Limited-Information Event-Triggered Observer-Based Control of Nonlinear Systems

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***Abstract*—In this note, we propose a framework for the event-based semiglobal practical stabilization of nonlinear continuous-time systems based on limited amount of information. Following an emulation approach, it is required the availability of a convergent state observer and of a global asymptotic stabilizer in continuous time, generally depending on the estimated state and on the measured output. We then introduce time sampling, quantizations on the input, observed state and output signals, and a periodic event-triggered mechanism to update the control law only when necessary. The proposed methodology guarantees practical stabilization of the system with state convergence to an arbitrarily small neighborhood of the origin by using finite data and a reduced number of control updates, which is desirable in digital and networked contexts with limited bandwidth. Numerical simulations on an actuated inverted pendulum show the potential of the approach.**

***Index Terms*—Event-triggered control, nonlinear systems, quantization, sampled-data stabilization.**

I. Introduction

Event-triggered control is a recent popular framework for the control of dynamical systems, which has been proved to be successful in properly managing shared computation and communication resources in the digital world [1], [2]. The idea is to update the control law whenever the system really needs attention, by avoiding input modifications unless they are necessary. While early event-based approaches needed continuous-time monitoring of the state or output variables, the more recent concept of periodic event-triggered control (PETC) [3], [4] relaxed this assumption by allowing only measurements on a discrete subset of time instants, where strict periodicity of the sampling sequence can be relaxed.

A large part of event-triggered control literature is devoted to the case of state feedback, see, e.g., [5], [6], [7] for the case of nonlinear systems or [8], [9], [10], [11], [12] for the case of nonlinear time-delay systems. If only the output signal is available for control, some papers explicitly assume the availability of a state estimation (see, e.g., [13], [14], [15], [16], [17], [18]), while in other works this is not necessary (see, e.g., [19], [20], [21], [22], [23]). Some of these papers consider sampling-induced errors but, to the best of our knowledge, only very few formally address the quantization nonideality in the output-feedback event-triggered design (see, e.g., [22], which assumes linear dynamics, recently extended to the nonlinear case in [23] in a hybrid system formalism).

In this article, we pursue the observer-based approach to event-triggered control from outputs and consider this problem in the framework of the stabilization in the sample-and-hold sense [24], [25], [26], following an *emulation approach* [27], in which a controller is first designed in continuous time ignoring network and/or digital nonidealities and is later modified to take them into account, preserving to some extent the properties of the ideal closed-loop system.

In more detail, by building, in part, on previous results on this topic, we here propose a finite-information (sampled and quantized) feedback, assuming the availability of an asymptotically convergent state observer (for example designed by means of Luenberger-like [28] or high-gain [29] methods) and of a global stabilizer in continuous time, depending on the system output and on the observed state. Such a pure continuous-time setting is embedded into a digital framework, accounting for time sampling and quantization of input, state and output, so that the communication among the different components of the control loop is characterized by exchange of limited information [30] over a finite bandwidth, a problem tackled in the different framework of networked control systems, e.g., in [31]. Fig. 1 illustrates the control scheme considered in this article.

With respect to classical nonlinear state-feedback design, the proposed solution considers at once four major nonidealities, which, to the best of our knowledge, have not been addressed in a unified setting: sampled-data control, quantization on the input, output and observer state, event-triggered mechanism, and partial information (output measurements). A subset of these features have been separately addressed in previous works [32], [33], [34]. Furthermore, differently from [8], [11], considering the more general case of systems with state delays, the event-based control is here presented in absence of full state information.

The rest of this article is organized as follows. Section II recalls some notation and basic notions. Section III sets up the model and observer formulation and assumptions. Section IV includes the main result of the article, regarding the stabilization in the sample-and-hold sense of nonlinear systems with quantization, following an observer-based event-triggered approach. Section V shows an illustrative example of the developed method. Finally, Section VI concludes this article.

II. Notation and Preliminaries

$\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of nonnegative reals $[0, +\infty)$, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{Z}^+$ denotes the set of nonnegative integer numbers. $\mathbb{R}^{m\times n}$ denotes the set of real-valued matrixes with m rows and n columns. $\mathcal{I}_n$ is the identity

Manuscript received 18 May 2023; accepted 14 October 2023. Date of publication 26 October 2023; date of current version 29 February 2024. This work was supported by the Center of Excellence DEWS. Recommended by Associate Editor G. Besancon. *(Corresponding author: Alessandro Borri.)*

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Digital Object Identifier 10.1109/TAC.2023.3327938

