{\ell} + \frac{2^{\frac{p}{2}}}{\ell},\end{aligned}$$

namely

$$\int_{B_R} |Du_\ell(x) - Dv_\ell(x)|^p \, dx \leqslant \frac{\mathcal{F}(u_*, B_R)}{\ell^{p-1}} + \frac{2^{\frac{p}{2}}\text{meas}(B_R)}{\ell^p} + \frac{2^{\frac{p}{2}}}{\ell^p}.$$

So we have that $||Du_\ell - Dv_\ell||_{L^p(B_R)} \to 0$ as $\ell \to +\infty$. We keep in mind (3.18) that guarantees $||Dv_\ell - Du_*||_{L^p(B_R)} \to 0$, then $Du_\ell \to Du_*$ in $L^p(B_R)$. Since $u_\ell = u_*$ on $\partial\Omega$ we have $u_\ell \to u_*$ in $W^{1,p}(B_R, \mathbb{R}^N)$. Consequently, by the lower semicontinuity of $\mathcal{F}$ we can write

$$\mathcal{F}(u_*, B_R) \leqslant \liminf_{\ell \to +\infty} \mathcal{F}(u_\ell, B_R).$$

Taking into account (3.20) we get

$$\limsup_{\ell \to +\infty} \mathcal{F}(u_\ell, B_R) \leqslant \mathcal{F}(u_*, B_R) \leqslant \liminf_{\ell \to +\infty} \mathcal{F}(u_\ell, B_R),$$

that is

$$\mathcal{F}(u_\ell, B_R) \to \mathcal{F}(u_*, B_R).$$

This result holds true for all $u_* \in W^{1,p}(B_R, \mathbb{R}^N)$ such that $\mathcal{F}(u_*, B_R) < +\infty$. This ends the proof. □

Remark 3.2.6. Let us highlight the fact that we show more than what is needed to have the absence of the gap $\mathcal{L}$ between the spaces X and Y. Indeed, for every $u \in W^{1,p}(B_R, \mathbb{R}^N)$ we find a sequence u_ℓ which coincides with u on the boundary of B_R, i.e.

$$u_\ell \in (u + W_0^{1,p}(B_R, \mathbb{R}^N)) \cap W^{1,q}_{\text{loc}}(B_R, \mathbb{R}^N),$$

and converges strongly to u in $W^{1,p}(B_R, \mathbb{R}^N)$.

3.3 Application of the penalization technique to other functionals

In this section we want to examine if a suitable penalisation can be applied to other functionals in order to obtain $\mathcal{L}(u) \equiv 0$ between suitable spaces X and Y. In particular we analyze two situations. The first one concerns a functional in the vectorial case, with the density $f = f(x, z)$ satisfying the hypothesis of the paper [70]. In this setting the approximating sequence belongs to Morrey space. The second framework regards the scalar case and deals with densities like $f(x, u, z) = d(z) + h(x, u)$ that verify the assumptions of [37]. Here the approximating sequence has bounded gradients.

3.3.1 Vectorial functionals and Morrey space

In the following we take into account again

$$\mathcal{F}(u, \Omega) = \int_\Omega f(x, Du(x)) \, dx,$$

with $u : \Omega \to \mathbb{R}^N$, where $\Omega \subset \mathbb{R}^n$ is open and bounded, $n \geqslant 2$ and $N \geqslant 1$. We will deal with the notions of Morrey space and Morrey - Sobolev space; we recall here the definitions.

Definition 3.3.1 (Morrey space)**.** For each $p \in [1, \infty)$ and $0 \leqslant \lambda \leqslant n$, we define the Morrey space

$$L^{p,\lambda}(\Omega, \mathbb{R}^N) := \left\{ u \in L^p(\Omega, \mathbb{R}^N) : \sup_{x \in \Omega, \rho > 0} \frac{1}{\rho^\lambda} \int_{\Omega \cap B_\rho(x)} |u|^p \, dx < \infty \right\}.$$

Definition 3.3.2 (Sobolev-Morrey space)**.** For each $p \in [1, +\infty)$ and $0 \leqslant \lambda \leqslant n$, we say that a mapping $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ belongs to the Sobolev-Morrey space $W^{1,(p,\lambda)}(\Omega, \mathbb{R}^N)$ if $u \in L^{p,\lambda}(\Omega, \mathbb{R}^N)$ and $Du \in L^{p,\lambda}(\Omega, \mathbb{R}^{N \times n})$.

We consider $f : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}$ as in Theorem 1.1 of Fey - Foss [70], namely: there exist numbers $1 < p < q$, $0 < \lambda < n$, $L \geqslant 1$, and a convex function $\tilde{f} : [0, +\infty) \to [0, +\infty)$ satisfying, for all $R > 0$,

$$\begin{cases} \tilde{f} \in C^1([0, +\infty)) \cap C^2((0, +\infty)), \tilde{f}'' \in L^1((0, R)) \\ (p-1)\dfrac{\tilde{f}'(t)}{t} \leqslant \tilde{f}''(t) \leqslant (q-1)\dfrac{\tilde{f}'(t)}{t}, \quad t > 0 \\ \tilde{f}(0) = \tilde{f}'(0) = 0, \ \tilde{f}(1) > 0 \end{cases} \tag{3.21}$$

such that for every $\varepsilon > 0$ and $x \in \Omega$ there exists $\sigma_\varepsilon(x) \in [0, +\infty)$:

$$|f(x, z) - \tilde{f}(|z|)| < \varepsilon \tilde{f}(|z|), \tag{3.22}$$

whenever $|z| > \sigma_\varepsilon(x)$. Moreover, there exists $a(x) \in [0, +\infty)$ such that

$$|f(x, z)| \leqslant L|z|^q + a(x), \tag{3.23}$$

for every $x \in \Omega$ and $z \in \mathbb{R}^{N \times n}$. In addition, $\sigma_\varepsilon \in L^{q,\lambda}(\Omega, \mathbb{R})$ and $a \in L^{1,\lambda}(\Omega, \mathbb{R})$. We strengthen the hypothesis of asymptotically convexity of f given by (3.22) assuming that $z \mapsto f(x, z)$ is convex. Moreover, we impose $f(x, z) \geqslant 0$ for all $x \in \Omega$ and $z \in \mathbb{R}^{N \times n}$. Note that the previous assumptions imply, by Lemma 3.1 in [70], that the function $\tilde{f}$ is increasing and

$$\tilde{f}(t_1 + t_2) \leqslant 2^q(\tilde{f}(t_1) + \tilde{f}(t_2)), \quad \text{for all } t_1, t_2 \geqslant 0, \tag{3.24}$$

$$\tilde{f}(ct) \leqslant c^q \tilde{f}(t), \quad \text{for all } t \geqslant 0 \text{ and } c \geqslant 1. \tag{3.25}$$

$$\tilde{f}(t) \leqslant \tilde{f}(1)(1 + t^q), \quad \text{for all } t \geqslant 0. \tag{3.26}$$

Let us add another assumption, that is a slight modification of [70, Lemma 3.1 (iv)],

$$L^{-1} t^p \leqslant \tilde{f}(t). \tag{3.27}$$

We aim to approximate every $u_* \in W^{1,p}(B_R, \mathbb{R}^N)$ with a sequence in the Morrey space $\{u_\ell\}_{\ell \in \mathbb{N}} \subset (u_* + W_0^{1,p}(B_\rho, \mathbb{R}^N)) \cap W^{1,(p,\lambda)}_{\mathrm{loc}}(B_\rho, \mathbb{R}^N)$, $\rho < R$, showing also the convergence in energy. For this purpose let us take any $u_* \in W^{1,p}(B_R, \mathbb{R}^N)$ such that $\mathcal{F}(u_*, B_R) < +\infty$. First of all, we prove that for any $\rho \in (0, R)$ there exists a suitable sequence $\{v_\ell\}_{\ell\in\mathbb{N}} \in \mathrm{Lip}(\overline{B}_\rho, \mathbb{R}^N)$ such that $v_\ell \to u_*$ strongly in $W^{1,p}(B_\rho, \mathbb{R}^N)$ and

$$\int_{B_\rho} \tilde{f}(|Du_*(x)) - Dv_\ell(x)|)\ dx \leqslant \frac{1}{\ell^p}. \tag{3.28}$$

Indeed, we start defining

$$S_1 \coloneqq B_R \cap \left\{x : |Du_*(x)| \leqslant \sigma_{\frac{1}{2}}(x)\right\}$$

and

$$S_2 \coloneqq B_R \cap \left\{x : |Du_*(x)| > \sigma_{\frac{1}{2}}(x)\right\}.$$

Now, using (3.26) and (3.22) with $\varepsilon = \frac{1}{2}$ we get

$$\begin{aligned}\frac{1}{2}\int_{B_R} \tilde{f}(|Du_*(x)|)\ dx &= \frac{1}{2}\left(\int_{S_1} \tilde{f}(|Du_*(x)|)\ dx + \int_{S_2} \tilde{f}(|Du_*(x)|)\ dx\right)\\ &\leqslant \frac{1}{2}\left(\tilde{f}(1)\int_{S_1} (\sigma_{\frac{1}{2}}(x))^q\ dx + \mathrm{meas}(B_R)\right) + \int_{B_R} f(x, Du_*(x))\ dx.\end{aligned}$$

Recalling that $\sigma_\varepsilon \in L^{q,\lambda}$ and $\int_{B_R} f(x, Du_*(x))\ dx < +\infty$ we obtain

$$\int_{B_R} \tilde{f}(|Du_*(x)|)\ dx < +\infty.$$

Now, as in the proof of Theorem 3.2.5, for every $0 < \delta < \frac{R-\rho}{2}$ we mollify u_* and we get $w_\delta \in \mathrm{Lip}(\overline{B}_\rho, \mathbb{R}^N)$ such that $w_\delta \to u_*$ strongly in $W^{1,p}(B_\rho, \mathbb{R}^N)$. Then, Jensen's inequality and Fubini's theorem lead to

$$\int_{B_\rho} \tilde{f}(|Dw_\delta(x)|)\ dx \leqslant \int_{B_{\rho+\delta}} \tilde{f}(|Du_*(x)|)\ dx. \tag{3.29}$$

We pick a subsequence $\{w_{\delta_j}\}_{j\in\mathbb{N}}$ such that

$$Dw_{\delta_j}(x) \to Du_*(x) \quad \text{for a.e. } x \in B_\rho.$$

Using (3.29) together with Fatou's lemma we obtain

$$\begin{aligned}\int_{B_\rho} \tilde{f}(|Du_*(x)|)\ dx &\leqslant \liminf_{j\to+\infty} \int_{B_\rho} \tilde{f}(|Dw_{\delta_j}(x)|)\ dx\\ &\leqslant \limsup_{j\to+\infty} \int_{B_\rho} \tilde{f}(|Dw_{\delta_j}(x)|)\ dx\\ &\leqslant \limsup_{j\to+\infty} \int_{B_{\rho+\delta_j}} \tilde{f}(|Du_*(x)|)\ dx\\ &= \lim_{j\to+\infty} \int_{B_{\rho+\delta_j}} \tilde{f}(|Du_*(x)|)\ dx\\ &= \int_{B_\rho} \tilde{f}(|Du_*(x)|)\ dx,\end{aligned}$$

since $\delta_j \to 0$ as $j \to +\infty$ and $\tilde{f}(|Du_*|) \in L^1(B_R)$. Namely

$$\int_{B_\rho} \tilde{f}(|Dw_{\delta_j}(x)|)\, dx \to \int_{B_\rho} \tilde{f}(|Du_*(x)|)\, dx.$$

At this point we can apply the generalized dominate convergence theorem to get

$$\int_{B_\rho} \tilde{f}(|Du_*(x) - Dw_{\delta_j}(x)|)\, dx \to 0.$$

Then there exists $\{w_{\delta_{j_\ell}}\}_{\ell\in\mathbb{N}}$ such that

$$\int_{B_\rho} \tilde{f}(|Du_*(x)) - Dw_{\delta_{j_\ell}}(x)|)\, dx \leqslant \frac{1}{\ell^p}.$$

Hence, the sequence $\{v_\ell\}_{\ell\in\mathbb{N}}$ we were looking for is defined by

$$v_\ell \coloneqq w_{\delta_{j_\ell}}.$$

At this point we consider the following perturbed functional

$$\tilde{\mathcal{G}}_\ell(u) \coloneqq \int_{B_\rho} \Big[f(x, Du(x)) + \ell\tilde{f}(|Du(x) - Dv_\ell(x)|) \Big]\, dx.$$

Let us set

$$\tilde{g}_\ell(x,z) \coloneqq f(x,z) + \ell\tilde{f}(|z - Dv_\ell(x)|).$$

It is easy to see that, by the direct method of the Calculus of Variations, for any $\ell \in \mathbb{N}$ there exists $u_\ell \in u_* + W_0^{1,p}(B_\rho, \mathbb{R}^N)$ such that

$$\min_{u\in u_* + W_0^{1,p}(B_\rho,\mathbb{R}^N)} \tilde{\mathcal{G}}_\ell(u) = \tilde{\mathcal{G}}_\ell(u_\ell).$$

To obtain Morrey regularity for the minimizer u_ℓ we aim to verify that $\tilde{g}_\ell$ satisfies the same hypothesis of f, in order to apply Theorem 1.1 in [70]. Recalling that $\tilde{f}$ is increasing and using (3.24) we achive the existence of a constant $L_\ell \geqslant 1$ and a function $a_\ell \in L^{1,\lambda}$ both depending on ℓ such that

$$\tilde{g}_\ell(x,z) \leqslant L_\ell |z|^q + a_\ell(x),$$

for all $x \in \Omega$ and $z \in \mathbb{R}^{N\times n}$. Now, by (3.25) we obtain that for all $\tilde{\varepsilon} \in (0,1]$

$$(1-\tilde{\varepsilon})^q \tilde{f}(|z|) \leqslant \tilde{f}((1-\tilde{\varepsilon})|z|) \leqslant \tilde{f}(|z - Dv_\ell(x)|) \leqslant \tilde{f}((1+\tilde{\varepsilon})|z|) \leqslant (1+\tilde{\varepsilon})^q(|z|),$$

whenever $|z| > \frac{|Dv_\ell(x)|}{\tilde{\varepsilon}}$. Then, for all $\varepsilon, \tilde{\varepsilon} \in (0,1]$ there is $\tilde{\sigma}_{\varepsilon,\tilde{\varepsilon}}(x) \coloneqq \sigma_\varepsilon(x) + \frac{|Dv_\ell(x)|}{\tilde{\varepsilon}}$ such that if $|z| > \tilde{\sigma}_{\varepsilon,\tilde{\varepsilon}}(x)$ we have

$$\begin{aligned} \tilde{g}_\ell(x,z) &= f(x,z) + \ell\tilde{f}(|Du(x) - Dv_\ell(x)|) \\ &\leqslant (1+\varepsilon)\tilde{f}(|z|) + \ell(1+\tilde{\varepsilon})^q \tilde{f}(|z|) \\ &= [1 + \varepsilon + \ell(1+\tilde{\varepsilon})^q]\tilde{f}(|z|), \end{aligned}$$

and, in similar fashion,

$$\tilde{g}_\ell(x,z) \geqslant [(1-\varepsilon) + \ell(1-\tilde{\varepsilon})^q]\tilde{f}(|z|).$$

Taking

$$\tilde{\varepsilon} = \min\left\{1-(1-\varepsilon)^{\frac{1}{q}}, (1+\varepsilon)^{\frac{1}{q}} - 1\right\},$$

we get

$$(1-\varepsilon)(1+\ell)\tilde{f}(|z|) \leqslant \tilde{g}_\ell(x,z) \leqslant (1+\varepsilon)(1+\ell)\tilde{f}(|z|).$$

Let us set

$$\tilde{f}_\ell(t) := (1+\ell)\tilde{f}(t).$$

It is straightfoward to check that $\tilde{f}_\ell$ is convex and satisfies (3.21). Therefore, by [70, Theorem 1.1] the minimizer u_ℓ enjoys

$$u_\ell \in (u_* + W_0^{1,p}(B_\rho, \mathbb{R}^N)) \cap W^{1,(p,\lambda)}_{\mathrm{loc}}(B_\rho, \mathbb{R}^N).$$

Now, bearing in mind (3.27) and (3.28) and following Step 2 of Theorem 3.2.5 we get that $u_\ell \to u_*$ strongly in $W^{1,p}(B_\rho, \mathbb{R}^N)$ and $\mathcal{F}(u_\ell, B_\rho) \to \mathcal{F}(u_*, B_\rho)$. We have just proved the following theorem.

Theorem 3.3.3. *Let $f : \Omega \times \mathbb{R}^{N\times n} \to [0,+\infty)$ be a Carathéodory function such that $z \mapsto f(x,z)$ is convex and (3.22), (3.23) are satisfied with $\tilde{f}$ as in (3.21) and (3.27). Let u_* be a function in $W^{1,p}(B_R, \mathbb{R}^N)$ such that $\mathcal{F}(u_*, B_R) < +\infty$, where $B_R \subset \Omega$. Fix $\rho < R$. Then there exists a sequence*

$$\{u_\ell\}_{\ell\in\mathbb{N}} \subset (u_* + W_0^{1,p}(B_\rho, \mathbb{R}^N)) \cap W^{1,(p,\lambda)}_{loc}(B_\rho, \mathbb{R}^N)$$

such that $u_\ell \to u$ strongly in $W^{1,p}(B_\rho, \mathbb{R}^N)$ and

$$\mathcal{F}(u_\ell, B_\rho) \to \mathcal{F}(u_*, B_\rho).$$

An example of function that satisfies the hypothesis of Theorem 3.3.3 is

$$f(x,z) := \tilde{f}(|z| + b(x)),$$

for a suitable $b(x) \geqslant 0$, where

$$\tilde{f}(t) := t^p \ \ln(e+t).$$

3.3.2 Scalar functionals and bounded gradients

In this last section we consider an example in the multidimensional scalar case. We deal with the following functional

$$\mathcal{F}(u,\Omega) := \int_\Omega f(x, u(x), Du(x)) \ dx,$$

with $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is open and bounded, $n \geqslant 2$. We assume that the density f has the following form

$$f(x, u, z) := d(z) + h(x, u),$$

and satisfies the hypothesis of Theorem 1.1 in [37], that we recall here for the convenience of the reader. Let p and q be such that

$$1 < p < q < p\left(\frac{n+1}{n}\right). \tag{3.30}$$

The function $d : \mathbb{R}^n \to [0, +\infty)$ has the (p, q)-growth (H2) and is p-uniformly convex at infinity, that is, there exist $\nu, R > 0$ such that

$$d\left(\frac{z_1 + z_2}{2}\right) \leqslant \frac{1}{2}d(z_1) + \frac{1}{2}d(z_2) - \nu(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_1 - z_2|^2 \tag{3.31}$$

when the line segment joining z_1 and z_2 is all outside $B(0, R)$. The function $h : \Omega \times \mathbb{R} \to [0, +\infty)$ is of Carathéodory type and $x \mapsto h(x, 0) \in L^1(\Omega, \mathbb{R})$. Moreover, there exists a function $a \in L^\infty_{\text{loc}}(\Omega, \mathbb{R})$ such that

$$|h(x, u_1) - h(x, u_2)| \leqslant a(x)|u_1 - u_2|, \quad \text{for a.e. } x \in \Omega, \text{ for all } u_1, u_2 \in \mathbb{R}. \tag{3.32}$$

We stress that, unlike [37], we require that d is also convex everywhere. Morevover, here we ask that $h(x, u) \geqslant 0$.

Our target is to approximate every $u_* \in W^{1,p}(B_R, \mathbb{R})$ with a sequence $\{u_\ell\}_{\ell \in \mathbb{N}} \subset (u_* + W_0^{1,p}(B_R, \mathbb{R})) \cap W^{1,\infty}_{\text{loc}}(B_R, \mathbb{R})$ and show the approximation in energy. So, let $u_* \in W^{1,p}(B_R, \mathbb{R})$ be a function such that $\mathcal{F}(u_*, B_R) < +\infty$.

We consider the following penalization of $\mathcal{F}$

$$\tilde{\tilde{\mathcal{G}}}_\ell(u, B_R) := \int_{B_R} [f(x, u(x), Du(x)) + \ell|u(x) - u_*(x)|] \, dx.$$

By the direct method of the Calculus of Variations we get, for every $\ell \in \mathbb{N}$, a minimizer $u_\ell \in u_* + W_0^{1,p}(B_R, \mathbb{R})$ of $\tilde{\tilde{\mathcal{G}}}_\ell$. Now, in order to apply [37, Theorem 1.1] to $\tilde{\tilde{\mathcal{G}}}_\ell$, we just need to observe that $h(x, u) + \ell|u(x) - u_*(x)|$ keeps checking (3.32) and that $x \mapsto h(x, 0) + \ell|u_*(x)| \in L^1(B_R, \mathbb{R})$. Then the minimizer u_ℓ enjoys

$$u_\ell \in (u_* + W_0^{1,p}(B_R, \mathbb{R})) \cap W^{1,\infty}_{\text{loc}}(B_R, \mathbb{R}).$$

At this point, following Step 2 of Theorem 3.2.5, we get that $u_\ell \to u_*$ strongly in $L^1(B_R, \mathbb{R})$. But, by (H2), we have

$$L^{-1}||Du_\ell||^p_{L^p(B_R)} \leqslant \mathcal{F}(u_\ell, B_R) \leqslant \tilde{\tilde{\mathcal{G}}}_\ell(u_\ell, B_R) \leqslant \tilde{\tilde{\mathcal{G}}}_\ell(u_*, B_R) = \mathcal{F}(u_*, B_R),$$

then, by the previous inequality, up to not relabelled subsequences, we may suppose that there exists $u_\infty \in u_* + W_0^{1,p}(B_R, \mathbb{R})$ such that

$$u_\ell \to u_\infty \quad \text{strongly in } L^p(B_R, \mathbb{R})$$

$$Du_\ell \rightharpoonup Du_\infty \quad \text{weakly in } L^p(B_R, \mathbb{R}^n).$$

Therefore, due to the uniqueness of the limit, we can say that $u_\infty = u_*$ and

$$Du_\ell \rightharpoonup Du_* \quad \text{weakly in } L^p(B_R, \mathbb{R}).$$

Thus, the following theorem holds true.

Theorem 3.3.4. *Let p and q be as in (3.30). Let $d : \mathbb{R}^n \to [0, +\infty)$ be a convex function which verifies (H2) and (3.31) and let $h : \Omega \times \mathbb{R} \to [0, +\infty)$ be a Carathéodory function such that $x \mapsto h(x, 0) \in L^1(\Omega, \mathbb{R})$ and (3.32) is satisfied. Let us set*

$$\mathcal{F}(u, B_R) \coloneqq \int_{B_R} [d(Du(x)) + h(x, u(x))] \, dx.$$

Let u_* be a function in $W^{1,p}(B_R, \mathbb{R}^N)$ such that $\mathcal{F}(u_*, B_R) < +\infty$, where $B_R \subset \Omega$. Then there exists a sequence

$$\{u_\ell\}_{\ell \in \mathbb{N}} \subset (u_* + W_0^{1,p}(B_R, \mathbb{R})) \cap W^{1,\infty}_{loc}(B_R, \mathbb{R})$$

such that $u_\ell \rightharpoonup u$ weakly in $W^{1,p}(B_R, \mathbb{R})$ and

$$\mathcal{F}(u_\ell, B_R) \to \mathcal{F}(u_*, B_R).$$

3.4 Construction of the approximating densities

We now face the problem of how to construct the approximating functions f_k. We require that the function f is radial, i.e.

$$f(x, z) = \tilde{f}(x, |z|), \quad \text{for all } (x, z) \in \Omega \times \mathbb{R}^{N \times n}, \tag{H5}$$

for some $\tilde{f} : \Omega \times [0, +\infty) \to \mathbb{R}$. Moreover, we assume that f satisfies the following

$$L^{-1}(\mu^2 + |z|^2)^{\frac{p}{2}} \leqslant f(x, z) \leqslant L(1 + |z|^q) \tag{H2'}$$

in place of (H2). Let us observe that if we consider a function f that satisfies (H2) and (H3) then, if $p \geqslant 2$ it obviously satisfies (H3) and (H2') with $\mu = 0$. In the case where $p < 2$ we can consider $\hat{f} = f + c$, where c is a suitable constant and now $\hat{f}$ satisfies (H2') and (H3). Moreover, changing f with $\hat{f}$ does not affect our problem. In any case, we prefer to use (H2') because of the calculations involved in the proof of the next theorem. We point out that for the approximation we gain benefit from [45, Theorem 2.5 (ii)].

Theorem 3.4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\alpha \in (0, 1]$ and p, q be such that*

$$1 < p < q < p\left(\frac{n + \alpha}{n}\right).$$

Let $f : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}$ be a Carathéodory function. Assume that there exist $L \in [1, +\infty)$ and $\mu \in [0, 1]$ such that f satisfies (H1), (H2'), (H3), (H4) and (H5).

