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On the design and the digital implementation of observer-based controllers for tracking of nonlinear time-delay systems

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Abstract

In this article, the tracking control problem for a class of nonlinear time-delay systems is investigated. In particular, a new methodology for the design and the digital implementation of observer-based tracking controllers is provided for a class of control-affine nonlinear systems with state delays. First, a procedure for the design of continuous-time observer-based tracking controllers ensuring the global asymptotic stability of the corresponding closed-loop tracking error system is provided for the considered class of systems. Then, sufficient conditions are provided for the existence of a suitably fast sampling and of an accurate quantization of the input/output channels such that the digital implementation of the proposed continuous-time observer-based tracking controller ensures the semi-global practical stability property of the related sampled-data quantized closed-loop tracking error system, with arbitrarily small final target ball of the origin. Moreover, it is shown that, in the special case of delay-free nonlinear systems, the sufficient conditions provided for the digital implementation of the proposed continuous-time observer-based tracking controller can be strongly relaxed. In the theory here developed, time-varying sampling periods and nonuniform quantization of the input/output channels are taken into account. The proposed results are validated through examples concerning a class of neural networks systems and a class of time-delay systems including, as a special case, a delay-free actuated inverted pendulum.

KEYWORDS

input-to-state stability, Lyapunov–Krasovskii functionals, nonlinear time-delay systems, observer-based tracking control, quantized sampled-data controllers, stabilization in the sample-and-hold sense

1 | INTRODUCTION

In the last years, many efforts have been devoted to the development of methodologies for the design of observers and static state/dynamic output feedback stabilizers for nonlinear time-delay systems.[1–6] As far as tracking control systems are concerned, it is well-known that, in the literature, tracking control problems are commonly addressed by considering the equivalent problem concerning the stabilization of the origin for a nonlinear time-varying system.[7,8] Methodologies for the design of observer-based tracking controllers are very few in the literature of systems with state delays. Recently, a procedure for the design of dynamic output feedback tracking controllers has been provided for a class of continuous-time periodic linear systems affected by time-varying state delay, uncertainties and external disturbances.[9] On the other hand, in the case of nonlinear systems with state delays, to our best knowledge, methodologies for the design of continuous-time observer-based tracking controllers have never been provided in the literature.

The first contribution of this article is to fill this gap by providing sufficient conditions for the existence of continuous-time observer-based tracking controllers for an important class of nonlinear systems in control affine form and affected by state delays. In particular, the methodology proposed by Germani et al. for the design of continuous-time observer-based stabilizers[1] is here extended to the case of tracking control. A geometric approach and the notion of input-to-state stability (ISS) are here suitably revised in order to deal with the tracking control problem and used as tools to provide asymptotic stability results for the related closed-loop tracking error system.

Nowadays, it is well-known that, in engineering applications, the use of digital devices for the practical implementation of proposed control strategies is more and more growing leading to an increasing attention on the study of digital control systems.[10–18] In this context, an important aspect to take into account is the unavoidable presence of sampling and quantization in the devices implementing the proposed control scheme. As far as tracking control problems are concerned, few results are available for nonlinear systems with state delays concerning the digital implementation of tracking controllers. Recently, a methodology for the design of sampled-data dynamic output feedback tracking controllers has been proposed for a class of control affine nonlinear systems with state delays and practical stability results are provided for the related closed-loop system.[19] On the other hand, the presence of quantization in the input/output channels as well as the digital implementation of the dynamical part of the proposed controller have not been addressed and theoretical results are provided by considering that the designed observer evolves on a continuous-time basis.[19] To our best knowledge, results concerning quantized sampled-data observer-based tracking controllers, fully described by discrete-time equations, are not available in the literature of nonlinear systems with state delays.

As a second contribution of this article, we provide sufficient Lyapunov–Krasovskii like conditions for the existence of a suitably fast sampling and of an accurate quantization of the input/output channels such that the digital implementation of the proposed continuous-time observer-based tracking controller ensures the semi-global practical stability property of the related quantized sampled-data closed-loop tracking error system with arbitrarily small final target ball of the origin. The stabilization in the sample-and-hold sense theory[20–23] and the notion of dynamic output steepest descent feedbacks (DOSDFs)[22,24] are used as a tool to prove the results. Time-varying sampling periods as well as the nonuniform quantization of the input/output channels are taken into account. Furthermore, the stable inter-sampling system behavior is proved. In the theory here developed, the special case of delay-free system is included. In particular, in such case, by exploiting the converse Lyapunov theorems,[25] it is shown that the sufficient conditions provided for the digital implementation of the proposed continuous-time observer-based tracking controller can be strongly relaxed. We highlight here that, to our best knowledge, it is the first time in the literature of nonlinear systems with state delays that a methodology for the design of quantized sampled-data observer-based tracking controllers, fully described by discrete-time equations, is provided. The proposed methodology is validated through applications concerning a class of neural networks systems[26] and a class of time-delay systems including, as a special case, a delay-free actuated inverted pendulum.[27]

Notation $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^\star$ denotes the extended real line $[-\infty,+\infty]$, $\mathbb{R}^+$ denotes the set of nonnegative reals $[0,+\infty)$. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. For a given positive integer n and for a symmetric, positive definite matrix $P \in \mathbb{R}^{n\times n}$, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and the minimum eigenvalue of P, respectively. For a given positive integer n and a given positive real H, the symbol $\mathcal{B}^n_H$ denotes the subset $\{x \in \mathbb{R}^n : |x| \le H\}$. The essential supremum norm of an essentially bounded function is indicated with the symbol $\|\cdot\|_\infty$. For a positive integer n, for a positive real Δ (maximum involved time-delay): $\mathcal{C}^n$ and $W^{1,\infty}_n$ denote the space of the continuous functions mapping $[-\Delta,0]$ into $\mathbb{R}^n$ and the space of the absolutely continuous functions, with essentially bounded derivative, mapping $[-\Delta,0]$ into $\mathbb{R}^n$, respectively. Notice that, when $\Delta = 0$, the spaces $\mathcal{C}^n$ and $\mathbb{R}^n$ are isomorphic and, for any $\phi \in \mathcal{C}^n$, $\|\phi\|_\infty = |\phi(0)|$. For a positive real H, for $\phi \in \mathcal{C}^n$, $\mathcal{C}^n_H(\phi) = \{\psi \in \mathcal{C}^n : \|\psi - \phi\|_\infty \le H\}$. The symbol $\mathcal{C}^n_H$ denotes $\mathcal{C}^n_H(0)$. For a continuous function $x : [-\Delta,c) \to \mathbb{R}^n$, with $0 < c \le +\infty$, for any real $t \in [0,c)$, x_t is the function in $\mathcal{C}^n$ defined as $x_t(\tau) = x(t+\tau),\ \tau \in [-\Delta,0]$. For a positive integer n, for $\mathbb{S} = \mathbb{R}^n$ (or $\mathbb{R}^+$): $C^1(\mathbb{S};\mathbb{R}^+)$ denotes the space of the continuous functions from $\mathbb{S}$ to $\mathbb{R}^+$, admitting continuous (partial) derivatives; $C^1_L(\mathbb{S};\mathbb{R}^+)$ denotes the subset of the functions in $C^1(\mathbb{S};\mathbb{R}^+)$ admitting locally Lipschitz (partial) derivatives. A continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{P}_0$ if $\gamma(0) = 0$; of class $\mathcal{N}$ if it is of class $\mathcal{P}_0$ and increasing (not necessarily strictly increasing); of class $\mathcal{P}$ if it is of class $\mathcal{P}_0$ and $\gamma(s) > 0$, $s > 0$; of class $\mathcal{K}$ if it is of class $\mathcal{P}$ and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded; of class $\mathcal{L}$ if it monotonically decreases to zero as its argument tends to $+\infty$. A continuous function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{KL}$ if, for each fixed $t \geq 0$, the function $s \to \beta(s,t)$ is of class $\mathcal{K}$ and, for each fixed $s \geq 0$, the function $t \to \beta(s,t)$ is of class $\mathcal{L}$. For positive integers n, l, m, for a function $f : C^n \times C^l \times C^m \to \mathbb{R}^n$, and for a locally Lipschitz functional $V : C^n \to \mathbb{R}^+$, the derivative (upper right-hand Dini directional derivative in the case $\Delta = 0$, and derivative in Driver's form in the case $\Delta > 0$[28,29]) $D^+V : C^n \times C^l \times C^m \to \mathbb{R}^\star$, of the functional V, is defined, for $\phi \in C^n$, $\phi_r \in C^l$, $\phi_d \in C^m$, as

$$D^+V(\phi, \phi_r, \phi_d) = \limsup_{h \to 0^+} \frac{V(\phi_{h,\phi_r,\phi_d}) - V(\phi)}{h}, \tag{1}$$

where, in the case $\Delta > 0$, for $0 \leq h < \Delta$, $\phi_{h,\phi_r,\phi_d} \in C^n$ is defined, for $s \in [-\Delta, 0]$, as

$$\phi_{h,\phi_r,\phi_d}(s) = \begin{cases} \phi(s+h), & s \in [-\Delta, -h) \\ \phi(0) + (s+h)f(\phi, \phi_r, \phi_d), & s \in [-h, 0], \end{cases} \tag{2}$$

and, for $\Delta = 0$ and $h \in [0,1)$, as

$$\phi_{h,\phi_r,\phi_d}(0) = \phi(0) + hf(\phi, \phi_r, \phi_d).$$

For given triple (f, g, h) of smooth functions, where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$, the symbols $L_f^i h(x)$, $L_g L_f^i h(x)$, $x \in \mathbb{R}^n$, $i = 0, 1, \ldots$, denote the repeated Lie derivatives, defined as[30]

$$\begin{aligned} &L_f^0 h(x) = h(x), \quad L_f^i h(x) = \frac{\partial L_f^{i-1} h(x)}{\partial x} f(x), \quad i = 1, 2, \ldots, \\ &L_g L_f^j h(x) = \frac{\partial L_f^j h(x)}{\partial x} g(x), \quad j = 0, 1, \ldots . \end{aligned} \tag{3}$$

A triple (f, g, h) has full relative degree in an open set $\mathbb{S} \subseteq \mathbb{R}^n$ if, for any $x \in \mathbb{S}$,

$$L_g L_f^i h(x) = 0, \quad i = 0, 1, \ldots, n-2, \quad L_g L_f^{n-1} h(x) \neq 0. \tag{4}$$

If $\mathbb{S} = \mathbb{R}^n$, then the triple is said to have full uniform relative degree. For positive integers n, m, the symbols $0_{n\times m}$ and I_n denote the zero matrix in $\mathbb{R}^{n\times m}$ and the identity matrix in $\mathbb{R}^{n\times n}$, respectively. For a positive integer n, $A_b \in \mathbb{R}^{n\times n}$, $B_b \in \mathbb{R}^n$ and $C_b^T \in \mathbb{R}^n$ denote the Brunovskii triple matrices, that is,

$$A_b = \begin{bmatrix} 0_{n-1\times 1} & I_{n-1} \\ 0 & 0_{1\times n-1} \end{bmatrix}, \quad B_b = \begin{bmatrix} 0_{n-1\times 1} \\ 1 \end{bmatrix}, \quad C_b = \begin{bmatrix} 1 & 0_{1\times n-1} \end{bmatrix}. \tag{5}$$

2 | PRELIMINARIES

In the following, some useful notions and results very helpful for the presentation of the proposed control methodology are introduced. In particular, the notion of ISS and some useful results related to such a notion[28,29,31,32] are properly revised in order to deal with tracking control problems.

Let us consider a nonlinear time-delay system described by the following RFDE

$$\begin{aligned} \dot{x}(t) &= f(x_t, r_t, d_t), & t \geq 0 \text{ a.e.} \\ x(\tau) &= x_0(\tau), & \tau \in [-\Delta, 0], \end{aligned} \tag{6}$$

where: $x(t) \in \mathbb{R}^n$; $x_0, x_t \in C^n$; $r_t \in C^l$ is a continuously differentiable reference signal satisfying $\|r_t\|_\infty \leq \gamma_r$, $\forall t \geq 0$, $\gamma_r \geq 0$ and admitting bounded continuous derivatives up to the order n; $\Delta \geq 0$ is the maximum involved time delay, assumed to be known; $d_t \in C^m$ is the input signal; $f : C^n \times C^l \times C^m \to \mathbb{R}^n$ is a function Lipschitz on bounded sets of $C^n \times C^l \times C^m$. It is assumed that $f(0, r_t, 0) = 0, \forall t \geq 0$. Furthermore, in the case $\Delta > 0$, it is assumed that the initial state $x_0 \in W_n^{1,\infty}$. In the following, the notion of ISS,[28,29,31,32] suitably modified in order to take into account the reference signal involved in the system dynamics (6), is recalled.

Definition 1. The system described by (6) is said to be ISS if there exist a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that, for any initial state $x_0 \in C^n$, for any measurable, locally essentially bounded input $d_t \in C^m$, $t \geq 0$, and for any continuously differentiable reference signal r_t satisfying $\|r_t\|_\infty \leq \gamma_r$, $\forall t \geq 0$, $\gamma_r \geq 0$, and admitting bounded continuous derivatives up to the order n, the solution exists for all $t \geq 0$ and, furthermore, it satisfies

$$|x(t)| \leq \beta(\|x_0\|_\infty, t) + \gamma(\underset{\tau\in[0,t)}{esssup}\|d_\tau\|_\infty). \tag{7}$$

Theorem 1. *If there exist a Lipschitz on bounded sets functional $V : C^n \to \mathbb{R}^+$, functions α_1, α_2 of class $\mathcal{K}_\infty$ and functions α_3, ρ of class $\mathcal{K}$ such that*:

$$\begin{aligned} &\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_a), \quad \forall \phi \in C^n, \\ &D^+V(\phi, \phi_r, \phi_d) \leq -\alpha_3(\|\phi\|_a), \quad \forall \phi \in C^n, \ \phi_r \in C^l_{\gamma_r}, \quad \phi_d \in C^m : \|\phi\|_a \geq \rho(\|\phi_d\|_\infty), \end{aligned} \tag{8}$$

where γ_r is a positive real denoting the bound of the reference signal (see (6)) and the symbol $\|\phi\|_a$ denotes any semi-norm in C^n such that, for some positive reals $\gamma_a, \overline{\gamma}_a$, the following inequalities hold,

$$\gamma_a|\phi(0)| \leq \|\phi\|_a \leq \overline{\gamma}_a\|\phi\|_\infty, \quad \forall \phi \in C^n, \tag{9}$$

then, system (6) is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Proof of Theorem 1. The same reasoning used in the work by Pepe et al.[28] for proving Theorem 3.1 can be here repeated in order to prove Theorem 1. On the other hand, suitable reformulations and considerations are here required in order to deal with continuously differentiable reference signal r_t and infinite dimensional inputs d_t introduced to address tracking control problems and which, in the work by Pepe et al.,[28] are not considered. Let the input $d_t \in C^m$ be such that $\underset{t\geq 0}{esssup}\|d_t\|_\infty = \nu$, for a suitable $\nu \in \mathbb{R}^+$. Let $c = \alpha_2(\rho(\nu))$ and introduce the set $S = \{\psi \in C^n : V(\psi) \leq c\}$. ■

Claim 1. If the solution $x(t)$ is such that, for a certain time $t_0 \geq 0$, $x_{t_0} \in S$, then $x_t \in S$ for $t \geq t_0$.

Proof of Claim 1. First, taking into account the Lipschitz on bounded sets property of the functional V, we notice that for any (component-wise) locally absolutely continuous solution $x(t)$ of system (6) over a maximal interval $[0, b)$, $0 < b \leq +\infty$ the following facts hold for the function $w : [0, b) \to \mathbb{R}^+$, given by $w(t) = V(x_t)$ (we recall that $x_0 \in W_n^{1,\infty}$):

1. the function w is locally absolutely continuous in $[0, b)$[29];
2. the upper right-hand derivative of the function w,

$$D^+w(t) = \limsup_{h\to 0^+} \frac{w(t+h) - w(t)}{h},$$

is such that, for almost all $t \in [0, b)$,[29]

$$D^+w(t) = D^+V(x_t, r_t, d_t). \tag{10}$$

It follows that the locally absolutely continuous function $w(t)$ is non-increasing when $D^+V(x_t, r_t, d_t)$ is non-positive almost everywhere. Thanks to such a property, the same reasoning used in the work by Sontag[32] for the case of delay-free systems can be applied here to prove that Claim 1 holds. For details, the reader can refer to the proof of the claim reported after (37) in the work by Sontag.[32] ■

Taking into account the first two inequalities in (8), it follows that when $x_t \in S$, $|x(t)| \leq \gamma(v)$ with

$$\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho. \tag{11}$$

Claim 2. There exists a $\mathcal{KL}$ function β such that, for each initial state x_0, with $V(x_0) > c$, and each bounded input d_t, there exists a time instant $T > 0$ (possibly $T = +\infty$) such that

$$\begin{aligned} &|x(t)| \leq \beta(\|x_0\|_\infty, t), \quad \forall t < T, \\ &x_t \in S, \quad \forall t \geq T, \end{aligned} \tag{12}$$

with $\beta(s,t) = \alpha_1^{-1}(\overline{\beta}(\alpha_2(\overline{\gamma}_a s), t))$.