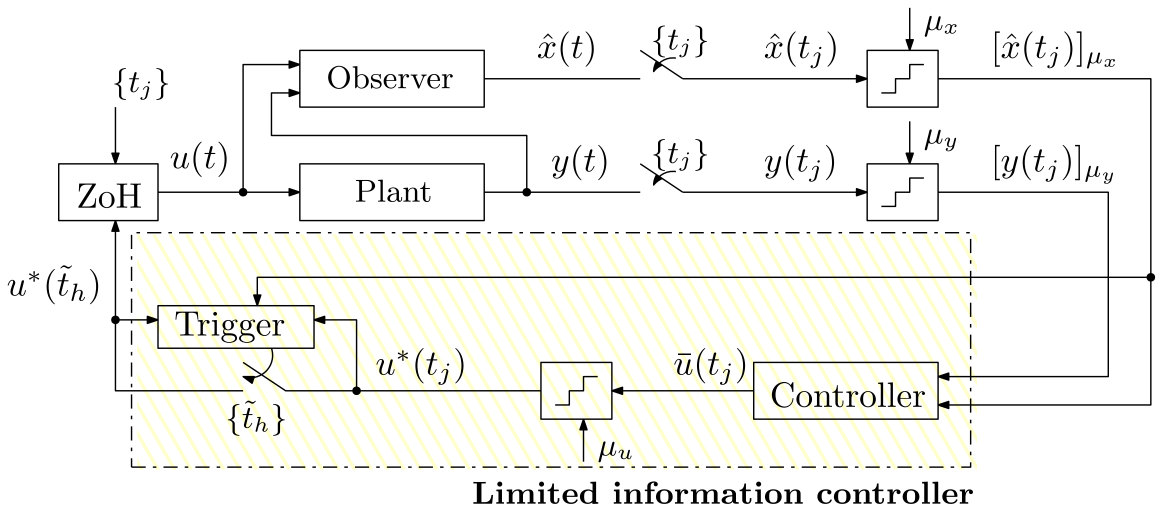

Fig. 1. Scheme of event-triggered observer-based control with limited information.

matrix in $\mathbb{R}^{n\times n}$. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix.

For a given positive integer n and a given positive real h, the symbol $\mathcal{B}_h^n$ denotes the subset $\{x \in \mathbb{R}^n :\ |x| \le h\}$. Let us here recall that a continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is: of class $\mathcal{P}_0$ if $\gamma(0) = 0$; of class $\mathcal{P}$ if it is of class $\mathcal{P}_0$ and $\gamma(s) > 0$, $s > 0$; of class $\mathcal{K}$ if it is of class $\mathcal{P}$ and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded; of class $\mathcal{L}$ if it is continuous and it monotonically decreases to zero as its argument tends to $+\infty$. A continuous function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \ge 0$ and $\beta(s, \cdot)$ is of class $\mathcal{L}$ for each $s \ge 0$. For a nonnegative real s, $[s]$ is the largest nonnegative integer smaller than or equal to s.

A continuous-time system is said to be forward complete (FC) if, for any initial condition and input signal, the corresponding solution is defined for all $t \ge 0$ [35]. Throughout the article, GAS stands for globally asymptotically stable or global asymptotic stability.

It is convenient to recall here the following direct and converse Lyapunov theorem (see [36, Th. 4.18] and [37]).

Theorem 1: The system described by

$$\dot{z}(t) = \bar{F}(z(t)) \tag{1}$$

with $\bar{F} : \mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{n}}$ locally Lipschitz and satisfying $\bar{F}(0) = 0$, is GAS if and only if there exists a smooth function $V : \mathbb{R}^{\bar{n}} \to \mathbb{R}^+$, functions α_1, α_2 of class $\mathcal{K}_\infty$ and α_3 of class $\mathcal{K}$, such that the following conditions hold for all $z \in \mathbb{R}^{\bar{n}}$:

$$\alpha_1(|z|) \le V(z) \le \alpha_2(|z|) \tag{2}$$

$$\frac{\partial V}{\partial z}\bar{F}(z) \le -\alpha_3(|z|). \tag{3}$$

III. Model and Observer Formulation and Assumptions

We consider a continuous-time system

$$\begin{cases} \dot{x}(t) &= f(x(t), u(t)), \qquad x(0) = x_0 \\ y(t) &= h(x(t)) \end{cases} \tag{4}$$

where $x_0,\ x(t) \in \mathbb{R}^n$; $u(t) \in \mathbb{R}^m$ is the input signal, assumed piecewise-continuous; $y(t) \in \mathbb{R}^q$ is the output signal; n, m, and q are positive integers; f is a locally Lipschitz function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$; h is a locally Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}^q$. It is assumed that $f(0, 0) = 0$ and $h(0) = 0$. We assume that system (4) is FC.

We assume the availability of a continuous-time observer (see, e.g., [28], [29]) in the form

$$\dot{\hat{x}}(t) = \hat{f}(\hat{x}(t), h(x(t)), u(t)), \qquad \hat{x}(0) = \hat{x}_0 \tag{5}$$

where $\hat{f} : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz and satisfies $\hat{f}(0, 0, 0) = 0$, $\hat{x}_0 \in \mathbb{R}^n$. We also assume for (5) (the term $h(x(t))$ can be considered here as another input) the FC property and the existence of a function β of class $\mathcal{KL}$ such that the observation error $e(t) := x(t) - \hat{x}(t)$ satisfies the inequality

$$|e(t)| \le \beta(|e(0)|, t) \qquad \forall t \ge 0 \tag{6}$$

independently of the piecewise-continuous input function $u(\cdot)$ (see, e.g., [38]).

Define the extended state

$$z = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}^T := \begin{bmatrix} \hat{x}^T & e^T \end{bmatrix}^T \in \mathbb{R}^{2n}$$

such that system (4)–(5) can be rewritten in the form

$$\dot{z}(t) = F(z(t), u(t)) \tag{7}$$

for a proper choice of F.

We introduce here the following assumption, which ensures the existence of a continuous-time observed-based control law stabilizing the closed-loop system.

Assumption 1: There exists a locally Lipschitz feedback $\bar{k} : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^m$, which is a function of the observed state $\hat{x}(t)$ and of the output $y(t)$, such that the closed-loop system (7), with the observer-based control law

$$u(t) = \bar{k}(\hat{x}(t), y(t)) = \bar{k}(z_1(t), h(z_1(t) + z_2(t)))$$

is GAS.