Then $\forall k \in \mathbb{N}$ *there exists a Carathéodory function* $f_k : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ *and a constant* $c_0(q) \in (1, +\infty)$ *such that*

$$z \mapsto f_k(x,z) \in C^1(\mathbb{R}^{N\times n}), \quad \textit{for all } x \in \Omega, \tag{AP1}$$

$$\frac{L^{-1}}{2\tilde{c}}(\mu^2+|z|^2)^{\frac{p}{2}} \leqslant f_k(x,z) \leqslant L(1+|z|^q), \tag{AP2}$$

$$\frac{L^{-1}}{2\tilde{c}} p \min\{p-1,1\} \left(\frac{1}{6}\right)^{p-1} \tag{AP3}$$

$$\times\, (\mu^2+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_1-z_2|^2 \leqslant \left\langle \frac{\partial f_k}{\partial z}(x,z_1) - \frac{\partial f_k}{\partial z}(x,z_2), z_1-z_2 \right\rangle,$$

$$\left|\frac{\partial f_k}{\partial z}(x,z) - \frac{\partial f_k}{\partial z}(y,z)\right| \leqslant L|x-y|^\alpha(1+|z|^{q-1}), \tag{AP4}$$

$$\left|\frac{\partial f_k}{\partial z}(x,z)\right| \leqslant 2[3c_0(q)L2^{\frac{q-1}{q}} + \frac{L^{-1}}{2\tilde{c}}p2^{\frac{p-1}{2}}](1+k^{q-1})(1+|z|^{p-1}), \tag{AP5}$$

$$\left|\frac{\partial f_k}{\partial z}(x,z) - \frac{\partial f_k}{\partial z}(y,z)\right| \leqslant L(1+k)^{q-1}|x-y|^\alpha(1+|z|^{p-1}), \tag{AP6}$$

$$f_k(x,z) \leqslant f(x,z), \quad \textit{for all } (x,z) \in \Omega \times \mathbb{R}^{N\times n}, \tag{AP7}$$

$$\lim_{k\to+\infty} f_k(x,z) = f(x,z), \quad \textit{for all } (x,z) \in \Omega \times \mathbb{R}^{N\times n}, \tag{AP8}$$

$$f_k(x,z) \leqslant f_{k+1}(x,z), \quad \textit{for all } (x,z) \in \Omega \times \mathbb{R}^{N\times n}, \tag{AP9}$$

where

$$\tilde{c} = \begin{cases} p\sqrt{10} & \textit{if } \ 1<p<2 \\ p(p-1)2^{\frac{p-2}{2}} & \textit{if } \ p \geqslant 2. \end{cases}$$

Proof. Let us start with the following preliminary observation. Let e^i_j be a versor of $\mathbb{R}^{N\times n}$, $i=1,\dots,N$, $j=1,\dots,n$, and note that, since $t \geqslant 0$,

$$\tilde{f}(x,t) = f(x,te^i_j).$$

We deduce that

$$\frac{\partial \tilde{f}}{\partial t}(x,t_0) = \frac{\partial f}{\partial z^i_j}(x,t_0e^i_j), \quad \forall t_0 \in [0,+\infty).$$

Moreover, since $f(x,z) = \tilde{f}(x,|z|)$, we have $f(x,-z) = f(x,z)$ and $\frac{\partial f}{\partial z_j^i}(x,0) = 0$. This implies

$$\frac{\partial \tilde{f}}{\partial t}(x,0) = 0.$$

Finally observe that

$$t \mapsto \tilde{f}(x,t) \in C^1([0,+\infty)).$$

We divide the proof into four steps.

Step 1: the function

$$g(x,z) = f(x,z) - \frac{L^{-1}}{2\tilde{c}}(\mu^2 + |z|^2)^{\frac{p}{2}} \geqslant 0$$

is radial and satisfies the following two properties

$$\left\langle \frac{\partial g}{\partial z}(x,z_1) - \frac{\partial g}{\partial z}(x,z_2), z_1 - z_2 \right\rangle \geqslant \frac{L^{-1}}{2}(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_1 - z_2|^2 \tag{3.33}$$

$$\left| \frac{\partial g}{\partial z}(x,z) \right| \leqslant [3c_0 L 2^{\frac{q-1}{q}} + \frac{L^{-1}}{\tilde{c}} p 2^{\frac{p-1}{2}}](1 + |z|^{q-1}) \tag{3.34}$$

where $\tilde{c}$ is as in the statement of the theorem and $c_0 = c_0(q) \in (1,+\infty)$

Let us define

$$f_\mu(z) := (\mu^2 + |z|^2)^{\frac{p}{2}},$$

we observe that

$$z \mapsto g(x,z) \in C^1(\mathbb{R}^{N\times n})$$

and

$$\frac{\partial g}{\partial z_j^i}(x,z) = \frac{\partial f}{\partial z_j^i}(x,z) - \frac{L^{-1}}{2\tilde{c}}\frac{\partial f_\mu}{\partial z_j^i}(z).$$

In order to prove (3.33) let us write the following inequalities:

$$\begin{aligned}
&\left\langle \frac{\partial g}{\partial z}(x,z_1) - \frac{\partial g}{\partial z}(x,z_2), z_1 - z_2 \right\rangle \\
&\quad = \left\langle \frac{\partial f}{\partial z}(x,z_1) - \frac{\partial f}{\partial z}(x,z_2), z_1 - z_2 \right\rangle - \frac{L^{-1}}{2\tilde{c}}\left\langle \frac{\partial f_\mu}{\partial z}(z_1) - \frac{\partial f_\mu}{\partial z}(z_2), z_1 - z_2 \right\rangle \\
&\quad \overset{(H3)}{\geqslant} L^{-1}(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_1 - z_2|^2 - \frac{L^{-1}}{2\tilde{c}}\underbrace{\left\langle \frac{\partial f_\mu}{\partial z}(z_1) - \frac{\partial f_\mu}{\partial z}(z_2), z_1 - z_2 \right\rangle}_{\mathrm{I}}.
\end{aligned} \tag{3.35}$$

Now we study I as p and μ change

- Case 1: $p \geqslant 2$ and $0 \leqslant \mu \leqslant 1$.

  First we deal with the case $p \geqslant 2$ and $0 < \mu \leqslant 1$. We observe that

$$
\begin{aligned}
&\sum_{i,l=1}^{N}\sum_{j,q=1}^{n}\frac{\partial^2 f_\mu}{\partial z_q^l \partial z_j^i}(z)\lambda_q^l\lambda_j^i\\
&= \sum_{i,l=1}^{N}\sum_{j,q=1}^{n}\frac{p}{2}\left[\frac{p-2}{2}(\mu^2+|z|^2)^{\frac{p-4}{2}}2z_q^l 2z_j^i+(\mu^2+|z|^2)^{\frac{p-2}{2}}2\delta^{il}\delta_{jq}\right]\lambda_q^l\lambda_j^i\\
&= \sum_{i,l=1}^{N}\sum_{j,q=1}^{n}p(\mu^2+|z|^2)^{\frac{p-4}{2}}[(p-2)z_q^l z_j^i+(\mu^2+|z|^2)\delta^{il}\delta_{jq}]\lambda_q^l\lambda_j^i\\
&= \sum_{i,l=1}^{N}\sum_{j,q=1}^{n}p(\mu^2+|z|^2)^{\frac{p-4}{2}}[(p-2)z_q^l\lambda_q^l z_j^i\lambda_j^i+(\mu^2+|z|^2)\delta^{il}\delta_{jq}\lambda_q^l\lambda_j^i]\\
&= p(\mu^2+|z|^2)^{\frac{p-4}{2}}[(p-2)\langle z,\lambda\rangle^2+(\mu^2+|z|^2)|\lambda|^2\\
&\leqslant p(\mu^2+|z|^2)^{\frac{p-4}{2}}[(p-2)|z|^2|\lambda|^2+(\mu^2+|z|^2)|\lambda|^2]\\
&\leqslant p(\mu^2+|z|^2)^{\frac{p-4}{2}}[(p-2)(\mu^2+|z|^2)|\lambda|^2+(\mu^2+|z|^2)|\lambda|^2]\\
&= p(p-1)(\mu^2+|z|^2)^{\frac{p-2}{2}}|\lambda|^2.
\end{aligned}
\tag{3.36}
$$

  Now,

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{n}\left(\frac{\partial f_\mu}{\partial z_j^i}(z_1)-\frac{\partial f_\mu}{\partial z_j^i}(z_2)\right)({z_1}_j^i-{z_2}_j^i)\\
&\quad= \sum_{i=1}^{N}\sum_{j=1}^{n}\left[\frac{\partial f_\mu}{\partial z_j^i}(z_2+t(z_1-z_2))\right]_{t=0}^{t=1}({z_1}_j^i-{z_2}_j^i)\\
&\quad= \sum_{i=1}^{N}\sum_{j=1}^{n}\int_0^1\frac{\partial}{\partial t}\left(\frac{\partial f_\mu}{\partial z_j^i}(z_2+t(z_1-z_2))\right)\,dt({z_1}_j^i-{z_2}_j^i)\\
&\quad= \int_0^1\sum_{i,l=1}^{N}\sum_{j,q=1}^{n}\frac{\partial^2 f_\mu}{\partial z_q^l\partial z_j^i}(z_2+t(z_1-z_2))({z_1}_q^l-{z_2}_q^l)({z_1}_j^i-{z_2}_j^i)\,dt\\
&\quad\overset{(3.36)}{\leqslant} p(p-1)\int_0^1(\mu^2+|z_2+t(z_1-z_2)|^2)^{\frac{p-2}{2}}\,dt|z_1-z_2|^2\\
&\quad\leqslant p(p-1)2^{\frac{p-2}{2}}(\mu^2+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_1-z_2|^2.
\end{aligned}
$$

  So we can conclude that

$$
\left\langle\frac{\partial f_\mu}{\partial z}(z_1)-\frac{\partial f_\mu}{\partial z}(z_2),z_1-z_2\right\rangle\leqslant p(p-1)2^{\frac{p-2}{2}}(\mu^2+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_1-z_2|^2.
$$

  Now we consider $p \geqslant 2$ and $\mu = 0$. Then, by a straightfoward consequence of the definition of $f_0(z) = |z|^p$ we can deduce that $f_0 \in C^1(\mathbb{R}^{N\times n})$ and

$$\lim_{\mu\to 0^+} \frac{\partial f_\mu}{\partial z_j^i}(z) = \frac{\partial f_0}{\partial z_j^i}(z), \quad \text{for all } z \in \mathbb{R}^{N\times n}.$$

We introduce the following functions which will be useful for writing the estimate for f_μ:

- if $\mu > 0$, then

$$h_\mu(z_1, z_2) := (\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} |z_1 - z_2|^2$$

- if $\mu = 0$, then

$$h_0(z_1, z_2) := \begin{cases} (|z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} |z_1 - z_2|^2 & \text{if } \ (z_1, z_2) \neq (0,0) \\ 0 & \text{if } \ (z_1, z_2) = (0,0). \end{cases}$$

It is easy to see that both h_0 and h_μ are $C^0(\mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n})$ and that

$$\lim_{\mu\to 0^+} h_\mu(z_1, z_2) = h_0(z_1, z_2), \quad \forall (z_1, z_2) \in \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n}.$$

Finally observing that

$$\sum_{i=1}^{N}\sum_{j=1}^{n} \left(\frac{\partial f_\mu}{\partial z_j^i}(z_1) - \frac{\partial f_\mu}{\partial z_j^i}(z_2) \right) ({z_1}_j^i - {z_2}_j^i) \leqslant p(p-1)2^{\frac{p-2}{2}} h_\mu(z_1, z_2)$$

and letting $\mu \to 0^+$ we get

$$\sum_{i=1}^{N}\sum_{j=1}^{n} \left(\frac{\partial f_0}{\partial z_j^i}(z_1) - \frac{\partial f_0}{\partial z_j^i}(z_2) \right) ({z_1}_j^i - {z_2}_j^i) = p(p-1)2^{\frac{p-2}{2}} h_0(z_1, z_2)$$

which implies the following: if $p \geqslant 2$ and $0 \leqslant \mu \leqslant 1$ then

$$\left\langle \frac{\partial f_\mu}{\partial z}(z_1) - \frac{\partial f_\mu}{\partial z}(z_2), z_1 - z_2 \right\rangle \leqslant p(p-1)2^{\frac{p-2}{2}} (\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} |z_1 - z_2|^2$$

where $(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} |z_1 - z_2|^2 = 0$ if $\mu = 0$ and $(z_1, z_2) = (0,0)$.

- Case 2: $1 < p < 2$ and $0 \leqslant \mu \leqslant 1$.

  If $z_1 \neq z_2$ we use Lemma 2.1.11 for f_μ, which gives us

$$\begin{aligned} \left| \left\langle \frac{\partial f_\mu}{\partial z}(z_1) - \frac{\partial f_\mu}{\partial z}(z_2), z_1 - z_2 \right\rangle \right| &\leqslant \left| \frac{\partial f_\mu}{\partial z}(z_1) - \frac{\partial f_\mu}{\partial z}(z_2) \right| |z_1 - z_2| \\ &\leqslant p\sqrt{10}(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} |z_1 - z_2|^2, \end{aligned}$$

  we underline that the constant $p\sqrt{10}$ is derived from the proof of the same lemma. The case $z_1 = z_2$ is trivial.

Summarizing both case 1 and case 2 we obtain

$$\left\langle \frac{\partial f_\mu}{\partial z}(z_1) - \frac{\partial f_\mu}{\partial z}(z_2), z_1 - z_2 \right\rangle \leqslant \tilde{c}(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} |z_1 - z_2|^2.$$

At this point we jump back and we put the last inequality into (3.35) in order to have

$$\left\langle \frac{\partial g}{\partial z}(x, z_1) - \frac{\partial g}{\partial z}(x, z_2), z_1 - z_2 \right\rangle \geqslant \frac{L^{-1}}{2}(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} |z_1 - z_2|^2,$$

notably, $z \mapsto g(x, z)$ is convex.

Now we pass examining (3.34). We observe that if $z \neq 0$ then

$$\begin{aligned} \left| \frac{\partial f_\mu}{\partial z}(z) \right| &= p(\mu^2 + |z|^2)^{\frac{p-2}{2}} |z| \\ &\leqslant p(\mu^2 + |z|^2)^{\frac{p-2}{2}} p(\mu^2 + |z|^2)^{\frac{1}{2}} \\ &\leqslant p(1 + |z|^2)^{\frac{p-1}{2}} \\ &\leqslant p2^{\frac{p-1}{2}}(1 + |z|^{p-1}) \\ &\leqslant p2^{\frac{p+1}{2}}(1 + |z|^{q-1}), \end{aligned} \tag{3.37}$$

where for the last inequality we have used that $1 + |z|^{p-1} \leqslant 2(1 + |z|^{q-1})$, we point out that the same estimate is in force also for $z = 0$. Accordingly

$$\begin{aligned} \left| \frac{\partial g}{\partial z}(x, z) \right| &\leqslant \left| \frac{\partial f}{\partial z}(x, z) \right| + \frac{L^{-1}}{2\tilde{c}} \left| \frac{\partial f_\mu}{\partial z}(z) \right| \\ &\leqslant 3c_0 L 2^{\frac{q-1}{q}}(1 + |z|^{q-1}) + \frac{L^{-1}}{2\tilde{c}} p2^{\frac{p+1}{2}}(1 + |z|^{q-1}) \\ &= [3c_0 L 2^{\frac{q-1}{q}} + \frac{L^{-1}}{\tilde{c}} p2^{\frac{p-1}{2}}](1 + |z|^{q-1}) \end{aligned}$$

where $c_0 = c_0(q) \in (1, +\infty)$ and $3c_0 L 2^{(q-1)/q}$ is the constant coming from the application of Lemma 2.1.10 to f. At last, we stress that g is radial, in fact

$$g(x, z) = f(x, z) - \frac{L^{-1}}{2\tilde{c}}(\mu^2 + |z|^2)^{\frac{p}{2}} = \tilde{f}(x, |z|) - \frac{L^{-1}}{2\tilde{c}} \tilde{f}_\mu(|z|) =: \tilde{g}(x, |z|)$$

where $\tilde{f}_\mu(t) = (\mu^2 + t^2)^{\frac{p}{2}}$. Like before we notice that $\frac{\partial \tilde{g}}{\partial t}(x, 0) = 0$ and $t \mapsto \tilde{g}(x, t) \in C^1([0, +\infty))$. Finally, we state the following lemma in order to get that $\tilde{g}$ is convex and increasing on the second variable.

Lemma 3.4.2. *Let $\varphi : \mathbb{R}^{N \times n} \to \mathbb{R}$ be convex and such that*

$$\varphi(z) = \tilde{\varphi}(|z|)$$

for some $\tilde{\varphi} : [0, +\infty) \to \mathbb{R}$. Then $\tilde{\varphi}$ is convex and increasing.

<u>Step 2</u>: let $k \in \{1, \dots, n\}$ and define

$$\tilde{g}_k(x,t) := \begin{cases} \tilde{g}(x,t) & \text{if } \ 0 \leqslant t \leqslant k \\ \tilde{g}(x,k) + \dfrac{\partial \tilde{g}}{\partial t}(x,k)(t-k) & \text{if } \ t > k, \end{cases}$$

then $\frac{\partial \tilde{g}_k}{\partial t}(x,0) = 0$ and $t \mapsto \tilde{g}_k(x,t)$ is $C^1([0,+\infty))$ and convex, moreover $\tilde{g}_k$ is such that

$$\left| \frac{\partial \tilde{g}_k}{\partial t}(x,t) - \frac{\partial \tilde{g}_k}{\partial t}(y,t) \right| \leqslant L|x-y|^\alpha (1 + t^{q-1}) \tag{3.38}$$

$$\left| \frac{\partial \tilde{g}_k}{\partial t}(x,t) - \frac{\partial \tilde{g}_k}{\partial t}(y,t) \right| \leqslant L|x-y|^\alpha (1 + k^{q-1}) \tag{3.39}$$

$$\left| \frac{\partial \tilde{g}_k}{\partial t}(x,t) \right| \leqslant [3c_0 L 2^{\frac{q-1}{q}} + \frac{L^{-1}}{\tilde{c}} p 2^{\frac{p-1}{2}}](1 + k^{q-1}) \tag{3.40}$$

$$0 \leqslant \tilde{g}_k(x,t) \leqslant \tilde{g}(x,t), \quad \text{for all } (x,t) \in \Omega \times [0,+\infty) \tag{3.41}$$

$$\lim_{k \to +\infty} \tilde{g}_k(x,t) = \tilde{g}(x,t), \quad \text{for all } (x,t) \in \Omega \times [0,+\infty) \tag{3.42}$$

$$\tilde{g}_k(x,t) \leqslant \tilde{g}_{k+1}(x,t), \quad \text{for all } (x,t) \in \Omega \times [0,+\infty) \tag{3.43}$$

It is straightfoward to check that $\frac{\partial \tilde{g}_k}{\partial t}(x,0) = 0$ and $t \mapsto \tilde{g}_k(x,t)$ is $C^1([0,+\infty))$ and convex. Now let us move on to show (3.38)-(3.43), we will divide every single proof according to the interval to which t belongs. First of all we stress that

$$\frac{\partial \tilde{g}_k}{\partial t}(x,t) = \begin{cases} \dfrac{\partial \tilde{g}}{\partial t}(x,t) & \text{if } \ 0 \leqslant t \leqslant k \\ \dfrac{\partial \tilde{g}}{\partial t}(x,k) & \text{if } \ t > k, \end{cases}$$

then, regarding (3.38) and (3.39), we have the following assertions:

- if $0 \leqslant t \leqslant k$, then

$$\begin{aligned} \left| \frac{\partial \tilde{g}_k}{\partial t}(x,t) - \frac{\partial \tilde{g}_k}{\partial t}(y,t) \right| &= \left| \frac{\partial \tilde{g}}{\partial t}(x,t) - \frac{\partial \tilde{g}}{\partial t}(y,t) \right| \\ &= \left| \frac{\partial \tilde{f}}{\partial t}(x,t) - \frac{\partial \tilde{f}}{\partial t}(y,t) \right| \\ &= \left| \frac{\partial f}{\partial z_j^i}(x,te_j^i) - \frac{\partial f}{\partial z_j^i}(y,te_j^i) \right| \\ &\leqslant L|x-y|^\alpha (1 + t^{q-1}) \\ &\leqslant L|x-y|^\alpha (1 + k^{q-1}); \end{aligned}$$

- if $t > k$, then, in a similar fashion,

$$\left| \frac{\partial \tilde{g}_k}{\partial t}(x,t) - \frac{\partial \tilde{g}_k}{\partial t}(y,t) \right| \leqslant L|x-y|^\alpha (1 + k^{q-1}) \leqslant L|x-y|^\alpha (1 + t^{q-1}).$$

Now bearing in mind (3.34) we show (3.40)

- if $0 \leqslant t \leqslant k$, then

$$\begin{aligned}\left|\frac{\partial \tilde{g}_k}{\partial t}(x,t)\right| = \left|\frac{\partial \tilde{g}}{\partial t}(x,t)\right| &= \left|\frac{\partial g}{\partial z_j^i}(x, te_j^i)\right| \\ &\leqslant [3c_0 L 2^{\frac{q-1}{q}} + \frac{L^{-1}}{\tilde{c}} p 2^{\frac{p-1}{2}}](1+t^{q-1}) \\ &\leqslant [3c_0 L 2^{\frac{q-1}{q}} + \frac{L^{-1}}{\tilde{c}} p 2^{\frac{p-1}{2}}](1+k^{q-1});\end{aligned}$$

- if $t > k$, then

$$\begin{aligned}\left|\frac{\partial \tilde{g}_k}{\partial t}(x,t)\right| = \left|\frac{\partial \tilde{g}}{\partial t}(x,k)\right| &= \left|\frac{\partial g}{\partial z_j^i}(x, ke_j^i)\right| \\ &\leqslant [3c_0 L 2^{\frac{q-1}{q}} + \frac{L^{-1}}{\tilde{c}} p 2^{\frac{p-1}{2}}](1+k^{q-1}).\end{aligned}$$

Regarding (3.41), we have the following assertions:

- if $0 \leqslant t \leqslant k$, then
$$\tilde{g}_k(x,t) = \tilde{g}(x,t) \geqslant 0;$$
- if $t > k$, then
$$\tilde{g}_k(x,t) = \tilde{g}(x,k) + \frac{\partial \tilde{g}}{\partial t}(x,k)(t-k) \leqslant \tilde{g}(x,t)$$
because $t \mapsto \tilde{g}(x,t)$ is $C^1([0,+\infty))$ and convex. Note that $\tilde{g}(x,k) \geqslant 0$ and $\frac{\partial \tilde{g}}{\partial t}(x,k) \geqslant 0$, then $\tilde{g}_k(x,t) \geqslant 0$.

For the property (3.42) we proceed differently, first we fix $t \in [0,+\infty)$ and then we consider $k_t = [t]+1$, where $[t]$ is the integer part of t. Thus $0 \leqslant t < k_t$ and for all $k \geqslant k_t$ we have $\tilde{g}_k(x,t) = \tilde{g}(x,t)$, i.e.

$$\lim_{k\to+\infty} \tilde{g}_k(x,t) = \tilde{g}(x,t).$$

Finally for the last property we analyze three possibilities

- if $0 \leqslant t \leqslant k$, then
$$\tilde{g}_k(x,t) = \tilde{g}(x,t) = \tilde{g}_{k+1}(x,t);$$
- if $k < t \leqslant k+1$, we recall that $t \mapsto \tilde{g}(x,t)$ is $C^1([0,+\infty))$ and convex, then

$$\begin{aligned}\tilde{g}_k(x,t) &= \tilde{g}(x,k) + \frac{\partial \tilde{g}}{\partial t}(x,k)(t-k) \\ &\leqslant \tilde{g}(x,t) = \tilde{g}_{k+1}(x,t);\end{aligned}$$

- if $t > k+1$, we use this lemma

  Lemma 3.4.3. *Let $\tilde{\varphi} : [0,+\infty) \to \mathbb{R}$ be $C^1([0,+\infty))$ and convex, let $t_1, t_2 \in [0,+\infty)$ be such that $t_1 < t_2$, then*

  $$\tilde{\varphi}(t_1) + \tilde{\varphi}'(t_1)(t - t_1) \leqslant \tilde{\varphi}(t_2) + \tilde{\varphi}'(t_2)(t - t_2), \quad \forall t \geqslant t_2.$$

  Hence we can conclude that

  $$\begin{aligned} \tilde{g}_k(x,t) &= \tilde{g}(x,k) + \frac{\partial \tilde{g}}{\partial t}(x,k)(t-k) \\ &\leqslant \tilde{g}(x,k+1) + \frac{\partial \tilde{g}}{\partial t}(x,k+1)(t-(k+1)) \\ &= \tilde{g}_{k+1}(x,t). \end{aligned}$$

Step 3: let $k \in \{1,\dots,n\}$ and define

$$g_k(x,z) \coloneqq \tilde{g}_k(x,|z|),$$

then $\frac{\partial g_k}{\partial z}(x,0) = 0$ and $z \mapsto g_k(x,z)$ is $C^1(\mathbb{R}^{N\times n})$ and convex. Moreover, g_k is such that

$$\left|\frac{\partial g_k}{\partial z}(x,z) - \frac{\partial g_k}{\partial z}(y,z)\right| \leqslant L|x-y|^{\alpha}(1+|z|^{q-1}) \tag{3.44}$$

$$\left|\frac{\partial g_k}{\partial z}(x,z) - \frac{\partial g_k}{\partial z}(y,z)\right| \leqslant L(1+k)^{q-1}|x-y|^{\alpha}(1+|z|^{p-1}) \tag{3.45}$$

$$\left|\frac{\partial g_k}{\partial z}(x,z)\right| \leqslant [3c_0 L 2^{\frac{q-1}{q}} + \frac{L^{-1}}{\tilde{c}} p 2^{\frac{p-1}{2}}](1+k^{q-1})(1+|z|^{p-1}) \tag{3.46}$$

$$g_k(x,z) \leqslant g(x,z), \quad \forall (x,z) \in \Omega \times \mathbb{R}^{N\times n} \tag{3.47}$$

$$\lim_{k\to+\infty} g_k(x,z) = g(x,z), \quad \forall (x,z) \in \Omega \times \mathbb{R}^{N\times n} \tag{3.48}$$

$$g_k(x,z) \leqslant g_{k+1}(x,z), \quad \forall (x,z) \in \Omega \times \mathbb{R}^{N\times n} \tag{3.49}$$

We use the following lemmas to deduce that $\frac{\partial g_k}{\partial z}(x,0) = 0$ and $z \mapsto g_k(x,z)$ is $C^1(\mathbb{R}^{N\times n})$ and convex.

Lemma 3.4.4. *Let $\tilde{\varphi} : [0,+\infty) \to \mathbb{R}$ be $C^1([0,+\infty))$, with $\tilde{\varphi}'(0) = 0$, and such that*

$$\tilde{\varphi}(|z|) = \varphi(z)$$

for some $\varphi : \mathbb{R}^{N\times n} \to \mathbb{R}$. Then φ is $C^1(\mathbb{R}^{N\times n})$ and

$$\frac{\partial \varphi}{\partial z^i_j}(z) = \begin{cases} \tilde{\varphi}'(|z|)\dfrac{z^i_j}{|z|} & \text{if } \ z \neq 0 \\ 0 & \text{if } \ z = 0. \end{cases}$$

Lemma 3.4.5. *Let $\tilde{\varphi} : [0, +\infty) \to \mathbb{R}$ be convex, increasing and such that*

$$\tilde{\varphi}(|z|) = \varphi(z)$$

for some $\varphi : \mathbb{R}^{N\times n} \to \mathbb{R}$. Then φ is convex.