Proof of Claim 2. Taking into account (9) and (10), for almost all t in some interval $[0, T)$ (where $V(x_t) > c$), by (8), the following inequalities hold, for $w(t) = V(x_t)$,

$$D^+ w(t) = D^+ V(x_t, r_t, d_t) \leq -\alpha_3(\|x_t\|_a) \leq -\alpha_3(\alpha_2^{-1}(V(x_t))) = -\alpha_3(\alpha_2^{-1}(w(t))), \quad \text{a.e. in } [0, T). \tag{13}$$

Then, from Lemma 4.4 in the work by Lin et al.,[25] there exists a $\mathcal{KL}$ function $\overline{\beta}$ such that for all t in the interval $[0, T)$ the following inequality holds

$$w(t) \leq \overline{\beta}(w(0), t). \tag{14}$$

From (14) and taking into account (8), it follows that for all t in the interval $[0, T)$ the following inequalities hold

$$V(x_t) \leq \overline{\beta}(V(x_0), t) \leq \overline{\beta}(\alpha_2(\|x_0\|_a), t) \leq \overline{\beta}(\alpha_2(\overline{\gamma}_a \|x_0\|_\infty), t). \tag{15}$$

Then, taking into account (8) and (15), we obtain

$$|x(t)| \leq \alpha_1^{-1}(\overline{\beta}(\alpha_2(\overline{\gamma}_a \|x_0\|_\infty), t)) = \beta(\|x_0\|_\infty, t). \tag{16}$$

The inequality (13) guarantees that the locally absolutely continuous function $w(t) = V(x_t)$ is non-increasing in $[0, T)$ and this, together with Claim 1, guarantees that the solution $x(t)$ is defined for all $t \geq 0$. Finally, (11) and (16) yield the result. ∎

In the following, a crucial result which will be very helpful in the article from a technical point of view is provided. In particular, Lemma 1 in the work by Germani et al.[1] is extended to the case of nonlinear time-delay systems affected by exogenous bounded signals.

Lemma 1. *Let us consider the nonlinear time-delay system described by*

$$\begin{aligned} &\dot{z}(t) = f_z(z_t, r_t, e_t), && t \geq 0 \text{ a.e.}, \\ &\dot{e}(t) = f_e(e_t, r_t, z_t), && t \geq 0 \text{ a.e.}, \\ &z(\tau) = z_0(\tau), \quad e(\tau) = e_0(\tau), && \tau \in [-\Delta, 0], \end{aligned} \tag{17}$$

where: z_t, $e_t \in C^n$ *are the state variables;* $r_t \in C^l$ *is a continuously differentiable reference signal satisfying* $\|r_t\|_\infty \leq \gamma_r$, $\forall t \geq 0$, $\gamma_r \geq 0$ *and admitting bounded continuous derivatives up to the order* n; $f_z : C^n \times C^l \times C^n \to \mathbb{R}^n$ *and* $f_e : C^n \times C^l \times C^n \to \mathbb{R}^n$ *are smooth functions such that*

1. *the following conditions hold for any* $\phi_z \in C^n$ *and* $\phi_r \in C^l_{\gamma_r}$

$$f_z(0, \phi_r, 0) = 0, \quad f_e(0, \phi_r, 0) = 0, \quad f_z(\phi_z, \phi_r, 0) = A_c \phi_z(0), \tag{18}$$

with A_c *a Hurwitz matrix;*

2. *the following condition holds for any* $\phi_z,\ \phi_e \in C^n$ *and* $\phi_r \in C^l_{\gamma_r}$

$$|f_z(\phi_z, \phi_r, \phi_e) - f_z(\phi_z, \phi_r, 0)| \leq \gamma_{f_z} \|\phi_e\|_\infty, \tag{19}$$

with some $\gamma_{f_z} \geq 0$.

Assume that system (17) is such that there exists a function β *of class* $\mathcal{KL}$ *such that*

$$e(t) \leq \beta(\|e_0\|_\infty, t), \quad t \geq 0. \tag{20}$$

Then, system (17) is globally asymptotically stable.

Proof of Lemma 1. Taking into account (18) and the reasoning exploited in the work by Yeganefar et al.[31] to prove Theorem 3.2, from the converse Lyapunov–Krasovskii theorem,[31] it follows that for the unperturbed system,

$$\dot{z}(t) = f_z(z_t, r_t, 0), \quad t \geq 0 \text{ a.e.}, \tag{21}$$

there exist a functional $V : C^n \to \mathbb{R}^+$, positive reals C_i, $i = 1, 2, 3, 4$, such that the following conditions hold (with respect to the system (21))

$$\begin{aligned}
&C_1\|\phi_z\|_\infty \leq V(\phi_z) \leq C_2\|\phi_z\|_\infty, \quad \forall \phi_z \in C^n, \\
&D^+V(\phi_z, \phi_r, 0) \leq -C_3\|\phi_z\|_\infty, \quad \forall \phi_z \in C^n, \quad \forall \phi_r \in C^l_{\gamma_r}, \\
&|V(\phi_{z_1}) - V(\phi_{z_2})| \leq C_4\|\phi_{z_1} - \phi_{z_2}\|_\infty, \quad \forall \phi_{z_1},\ \phi_{z_2} \in C^n.
\end{aligned} \tag{22}$$

Notice that, from a technical point of view, the converse Lyapunov–Krasovskii theorem has been here applied to a delay-free system dealt with as a delayed one (see (18) and (21)). By computing the upper right-hand Dini derivative of the functional V (see (1)) with respect to the perturbed system

$$\dot{z}(t) = f_z(z_t, r_t, e_t), \quad t \geq 0 \text{ a.e.}, \tag{23}$$

we obtain, for $\phi_z,\ \phi_e \in C^n$ and $\phi_r \in C^l_{\gamma_r}$

$$\begin{aligned}
D^+V(\phi_z, \phi_r, \phi_e) &= \limsup_{h\to 0^+} \frac{V(\phi_{h,\phi_r,\phi_e}) - V(\phi_z)}{h} = \limsup_{h\to 0^+} \frac{V(\phi_{h,\phi_r,\phi_e}) - V(\phi_{h,\phi_r,0}) + V(\phi_{h,\phi_r,0}) - V(\phi_z)}{h} \\
&\leq D^+V(\phi_z, \phi_r, 0) + \limsup_{h\to 0^+} \frac{V(\phi_{h,\phi_r,\phi_e}) - V(\phi_{h,\phi_r,0})}{h} \leq -C_3\|\phi_z\|_\infty + \limsup_{h\to 0^+} \frac{V(\phi_{h,\phi_r,\phi_e}) - V(\phi_{h,\phi_r,0})}{h}.
\end{aligned} \tag{24}$$

Taking into account (19), the following inequalities hold

$$\begin{aligned}
|V(\phi_{h,\phi_r,\phi_e}) - V(\phi_{h,\phi_r,0})| &\leq C_4\|\phi_{h,\phi_r,\phi_e} - \phi_{h,\phi_r,0}\| = C_4 \sup_{s\in[-\Delta,0]} |\phi_{h,\phi_r,\phi_e}(s) - \phi_{h,\phi_r,0}(s)| \\
&= C_4 \sup_{s\in[-h,0]} |s+h||f_z(\phi_z, \phi_r, \phi_e) - f_z(\phi_z, \phi_r, 0)| \leq C_4|h|\gamma_{f_z}\|\phi_e\|_\infty.
\end{aligned} \tag{25}$$

Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be the function of class $\mathcal{K}_\infty$ defined as $\omega(s) = \theta s$, where $0 < \theta < \dfrac{C_3}{C_4\gamma_{f_z}}$. Then, if $\|\phi_z\|_\infty \geq \omega^{-1}(\|\phi_e\|_\infty)$, the following inequalities hold:

$$D^+V(\phi_z, \phi_r, \phi_e) \leq -C_3\|\phi_z\|_\infty + C_4\gamma_{f_z}\theta\|\phi_z\|_\infty \leq -\delta\|\phi_z\|_\infty, \tag{26}$$

where $\delta = C_3 - C_4\gamma_{f_z}\theta > 0$. Let us choose $\theta = \dfrac{C_3}{2C_4\gamma_{f_z}}$ so that $\delta = C_3/2$. Hence, the conditions in Theorem 1 (see (8)) are satisfied with functions $\alpha_1(s) = C_1 s$, $\alpha_2(s) = C_2 s$, $\alpha_3(s) = \delta s$ and $\rho(s) = \omega^{-1}(s)$. Thus, it follows that the perturbed system

$$\dot{z}(t) = f_z(z_t, r_t, e_t), \quad t \geq 0 \text{ a.e.},$$
$$z(\tau) = z_0(\tau), \quad \tau \in [-\Delta, 0], \tag{27}$$

is ISS (see Definition 1). From the same reasoning used in the work by Chaillet et al.[29] for the case of nonlinear time-delay systems without exogenous disturbances, taking into account (20), the following implication holds

$$\lim_{t\to+\infty} \|e_t\|_\infty = 0 \Rightarrow \lim_{t\to+\infty} \|z_t\|_\infty = 0. \tag{28}$$

Then, from (20), (28) and the ISS property of the perturbed system (27), it follows that system (17) is globally asymptotically stable. ∎

3 PROBLEM STATEMENT

Let us consider a nonlinear time-delay system described by

$$\dot{x}(t) = f(x(t)) + g(x(t))(p_1(x_t)u(t) + p_2(x_t)), \quad t \geq 0 \text{ a.e.}$$
$$y_t(\tau) = h(x_t(\tau)),$$
$$x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0], \tag{29}$$

where: $x(t) \in \mathbb{R}^n$; $x_0, x_t \in C^n$; $\Delta \geq 0$ is the maximum involved time delay; $u(t) \in \mathbb{R}$ is the input signal, $y_t \in C$ is the output signal; $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are smooth functions admitting continuous partial derivatives of any order; $p_i : C^n \to \mathbb{R}$, $i = 1, 2$, are continuously Frechet differentiable functionals. Furthermore, in the case $\Delta > 0$, it is assumed that the initial state $x_0 \in W_n^{1,\infty}$.[23] Let us introduce the following assumption for the system (29).[1]

Assumption 1.

(H_1) The triple (f, g, h) has full uniform relative degree.

(H_2) The function $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, defined, for $x \in \mathbb{R}^n$, by

$$\Phi(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}, \tag{30}$$

is a diffeomorphism in $\mathbb{R}^n$, and there exist positive reals γ_Φ and $\gamma_{\Phi^{-1}}$ such that, for any $z_1, z_2 \in \mathbb{R}^n$, the following inequalities hold

$$|\Phi(z_1) - \Phi(z_2)| \leq \gamma_\Phi |z_1 - z_2|, \qquad |\Phi^{-1}(z_1) - \Phi^{-1}(z_2)| \leq \gamma_{\Phi^{-1}} |z_1 - z_2|. \tag{31}$$

(H_3) There exist positive reals $\gamma_{L_f}, \gamma_{L_g}$ such that

$$|L_f^n h(\Phi^{-1}(z_1)) - L_f^n h(\Phi^{-1}(z_2))| \leq \gamma_{L_f} |z_1 - z_2|, \quad \forall z_1, \ z_2 \in \mathbb{R}^n,$$
$$|L_g L_f^{n-1} h(\Phi^{-1}(\phi_1(0))) p_2(\Psi^{-1}(\phi_1)) - L_g L_f^{n-1} h(\Phi^{-1}(\phi_2(0))) p_2(\Psi^{-1}(\phi_2))| \leq \gamma_{L_g} \|\phi_1 - \phi_2\|_\infty,$$
$$\forall \phi_1, \ \phi_2 \in C^n, \tag{32}$$

where $\Psi : C^n \to C^n$ is the function defined, for any $\phi \in C^n$, as $\Psi(\phi)(\tau) = \Phi(\phi(\tau))$, $\tau \in [-\Delta, 0]$.

(H_4) There exists a function $\overline{G} : \mathcal{C} \to \mathbb{R}$, such that, for any $\phi \in \mathcal{C}^n$

$$L_g L_f^{n-1} h(\phi(0)) p_1(\phi) = \overline{G}(H(\phi)), \tag{33}$$

where $H : \mathcal{C}^n \to \mathcal{C}$ is the function defined as $H(\phi)(\tau) = h(\phi(\tau))$, $\tau \in [-\Delta, 0]$.

(H_5) There exists a real $\overline{p} > 0$ such that $|p_1(\phi)| \geq \overline{p}$, $\forall \phi \in \mathcal{C}^n$.

In the following, the problems addressed in this article are introduced.

Problem 1. Under Assumption 1, for a given bounded continuously differentiable reference signal $y_{d,t} \in \mathcal{C}$, $t \in \mathbb{R}^+$, admitting bounded continuous derivatives up to the order n, the problems addressed in this article are the following ones:

(i) design an observer-based dynamic output feedback tracking controller for system (29) such that, for any $x_0 \in \mathcal{C}^n$, the solution $x(t)$ of the corresponding continuous-time closed-loop system is bounded and the following condition holds

$$\lim_{t\to\infty} |y_t(0) - y_{d,t}(0)| = 0; \tag{34}$$

(ii) provide sufficient conditions for the existence of a suitably fast sampling and of an accurate quantization of the input/output channels such that the digital implementation of the dynamic output feedback tracking controller designed in point (i) ensures the condition (34) in a semi-global practical sense with arbitrarily small steady state tracking error.

Let us introduce the function $Q : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ defined for any $x \in \mathbb{R}^n$ as follows

$$Q(x) = \frac{\partial \Phi(x)}{\partial x}. \tag{35}$$

Notice that H_2 in Assumption 1 implies that $Q(x)$ is nonsingular in all $\mathbb{R}^n$.

4 | DESIGN OF THE CONTINUOUS-TIME OBSERVER-BASED TRACKING CONTROLLER

In order to address Problem 1, let us introduce the following extended tracking output error variable

$$z(\theta) = \Phi(x(\theta)) - r(\theta), \quad r(\theta) = \begin{bmatrix} y_d(\theta) \\ \dfrac{dy_d(\theta)}{d\theta} \\ \dfrac{d^2 y_d(\theta)}{d\theta^2} \\ \vdots \\ \dfrac{d^{n-1} y_d(\theta)}{d\theta^{n-1}} \end{bmatrix} = \begin{bmatrix} y_d(\theta) \\ y_d^{(1)}(\theta) \\ y_d^{(2)}(\theta) \\ \vdots \\ y_d^{(n-1)}(\theta) \end{bmatrix}, \quad \theta \in [-\Delta, \infty). \tag{36}$$

From (29), taking into account Assumption 1 and (36), we obtain the following tracking error system

$$\begin{aligned} \dot{z}(t) &= A_b z(t) + B_b(F(z_t, r_t) + G(H_z(z_t), r_t)u(t) - y_d^{(n)}(t)), \quad t \geq 0 \text{ a.e.} \\ y_{z,t}(\tau) &= H_z(z_t)(\tau) = C_b z_t(\tau), \quad \tau \in [-\Delta, 0], \end{aligned} \tag{37}$$

where: the function $H_z : \mathcal{C}^n \to \mathcal{C}$ is defined, for $\phi \in \mathcal{C}^n$ as, $H_z(\phi)(\tau) = C_b\phi(\tau)$, $\tau \in [-\Delta, 0]$; $z_t \in \mathcal{C}^n$; $y_{z,t} \in \mathcal{C}$; $r_t \in \mathcal{C}^n$, $r_t(\tau) = r(t+\tau)$, $\tau \in [-\Delta, 0]$; the function $G : \mathcal{C} \times \mathcal{C}^n \to \mathbb{R}$ is defined as (see (H_4) in Assumption 1)

$$G(H_z(z_t), r_t) = L_g L_f^{n-1} h(\Psi^{-1}(z_t + r_t)(0)) p_1(\Psi^{-1}(z_t + r_t)) = \overline{G}(H(\Psi^{-1}(z_t + r_t))); \tag{38}$$

the function $F : \mathcal{C}^n \times \mathcal{C}^n \to \mathbb{R}$ is defined for any $\phi_z,\ \phi_r \in \mathcal{C}^n$ as

$$F(\phi_z, \phi_r) = L_f^n h(\Phi^{-1}(\phi_z(0) + \phi_r(0))) + L_g L_f^{n-1} h(\Phi^{-1}(\phi_z(0) + \phi_r(0))) p_2(\Psi^{-1}(\phi_z + \phi_r)). \tag{39}$$

We highlight here that, in the following, point (i) in Problem 1 will be addressed by considering the problem of designing a continuous-time global asymptotic observer-based stabilizer for the system described by (37). Indeed, point (i) in Problem 1 follows from the global asymptotic stability (GAS) property of the system (37).