For the sake of a more compact notation, we define the locally Lipschitz function $k : \mathbb{R}^{2n} \to \mathbb{R}^m$ as

$$k(z) = \bar{k}(z_1, h(z_1 + z_2)) \qquad \forall z = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}^T \in \mathbb{R}^{2n}.$$

Note that, by virtue of Assumption 1, Theorem 1 holds for the closed-loop system $\dot{z}(t) = \bar{F}(z(t)) := F(z(t), k(z(t)))$, implying from (3)

$$\alpha_1(|z|) \le V(z) \le \alpha_2(|z|) \tag{8}$$

$$\frac{\partial V}{\partial z}F(z, k(z)) \le -\alpha_3(|z|) \tag{9}$$

for any $z \in \mathbb{R}^{2n}$, for some class-$\mathcal{K}_\infty$ functions α_1 and α_2, for some class-$\mathcal{K}$ function α_3, and for some smooth Lyapunov function $V : \mathbb{R}^{\bar{n}} \to \mathbb{R}^+$, with $\bar{n} = 2n$.

We also define, for simplicity of notation, for all $z \in \mathbb{R}^{2n}$, $u \in \mathbb{R}^m$, the quantity

$$D^+V(z,u) := \frac{\partial V(z)}{\partial z} F(z,u) \tag{10}$$

denoting the directional derivative $D^+V : \mathbb{R}^{2n} \times \mathbb{R}^m \to \mathbb{R}$ of the function V along the augmented dynamics F. Hence (9) can be rewritten as

$$D^+V(z,k(z)) \le -\alpha_3(|z|)$$

for any $z \in \mathbb{R}^{2n}$.

IV. Sample-and-Hold Quantized Observer-Based Event-Triggered Stabilization

We recall here the notion of partition of $[0,+\infty)$ (see [8] and [24]).

Definition 1: A partition $\pi = \{t_i\}_{i\in\mathbb{Z}^+}$ of $[0,+\infty)$ is a countable, strictly increasing sequence t_i, with $t_0 = 0$, such that $t_i \to +\infty$ as $i \to +\infty$. The diameter of π, denoted diam(π), is defined as $\sup_{i\ge 0}(t_{i+1}-t_i)$. The dwell time of π, denoted dwell(π), is defined as $\inf_{i\ge 0}(t_{i+1}-t_i)$. For any positive real $a \in (0,1]$, $\delta > 0$, $\pi_{a,\delta}$ is any partition π with $a\delta \le \text{dwell}(\pi) \le \text{diam}(\pi) \le \delta$.

The real $a \in (0,1]$, in Definition 1, is introduced in order to allow for nonuniform sampling (see [8] and [11]).

We now define (possibly nonuniform) input, state, and output quantizer operators as

$$[\cdot]_{\mu_u} : \mathbb{R}^m \to \mathcal{Q}_u$$
$$[\cdot]_{\mu_x} : \mathbb{R}^n \to \mathcal{Q}_x$$
$$[\cdot]_{\mu_y} : \mathbb{R}^q \to \mathcal{Q}_y$$

where $\mathcal{Q}_u$, $\mathcal{Q}_x$, and $\mathcal{Q}_y$ are suitable bounded subsets of $\mathbb{R}^m$, $\mathbb{R}^n$, and $\mathbb{R}^q$, respectively. These quantizers are characterized by the following implications (see [11], [12]):

$$|u| \le E_u \Longrightarrow |u - [u]_{\mu_u}| \le \mu_u$$
$$|x| \le E_x \Longrightarrow |x - [x]_{\mu_x}| \le \mu_x$$
$$|y| \le E_y \Longrightarrow |y - [y]_{\mu_y}| \le \mu_y$$

for some positive reals E_u, E_x, E_y and μ_u, μ_x, μ_y, called ranges and error bounds of the quantizers, respectively (see [30] and [34]). Strictly positive error bounds result in finite codomains of the quantizer functions. On the other hand, the cases $\mu_u = 0$, $\mu_x = 0$, or $\mu_y = 0$ are degenerate conditions in which no quantization is applied to input, state, or output, respectively.

Before introducing the main result, we consider the following more compact notation for the sake of readability:

$$\bar{u}(t) = \bar{k}([\hat{x}(t)]_{\mu_x}, [y(t)]_{\mu_y}) \tag{11}$$

$$u^*(t) = [\bar{u}(t)]_{\mu_u}. \tag{12}$$

In the following, the quantized sampled-data event-based controller is presented. For given positive reals r, R, with $0 < r < R$, let E, $\bar{E}$, E_Y, $\bar{E}_Y$, E_U, $\bar{E}_U$ be positive reals such that

$$0 < r < R < E, \qquad \alpha_1(E) > \alpha_2(R) \tag{13}$$

$$\bar{E} = E + 1, \quad E_Y = \sup_{x\in\mathcal{B}^n_{2E}} |h(x)|, \quad \bar{E}_Y = E_Y + 1 \tag{14}$$

$$E_U = \sup_{\substack{\hat{x}\in\mathcal{B}^n_{\bar{E}}\\ y\in\mathcal{B}^q_{\bar{E}_Y}}} |\bar{k}(\hat{x},y)|, \qquad \bar{E}_U = E_U + 1 \tag{15}$$

where functions α_1 and α_2 are defined in (8).