In order to prove (3.44)-(3.46), we observe that

$$\frac{\partial g_k}{\partial z_j^i}(x,z) = \begin{cases} \dfrac{\partial \tilde{g}_k}{\partial t}(x,|z|)\dfrac{z_j^i}{|z|} & \text{if } \ z \neq 0 \\ 0 & \text{if } \ z = 0, \end{cases}$$

then, for $z \neq 0$

$$\begin{aligned} \left|\frac{\partial g_k}{\partial z}(x,z) - \frac{\partial g_k}{\partial z}(y,z)\right| &= \left|\frac{\partial \tilde{g}_k}{\partial t}(x,|z|) - \frac{\partial \tilde{g}_k}{\partial t}(y,|z|)\right| \\ &\overset{(3.38)}{\leqslant} L|x-y|^{\alpha}(1+|z|^{q-1}), \end{aligned}$$

$$\begin{aligned} \left|\frac{\partial g_k}{\partial z}(x,z) - \frac{\partial g_k}{\partial z}(y,z)\right| &= \left|\frac{\partial \tilde{g}_k}{\partial t}(x,|z|) - \frac{\partial \tilde{g}_k}{\partial t}(y,|z|)\right| \\ &\overset{(3.39)}{\leqslant} L(|x-y|^{\alpha}(1+k^{q-1}) \\ &\leqslant L(1+k^{q-1})|x-y|^{\alpha}(1+|z|^{p-1}) \end{aligned}$$

and

$$\begin{aligned} \left|\frac{\partial g_k}{\partial z}(x,z)\right| = \left|\frac{\partial \tilde{g}_k}{\partial t}(x,|z|)\right| &\overset{(3.40)}{\leqslant} [3c_0 L 2^{\frac{q-1}{q}} + \frac{L^{-1}}{\tilde{c}} p 2^{\frac{p-1}{2}}](1+k^{q-1}) \\ &\leqslant [3c_0 L 2^{\frac{q-1}{q}} + \frac{L^{-1}}{\tilde{c}} p 2^{\frac{p-1}{2}}](1+k^{q-1})(1+|z|^{p-1}), \end{aligned}$$

and the same estimates are valid also for $z = 0$. Now to conclude we note that using Step 2 again we achieve

$$g_k(x,z) = \tilde{g}_k(x,|z|) \leqslant \tilde{g}(x,|z|) = g(x,z),$$

$$\lim_{k\to+\infty} g_k(x,z) = \lim_{k\to+\infty} \tilde{g}_k(x,|z|) = \tilde{g}(x,|z|) = g(x,z)$$

and

$$g_k(x,z) = \tilde{g}_k(x,|z|) \leqslant \tilde{g}_{k+1}(x,|z|) = g_{k+1}(x,z).$$

Step 4: let $k \in \{1,\dots,n\}$ and define the approximating function

$$f_k(x,z) \coloneqq g_k(x,z) + \frac{L^{-1}}{2\tilde{c}} f_\mu(z),$$

then f_k satisfies (AP1)-(AP9)

Once noted that $z \mapsto f_k(x,z) \in C^1(\mathbb{R}^{N\times n})$, let us move on to demonstrate (AP2)-(AP9) making use of the previous steps. For (AP2) we can directly observe that since $g_k \geqslant 0$

$$\frac{L^{-1}}{2\tilde{c}}(\mu^2+|z|^2)^{\frac{p}{2}} \leqslant f_k(x,z) \overset{(3.47)}{\leqslant} g(x,z) + \frac{L^{-1}}{2\tilde{c}} f_\mu(x,z) \overset{(\mathrm{H2'})}{\leqslant} L(1+|z|^q).$$

For (AP3) we have to make a further calculation. Following the Step 1 we write

$$\begin{aligned}\sum_{i,l=1}^{N}\sum_{j,q=1}^{n} \frac{\partial^2 f_\mu}{\partial z_q^l \partial z_j^i}(z)\lambda_q^l\lambda_j^i &= p(\mu^2+|z|^2)^{\frac{p-4}{2}}[(p-2)\langle z,\lambda\rangle^2 + (\mu^2+|z|^2)|\lambda|^2] \\ &\geqslant p\min\{p-1,1\}(\mu^2+|z|^2)^{\frac{p-2}{2}}|\lambda|^2,\end{aligned}$$

where, in order to have the last inequality, we have just distinguished the cases $1<p<2$ and $p \geqslant 2$. After using a calculation similar to the one that led to (3.36), we have

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{j=1}^{n}\left(\frac{\partial f_\mu}{\partial z_j^i}(z_1) - \frac{\partial f_\mu}{\partial z_j^i}(z_2)\right)({z_1}_j^i - {z_2}_j^i) = \\ &\int_0^1 \sum_{i,l=1}^{N}\sum_{j,q=1}^{n}\frac{\partial^2 f_\mu}{\partial z_q^l \partial z_j^i}(z_2+t(z_1-z_2))({z_1}_q^l - {z_2}_q^l)({z_1}_j^i - {z_2}_j^i)\, dt \geqslant \\ &p\min\{p-1,1\}\underbrace{\int_0^1 (\mu^2+|z_2+t(z_1-z_2)|^2)^{\frac{p-2}{2}}\, dt}_{\mathrm{II}}\,|z_1-z_2|^2.\end{aligned}$$

Now, if $p \geqslant 2$ and $0<\mu\leqslant 1$, let us control II from below by considering two cases:

- if $|z_1| \geqslant |z_2|$ let $a \in (0,1)$ and $t \in [1-a,1]$. Then, see [73]

$$\begin{aligned}|z_2+t(z_1-z_2)| = |tz_1+(1-t)z_2| &\geqslant t|z_1| - (1-t)|z_2| \\ &\geqslant (1-a)|z_1| - a|z_2| \\ &\geqslant \frac{1-a}{2}|z_1| + \frac{1-3a}{2}|z_2| \\ &\geqslant \frac{1-3a}{2}(|z_1|+|z_2|),\end{aligned}$$

  now taking $a \in \left(0, \frac{1}{3}\right)$ we get

$$|z_2+t(z_1-z_2)|^2 \geqslant \left(\frac{1-3a}{2}\right)^2(|z_1|^2+|z_2|^2),$$

  and so

$$\begin{aligned}\mathrm{II} &\geqslant \int_{1-a}^1 \left[\left(\frac{1-3a}{2}\right)^2(\mu^2+|z_1|^2+|z_2|^2)\right]^{\frac{p-2}{2}} dt \\ &= a\left(\frac{1-3a}{2}\right)^{p-2}(\mu^2+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}};\end{aligned}$$

- if $|z_1| \leqslant |z_2|$ let $b \in (0,1)$ and $t \in [0,b]$. Then

$$
\begin{aligned}
|z_2 + t(z_1 - z_2)| = |tz_1 + (1-t)z_2| &\geqslant (1-t)|z_2| - t|z_1| \\
&\geqslant (1-b)|z_2| - b|z_1| \\
&\geqslant \frac{1-b}{2}|z_2| + \frac{1-3b}{2}|z_1| \\
&\geqslant \frac{1-3b}{2}(|z_1| + |z_2|).
\end{aligned}
$$

Now we pick $b \in \left(0, \frac{1}{3}\right)$ and in the same way as in the previous case we get

$$
\begin{aligned}
\mathrm{II} &\geqslant \int_0^b \left[\left(\frac{1-3b}{2}\right)^2 (\mu^2 + |z_1|^2 + |z_2|^2)\right]^{\frac{p-2}{2}} dt \\
&= b\left(\frac{1-3b}{2}\right)^{p-2} (\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}.
\end{aligned}
$$

Now choosing $a = b = \frac{1}{6}$ we obtain

$$
\mathrm{II} \geqslant \left(\frac{1}{6}\right)^{p-1} (\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}, \quad \forall z_1, z_2 \in \mathbb{R}^{N\times n}.
$$

If $1 < p < 2$ and $0 < \mu \leqslant 1$ we use the convexity of $z \mapsto |z|^2$ and the fact that $p - 2 < 0$, consequently $\mathrm{II} \geqslant (\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}$, see [2]).

Hence summarizing, we achieve

$$
\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{n} \left(\frac{\partial f_\mu}{\partial z_j^i}(z_1) - \frac{\partial f_\mu}{\partial z_j^i}(z_2)\right)({z_1}_j^i - {z_2}_j^i) &\geqslant p\min\{p-1, 1\}\left(\frac{1}{6}\right)^{p-1} \\
&\quad \times (\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_1 - z_2|^2,
\end{aligned}
$$

when $p > 1$ and $0 < \mu \leqslant 1$. For the case $\mu = 0$ we introduce the functions h_0 and h_μ defined as in the Step 1 and therefore in a similar fashion we get

$$
\begin{aligned}
\left\langle \frac{\partial f_\mu}{\partial z}(z_1) - \frac{\partial f_\mu}{\partial z}(z_2), z_1 - z_2 \right\rangle &\geqslant p\min\{p-1, 1\}\left(\frac{1}{6}\right)^{p-1} \\
&\quad \times (\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_1 - z_2|^2, \qquad (3.50)
\end{aligned}
$$

where $(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_1 - z_2|^2 = 0$ if $\mu = 0$ and $(z_1, z_2) = (0,0)$.

We are ready to prove (AP3)

$$
\begin{aligned}
\left\langle \frac{\partial f_k}{\partial z}(x,z_1) - \frac{\partial f_k}{\partial z}(x,z_2), z_1 - z_2 \right\rangle &= \left\langle \frac{\partial g_k}{\partial z}(x,z_1) - \frac{\partial g_k}{\partial z}(x,z_2), z_1 - z_2 \right\rangle \\
&\quad + \frac{L^{-1}}{2\tilde{c}} \left\langle \frac{\partial f_\mu}{\partial z}(z_1) - \frac{\partial f_\mu}{\partial z}(z_2), z_1 - z_2 \right\rangle \\
&\geqslant \frac{L^{-1}}{2\tilde{c}} \left\langle \frac{\partial f_\mu}{\partial z}(z_1) - \frac{\partial f_\mu}{\partial z}(z_2), z_1 - z_2 \right\rangle \\
&\overset{(3.50)}{\geqslant} \frac{L^{-1}}{2\tilde{c}} p \min\{p-1, 1\} \left(\frac{1}{6}\right)^{p-1} \\
&\quad \times (\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} |z_1 - z_2|^2,
\end{aligned}
$$

where we have used the fact that $\left\langle \frac{\partial g_k}{\partial z}(x,z_1) - \frac{\partial g_k}{\partial z}(x,z_2), z_1 - z_2 \right\rangle \geqslant 0$ as a consequence of the convexity of $z \mapsto g_k(x,z)$. As far as (AP4)-(AP6) are concerned, we observe that

$$
\left| \frac{\partial f_k}{\partial z}(x,z) - \frac{\partial f_k}{\partial z}(y,z) \right| = \left| \frac{\partial g_k}{\partial z}(x,z) - \frac{\partial g_k}{\partial z}(y,z) \right| \overset{(3.44)}{\leqslant} L|x-y|^\alpha (1 + |z|^{q-1}),
$$

by (3.37) and (3.46)

$$
\begin{aligned}
\left| \frac{\partial f_k}{\partial z}(x,z) \right| &\leqslant \left| \frac{\partial g_k}{\partial z}(x,z) \right| + \frac{L^{-1}}{2\tilde{c}} \left| \frac{\partial f_\mu}{\partial z}(z) \right| \\
&\leqslant [(3c_0 L 2^{\frac{q-1}{q}} + \frac{L^{-1}}{2\tilde{c}} p 2^{\frac{p+1}{2}})(1 + k^{q-1}) + \frac{L^{-1}}{2\tilde{c}} p 2^{\frac{p-1}{2}}](1 + |z|^{p-1}) \\
&\leqslant 2[3c_0 L 2^{\frac{q-1}{q}} + \frac{L^{-1}}{2\tilde{c}} p 2^{\frac{p+1}{2}}](1 + k^{q-1})(1 + |z|^{p-1}),
\end{aligned}
$$

finally

$$
\begin{aligned}
\left| \frac{\partial f_k}{\partial z}(x,z) - \frac{\partial f_k}{\partial z}(y,z) \right| &= \left| \frac{\partial g_k}{\partial z}(x,z) - \frac{\partial g_k}{\partial z}(y,z) \right| \\
&\overset{(3.45)}{\leqslant} L(1+k)^{q-1} |x-y|^\alpha (1 + |z|^{p-1}).
\end{aligned}
$$

It remains to prove (AP7)-(AP9), which, as we will see, are a direct application of the Step 3

$$
f_k(x,z) = g_k(x,z) + \frac{L^{-1}}{2\tilde{c}} f_\mu(z) \overset{(3.47)}{\leqslant} g(x,z) + \frac{L^{-1}}{2\tilde{c}} f_\mu(z) = f(x,z),
$$

$$
\lim_{k \to +\infty} f_k(x,z) = \lim_{k \to +\infty} g_k(x,z) + \frac{L^{-1}}{2\tilde{c}} f_\mu(z) \overset{(3.48)}{=} g(x,z) + \frac{L^{-1}}{2\tilde{c}} f_\mu(z) = f(x,z)
$$

and finally

$$
f_k(x,z) = g_k(x,z) + \frac{L^{-1}}{2\tilde{c}} f_\mu(z) \overset{(3.49)}{\leqslant} g_{k+1}(x,z) + \frac{L^{-1}}{2\tilde{c}} f_\mu(z) = f_{k+1}(x,z).
$$

The proof is finished. □

A direct consequence of Theorems 3.2.5 and 3.4.1 is the following corollary.

Corollary 3.4.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\alpha \in (0,1]$ and p, q be such that*

$$1 < p < q < p\left(\frac{n+\alpha}{n}\right).$$

Let $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Carathéodory function such that there exist constants $L \in [1,+\infty)$ and $\mu \in [0,1]$: f satisfies (H1), (H2'), (H3), (H4), (H5). Let u_* be a function in $W^{1,p}(B_R, \mathbb{R}^N)$ such that $\mathcal{F}(u_*, B_R) < +\infty$. Then

$$\mathcal{L}(u_*) = 0.$$

Proof. We observe that by Theorem 3.4.1 we get that $\forall k \in \mathbb{N}$ there exist a Carathéodory function $f_k : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ and positive constants $\tilde{L}$ and $c(k)$ such that for all $(x, z) \in \Omega \times \mathbb{R}^{N\times n}$ the hypothesis (AP1)-(AP9) of Theorem 3.2.1 are satisfied. The proof proceeds in the same way as that of Theorem 3.2.5. □

3.5 Example

Now we give an example covered by Theorem 3.2.1. Let Ω and p, q, α be as in the hypothesis of the theorem. We define

$$f(x, z) := g(x, z) + (\mu^2 + |z|^2)^{\frac{p}{2}},$$

where $g(x, z) := \tilde{g}(x, |z|)$ and $\tilde{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ is defined in the following way

$$\tilde{g}(x,t) := \begin{cases} |x_1|^\alpha (t - |x_1|^\alpha)^q & \text{if } \ t > |x_1|^\alpha \\ 0 & \text{if } \ t \leqslant |x_1|^\alpha, \end{cases}$$

where $x = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n$. The function $\tilde{g}$ behaves as in Figure 3.1. Firstly, note that, since Ω is bounded, there exists $M > 0$ such that $|x_1| \leqslant M$, consequently

$$0 \leqslant \tilde{g}(x,t) \leqslant M^\alpha |t|^q \leqslant M^\alpha (1 + |t|^q)$$

so

$$0 \leqslant g(x, z) \leqslant M^\alpha (1 + |z|^q). \tag{3.51}$$

Now we observe that $t \mapsto \tilde{g}(x, t) \in C^0(\mathbb{R})$ and that if $x_1 \neq 0$ then

$$\frac{\partial \tilde{g}}{\partial t}(x,t) = \begin{cases} |x_1|^\alpha q (t - |x_1|^\alpha)^{q-1} & \text{if } \ t > |x_1|^\alpha \\ 0 & \text{if } \ t \leqslant |x_1|^\alpha, \end{cases}$$

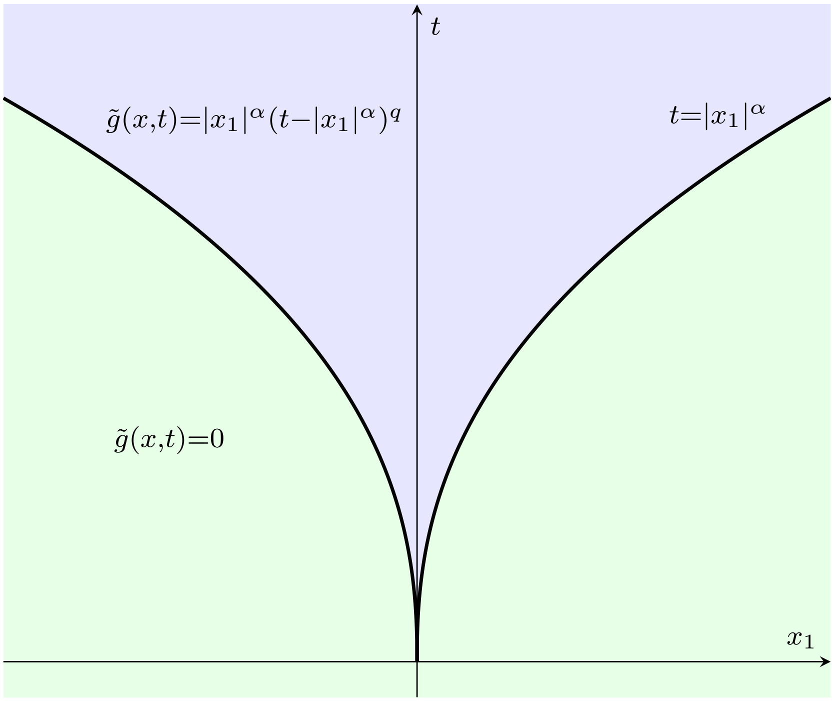

Figure 3.1: Strictly above $t = |x_1|^\alpha$ the function $\tilde{g}$ is equal to $|x_1|^\alpha(t - |x_1|^\alpha)^q$ while below $\tilde{g}$ is zero

we point out that this formula is in force also for $x_1 = 0$. Hence $t \mapsto \tilde{g}(x,t) \in C^1(\mathbb{R})$. We also note that $\frac{\partial \tilde{g}}{\partial t}(x,0) = 0,\ \forall x \in \mathbb{R}$. Using Lemma 3.4.4 we can conclude that

$$\frac{\partial g}{\partial z^i_j}(x,z) = \begin{cases} \dfrac{\partial \tilde{g}}{\partial t}(x,|z|)\dfrac{z^i_j}{|z|} & \text{if } \ z \neq 0 \\ 0 & \text{if } \ z = 0 \end{cases}$$

and that $z \mapsto g(x,z) \in C^1(\mathbb{R}^{N\times n})$. The last property we want to verify is the following

$$\left|\frac{\partial g}{\partial z}(x,z) - \frac{\partial g}{\partial z}(y,z)\right| \leqslant q \max\Big\{(2M)^{\max\{0,\alpha(q-2)\}}M^\alpha, (2M)^{\max\{0,\alpha(2-q)\}} \\ + \max\{1, q-1\}M^\alpha\Big\}\, |x-y|^{\min\{\alpha,\alpha(q-1)\}}(1+|z|^{q-1}). \tag{3.52}$$

We start observing that for $z \neq 0$

$$\left|\frac{\partial g}{\partial z}(x,z) - \frac{\partial g}{\partial z}(y,z)\right| = \underbrace{\left|\frac{\partial \tilde{g}}{\partial t}(x,|z|) - \frac{\partial \tilde{g}}{\partial t}(y,|z|)\right|}_{\mathrm{I}}$$

and we go on estimating I according to the interval to which $|z|$ belongs:

- if $|z| \leqslant |x_1|^\alpha$ and $|z| \leqslant |y_1|^\alpha$, then
$$\mathrm{I} = 0;$$
- if $|x_1|^\alpha < |z| \leqslant |y_1|^\alpha$, we have
$$\begin{aligned}\mathrm{I} &= \Big||x_1|^\alpha q(|z| - |x_1|^\alpha)^{q-1}\Big| \\ &\leqslant |x_1|^\alpha q(|y_1|^\alpha - |x_1|^\alpha)^{q-1} \\ &\leqslant M^\alpha q|y-x|^{\alpha(q-1)}(1+|z|^{q-1}) \\ &\leqslant (2M)^{\alpha(q-1)-\min\{\alpha,\alpha(q-1)\}} M^\alpha q|y-x|^{\min\{\alpha,\alpha(q-1)\}}(1+|z|^{q-1}),\end{aligned}$$
where we have used the fact that for $a, b \in \mathbb{R}^n$ and $0 < \sigma \leqslant 1$
$$||a|^\sigma - |b|^\sigma| \leqslant |a-b|^\sigma;$$
- if $|y_1|^\alpha < |z| \leqslant |x_1|^\alpha$, then, like before,
$$\mathrm{I} \leqslant (2M)^{\alpha(q-1)-\min\{\alpha,\alpha(q-1)\}} M^\alpha q|x-y|^{\min\{\alpha,\alpha(q-1)\}}(1+|z|^{q-1});$$
- if $|z| > |x_1|^\alpha$ and $|z| > |y_1|^\alpha$, then,
$$\begin{aligned}\mathrm{I} &= ||x_1|^\alpha q(|z| - |x_1|^\alpha)^{q-1} - |y_1|^\alpha q(|z| - |y_1|^\alpha)^{q-1}| \\ &= ||x_1|^\alpha q(|z| - |x_1|^\alpha)^{q-1} - |y_1|^\alpha q(|z| - |x_1|^\alpha)^{q-1} \\ &\quad + |y_1|^\alpha q(|z| - |x_1|^\alpha)^{q-1} - |y_1|^\alpha q(|z| - |y_1|^\alpha)^{q-1}| \\ &\leqslant ||x_1|^\alpha - |y_1|^\alpha| q(|z| - |x_1|^\alpha)^{q-1} \\ &\quad + |y_1|^\alpha q|(|z| - |x_1|^\alpha)^{q-1} - (|z| - |y_1|^\alpha)^{q-1}| \\ &\leqslant q|x-y|^\alpha(1+|z|^{q-1}) + \underbrace{M^\alpha q|(|z| - |x_1|^\alpha)^{q-1} - (|z| - |y_1|^\alpha)^{q-1}|}_{\mathrm{II}}.\end{aligned}$$
We observe that for $1 < q \leqslant 2$ we have
$$\begin{aligned}\mathrm{II} &\leqslant M^\alpha q|(|z| - |x_1|^\alpha) - (|z| - |y_1|^\alpha)|^{q-1} \\ &\leqslant M^\alpha q|y-x|^{\alpha(q-1)}(1+|z|^{q-1}).\end{aligned}$$
When $q > 2$ we use the mean value theorem, then there exists $\xi \in [\min\{|z| - |x_1|^\alpha, |z| - |y_1|^\alpha\}, \max\{|z| - |x_1|^\alpha, |z| - |y_1|^\alpha\}]$ such that
$$\begin{aligned}\mathrm{II} &\leqslant M^\alpha q(q-1)\xi^{q-2}||z| - |x_1|^\alpha - (|z| - |y_1|^\alpha)| \\ &\leqslant M^\alpha q(q-1)\xi^{q-2}|y-x|^\alpha \\ &\leqslant M^\alpha q(q-1)|z|^{q-2}|y-x|^\alpha \\ &\leqslant M^\alpha q(q-1)|y-x|^\alpha(1+|z|^{q-1}).\end{aligned}$$
Hence for $|z| > |x_1|^\alpha$ and $|z| > |y_1|^\alpha$ we get
$$\begin{aligned}\mathrm{I} \leqslant q &\left[(2M)^{\alpha-\min\{\alpha,\alpha(q-1)\}} + \max\{1, q-1\}M^\alpha\right] \\ &\times |x-y|^{\min\{\alpha,\alpha(q-1)\}}(1+|z|^{q-1}).\end{aligned}$$

Summarizing our results we can conclude that

$$\begin{aligned}&\left|\frac{\partial g}{\partial z}(x,z)-\frac{\partial g}{\partial z}(y,z)\right|\\&\quad\leqslant q\max\Big\{(2M)^{\alpha(q-1)-\min\{\alpha,\alpha(q-1)\}}M^{\alpha},(2M)^{\alpha-\min\{\alpha,\alpha(q-1)\}}\\&\qquad+\max\{1,q-1\}M^{\alpha}\Big\}\,|x-y|^{\min\{\alpha,\alpha(q-1)\}}(1+|z|^{q-1}),\end{aligned}$$

we point out that this estimate is still valid also for $z=0$.

Now let us briefly check that the function f verifies (H1)-(H5). Conditions (H1) and (H5) are easy to be checked. As far as (H2') is concerned, we observe that, since $g\geqslant 0$, from below we can directly write

$$f(x,z)\geqslant(\mu^2+|z|^2)^{\frac{p}{2}};$$

we estimate f from above as follows

$$\begin{aligned}f(x,z)&\overset{(3.51)}{\leqslant} M^{\alpha}(1+|z|^q)+(1+|z|^2)^{\frac{q}{2}}\\&\leqslant M^{\alpha}(1+|z|^q)+2^{\frac{q}{2}}(1+|z|^q)\\&= (M^{\alpha}+2^{\frac{q}{2}})(1+|z|^q).\end{aligned}$$

Now we prove (H3): since $t\mapsto\tilde g(x,t)$ is convex and increasing then $z\mapsto\tilde g(x,|z|)$ is convex and so $\left\langle\frac{\partial g}{\partial z}(x,z_1)-\frac{\partial g}{\partial z}(x,z_2),z_1-z_2\right\rangle\geqslant 0$, hence

$$\begin{aligned}\left\langle\frac{\partial f}{\partial z}(x,z_1)-\frac{\partial f}{\partial z}(x,z_2),z_1-z_2\right\rangle&\overset{(3.50)}{\geqslant} p\min\{p-1,1\}\left(\frac{1}{6}\right)^{p-1}\\&\qquad\times(\mu^2+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_1-z_2|^2.\end{aligned}$$

Finally, in order to conclude, we observe that

$$\begin{aligned}\left|\frac{\partial f}{\partial z}(x,z)-\frac{\partial f}{\partial z}(y,z)\right|&=\left|\frac{\partial g}{\partial z}(x,z)-\frac{\partial g}{\partial z}(y,z)\right|\\&\overset{(3.52)}{\leqslant}|x-y|^{\min\{\alpha,\alpha(q-1)\}}(1+|z|^{q-1})\\&\times q\max\Big\{(2M)^{\max\{0,\alpha(q-2)\}}M^{\alpha},(2M)^{\max\{0,\alpha(2-q)\}}+\max\{1,q-1\}M^{\alpha}\Big\}.\end{aligned}$$

Let us show that our example fits neither all hypothesis of [68, Theorem 3.1] nor all of [44, Theorem 1.1]. Regarding the first-mentioned theorem we verify that assumption [68, Theorem 3.1 (4)] is not in force and we write it for the convenience of the reader:

- for $B_R\Subset\Omega,\varepsilon_0\in(0,1]$ with $B_{R+2\varepsilon_0}\Subset\Omega, x\in B_R$ and $\varepsilon\in(0,\varepsilon_0)$, there exists $\tilde y=\tilde y(x,\varepsilon)\in\overline{B(x,\varepsilon)}$ such that for $z\in\mathbb{R}^{N\times n}$ and $y\in\overline{B(x,\varepsilon)}$, we have $f(\tilde y,z)\leqslant f(y,z)$.