Remark 1. Notice that, system (37) is in a lower triangular form with trivial zero dynamics. Many methodologies have been provided in the literature for the design of continuous-time stabilizers for class of systems in lower triangular form with trivial/non-trivial zero dynamics.[6,33] For instance, in the work by Lin and Zhang,[6] a methodology for the design of continuous-time dynamic output feedback stabilizers is provided for a class of lower triangular systems with discrete-time delays and non-trivial zero dynamics. The methodologies provided in the work by Lin and Zhang[6] and in the work by Zhao and Lin[33] cannot be directly applied here due to the presence: (i) of known exogenous disturbances mimicking the chosen reference signal as common in the tracking control framework; (ii) of possible distributed time-delays in the functions describing the system at hand (see (37)). In particular, we highlight that, to our best knowledge, methodologies for the design of dynamic output feedback stabilizers for the class of systems reported in (37) have never been provided in the literature. Here, for the first time in the literature, a methodology for the design of dynamic output feedback stabilizers and results concerning their quantized sampled-data implementation are provided for the class of systems described by (37) in order to solve the tracking control problem considered in Problem 1. On the other hand, the study of a methodology for the design of tracking controllers taking into account the presence of possible non-trivial zero dynamics in the corresponding tracking error system (see (37)) is beyond the aims of this article and is left for future investigations. Such an interesting topic could be addressed, for instance, by exploiting the techniques provided in the work by Lin and Zhang.[6]

Remark 2. Notice that, in (29), the functions $p_1(\cdot)$ and $p_2(\cdot)$ are affected by state-delays. On the other hand, the triple (f, g, h) is required to be delay-free. Such requirement together with the conditions (H_1) and (H_2) in Assumption 1 ensure the existence of a suitable change of coordinates (see (30) and (36)) transforming the nonlinear system (29) in the new structural form (37) where the top $n-1$ equations are linear delay-free dynamics and the time-delay nonlinearities together with the control input only appear in the last equation. In the literature of nonlinear systems, such a configuration (see (37)) is commonly called matching condition,[34,35] which is both general and practical for control synthesis and, moreover, includes a large class of systems as, for instance, vehicle systems and delay-free actuated inverted pendulum systems.[36] We highlight also that, the conditions (H_2) and (H_3) concern the globally Lipschitz property of the diffeomorphic transformation $\Phi(\cdot)$, of its inverse $\Phi^{-1}(\cdot)$ and of the functions $L_f^n h(\cdot)$, $L_g L_f^{n-1} h(\cdot) p_2(\cdot)$. Even though such conditions could appear demanding, they are satisfied by many classes of practical systems as, for instance, the ones cited above. Moreover, the results in the forthcoming Theorem 3 provide a solution to relax the condition (H_3) allowing the application of the proposed observer-based tracking controller also to particular classes of locally Lipschitz nonlinear time-delay systems as the one studied in Section 6.

Notice that, from (H_3) in Assumption 1, there exists a positive real γ_F such that, for any $\phi_{z_i},\ \phi_{r_i} \in \mathcal{C}^n$, $i = 1, 2$, the following inequality holds

$$|F(\phi_{z_1}, \phi_{r_1}) - F(\phi_{z_2}, \phi_{r_2})| \le \gamma_F(\|\phi_{z_1} - \phi_{z_2}\|_\infty + \|\phi_{r_1} - \phi_{r_2}\|_\infty). \tag{40}$$

In the following, the continuous-time observer-based stabilizer provided in the work by Germani et al.[1] is suitably revised in order to deal with point (i) in Problem 1. In particular, the proposed continuous-time observer-based tracking controller for the system (29), solving point (i) in Problem 1, is here described by

$$\begin{aligned}
\dot{\hat{z}}(t) &= A_b\hat{z}(t) + B_b(F(\hat{z}_t, r_t) + G(H_z(z_t), r_t)u(t) - y_d^{(n)}(t)) - KC_b(\hat{z}(t) - z(t)),\\
u(t) &= \frac{-F(\hat{z}_t, r_t) + y_d^{(n)}(t) + \Gamma\hat{z}(t)}{G(H_z(z_t), r_t)},\\
\hat{z}(\tau) &= \hat{z}_0(\tau), \quad \tau \in [-\Delta, 0],
\end{aligned} \tag{41}$$

where: $\hat{z}(t) = \Phi(\hat{x}(t)) - r(t) \in \mathbb{R}^n$ with $\hat{x}(t) \in \mathbb{R}^n$ denoting the estimation of the system state described by (29) (i.e., the estimation of $x(t)$ in (29)); $\hat{z}_t,\ \hat{z}_0 \in \mathcal{C}^n$; F and G are the functions in (37); $y_d^{(n)}(t)$ is given in (37); $K,\ \Gamma^T \in \mathbb{R}^n$ are suitable tuning parameters.

Remark 3. Notice that, the proposed continuous-time observer-based tracking controller (41) is inspired by the continuous-time observer-based stabilizer provided in the work by Germani et al.[1] In particular, the equation describing the observer dynamics are exactly the same proposed in (18) of the work by Germani et al.[1] here rewritten with respect to the tracking error variable $z(t)$ for simplicity in the notation which will be also used for the presentation of the proposed digital framework in the next section. On the other hand, the control input (19) in the work by Germani et al.[1] has been here suitably revised in order to address point (i) in Problem 1 (see (41)). It is here highlighted also that for the practical implementation of the proposed controller the explicit knowledge of the system state estimation $\hat{x}(t)$ is not required (see (41)).

In the following, the first main result of the article is provided. In particular, it is proved that there exist suitable control tuning parameters K and Γ such that point (i) in Problem 1 is solved with the proposed continuous-time observer-based tracking controller (41).

Theorem 2. *Let Assumption 1 hold. Let $y_{d,t} \in \mathcal{C}$, $t \in \mathbb{R}^+$, be a chosen reference signal as in Problem 1. Then, there exist parameters K and Γ such that point (i) in Problem 1 is solved with the proposed continuous-time observer-based tracking controller (41).*

Proof of Theorem 2. In order to prove Theorem 2, we will show that the continuous-time closed-loop system described by (37)–(41) is GAS. Indeed, the GAS property of the continuous-time closed-loop system (37)–(41) implies point (i) in Problem 1. First, we notice that the continuous-time closed-loop system (37)–(41) is described by

$$\begin{aligned}
\dot{z}(t) &= A_b z(t) + B_b(F(z_t, r_t) - F(\hat{z}_t, r_t) + \Gamma\hat{z}(t)),\\
\dot{\hat{z}}(t) &= (A_b + B_b\Gamma)\hat{z}(t) - KC_b(\hat{z}(t) - z(t)).
\end{aligned} \tag{42}$$

Let us now consider the estimation error variable defined as $e_t = \hat{z}_t - z_t \in \mathcal{C}^n$. Taking into account that $\hat{z}(t) = e(t) + z(t)$, from (42), we obtain

$$\begin{aligned}
\dot{z}(t) &= A_b z(t) + B_b(F(z_t, r_t) - F(z_t + e_t, r_t) + \Gamma(z(t) + e(t))),\\
\dot{e}(t) &= (A_b - KC_b)e(t) + B_b(F(z_t + e_t, r_t) - F(z_t, r_t)).
\end{aligned} \tag{43}$$

Notice that, the GAS property of the continuous-time closed-loop system (42) follows from the GAS property of the corresponding closed-loop system (43). In the following, we will prove the GAS property of system (43) by making use of the results in Lemma 1. Notice that, system (43) is in the form (17) with

$$\begin{aligned}
f_z(\phi_z, \phi_r, \phi_e) &= A_b\phi_z(0) + B_b(F(\phi_z, \phi_r) - F(\phi_z + \phi_e, \phi_r) + \Gamma(\phi_z(0) + \phi_e(0))),\\
f_e(\phi_e, \phi_r, \phi_z) &= (A_b - KC_b)\phi_e(0) + B_b(F(\phi_z + \phi_e, \phi_r) - F(\phi_z, \phi_r)).
\end{aligned} \tag{44}$$

First, we prove the existence of the function β. Let $\lambda = \begin{bmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_n\end{bmatrix}$ be a n-tuple of negative real eigenvalues, with $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. Let K be such that the matrix $A - KC_b$ has the n-tuple of negative real eigenvalues $\lambda_i, i = 1, \ldots, n$. Let $E(t) = V(\lambda)e(t), t \geq -\Delta$, where $V(\lambda)$ is the Vandermonde Matrix (see Lemma 1 in the work by Ciccarella et al.[37]). From system (43), taking into account (40) and that $|V(\lambda)B_b| = \sqrt{n}$, we obtain, following the reasoning exploited in the work by Ciccarella et al.,[37]

$$\begin{aligned}|E(t)| &\leq e^{\lambda_1 t}|E(0)| + \int_0^t e^{\lambda_1(t-\tau)}|V(\lambda)B_b||F(z_\tau + e_\tau, r_\tau) - F(z_\tau, r_\tau)|d\tau \\ &\leq e^{\lambda_1 t}|E(0)| + \int_0^t e^{\lambda_1(t-\tau)}|V(\lambda)B_b|\gamma_F\|e_\tau\|_\infty d\tau \\ &\leq e^{\lambda_1 t}|E(0)| + \int_0^t e^{\lambda_1(t-\tau)}\sqrt{n}\gamma_F|V^{-1}(\lambda)|(\|E_\tau\|_\infty + |E(\tau)|)d\tau, \end{aligned} \tag{45}$$

where $E_\tau : [-\Delta, 0] \to \mathbb{R}^n$ is defined as $E_\tau(\theta) = E(\tau + \theta)$, $\theta \in [-\Delta, 0]$, $\tau \geq 0$. From here on steps (3.12)–(3.20), provided in work by Germani and Pepe for proving Theorem 3.3,[2] can be repeated to prove that there exists a negative real c such that

$$|E(t)| \leq |V(\lambda)|\|e_0\|_\infty e^{ct}, \quad t \geq 0. \tag{46}$$

Then, we have that

$$|e(t)| = |V^{-1}(\lambda)V(\lambda)e(t)| = |V^{-1}(\lambda)E(t)| \leq |V^{-1}(\lambda)||V(\lambda)|\|e_0\|_\infty e^{ct}. \tag{47}$$

Let Γ be such that the matrix $A_b + B_b\Gamma$ is Hurwitz. Then, all the hypotheses in Lemma 1 are here satisfied and, consequently, the system described by (43) is GAS. The proof of the theorem is complete. ∎

In the forthcoming theorem, the continuous-time observer-based tracking controller provided in (41) is suitably adapted to the case in which Assumption 1 is partially satisfied.

Theorem 3. *Let H_1, H_2, H_4, and H_5 of Assumption 1 be satisfied. Let the first inequality in (32) be satisfied for any $z_1, z_2 \in \mathbb{R}^n$ such that $C_b z_1 = C_b z_2$. Let the second inequality in (32) be satisfied for any $\phi_1, \phi_2 \in C^n$ such that $C_b\phi_1(\tau) = C_b\phi_2(\tau)$, $\tau \in [-\Delta, 0]$. Let $y_{d,t} \in C$, $t \in \mathbb{R}^+$, be a chosen reference signal as in Problem 1. Then, there exist parameters K and Γ such that point (i) in Problem 1 is solved with the continuous-time observer-based tracking controller described by*

$$\begin{aligned}\dot{\hat{z}}(t) &= A_b\hat{z}(t) + B_b(F(\tilde{z}_t, r_t) + G(H_z(z_t), r_t)u(t) - y_d^{(n)}(t)) - KC_b(\hat{z}(t) - z(t)), \\ u(t) &= \frac{-F(\tilde{z}_t, r_t) + y_d^{(n)}(t) + \Gamma\hat{z}(t)}{G(H_z(z_t), r_t)}, \\ \hat{z}(\tau) &= \hat{z}_0(\tau), \quad \tau \in [-\Delta, 0], \end{aligned} \tag{48}$$

where: $\hat{z}(t) = \Phi(\hat{x}(t)) - r(t) = \begin{bmatrix}\hat{z}_1(t) \\ \vdots \\ \hat{z}_n(t)\end{bmatrix} \in \mathbb{R}^n$ with $\hat{x}(t) \in \mathbb{R}^n$ denoting the estimation of the system state described by (29) (i.e., the estimation of $x(t)$ in (29)); $\hat{z}_t$, $\hat{z}_0 \in C^n$;

$$\tilde{z}(t) = \begin{bmatrix} z_1(t) \\ \hat{z}_2(t) \\ \vdots \\ \hat{z}_n(t)\end{bmatrix};$$

F and G are the functions in (37); $y_d^{(n)}(t)$ is given in (37).

Proof of Theorem 3. The proof is similar to the one of Theorem 2 and, for this reason, it is here omitted. ∎

5 | DIGITAL IMPLEMENTATION OF THE PROPOSED CONTINUOUS-TIME OBSERVER-BASED TRACKING CONTROLLER

In this section, sufficient conditions for the digital implementation of the continuous-time observer-based controller (41) (see also (48)) are provided. In particular, under suitable conditions, it is proved that there exist a suitably fast sampling and an accurate quantization of the input/output channels such that the digital implementation of the continuous-time observer-based tracking controller (41) (see also (48)) ensures the semi-global practical stability of the related quantized sampled-data closed-loop tracking error system, with arbitrarily small final target ball of the origin. The notion of DOSDF[22,24] and the stabilization in the sample-and-hold sense theory will be used as tools to provide the sufficient conditions for the digital implementation of the continuous-time observer-based tracking controller (41) (see also (48)). First, we notice that system (37) is in the following form

$$\begin{aligned} \dot{z}(t) &= \bar{f}_z(z_t, \tilde{r}_t, u(t)), \quad t \geq 0 \text{ a.e.} \\ y_{z,t}(\tau) &= H_z(z_t)(\tau) = C_b z_t(\tau), \\ z(\tau) &= z_0(\tau), \quad \tau \in [-\Delta, 0], \end{aligned} \tag{49}$$

where: $z_t \in C^n$; $\tilde{r}_t = \begin{bmatrix} r_t \\ y^{(n)}_{d,t} \end{bmatrix} \in C^{n+1}$, $y^{(n)}_{d,t}(\tau) = y^{(n)}_d(t+\tau)$, $\tau \in [-\Delta, 0]$, $t \geq 0$; $u(t) \in \mathbb{R}$ is the input signal in (37) (see also (29)); $y_{z,t} \in C$ is the output signal in (37); H_z is the function in (37); $\bar{f}_z : C^n \times C^{n+1} \times \mathbb{R} \to \mathbb{R}^n$ is the function defined, for any $\phi_z \in C^n$, $\phi_{\tilde{r}} = \begin{bmatrix} \phi_r \\ \phi_{y^{(n)}} \end{bmatrix} \in C^{n+1}$, $\phi_r \in C^n$, $\phi_{y^{(n)}} \in C$ and $u \in \mathbb{R}$, as follows

$$\bar{f}_z(\phi_z, \phi_{\tilde{r}}, u) = A_b \phi_z(0) + B_b(F(\phi_z, \phi_r) + G(H_z(\phi_z), \phi_r)u - \phi_{y^{(n)}}(0)). \tag{50}$$

Remark 4. We highlight here that, taking into account Problem 1 and (37), $\|\tilde{r}_t\|_\infty \leq \gamma_{\tilde{r}}, \forall t \geq 0$ with $\gamma_{\tilde{r}}$ a positive real.

Assumption 2. It is assumed that the reference signal $y_{d,t}$ in Problem 1 is such that there exists a function ρ of class $\mathcal{N}$ satisfying

$$\|\tilde{r}_{t_1} - \tilde{r}_{t_2}\|_\infty \leq \rho(|t_1 - t_2|), \quad \forall t_1,\ t_2 \in \mathbb{R}^+. \tag{51}$$

In order to provide sufficient conditions for the digital implementation of the continuous-time observer-based controller (41) (see also (48)), let $F_z : C^{2n} \times C^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{2n}$ be the function defined, for any $\phi = \begin{bmatrix} \phi_z \\ \phi_{\hat{z}} \end{bmatrix} \in C^{2n}$, $\phi_z,\ \phi_{\hat{z}} \in C^n$, $\phi_{\tilde{r}} \in C^{n+1}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^{n+1}$, $v_1 \in \mathbb{R}$, $v_2 \in \mathbb{R}^n$, as[24]

$$F_z(\phi, \phi_{\tilde{r}}, v) = \begin{bmatrix} \bar{f}_z(\phi_z, \phi_{\tilde{r}}, v_1) \\ v_2 \end{bmatrix}, \tag{52}$$

where $\bar{f}_z$ is the function in (50).