Furthermore, for any $\sigma \in (0,1)$, and any partition $\pi_{a,\delta}$ with $a \in (0,1]$ and $\delta > 0$, define the event-based control law

$$u(t) = u^*(\tilde{t}_h), \quad \tilde{t}_h \le t < \tilde{t}_{h+1}, \quad h = 0,1,\ldots, \tag{16}$$

and the sequence $\{\tilde{t}_h\}_{h\in\mathbb{Z}^+}$ as

$$\tilde{t}_{h+1} = \min\Big\{ t > \tilde{t}_h \,|\, -D^+V(\xi(t), u^*(\tilde{t}_h)) + \sigma D^+V(\xi(t), u^*(t)) \le (1+\sigma)K\beta(\bar{e}_0, t), \quad t = t_j, \quad j = 0,1,\ldots \Big\} \tag{17}$$

where $\xi(t) := \Big[[\hat{x}^T(t)]_{\mu_x} \quad 0^T \Big]^T$, with $\bar{e}_0 \ge |z_2(0)|$ being an upper bound on the initial observation error, and where the map D^+V is defined in (10).

Fig. 1 shows an illustration of the closed-loop system described so far. In synthesis, the plant (4) evolves in continuous time with the control input function $u(\cdot)$ in (16) resulting from a piecewise-constant interpolation [provided by a zero-order-hold, (ZoH)] of the samples $u^*(\tilde{t}_h)$. These inputs are computed by sampling (11) and quantization (12) of the feedback function k (see Assumption 1), taking as inputs the quantized versions of the estimated state and plant output, sampled over the sequence $\{t_j\}$. Notice that the value of the event-based control input function $u(\cdot)$ in (16) is changed on a subsequence $\{\tilde{t}_h\}$ of the original sampling sequence $\{t_j\}$, determined from (17).

We here state the main result of the article.

Theorem 1: Consider the system (4), for which it is assumed the availability of the observer (5) with error bound (6), and for which a locally Lipschitz feedback exists such that the closed-loop system is GAS in continuous time, according to Assumption 1. Let $\sigma \in (0,1)$, $a \in (0,1]$. Then, for all $r, R \in \mathbb{R}^+$, $0 < r < R$, for any E, E_U, E_Y satisfying (13)–(15), there exist positive reals δ, T, μ_u, μ_x, μ_y such that: for any partition $\pi_{a,\delta} = \{t_j,\ j = 0,1,\ldots\}$ of $[0,+\infty)$, for any input, state and output quantizers with error bounds μ_u, μ_x, μ_y and ranges $E_u = E_U$, $E_x = E$, $E_y = E_Y$, respectively, for any $z_0 \in \mathcal{B}^{2n}_R$, the solution starting from z_0 with the sampled-data observer-based event-triggered control law (16)–(17), with L and K satisfying (20)–(21), exists $\forall t \ge 0$ and, furthermore, satisfies

$$|z(t)| \le E \ \forall t \ge 0; \qquad |z(t)| \le r \ \forall t \ge T. \tag{18}$$

Remark 1: We highlight that the control law in (16) is updated only when the condition

$$-D^+V(\xi(t), u^*(\tilde{t}_h)) + \sigma D^+V(\xi(t), u^*(t)) \le (1+\sigma)K\beta(\bar{e}_0, t) \tag{19}$$

holds, so (19) will be referred to as *triggering condition*. Note that this condition only depends on the quantized versions of estimated state $[\hat{x}(t)]_{\mu_x}$, currently actuated input $u^*(\tilde{t}_h)$, and most recently computed input $u^*(t)$; furthermore, the triggering condition is not checked continuously in time but just at times t_j, $j = 0,1,\ldots$, with a guaranteed minimum dwell-time $a\delta$ between two consecutive sampling instants (absence of Zeno behavior) [11]. Hence, no continuous-time monitoring of the state variable is needed, which is a distinguishing feature of PETC, see, e.g., [3], [4], [9], [22], and [23].

Remark 2: Note that the triggering condition (19) gets less conservative as time goes by, namely, it is more difficult to satisfy for increasing times (for any given value on the left-hand side), since the observation error goes to zero according to the function $\beta(\bar{e}_0, t)$. Furthermore, although the state variables are often unavailable at all times, the initial state of the plant can be known in some applications, for example in the control of mechanical systems, where the state usually evolves from resting conditions. In these cases, one can assume $\bar{e}_0 = 0$ and the triggering condition (19) is easier to fulfill with respect to the case of unknown initial state. Note also that the knowledge of the initial state does not allow to remove the observer algorithm. As a matter of fact, the state estimator exploits, at all times, a feedback correction based on the measured output which guarantees robustness with respect to possible uncertainties and unmodeled dynamics, even in the case of known initial state. On the other hand, in the ideal case without uncertainties, we can state that the estimation error is always zero if the initial state is known (see [28]). We finally highlight that our construction is different from the one proposed in [23], which does not include a state estimation but assumes additional conditions on the Lyapunov function of the system, see, e.g., [23, eq. (24)].

Proof: Taking into account Assumption 1 and Theorem 1, let $V : \mathbb{R}^{2n} \to \mathbb{R}^+$ be a smooth function, α_1 and α_2 be functions of class $\mathcal{K}_\infty$, and α_3 be a function of class $\mathcal{K}$, such that conditions (8)–(9) hold.