This means that the minimum point $\tilde{y}$ of $y \mapsto f(y,z)$ on small balls does not depend on z. We are going to show that in our example the minimum point $\tilde{y} \in \overline{B(x,\varepsilon)}$ depends on z.

For our purpose let us consider $x_1 > 0$ and $t > |x_1|^\alpha$, in this range $\tilde{g} = |x_1|^\alpha (t - |x_1|^\alpha)^q$. Now we want to understand when

$$\frac{\partial \tilde{g}}{\partial x_1}(x,t) \lesseqgtr 0$$

which some computations reveal to be equivalent to

$$x_1 \gtreqless \left(\frac{t}{1+q}\right)^{\frac{1}{\alpha}}.$$

We can also observe that $\left(\frac{t}{1+q}\right)^{\frac{1}{\alpha}} < \frac{t^{\frac{1}{\alpha}}}{2}$ and that for $x_1 = 0$ and $x_1 = t^{\frac{1}{\alpha}}$ the function $\tilde{g}$ is equal to zero. To make the ideas more clear we draw two graphs, see figure 3.2, of $s \mapsto s^\alpha (t - s^\alpha)^q$ which differ for suitably different t_1 and t_2, we also highlight in the same graphic the interval $(x_1 - \varepsilon, x_1 + \varepsilon)$, for some fixed x_1 and ε. Now we would like to

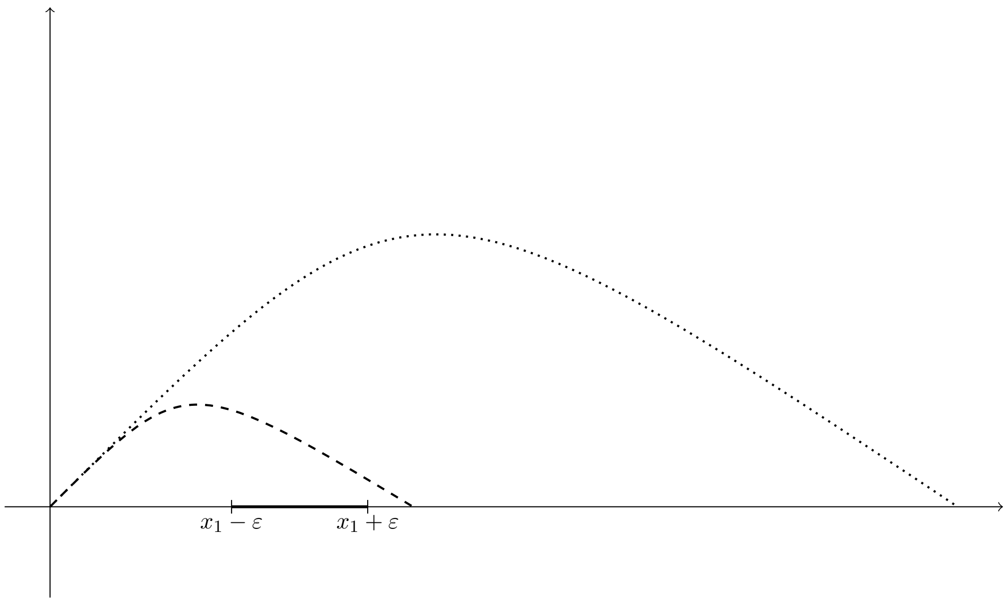

Figure 3.2: plots of $s \mapsto s^\alpha (t - s^\alpha)^q$ for suitable t_1 (dashed) and t_2 (dotted) with $0 < t_1 < t_2$

understand in which way t_1 and t_2 should be selected once fixed $x \in B_R$ and chosen an appropriate $\varepsilon \in (0, \varepsilon_0)$. We observe that the following conditions must be met:

$$x_1 - \varepsilon \geqslant \left(\frac{t_1}{1+q}\right)^{\frac{1}{\alpha}}, \tag{3.53}$$

$$x_1+\varepsilon \leqslant t_1^{\frac{1}{\alpha}} = \left(\frac{t_2}{1+q}\right)^{\frac{1}{\alpha}}. \tag{3.54}$$

To satisfy (3.53) we can pick $\varepsilon \leqslant \frac{x_1}{2}$, so for $t_1 \leqslant (1+q)\left(\frac{x_1}{2}\right)^\alpha$ the condition is verified. Moreover, taking $\varepsilon \leqslant \theta\frac{x_1}{2}$, where $\theta > 0$ will be chosen later, we can easily see that for $t_1 \geqslant (2+\theta)^\alpha \left(\frac{x_1}{2}\right)^\alpha$ the condition (3.54) works. So choosing $\theta := (1+q)^{\frac{1}{\alpha}} - 2 > 0$ and $\varepsilon \leqslant \min\left\{\frac{x_1}{2}, \theta\frac{x_1}{2}\right\}$, $\varepsilon < \varepsilon_0$, we can select $t_1 := (1+q)\left(\frac{x_1}{2}\right)^\alpha$. Now, bearing in mind the equality in (3.54), to conclude it is sufficient to take $t_2 := (1+q)t_1$. Therefore

$$\tilde{g}(\tilde{y}, t_1) := \min_{y\in B(x,\varepsilon)} \tilde{g}(y,t_1), \quad \tilde{y} = (x_1+\varepsilon, x_2, \dots, x_n),$$

$$\tilde{g}(\tilde{\tilde{y}}, t_2) := \min_{y\in B(x,\varepsilon)} \tilde{g}(y,t_2), \quad \tilde{\tilde{y}} = (x_1-\varepsilon, x_2, \dots, x_n).$$

So the minimum point of $y \mapsto \tilde{g}(y,t)$ depends on t. Since $f(x,z) = g(x,z) + (\mu^2+|z|^2)^{\frac{p}{2}} = \tilde{g}(x,|z|) + (\mu^2+|z|^2)^{\frac{p}{2}}$, then the minimum point of $y \mapsto f(y,z)$ depends on z, but this contradicts [68, Theorem 3.1 (4)].

Now let us move on to [44, Theorem 1.1]. Let p, q and σ be as in [44, Theorem 1.1 (1.8)], namely

$$2 \leqslant p \leqslant q < p+1, \quad \sigma \geqslant \frac{p+2}{p+1-q},$$

we will show that [44, Theorem 1.1 (1.6)] is not verified, i.e. $\frac{\partial^2 f}{\partial x \partial z}(x,z) \notin L^\sigma_{\mathrm{loc}}(\Omega)$.

We consider $t > x_1^\alpha > 0$ and $t > qx_1^\alpha$, and therefore $x_1 < \left(\frac{t}{q}\right)^{1/\alpha}$. Let us begin proving that there exists $\alpha \in (0,1]$ such that $\frac{\partial^2 \tilde{g}}{\partial x_1 \partial t} \notin L^\sigma(Q)$, where $Q := \underbrace{(0,\varepsilon)\times\cdots\times(0,\varepsilon)}_{n-times}$ and $0 < \varepsilon < \left(\frac{t}{q}\right)^{\frac{1}{\alpha}}$. We start observing that, when $0 < x_1 < \varepsilon$, the following assertions hold:

- if $q \geqslant 2$ then $(t-x_1^\alpha)^{q-2} \geqslant (t-\varepsilon^\alpha)^{q-2}$,
- if $1 < q < 2$ then $(t-x_1^\alpha)^{q-2} \geqslant t^{q-2}$

and that

$$\min\{(t-\varepsilon^\alpha)^{q-2}, t^{q-2}\} := \begin{cases} (t-\varepsilon^\alpha)^{q-2} & \text{if } \ q \geqslant 2 \\ t^{q-2} & \text{if } \ 1 < q < 2. \end{cases}$$

Hence,

$$\begin{aligned}\int_Q \left|\frac{\partial^2 \tilde{g}}{\partial x_1 \partial t}(x,t)\right|^\sigma dx &= \int_Q [\alpha x_1^{\alpha-1}(t-x_1^\alpha)^{q-2} q(t-qx_1^\alpha)]^\sigma\, dx \\ &\geqslant \int_Q [\alpha x_1^{\alpha-1} \min\{(t-\varepsilon^\alpha)^{q-2}, t^{q-2}\} q(t-q\varepsilon^\alpha)]^\sigma\, dx \\ &= \varepsilon^{n-1}[\alpha \min\{(t-\varepsilon^\alpha)^{q-2}, t^{q-2}\} q(t-q\varepsilon^\alpha)]^\sigma \int_0^\varepsilon x_1^{(\alpha-1)\sigma}\, dx_1.\end{aligned}$$

Now we observe that $\sigma > 1$ then choosing α such that

$$0 < \alpha \leqslant 1 - \frac{1}{\sigma}$$

we have $\frac{\partial^2 \tilde{g}}{\partial x_1 \partial t} \notin L^{\sigma}_{\mathrm{loc}}(\Omega)$ and so by definition $\frac{\partial^2 f}{\partial x \partial z} \notin L^{\sigma}_{\mathrm{loc}}(\Omega)$. So we can conclude that our example is not covered by [44, Theorem 1.1] when f is defined by $\alpha \in \left(0, 1 - \frac{1}{\sigma}\right]$.

Now we want to show that modifying slightly our lagrangian f also assumption (2) of [23] is not verified. For the convenience of the reader, we recall assumption (2) of [23]: there exist $0 < \nu, \beta < 1 < \tilde{\nu} < +\infty$ such that

$$\nu M(x, \beta z) \leqslant f(x,z) \leqslant \tilde{\nu}(M(x,z) + 1), \quad \forall (x,z) \in \Omega \times \mathbb{R}^{N \times n}, \tag{3.55}$$

where $M : \Omega \times \mathbb{R}^{N \times n} \to [0, +\infty)$ is a weak N-function. We take a g which depends on z_n^N, that is the last entry of the matrix

$$z = \begin{pmatrix} z_1^1 & \dots & z_n^1 \\ \vdots & \ddots & \vdots \\ z_1^N & \dots & z_n^N \end{pmatrix}.$$

Our new density is the following

$$f(x,z) := g(x,z) + (\mu^2 + |z|^2)^{\frac{p}{2}},$$

with $g(x,z) := \tilde{g}(x, z_n^N)$ and $\tilde{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$\tilde{g}(x, z_n^N) := \begin{cases} |x_1|^{\alpha}(z_n^N - |x_1|^{\alpha})^q & \text{if } \; z_n^N > |x_1|^{\alpha} \\ 0 & \text{if } \; z_n^N \leqslant |x_1|^{\alpha}, \end{cases}$$

The lagrangian f still satisfies hypothesis (H1)-(H4) of Theorem 3.1.1. We point out that f verifies neither hypothesis (4) of [68, Theorem 3.1] nor hypothesis (1.8) of [44, Theorem 1.1]. Let us check that (3.55) does not hold true for the present lagrangian f. Since M is a weak N- function we have

$$M(x,z) = M(x,-z), \tag{3.56}$$

for almost every $x \in \Omega$ and every $z \in \mathbb{R}^{N \times n}$. Now, if $z_n^N \leqslant |x_1|^{\alpha}$ then

$$\nu M(x, \beta z) \leqslant f(x,z) = (\mu^2 + |z|^2)^{\frac{p}{2}},$$

accordingly

$$M(x, \xi) \leqslant \frac{1}{\nu}\left(\mu^2 + \frac{1}{\beta^2}|\xi|^2\right)^{\frac{p}{2}},$$

for all $\xi \in \mathbb{R}^{N\times n}$ such that $\xi_n^N \leqslant \beta|x_1|^\alpha$. If $\xi_n^N > \beta|x_1|^\alpha$ then, by (3.55) and (3.56), we still get

$$M(x,\xi) = M(x,-\xi) \leqslant \frac{1}{\nu}\left(\mu^2 + \frac{1}{\beta^2}|\xi|^2\right)^{\frac{p}{2}}.$$

Hence for $z_n^N > |x_1|^\alpha$

$$\begin{aligned} f(x,z) &= |x_1|^\alpha(z_n^N - |x_1|^\alpha)^q + (\mu^2+|z|^2)^{\frac{p}{2}} \\ &\leqslant \tilde{\nu}(M(x,z)+1) \\ &\leqslant \tilde{\nu}\left[\frac{1}{\nu}\left(\mu^2 + \frac{1}{\beta^2}|z|^2\right)^{\frac{p}{2}} + 1\right]. \end{aligned}$$

Recalling that $q > p$, the above inequality is not in force if $z_n^N \to +\infty$, thus assumption (3.55) is not valid for f.

Chapter 4

The Sobolev class where a weak solution is a local minimizer

In this chapter we consider the following non-autonomous energy integral

$$\mathcal{F}_b(u,\Omega) = \int_\Omega \left[f(x, Du(x)) + \langle b(x), u(x)\rangle \right] \, dx, \tag{4.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $u : \Omega \to \mathbb{R}^N$, $n \geqslant 2$ and $N \geqslant 1$. The function $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is measurable with respect to $x \in \Omega \subset \mathbb{R}^n$ and it is convex and C^1 with respect to $z \in \mathbb{R}^{N\times n}$. Moreover,

$$\langle b(x), u\rangle = \sum_{\beta=1}^{N} b^\beta(x) u^\beta(x)$$

represents the scalar product. For $1 < p \leqslant q$ we assume the (p,q)-growth condition

$$c_1 |z|^p - c_2 \leqslant f(x,z) \leqslant c_3 |z|^q + c_4 \,, \tag{4.2}$$

for some positive constants $c_1, \ldots, c_4$; we also assume the following summability condition on $b = (b^\beta)_{\beta=1,2,\ldots,N}$

$$b \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^N) \,. \tag{4.3}$$

The coercivity condition at the left hand side of (4.2) ensures the existence of global minimizers $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ of $\mathcal{F}_b$ when suitable boundary values have been fixed.

The main aim of this chapter is to give conditions to rule out the Lavrentiev phenomenon in the general case $p \leqslant q$. More precisely, under growth conditions on p and q we propose cases with

$$\inf\left\{\mathcal{F}_b(v) : v \in u + W_0^{1,p}(\Omega',\mathbb{R}^N)\right\} = \inf\left\{\mathcal{F}_b(v) : v \in u + W_0^{1,q}(\Omega',\mathbb{R}^N)\right\},$$

for every $\Omega' \Subset \Omega$, where u is a local minimizer of $\mathcal{F}_b$; see Theorem 4.1.2 for vector-valued integrals and Theorem 4.2.2 in the scalar case. In order to give some details we start to consider the standard growth $p = q$ and we remark that, since $z \mapsto f(x,z)$ is convex and C^1, growth conditions (4.2) and Lemma 2.1 in [93] imply

$$\left| \frac{\partial f}{\partial z_i^\beta}(x,z) \right| \leqslant c_5 \left(1 + |z|^{q-1} \right), \tag{4.4}$$

see also Step 2 of section 2 in [91]. When $p = q$, (4.4) implies

$$x \mapsto \frac{\partial f}{\partial z_i^\beta}(x, Du(x)) \in L^{\frac{p}{p-1}}(\Omega); \tag{4.5}$$

then we can write Euler equation in weak form, which in fact is a system of N differential equations in divergence form, when we see each equation corresponding to each separate component of the test vector-valued map $\varphi(x) = (\varphi^\beta(x))_{\beta=1,2,\dots,N}$. For every $\varphi \in W_0^{1,p}(\Omega, \mathbb{R}^N)$ we have

$$\int_\Omega \sum_{\beta=1}^N \sum_{i=1}^n \frac{\partial f}{\partial z_i^\beta}(x, Du(x)) D_i \varphi^\beta(x)\ dx + \int_\Omega \sum_{\beta=1}^N b^\beta(x) \varphi^\beta(x)\ dx = 0\,. \tag{4.6}$$

Also note that, when $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ and $p = q$, then (4.2) implies that $x \mapsto f(x, Du(x)) \in L^1(\Omega)$. Moreover the convexity of $z \mapsto f(x,z)$ guarantees that, if $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, then

$$u \text{ globally minimizes } \mathcal{F}_b \iff u \text{ solves Euler equation.} \tag{4.7}$$

The case $p < q$ is more delicate. Indeed, in such a situation, if $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, (4.5) changes into

$$x \mapsto \frac{\partial f}{\partial z_i^\beta}(x, Du(x)) \in L^{\frac{p}{q-1}}(\Omega)\,. \tag{4.8}$$

Since $\frac{p}{q-1} < \frac{p}{p-1}$, then we cannot test (4.6) with $\varphi \in W_0^{1,p}(\Omega, \mathbb{R}^N)$ any longer. On the contrary, if $u \in W^{1,q}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$, then (4.4) implies

$$x \mapsto \frac{\partial f}{\partial z_i^\beta}(x, Du(x)) \in L^{\frac{q}{q-1}}_{\mathrm{loc}}(\Omega) \tag{4.9}$$

and, being q and $\frac{q}{q-1}$ conjugate exponents, (4.6) can be tested by any $\varphi \in W^{1,q}(\Omega, \mathbb{R}^N)$ with compact support in Ω. In this chapter we use this approach to obtain existence and regularity of minimizers. In fact the idea is to achieve existence in a Sobolev class by means of regularity results, precisely by means of the higher integrability for the gradient Du of solutions of the Euler equation or system. A weak solution of (4.6), with a high degree of integrability, can be found and such a weak solution turns out to be a local minimizer of $\mathcal{F}_b$ also in a larger Sobolev class. Main steps in the proofs are the results due by Cupini, Leonetti, Mascolo [46] in the vector-valued case $N \geqslant 1$ and, in the more strict scalar case $N = 1$ but with better exponents, by Marcellini [94, Section 4] and by Cupini, Marcellini, Mascolo [48].

4.1 Vectorial case

Consider the Dirichlet problem in $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(A_i^\beta(x, Du(x))) = b^\beta(x) & \text{in } \ \Omega, \quad \beta = 1, \dots, N \\ u(x) = \tilde{u}(x) & \text{on } \ \partial\Omega. \end{cases} \tag{4.10}$$

The following theorem holds.

Theorem 4.1.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geqslant 2$ and $A_i^\beta : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}, i = 1, \dots, n,\ \ \beta = 1, \dots, N$, be continuous with respect to z. We assume that there exists $\ 0 < \alpha \leqslant 1$ and $\nu, L, H > 0$ such that*

$$\nu(|z|^2 + |\tilde{z}|^2)^{\frac{p-2}{2}}|z - \tilde{z}|^2 \leqslant \sum_{\beta=1}^{N}\sum_{i=1}^{n}[A_i^\beta(x, z) - A_i^\beta(x, \tilde{z})][z_i^\beta - \tilde{z}_i^\beta], \ \ z, \tilde{z} \in \mathbb{R}^{N\times n} \tag{4.11}$$

$$|A_i^\beta(x, z)| \leqslant L(1 + |z|)^{q-1}; \tag{4.12}$$

$$\sum_{\beta=1}^{N}\sum_{i=1}^{n}|A_i^\beta(x, z) - A_i^\beta(\tilde{x}, z)| \leqslant H|x - \tilde{x}|^\alpha(1 + |z|)^{q-1}, \ \ x, \tilde{x} \in \Omega \tag{4.13}$$

with p and q satisfying

$$2 \leqslant p \leqslant q < p\left(\frac{n+\alpha}{n}\right). \tag{4.14}$$

Let b be a function in $L^{p/(p-1)}(\Omega, \mathbb{R}^N)$. Then for all $\tilde{u} \in W^{1,p\frac{q-1}{p-1}}(\Omega, \mathbb{R}^N)$ there exists a weak solution

$$u \in (\tilde{u} + W_0^{1,p}(\Omega, \mathbb{R}^N)) \cap W_{loc}^{1,s}(\Omega, \mathbb{R}^N)$$

of the Dirichlet problem (4.10), for all $q \leqslant s < p\left(\frac{n}{n-\alpha}\right)$.

The proof of the Theorem 4.1.1 proceeds basically in the same way as the one of Theorem 1.1 in [46] where $b(x) \equiv 0$; for the sake of clarity of exposition we postpone the proof at Section 4.3.

Under the previous assumptions a vector-valued map $u \in W_{\text{loc}}^{1,q}(\Omega, \mathbb{R}^N)$ is a weak solution to the system if

$$\int_\Omega \sum_{\beta=1}^{N}\sum_{i=1}^{n} A_i^\beta(x, Du(x))D_i\varphi^\beta(x)\ dx + \int_\Omega \sum_{\beta=1}^{N} b^\beta(x)\varphi^\beta(x)\ dx = 0, \tag{4.15}$$

for all $\varphi \in W^{1,q}(\Omega,\ \mathbb{R}^N)$ with supp$\varphi \Subset \Omega$. We focus our attention on growth condition (4.12). First, we observe that since $|Du|^{q-1} \in L_{\text{loc}}^{\frac{s}{q-1}}$ we obtain

$$A(x, Du(x)) \in L_{\text{loc}}^{\frac{s}{q-1}},$$

for s such that $q \leqslant s < p\left(\frac{n}{n-\alpha}\right)$. The question is now the following: if we assume $\varphi \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $\mathrm{supp}\varphi \Subset \Omega$, do we have that $A(x, Du(x)) \in L^{\frac{p}{p-1}}(\mathrm{supp}\varphi)$? If $q < \frac{np-\alpha}{n-\alpha}$ the answer is affirmative; indeed in this case it is easy to check that

$$q < \frac{np-\alpha}{n-\alpha} \;\Rightarrow\; \exists\, s \quad \text{such that } q < s < p\left(\frac{n}{n-\alpha}\right) \text{ and } \frac{p}{p-1} < \frac{s}{q-1}.$$

Moreover, when $p \leqslant \frac{n}{\alpha}$ we have that

$$\frac{np-\alpha}{n-\alpha} \leqslant p\left(\frac{n+\alpha}{n}\right) < p\left(\frac{n}{n-\alpha}\right).$$

Let us assume $q < \frac{np-\alpha}{n-\alpha}$, therefore, the weak solution u given by Theorem 4.1.1 satisfies (4.15) for all $\varphi \in W^{1,p}(\Omega,\ \mathbb{R}^N)$ with $\mathrm{supp}\varphi \Subset \Omega$. Let us consider functional $\mathcal{F}_b$ in (4.1) under the (p, q)-growth (4.2) and assumption (4.3) on b. We assume that

$$A_i^\beta(x, z) = \frac{\partial f}{\partial z_i^\beta}(x, z)$$

satisfies (4.11), (4.12), (4.13), (4.14); we assume also

$$q < \frac{np-\alpha}{n-\alpha} \tag{4.16}$$

and $\tilde{u} \in W^{1,p\frac{q-1}{p-1}}(\Omega, \mathbb{R}^N)$. By Theorem 4.1.1 we have that there exists

$$u \in (\tilde{u} + W_0^{1,p}(\Omega, \mathbb{R}^N)) \cap W^{1,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$$

satisfying Euler equation (4.6) of functional $\mathcal{F}_b$, i.e. for every test function $\varphi \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $\mathrm{supp}\varphi \Subset \Omega$.

On the other hand, since $z \mapsto f(x, z)$ is convex we have

$$f(x, Du(x) + D\varphi(x)) \geqslant f(x, Du(x)) + \sum_{\beta=1}^{N}\sum_{i=1}^{n} \frac{\partial f}{\partial z_i^\beta}(x, Du(x)) D_i\varphi^\beta(x).$$

Consequently by (4.15), for all $\varphi \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $\mathrm{supp}\varphi \Subset \Omega$, we get

$$\begin{aligned}
\int_{\mathrm{supp}\varphi} f(x, Du(x) + D\varphi(x))\, dx &\geqslant \int_{\mathrm{supp}\varphi} f(x, Du(x))\, dx \\
&\quad + \int_{\mathrm{supp}\varphi} \sum_{\beta=1}^{N}\sum_{i=1}^{n} \frac{\partial f}{\partial z_i^\beta}(x, Du(x)) D_i\varphi^\beta(x)\, dx \\
&= \int_{\mathrm{supp}\varphi} \left[f(x, Du(x)) - \sum_{\beta=1}^{N} b^\beta(x)\varphi^\beta(x) \right] dx. \qquad (4.17)
\end{aligned}$$

The information in the last display shows that u is also a local minimizer of the functional (4.1).

Thus, the following theorem holds true.

Theorem 4.1.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geqslant 2$. Let $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a C^1 function with respect to z and $f(x,0)$ is measurable, such that*

$$c_1|z|^p - c_2 \leqslant f(x,z) \leqslant c_3|z|^q + c_4,$$

where $0 < c_1 \leqslant c_3$, $0 \leqslant c_2, c_4$. Assume that $\frac{\partial f}{\partial z_i^\beta}(x,z)$ satisfies (4.11), (4.12) and (4.13) , with p and q as in (4.14) and (4.16). Let b be a $L^{p/(p-1)}(\Omega,\mathbb{R}^N)$ function, then for all $\tilde{u} \in W^{1,p\frac{q-1}{p-1}}(\Omega,\mathbb{R}^N)$ there exists a local minimizer

$$u \in (\tilde{u} + W_0^{1,p}(\Omega,\mathbb{R}^N)) \cap W_{loc}^{1,s}(\Omega,\mathbb{R}^N)$$

of the functional

$$\mathcal{F}_b(u,\Omega) = \int_\Omega \left[f(x,Du(x)) + \sum_{\beta=1}^N b^\beta(x)u^\beta(x) \right] \, dx,$$

for all $q \leqslant s < p\left(\frac{n}{n-\alpha}\right)$. Moreover, for every $\Omega' \Subset \Omega$ it holds

$$\inf_{v \in u + W_0^{1,p}(\Omega')} \mathcal{F}_b = \inf_{v \in u + W_0^{1,q}(\Omega')} \mathcal{F}_b.$$

Remark 4.1.3. Let us assume that $b = 0$ and $f = f(z)$ in (4.1); in [66] it is shown that every local minimizer $u \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ enjoys higher integrability: $u \in W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \cap W^{1,\frac{np}{n-2}}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, provided

$$2 \leqslant p < q < p + 2\min\left\{1, \frac{p}{n}\right\}. \tag{4.18}$$

When $p \leqslant n$, the previous restriction becomes

$$2 \leqslant p < q < p\left(\frac{n+2}{n}\right). \tag{4.19}$$

On the other hand, the best result in the case $f(x,z)$ is obtained in our assumptions when $\alpha = 1$ and (4.14) becomes

$$2 \leqslant p < q < p\left(\frac{n+1}{n}\right). \tag{4.20}$$

We remark that the gap between $\frac{n+1}{n}$ and $\frac{n+2}{n}$ is due to the basic difference between the non autonomous case $f(x,z)$ and the autonomous one $f(z)$. Indeed, as far as p, q, n, α satisfy $p < n < n + \alpha < q$, in [67] an example of $f(x,z)$ is given for which a global minimizer is not in $W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$. Note that such an example is the double phase functional

$$f(x,z) = |z|^p + a(x)|z|^q, \tag{4.21}$$

For the double phase functional we refer to M. Colombo - Mingione [42] and M. Colombo - Mingione - Baroni [11]. For more general structure of the energy function we recall [8, 23, 28, 40, 41, 44, 57, 58, 63, 68, 81]. The gap between $\frac{n+1}{n}$ and $\frac{n+2}{n}$ shows up when dealing with the autonomous case $A_i^\beta(z)$, $f(z)$ and comparing weak solutions of (4.22) with minimizers of (4.1): for weak solutions we need

$$q < p\left(\frac{n+1}{n}\right),$$

for minimizers we need

$$q < p\left(\frac{n+2}{n}\right),$$

see, for instance, the introduction of [96] and Theorems 1.2, 1.3, 1.17 in [13]. In the scalar case we have

$$q < p\left(\frac{n+2}{n}\right)$$

also for weak solutions too, provided an additional restriction on A_i is assumed: see next section for details.
Let us remark that the recent work of Shäffner [105] shows that the higher integrability $W^{1,\frac{np}{n-2}}$ for minimizers holds true under the restriction

$$2 \leqslant p < q < p\frac{n+1}{n-1}.$$

Note that since

$$\frac{n+2}{n} < \frac{n+1}{n-1}$$

then Shäffner's result improves on bound (4.19). See also [15, 16] and [82].
For details and references on problems with (p, q)-growth we quote the classical starting results in [93, 94], the well known article by Mingione [102] and the recent surveys [95, 96, 103]; see also [47, 49, 54, 64, 97].