First, for the reader's convenience, the notion of smoothly separable functionals is recalled.[23]

Definition 2. A functional $V : C^{2n} \to \mathbb{R}^+$ is said to be smoothly separable if there exist a function $V_1 \in C^1_L(\mathbb{R}^{2n}; \mathbb{R}^+)$, a locally Lipschitz functional $V_2 : C^{2n} \to \mathbb{R}^+$, functions β_i of class $\mathcal{K}_\infty$, $i = 1, 2$, such that, for any $\phi \in C^{2n}$, the following equality/inequalities hold

$$\begin{aligned} V(\phi) &= V_1(\phi(0)) + V_2(\phi), \\ \beta_1(|\phi(0)|) &\leq V_1(\phi(0)) \leq \beta_2(|\phi(0)|). \end{aligned} \tag{53}$$

In the following, for a given positive integer n, for a function $F_z : C^{2n} \times C^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{2n}$, and for a locally Lipschitz functional $V : C^{2n} \to \mathbb{R}^+$, the derivative (upper right-hand Dini directional derivative in the case $\Delta = 0$, and derivative in Driver's form in the case $\Delta > 0$[28]) $D^+V : C^{2n} \times C^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^\star$, of the functional V, is defined, for $\phi \in C^{2n}$, $\phi_{\tilde{r}} \in C^{n+1}$,

$\nu \in \mathbb{R}^{n+1}$, as

$$D^+V(\phi, \phi_{\bar{r}}, \nu) = \limsup_{h\to 0^+} \frac{V(\phi_{h,\phi_{\bar{r}},\nu}) - V(\phi)}{h}, \tag{54}$$

where, in the case $\Delta > 0$, for $0 \leq h < \Delta$, $\phi_{h,\phi_{\bar{r}},\nu} \in C^n$ is defined, for $s \in [-\Delta, 0]$, as

$$\phi_{h,\phi_{\bar{r}},\nu}(s) = \begin{cases} \phi(s+h), & s \in [-\Delta, -h) \\ \phi(0) + (s+h)F_z(\phi, \phi_{\bar{r}}, \nu), & s \in [-h, 0], \end{cases}$$

and, for $\Delta = 0$ and $h \in [0, 1)$, as

$$\phi_{h,\phi_{\bar{r}},\nu}(0) = \phi(0) + hF_z(\phi, \phi_{\bar{r}}, \nu).$$

We denote here with $\mathcal{V}$ the set of Lyapunov–Krasovskii functionals $V : C^{2n} \to \mathbb{R}^+$ with the following properties[22,24]:

1. V is smoothly separable, with $V_2 \equiv 0$ in the case $\Delta = 0$, according to Definition 2 (recall $V = V_1 + V_2$);
2. in the case $\Delta > 0$, the function $(\phi, \phi_{\bar{r}}, \nu) \to D^+V_2(\phi, \phi_{\bar{r}}, \nu)$, $\phi \in C^{2n}$, $\phi_{\bar{r}} \in C^{n+1}$, $\nu \in \mathbb{R}^{n+1}$, is Lipschitz on bounded subsets of $C^{2n} \times C^{n+1} \times \mathbb{R}^{n+1}$, where the derivative in Driver's form (see (54)) of the functional V_2 is computed with respect to the function F_z in (52);
3. there exist functions γ_1, γ_2 of class $\mathcal{K}_\infty$ such that for any $\phi \in C^{2n}$, the following inequalities hold:

$$\gamma_1(|\phi(0)|) \leq V(\phi) \leq \gamma_2(\|\phi\|_\infty). \tag{55}$$

Let us consider $k : C^n \times C^{n+1} \times C \to \mathbb{R}^{n+1}$ be the function defined, for any $\phi = \begin{bmatrix} \phi_z \\ \phi_{\hat{z}} \end{bmatrix} \in C^{2n}$, $\phi_z, \ \phi_{\hat{z}} \in C^n$, $\phi_{\bar{r}} = \begin{bmatrix} \phi_r \\ \phi_{y^{(n)}} \end{bmatrix} \in C^{n+1}$, $\phi_r \in C^n$, $\phi_{y^{(n)}} \in C$ (see (41) and also (48))

$$k(\phi_{\hat{z}}, \phi_{\bar{r}}, H_z(\phi_z)) = \begin{bmatrix} \dfrac{-F(\phi_{\hat{z}}, \phi_r) + \phi_{y^{(n)}}(0) + \Gamma\phi_{\hat{z}}(0)}{G(H_z(\phi_z), \phi_r)} \\ \\ (A_b + B_b\Gamma)\phi_{\hat{z}}(0) - KC_b\phi_{\hat{z}}(0) + KH_z(\phi_z)(0) \end{bmatrix}. \tag{56}$$

In the following, sufficient conditions for the quantized sampled-data implementation of the continuous-time observer-based controller (41) (see also (48)) are provided.

Assumption 3. There exist a Lyapunov–Krasovskii functional $V \in \mathcal{V}$, positive reals η, μ, a function p in $C^1_L(\mathbb{R}^+; \mathbb{R}^+)$, of class $\mathcal{K}_\infty$, a function $\overline{\alpha}$ of class $\mathcal{K}$ such that $\overline{I}_d - \overline{\alpha}$ is of class $\mathcal{K}_\infty$, a real $\nu \in \{0, 1\}$, such that for any $\phi = \begin{bmatrix} \phi_z \\ \phi_{\hat{z}} \end{bmatrix} \in C^{2n}$, $\phi_z, \phi_{\hat{z}} \in C^n$, $\phi_{\bar{r}} \in C^{n+1}$, the following inequality holds:

$$\begin{aligned} &\nu D^+V(\phi, \phi_{\bar{r}}, k(\phi_{\hat{z}}, \phi_{\bar{r}}, H_z(\phi_z))) + \eta \max\{0, D^+p\circ V_1(\phi, \phi_{\bar{r}}, k(\phi_{\hat{z}}, \phi_{\bar{r}}, H_z(\phi_z))) + \mu p\circ V_1(\phi(0))\} \\ &\quad \leq \overline{\alpha}(\eta\mu e^{-\mu\Delta} p\circ\beta_1(\|\phi\|_\infty)), \end{aligned} \tag{57}$$

where: k is the function in (56), β_1 is the function of class $\mathcal{K}_\infty$ in Definition 2; the derivative in Driver's form (see (1)) of the functional V is computed with respect to the function F_z in (52).

Proposition 1. *In the case $\Delta = 0$, if there exists a function $\widetilde{F} : C^n \times C^n \to \mathbb{R}$ such that for any $\phi_z, \ \phi_{\hat{z}}, \ \phi_r \in C^n$, the following condition holds*

$$F(\phi_z, \phi_r) - F(\phi_{\hat{z}}, \phi_r) = \widetilde{F}(\phi_z, \phi_{\hat{z}}), \tag{58}$$

where F is the function in (39), then Assumption 3 holds.

Proof of Proposition 1. First, we notice that from (58), it follows that the continuous–time closed–loop system (37)–(41) (or alternatively (37)–(48)) does not involve the reference signal r_t. Then, the proof follows from the use of the converse Lyapunov theorem[25] and the reasoning provided in Remark 1 of the work by Di Ferdinando et al.[22] ■

In the following, the proposed digital implementation of the observer-based continuous-time tracking controller (41) (see also (48)) is presented. First, in order to introduce the considered quantized sampled-data framework, we recall the notion of partition,[20,23] the notion of spline approximation[23] and the notion of quantizers.[11,12]

Definition 3. For a positive integer l, a partition $\pi = \{t_j,\ \ j = -l, -l+1, \dots \}$ of $[-l\Delta, +\infty)$ is a countable, strictly increasing sequence $t_j \in [-l\Delta, +\infty)$, with $t_0 = 0$, such that $t_j \to +\infty$ as $j \to +\infty$. The diameter of π, denoted diam(π), is defined as $\sup_{j\geq -l} t_{j+1} - t_j$. The dwell time of π, denoted dwell(π), is defined as $\inf_{j\geq -l} t_{j+1} - t_j$. For a given $a \in (0, 1]$, $\delta > 0$, $\pi_{a,\delta}$ is any partition π with $a\delta \leq \text{dwell}(\pi) \leq \text{diam}(\pi) \leq \delta$.

For given $\delta < \Delta$ ($\Delta > 0$), $a \in (0, 1]$, let l be the smallest positive integer such that $la\delta \geq \Delta$. Let $\mathcal{T}_{l,a,\delta} \subset \mathbb{R}^{l+1}$ be the set defined as follows[23]

$$\mathcal{T}_{l,a,\delta} = \left\{ w = \begin{pmatrix} w_0 \\ \vdots \\ w_l \end{pmatrix} \in \mathbb{R}^{l+1}, \quad w_k \in [-l\delta, 0], \quad k = 0, 1, \dots, l, \quad w_0 = 0, \quad w_0 - w_l \geq \Delta, \right.$$
$$\left. \delta \geq w_k - w_{k+1} \geq a\delta, \quad k = 0, 1, \dots, l-1 \right\}. \tag{59}$$

Let $P_{l,a,\delta} : \mathbb{R}^{l+1} \times \mathcal{T}_{l,a,\delta} \to C$ be the map defined, for $z = \begin{pmatrix} z_0 \\ \vdots \\ z_l \end{pmatrix} \in \mathbb{R}^{l+1}$, $w = \begin{pmatrix} w_0 \\ \vdots \\ w_l \end{pmatrix} \in \mathcal{T}_{l,a,\delta}$ and $\tau \in [-\Delta, 0]$, as follows[23]

$$(P_{l,a,\delta}(z, w))(\tau) = z_{k+1} + \frac{\tau - w_{k+1}}{w_k - w_{k+1}}(z_k - z_{k+1}), \tag{60}$$

where k is the smallest integer in $\{0, 1, \dots, l-1\}$ such that $w_k \geq \tau \geq w_{k+1}$.

We recall that an output *quantizer* and an input *quantizer* are piece-wise constant functions $q_y : \mathbb{R} \to Q_y$ and $q_u : \mathbb{R} \to Q_u$, where Q_y, Q_u, are suitable finite subsets of $\mathbb{R}^q$ and $\mathbb{R}$, respectively. These quantizers are characterized, for some given positive reals E_y, U_1, μ_y, μ_u, by the following implications[11,12]

$$\begin{aligned} |y| \leq E_y &\Rightarrow |q_y(y) - y| \leq \mu_y, \\ |u| \leq U_1 &\Rightarrow |q_u(u) - u| \leq \mu_u, \end{aligned} \tag{61}$$

where: the positive reals E_y, U_1 are called ranges of the output and input quantizers; the positive reals μ_y, μ_u, are called error bounds of the output and input quantizers.[11,12]

Under Assumptions 1 and 3, for a given partition $\pi_{a,\delta}$, a given output quantizer q_y and a given input quantizer q_u, we propose here the following quantized sampled-data observer-based tracking controller for the system (29)

$$\begin{aligned} u(t) &= q_u\left(\begin{bmatrix} 1 & 0_{1\times n} \end{bmatrix} k\left(\hat{z}_{t_j}, \tilde{r}_{t_j}, P_j^{q_y}\right)\right) = q_u\left(\frac{-F(\hat{z}_{t_j}, r_{t_j}) + y_d^{(n)}(t_j) + \Gamma\hat{z}(t_j)}{G\left(P_j^{q_y}\right), r_{t_j})}\right), \\ t &\in [t_j, t_{j+1}), \quad t_j \in \pi_{a,\delta}, \quad \delta_j = t_{j+1} - t_j, \quad j = 0, 1, \dots, \\ \hat{z}_{t_{j+1}}(\theta) &= \begin{cases} \hat{z}_{t_j}(\theta + \delta_j), & \theta \in [-\Delta, -\delta_j), \\ \hat{z}_{t_j}(0) + (\theta + \delta_j)\begin{bmatrix} 0_{n\times 1} & I_n \end{bmatrix} k\left(\hat{z}_{t_j}, \tilde{r}_{t_j}, P_j^{q_y}\right), & \theta \in [-\delta_j, 0],\ \Delta > 0, \end{cases} \\ \hat{z}(t_{j+1}) &= \hat{z}(t_j) + \delta_j \begin{bmatrix} 0_{n\times 1} & I_n \end{bmatrix} k\left(\hat{z}_{t_j}, \tilde{r}_{t_j}, P_j^{q_y}\right), \quad \Delta = 0, \end{aligned} \tag{62}$$

where: k is the function in (56);

$$P_j^{q_y} = P_{l,a,\delta}(B_S^{q_y}(j), B_{\mathcal{T}}(j));$$

$P_{l,a,\delta}$ is the map defined in (60); $B_S^{q_y} : \mathbb{N} \to \mathbb{R}^{l+1}$ and $B_{\mathcal{T}} : \mathbb{N} \to \mathbb{R}^{l+1}$ are defined (recursively) as

$$B_S^{q_y}(0) = \begin{pmatrix} q_y(\overline{y}_{z,0}(0)) \\ \vdots \\ q_y(\overline{y}_{z,0}(t_{-l})) \end{pmatrix} = \begin{pmatrix} q_y(\overline{H}_z(z_0)(0)) \\ \vdots \\ q_y(\overline{H}_z(z_0)(t_{-l})) \end{pmatrix}, \quad \overline{y}_{z,0}(\tau) = \overline{H}_z(z_0)(\tau) = \begin{cases} y_{z,0}(\tau) = H_z(z_0)(\tau) & \tau \in [-\Delta, 0] \\ y_{z,0}(-\Delta) = H_z(z_0)(-\Delta) & \tau \in [t_{-l}, -\Delta] \end{cases}$$

$$B_S^{q_y}(j) = \begin{pmatrix} q_y(H_z(z_{t_j})(0)) \\ 0_{l\times 1} \end{pmatrix} + \begin{pmatrix} 0_{1\times l} & 0 \\ I_l & 0_{l\times 1} \end{pmatrix} B_S^{q_y}(j-1), \quad B_{\mathcal{T}}(0) = \begin{pmatrix} 0 \\ t_{-1} \\ \vdots \\ t_{-l} \end{pmatrix}, \tag{63}$$

$$B_{\mathcal{T}}(j) = \begin{pmatrix} 0_{1\times l} & 0 \\ I_l & 0 \end{pmatrix} \left(B_{\mathcal{T}}(j-1) - (t_j - t_{j-1}) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right), \quad j = 1, 2, \ldots ;$$

$u(t) \in \mathbb{R}$ is the input signal in (29).

Remark 5. We highlight here that, the observer-based tracking controller proposed in this article (see (41), (48), and (62)) is of delay-dependent type. From a practical point of view, in the continuous-time framework, the implementation of the observer-based tracking controller (41) necessarily requires the knowledge of the infinite dimensional variables $\hat{z}_t$ and $y_{z,t}$ which, in real practice, are often unavailable due to technological constraints. On the other hand, from a theoretical point of view, the knowledge of the function y_t and, consequently, of the function $y_{z,t}$ (see (29) and (37)) is introduced in order to potentially consider further informations on the system at hand which, in the case of their availability, could turn out to be very helpful for the design of the proposed dynamic output feedback controller. For instance, the knowledge of the delayed output measurements: (i) could increase the possibility to satisfy the condition (H_4) in Assumption 1; (ii) could be helpful to relax the requirements in (H_3) (see Theorem 3). In the literature concerning nonlinear systems in lower triangular form, many approaches have been provided for the design of delay-free dynamic output feedback stabilizers also in the sampled-data context.[33,38] On the other hand, such design procedures cannot be directly applied here due to: (i) the presence of possible distributed time-delays in the functions describing the system at hand; (ii) the presence of known exogenous disturbances mimicking the chosen reference signal; (iii) the consideration, in the digital context, of quantization in both input/output channels. To our best knowledge, such a framework has never been investigated in the literature. In this article, for the first time in the literature of nonlinear systems with state delays, a methodology for the design of observer-based tracking controllers is proposed and results are provided in both continuous-time and digital frameworks (see Theorems 2, 3, and forthcoming Theorem 4). We highlight also that, in the digital framework here proposed, the drawback concerning the knowledge of the infinite dimensional variables $y_{z,t}$ and $\hat{z}_t$ is overcome (see (62), (63)). In particular, the problems related to the knowledge of the signal $y_{z,t}$ are here overcome by exploiting a spline approximation approach[23] to obtain, from the available quantized sampled-data output measurements $y_z(t_j)$, an approximation of the function y_{z,t_j} required for the implementation of the controller. Indeed, in (63), $B_S^{q_y}$ and $B_{\mathcal{T}}$ describe buffers of length $l+1$ collecting the quantized sampled-data measurements $y_z(t_j)$ and the lengths of the times elapsed between a sampling and the following. The informations in $B_S^{q_y}$ and $B_{\mathcal{T}}$ (see (63)) are used in order to obtain an approximation of the output signal y_{z,t_j} via (60). Moreover, within the proposed digital framework, the function $\hat{z}_t$ which characterizes the estimation of the system state z_t can be easily computed, without the introduction of particular computational devices and sensors, because its evolution is described by simple difference equations (see (62)). Then, the problems related to the practical implementation of the proposed delay-dependent type controller are overcome in the digital framework making no more an issue the presence of delayed terms in the function describing the controller which could be useful from a practical point of view (see, for instance, Section 6).

In the following theorem, we provide results concerning the semi-global practical stability property of the quantized sampled-data closed-loop system described by (49) (see also (37)) with (62). In particular, it is shown that there exist a suitably fast sampling δ and an accurate quantization of the input/output channels (i.e., ranges E_y, U_1 and error bounds μ_y, μ_u for the quantizers q_y and q_u in (62)) such that, the semi-global practical stability property, with arbitrarily small final target ball, of the quantized sampled-data closed-loop system described by (49) (see also (37)) with (62) is ensured. We highlight that, from the semi-global practical stability property of the quantized sampled-data closed-loop system (49) (see also (37)) with (62), it follows that point (ii) in Problem 1 is solved.