Let r and R be any positive reals, $0 < r < R$. Let $a \in (0, 1]$ be arbitrarily fixed. Let $z_0 = \begin{bmatrix} z_{10}^T & z_{20}^T \end{bmatrix}^T \in \mathcal{B}_R^{2n}$, let e_1 and e_2 be positive reals satisfying $e_2 < e_1 < r$ and $\alpha_1(r) > \alpha_2(e_1)$, and let E be a positive real satisfying the inequalities in (13)–(15), where the increased bounds $\bar{E}$, $\bar{E}_Y$, and $\bar{E}_U$ are defined to account for the further uncertainty involved in the quantization of observed state, output and input, respectively. In particular, it can be readily seen that $z = \begin{bmatrix} \hat{x}^T & e^T \end{bmatrix}^T \in \mathcal{B}_E^{2n}$ implies that $\hat{x} \in \mathcal{B}_E^n$ and $x = \hat{x} + e \in \mathcal{B}_{2E}^n$, implying in turn that $[\hat{x}]_{\mu_x} \in \mathcal{B}_{\bar{E}}^n$ and $[y]_{\mu_y} \in \mathcal{B}_{\bar{E}_Y}^q$, finally leading to $u^* \in \mathcal{B}_{\bar{E}_U}^m$ (see also [33]).

Taking into account the locally Lipschitz property of the involved functions and the smoothness of V, let L, K, and M be positive reals such that the following inequalities hold:

$$\left|\bar{k}(\hat{x}_1, y_1) - \bar{k}(\hat{x}_2, y_2)\right| \leq L(|\hat{x}_1 - \hat{x}_2| + |y_1 - y_2|) \tag{20}$$

$$|D^+V(z_1, u_1) - D^+V(z_2, u_2)| \leq K\left(|z_1 - z_2| + |u_1 - u_2|\right) \tag{21}$$

$$|F(z_1, u_1)| \leq M \tag{22}$$

$\forall \hat{x}_1, \hat{x}_2 \in \mathcal{B}_E^n, \forall y_1, y_2 \in \mathcal{B}_{\bar{E}_Y}^q, \forall z_1, z_2 \in \mathcal{B}_E^{2n}, \forall u_1, u_2 \in \mathcal{B}_{\bar{E}_U}^m$.

Let $\eta = \sigma\alpha_3(e_2)$. Let δ, μ_u, μ_x, and μ_y be nonnegative reals such that

$$0 < \delta \leq 1, \qquad e_2 + \delta M < e_1, \qquad R + \delta M < E \tag{23}$$

$$0 \leq \mu_u \leq 1, \qquad 0 \leq \mu_x \leq 1, \qquad 0 \leq \mu_y \leq 1 \tag{24}$$

$$\alpha_1(r) > \alpha_2(e_1) + \frac{2}{3}\eta\delta \tag{25}$$

$$\frac{\eta}{3} \geq K\left(M\delta + (2 + \sigma)\left(\mu_x + \mu_u + L(\mu_x + \mu_y)\right)\right). \tag{26}$$

Let us consider a partition $\pi_{a,\delta} = \{t_j\}_{j \in \mathbb{Z}^+}$ (see Definition 1). Let $\{\tilde{t}_h\}$ be the sequence of event times defined in (17), and define the sequence $i_j = \max\{g \in \mathbb{Z}^+ | g \leq j, t_g \in \{\tilde{t}_h\}\}, j = 0, 1, \ldots$

Then, from (16)–(17), we have

$$u(t) = u^*(t_{i_j}), \qquad t_j \leq t < t_{j+1},\ j = 0, 1, \ldots$$

It is readily seen that the solution exists in $[0, t_1]$ otherwise, by contradiction, if the solution blows up, there exists a time $\tau \in [0, t_1)$ such that $|z(t)| < E, t \in [0, \tau)$, and $|z(\tau)| = E$. But, by (22) and (23), for $t \in [0, \tau]$, the following inequalities hold:

$$|z(t)| \leq |z_0| + \int_0^t |F(z(\theta), u^*(0))|\, d\theta \leq R + \delta M < E. \tag{27}$$

Thus, taking $t = \tau$, the absurd inequality arises $E < E$. Therefore, the solution exists in $[0, t_1]$ and, by (27) for $t \in [0, t_1]$, it follows that $z(t) \in \mathcal{B}_E^{2n}, t \in [0, t_1]$.

Let

$$w(t) = V(z(t)) \tag{28}$$

where $z(t)$ is the solution of the closed-loop system described by the extended state (7) with control law (16)–(17).

Then from (28), for $t \in (0, t_1]$, by virtue of the mean value theorem for integrals, we have, for some $t^\star \in [0, t]$,

$$\begin{aligned} w(t) - w(0) &= \int_0^t D^+V(z(\theta), u^*(0))d\theta \\ &\leq D^+V(z(t^\star), u^*(0))t \\ &= D^+V(z(t^\star), u^*(0))t \\ &\quad + \sigma D^+V(z_0, k(z_0))t - \sigma D^+V(z_0, k(z_0))t \\ &\quad + D^+V(z_0, k(z_0))t - D^+V(z_0, k(z_0))t. \end{aligned} \tag{29}$$