4.2 Scalar case

Consider the Dirichlet problem

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(A^i(x, Du(x))) = b(x) & \text{in } \Omega \\ u(x) = \tilde{u}(x) & \text{on } \partial\Omega. \end{cases} \tag{4.22}$$

In the scalar case $N = 1$, to solve (4.22) we refer to the existence and regularity results in Theorem 4.1 in [94] and Theorem 2.1 in [48] that we merge into the following:

Theorem 4.2.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geqslant 2$. Let $A^i : \Omega \times \mathbb{R}^n \to \mathbb{R}, i = 1, \dots, n$, be locally Lipschitz continuous functions in $\Omega \times \mathbb{R}^n$ such that there exist $\mu, M > 0$: for a.e. $x \in \Omega$ and $\forall z, \tilde{z} \in \mathbb{R}^n$*

$$\mu(1 + |z|^2)^{(p-2)/2}|\tilde{z}|^2 \leqslant \sum_{i,j=1}^{n} A^i_{z_j}(x, z)\tilde{z}_i\tilde{z}_j; \tag{4.23}$$

$$|A^i_{z_j}(x, z)| \leqslant M(1 + |z|^2)^{(q-2)/2}; \tag{4.24}$$

$$|A^i_{z_j}(x, z) - A^j_{z_i}(x, z)| \leqslant M(1 + |z|^2)^{(p+q-4)/4}; \tag{4.25}$$

$$|A^i_{x_s}(x, z)| \leqslant M(1 + |z|^2)^{(p+q-2)/4}, \quad s = 1, \dots, n; \tag{4.26}$$

$$|A^i(x, 0)| \leqslant M, \quad \forall x \in \Omega; \tag{4.27}$$

with p and q such that

$$\left\{ \begin{array}{ll} p \leqslant q \leqslant p+1 \text{ and } q < p\left(\frac{n-1}{n-p}\right) & \text{if } 1 < p < 2 \\ p \leqslant q \leqslant p+1 \text{ and } q < p\left(\frac{n-1}{n-p}\right) & \text{if } n > 4 \text{ and } \frac{3n}{n+2} < p \leqslant \frac{n}{2} \\ 2 \leqslant p \leqslant q < p\left(\frac{n+2}{n}\right) & \text{otherwise} \end{array} \right\} \tag{4.28}$$

Assume $b \in L^{p/(p-1)}(\Omega, \mathbb{R}) \cap L^\infty_{loc}(\Omega, \mathbb{R})$.

Then for all $\tilde{u} \in W^{1,p\frac{q-1}{p-1}}(\Omega, \mathbb{R})$ there exists a weak solution

$$u \in (\tilde{u} + W^{1,p}_0(\Omega, \mathbb{R})) \cap W^{1,\infty}_{loc}(\Omega, \mathbb{R}) \cap W^{2,2}_{loc}(\Omega, \mathbb{R})$$

of the Dirichlet problem (4.22).

A function $u \in W^{1,q}_{\text{loc}}(\Omega, \mathbb{R})$ is a weak solution to the equation when

$$\int_\Omega \sum_{i=1}^{n} A^i(x, Du(x))D_i\varphi(x)\ dx + \int_\Omega b(x)\varphi(x)\ dx = 0, \tag{4.29}$$

for all $\varphi \in W^{1,q}(\Omega,\ \mathbb{R})$ with $\operatorname{supp}\varphi \Subset \Omega$.
We show that the weak solution u is also a local minimizer of the functional

$$\tilde{\mathcal{F}}_b(u, \Omega) = \int_\Omega [f(x, Du(x)) + b(x)u(x)]\ dx. \tag{4.30}$$

For this purpose we observe that (4.24)-(4.28) imply that there exists $\tilde{M} \in (0, +\infty)$ such that

$$|A^i(x, Du(x))| \leqslant \tilde{M}(1 + |Du(x)|^{q-1}).$$

Since $|Du|^{q-1} \in L^\infty_{\text{loc}}$ we get $A(x, Du(x)) \in L^\infty_{\text{loc}}$. Therefore: if $\varphi \in W^{1,p}(\Omega, \mathbb{R})$ with $\text{supp}\varphi \Subset \Omega$ we have that

$$A(x, Du(x)) \in L^{\frac{p}{p-1}}(\text{supp}\varphi).$$

Hence we can repeat the same argument as above and obtain (4.29) for all $\varphi \in W^{1,p}(\Omega,\mathbb{R})$ with $\mathrm{supp}\varphi \Subset \Omega$.
Now we consider the functional (4.30), where f satisfies (4.2) and $f(x,z)$ is C^2 with respect to z. We assume that

$$b \in L^{p/(p-1)}(\Omega,\mathbb{R}) \cap L^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}).$$

Moreover,

$$A^i(x,z) = \frac{\partial f}{\partial z_i}(x,z)$$

is locally Lipschitz continuous in $\Omega \times \mathbb{R}^n$ and satisfies (4.23)-(4.27), with p,q,n as in (4.28). We observe that (4.23) implies the convexity of $z \mapsto f(x,z)$.
For a fixed boundary value $\tilde{u} \in W^{1,p\frac{q-1}{p-1}}(\Omega,\mathbb{R}^N)$, by Theorem 4.2.1 there exists

$$u \in (\tilde{u} + W_0^{1,p}(\Omega,\mathbb{R})) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}) \cap W^{2,2}_{\mathrm{loc}}(\Omega,\mathbb{R})$$

verifying (4.29). By (4.29) we have now for the scalar case

$$\int_{\mathrm{supp}\varphi} f(x, Du(x) + D\varphi(x))\, dx \geqslant \int_{\mathrm{supp}\varphi} \left[f(x,Du(x)) - b(x)\varphi(x)\right]\, dx,$$

for all $\varphi \in W^{1,p}(\Omega,\mathbb{R})$ with $\mathrm{supp}\varphi \Subset \Omega$. Then we have just obtained that $u \in (\tilde{u} + W_0^{1,p}(\Omega,\mathbb{R})) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}) \cap W^{2,2}_{\mathrm{loc}}(\Omega,\mathbb{R})$ is a local minimizer of the functional (4.30).

Thus we have proved the following theorem.

Theorem 4.2.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geqslant 2$. Let $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$, $f(x,0)$ is measurable and such that*

$$c_1|z|^p - c_2 \leqslant f(x,z) \leqslant c_3|z|^q + c_4,$$

where $0 < c_1 \leqslant c_3$, $0 \leqslant c_2, c_4$. Moreover, $f(x,z)$ is C^2 with respect to z. Assume that $A^i = \frac{\partial f}{\partial z^i}$ is locally Lipschitz continuous in $\Omega \times \mathbb{R}^n$ and satisfies (4.23)-(4.27), with p,q,n as in (4.28) of Theorem 4.2.1.
Let b be a $L^{p/(p-1)}(\Omega,\mathbb{R}) \cap L^\infty_{loc}(\Omega,\mathbb{R})$ function. Then for all $\tilde{u} \in W^{1,p\frac{q-1}{p-1}}(\Omega,\mathbb{R})$ there exists a local minimizer

$$u \in (\tilde{u} + W_0^{1,p}(\Omega,\mathbb{R})) \cap W^{1,\infty}_{loc}(\Omega,\mathbb{R}) \cap W^{2,2}_{loc}(\Omega,\mathbb{R})$$

of the functional

$$\tilde{\mathcal{F}}_b(u,\Omega) = \int_\Omega \left[f(x,Du(x)) + b(x)u(x)\right]\, dx$$

and for every $\Omega' \Subset \Omega$ it holds

$$\inf_{v \in u + W_0^{1,p}(\Omega')} \tilde{\mathcal{F}}_b = \inf_{v \in u + W_0^{1,\infty}(\Omega')} \tilde{\mathcal{F}}_b.$$

Remark 4.2.3. The relation between minimizer and weak solution of the Euler equation has been studied in [33, 34], for $N \geqslant 1$ when $f = f(z)$ and in [19], for $N = 1$ when $f = f(x,u,z)$ and u is locally bounded.

4.3 Proof of Theorem 4.1.1

This section is devoted to the proof of Theorem 4.1.1. We start recalling that $u \in (\tilde{u} + W_0^{1,p}(\Omega, \mathbb{R}^N)) \cap W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a weak solution of (4.10) if

$$\int_\Omega \sum_{\beta=1}^N \sum_{i=1}^n A_i^\beta(x, Du(x)) D_i\varphi^\beta(x)\, dx + \int_\Omega \sum_{\beta=1}^N b^\beta(x)\varphi^\beta(x)\, dx = 0,$$

for all $\varphi \in W^{1,q}(\Omega, \mathbb{R}^N)$ with $\mathrm{supp}\varphi \Subset \Omega$. We observe that (4.11) and (4.12) imply that there exist $L, \tilde{L}$ such that

$$\frac{\nu}{2}|z|^p - L \leqslant \sum_{\beta=1}^N \sum_{i=1}^n A_i^\beta(x, z) z_i^\beta \leqslant \tilde{L}(1 + |z|)^q. \tag{4.31}$$

As in [46] we consider the following approximation of A_i^β:

$$A_{\varepsilon,i}^\beta(z) := A_i^\beta(z) + \varepsilon|z|^{q-2} z_i^\beta, \tag{4.32}$$

where $\varepsilon \in (0,1)$, $A_{\varepsilon,i}^\beta : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies

$$\sum_{\beta=1}^N \sum_{i=1}^n A_{\varepsilon,i}^\beta(x, z) z_i^\beta \geqslant \varepsilon|z|^q - \lambda; \tag{4.33}$$

$$|A_{\varepsilon,i}^\beta(x, z)| \leqslant (1 + M)(1 + |z|)^{q-1}; \tag{4.34}$$

$$\nu(|z|^2 + |\tilde{z}|^2)^{\frac{p-2}{2}} |z - \tilde{z}|^2 \leqslant \sum_{\beta=1}^N \sum_{i=1}^n [A_{\varepsilon,i}^\beta - A_{\varepsilon,i}^\beta(x, \tilde{z})][z_i^\beta - \tilde{z}_i^\beta]. \tag{4.35}$$

Let us consider the following approximation of the system in (4.10)

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}(A_{\varepsilon,i}^\beta(x, Du(x))) = b^\beta, \tag{4.36}$$

by the classical theory of monotone operators, see [87], there exists an unique $u_\varepsilon \in \tilde{u} + W_0^{1,q}(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega \sum_{\beta=1}^N \sum_{i=1}^n A_{\varepsilon,i}^\beta(x, Du_\varepsilon(x)) D_i\varphi^\beta(x)\, dx + \int_\Omega \sum_{\beta=1}^N b^\beta(x)\varphi^\beta(x)\, dx = 0. \tag{4.37}$$

Now we show an estimate for the L^p norms of Du_ε.

Proposition 4.3.1. *Assume (4.11) and (4.12). Let $u_\varepsilon \in \tilde{u} + W_0^{1,q}(\Omega, \mathbb{R}^N)$, with $\tilde{u} \in W^{1,p\frac{q-1}{p-1}}(\Omega, \mathbb{R}^N)$, be a solution of (4.37) for every $\varepsilon \in (0,1)$. Then there exist positive constants c_1 and $c_2 = c_2(p, n, \Omega)$, independent of ε, such that*

$$\int_\Omega |Du_\varepsilon(x)|^p \, dx \leqslant \left[c_1 c_2 \sum_\beta \left(\int_\Omega |b^\beta(x)|^{\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}} \right]^{\frac{p}{p-1}} + \frac{p}{p-1} \left\{ c_1 c_2 \sum_\beta \left(\int_\Omega |b^\beta|^{\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |D\tilde{u}|^p \, dx \right)^{\frac{1}{p}} + c_1 \int_\Omega (1 + |D\tilde{u}(x)|)^{p\frac{q-1}{p-1}} \, dx \right\}. \tag{4.38}$$

Proof. Let $\varphi = u_\varepsilon - \tilde{u}$, then (4.37) reads as

$$\int_\Omega \sum_{\beta=1}^N \sum_{i=1}^n A^\beta_{\varepsilon,i}(x, Du_\varepsilon(x)) D_i(u^\beta_\varepsilon - \tilde{u}^\beta)(x) \, dx + \int_\Omega \sum_{\beta=1}^N b^\beta(x)(u^\beta_\varepsilon - \tilde{u}^\beta)(x) \, dx = 0.$$

Let us denote $B := \sum_\beta \left(\int_\Omega |b^\beta(x)|^{\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}}$, with the choice of $\xi = Du_\varepsilon$ and $\eta = D\tilde{u}$ Lemma 2.1.9 implies that there exists $c_1 > 0$ such that

$$\begin{aligned}
\int_\Omega |Du_\varepsilon(x)|^p \, dx &\leqslant c_1 \int_\Omega \left\{ \sum_{\beta,i} A^\beta_{\varepsilon,i}(x, Du_\varepsilon) D_i(u^\beta_\varepsilon - \tilde{u}^\beta) + (1 + |D\tilde{u}(x)|)^{p\frac{q-1}{p-1}} \right\} dx \\
&\overset{(4.37)}{=} c_1 \int_\Omega \left\{ -\sum_\beta b^\beta(x)(u^\beta_\varepsilon - \tilde{u}^\beta)(x) + (1 + |D\tilde{u}(x)|)^{p\frac{q-1}{p-1}} \right\} dx \\
&\leqslant c_1 \left\{ \sum_\beta \left(\int_\Omega |b^\beta(x)|^{\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |u^\beta_\varepsilon(x) - \tilde{u}^\beta(x)|^p \, dx \right)^{\frac{1}{p}} \right. \\
&\qquad \left. + \int_\Omega (1 + |D\tilde{u}(x)|)^{p\frac{q-1}{p-1}} \, dx \right\} \quad \text{(Hölder inequality)} \\
&\leqslant c_1 \left\{ \sum_\beta \left(\int_\Omega |b^\beta|^{\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}} c_2 \left(\int_\Omega |Du^\beta_\varepsilon - D\tilde{u}^\beta|^p \, dx \right)^{\frac{1}{p}} \right. \\
&\qquad \left. + \int_\Omega (1 + |D\tilde{u}(x)|)^{p\frac{q-1}{p-1}} \, dx \right\} \quad \text{(Poincaré inequality)}
\end{aligned}$$

$$\begin{aligned}
&\leqslant c_1\Bigg\{Bc_2\left[\left(\int_\Omega |Du_\varepsilon(x)|^p\,dx\right)^{\frac{1}{p}}+\left(\int_\Omega |D\tilde{u}(x)|^p\,dx\right)^{\frac{1}{p}}\right]\\
&\qquad\qquad+\int_\Omega (1+|D\tilde{u}(x)|)^{p\frac{q-1}{p-1}}\,dx\Bigg\}\\
&\leqslant \frac{1}{p}\int_\Omega |Du_\varepsilon|^p\,dx+\frac{p-1}{p}(c_1c_2B)^{\frac{p}{p-1}}+c_1c_2B\left(\int_\Omega |D\tilde{u}|^p\,dx\right)^{\frac{1}{p}}\\
&\qquad+c_1\int_\Omega (1+|D\tilde{u}(x)|)^{p\frac{q-1}{p-1}}\,dx, \quad \text{(Young inequality)}
\end{aligned}$$

where $c_2 = c_2(p, n, \Omega)$. Then

$$\begin{aligned}
\left(1-\frac{1}{p}\right)\int_\Omega |Du_\varepsilon(x)|^p\,dx \leqslant \frac{p-1}{p}(c_1c_2B)^{\frac{p}{p-1}}+c_1c_2B\left(\int_\Omega |D\tilde{u}(x)|^p\,dx\right)^{\frac{1}{p}}\\
+c_1\int_\Omega (1+|D\tilde{u}(x)|)^{p\frac{q-1}{p-1}}\,dx.
\end{aligned}$$

□

Remark 4.3.2. We observe that

$$\int_\Omega |Du_\varepsilon(x)-D\tilde{u}(x)|^p\,dx \leqslant 2^{p-1}\int_\Omega (|Du_\varepsilon(x)|^p+|D\tilde{u}(x)|^p)\,dx,$$

then (4.38) implies that

$$||u_\varepsilon-\tilde{u}||_{W_0^{1,p}(\Omega)} \leqslant c,$$

with c independent of ε.

Let us now prove that the L^s norms of Du_ε are boundend with respect to ε, for every $s \in \left[q, p\frac{n}{n-\alpha}\right)$.

Proposition 4.3.3. *Assume* (4.11),(4.12) *and* (4.13). *Let* $u_\varepsilon \in \tilde{u}+W_0^{1,q}(\Omega,\mathbb{R}^N)$, *with* $\tilde{u} \in W^{1,p\frac{q-1}{p-1}}(\Omega,\mathbb{R}^N)$, *be a solution of (4.37) for every* $\varepsilon \in (0,1)$. *Then for all* $s \in \left[q, p\frac{n}{n-\alpha}\right)$ *there exist* $\sigma_1, \sigma_2, \tau > 0$, *independent of* ε, *such that for all* $B_R \Subset \Omega$, $R \in (0,1]$, *and for all* $\rho < R$

$$\begin{aligned}
\int_{B_\rho} |Du_\varepsilon(x)|^s\,dx \leqslant \frac{\tilde{c}_1}{(R-\rho)^\tau}\Bigg\{\left(\int_{B_R} (1+|Du_\varepsilon(x)|)^p\,dx\right)^{\sigma_1}\\
+\left(\int_{B_R} |b(x)|^{p/(p-1)}\,dx\right)^{\sigma_2}\Bigg\}+\tilde{c}_2R^n, \qquad (4.39)
\end{aligned}$$

where $\tilde{c}_1, \tilde{c}_2$ *are positive constants independent of* ε.

Proof. From now on we write u in place of u_ε. Let us start considering $\rho \leqslant r < d \leqslant R$ and defining $\eta \in C_c^\infty(\Omega, \mathbb{R}^N)$ such that supp$\eta \subset B_{\frac{d+r}{2}}$, $\eta \equiv 1$ in B_r and $|D\eta| \leqslant \frac{4}{d-r}$. Let $\varphi = \tau_{s,-h}(\eta^2 \tau_{s,h} u)$, where $|h| < \frac{d-r}{2}$. Then (4.37) reads as

$$\begin{aligned} \mathrm{I} + \mathrm{II} + \mathrm{III} = & \sum_{\beta,i} \int_\Omega \eta^2 \tau_{s,h}(A_{\varepsilon,i}^\beta(x, Du(x)))\tau_{s,h} D_i u^\beta(x)\ dx + \sum_\beta \int_\Omega b^\beta(x)\varphi^\beta(x)\ dx \\ & + \sum_{\beta,i} \int_\Omega \tau_{s,h}(A_{\varepsilon,i}^\beta(x, Du(x))) 2\eta D_i \eta \tau_{s,h} u^\beta(x)\ dx = 0. \end{aligned} \tag{4.40}$$

We divide the proof into five steps.

<u>Step 1</u>: for positive costants $\tilde{c}_2$ and $\tilde{c}_3$ independent of ε the following estimate is valid

$$\int_{B_d} \eta^2 |\tau_{s,h}(|Du|^2)^{\frac{p-2}{4}} Du|^2\ dx \leqslant \left[\frac{\tilde{c}_2}{(d-r)^p} \int_{B_d} (1+|Du|)^q\ dx + \tilde{c}_3 B^{\frac{p}{p-1}}\right] |h|^\alpha.$$

We start observing that

$$\begin{aligned} \mathrm{I} = & \sum_{\beta,i} \int_\Omega \eta^2 [A_{\varepsilon,i}^\beta(x+he_s, Du(x+he_s)) - A_{\varepsilon,i}^\beta(x, Du(x+he_s))] \tau_{s,h} D_i u^\beta\ dx \\ & + \sum_{\beta,i} \int_\Omega \eta^2 [A_{\varepsilon,i}^\beta(x, Du(x+he_s)) - A_{\varepsilon,i}^\beta(x, Du(x))] \tau_{s,h} D_i u^\beta\ dx \\ = & \ \mathrm{I}_1 + \mathrm{I}_2 \end{aligned}$$

Now we study II :

$$\begin{aligned} |\mathrm{II}| \leqslant & \sum_\beta \int_\Omega |b^\beta(x)||\varphi^\beta(x)|\ dx \\ \leqslant & \sum_\beta \left(\int_{B_R} |b^\beta(x)|^{\frac{p}{p-1}}\ dx\right)^{\frac{p-1}{p}} \left(\int_{B_d} |\tau_{s,-h}(\eta^2 \tau_{s,h} u(x))|^p\ dx\right)^{\frac{1}{p}} \quad \text{(Hölder inequality)} \\ \leqslant & \ B|h| \left(\int_{B_{\frac{r+d}{2}}} |D_s(\eta^2 \tau_{s,h} u(x))|^p\ dx\right)^{\frac{1}{p}} \quad \text{(Lemma 2.1.2)} \\ = & \ B|h| \left(\int_{B_{\frac{r+d}{2}}} |2\eta D_s \eta \tau_{s,h} u(x) + \eta^2 \tau_{s,h} D_s u(x)|^p\ dx\right)^{\frac{1}{p}} \\ \leqslant & \ B|h| \left\{\left(\int_{B_{\frac{r+d}{2}}} |2\eta D_s \eta \tau_{s,h} u|^p\ dx\right)^{\frac{1}{p}} + \left(\int_{B_{\frac{r+d}{2}}} |\eta^2 \tau_{s,h} D_s u|^p\ dx\right)^{\frac{1}{p}}\right\}. \end{aligned} \tag{4.41}$$

Now let us study the right hand side of the above inequality; we start with the first term

$$
\begin{aligned}
B|h|\left(\int_{B_{\frac{r+d}{2}}}|2\eta D_s\eta\tau_{s,h}u(x)|^p\,dx\right)^{\frac{1}{p}} &\leqslant B|h|\frac{8}{d-r}\left(\int_{B_{\frac{r+d}{2}}}|\tau_{s,h}u(x)|^p\,dx\right)^{\frac{1}{p}}\\
&\leqslant B|h|^2\frac{8}{d-r}\left(\int_{B_d}|D_su(x)|^p\,dx\right)^{\frac{1}{p}} \text{ (Lemma 2.1.2)}\\
&\leqslant \frac{1}{p}\left(\frac{8}{d-r}\right)^p|h|^p\int_{B_d}|D_su(x)|^p\,dx\\
&\quad+\frac{p-1}{p}(B|h|)^{\frac{p}{p-1}} \quad \text{(Young inequality)}\\
&\leqslant \frac{1}{p}\left(\frac{8}{d-r}\right)^p\int_{B_d}(1+|Du(x)|)^q\,dx|h|^\alpha\\
&\quad+\frac{p-1}{p}(B|h|)^{\frac{p}{p-1}}.
\end{aligned}
$$

As far as the second term of the right hand side of (4.41) is concerned, we observe that by Young inequality

$$
\begin{aligned}
B|h|\left(\int_{B_{\frac{r+d}{2}}}|\eta^2\tau_{s,h}D_su(x)|^p\,dx\right)^{\frac{1}{p}} &\leqslant \frac{\delta}{p}\int_{B_{\frac{r+d}{2}}}|\tau_{s,h}D_su(x)|^p\eta^{2p}\,dx\\
&\quad+\frac{p-1}{p}\delta^{-\frac{1}{p-1}}B^{\frac{p}{p-1}}|h|^{\frac{p}{p-1}}\\
\leqslant 2^{\frac{p-2}{2}}\frac{\delta}{p}\int_{B_{\frac{r+d}{2}}}|\tau_{s,h}Du(x)|^2\left[|Du(x+he_s)|^2+|Du(x)|^2\right]^{\frac{p-2}{2}}\eta^2\,dx&\\
&\quad+\frac{p-1}{p}\delta^{-\frac{1}{p-1}}B^{\frac{p}{p-1}}|h|^{\frac{p}{p-1}},
\end{aligned}
$$

for every $\delta>0$. Now let us study III, by (4.34)

$$
\begin{aligned}
|\mathrm{III}| &\leqslant \tilde{c}\int_\Omega\eta|D\eta|(1+|Du(x)|+|Du(x+he_s)|)^{q-1}|\tau_{s,h}u(x)|\,dx\\
&\leqslant \frac{\tilde{c}}{d-r}\left(\int_{B_{\frac{r+d}{2}}}(1+|Du|+|Du(x+he_s)|)^q\,dx\right)^{\frac{q-1}{q}}\left(\int_{B_{\frac{r+d}{2}}}|\tau_{s,h}u|^q\,dx\right)^{\frac{1}{q}}\\
&\leqslant \frac{\tilde{c}}{d-r}\int_{B_d}(1+|Du(x)|)^q\,dx\;|h|^\alpha, \quad \text{(Lemma 2.1.2)}
\end{aligned}
$$

where we have used the fact that $|h|<1$. As far as I_2 is concerned, we observe that by the Hölder continuity of $A(\cdot, z)$ in (4.13)

$$
\begin{aligned}
|\mathrm{I}_2| &\leqslant \tilde{c} \int_\Omega \eta^2 (1 + |Du(x + he_s)|)^{q-1} |\tau_{s,h} Du(x)| \; dx \; |h|^\alpha \\
&\leqslant \tilde{c} \int_{B_{\frac{d+r}{2}}} (1 + |Du(x)| + |Du(x + he_s)|)^q \; dx \; |h|^\alpha \\
&\leqslant \tilde{c} \int_{B_d} (1 + |Du(x)|)^q \; dx \; |h|^\alpha,
\end{aligned}
$$

we point out that the costant $\tilde{c}$ does not depend on ε. By (4.35) we estimate I_1 in the following way

$$
\mathrm{I}_1 \geqslant \nu \int_{B_{\frac{r+d}{2}}} |\tau_{s,h} Du(x)|^2 \left[|Du(x + he_s)|^2 + |Du(x)|^2 \right]^{\frac{p-2}{2}} \eta^2 \; dx.
$$

Now we observe that $\mathrm{I}_1 = -\mathrm{II} - \mathrm{III} - \mathrm{I}_2$, then for $\delta = \frac{p\nu}{2^{\frac{p}{2}}}$ we get

$$
\begin{aligned}
&\frac{\nu}{2} \int_{B_{\frac{r+d}{2}}} |\tau_{s,h} Du(x)|^2 \left[|Du(x + he_s)|^2 + |Du(x)|^2 \right]^{\frac{p-2}{2}} \eta^2 \; dx \\
&\leqslant \frac{p-1}{p} \left[1 + \left(\frac{p\nu}{2^{\frac{p}{2}}} \right)^{-\frac{1}{p-1}} \right] (B|h|)^{\frac{p}{p-1}} + \frac{1}{p} \left(\frac{8}{d-r} \right)^p \int_{B_d} (1 + |Du(x)|)^q \; dx \; |h|^\alpha \\
&\quad + \frac{\tilde{c}}{d-r} \int_{B_d} (1 + |Du(x)|)^q \; dx \; |h|^\alpha + \tilde{c} \int_{B_d} (1 + |Du(x)|)^q \; dx \; |h|^\alpha.
\end{aligned}
$$

Now by Lemma 2.1.5 we get that the left hand side of the above inequality is greater than

$$
\tilde{c}_1 \int_{B_d} \eta^2 |\tau_{s,h} (|Du(x)|^2)^{\frac{p-2}{4}} Du(x)|^2 \; dx,
$$

for a suitable positive constant $\tilde{c}_1$ independent of ε. With some easy extra calculation we get

$$
\int_{B_{\frac{r+d}{2}}} \eta^2 |\tau_{s,h} (|Du|^2)^{\frac{p-2}{4}} Du|^2 \; dx \leqslant \left[\frac{\tilde{c}_2}{(d-r)^p} \int_{B_d} (1 + |Du|)^q \; dx + \tilde{c}_3 B^{\frac{p}{p-1}} \right] |h|^\alpha,
$$

where $\tilde{c}_2$ and $\tilde{c}_3$ are positive and independent of ε.