Theorem 4. *Let Assumptions 1–3 hold. Let $\gamma_{\tilde r}$ be the positive real in Remark 4. Let a be an arbitrary real in* $(0,1]$. *Then, for any positive reals $\tilde q$, r, R with $0 < r < R$, there exist positive reals δ, T, E, E_y, U_1, μ_y, and μ_u such that: for any partition $\pi_{a,\delta}$, for any output quantizer $q_y : \mathbb{R} \to \mathcal{Q}_y$ with range E_y and error bound μ_y, for any input quantizer $q_u : \mathbb{R} \to \mathcal{Q}_u$ with range U_1 and error bound μ_u, for any initial states $z_0 \in \mathcal{C}^n_R \cap W^{1,\infty}_n$, $\hat z_0 \in \mathcal{C}^n_R \cap W^{1,\infty}_n$ and, in the case $\Delta > 0$, satisfying* $\operatorname{esssup}_{\theta\in[-\Delta,0]} \left\| \begin{bmatrix} \frac{dz_0(\theta)}{d\theta} \\ \frac{d\hat z_0(\theta)}{d\theta} \end{bmatrix} \right\| \le \tilde q$, *the corresponding solution of the quantized sampled-data closed-loop system described by (49) (see also (37)) with (62) exists for all $t \in \mathbb{R}^+$, and, furthermore, the following inequalities hold*:

$$
\begin{aligned}
&\left\| \begin{bmatrix} z_t \\ \hat z_{t_j} \end{bmatrix} \right\|_\infty \le E, \quad \forall t \in \mathbb{R}^+, \quad j = 0, 1, \dots ; \\
&\left\| \begin{bmatrix} z_t \\ \hat z_{t_j} \end{bmatrix} \right\|_\infty \le r, \quad \forall t \ge T, \quad \forall j \in \{ i \in \mathbb{N} | t_i \ge T,\ t_i \in \pi_{a,\delta} \}.
\end{aligned} \tag{64}
$$

Remark 6. Notice that, from conditions (64) in Theorem 4, it follows that the solution of the quantized sampled-data closed-loop system described by (29)–(62) exists for all $t \in \mathbb{R}^+$, and, furthermore, satisfies

$$
\begin{aligned}
&|y(t) - y_d(t)| \le E, \quad \forall t \ge 0, \\
&|y(t) - y_d(t)| \le r, \quad \forall t \ge T,
\end{aligned} \tag{65}
$$

that is, point (ii) in Problem 1 is solved.

5.1 | Proof of Theorem 4

In the following, the stabilization in the sample-and-hold theory[20–24] and the notion of DOSDF[22,24] are used as tools in order to prove the results. First, for the reader's convenience, the notion of DOSDF is recalled.[22,24] Such a notion is inherited by the definition of steepest descent feedback[20,21,23] which is directly connected with the well-known Artstein's methodologies exploiting control Lyapunov–Krasovskii functionals V for the design of controllers.[20–24,39,40] We highlight that, in the following definition, in the case $\Delta = 0$, the spaces $\mathcal{C}^n$, $\mathcal{C}^{n+1}$, $\mathcal{C}^{2n}$ are isomorphic with $\mathbb{R}^n$, $\mathbb{R}^{n+1}$, $\mathbb{R}^{2n}$, respectively.

Definition 4. Let $V \in \mathcal{V}$. A locally bounded function $k : \mathcal{C}^n \times \mathcal{C}^{n+1} \times \mathcal{C} \to \mathbb{R}^{n+1}$, continuous or not, is said to be a DOSDF for the system described by (49) (see also (37)), induced by V, if there exist positive reals η, μ, a function p in $C^1_L(\mathbb{R}^+;\mathbb{R}^+)$, of class $\mathcal{K}_\infty$, a function $\overline{\alpha}$ of class $\mathcal{K}$ such that $\overline{I}_d - \overline{\alpha}$ is of class $\mathcal{K}_\infty$, a real $\nu \in \{0,1\}$, such that, for any $\phi = \begin{bmatrix} \phi_z \\ \phi_{\hat z} \end{bmatrix} \in \mathcal{C}^{2n}$, $\phi_z, \phi_{\hat z} \in \mathcal{C}^n$, $\phi_{\tilde r} \in \mathcal{C}^{n+1}$, the following inequality holds:

$$
\nu D^+ V(\phi, \phi_{\tilde r}, k(\phi_{\hat z}, \phi_{\tilde r}, H_z(\phi_z))) + \eta \max\{0, D^+ p \circ V_1(\phi, \phi_{\tilde r}, k(\phi_{\hat z}, \phi_{\tilde r}, H_z(\phi_z))) + \mu p \circ V_1(\phi(0))\} \le \overline{\alpha}(\eta \mu e^{-\mu\Delta} p \circ \beta_1(\|\phi\|_\infty)), \tag{66}
$$

where: β_1 is the function of class $\mathcal{K}_\infty$ in Definition 2; the derivative in Driver's form (see (54)) of the functional V is computed with respect to the function F_z in (52).

The following proof is based on the results recently provided in the work by Di Ferdinando et al.[24] where the stabilization in the sample-and-hold sense theory[20–24] is used as a tool in order to show that: there exists a suitably small sampling and an accurate quantization of the input/output channels such that the digital implementation of DOSDFs (continuous or not) guarantees the semi-global practical stability property of the related quantized sampled-data closed-loop system, with arbitrarily small final target ball of the origin (see Theorem 1 in the work by Di Ferdinando et al.[24]). It is here highlighted that, the proof of Theorem 1 provided in the work by Di Ferdinando et al.[24] cannot be directly applied here due to new considerations regarding: (i) the case of observer-based tracking controllers; (ii) the problems related to the possible non-availability in the buffer of suitable past values of the output signal required for the correct implementation of a proposed delay-dependent observer-based tracking controller. Then, a new devoted proof is required to cope with tracking control problems and the use of spline methodologies for the approximation of the infinite dimensional output signal $y_{z,t}$ which, in the work by Di Ferdinando et al.,[24] are not considered. In the following, by taking into account that, under Assumption 3, the function k in (56), derived from the proposed observer-based continuous-time tracking controller (41) (see also (48)), is a DOSDF for the system described by (49) (see Definition 4), the stabilization in the sample-and-hold sense theory is properly reformulated to show that there exist a suitably fast sampling δ and an accurate quantization of the input/output channels (i.e., ranges E_y, U_1 and error bounds μ_y, μ_u for the quantizers q_y and q_u in (62)) such that, the semi-global practical stability property, with arbitrarily small final target ball, of the quantized sampled-data closed-loop system described by (49) (see also (37)) with (62) is ensured (i.e., the results in Theorem 4 hold). First, we recall a result provided in the work by Pepe,[23] which is very helpful in the forthcoming proof. In particular, Theorem 2.3 in the work by Pepe[23] is here suitably adapted in order to cope with observer-based tracking controllers (see forthcoming Lemma 2).

Lemma 2. *Let Assumptions 1–3 hold. Let $\gamma_{\tilde{r}}$ be the positive real in Remark 4. Let α_i, $i = 1,2,3$, be the functions of class $\mathcal{K}_\infty$, defined for $s \in \mathbb{R}^+$, as $\alpha_1(s) = \eta e^{-\mu\Delta} p \circ \beta_1(s)$, $\alpha_2(s) = \nu\gamma_2(s) + \eta p \circ \beta_2(s)$, $\alpha_3(s) = (\overline{I}_d - \overline{\alpha})(\eta\mu e^{-\mu\Delta} p \circ \beta_1(s))$. Let $V_3 : \mathcal{C}^{2n} \to \mathbb{R}^+$, $V_\infty : \mathcal{C}^{2n} \to \mathbb{R}^+$ be the functionals defined, for $\phi = \begin{bmatrix}\phi_z \\ \phi_{\hat{z}}\end{bmatrix} \in \mathcal{C}^{2n}$, $\phi_z, \phi_{\hat{z}} \in \mathcal{C}^n$, as*

$$V_3(\phi) = \sup_{\theta\in[-\Delta,0]} e^{\mu\theta} p \circ V_1(\phi(\theta)), \quad V_\infty(\phi) = \nu V(\phi) + \eta V_3(\phi).$$

Let $\mathcal{D}_\infty : \mathcal{C}^{2n} \times \mathcal{C}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be the functional defined, for $\phi = \begin{bmatrix}\phi_z \\ \phi_{\hat{z}}\end{bmatrix} \in \mathcal{C}^{2n}$, $\phi_z, \phi_{\hat{z}} \in \mathcal{C}^n$, $\phi_{\tilde{r}} \in \mathcal{C}^{n+1}$, $v \in \mathbb{R}^{n+1}$, as

$$\mathcal{D}_\infty(\phi, \phi_{\tilde{r}}, v) = \nu D^+V(\phi, \phi_{\tilde{r}}, v) - \eta\mu V_3(\phi) + \eta \max\{0, D^+ p \circ V_1(\phi, \phi_{\tilde{r}}, v) + \mu p \circ V_1(\phi(0))\}.$$

Then, the following hold

(i) $\alpha_1(\|\phi\|_\infty) \leq V_\infty(\phi) \leq \alpha_2(\|\phi\|_\infty)$, $\forall \phi \in \mathcal{C}^{2n}$;

(ii) the function $(\phi, \phi_{\tilde{r}}, v) \to \mathcal{D}_\infty(\phi, \phi_{\tilde{r}}, v)$, $\phi \in \mathcal{C}^{2n}$, $\phi_{\tilde{r}} \in \mathcal{C}^{n+1}$, $v \in \mathbb{R}^{n+1}$, *is Lipschitz on bounded subsets of* $\mathcal{C}^{2n} \times \mathcal{C}^{n+1} \times \mathbb{R}^{n+1}$;

(iii) $D^+V_\infty(\phi, \phi_{\tilde{r}}, v) \leq \mathcal{D}_\infty(\phi, \phi_{\tilde{r}}, v)$, $\forall \phi \in \mathcal{C}^{2n}$, $\forall \phi_{\tilde{r}} \in \mathcal{C}^{n+1}_{\gamma_{\tilde{r}}}$, $\forall v \in \mathbb{R}^{n+1}$;

(iv) $\mathcal{D}_\infty(\phi, \phi_{\tilde{r}}, k(\phi_{\hat{z}}, \phi_{\tilde{r}}, H_z(\phi_z))) \leq -\alpha_3(\|\phi\|_\infty)$, $\forall \phi = \begin{bmatrix}\phi_z \\ \phi_{\hat{z}}\end{bmatrix} \in \mathcal{C}^{2n}$, $\phi_z, \phi_{\hat{z}} \in \mathcal{C}^n$, $\forall \phi_{\tilde{r}} \in \mathcal{C}^{n+1}_{\gamma_{\tilde{r}}}$.

Let us consider the open-loop system described by Di Ferdinando et al.[22] and Di Ferdinando and Pepe[41]

$$\begin{aligned} \dot{z}(t) &= \overline{f}_z(z_t, \tilde{r}_t, v_1(t)), \quad t \geq 0 \quad \text{a.e.}, \\ \dot{\hat{z}}(t) &= v_2(t), \\ y_{z,t}(\tau) &= H_z(z_t)(\tau) = C_b z_t(\tau), \\ z(\tau) &= z_0(\tau), \quad \hat{z}(\tau) = \hat{z}_0(\tau), \quad \tau \in [-\Delta, 0], \end{aligned} \tag{67}$$

where: z_0 is the initial state in (49) (see also (37)); $z_t \in \mathcal{C}^n$; $z(t) \in \mathbb{R}^n$; $\tilde{r}_t \in \mathcal{C}^{n+1}_{\gamma_{\tilde{r}}}$, $\forall t \in \mathbb{R}^+$ (see (49) and Remark 4); $\hat{z}_t \in \mathcal{C}^n$; $\hat{z}(t) \in \mathbb{R}^n$; $\hat{z}_0 \in W^{1,\infty}_l$ is the initial state related to the new variable $\hat{z}(t)$; $\overline{f}_z$ is the function in (49), (50); $v_1(t) = u(t) \in \mathbb{R}$ is the input in (49) (see also (37) and (29)); $v_2(t) \in \mathbb{R}^n$ is a new input (Lebesgue measurable and locally essentially bounded);

$y_{z,t} \in C$ is the output in (49); H_z is the function in (49). Let (as long as the solution of (67) exists) $\chi(t) = \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix} \in \mathbb{R}^{2n}$, $\chi_t = \begin{bmatrix} z_t \\ \hat{z}_t \end{bmatrix} \in C^{2n}$, $v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \in \mathbb{R}^{n+1}$. The open-loop system (67) can be rewritten as follows

$$\dot{\chi}(t) = \begin{bmatrix} \dot{z}(t) \\ \dot{\hat{z}}(t) \end{bmatrix} = \begin{bmatrix} \overline{f}_z(z_t, \tilde{r}_t, v_1(t)) \\ v_2(t) \end{bmatrix} = F_z(\chi_t, \tilde{r}_t, v(t)),$$
$$\chi(\tau) = \chi_0(\tau) = \begin{bmatrix} z_0(\tau) \\ \hat{z}_0(\tau) \end{bmatrix}, \quad \tau \in [-\Delta, 0], \tag{68}$$

where F_z is the function defined in (52). Let:

1. the functionals $V_\infty : C^{2n} \to \mathbb{R}^+$ and $\mathcal{D}_\infty : C^{2n} \times C^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be as in Lemma 2;
2. α_i, $i = 1, 2, 3$, be the functions of class $\mathcal{K}_\infty$ as in Lemma 2;
3. r, R, be any positive reals, $0 < r < R$;
4. $\overline{R} = \sqrt{2}R$;
5. e_1, e_2, E be positive reals satisfying:

$$0 < e_2 < e_1 < r < \overline{R} < E, \quad \alpha_1(r) > \alpha_2(e_1), \quad \alpha_1(E) > \alpha_2(\overline{R});$$

6. (involved $\phi = \begin{bmatrix} \phi_z \\ \phi_{\hat{z}} \end{bmatrix} \in C^{2n}$, $\phi_z, \phi_{\hat{z}} \in C^n$)

$$E_y = \sup_{\phi \in C_E^{2n}} \|H_z(\phi_z)\|_\infty, \quad H = E_y + 1,$$
$$U_1 = \sup_{\phi \in C_E^{2n},\ \phi_{\tilde{r}} \in C_{\gamma_{\tilde{r}}}^{n+1},\ y \in C_H} \left| \begin{bmatrix} 1 & 0_{1\times n} \end{bmatrix} k(\phi_{\hat{z}}, \phi_{\tilde{r}}, y) \right|, \quad U_2 = U_1 + 1, \quad U = \sup_{\phi \in C_E^{2n},\ \phi_{\tilde{r}} \in C_{\gamma_{\tilde{r}}}^{n+1},\ y \in C_H} \left\| \begin{bmatrix} U_2 \\ \begin{bmatrix} 0_{n\times 1} & I_n \end{bmatrix} |k(\phi_{\hat{z}}, \phi_{\tilde{r}}, y)| \end{bmatrix} \right\|, \tag{69}$$

where k is the function defined in (56);

7. M, L_k K be positive reals such that, for any $\phi_1 = \begin{bmatrix} \phi_{z_1} \\ \phi_{\hat{z}_1} \end{bmatrix}$, $\phi_2 = \begin{bmatrix} \phi_{z_2} \\ \phi_{\hat{z}_2} \end{bmatrix} \in C_E^{2n}$, ϕ_{z_i}, $\phi_{\hat{z}_i} \in C^n$, $i = 1, 2$, $\phi_{\tilde{r}_1}$, $\phi_{\tilde{r}_2} \in C_{\gamma_{\tilde{r}}}^{n+1}$, v_1, $v_2 \in \mathcal{B}_U^{n+1}$ and y_1, $y_2 \in C_H$, the following conditions hold:

$$|F_z(\phi_1, \phi_{\tilde{r}_1}, v_1)| \le M; \tag{70}$$

$$|k(\phi_{\hat{z}_1}, \phi_{\tilde{r}_1}, y_1) - k(\phi_{\hat{z}_2}, \phi_{\tilde{r}_2}, y_2)| \le L_k(\|\phi_1 - \phi_2\|_\infty + \|\phi_{\tilde{r}_1} - \phi_{\tilde{r}_2}\|_\infty + |y_1 - y_2|); \tag{71}$$

$$|\mathcal{D}_\infty(\phi_1, \phi_{\tilde{r}_1}, v_1) - \mathcal{D}_\infty(\phi_2, \phi_{\tilde{r}_2}, v_2)| \le K(\|\phi_1 - \phi_2\|_\infty + \|\phi_{\tilde{r}_1} - \phi_{\tilde{r}_2}\|_\infty + |v_1 - v_2|), \tag{72}$$

where k is the function defined in (56);

8. $\beta = \alpha_3(e_2)$;
9. $a \in (0, 1]$ be arbitrarily fixed;
10. $T = \dfrac{3\alpha_2(\overline{R})}{\beta a} + 1$;
11. $\tilde{q}$ be any positive real;
12. $\overline{q} = \begin{cases} \max\{\tilde{q},\ M\}, & \Delta > 0, \\ 0, & \Delta = 0; \end{cases}$ $\quad \overline{q}_H = \begin{cases} \sup_{\phi \in C_E^{2n}} \left(esssup_{\theta \in [-\Delta, 0]} \left| \frac{dH_z(\phi_z)(\theta)}{d\theta} \right| \right), & \Delta > 0, \\ 0, & \Delta = 0; \end{cases}$

13. $\delta,\ \mu_y,\ \mu_u \in \mathbb{R}^+$ such that:

$$\delta < \max\{1, \Delta\}, \quad 0 < \mu_y \leq 1, \quad 0 < \mu_u \leq 1, \quad e_2 + \delta M < e_1,$$
$$\overline{R} + \delta M < E, \quad \alpha_1(r) > \alpha_2(e_1) + \frac{2}{3}\beta\delta, \quad \frac{\beta}{3} > K(2\overline{q}\delta + \rho(\delta) + \mu_u + 3L_k\mu_y + 2L_k\overline{q}_H\delta), \tag{73}$$

where ρ is the function of class $\mathcal{N}$ in (51);

14. $\chi_0 = \begin{bmatrix} z_0 \\ \hat{z}_0 \end{bmatrix} \in C^{2n} \cap W^{1,\infty}_{2n}$, $z_0 \in C^n_R \cap W^{1,\infty}_n$, $\hat{z}_0 \in C^n_R \cap W^{1,\infty}_n$ and, in the case $\Delta > 0$, such that $\text{esssup}_{\theta\in[-\Delta,0]} \left\| \begin{bmatrix} \frac{dz_0(\theta)}{d\theta} \\ \frac{d\hat{z}_0(\theta)}{d\theta} \end{bmatrix} \right\| \leq \tilde{q}$;
15. an output quantizer $q_y : \mathbb{R} \to \mathcal{Q}_y$ and an input quantizer $q_u : \mathbb{R} \to \mathcal{Q}_u$ such that, $\forall y \in \mathcal{B}_{E_y}$ and $\forall u \in \mathcal{B}_{U_1}$, inequalities (61) are satisfied.