By conditions (9) and (21), we have

$$D^+V(z_0, k(z_0)) \leq -\alpha_3(|z_0|) \tag{30}$$

$$(1 - \sigma)D^+V(z_0, k(z_0)) \leq 0 \tag{31}$$

$$\begin{aligned} &|D^+V(z(t^\star), u^*(0)) - D^+V(z_0, k(z_0))| \\ &\quad \leq K\left(|z(t^\star) - z_0| + |u^*(0) - k(z_0)|\right) \\ &\quad \leq K\left(|z(t^\star) - z_0| + |u^*(0) - \bar{u}(0)| + |\bar{u}(0) - k(z_0)|\right) \end{aligned} \tag{32}$$

where

$$|z(t^\star) - z_0| \leq Mt^\star \leq M\delta \tag{33}$$

is implied by (22) and by $t^\star \leq \delta$. Furthermore

$$|u^*(0) - \bar{u}(0)| = |\left[\bar{u}(0)\right]_{\mu_u} - \bar{u}(0)| \leq \mu_u \tag{34}$$

$$\begin{aligned} |\bar{u}(0) - k(z_0)| &= |\bar{k}([\hat{x}(0)]_{\mu_x}, [y(0)]_{\mu_y}) - \bar{k}(\hat{x}(0), y(0))| \\ &\leq L\left(|\hat{x}(0) - [\hat{x}(0)]_{\mu_x}| + |y(0) - [y(0)]_{\mu_y}|\right) \\ &\leq L\left(\mu_x + \mu_y\right). \end{aligned} \tag{35}$$

Hence, from (29)–(35), we obtain

$$\begin{aligned} w(t) - w(0) &\leq -\sigma\alpha_3(|z_0|)t \\ &\qquad + \left(KM\delta + K\mu_u + KL\left(\mu_x + \mu_y\right)\right)t \\ &\leq -\sigma\alpha_3(|z_0|)t + \frac{1}{3}\eta t \end{aligned}$$

where the last line is implied by (26).

Claim 1: The solution exists in $[0, +\infty)$ and $z(t) \in \mathcal{B}_E^{2n}, t \geq 0$.

Then, in any interval $[t_j, t_{j+1}]$, $j = 1, 2, \ldots$, the same reasoning used in the interval $[0, t_1]$ can be repeated. In particular, for any fixed $t \in (t_j, t_{j+1}]$, $j \geq 1$, for some $t^\star \in [t_j, t]$, one can write

$$\begin{aligned} w(t) - w(t_j) &\leq D^+V(z(t^\star), u^*(t_{i_j}))(t - t_j) \\ &= D^+V(z(t^\star), u^*(t_{i_j}))(t - t_j) \\ &\quad + \sigma D^+V(z(t_j), k(z(t_j)))(t - t_j) \end{aligned}$$

$$\begin{aligned}&-\sigma D^+V(z(t_j),k(z(t_j)))(t-t_j)\\&+D^+V(z(t_j),k(z(t_{i_j})))(t-t_j)\\&-D^+V(z(t_j),k(z(t_{i_j})))(t-t_j)\end{aligned}\tag{36}$$

and, with analogous simplifications to those used in (30)–(35), we get

$$D^+V(z(t_j),k(z(t_j)))\le -\alpha_3(|z(t_j)|)\tag{37}$$

$$\begin{aligned}&|D^+V(z(t^\star),u^*(t_{i_j}))-D^+V(z(t_j),k(z(t_{i_j})))|\\&\le KM\delta+K\mu_u+KL(\mu_x+\mu_y)\end{aligned}\tag{38}$$

$$D^+V(z(t_j),k(z(t_{i_j})))-\sigma D^+V(z(t_j),k(z(t_j)))\tag{39}$$

$$=\begin{cases}(1-\sigma)D^+V(z(t_j),k(z(t_j))) & \text{if } i_j=j \quad (40)\\ D^+V(z(t_j),k(z(t_{i_{j-1}}))) & \\ -\sigma D^+V(z(t_j),k(z(t_j))) & \text{if } i_j=i_{j-1}. \quad (41)\end{cases}$$

We now analyze the two subcases in (40)–(41) as follows.

1) If $i_j=j$ (trigger), from (40), it is seen readily that

$$(1-\sigma)D^+V(z(t_j),k(z(t_j)))\le 0.\tag{42}$$

2) If $i_j=i_{j-1}$ (no trigger), the triggering condition in (19), evaluated at $t=t_j$ and $\tilde{t}_h=t_{i_{j-1}}$ is false, i.e.,

$$\begin{aligned}&-D^+V(\xi(t_j),u^*(t_{i_{j-1}}))+\sigma D^+V(\xi(t_j),u^*(t_j))\\&\quad>(1+\sigma)K\beta(\bar{e}_0,t_j)\end{aligned}\tag{43}$$

implying

$$\begin{aligned}&D^+V(z(t_j),k(z(t_{i_{j-1}})))-\sigma D^+V(z(t_j),k(z(t_j)))\\&\le D^+V(\xi(t_j),u^*(t_{i_{j-1}}))-\sigma D^+V(\xi(t_j),u^*(t_j))\\&\quad+K\left(|z(t_j)-\xi(t_j)|+|k(z(t_{i_{j-1}}))-u^*(t_{i_{j-1}})|\right)\\&\quad+\sigma K\left(|z(t_j)-\xi(t_j)|+|k(z(t_j))-u^*(t_j)|\right)\\&\le D^+V(\xi(t_j),u^*(t_{i_{j-1}}))-\sigma D^+V(\xi(t_j),u^*(t_j))\\&\quad+(1+\sigma)K\left(\mu_x+\beta(\bar{e}_0,t_j)+\mu_u+L(\mu_x+\mu_y)\right)\\&<-(1+\sigma)K\beta(\bar{e}_0,t_j)\\&\quad+(1+\sigma)K\left(\mu_x+\beta(\bar{e}_0,t_j)+\mu_u+L(\mu_x+\mu_y)\right)\\&=(1+\sigma)K\left(\mu_x+\mu_u+L(\mu_x+\mu_y)\right)\end{aligned}\tag{44}$$