Step 2: we claim that $|Du| \in L^{q\delta}_{loc}(\Omega)$, where $\delta = \frac{pn}{q(n-\theta)}$ and $\theta \in (0, \alpha)$

We observe that since $\frac{q}{p} < \frac{n+\alpha}{n}$ we can choose $\theta \in (0, \alpha)$ such that $\frac{q}{p} < \frac{n+\theta}{n}$, this choice is possible if $0 \leqslant n \left(\frac{q}{p} - 1 \right) < \theta < \alpha$. Moreover, we note that $p < q\delta < p\frac{n}{n-\alpha}$. Now we fix $B_R \Subset \Omega$ with $\rho \leqslant r < d \leqslant R \leqslant 1$. By Step 1 and Lemma 2.1.4, applied with $b = \frac{\theta}{2}$, we have $|Du|^{\frac{p}{2}} \in L^{\frac{2n}{n-\theta}}(B_r)$ and

$$\| \, |Du|^{\frac{p}{2}} \|_{L^{\frac{2n}{n-\theta}}(B_r)} \leqslant \frac{\tilde{c}_4}{(d-r)^{2+\theta+\alpha}} \left\{ \frac{1}{(d-r)^{\frac{p}{2}}} \left(\int_{B_d} (1+|Du|)^q \, dx \right)^{\frac{1}{2}} + B^{\frac{p}{2(p-1)}} + \left(\int_{B_d} |Du|^p \, dx \right)^{\frac{1}{2}} \right\}$$
$$= \frac{2\tilde{c}_4}{(d-r)^{2+\theta+\alpha+p/2}} \left(\int_{B_d} (1+|Du|)^q \, dx + B^{\frac{p}{p-1}} \right)^{\frac{1}{2}},$$

for a suitable positive constant $\tilde{c}_4$ independent of ε. For $\beta = \left(2+\theta+\alpha+\frac{p}{2}\right)\frac{2n}{n-\theta}$ we get

$$\int_{B_r} |Du(x)|^{q\delta} \, dx \leqslant \frac{\tilde{c}_5}{(d-r)^{\beta}} \left(\int_{B_d} (1+|Du|)^q \, dx + B^{\frac{p}{p-1}} \right)^{\frac{q\delta}{p}}, \tag{4.42}$$

with $\tilde{c}_5$ independent of ε. Since r and d are arbitrary we get the claim.

If $p = q$ then we go to Step 5. If $p < q$ we need Step 3 and Step 4.

<u>Step 3</u>: we prove that there exist positive constants $t, \sigma, \tilde{c}_6$ independent of ε, satisfying $\frac{q\delta}{p} < t \leqslant \frac{q\delta-p}{q-p}$, $\sigma \in (0,1)$, such that

$$\int_{B_r} |Du(x)|^{q\delta} \, dx \leqslant \frac{\tilde{c}_6}{(d-r)^{\frac{\beta}{1-\sigma}}} \left(\int_{B_d} (1+|Du(x)|)^p \, dx \right)^{\frac{\sigma(t-1)}{1-\sigma}}$$
$$+ \frac{1}{2} \int_{B_d} |Du(x)|^{q\delta} \, dx + \frac{\tilde{c}_5 B^{\frac{q\delta}{p-1}}}{(d-r)^{\beta}} + \tilde{c}_6 R^n. \tag{4.43}$$

First of all we observe that since $\frac{q}{p} < \frac{n+\theta}{n}$ then $\frac{q\delta}{p} < \frac{q\delta-p}{q-p}$, moreover $t \in \left(\frac{q\delta}{p}, \frac{q\delta-p}{q-p}\right]$ implies

$$1 < \frac{q\delta}{p} < t, \quad p \geqslant q\frac{t-\delta}{t-1}. \tag{4.44}$$

Now by Hölder inequality we get

$$\left(\int_{B_d} (1+|Du(x)|)^q \, dx \right)^{\frac{q\delta}{p}} = \left(\int_{B_d} (1+|Du(x)|)^{\frac{q\delta}{t}} (1+|Du(x)|)^{q\left(1-\frac{\delta}{t}\right)} \, dx \right)^{\frac{q\delta}{p}}$$
$$= \left(\int_{B_d} (1+|Du(x)|)^{q\delta} \, dx \right)^{\frac{q\delta}{pt}} \left(\int_{B_d} (1+|Du(x)|)^{q\frac{t-\delta}{t-1}} \, dx \right)^{\frac{q\delta(t-1)}{pt}}.$$

Then for $\sigma = \frac{q\delta}{pt}$ and $b = \frac{t-\delta}{t-1}$ the above inequality and (4.42) imply

$$\begin{aligned}
\int_{B_r} |Du(x)|^{q\delta}\, dx &\leqslant \frac{\tilde{c}_5}{(d-r)^\beta} \left(\int_{B_d} (1+|Du(x)|)^{q\delta}\, dx \right)^\sigma \\
&\quad \times \left(\int_{B_d} (1+|Du(x)|)^{qb}\, dx \right)^{\sigma(t-1)} + \frac{\tilde{c}_5 B^{\frac{q\delta}{p-1}}}{(d-r)^\beta} \\
&\leqslant \frac{\tilde{c}_6}{(d-r)^{\frac{\beta}{1-\sigma}}} \left(\int_{B_d} (1+|Du(x)|)^{qb}\, dx \right)^{\frac{\sigma(t-1)}{1-\sigma}} \\
&\quad + \frac{1}{2^{q\delta}} \int_{B_d} (1+|Du(x)|)^{q\delta}\, dx + \frac{\tilde{c}_5 B^{\frac{q\delta}{p-1}}}{(d-r)^\beta} \quad \text{(Young inequality)} \\
&\leqslant \frac{\tilde{c}_6}{(d-r)^{\frac{\beta}{1-\sigma}}} \left(\int_{B_d} (1+|Du(x)|)^{p}\, dx \right)^{\frac{\sigma(t-1)}{1-\sigma}} \\
&\quad + \frac{1}{2} \int_{B_d} |Du(x)|^{q\delta}\, dx + \frac{\tilde{c}_5 B^{\frac{q\delta}{p-1}}}{(d-r)^\beta} + \tilde{c}_6 R^n,
\end{aligned}$$

where we have used the fact that $(|x|+|y|)^{q\delta} \leqslant 2^{q\delta-1}(|x|^{q\delta}+|y|^{q\delta})$. We point out that by Step 2 and (4.44) the right hand side is finite.

Step 4:

Now we use Lemma 2.1.8 with $Z(r) = \int_{B_r} |Du(x)|^{q\delta}\, dx$, then (4.43) implies

$$\int_{B_\rho} |Du(x)|^{q\delta}\, dx \leqslant \frac{\tilde{c}_7}{(R-\rho)^{\frac{\beta}{1-\sigma}}} \left(\int_{B_R} (1+|Du|)^p\, dx \right)^{\frac{\sigma(t-1)}{1-\sigma}} + \frac{\tilde{c}_7 B^{\frac{q\delta}{p-1}}}{(R-\rho)^\beta} + \tilde{c}_7 R^n,$$

for some positive constant $\tilde{c}_7$ independent of ε.

Step 5:

We observe that since θ is any number in $\left(n\left(\frac{q}{p}-1\right), \alpha\right)$ then $s = q\delta$ can be any number in $\left(\frac{p^2}{2p-q}, p\frac{n}{n-\alpha}\right)$. We note also that $q \leqslant \frac{p^2}{2p-q}$. When $p<q$, looking at Step 4 we see that estimate (4.39) works when s belongs to $\left(\frac{p^2}{2p-q}, p\frac{n}{n-\alpha}\right)$. Instead if $s \in \left[q, \frac{p^2}{2p-q}\right]$ then (4.39) is obtained by means of Hölder inequality. When $p=q$ we look at (4.42). □

Now we are ready to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. For all $\varepsilon \in (0,1)$ let us consider the operator $A_{\varepsilon,i}$ defined in (4.32) and let $u_\varepsilon \in \tilde{u} + W_0^{1,q}(\Omega, \mathbb{R}^N)$ be the solution of (4.37) for all $\varphi \in W_0^{1,q}(\Omega, \mathbb{R}^N)$.

We divide the proof in three steps.

Step 1: there exists $u_0 \in \tilde{u} + W_0^{1,p}(\Omega, \mathbb{R}^N)$ such that, up to subsequences, $Du_\varepsilon \to Du_0$ strongly in $L^p_{loc}(\Omega)$

By Remark 4.3.1 the $W_0^{1,p}(\Omega)$ norm of $u_\varepsilon - \tilde{u}$ is bounded with respect to ε. Proposition 4.3.3 and estimate (4.38) imply that, once fixed $\Omega' \Subset \Omega$, Du_ε is bounded in $L^q(\Omega', \mathbb{R}^{N\times n})$, uniformly with respect to ε. Then there exists $u_0 \in \tilde{u} + W_0^{1,p}(\Omega, \mathbb{R}^N)$ such that $Du_0 \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$ and

$$u_\varepsilon - \tilde{u} \to u_0 - \tilde{u} \quad \text{weakly in } W_0^{1,p}(\Omega, \mathbb{R}^N)$$

$$Du_\varepsilon \to Du_0 \quad \text{weakly in } L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$$

up to subsequences. Moreover, since $q < p\left(\frac{n+\alpha}{n}\right)$ the Rellich theorem implies that $u_\varepsilon \to u_0$ in $L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Now let us consider the test function $\varphi = (u_\varepsilon - u_0)\eta$, where $\eta \in C_c^\infty(\Omega, \mathbb{R}^N)$. By (4.37)

$$\begin{aligned}\int_\Omega \sum_{\beta,i} A^\beta_{\varepsilon,i}(x, Du_\varepsilon) D_i(u^\beta_\varepsilon - u^\beta_0)\eta \; dx &= -\int_\Omega \sum_{\beta,i} A^\beta_{\varepsilon,i}(x, Du_\varepsilon)(u^\beta_\varepsilon - u^\beta_0) D_i\eta^\beta \; dx \\ &\quad - \int_\Omega \sum_\beta b^\beta(x)\varphi^\beta(x) \; dx \\ &\leqslant (M+1)\int_\Omega (1 + |Du_\varepsilon|)^{q-1}|u_\varepsilon - u_0||D\eta| \; dx \\ &\quad + \sum_\beta \int_\Omega |b^\beta \varphi^\beta| \; dx. \end{aligned} \tag{4.45}$$

We observe that by (4.35)

$$\begin{aligned}\frac{2^{\frac{p-2}{2}}}{\nu} \sum_{\beta,i} [A^\beta_{\varepsilon,i}(x, Du_\varepsilon) - A^\beta_{\varepsilon,i}(x, Du_0)] D_i[u^\beta_\varepsilon - u^\beta_0] &\geqslant 2^{\frac{p-2}{2}}(|Du_\varepsilon|^2 + |Du_0|^2)^{\frac{p-2}{2}} \\ &\quad \times |Du_\varepsilon - Du_0|^2 \\ &\geqslant (|Du_\varepsilon - Du_0|^2)^{\frac{p-2}{2}} |Du_\varepsilon - Du_0|^2 \\ &= |Du_\varepsilon - Du_0|^p. \end{aligned}$$

Thus (4.45) and the definition of $A^\beta_{\varepsilon,i}$ imply

$$\begin{aligned}
2^{\frac{2-p}{2}}\nu\int_\Omega |Du_\varepsilon - Du_0|^p\eta\;dx &\leqslant \sum_{\beta,i}[A^\beta_{\varepsilon,i}(x,Du_\varepsilon) - A^\beta_{\varepsilon,i}(x,Du_0)]D_i[u^\beta_\varepsilon - u^\beta_0]\eta\;dx\\
&= \int_\Omega \sum_{\beta,i} A^\beta_{\varepsilon,i}(x,Du_\varepsilon)D_i(u^\beta_\varepsilon - u_0)\eta\;dx\\
&\quad - \int_\Omega \sum_{\beta,i} A^\beta_i(x,Du_0)D_i(u^\beta_\varepsilon - u^\beta_0)\eta\;dx\\
&\quad - \varepsilon\int_\Omega \sum_{\beta,i} |D_iu^\beta_0|^{q-2}D_iu^\beta_0 D_i(u^\beta_\varepsilon - u^\beta_0)\eta\;dx\\
&\leqslant (M+1)\int_\Omega (1+|Du_\varepsilon|)^{q-1}|u_\varepsilon - u_0||D\eta|\;dx\\
&\quad + \int_\Omega |b(x)||u_\varepsilon - u_0||\eta|\;dx\\
&\quad - \int_\Omega \sum_{\beta,i} A^\beta_i(x,Du_0)D_i(u^\beta_\varepsilon - u^\beta_0)\eta\;dx\\
&\quad - \varepsilon\int_\Omega \sum_{\beta,i} |D_iu^\beta_0|^{q-2}D_iu^\beta_0 D_i(u^\beta_\varepsilon - u^\beta_0)\eta\;dx.
\end{aligned}$$

Hence for $\varepsilon \to 0$ the right hand side of the above inequality goes to 0, then

$$Du_\varepsilon \to Du_0 \quad \text{in } L^p_{\text{loc}}(\Omega). \tag{4.46}$$

<u>*Step 2*</u>*: up to subsequences $A^\beta_{\varepsilon,i}(x,Du_\varepsilon) \to A^\beta_i(x,Du_0)$ in $L^1_{loc}(\Omega)$*

For all $\Omega' \Subset \Omega$, adding and subtracting $A^\beta_i(x,Du_\varepsilon)$, we get

$$\begin{aligned}
\int_{\Omega'} |A^\beta_{\varepsilon,i}(x,Du_\varepsilon) - A^\beta_i(x,Du_0)|\;dx &\leqslant \int_{\Omega'} |A^\beta_i(x,Du_\varepsilon) - A^\beta_i(x,Du_0)|\;dx\\
&\quad + \int_{\Omega'} \varepsilon|Du_\varepsilon|^{q-1}\;dx.
\end{aligned} \tag{4.47}$$

We observe that since Du_ε in uniformly bounded in $L^q(\Omega')$ the second term at the right hand side of (4.47) goes to zero. Now, let us show that the first term in the right hand side of (4.47) goes to zero. For this purpose we consider the following functions

$$f_\varepsilon(x) := |A^\beta_i(x,Du_\varepsilon) - A^\beta_i(x,Du_0)|,$$

$$g_\varepsilon(x) := M\{(1+|Du_\varepsilon(x)|)^{q-1} + (1+|Du_0(x)|)^{q-1}\}$$

and

$$g_0(x) := 2M(1+|Du_0(x)|)^{q-1}.$$

Now, by (4.12) we get that $f_\varepsilon(x) \leqslant g_\varepsilon(x)$ for a.e. $x \in \Omega$. Moreover, by Step 1 $g_\varepsilon \to g_0$ and, by (4.46) and $q \leqslant p+1$, $\int_{\Omega'} g_\varepsilon\;dx \to \int_{\Omega'} g_0\;dx$ for $\varepsilon \to 0$. We conclude by the generalized Lebesgue convergence theorem.

Step 3: u_0 is a weak solution of (4.15)

By (4.37) and Step 2 we get

$$\begin{aligned}
&\int_\Omega \sum_{\beta=1}^N \sum_{i=1}^n A_i^\beta(x, Du_0(x)) D_i\varphi^\beta(x)\ dx + \int_\Omega \sum_{\beta=1}^N b^\beta(x)\varphi^\beta(x)\ dx \\
&= \lim_{\varepsilon\to 0} \int_\Omega \sum_{\beta=1}^N \sum_{i=1}^n A_{\varepsilon,i}^\beta(x, Du_\varepsilon(x)) D_i\varphi^\beta(x)\ dx + \int_\Omega \sum_{\beta=1}^N b^\beta(x)\varphi^\beta(x)\ dx \\
&= 0
\end{aligned}$$

for all $\varphi \in C_0^1(\Omega, \mathbb{R}^N)$. Since $A_i^\beta(x, Du_0(x))$ and $b^\beta(x)$ are in $L_{\text{loc}}^{\frac{q}{q-1}}(\Omega)$, then we get (4.15), with $u = u_0$, for every $\varphi \in W^{1,q}(\Omega, \mathbb{R}^N)$ with supp$\varphi \Subset \Omega$. Namely, u_0 is a weak solution of (4.15).

$\square$

Chapter 5

Absence and presence of Lavrentiev phenomenon for double phase functionals for every choice of exponents

In this chapter we study classes of weights ensuring the absence and presence of the Lavrentiev phenomenon for double phase functionals upon every choice of exponents. We introduce a new sharp scale for weights for which there is no Lavrentiev phenomenon up to a counterexample we provide. This scale embraces the sharp range for α-Hölder continuous weights. Moreover, it allows excluding the gap for every choice of exponents $q, p > 1$.

We consider the double-phase functional

$$\mathcal{P}(u) = \int_\Omega (|Du(x)|^p + a(x)|Du(x)|^q)\, dx\,, \tag{5.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $n > 1$, $1 \leqslant p, q < \infty$ and weight $a : \Omega \to [0, +\infty)$ is bounded. The functional is designed to model the transition between the region where the gradient is integrable with p-th power and the region where it has the higher integrability with q-th power. Therefore, we are interested only in the situation when $p < q$ and a vanishes on some subset of Ω, but $a \not\equiv 0$. The double phase $\mathcal{P}$ and various kinds of its minimizers have been studied since [93, 107], continued in a vast range of contributions including [12, 14, 42, 43, 53, 67, 68, 79] with sharpness discussed in [8, 67, 71, 106, 107]. More recent developments in this matter may be found in [7, 9, 24, 27, 32, 58, 84].

Let us recall the energy space W

$$W(\Omega, \mathbb{R}) = \left\{ u \in W_0^{1,1}(\Omega, \mathbb{R}) : \int_\Omega M(x, |Du(x)|)\, dx < \infty \right\},$$

where

$$M(x, t) = t^p + a(x)t^q.$$

It is known that if $a \in C^{0,\alpha}(\Omega)$, with $\alpha \in (0,1]$ then for

$$1 \leqslant p < q \leqslant p + \alpha$$

there is no Lavrentiev phenomenon between the spaces

$$X = u_0 + W(\Omega, \mathbb{R}) \quad \text{and} \quad Y = u_0 + C_c^\infty(\Omega, \mathbb{R}),$$

where $u_0 \in W(\Omega, \mathbb{R})$, see [23, 28]. Our main focus is to extend the gap between p and q defining a new class for the weight a. In other words, we want to say that if $a \in \mathcal{Z}^\varkappa$, for $\varkappa \in (0, +\infty)$, and p and q are in the following relation

$$1 \leqslant p < q \leqslant p + \varkappa \tag{5.2}$$

then

$$\inf_{u \in X} \mathcal{P}(u) = \inf_{u \in Y} \mathcal{P}(u).$$

The definition of $\mathcal{Z}^\varkappa$ reads as follows.

Definition 5.0.1 (Class $\mathcal{Z}^\varkappa(\Omega)$)**.** Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$. A function $a : \Omega \to [0, +\infty)$ belongs to $\mathcal{Z}^\varkappa(\Omega)$, for $\varkappa \in (0, +\infty)$, if there exists a positive constant C such that

$$a(x) \leqslant C\left(a(y) + |x-y|^\varkappa\right) \tag{5.3}$$

for all $x, y \in \Omega$.

Of course, α-Hölder continuous functions for $\alpha \in (0,1]$ belong to $\mathcal{Z}^\alpha$, but $\mathcal{Z}^\varkappa$ with $\varkappa \in (0, +\infty)$ is an essentially broader class of functions. To provide better understanding of this new scale, we set down its main properties.

Remark 5.0.2 (Basic properties of $\mathcal{Z}^\varkappa(\Omega)$)**.** If $\Omega \subset \mathbb{R}^n$ is an open set, then the following holds.

1. Let $\varkappa \in (0,1]$, then $a \in \mathcal{Z}^\varkappa(\Omega)$ if and only if there exists $\widetilde{a} \in C^{0,\varkappa}(\Omega)$, such that a is comparable to $\widetilde{a}$; i.e., there exists a positive constant c such that $\widetilde{a} \leqslant a \leqslant c\widetilde{a}$.

2. Let $\varkappa, \beta \in (0, +\infty)$, then $a \in \mathcal{Z}^\varkappa(\Omega)$ if and only if $a^\beta \in \mathcal{Z}^{\beta\varkappa}(\Omega)$.

3. Let $\varkappa \in (0, +\infty)$, then $a \in \mathcal{Z}^\varkappa(\Omega)$ if and only if there exists $\widetilde{a}$ comparable to a such that $\widetilde{a}^{\frac{1}{\varkappa}}$ is Lipschitz.

4. Suppose Ω is bounded, then $0 < \varkappa_1 \leqslant \varkappa_2$ if and only if $\mathcal{Z}^{\varkappa_2}(\Omega) \subset \mathcal{Z}^{\varkappa_1}(\Omega)$.

Let us prove these assertions.