Notice that, $\chi_0 = \begin{bmatrix} z_0 \\ \hat{z}_0 \end{bmatrix} \in C^{2n}_R \cap W^{1,\infty}_{2n}$. Let us consider a partition $\pi_{a,\delta}$. Let us consider the system described by (68) with (as long as the related solution exists)

$$v(t) = \begin{bmatrix} q_u\left(\begin{bmatrix} 1 & 0_{1\times n} \end{bmatrix} k\left(\hat{z}_{t_j}, \tilde{r}_{t_j}, P^{q_y}_j\right)\right) \\ \begin{bmatrix} 0_{n\times 1} & I_n \end{bmatrix} k\left(\hat{z}_{t_j}, \tilde{r}_{t_j}, P^{q_y}_j\right) \end{bmatrix} = k^{q_u}\left(\hat{z}_{t_j}, \tilde{r}_{t_j}, P^{q_y}_j\right),$$
$$t_j \leq t < t_{j+1}, \quad t_j \in \pi_{a,\delta}, \quad j = 0, 1, \ldots, \tag{74}$$

where k is the function defined in (56) and $k^{q_u} : C^n \times C^{n+1} \times C \to C^{n+1}$ is the function readily defined by (74). Notice that, taking into account steps (6), (13), and (15), $\forall \phi = \begin{bmatrix} \phi_z \\ \phi_{\hat{z}} \end{bmatrix} \in C^{2n}_E$, $\phi_z,\ \phi_{\hat{z}} \in C^n$, $H_z(\phi_z) \in C_{E_y}$ and, consequently, $q_y(H_z(\phi_z)(\tau)) \in \mathcal{B}_H$, $\forall \tau \in [-\Delta, 0]$. From such a consideration, it follows that, $P^{q_y}_0 \in C_H$. Then, $k^{q_u}(\hat{z}_0, \tilde{r}_0, P^{q_y}_0) \in \mathcal{B}^{n+1}_U$. Let $\chi(t) = \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix}$ be the solution of the quantized sampled-data closed-loop system (68), (74), in a maximal time interval $[0, b)$, $0 < b \leq +\infty$. Let $B_S : \mathbb{N} \to \mathbb{R}^{l+1}$ be defined (recursively) as

$$B_S(0) = \begin{pmatrix} \overline{y}_{z,0}(0) \\ \vdots \\ \overline{y}_{z,0}(t_{-l}) \end{pmatrix} = \begin{pmatrix} \overline{H}_z(z_0)(0) \\ \vdots \\ \overline{H}_z(z_0)(t_{-l}) \end{pmatrix}, \quad B_S(j) = \begin{pmatrix} H_z(z_{t_j})(0) \\ 0_{l\times 1} \end{pmatrix} + \begin{pmatrix} 0_{1\times l} & 0 \\ I_l & 0_{l\times 1} \end{pmatrix} B_S(j-1), \quad j = 1, 2, \ldots. \tag{75}$$

In order to simplify the involved notation, in the following, for $t_j \in \pi_{a,\delta}, j = 0, 1, \ldots$, we will denote with: (i) P_j the function $P_{l,a,\delta}(B_S(j), B_{\mathcal{T}}(j))$; (ii) k^*_j the function $k^{q_u}(\hat{z}_{t_j}, \tilde{r}_{t_j}, P^{q_y}_j)$; (iii) $\tilde{k}_j$ the function $k(\hat{z}_{t_j}, \tilde{r}_{t_j}, P^{q_y}_j)$; (iv) $\overline{k}_j$ the function $k(\hat{z}_{t_j}, \tilde{r}_{t_j}, P_j)$; (v) k_j the function $k(\hat{z}_{t_j}, \tilde{r}_{t_j}, H_z(z_{t_j}))$. We show first that the solution exists in $[0, t_1]$. Otherwise, by contradiction, if the solution blows up, there exists a time $\tau \in [0, t_1)$ such that $|\chi(t)| < E$, $t \in [0, \tau)$, and $|\chi(\tau)| = E$. But, from (70), (73), for $t \in [0, \tau]$, the inequalities hold:

$$|\chi(t)| \leq |\chi_0(0)| + \int_0^t |F_z(\chi_\theta, \tilde{r}_\theta, k^*_0)| d\theta \leq \overline{R} + \delta M < E. \tag{76}$$

Thus, taking $t = \tau$, the absurd inequality arises $E < E$. Therefore, the solution exists in $[0, t_1]$ and, by (76), it follows that $\chi_t \in C^{2n}_E$, $t \in [0, t_1]$. Taking into account (70) and $\overline{q}$, $\overline{q}_H$ provided in step (12), for any $t \in [0, t_1]$, the following inequality holds in the case $\Delta > 0$: $\text{esssup}_{\theta\in[-\Delta,0]} \left\| \begin{bmatrix} \frac{dz_t(\theta)}{d\theta} \\ \frac{d\hat{z}_t(\theta)}{d\theta} \end{bmatrix} \right\| \leq \overline{q}$; $\text{esssup}_{\theta\in[-\Delta,0]} \left| \frac{dH_z(z_t)(\theta)}{d\theta} \right| \leq \overline{q}_H$. Let $W(t) = V_\infty(\chi_t)$, $t \in [0, t_1]$, with $V_\infty : C^{2n} \to \mathbb{R}^+$ given in step (1). Taking into account point (iii) and (iv) in Lemma 2, (51) and (72) for any fixed $t \in (0, t_1]$, for some $t^* \in [0, t]$, the following equalities/inequalities hold:

$$W(t) - W(0) = \int_0^t D^+ V_\infty(\chi_\tau, \tilde{r}_\tau, k^*_0) d\tau \leq t\left(\frac{1}{t}\int_0^t \mathcal{D}_\infty(\chi_\tau, \tilde{r}_\tau, k^*_0) d\tau\right)$$

$$
\begin{aligned}
&= t\mathcal{D}_\infty(\chi_{t^*}, \tilde{r}_{t^*}, k_0^*) - t\mathcal{D}_\infty(\chi_0, \tilde{r}_0, \tilde{k}_0) + t\mathcal{D}_\infty(\chi_0, \tilde{r}_0, \tilde{k}_0) \\
&\le tK(\|\chi_{t^*} - \chi_0\|_\infty + \|\tilde{r}_{t^*} - \tilde{r}_0\|_\infty + \mu_u) + t\mathcal{D}_\infty(\chi_0, \tilde{r}_0, \tilde{k}_0) - t\mathcal{D}_\infty(\chi_0, \tilde{r}_0, \overline{k}_0) + t\mathcal{D}_\infty(\chi_0, \tilde{r}_0, \overline{k}_0) \\
&\le tK(\|\chi_{t^*} - \chi_0\|_\infty + \rho(\delta) + \mu_u + 3L_k\mu_y) + t\mathcal{D}_\infty(\chi_0, \tilde{r}_0, \overline{k}_0) - t\mathcal{D}_\infty(\chi_0, \tilde{r}_0, k_0) + t\mathcal{D}_\infty(\chi_0, \tilde{r}_0, k_0) \\
&\le tK(\|\chi_{t^*} - \chi_0\|_\infty + \rho(\delta) + \mu_u + 3L_k\mu_y + L_k\|P_0 - H_z(z_0)\|_\infty) + t\mathcal{D}_\infty(\chi_0, \tilde{r}_0, k_0) \\
&\le tK(\|\chi_{t^*} - \chi_0\|_\infty + \rho(\delta) + \mu_u + 3L_k\mu_y + L_k\|P_0 - H_z(z_0)\|_\infty) - t\alpha_3(\|\chi_0\|_\infty).
\end{aligned} \tag{77}
$$

Taking into account (70) and $\overline{q}$ provided in step (12), the following inequality holds in both cases $\Delta = 0$ and $\Delta > 0$: $\|\chi_{t^*} - \chi_0\|_\infty \le 2\overline{q}\delta$. Moreover, for given $\tau \in [-\Delta, 0]$, let $\bar{j}$ be the smallest integer in $\{1, 2, \ldots, l\}$ such that $B_\mathcal{T}(0)_{\bar{j}} \ge \tau \ge B_\mathcal{T}(0)_{\bar{j}+1}$. Thus, the equality/inequalities hold (see (60), (63))

$$
\begin{aligned}
|(P_0)(\tau) - H_z(z_0)(\tau)| &= \left| B_S(0)_{\bar{j}+1} + \frac{\tau - B_\mathcal{T}(0)_{\bar{j}+1}}{B_\mathcal{T}(0)_{\bar{j}} - B_\mathcal{T}(0)_{\bar{j}+1}} (B_S(0)_{\bar{j}} - B_S(0)_{\bar{j}+1}) - H_z(z_0)(\tau) \right| \\
&\le |B_S(0)_{\bar{j}+1} - H_z(z_0)(\tau)| + \left| \frac{\tau - B_\mathcal{T}(0)_{\bar{j}+1}}{B_\mathcal{T}(0)_{\bar{j}} - B_\mathcal{T}(0)_{\bar{j}+1}} (B_S(0)_{\bar{j}} - B_S(0)_{\bar{j}+1}) \right| \\
&\le \left| B_S(0)_{\bar{j}+1} - B_S(0)_{\bar{j}+1} - \int_{B_\mathcal{T}(0)_{\bar{j}+1}}^{\tau} \frac{d\overline{H}_z(z_0)(\theta)}{d\theta} d\theta \right| + \left| B_S(0)_{\bar{j}+1} + \int_{B_\mathcal{T}(0)_{\bar{j}+1}}^{B_\mathcal{T}(0)_{\bar{j}}} \frac{d\overline{H}_z(z_0)(\theta)}{d\theta} d\theta - B_S(0)_{\bar{j}+1} \right| \le 2\overline{q}_H\delta.
\end{aligned} \tag{78}
$$

From (77) and taking into account (73), (78), we obtain

$$
W(t) \le W(0) + tK(2\overline{q}\delta + \rho(\delta) + \mu_u + 3L_k\mu_y + 2L_k\overline{q}_H\delta) - t\alpha_3(\|\chi_0\|_\infty) \le W(0) + t\frac{\beta}{3} - t\alpha_3(\|\chi_0\|_\infty). \tag{79}
$$

Let us now consider the following two cases[20]: (1) $\|\chi_0\|_\infty \le e_2$; (2) $\|\chi_0\|_\infty > e_2$. As far as case (1) is concerned, by using again the first inequality in (76) and from (73), the following inequality holds, for any $t \in [0, t_1]$, $|\chi(t)| \le e_2 + \delta M < e_1$. From point (i) in Lemma 2, it follows $W(t) \le \alpha_2(e_1), \quad t \in [0, t_1]$. As far as case (2) is concerned, taking into account β given in step (8), we have that $\beta < \alpha_3(\|\chi_0\|_\infty)$. Therefore, from (79), we have, for any $t \in [0, t_1]$,

$$
W(t) \le W(0) - \frac{2}{3}\beta t.
$$

Let us introduce the following claim, which will be proved later.

Claim 3. The solution $\chi(t)$ of (68), (74), exists in $[0, +\infty)$ and, furthermore, $\chi_t \in C_E^{2n}, \quad \forall t \ge 0$.

Notice that, taking into account steps (6), (13), and (15), Claim 3 and the same reasoning used in the first interval $[0, t_1]$, $P_j^{q_y} \in C_H$ and, consequently, $k_j^* \in \mathcal{B}_U^{n+1}, j = 0, 1, \ldots$. Let $W(t) = V_\infty(\chi_t), t \in \mathbb{R}^+$. Then, in any interval $[t_j,\ t_{j+1}], j = 0, 1, \ldots$, by the same reasoning used in the interval $[0, t_1]$, we have, for $t \in [t_j,\ t_{j+1}]$ (see (77)–(79)),

$$
W(t) - W(t_j) \le (t - t_j)K(\|\chi_{t^*} - \chi_{t_j}\|_\infty + \rho(\delta) + \mu_u + 3L_k\mu_y + L_k\|P_j - y_{z,t_j}\|_\infty) - (t - t_j)\alpha_3(\|\chi_{t_j}\|_\infty). \tag{80}
$$

Let $\overline{y}_{z,j} : [B_\mathcal{T}(j)_{l+1}, 0] \to \mathbb{R}, j = 1, 2, \ldots$, be defined, for $\tau \in [B_\mathcal{T}(j)_{l+1}, 0]$, as follows

$$
\overline{y}_{z,j}(\tau) = \begin{cases} y_z(t_j + \tau), & \tau \in [-\Delta, 0], \\ y_z(t_j + \tau), & \tau \in [B_\mathcal{T}(j)_{l+1},\ -\Delta],\ t_j + \tau \in [0, t_j], \\ \overline{y}_{z,0}(t_j + \tau), & \tau \in [B_\mathcal{T}(j)_{l+1},\ -\Delta],\ t_j + \tau \in [t_{-l}, 0]. \end{cases}
$$

where $y_z(t) = H_z(z_t)(0)$ and $\overline{y}_{z,0}$ is the function defined in (63). The function $\overline{y}_{z,j}$ is absolutely continuous with essentially bounded derivative, a bound given by $\overline{q}_H$. Thus, by the same reasoning used in (78), the following inequality holds (see (60), (63)): $|(P_j)(\tau) - y_{z,t_j}(\tau)| \le 2\overline{q}_H\delta$. From (80), taking into account (73), we obtain

$$
W(t) \le W(t_j) + \frac{\beta}{3}(t - t_j) - \alpha_3(\|\chi_{t_j}\|_\infty)(t - t_j).
$$

Taking into account of both cases $\|\chi_{t_j}\|_\infty \leq e_2$ and $\|\chi_{t_j}\|_\infty > e_2$ we obtain, for $j \geq 0$:

$$W(t) \leq (W(t_j) - \frac{2}{3}\beta(t-t_j))H(\|\chi_{t_j}\|_\infty - e_2) + \alpha_2(e_1)H_0(e_2 - \|\chi_{t_j}\|_\infty).$$

The symbols H_0 and H denote Heaviside functions defined, for $s \in \mathbb{R}$, as follows: $H_0(s) = 1$ if $s \geq 0$, $H_0(s) = 0$ if $s < 0$; $H(s) = 1$ if $s > 0$, $H(s) = 0$ if $s \leq 0$. In particular, for $t = t_{j+1}, j = 0, 1, \ldots$, the inequality holds:

$$W(t_{j+1}) \leq (W(t_j) - \frac{2}{3}\beta(t_{j+1}-t_j))H(\|\chi_{t_j}\|_\infty - e_2) + \alpha_2(e_1)H_0(e_2 - \|\chi_{t_j}\|_\infty).$$

Notice that, for any integer $j \geq 0$, the inequality holds $W(t_j) \leq \alpha_2(\overline{R})$. From here on, the same steps used in the work by Pepe[23] for the proof of Theorem 4.1 can be suitably repeated here, in order to prove that: (i) the solution $\chi(t)$ of the closed–loop system (68), (74), exists for all $t \in \mathbb{R}^+$ and, furthermore, satisfies $\chi_t = \begin{bmatrix} z_t \\ \hat{z}_t \end{bmatrix} \in C_E^{2n}$, $\forall t \in \mathbb{R}^+$, $\chi_t = \begin{bmatrix} z_t \\ \hat{z}_t \end{bmatrix} \in C_r^{2n}$, $\forall t \geq T$, with T the positive real given in step (10); (ii) Claim 1 holds true. The reader can refer to steps from (4.26) to (4.32) in the work by Pepe[23] (taking, in such steps, $k_2 = \left[\frac{3\alpha_2(\overline{R})}{\beta a \delta}\right] + 1$). Now, from (68), (74), it follows that $\chi_t = \begin{bmatrix} z_t \\ \hat{z}_t \end{bmatrix}$ is the solution, for $t \in \mathbb{R}^+$, of the closed-loop system described by the equations

$$\begin{aligned}
&\dot{z}(t) = \overline{f}_z(z_t, \tilde{r}_t, q_u(\begin{bmatrix} 1 & 0_{1\times n} \end{bmatrix} k(\hat{z}_{t_j}, \tilde{r}_{t_j}, q_y(H_z(z_{t_j}))))),\\
&\dot{\hat{z}}(t) = \begin{bmatrix} 0_{n\times 1} & I_n \end{bmatrix} k(\hat{z}_{t_j}, \tilde{r}_{t_j}, q_y(H_z(z_{t_j}))),\\
&t \in [t_j, t_{j+1}), \quad t_j \in \pi_{a,\delta}, \quad j = 0, 1, \ldots,\\
&z(\tau) = z_0(\tau), \quad \hat{z}(\tau) = \hat{z}_0(\tau), \quad \tau \in [-\Delta, 0].
\end{aligned} \tag{81}$$

From (81), it follows that $\begin{bmatrix} z_t \\ \hat{z}_{t_j} \end{bmatrix}$ is the solution, for $t \in \mathbb{R}^+$, $t_j \in \pi_{a,\delta}$, of the system described by (49)–(62). It follows that (64) holds. The proof of the theorem is complete.