where we exploited the fact that

$$z(t)=\begin{bmatrix}\hat{x}^T(t) & e^T(t)\end{bmatrix}^T$$

so

$$\begin{aligned}|z(t)-\xi(t)|&=\left|\begin{bmatrix}\hat{x}^T(t) & e^T(t)\end{bmatrix}^T-\begin{bmatrix}[\hat{x}^T(t)]_{\mu_x} & 0^T\end{bmatrix}^T\right|\\&\le\max\{\mu_x,|e(t)|\}\le\mu_x+\beta(\bar{e}_0,t).\end{aligned}$$

As a consequence of (42)–(44), taking the worst case between (40) and (41), one gets [from (39)]

$$\begin{aligned}&D^+V(z(t_j),k(z(t_{i_j})))-\sigma D^+V(z(t_j),k(z(t_j)))\\&\quad\le(1+\sigma)K\left(\mu_x+\mu_u+L(\mu_x+\mu_y)\right).\end{aligned}\tag{45}$$

Hence, from (36)–(45), we obtain that, for any integer $j\ge 0$, for any $t\in(t_j,t_{j+1}]$, the following inequalities hold:

$$\begin{aligned}&w(t)-w(t_j)\le-\sigma\alpha_3(|z(t_j)|)(t-t_j)\\&\quad+\left(KM\delta+K\mu_u+KL\left(\mu_x+\mu_y\right)\right)(t-t_j)\\&\quad+(1+\sigma)K\left(\mu_x+\mu_u+L(\mu_x+\mu_y)\right)(t-t_j)\\&\le-\sigma\alpha_3(|z(t_j)|)(t-t_j)+\frac{1}{3}\eta(t-t_j)\end{aligned}$$

where the last line is again implied by (26).

From here on, the proof continues as the one of [39, Th. 1] (see also [11] and [33]), in order to prove that Claim 1 holds true and that there exists T such that $z(t)\in\mathcal{B}_r^{2n}$ for any $t\ge T$, with the time T given by [33]

$$T=\frac{3\alpha_2(R)}{\eta a}+1.$$

The interested reader can see steps from [26, (5.15) to (5.23)] with $k_2=[\frac{3\alpha_2(R)}{\eta a\delta}]+1$. The proof of the theorem is complete. ■

V. Application: Fully Actuated Inverted Pendulum

We consider the model of an actuated inverted pendulum

$$\begin{cases}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= \frac{g}{l}\sin(x_1(t))-\frac{k_0}{\bar{m}l^2}x_2(t)+\frac{1}{\bar{m}l^2}u(t)\\ y(t) &= x_1(t)\end{cases}\tag{46}$$

where $x_1(t)$ is the angular position and $x_2(t)$ is the angular velocity of the pendulum; the parameters are $\bar{l}=1$, $\bar{m}=0.1$, $k_0=0.2$, and $g=9.8$, taken from [36, p. 556], see also [40]. Note that in this formulation, the origin corresponds to the upright unstable equilibrium of the pendulum.

Consider the observer

$$\begin{bmatrix}\dot{\hat{x}}_1(t)\\ \dot{\hat{x}}_2(t)\end{bmatrix}=\begin{bmatrix}\hat{x}_2(t)\\ \frac{g}{l}\sin(y(t))-\frac{k_0}{\bar{m}l^2}\hat{x}_2(t)+\frac{1}{\bar{m}l^2}u(t)\end{bmatrix}+G(y(t)-\hat{x}_1(t))$$

implying from (46) that the error dynamics is linear and exponentially stable

$$\dot{e}(t)=(\tilde{A}-GC)e(t)$$

with $\tilde{A}=\begin{bmatrix}0 & 1\\ 0 & -\frac{k_0}{\bar{m}l^2}\end{bmatrix}$ and $C=[1\quad 0]$, hence satisfying (6) for all u, provided that $G=[g_1\quad g_2]^T$ is chosen such that matrix $\tilde{A}-GC$ is Hurwitz.

The equation of the extended dynamics $z=[\hat{x}^T\quad e^T]^T=[\hat{x}_1\quad\hat{x}_2\quad e_1\quad e_2]^T$ is

$$\dot{z}(t)=F(z(t),u(t))=\tilde{F}(z(t))+\tilde{B}u(t)$$

with

$$\tilde{F}(z)=\begin{bmatrix}\hat{x}_2+g_1e_1\\ -\frac{k_0}{\bar{m}l^2}\hat{x}_2+\frac{g}{l}\sin(\hat{x}_1+e_1)+g_2e_1\\ -g_1e_1+e_2\\ -g_2e_1-\frac{k_0}{\bar{m}l^2}e_2\end{bmatrix}$$

$$\tilde{B}=\begin{bmatrix}0 & \frac{1}{\bar{m}l^2} & 0 & 0\end{bmatrix}^T.$$

By imposing the linearizing feedback

$$u(t)=\bar{k}(\hat{x}(t),y(t))=-\bar{m}\bar{l}g\sin(y(t))+k_0\hat{x}_2(t)+H\hat{x}(t)$$

with $H=[h_1\quad h_2]$, the closed-loop dynamics becomes

$$\dot{z}(t)=\begin{bmatrix}\dot{\hat{x}}(t)\\ \dot{e}(t)\end{bmatrix}=\begin{bmatrix}(A+BH)\hat{x}(t)+GCe(t)\\ (\tilde{A}-GC)e(t)\end{bmatrix}$$