<u>Assertion 1.</u> Let us assume at first that $\widetilde{a} \in C^{0,\varkappa}$ and a is such that $\widetilde{a} \leqslant a \leqslant c\widetilde{a}$. Then, for every $x, y \in \Omega$ it holds that

$$a(x) \leqslant c\widetilde{a}(x) \leqslant c(\widetilde{a}(y) + [\widetilde{a}]_{0,\varkappa}|x-y|^\varkappa) \leqslant c(1 + [\widetilde{a}]_{0,\varkappa})(a(y) + |x-y|^\varkappa)\,,$$

which means that $a \in \mathcal{Z}^{\varkappa}(\Omega)$. This proves one implication of the assertion.
Let us now take any $a \in \mathcal{Z}^{\varkappa}(\Omega)$, $\varkappa \in (0,1]$. Let us define functions $\widetilde{a}_y$, for every $y \in \Omega$, and function $\widetilde{a}$, by

$$\widetilde{a}_y(x) := a(y) + |x-y|^{\varkappa} \quad \text{and} \quad \widetilde{a}(x) := \inf_{y\in\Omega} \widetilde{a}_y(x)\,.$$

Note that as $a \geqslant 0$, we have that $\widetilde{a}_y \geqslant 0$ for every $y \in \Omega$ and it follows that $\widetilde{a}$ is finite-valued. By $a \in \mathcal{Z}^{\varkappa}(\Omega)$ and the definition of the function $\widetilde{a}$, we have

$$\widetilde{a}(x) \leqslant \widetilde{a}_x(x) = a(x) \leqslant C\widetilde{a}(x),$$

for every $x \in \Omega$, which means that a is comparable to $\widetilde{a}$. It remains to prove that $\widetilde{a} \in C^{0,\varkappa}$. Note that for a fixed $y \in \Omega$, function $\widetilde{a}_y$ is a translation of the function $|\cdot|^{\varkappa}$, and therefore every $\widetilde{a}_y$ belongs to $C^{0,\varkappa}$ with $[\widetilde{a}_y]_{0,\varkappa} = 1$. Let us now take any $x, y \in \Omega$ and assume without loss of generality that $\widetilde{a}(x) \geqslant \widetilde{a}(y)$. For any $\epsilon > 0$, we take y_ϵ such that $\widetilde{a}_{y_\epsilon}(y) - \widetilde{a}(y) < \epsilon$. It holds that

$$\widetilde{a}(x) - \widetilde{a}(y) \leqslant \widetilde{a}_{y_\epsilon}(x) - \widetilde{a}(y) \leqslant \widetilde{a}_{y_\epsilon}(x) - \widetilde{a}_{y_\epsilon}(y) + \epsilon \leqslant |x-y|^{\varkappa} + \epsilon\,,$$

where the first inequality follows from the definition of $\widetilde{a}$, and in the last inequality we used Hölder continuity of $\widetilde{a}_{y_\epsilon}$. As $\epsilon > 0$ is arbitrary and the role of x and y is symmetric, we have $|\widetilde{a}(x) - \widetilde{a}(y)| \leqslant |x-y|^{\varkappa}$, which proves that $\widetilde{a} \in C^{0,\varkappa}$. This ends the proof of the assertion.
<u>Assertion 2.</u> For any $\beta > 0$ there exists a constant $c > 0$ such that for every $s, t \geqslant 0$ it holds that $(t+s)^\beta \leqslant c(t^\beta + s^\beta)$. If $a \in \mathcal{Z}^{\varkappa}$, $\varkappa > 0$, then for some $C > 0$ we have

$$a^\beta(x) \leqslant C^\beta (a(y) + |x-y|^{\varkappa})^\beta \leqslant Cc(a(y)^\beta + |x-y|^{\beta\varkappa})\,,$$

for any $x, y \in \Omega$, which means that $a^\beta \in \mathcal{Z}^{\beta\varkappa}$. The first implication of the assertion is proven, the reverse one follows using the same argument.
<u>Assertion 3.</u> By assertion 2 we know that $a \in \mathcal{Z}^{\varkappa}(\Omega)$ if and only if $a^{\frac{1}{\varkappa}} \in \mathcal{Z}^1(\Omega)$. This, by assertion 1, is equivalent to the existence of $\widetilde{b}$ of Lipschitz type and comparable to $a^{\frac{1}{\varkappa}}$. Putting $\widetilde{a} = (\widetilde{b})^{\varkappa}$ completes the proof.
<u>Assertion 4.</u> If $0 < \varkappa_1 \leqslant \varkappa_2$, we take any $a \in \mathcal{Z}^{\varkappa_2}(\Omega)$. As Ω is bounded for every $x, y \in \Omega$ it holds that $|x-y| \leqslant \operatorname{diam}\Omega$, and therefore $|x-y|^{\varkappa_2} \leqslant (\operatorname{diam}\Omega)^{\varkappa_2-\varkappa_1}|x-y|^{\varkappa_1}$. Therefore, for some constant $C > 0$ we have

$$a(x) \leqslant C(a(y) + |x-y|^{\varkappa_2}) \leqslant C\max\{1, (\operatorname{diam}\Omega)^{\varkappa_2-\varkappa_1}\}(a(y) + |x-y|^{\varkappa_1})\,,$$

which implies that $a \in \mathcal{Z}^{\varkappa_1}(\Omega)$. For the vice versa, we assume that $0 < \varkappa_2 < \varkappa_1$. Then, we observe that $|x-x_0|^{\varkappa_2} \in \mathcal{Z}^{\varkappa_2} \setminus \mathcal{Z}^{\varkappa_1}$, with $x_0 \in \Omega$. Indeed, in the view of assertion 3, there does not exist a Lipschitz function comparable with $|x-x_0|^{\frac{\varkappa_2}{\varkappa_1}}$.

□

The first point of Remark 5.0.2 says that for $\varkappa \in (0, 1]$, class $\mathcal{Z}^{\varkappa}(\Omega)$ is similar to Hölder continuity, but it is actually requiring admissible decay rate near regions where a vanishes. The second point of the remark allows extending this intuition to $\varkappa > 1$, as we can look at some power of a. In particular, according to the third point, the $\varkappa$-th roots of functions in $\mathcal{Z}^{\varkappa}$ are comparable to Lipschitz continuous functions. We show examples of functions in $\mathcal{Z}^{\varkappa}$ on an interval for large and small values of parameter $\varkappa$ on Figure 5.1. In both of these cases there is no reason for the smoothness or the continuity of functions from $\mathcal{Z}^{\varkappa}$. The controlled property is the rate of decay in the transition region, which is comparable to a power function with an exponent $\varkappa$.

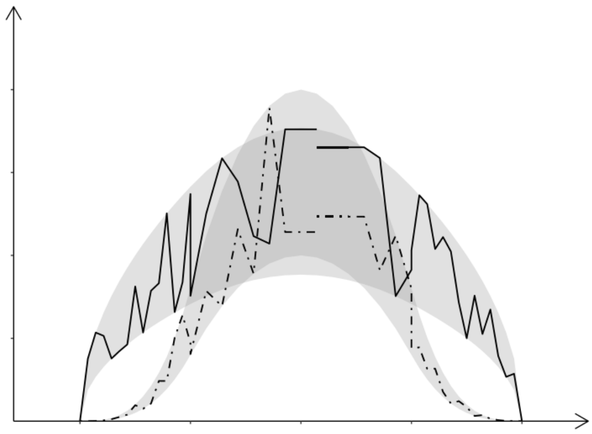

Figure 5.1: Solid line represents an example of $a_1 \in \mathcal{Z}^{\varkappa_1}$ for $\varkappa_1 \in (0, 1)$, while dot-dashed line $a_2 \in \mathcal{Z}^{\varkappa_2}$ for $\varkappa_2 > 1$. We stress that $a_2 \in \mathcal{Z}^{\varkappa_2} \subset \mathcal{Z}^{\varkappa_1}$.

We observe that in order to ensure the absence of the Lavrentiev gap for any q and p one can take a weight a decaying like e^{-1/t^2} that is faster than any polynomial. In particular, for every $\varkappa \in (0, +\infty)$, we have that the function $x \mapsto |x|^{\varkappa}$ belongs to $\mathcal{Z}^{\varkappa}(\mathbb{R}^n)$.

We also show that if

$$1 < p < n < n + \varkappa < q$$

there exist a domain, a boundary condition, and weight $a \in \mathcal{Z}^{\varkappa}$ for which the infima of $\mathcal{P}$ differ, see Section 5.2. The method is inspired by the two-dimensional checkerboard constructions of Zhikov [107] and its extension in [67]. In detail, we modify a weight $a \in C^{0,\alpha}$ from [67] to allow $a^{\frac{1}{\varkappa}}$ being comparable to a Lipschitz function, so that $a \in \mathcal{Z}^{\varkappa}$.

The present chapter is divided into three section. We start proving the absence of Lavrentiev phenomenon for (5.1). Then we pass to exhibit the counterexample in Section 5.2. In this part we also show how the smoothness of the weight does not help increasing the gap between p and q.

5.1 Approximation and absence of Lavrentiev phenomenon

In this section we prove the absence of Lavrentiev phenomenon. Let us briefly summarize the methods. We first establish the density of smooth functions with compact support in the energy space W. To this purpose we make use of the convolution with shrinking, explained in preliminaries 2.2, in order to construct a sequence $\{S_\delta\varphi\}_\delta \in C_c^\infty$. At this point the goal consist in showing that the family

$$\left\{|D(S_\delta\varphi)(x))|^p + a(x)|D(S_\delta\varphi)(x))|^q\right\}_\delta$$

is uniformly integrable. The proof proceeds basically as the one of [23, Theorem 2], but (5.7) highlights that there is no reason for p and q to be close if only one can adjust the decay of the weight to compensate it. The absence of the Lavrentiev phenomenon, stated in Theorem 5.1.3, is a consequence of the density of smooth functions via the ideas inspired by [23, 28] applying the Vitali convergence theorem. We simultaneously show that if $u \in W \cap C^{0,\gamma}$, $\gamma \in (0,1]$, we can relax the bound (5.2) even further. In fact, for $\gamma = 1$ there is no gap for arbitrary p and q. Moreover, to exclude the gap between $W \cap C^{0,\gamma}$ and C_c^∞, it suffices to take

$$q \leqslant p + \frac{\varkappa}{1-\gamma}, \quad \varkappa \in (0,\infty)\,. \tag{5.4}$$

In the next subsection we prove the density result, which is applied in Subsection 5.1.2 to get the absence of the Lavrentiev phenomenon.

5.1.1 Approximation

Let us establish the density of smooth functions with compact support in the energy space W. We divide the proof into three steps according to the property of the domain Ω. We initially take $\Omega \subset \mathbb{R}^n$ as a star-shaped domain with respect to the ball centered in zero, then with respect to a ball centered in a point different that zero and finally Ω is assumed to be an arbitrary Lipschitz domain. The result reads as follows.

Theorem 5.1.1 (Density of smooth functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, let $1 < p < q < +\infty$ and $a : \Omega \to [0,\infty)$ be such that $a \in \mathcal{Z}^\varkappa(\Omega)$, $\varkappa > 0$. Then the following assertions hold true.*

(i) If $\varkappa \geqslant q - p$, then for any $\varphi \in W(\Omega,\mathbb{R})$ there exists a sequence $\{\varphi_\delta\}_\delta \subset C_c^\infty(\Omega,\mathbb{R})$, such that $\varphi_\delta \to \varphi$ in $W(\Omega)$.

(ii) Let $\gamma \in (0,1]$. If $\varkappa \geqslant (q-p)(1-\gamma)$, then for any $\varphi \in W(\Omega,\mathbb{R}) \cap C^{0,\gamma}(\Omega,\mathbb{R})$ there exists a sequence $\{\varphi_\delta\}_\delta \subset C_c^\infty(\Omega,\mathbb{R})$, such that $\varphi_\delta \to \varphi$ in $W(\Omega)$.

Moreover, in both above cases, if $\varphi \in L^\infty(\Omega,\mathbb{R})$, then there exists $c = c(\Omega) > 0$, such that $||\varphi_\delta||_{L^\infty(\Omega)} \leqslant c||\varphi||_{L^\infty(\Omega)}$.

Proof. Let us at first notice that by Lemma 2.2.1, we have that $W(\Omega,\mathbb{R})\cap L^\infty(\Omega,\mathbb{R})$ is dense in $W(\Omega,\mathbb{R})$. Therefore, for the assertion (i), it suffices to consider the density of $C_c^\infty(\Omega,\mathbb{R})$ in $W(\Omega,\mathbb{R})\cap L^\infty(\Omega,\mathbb{R})$. Let us assume that in case of (i), we have $\gamma=0$. We shall prove the claims (i) and (ii) simultaneously. To this aim, let us take any $\varphi\in W(\Omega,\mathbb{R})\cap L^\infty(\Omega,\mathbb{R})$ in the case of $\gamma=0$ and $\varphi\in W(\Omega,\mathbb{R})\cap C^{0,\gamma}(\Omega,\mathbb{R})$ otherwise.

We divide the proof into three steps.

Step 1: Ω is a star-shaped domain with respect to a ball centred in zero and with radius $R>0$, that is $B(0,R)$

Recall the definition of $S_\delta\varphi$, given in (2.13), where we take $x_0=0$ and $\delta<R/4$. Our aim now is to prove that $DS_\delta\varphi$ converges to $D\varphi$ in $W(\Omega,\mathbb{R})$. Due to (2.12), it is enough to show that $S_\delta\varphi\to\varphi$ in L^1 and $DS_\delta\varphi\xrightarrow{M}D\varphi$ modularly in L_M. We observe that by (2.14) and Lemma 2.2.2, we have this first convergence as well as the fact that $D(S_\delta(\varphi))$ converges to $D\varphi$ in measure. Therefore, by (2.11), it suffices to prove that

$$\text{the family } \left\{|D(S_\delta\varphi)(x))|^p+a(x)|D(S_\delta\varphi)(x))|^q\right\}_\delta \text{ is uniformly integrable.} \tag{5.5}$$

Observe that by Lemma 2.2.3, for sufficiently small $\delta>0$, there exists a constant $C_S>0$, independent of δ, such that

$$||D(S_\delta\varphi)||_{L^\infty}\leqslant C_S\delta^{\gamma-1}\,. \tag{5.6}$$

Indeed, if $\gamma=0$, then by using assertion (2.15) and the fact that $\varphi\in L^\infty(\Omega,\mathbb{R})$, we can set

$$C_S:=||\varphi||_{L^\infty}||D\rho||_{L^1}$$

in (5.6). In the case of $\gamma\in(0,1]$, (2.16) provides that

$$||DS_\delta(\varphi)||_{L^\infty}\leqslant\frac{\delta^{\gamma-1}}{\kappa_\delta^\gamma}[\varphi]_{0,\gamma}||D\rho||_{L^1}.$$

As $\varphi\in C^{0,\gamma}(\Omega,\mathbb{R})$ and $\kappa_\delta\xrightarrow{\delta\to0}1$, we obtain inequality (5.6) with constant

$$C_S:=2[\varphi]_{0,\gamma}||D\rho||_{L^1},$$

for sufficiently small δ. We therefore have (5.6) for all $\gamma\in[0,1]$.

As $a\in\mathcal{Z}^\varkappa$, there exists a constant $C_a>1$ such that for any $x,y\in\Omega$ we have

$$a(x)\leqslant C_a(a(y)+|x-y|^\varkappa).$$

Let us take any $x,y\in\Omega,\tau>0,\delta\in(0,1)$ such that $|x-y|\leqslant\tau\delta$. We have

$$\begin{aligned}
&|DS_\delta(\varphi)(x)|^p+a(x)|DS_\delta(\varphi)(x)|^q && (5.7)\\
&\quad=|DS_\delta(\varphi)(x)|^p(1+a(x)|DS_\delta(\varphi)(x)|^{q-p})\\
&\quad\leqslant|DS_\delta(\varphi)(x)|^p(1+C_a(a(y)+\tau^\varkappa\delta^\varkappa)|DS_\delta(\varphi)(x)|^{q-p})\\
&\quad\leqslant C_a|DS_\delta(\varphi)(x)|^p(1+a(y)|DS_\delta(\varphi)(x)|^{q-p}+\tau^\varkappa\delta^\varkappa|DS_\delta(\varphi)(x)|^{q-p})\,. && (5.8)
\end{aligned}$$

By the inequality (5.6), we obtain that

$$\delta^{\varkappa}|DS_\delta(\varphi)(x)|^{q-p} \leqslant C_S^{q-p}\delta^{\varkappa}\delta^{(q-p)(\gamma-1)} \leqslant C_S^{q-p}\,, \tag{5.9}$$

where in the last inequality we used that $\delta \in (0,1)$ and $\varkappa + (q-p)(\gamma-1) \geqslant 0$. By (5.7) and (5.9), we have that there exists a constant $C_\tau > 0$, not depending on δ, such that

$$|DS_\delta(\varphi)(x)|^p + a(x)|DS_\delta(\varphi)(x)|^q \leqslant C_\tau\left(|DS_\delta(\varphi)(x)|^p + \left(\inf_{z\in B_{\tau\delta}(x)} a(z)\right)|DS_\delta(\varphi)(x)|^q\right). \tag{5.10}$$

Let us recall (2.14), that is $DS_\delta(\varphi) = \frac{1}{\kappa_\delta}S_\delta(D\varphi)$. By using Jensen's inequality in conjunction with the fact that $\kappa_\delta \geqslant 1/2$ for sufficiently small δ, we may write

$$\begin{aligned}|DS_\delta(\varphi)(x)|^p &= \frac{1}{\kappa_\delta^p}\left|\int_{B_\delta(0)}\rho_\delta(y)(D\varphi)((x-y)/\kappa_\delta)\,dy\right|^p\\ &\leqslant 2^p\int_{B_\delta(0)}\rho_\delta(y)|(D\varphi)((x-y)/\kappa_\delta)|^p\,dy = 2^pS_\delta(|D\varphi|^p)(x)\,,\end{aligned} \tag{5.11}$$

for sufficiently small $\delta > 0$. Analogously, it holds that

$$\begin{aligned}\left(\inf_{z\in B_{\tau\delta}(x)} a(z)\right)|DS_\delta(\varphi)(x)|^q &\leqslant 2^q\int_{B_\delta(0)}\rho_\delta(y)\left(\inf_{z\in B_{\tau\delta}(x)} a(z)\right)|(D\varphi)((x-y)/\kappa_\delta)|^q\,dy\\ &\leqslant 2^q\int_{B_\delta(0)}\rho_\delta(y)a((x-y)/\kappa_\delta)|(D\varphi)((x-y)/\kappa_\delta)|^q\,dy\\ &= 2^qS_\delta(a|D\varphi|^q)(x)\,,\end{aligned} \tag{5.12}$$

where τ is fixed such that for sufficiently small $\delta > 0$ we have

$$\left|\frac{x-y}{\kappa_\delta} - x\right| \leqslant \frac{|y|}{\kappa_\delta} + \frac{1-\kappa_\delta}{\kappa_\delta}|x| \leqslant \frac{\delta}{\kappa_\delta} + \frac{\delta}{2R\kappa_\delta}(\operatorname{diam}\Omega) \leqslant \tau\delta\,.$$

Observe that by (5.10) and by estimates (5.11) and (5.12), we have

$$\begin{aligned}M(x,|DS_\delta\varphi(x)|) &\leqslant 2^qC_\tau(S_\delta(|D\varphi|^p)(x) + S_\delta(a|D\varphi|^q)(x))\\ &= 2^qC_\tau S_\delta\left(M\left(\cdot,|D\varphi(\cdot)|\right)\right)(x)\,.\end{aligned} \tag{5.13}$$

The fact that $\varphi \in W(\Omega,\mathbb{R})$ implies that $M(x,|D\varphi(x)|) \in L^1(\Omega,\mathbb{R})$. Therefore, Lemma 2.2.2 gives us that the sequence $\{S_\delta\left(M\left(\cdot,|D\varphi(\cdot)|\right)\right)\}_\delta$ converges in L^1. By the Vitali Convergence Theorem, it means that the family $\{S_\delta\left(M\left(\cdot,|D\varphi(\cdot)|\right)\right)\}_\delta$ is uniformly integrable. Using the estimate (5.13), we deduce that the family $\{M(x,|D(S_\delta\varphi)(x)|)\}_\delta$ is uniformly integrable, which is (5.5). Therefore, the proof is completed for Ω being a bounded star-shaped domain with respect to a ball centred in zero.

Step 2: *Ω is a star-shaped with respect to a ball centred in point other than zero*

We translate the problem, obtaining the set being a star-shaped domain with respect to a ball centred in zero. Then, proceeding with the proof above and reversing translation of Ω gives the desired result.

Step 3: Ω is an arbitrary bounded Lipschitz domain

By [36, Lemma 8.2], a set $\overline{\Omega}$ can be covered by a finite family of sets $\{U_i\}_{i=1}^K$ such that each $\Omega_i := \Omega \cap U_i$ is a star-shaped domain with respect to some ball. Then

$$\Omega = \bigcup_{i=1}^{K} \Omega_i \,.$$

By [104, Proposition 2.3, Chapter 1], there exists the partition of unity related to the partition $\{U_i\}_{i=1}^K$, i.e., the family $\{\theta_i\}_{i=1}^K$ such that

$$0 \leqslant \theta_i \leqslant 1, \quad \theta_i \in C_c^\infty(U_i), \quad \sum_{i=1}^{K} \theta_i(x) = 1 \ \text{ for } \ x \in \Omega \,.$$

By the previous paragraph for every $i = 1, 2, \ldots, K$, as Ω_i is a star-shaped domain with respect to some ball, and $\theta_i \varphi \in W(\Omega_i)$, there exists a sequence $\{\varphi_\delta^i\}_\delta$ such that $\varphi_\delta^i \xrightarrow{\delta \to 0} \theta_i \varphi$ in $W(\Omega_i)$. Let us now consider the sequence $\{I_\delta\}_\delta$ defined as

$$I_\delta := \sum_{i=1}^{K} \varphi_\delta^i.$$

We shall show that $I_\delta \to \varphi$ in $W(\Omega)$. As we have that $\varphi_\delta^i \to \theta_i \varphi$ in L^1 for every i, we have $I_\delta \to \varphi$ in L^1. It suffices to prove that $DI_\delta \to D\varphi$ in $L_M(\Omega)$. Since the sequence $\{D\varphi_\delta^i\}_\delta$ converges to $D(\theta_i\varphi)$ in measure and $\sum_{i=1}^K D(\theta_i \varphi) = D\varphi$, it holds that

$$\{DI_\delta\}_\delta \to D\varphi \text{ in measure.} \tag{5.14}$$

Moreover, for any $x \in \Omega$ we have that

$$\begin{aligned} |DI_\delta(x)|^p + a(x)\, |DI_\delta(x)|^q &\leqslant \sum_{i=1}^{K} \left(K^{p-1} |D(\varphi_\delta^i)(x)|^p + K^{q-1} a(x) |D(\varphi_\delta^i)(x)|^q \right) \\ &\leqslant K^{q-1} \sum_{i=1}^{K} \left(|D(\varphi_\delta^i)(x)|^p + a(x) |D(\varphi_\delta^i)(x)|^q \right) . \end{aligned} \tag{5.15}$$

As for all $i = 1, 2, \ldots, K$, we have that $\{\varphi_\delta^i\}_\delta$ converges in $W(\Omega_i)$, it holds that the family $\{|D(\varphi_\delta^i)(x)|^p + a(x)|D(\varphi_\delta^i)(x)|^q\}_\delta$ is uniformly integrable. Therefore, the estimate (5.15) gives us that

$$\text{the family} \quad \left\{ \left| \sum_{i=1}^{K} D(\varphi_\delta^i)(x) \right|^p + a(x) \left| \sum_{i=1}^{K} D(\varphi_\delta^i)(x) \right|^q \right\}_\delta \quad \text{is uniformly integrable.}$$

This together with (5.14) and (2.11), as well as the fact that $I_\delta \to \varphi$ in L^1, gives us the result for an arbitrary bounded Lipschitz domain Ω. $\square$

5.1.2 Absence of the Lavrentiev phenomenon

As a direct consequence of Theorem 5.1.1 we infer the absence of the Lavrentiev phenomenon. We start with a simple formulation for the double phase functional (5.1) reading as follows.

Theorem 5.1.2 (Absence of the Lavrentiev phenomenon for a model functional)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, let $1 < p < q < +\infty$ and $a : \Omega \to [0, \infty)$ be such that $a \in \mathcal{Z}^{\varkappa}(\Omega)$, $\varkappa > 0$. Assume that u_0 satisfies*

$$\mathcal{P}(u_0) < +\infty.$$

Then the following assertions hold true.

(i) If $\varkappa \geqslant q - p$, then

$$\inf_{u \in u_0 + W(\Omega,\mathbb{R})} \mathcal{P}(u) = \inf_{u \in u_0 + C_c^\infty(\Omega,\mathbb{R})} \mathcal{P}(u)\,. \tag{5.16}$$

(ii) Let $\gamma \in (0, 1]$. If $\varkappa \geqslant (q - p)(1 - \gamma)$, then

$$\inf_{u \in u_0 + W(\Omega,\mathbb{R}) \cap C^{0,\gamma}(\Omega,\mathbb{R})} \mathcal{P}(u) = \inf_{u \in u_0 + C_c^\infty(\Omega,\mathbb{R})} \mathcal{P}(u)\,. \tag{5.17}$$

The above theorem is a special case of the following more general result. Let us consider the following variational functional

$$\mathcal{G}(u) = \int_\Omega g(x, u(x), Du(x))\, dx\,, \tag{5.18}$$

over an open and bounded set $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, where $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous with respect to the second and the third variable and $z \mapsto g(x, u, z)$ is convex. We suppose that there exist constants $0 < \nu < 1 < L$ and a nonnegative $\Lambda \in L^1(\Omega, \mathbb{R})$ such that

$$\nu\left(|z|^p + a(x)|z|^q\right) \leqslant g(x, u, z) \leqslant L\left(|z|^p + a(x)|z|^q + \Lambda(x)\right), \tag{5.19}$$

for all $x \in \Omega,\ u \in \mathbb{R},\ z \in \mathbb{R}^n$.

Theorem 5.1.3 (Absence of Lavrentiev phenomenon for general functionals)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, let $1 < p < q < +\infty$ and $a : \Omega \to [0, \infty)$ be such that $a \in \mathcal{Z}^{\varkappa}(\Omega)$, $\varkappa > 0$. Assume that u_0 satisfies*

$$\mathcal{G}(u_0) < +\infty.$$

Then the following assertions hold true.

(i) If $\varkappa \geqslant q - p$, then

$$\inf_{u \in u_0 + W(\Omega,\mathbb{R})} \mathcal{G}(u) = \inf_{u \in u_0 + C_c^\infty(\Omega,\mathbb{R})} \mathcal{G}(u)\,. \tag{5.20}$$

(ii) Let $\gamma \in (0,1]$. If $\varkappa \geqslant (q-p)(1-\gamma)$, then

$$\inf_{u\in u_0+W(\Omega,\mathbb{R})\cap C^{0,\gamma}(\Omega,\mathbb{R})} \mathcal{G}(u) = \inf_{u\in u_0+C_c^\infty(\Omega,\mathbb{R})} \mathcal{G}(u)\,. \tag{5.21}$$

Proof. Since $C_c^\infty(\Omega,\mathbb{R}) \subset W(\Omega,\mathbb{R})$, it holds that

$$\inf_{u\in u_0+W(\Omega,\mathbb{R})} \mathcal{G}(u) \leqslant \inf_{u\in u_0+C_c^\infty(\Omega,\mathbb{R})} \mathcal{G}(u)\,.$$

Let us concentrate on showing the opposite inequality. By direct methods of Calculus of Variation, there exists a minimizer, i.e., a function $u \in W(\Omega,\mathbb{R})$ such that

$$\mathcal{G}(u_0+u) = \inf_{u\in u_0+W(\Omega,\mathbb{R})} \mathcal{G}(u)\,.$$

By assertion (i) of Theorem 5.1.1, there exists $\{u_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\Omega,\mathbb{R})$ such that $u_k \to u-u_0$ in $W(\Omega,\mathbb{R})$. Since g is continuous with respect to the second and the third variable, we infer that

$$g(x,u_0(x)+u_k(x),Du_0(x)+Du_k(x)) \xrightarrow[k\to+\infty]{} g(x,u(x),Du(x)) \text{ in measure.}$$

We shall now show that

$$\text{the family } \{g(x,u_0(x)+u_k(x),Du_0(x)+Du_k(x))\}_{k\in\mathbb{N}} \text{ is uniformly integrable.} \tag{5.22}$$

By assumption (5.19), we notice that

$$\begin{aligned} &g(x,u_0(x)+u_k(x),Du_0(x)+Du_k(x)) \\ &\quad\leqslant L\left(|Du_0(x)+Du_k(x)|^p + a(x)|Du_0(x)+Du_k(x)|^q\right) + L\Lambda(x) \\ &\quad\leqslant C\left(|Du_k(x)|^p + a(x)|Du_k(x)|^q\right) + C\left(|Du_0(x)|^p + a(x)|Du_0(x)|^q\right) + L\Lambda(x)\,, \end{aligned}$$

where C is a positive constant, for every fixed $k \geqslant 1$. Note that $\Lambda \in L^1(\Omega,\mathbb{R})$ and as $\mathcal{G}(u_0) < +\infty$, by (5.19), we have

$$\int_\Omega \left(|Du_0(x)|^p + a(x)|Du_0(x)|^q\right) dx < +\infty.$$

Moreover, since $\{Du_k\}_k$ converges in $W(\Omega)$, we infer that

$$\text{the family} \quad \{|Du_k(x)|^p + a(x)|Du_k(x)|^q\}_{k\in\mathbb{N}} \quad \text{is uniformly integrable.}$$

Thus, (5.22) is justified. In turn, by Vitali Convergence Theorem, we have that

$$\mathcal{G}(u_0+u_k) \xrightarrow{k\to+\infty} \mathcal{G}(u+u_0). \tag{5.23}$$

Therefore, we get

$$\inf_{u\in u_0+C_c^\infty(\Omega,\mathbb{R})} \mathcal{G}(u) \leqslant \mathcal{G}(u_0+u) = \inf_{u\in u_0+W(\Omega,\mathbb{R})} \mathcal{G}(u).$$

Consequently, (5.20) is proven.