Remark 7. Notice that, in the proof of Theorem 4, a methodology for the computation of an upper bound for the sampling period δ, of upper bounds for the quantization errors μ_y, μ_u, of quantizers ranges E_y, U_1 and of a settling time T is provided (see steps (1)–(15)). According to our experience, such steps may well provide a conservative upper bound for the sampling period as well as a conservative quantization of the input/output channels. The source of such conservatism may be the use of Lipschitz constants of many involved functions as well as lower and upper bounds of Lyapunov–Krasovskii functionals and derivatives. On the other hand, the results provided in Theorem 4 are of the existence type, and the study of the conservativeness of the sampling frequency as well as of the quantization in the input/output channels is beyond the aim of this work, and is left for future investigations. We highlight here that, to our best knowledge, it is the first time in the literature of nonlinear systems with state-delays that results concerning the design of quantized sampled-data observer-based tracking controllers ensuring the semi-global practical stability property of the related closed-loop system are provided.

6 | APPLICATION TO A CLASS OF NONLINEAR TIME-DELAY SYSTEMS

In this section, we will show a class of nonlinear systems with state-delays for which all the assumptions needed to apply the main results of the article (i.e., Theorems 3 and 4) are satisfied. Let us consider a nonlinear time-delay system described by

$$\begin{aligned}
&\dot{x}_1(t) = p_1 x_1(t) + p_2 x_2(t) + p_3,\\
&\dot{x}_2(t) = p_4 x_1(t) + p_5 x_2(t) + f_1(x_{1,t}) + g_1(x_1(t))u(t),\\
&y_t(\tau) = x_{1,t}(\tau), \quad x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0],
\end{aligned} \tag{82}$$

where: $x_1(t), x_2(t) \in \mathbb{R}$; Δ is the maximum involved time delay; $x_t \in \mathcal{C}^2$, $x_{1,t}$, $x_{2,t} \in \mathcal{C}$; $u(t) \in \mathbb{R}$ is the input signal; $y_t \in \mathcal{C}$ is the output signal; $p_i \in \mathbb{R}$, $i = 1, \dots, 5$ are the involved parameters with $p_2 \neq 0$; $g_1 : \mathbb{R} \to \mathbb{R}$ is a smooth function such that, for any $x \in \mathbb{R}$, $g_{\min} \leq g_1(x) \leq g_{\max}$, with $g_{\min}$ and $g_{\max}$ given positive reals; $f_1 : \mathcal{C} \to \mathbb{R}$ is a function Lipschitz on bounded subsets of $\mathcal{C}$. Notice that, the class of systems in (82) includes many mathematical models describing, for instance, neural networks systems[26] (see Section 6.1) and, as a special case, a delay-free actuated inverted pendulum[27] (see Section 6.2).

Chosen a desired reference signal $y_d(t) \in \mathbb{R}$ as in Problem 1 and taking into account (36)–(39), in this case

$$
\begin{aligned}
\dot{z}_1(t) &= z_2(t),\\
\dot{z}_2(t) &= F(z_t, r_t) + p_2 g_1(z_1(t) + y_d(t))u(t) - y_d^{(2)}(t),\\
y_{z,t}(\tau) &= z_{1,t}(\tau), z(\tau) = z_0(\tau), \qquad \tau \in [-\Delta, 0],
\end{aligned} \tag{83}
$$

where: $z_1(t), z_2(t) \in \mathbb{R}$; $z_t \in \mathcal{C}^2$; $u(t) \in \mathbb{R}$ is the input signal in (82); $y_{z,t} \in \mathcal{C}$ is the output signal deriving from the output signal $y(t)$ in (82); $r_t = \begin{bmatrix} y_{d,t} \\ y_{d,t}^{(1)} \end{bmatrix} \in \mathcal{C}^2$; $F : \mathcal{C}^2 \times \mathcal{C}^2 \to \mathbb{R}$ is the function defined, for $\phi_z = \begin{bmatrix} \phi_{z_1} \\ \phi_{z_2} \end{bmatrix} \in \mathcal{C}^2$, $\phi_{z_1}, \ \phi_{z_2} \in \mathcal{C}$ and $\phi_r = \begin{bmatrix} \phi_{r_1} \\ \phi_{r_2} \end{bmatrix} \in \mathcal{C}^2$, $\phi_{r_1}, \ \phi_{r_2} \in \mathcal{C}$, as follows

$$
F(\phi_z, \phi_r) = (p_1 + p_5)(\phi_{z_2}(0) + \phi_{r_2}(0)) + (p_2 p_4 - p_1 p_5)(\phi_{z_1}(0) + \phi_{r_1}(0)) - p_3 p_5 + p_2 f_1(\phi_{z_1} + \phi_{r_1}). \tag{84}
$$

Notice that all the assumptions in Theorem 3 are satisfied for the system (82). From (48), the continuous-time observer-based tracking controller is here described by

$$
\begin{aligned}
\dot{\hat{z}}_1(t) &= \hat{z}_2(t) - K_1(\hat{z}_1(t) - z_1(t)),\\
\dot{\hat{z}}_2(t) &= F(\tilde{z}_t, r_t) + p_2 g(z_1(t) + y_d(t))u(t) - y_d^{(2)}(t) - K_2(\hat{z}_1(t) - z_1(t)),\\
u(t) &= \frac{y_d^{(2)}(t) - F(\tilde{z}_t, r_t) + \Gamma_1 \hat{z}_1(t) + \Gamma_2 \hat{z}_2(t)}{p_2 g(z_1(t) + y_d(t))},\\
\hat{z}(\tau) &= \hat{z}_0(\tau), \qquad \tau \in [-\Delta, 0],
\end{aligned} \tag{85}
$$

where: $\hat{z}_1(t), \hat{z}_2(t) \in \mathbb{R}$; $\hat{z}_t \in \mathcal{C}^2$; $u(t) \in \mathbb{R}$ is the input signal in (82) (see also (83)); K_i, Γ_i are the control tuning parameters (see (48) and Theorem 3); $\tilde{z}_t(\tau) = \begin{bmatrix} z_1(t+\tau) \\ \hat{z}_2(t+\tau) \end{bmatrix}$, $\tau \in [-\Delta, 0]$. In order to apply the results stated in Theorem 4, we have to check that Assumption 3 is satisfied for the example under exam. We first notice that, from (83), (85), in this case, $F_z : \mathcal{C}^4 \times \mathcal{C}^3 \times \mathbb{R}^3 \to \mathbb{R}^4$ and $k : \mathcal{C}^2 \times \mathcal{C}^3 \times \mathcal{C} \to \mathbb{R}^3$ are the functions defined, for any $\phi = \begin{bmatrix} \phi_z \\ \phi_{\hat{z}} \end{bmatrix} \in \mathcal{C}^4$, $\phi_z = \begin{bmatrix} \phi_{z_1} \\ \phi_{z_2} \end{bmatrix} \in \mathcal{C}^2$, $\phi_{\hat{z}} = \begin{bmatrix} \phi_{\hat{z}_1} \\ \phi_{\hat{z}_2} \end{bmatrix} \in \mathcal{C}^2$, $\phi_{z_i}, \ \phi_{\hat{z}_i} \in \mathcal{C}$, $i = 1, 2$, $\phi_{\tilde{r}} = \begin{bmatrix} \phi_r \\ \phi_{y_d^{(2)}} \end{bmatrix} \in \mathcal{C}^3$, $\phi_{\tilde{z}} = \begin{bmatrix} \phi_{z_1} \\ \phi_{\hat{z}_2} \end{bmatrix} = \begin{bmatrix} H_z(\phi_z) \\ \phi_{\hat{z}_2} \end{bmatrix} \in \mathcal{C}^2$, $\phi_r = \begin{bmatrix} \phi_{y_d} \\ \phi_{y_d^{(1)}} \end{bmatrix}$, $\phi_{y_d}, \ \phi_{y_d^{(1)}}, \ \phi_{y_d^{(2)}} \in \mathcal{C}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^3$, $v_1 \in \mathbb{R}$, $v_2 \in \mathbb{R}^2$, as (see (52) and (56))

$$
F_z(\phi, \phi_{\tilde{r}}, v) = \begin{bmatrix} F(\phi_z, \phi_r) + p_2 g(\phi_{z_1} + \phi_{y_d}) v_1 - \phi_{y_d^{(2)}}(0) \\ \\ v_2 \end{bmatrix},
$$

$$
k(\phi_{\hat{z}}, \phi_{\tilde{r}}, H_z(\phi_z)) = \begin{bmatrix} \dfrac{-F(\phi_{\tilde{z}}, \phi_r) + \phi_{y_d^{(2)}}(0) + \Gamma_1 \phi_{\hat{z}_1}(0) + \Gamma_2 \phi_{\hat{z}_2}(0)}{p_2 g(\phi_{z_1} + \phi_{y_d})} \\ \\ \phi_{\hat{z}_2}(0) - K_1(\phi_{\hat{z}_1}(0) - \phi_{z_1}(0)) \\ \\ \Gamma_1 \phi_{\hat{z}_1}(0) + \Gamma_2 \phi_{\hat{z}_2}(0) - K_2(\phi_{\hat{z}_1}(0) - \phi_{z_1}(0)) \end{bmatrix}. \tag{86}
$$

Then, taking into account (86), for any $\phi = \begin{bmatrix} \phi_z \\ \phi_{\hat{z}} \end{bmatrix} \in C^4$, $\phi_z = \begin{bmatrix} \phi_{z_1} \\ \phi_{z_2} \end{bmatrix} \in C^2$, $\phi_{\hat{z}} = \begin{bmatrix} \phi_{\hat{z}_1} \\ \phi_{\hat{z}_2} \end{bmatrix} \in C^2$, ϕ_{z_i}, $\phi_{\hat{z}_i} \in C$, $i = 1, 2$ and $\phi_{\tilde{r}} \in C^3$,

$$F_z(\phi, \phi_{\tilde{r}}, k(\phi_{\hat{z}}, \phi_{\tilde{r}}, H_z(\phi_z))) = A\phi(0), \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & (p_1 + p_5) & \Gamma_1 & \Gamma_2 - (p_1 + p_5) \\ K_1 & 0 & -K_1 & 1 \\ K_2 & 0 & \Gamma_1 - K_2 & \Gamma_2 \end{bmatrix}.$$

Notice that, from Theorem 3, A is an Hurwitz matrix. In the delay-free case (i.e., $\Delta = 0$), Assumption 3 holds for this example. Indeed, taking into account (84), condition (58) holds and, consequently, by applying Proposition 1, Assumption 3 follows. In the case $\Delta > 0$, Assumption 3 is here satisfied by choosing, for instance:

(i) the Lyapunov–Krasovskii functional $V : C^4 \to \mathbb{R}^+$, defined, for $\phi \in C^4$, as $V(\phi) = V_1(\phi(0)) + V_2(\phi)$, where, $V_1 : \mathbb{R}^4 \to \mathbb{R}^+$ is defined, for $\chi \in \mathbb{R}^4$, as $V_1(\chi) = \chi^T P \chi$ with P the symmetric positive definite matrix satisfying $A^T P + PA = -0.5 I_4$ and $V_2 : C^4 \to \mathbb{R}^+$ is defined, for $\phi \in C^4$, as $V_2(\phi) = 0$;
(ii) functions β_i, γ_i of class $\mathcal{K}_\infty$, $i = 1, 2$, defined, for $s \in \mathbb{R}^+$, as $\beta_1(s) = \gamma_1(s) = \lambda_{\min}(P)s^2$ and $\beta_2(s) = \gamma_2(s) = \lambda_{\max}(P)s^2$;
(iii) $\nu = 1, \eta = 1, \mu \leq \dfrac{1}{2\lambda_{\max}(P)}$;
(iv) $p = \overline{I}_d$ and $\overline{\alpha} = 0$.

It follows that all the assumptions required to apply Theorem 4 hold for the case under study. In the following, numerical examples are provided concerning: (i) a particular class of neural network systems with various activation functions and time-delays[26]; (ii) a class of time-delay systems including, as a special case, a delay-free actuated inverted pendulum.[27]

6.1 | Example 1

Inspired by the mathematical models describing neural networks systems with various activation functions and time-delays,[26] let us consider the following nonlinear time-delay system

$$\begin{aligned} \dot{x}_1(t) &= -1.2x_1(t) + x_2(t), \\ \dot{x}_2(t) &= 0.39x_2(t) + \operatorname{sech}(x_1(t - \Delta_1)) + 0.2\int_{t-\Delta_2}^{t} \arctan(x_1(\tau))d\tau + 4u(t), \\ y_t(\tau) &= x_{1,t}(\tau), \quad x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0], \end{aligned} \tag{87}$$

where: $x_1(t), x_2(t) \in \mathbb{R}$; $\Delta_1 = 1$, $\Delta_2 = 2$ are the involved time delays; $x_t \in C^2$, $x_{1,t}$, $x_{2,t} \in C$; $u(t) \in \mathbb{R}$ is the input signal; $y_t \in C$ is the output signal. Notice that, system (87) is in the form (83) and, consequently, we can apply Theorems 3 and 4. Taking into account Remark 7 by choosing, for instance, the controller parameters equal to $\Gamma_1 = -6$, $\Gamma_2 = -5$, $K_1 = 3$, $K_2 = 3$ and $\tilde{q} = 1$, $r = 1$, $R = 2$ and $a = 1$, by the use of steps (1)–(15) we obtain: $\delta = 4.76 \times 10^{-6}$, $E_y = 20.25$, $U_1 = 63.39$, $\mu_y = 2.8 \times 10^{-6}$ and $\mu_u = 5.2 \times 10^{-6}$. Taking into account Remark 7, as expected, the use of steps (1)–(15) provides conservative upper bounds for the sampling period (i.e., $\delta = 4.76 \times 10^{-6}$) and for the quantization errors (i.e., $\mu_y = 2.8 \times 10^{-6}$ and $\mu_u = 5.2 \times 10^{-6}$). On the other hand, taking into account Remark 7, a campaign of simulations has been performed by choosing sampling periods and quantization errors greater than the ones obtained by the use of steps (1)–(15) and good performances of the proposed digital tracking controller have been observed for sampling periods δ equal to 10^{-6}, 10^{-3}, 10^{-1}, 0.2, output quantization errors μ_y equal to 2×10^{-6}, 10^{-3}, 10^{-2} and input quantization errors μ_u equal to 5×10^{-6}, 10^{-2}, 10^{-1}. Figure 1 shows the simulations results for both cases of continuous-time and digital controller in which the following choices have been performed: $y_d(t) = 0.1\sin(2t)$, $t \in [-\Delta, \infty)$; the initial states of the system and of