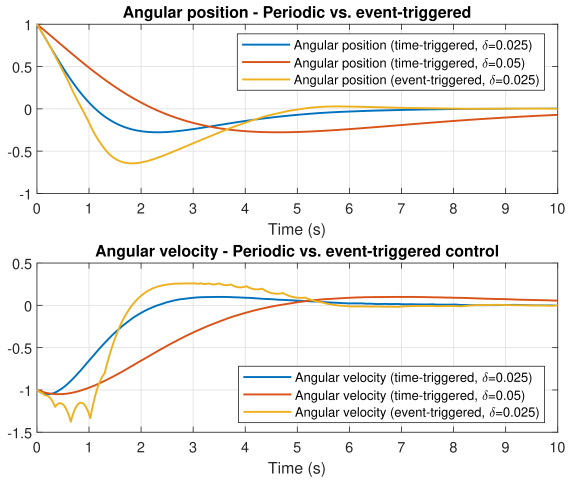

Fig. 2. Comparison of closed-loop state trajectories, in the case of sampled-data time-triggered control (with two different sampling intervals) and in the case of sampled-data event-triggered control.

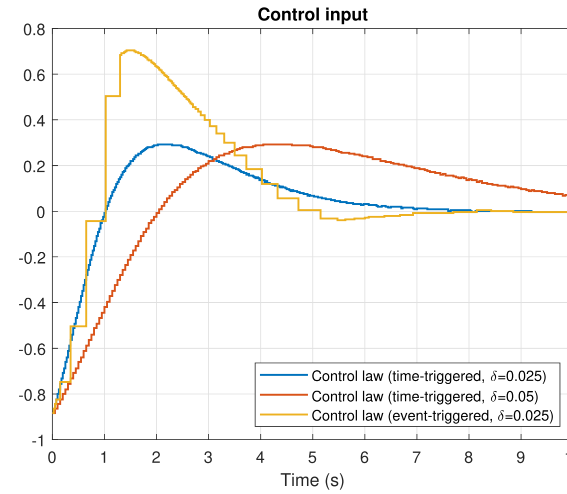

Fig. 3. Control input (sampled-data time-triggered versus event-triggered control).

$$= \begin{bmatrix} A+BH & GC \\ 0 & \tilde{A}-GC \end{bmatrix} z(t)$$

with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$, satisfying Assumption 1, provided that both matrixes $\tilde{A} - GC$ and $A + BH$ are Hurwitz.

Assume the eigenvalues to be placed in $\{-1, -2\}$ for both $\tilde{A} - GC$ and $A + BH$. For matrix $P = \mathcal{I}_4$, let $Q \in \mathbb{R}^{4\times 4}$ be the solution of the Lyapunov equation $A_{cl}^T Q + QA_{cl} = -P$, with $A_{cl} = \begin{bmatrix} A+BH & GC \\ 0 & \tilde{A}-GC \end{bmatrix}$. Hence, a quadratic Lyapunov function for the closed-loop system is $V(z) = \frac{1}{2} z^T Q z$.

In the following, we discuss some numerical simulations. The closed-loop eigenvalues assignment results in the error function $\beta(e(0), t) = e^{-t}|e(0)|$ in (6), with $|e(0)| \leq \bar{e}_0$ and $\bar{e}_0$ is a known upper bound. Since the angular position is measurable, we consider a null initial observation error for the position and a nonzero initial error for the velocity $z(0) = [1 \quad -0.5 \quad 0 \quad -0.5]^T$, namely, the observer initial velocity is $\hat{x}_2(0) = z_2(0) = -0.5$ while the unknown initial velocity is $x_2(0) = \hat{x}_2(0) + e_2(0) = z_2(0) + z_4(0) = -1$, for which we consider a conservative error bound $\bar{e}_0 = 1$.

We consider a periodic sampling ($a = 1$) with $\delta = 0.025$, equal quantization values $\mu_u = \mu_x = \mu_y = 0.002$, while the parameter σ affecting the triggering frequency is set to 0.1.

The simulation results are shown in Figs. 2 and 3. Fig. 2 compares the performances of the quantized sampled-data time-triggered control, with two different sampling intervals, with those exhibited by the quantized sampled-data event-based control. The event-triggered solution with $\delta = 0.025$ achieves practical stabilization within an horizon of 10 s, in spite of the much lower average frequency of control updates (around 30% of sampling intervals) with respect to the sampled-data time-triggered controller with the same sampling interval; furthermore, the tracking is faster than the one exhibited by the time-based sampled-data controller with double sampling interval $\delta = 0.05$. Fig. 3 shows the control input functions in the different cases. The performed simulations fully validate the theoretical results.

VI. Conclusion

In this work, we addressed the topic of quantized sample-and-hold stabilization of nonlinear systems from quantized output measurements, following an event-triggered observer-based approach. In the spirit of the emulation approach, the existence of a continuous-time globally asymptotically stabilizing control law is assumed, and conditions for preserving stability of the closed-loop system in a semiglobal practical sense in presence of digital nonidealities are derived. The extension of this framework to the infinite-dimensional case of nonlinear systems with possibly time-varying state delays is a current topic of investigation.

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