By repeating the same procedure for $u \in W(\Omega,\mathbb{R}) \cap C^{0,\gamma}(\Omega,\mathbb{R})$ with the use of Theorem 5.1.1 *(ii)* instead of *(i)*, one gets (5.21). □

Remark 5.1.4. It is easy to observe that convergence (5.23) holds for every $u \in u_0 + W(\Omega,\mathbb{R})$, that is to say: not necessarily for the minimizer.

5.2 Sharpness

By sharpness, we mean that if p, q and $\varkappa$ are outside the proper range (5.2), it is possible to find a Lipschitz domain Ω, a weight $a \in \mathcal{Z}^{\varkappa}(\Omega)$ and a boundary datum $u_0 \in W(\Omega, \mathbb{R})$ such that the Lavrentiev phenomenon occurs, i.e.

$$\inf_{u \in X} \mathcal{P}(u) < \inf_{u \in Y} \mathcal{P}(u).$$

We point out that, for our example, we modify the construction from [67] based on the seminal idea of Zhikov's checkerboard [107].

Theorem 5.2.1 (Sharpness)**.** *Let $p, q, \varkappa > 0$ be such that*

$$1 < p < n < n + \varkappa < q\,.$$

Then there exist a Lipschitz domain $\Omega \subset \mathbb{R}^n$, a function $a \in \mathcal{Z}^{\varkappa}$ and u_0 satisfying $\mathcal{P}(u_0) < +\infty$ such that

$$\inf_{u \in u_0 + W(\Omega, \mathbb{R})} \mathcal{P}(u) < \inf_{u \in u_0 + C_c^{\infty}(\Omega, \mathbb{R})} \mathcal{P}(u)\,. \tag{5.24}$$

In order to show the presence of the Lavrentiev phenomenon we first define the Lipschitz domain Ω, the function a and the boundary datum u_0. We choose Ω as the ball of center 0 and radius 1, i.e.,

$$\Omega = B_1 \coloneqq B_1(0)\,. \tag{5.25}$$

Now let us define the following set

$$V \coloneqq \left\{ x \in B_1 : \; x_n^2 - \sum_{i=1}^{n-1} x_i^2 > 0 \right\}\,. \tag{5.26}$$

Regarding the weight a we introduce the function $\ell : \Omega \to \mathbb{R}$ via the following formula

$$\ell(x) \coloneqq \max\left\{ x_n^2 - \sum_{i=1}^{n-1} x_i^2, 0 \right\} |x|^{-1}\,, \qquad x = (x_1, \dots, x_n)\,.$$

The weight is defined as

$$a \coloneqq \ell^{\varkappa}\,. \tag{5.27}$$

Computing the partial derivative of ℓ in V we get

$$\frac{\partial \ell}{\partial x_i} = \begin{cases} -\frac{x_i}{|x|^3}\left(\sum_{j=1}^{n-1} x_j^2 + 3x_n^2\right) & \text{if} \quad i = 1, \dots, n-1\,, \\ \frac{x_i}{|x|^3}\left(\sum_{j=1}^{n-1} 3x_j^2 + x_i^2\right) & \text{if} \quad i = n\,. \end{cases}$$

We can observe that $||D\ell||_{L^\infty(B_1)}$ is bounded. In turn, ℓ is Lipschitz continuous and consequently $a \in \mathcal{Z}^{\varkappa}(B_1)$. We note that $\operatorname{supp} a \subset V$, so the set V shall include whole q-phase, while p-phase will be in $B_1 \setminus V$.

Let us state and prove a lemma that we will use in the proof of Theorem 5.2.1.

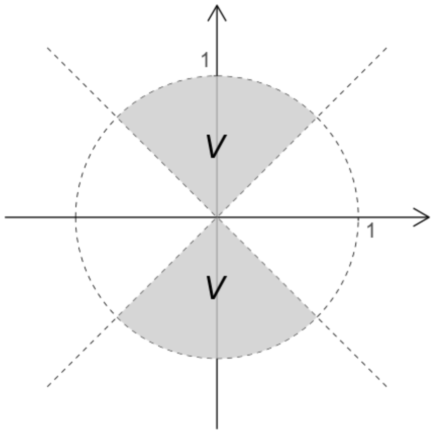

Figure 5.2: The main properties of $a \in \mathcal{Z}^{\varkappa}(B_1)$ and $u_* \in W(B_1)$ that produce the Lavrentiev gap for our counterexample are the facts that $\operatorname{supp} a \subset V$, and $Du_* \equiv 0$ in $B_1 \setminus V$.

Lemma 5.2.2. *Let a be defined by (5.27) and V be defined as in (5.26). Then*

$$r_1 := \int_V |x|^{-\frac{q(n-1)}{q-1}} a(x)^{-\frac{1}{q-1}}\, dx \ < \infty\,. \tag{5.28}$$

Proof. We use the spherical coordinates. The proof is presented in two cases – for $n = 2$ and $n > 2$.

For $n = 2$ we take

$$x_1 := \rho\cos\theta \quad \text{and} \quad x_2 := \rho\sin\theta\,,$$

consequently

$$a = \rho^{\varkappa}\max(-\cos 2\theta, 0)^{\varkappa},$$

where $\theta \in [0, 2\pi)$. After this change of variables V is mapped into $S := (0,1) \times \left[(\frac{\pi}{4}, \frac{3\pi}{4}) \cup (\frac{5\pi}{4}, \frac{7\pi}{4})\right]$, so (5.28) reads as

$$r_1 = \int_S \rho^{1-\frac{q+\varkappa}{q-1}} |\cos(2\theta)|^{-\frac{\varkappa}{q-1}}\ d\rho\, d\theta\,.$$

As $q > \varkappa + 2$, we have $1 - \frac{q+\varkappa}{q-1} > -1$, which implies that $\int_0^1 \rho^{1-\frac{q(1+\varkappa)}{q-1}}\, d\rho < \infty$. As far as $-\cos(2\theta)^{-\frac{\varkappa}{q-1}}$ is concerned, we observe at first that over the set that we integrate on, it holds that $\cos(2\theta) = 0$ only for $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. Therefore, it suffices to prove the integrability of $|\cos(2\theta)|^{-\frac{\varkappa}{q-1}}$ near these points. Observe that for sufficiently small $\theta_0 > 0$ we have

$$\left|\cos\left(2(\theta_0 + \tfrac{\pi}{4})\right)\right| \geqslant 2\theta_0\left(1 - \frac{2\theta_0}{\pi}\right) \geqslant \theta_0\,,$$

which means that for $\theta = \theta_0 + \frac{\pi}{4}$ we get

$$|\cos(2\theta)|^{-\frac{\varkappa}{q-1}} \leqslant \left(\theta - \frac{\pi}{4}\right)^{-\frac{\varkappa}{q-1}} . \tag{5.29}$$

Since $q > \varkappa + 1$, we have $-\frac{\varkappa}{q-1} > -1$, and therefore, we have the integrability of $|\cos(2\theta)|^{-\frac{\varkappa}{q-1}}$ near $\frac{\pi}{4}$, and by analogy, also in the points $\frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. Therefore, we showed that r_1 is finite for $n = 2$.

For $n > 2$ we set

$$x_1 := \rho\cos\theta \prod_{k=1}^{n-2} \sin\theta_k\,, \quad x_2 := \rho\sin\theta \prod_{k=1}^{n-2} \sin\theta_k\,, \quad x_i := \rho\cos\theta_{n-2} \prod_{k=i-1}^{n-2} \sin\theta_k\,, \text{ for } i \geqslant 3$$

and so

$$a = \rho^{\varkappa} \max(\cos 2\theta_{n-2}, 0)^{\varkappa}\,,$$

with $\rho > 0$ and $\theta_i \in [0, \pi]$ for $i = 1, \ldots, n-2$. We observe that V is mapped to $S = (0,1) \times (0, 2\pi) \times (0, \pi)^{n-2} \times \left((0, \frac{\pi}{4}) \cap \left(\frac{3\pi}{4}, \pi\right)\right)$, that is, $\theta_{n-2} \in (0, \frac{\pi}{4}) \cap \left(\frac{3\pi}{4}, \pi\right)$ and the modulus of the determinant of the change of variable may be estimated by ρ^{n-1}. Therefore, we can estimate

$$r_1 \leqslant \int_S \rho^{n-1+\frac{q(1-n)-\varkappa}{q-1}} |\cos(2\theta_{n-2})|^{-\frac{\varkappa}{q-1}}\, d\rho\, d\theta_{n-2}\,.$$

As $q > \varkappa + n$, it follows that $n - 1 + \frac{q(1-n)-\varkappa}{q-1} > -1$, and therefore, $\int_0^1 \rho^{n-1+\frac{q(1-n)-\varkappa}{q-1}}\, d\rho < \infty$. Using analogous estimates as (5.29), one may also prove the integrability of $(\cos(2\theta_{n-2}))^{-\frac{\varkappa}{q-1}}$ in $(0, \frac{\pi}{4}) \cap \left(\frac{3\pi}{4}, \pi\right)$, obtaining the finiteness of r_1 in case of $n \geqslant 3$. $\square$

As far as the boundary datum is concerned we first define a function u_* and, after we establish some of its properties, we shall find u_0 such that $u_* \in (u_0 + W(B_1, \mathbb{R}))$, but $u_* \notin \overline{(u_0 + C_c^\infty(B_1, \mathbb{R}))}^W$. We set

$$u_*(x) := \begin{cases} \sin(2\theta) & \text{if } \; 0 \leqslant \theta \leqslant \frac{\pi}{4}\,, \\ 1 & \text{if } \; \frac{\pi}{4} \leqslant \theta \leqslant \frac{3\pi}{4}\,, \\ \sin(2\theta - \pi) & \text{if } \; \frac{3\pi}{4} \leqslant \theta \leqslant \frac{5\pi}{4}\,, \\ -1 & \text{if } \; \frac{5\pi}{4} \leqslant \theta \leqslant \frac{7\pi}{4}\,, \\ \sin(2\theta) & \text{if } \; \frac{7\pi}{4} \leqslant \theta \leqslant 2\pi, \end{cases} \tag{5.30}$$

for $n = 2$, and

$$u_*(x) := \begin{cases} 1 & \text{if } \; 0 \leqslant \vartheta_{n-2} \leqslant \frac{\pi}{4}\,, \\ \sin(2\vartheta_{n-2}) & \text{if } \; \frac{\pi}{4} \leqslant \vartheta_{n-2} \leqslant \frac{3\pi}{4}\,, \\ -1 & \text{if } \; \frac{3\pi}{4} \leqslant \vartheta_{n-2} \leqslant \pi\,, \end{cases} \tag{5.31}$$

for $n \geqslant 3$. The boundary datum u_0 is determined by the following expression

$$u_0(x) \coloneqq t_0|x|^2 u_*(x)\,, \tag{5.32}$$

where t_0 will be chosen. We have the following lemma.

Lemma 5.2.3. *The function u_* belongs to $u_0 + W(B_1, \mathbb{R})$. In particular*

$$r_2 \coloneqq \int_{B_1} |Du_*(x)|^p \, dx < +\infty\,.$$

Proof. We start observing that $\operatorname{supp} a \subset V$ and $Du_* \equiv 0$ in $\operatorname{supp} a$, i.e.,

$$\int_{B_1} (|Du_*(x)|^p + a(x)|Du_*(x)|^q) \, dx = \int_{B_1} |Du_*(x)|^p \, dx = r_2\,.$$

To justify that r_2 is finite, we notice that using spherical coordinates for $n = 2$ one gets

$$r_2 = \int_0^1 \rho \, d\rho \left[\int_0^{\frac{\pi}{4}} |2\cos(2\theta)|^p \, d\theta + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} |2\cos(2\theta - \pi)|^p \, d\theta + \int_{\frac{7\pi}{4}}^{2\pi} |2\cos(2\theta)|^p \, d\theta \right] < \infty\,,$$

whereas when $n > 2$, then

$$r_2 = \int_0^1 \int_0^{2\pi} \int_0^{\pi} |\det J| \, d\rho \, d\theta \prod_{i=1}^{n-3} d\theta_i \int_0^{\pi} |2\cos(2\theta_{n-2} - \pi)|^p \, d\theta_{n-2} < \infty\,,$$

where J is the Jacobian matrix of the spherical coordinate transformation. Now, since $p < n$ we can apply the Sobolev embedding theorem to obtain $u_* \in L^p(B_1, \mathbb{R})$. Then $u_* \in W^{1,1}(B_1, \mathbb{R})$ and $\mathcal{P}(u_*) < +\infty$, namely $u_* \in W(B_1, \mathbb{R})$. $\square$

We take

$$t_0 > \left[r_2 \left(\frac{q}{r_3}\right)^q \left(\frac{r_1}{q-1}\right)^{q-1} \right]^{\frac{1}{q-p}}\,, \tag{5.33}$$

with r_1 from Lemma 5.2.2, r_2 from Lemma 5.2.3, and

$$r_3 \coloneqq \mathcal{H}^{n-1}(\overline{V} \cap \partial B_1), \tag{5.34}$$

where $\mathcal{H}^{n-1}$ is the classical Hausdorff measure of dimension $n-1$, defined on $\mathbb{R}^n$. Now let us state the following observation made in [67]. The proof consists of calculations with the spherical coordinates in which Fubini's theorem and Jensen's inequality are used, see [67, p. 17] for details.

Lemma 5.2.4. *For any function $w \in u_0 + C_0^\infty(B_1, \mathbb{R})$ it holds*

$$t_0 \mathcal{H}^{n-1}(\overline{V} \cap \partial B_1) \leqslant \int_V \frac{1}{|x|^{n-1}} \left| \left\langle \frac{x}{|x|}, Dw(x) \right\rangle \right| dx\,,$$

for t_0 as in (5.33) and u_0 as in (5.32).

Now we are ready to prove the theorem.

Proof of Theorem 5.2.1. Bearing in mind the definition of u_* in (5.30)-(5.31) and of u_0 in (5.32) we start observing that

$$\begin{aligned}\inf_{u\in u_0+W(B_1)} \mathcal{P}(u) \leqslant \mathcal{P}(t_0u_*) &= t_0^p \int_{B_1} |Du_*(x)|^p\,dx + t_0^q \int_{B_1} a(x)|Du_*(x)|^q\,dx \\ &= t_0^p \int_{B_1} |Du_*(x)|^p\,dx = t_0^p r_2\,, \end{aligned}\tag{5.35}$$

which is finite by Lemma 5.2.3. Let us fix arbitrary $w \in u_0 + C_0^\infty(B_1,\mathbb{R})$ and $\lambda > 0$. In order to estimate from below $\mathcal{P}(w)$ we notice that Lemma 5.2.4 together with Young's inequality and Lemma 5.2.2 leads to

$$\begin{aligned} r_3\lambda t_0 &\leqslant \int_V \left(\frac{\lambda}{|x|^{n-1}}\frac{1}{a(x)}\right)\left|\left\langle \frac{x}{|x|}, Dw(x)\right\rangle\right| a(x)\,dx \\ &\leqslant \int_V \left(\frac{\lambda}{|x|^{n-1}}\frac{1}{a(x)}\right)^{\frac{q}{q-1}} a(x)\,dx + \int_V \left|\left\langle \frac{x}{|x|}, Dw(x)\right\rangle\right|^q a(x)\,dx \\ &\leqslant r_1\lambda^{\frac{q}{q-1}} + \int_V a(x)|Dw(x)|^q\,dx\,. \end{aligned}$$

where $\lambda > 0$ is fixed. Consequently,

$$r_3\lambda t_0 \leqslant r_1\lambda^{\frac{q}{q-1}} + \mathcal{P}(w).$$

Then for any $w \in u_0 + C_0^\infty(B_1,\mathbb{R})$ it holds

$$\mathcal{P}(w) \geqslant r_1 \sup_{\lambda>0}\left(\lambda t_0\frac{r_3}{r_1} - \lambda^{\frac{q}{q-1}}\right) = r_1 \sup_{\lambda\in\mathbb{R}}\left(\lambda t_0\frac{r_3}{r_1} - |\lambda|^{\frac{q}{q-1}}\right) = r_1\left(\frac{(q-1)t_0r_3}{qr_1}\right)^q \frac{1}{q-1}.$$

Now, bearing in mind (5.33) and using (5.35) we get

$$\inf_{u\in u_0+C_0^\infty(B_1,\mathbb{R})} \mathcal{P}(u) \geqslant \left(\frac{r_3}{q}\right)^q \left(\frac{q-1}{r_1}\right)^{q-1} t_0^q > r_2t_0^p \geqslant \inf_{u\in u_0+W(B_1,\mathbb{R})} \mathcal{P}(u). \tag{5.36}$$

Hence the occurrence of the Lavrentiev phenomenon, that is (5.24), is proven. □

5.2.1 Smoothness of the weight

In this section we want to stress the fact that $C^{1,\alpha}$-regularity for $\alpha \in (0,1]$ of the weight implies its $\mathcal{Z}^{1+\alpha}$-regularity, but smoothness of the weight does not give more than $\mathcal{Z}^2$. To state it precisely, we give the following proposition.

Proposition 5.2.5. *If $\Omega \subset \mathbb{R}^n$ is an open and bounded set, then the following holds.*

(i) If $0 \leqslant a \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1]$ and $a > 0$ on $\partial\Omega$, then $a \in \mathcal{Z}^{1+\alpha}(\Omega)$.

(ii) There exists $0 \leqslant a \in C^\infty(\overline{\Omega})$*, such that* $a > 0$ *on* $\partial\Omega$*,* $a \in \mathcal{Z}^2(\Omega)$*, and* $a \notin \mathcal{Z}^{2+\varepsilon}(\Omega)$ *for any* $\varepsilon > 0$*.*

Proof. We concentrate on *(i)*. Our reasoning is inspired by the proof of Glaeser-type inequality, see [61]. Suppose by contradiction that $a \notin \mathcal{Z}^{1+\alpha}$. This implies that there exist sequences $\{x_k\}_k, \{y_k\}_k \subset \Omega$ and $C_k \in \mathbb{R}$ with $\lim_{k\to+\infty} C_k = +\infty$ such that

$$a(x_k) \geqslant C_k(a(y_k) + |x_k - y_k|^{1+\alpha})\,. \tag{5.37}$$

As $\overline{\Omega}$ is compact, by taking subsequences if necessary, we may assume that $x_k \to \bar{x}, y_k \to \bar{y}$, where $\bar{x}, \bar{y} \in \overline{\Omega}$. Observe that taking limits in (5.37), we obtain that for every $C > 0$ we have $a(\bar{x}) > C(a(\bar{y}) + |\bar{x} - \bar{y}|^{1+\alpha})$. As a is bounded, we have that $a(\bar{y}) + |\bar{x} - \bar{y}|^{1+\alpha} = 0$. That is, we have $\bar{x} = \bar{y}$ and $a(\bar{x}) = 0$. We shall denote $x_0 \coloneqq \bar{x} = \bar{y}$. As $a(x_0) = 0$, by assumption, we have $x_0 \in \Omega$ and there exists $R > 0$ such that $B(x_0, R) \subseteq \Omega$.

Let us fix any $\nu \in \mathbb{R}^n$ such that $|\nu| = 1$. By Lagrange Mean Value Theorem, for arbitrary $z \in B(x_0, R)$ and $h \in \mathbb{R}$ such that $z + h\nu \in B(x_0, R)$, we have

$$a(z + h\nu) = a(z) + h\frac{\partial a}{\partial \nu}(z + \varsigma\nu), \tag{5.38}$$

where $\varsigma \in [-|h|, |h|]$. Using that $a \in C^{1,\alpha}(\Omega)$, we get that for some constant C, independent of ν, we have

$$\left|\frac{\partial a}{\partial \nu}(z + \varsigma\nu) - \frac{\partial a}{\partial \nu}(z)\right| \leqslant C|\varsigma|^\alpha \leqslant C|h|^\alpha\,,$$

and, consequently,

$$\frac{\partial a}{\partial \nu}(z) - C|h|^\alpha \leqslant \frac{\partial a}{\partial \nu}(z + \varsigma\nu) \leqslant \frac{\partial a}{\partial \nu}(z) + C|h|^\alpha\,.$$

Thus, for $h \geqslant 0$ it holds that

$$h\frac{\partial a}{\partial \nu}(z + \varsigma\nu) \leqslant h\frac{\partial a}{\partial \nu}(z) + Ch|h|^\alpha = h\frac{\partial a}{\partial \nu}(z) + C|h|^{\alpha+1}\,,$$

while for $h < 0$ we have

$$h\frac{\partial a}{\partial \nu}(z + \varsigma\nu) \leqslant h\frac{\partial a}{\partial \nu}(z) - Ch|h|^\alpha = h\frac{\partial a}{\partial \nu}(z) + C|h|^{\alpha+1}\,.$$

By (5.38) and the last two displays, it means that

$$a(z + h\nu) \leqslant a(z) + h\frac{\partial a}{\partial \nu}(z) + C|h|^{1+\alpha}\,.$$

As $a \geqslant 0$, we have

$$0 \leqslant a(z) + h\frac{\partial a}{\partial \nu}(z) + C|h|^{1+\alpha}\,, \tag{5.39}$$

as long as $h \in \mathbb{R}$ and $z, z + h\nu \in B(x_0, R)$. For any $z \in B(x_0, R)$, let us now denote

$$h_z := -c\left|\frac{\partial a}{\partial \nu}(z)\right|^{1/\alpha} \operatorname{sgn}\left(\frac{\partial a}{\partial \nu}(z)\right) \quad \text{with} \quad c = (2C)^{-1/\alpha}.$$

Note that as $a(x_0) = 0$, we also have $Da(x_0) = 0$, as $x_0 \in \Omega$ is a minimum of a. Since $a \in C^{1,\alpha}(\overline{\Omega})$, for any $z \in B(x_0, R)$ we have $\left|\frac{\partial a}{\partial \nu}(z)\right| \leqslant C|z - x_0|^\alpha$, which gives us

$$|z + h_z\nu - x_0| \leqslant |z - x_0| + |h_z| = |z - x_0| + c\left|\frac{\partial a}{\partial \nu}(z)\right|^{1/\alpha} \leqslant (1 + cC^{1/\alpha})|z - x_0|.$$

Therefore, if we take $r := \frac{R}{1+cC^{1/\alpha}}$, for any $z \in B(x_0, r)$ we have $z + h_z\nu \in B(x_0, R)$. Hence, by (5.39) we obtain

$$0 \leqslant a(z) + h_z\frac{\partial a}{\partial \nu}(z) + C|h_z|^{1+\alpha} = a(z) - \tfrac{1}{2}c\left|\frac{\partial a}{\partial \nu}(z)\right|^{1+1/\alpha},$$

which means that for some constant $C_a > 0$ it holds that

$$\left|\frac{\partial a}{\partial \nu}(z)\right| \leqslant C_a a(z)^{\frac{\alpha}{1+\alpha}}. \tag{5.40}$$

Note that by ambiguity of ν, estimate (5.40) holds for arbitrary $\nu \in \mathbb{R}^n$ such that $|\nu| = 1$. Let us take any $x, y \in B(x_0, r)$. Note that we can always find $\tilde{y} \in [y, x]$ such that $a(\tilde{y}) \leqslant a(y)$ and $a > 0$ on the segment (y, x). Indeed, if $a > 0$ on (y, x), then we can take $\tilde{y} = y$. In other case, we may define

$$\tilde{t} := \sup\{t \in [0, 1] : a(y + t(x - y)) = 0\}$$

and set $\tilde{y} := y + \tilde{t}(x - y)$. We see by the definition that $a(\tilde{y}) = 0 \leqslant a(y)$ and a is positive on (y, x). Therefore, if we set $\nu = \frac{x-\tilde{y}}{|x-\tilde{y}|}$, the function $t \mapsto a(\tilde{y} + t\nu)^{\frac{1}{1+\alpha}}$ is differentiable for $t \in (0, |x - \tilde{y}|)$, with derivative equal to $\frac{1}{1+\alpha}\left(\frac{\partial a}{\partial \nu}(\tilde{y} + t\nu)\right)(a(\tilde{y} + t\nu))^{-\frac{\alpha}{1+\alpha}}$. By the definition of $\tilde{y}$ and (5.40), we have

$$\begin{aligned} a(x)^{\frac{1}{1+\alpha}} - a(y)^{\frac{1}{1+\alpha}} &\leqslant a(x)^{\frac{1}{1+\alpha}} - a(\tilde{y})^{\frac{1}{1+\alpha}} \\ &= \int_0^{|x-\tilde{y}|}\left(\frac{\partial a}{\partial \nu}(\tilde{y} + t\nu)\right)(a(\tilde{y} + t\nu))^{-\frac{\alpha}{1+\alpha}}\,dt \\ &\leqslant C_a|x - \tilde{y}| \leqslant C_a|x - y|, \end{aligned}$$

which by symmetry means that $a^{\frac{1}{1+\alpha}}$ is Lipschitz on $B(x_0, r)$. By Remark 5.0.2, we have that $a \in \mathcal{Z}^{1+\alpha}(B(x_0, r))$, which contradicts (5.37), as $\{x_k\}_k$ and $\{y_k\}_k$ converge to x_0. Hence, $a \in \mathcal{Z}^{1+\alpha}(\Omega)$.

For *(ii)* it is enough to consider $x_0 \in \Omega \subset \mathbb{R}^n$ and $a(x) = |x - x_0|^2$, which is smooth, but only in $\mathcal{Z}^2$. $\square$

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