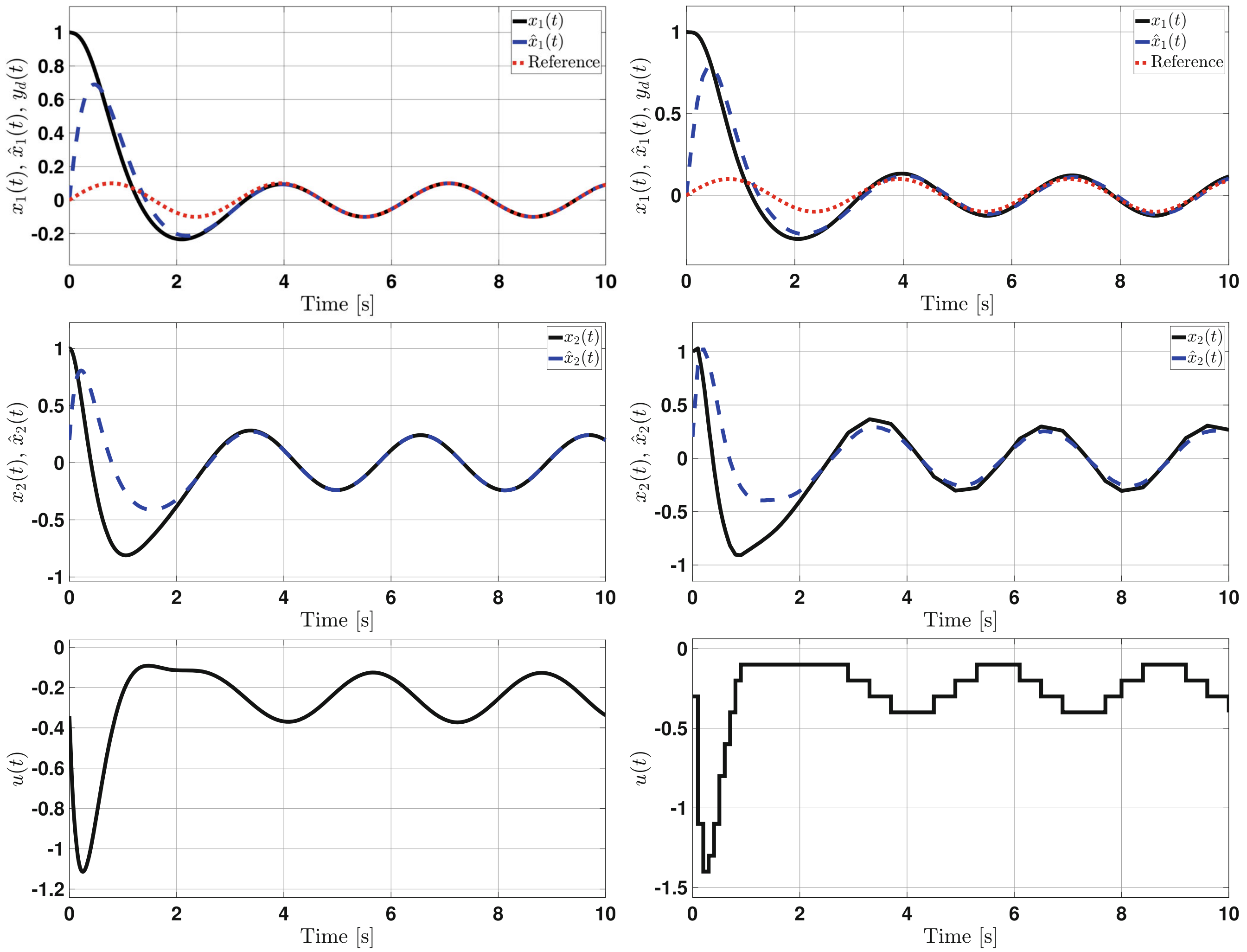

FIGURE 1 In the left column, the simulation results for the case of continuous-time controller are reported: (a) in the first two panels, the system state variables $x_1(t), x_2(t)$, the controller state variables $\hat{x}_1(t), \hat{x}_2(t)$ and the reference signal $y_d(t)$ are reported with continuous black lines, dashed blue lines and dashed red line, respectively; (b) in the third panel the control input signal is reported. In the left right column, the simulation results for the case of digital controller are reported: (a) in the first two panels, the system state variables $x_1(t), x_2(t)$, the controller state variables $\hat{x}_1(t), \hat{x}_2(t)$ and the reference signal $y_d(t)$ are reported with continuous black lines, dashed blue lines and dashed red line, respectively; (b) in the third panel the control input signal is reported.

the observer-based tracking controller equal to $\begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\hat{z}_0(\tau) = \begin{bmatrix} \hat{z}_1(\tau) \\ \hat{z}_2(\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\tau \in [-\Delta, 0]$; an output quantizer and an input quantizer with

$$Q_y = \{y \in \mathbb{R}|\ y_i = \pm 0.01\overline{k},\ i = 1, 2,\ \overline{k} = 0, 1, \dots, 10^3\};$$

$$Q_u = \{u \in \mathbb{R}|\ u = \pm 0.1\overline{k},\ \overline{k} = 0, 1, \dots, 10^2\};$$

an uniform sampling period $\delta = 0.1$[s]. In the left column of Figure 1, the system variables $x_1(t)$, $x_2(t)$, the controller variables $\hat{x}_1(t) = \hat{z}_1(t) + y_d(t)$, $\hat{x}_2(t) = \hat{z}_2(t) + 1 + y_d^{(1)}(t)$ and the control input signal $u(t)$ are plotted in the case of continuous-time controller (i.e., (87)–(85)). Simulations fully validate the theoretical results stated in Theorem 3. In the right column of Figure 1, the system variables $x_1(t)$, $x_2(t)$, the controller variables $\hat{x}_1(t) = \hat{z}_1(t) + y_d(t)$, $\hat{x}_2(t) = \hat{z}_2(t) + 1.2(\hat{z}_1(t) + y_d(t)) + y_d^{(1)}(t)$ (linear interpolations of discrete-time available values $\hat{x}_1(j\delta) = \hat{z}_1(j\delta) + y_d(j\delta)$, $\hat{x}_2(j\delta) =$

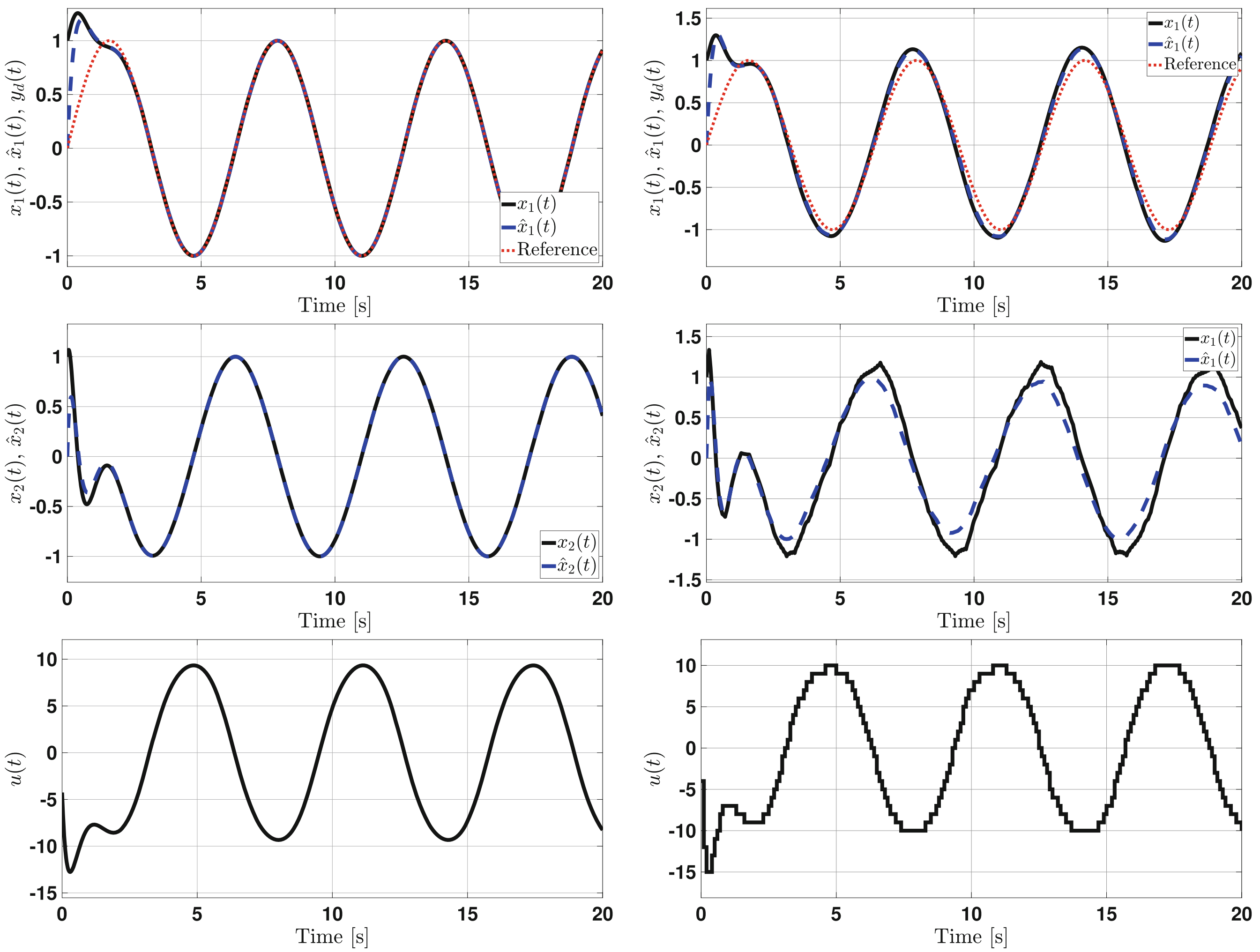

FIGURE 2 In the left column, the simulation results for the case of continuous-time controller are reported: (a) in the first two panels, the system state variables $x_1(t), x_2(t)$, the controller state variables $\hat{x}_1(t), \hat{x}_2(t)$ and the reference signal $y_d(t)$ are reported with continuous black lines, dashed blue lines and dashed red line, respectively; (b) in the third panel the control input signal is reported. In the left right column, the simulation results for the case of digital controller are reported: (a) in the first two panels, the system state variables $x_1(t), x_2(t)$, the controller state variables $\hat{x}_1(t), \hat{x}_2(t)$ and the reference signal $y_d(t)$ are reported with continuous black lines, dashed blue lines and dashed red line, respectively; (b) in the third panel the control input signal is reported.

$\hat{z}_2(j\delta) + 1 + y_d^{(1)}(j\delta)$, $j = 0, 1, \ldots$) and the control input signal $u(t)$ are plotted in the case of digital controller. In particular, the right column of Figure 1 shows the simulation results concerning the closed-loop system described by (87) with the proposed digital observer-based tracking controller (62) where, for the example under exam, according to (56), the function k is described by (86). Simulations fully validate the theoretical results stated in Theorem 4.

6.2 | Example 2

Let us consider the nonlinear time-delay system described by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= \gamma_1 \sin(x_1(t)) + \gamma_2 x_2(t) + \sigma x_1^2(t-\Delta) + \gamma_3 u(t), \\ y_t(\tau) &= x_{1,t}(\tau), \quad x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0], \end{aligned} \tag{88}$$

where: $x_1(t), x_2(t) \in \mathbb{R}$; $x_t \in C^2$, $x_{1,t},\ x_{2,t} \in C$; $\Delta = 0.5$ is the involved time delay; $\sigma, \gamma_i \in \mathbb{R}$ are the involved parameters; $u(t) \in \mathbb{R}$ is the input signal; $y_t \in C$ is the output signal. Notice that, system (88) is in the form (83) and, consequently, we can apply Theorems 3 and 4. We highlight also that, in the special case $\sigma = 0$, (88) describes the delay-free actuated inverted pendulum considered in the work of Khalil[27] where: $x_1(t)$ is the angular position and $x_2(t)$ is the angular velocity of the pendulum; $\gamma_1 = \frac{g}{l}$, $\gamma_2 = -\frac{K_0}{ml^2}$, and $\gamma_3 = \frac{1}{ml^2}$. In the following, the model parameters have been taken equal to the ones of the inverted pendulum considered in the work of Khalil[27] with $\sigma = \frac{g}{l}$ and $l = 1, m = 0.1, K_0 = 0.2, g = 9.8$. Taking into account Remark 7 by choosing, for instance, the controller parameters equal to $\Gamma_1 = -6, \Gamma_2 = -5, K_1 = 3, K_2 = 3$ and $\tilde{q} = 1$, $r = 1$, $R = 2$ and $a = 1$, by the use of steps (1)–(15) we obtain: $\delta = 1.12 \times 10^{-5}$, $E_y = 9.29$, $U_1 = 32.2$, $\mu_y = 9.35 \times 10^{-6}$, and $\mu_u = 9.85 \times 10^{-6}$. Taking into account Remark 7, as expected, the use of steps (1)–(15) provides conservative upper bounds for the sampling period (i.e., $\delta = 1.12 \times 10^{-5}$) and for the quantization errors (i.e., $\mu_y = 9.35 \times 10^{-6}$ and $\mu_u = 9.85 \times 10^{-6}$). On the other hand, taking into account Remark 7, a campaign of simulations has been performed by choosing sampling periods and quantization errors greater than the ones obtained by the use of steps (1)–(15) and good performances of the proposed digital tracking controller have been observed for sampling periods δ equal to 10^{-5}, 10^{-2}, 10^{-1}, 0.2, output quantization errors μ_y equal to 9×10^{-6}, 10^{-2}, 10^{-1} and input quantization errors μ_u equal to 9×10^{-6}, 10^{-1}, 1. Figure 2 shows the simulations results for both cases of continuous-time and digital controller in which the following choices have been performed: $y_d(t) = \sin(t)$; the initial states of the system and of the observer-based tracking controller equal to $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} \hat{z}_1(0) \\ \hat{z}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, an output quantizer and an input quantizer with

$$Q_y = \{y \in \mathbb{R} |\ y = \pm 0.01\overline{k},\ i = 1, 2,\ \overline{k} = 0, 1, \dots, 10^3\};$$

$$Q_u = \{u \in \mathbb{R} |\ u = \pm\overline{k},\ \overline{k} = 0, 1, \dots, 10^2\};$$

a uniform sampling period $\delta = 0.1[\mathrm{s}]$. In the left column of Figure 2, the system variables $x_1(t), x_2(t)$, the controller variables $\hat{x}_1(t) = \hat{z}_1(t) + y_d(t), \hat{x}_2(t) = \hat{z}_2(t) + y_d^{(1)}(t)$ and the control input signal $u(t)$ are plotted in the case of continuous-time controller (i.e., (88)–(85)). Simulations fully validate the theoretical results stated in Theorem 3. In the right column of Figure 2, the system variables $x_1(t), x_2(t)$, the controller variables $\hat{x}_1(t) = \hat{z}_1(t) + y_d(t), \hat{x}_2(t) = \hat{z}_2(t) + y_d^{(1)}(t)$ (linear interpolations of discrete-time available values $\hat{x}_1(j\delta) = \hat{z}_1(j\delta) + y_d(j\delta), \hat{x}_2(j\delta) = \hat{z}_2(j\delta) + y_d^{(1)}(j\delta), j = 0, 1, \dots$) and the control input signal $u(t)$ are plotted in the case of digital controller. In particular, the right column of Figure 2 shows the simulation results concerning the closed-loop system described by (88) with the proposed digital observer-based tracking controller (62) where, for the example under exam, according to (56), the function k is described by (86). Simulations fully validate the theoretical results stated in Theorem 4.

7 | CONCLUSION

In this article, the tracking control problem for a class of nonlinear time-delay systems has been studied. In particular: (i) a procedure for the design of continuous-time observer-based tracking controllers ensuring the GAS of the corresponding closed-loop tracking error system has been provided for a class of control-affine nonlinear systems with state delays; (ii) sufficient conditions have been provided for the existence of a suitably fast sampling and of an accurate quantization of the input/output channels such that the digital implementation of the proposed continuous-time observer-based tracking controller ensures the semi-global practical stability property of the related quantized sampled-data closed-loop tracking error system, with arbitrarily small final target ball of the origin. The result (i) has been proved by exploiting a geometric approach and the notion of the ISS. On the other hand, a Lyapunov–Krasovskii approach and, in particular, the stabilization in the sample-and-hold sense theory has been used as a tool to prove the result (ii). Moreover, by exploiting the converse Lyapunov theorems, it has been shown that, in the special case of delay-free nonlinear systems, the sufficient conditions provided for the digital implementation of the proposed continuous-time observer-based tracking controller can be strongly relaxed. In the theory here developed, time-varying sampling periods and the nonuniform quantization of the input/output channels have been taken into account. The proposed results have been validated through examples concerning a class of neural networks systems and a class of time-delay systems including, as a special case, a delay-free actuated inverted pendulum.

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CONFLICT OF INTEREST STATEMENT

None of the authors have any conflicts of interest.

DATA AVAILABILITY STATEMENT

Research data are not shared.

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