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Minimal graphs in three-dimensional Killing submersions

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ABSTRACT

We consider a connected and orientable Riemannian (resp. Lorentzian) three-manifold $\mathbb{E}$ admitting a never vanishing (resp. temporal) complete Killing vector field $\xi \in \mathfrak{X}(\mathbb{E})$ whose associated one-parameter group of isometries G of $\mathbb{E}$ acts freely and properly on $\mathbb{E}$. Then, there exists a Killing submersion $\pi\colon \mathbb{E} \to M = {}^{\mathbb{E}}/_{G}$ whose fibers are the integral curves of ξ. Killing submersions give rise to a natural notion of graph over a domain in M, that is, a smooth section of π over this domain.

In this setting we solve the Jenkins–Serrin problem for the minimal surface equation in $\mathbb{E}$ over a relatively compact open domain $\Omega \subset M$ with prescribed finite or infinite values on some arcs of the boundary under the only assumption that the same value $+\infty$ or $-\infty$ cannot be prescribed on two adjacent components of $\partial\Omega$ forming a convex angle. We show that the solution exists if and only if some generalized Jenkins–Serrin conditions (in terms of a conformal metric in M) are fulfilled. We develop further the theory of divergence lines to study the convergence of a sequence of minimal graphs. We solve the Dirichlet problem for minimal Killing graphs over certain unbounded domains of M, taking piecewise continuous boundary values.

We study the uniqueness of solutions of the Dirichlet problem over unbounded domains of M obtaining a general Collin-Krust type estimate. In the particular case of the Heisenberg group, we prove a uniqueness result for minimal Killing graphs with bounded boundary values over a strip.

Finally, we develop a conformal duality for spacelike graphs in Riemannian and Lorentzian Riemannian and Lorentzian Killing submersions. The duality swaps mean curvature and bundle curvature and sends the length of the Killing vector field to its reciprocal while keeping invariant the base surface. We obtain two consequences of this result. On the one hand, we find entire graphs in Lorentz–Minkowski space $\mathbb{L}^3$ with prescribed mean curvature a bounded function $H \in \mathcal{C}^\infty(\mathbb{R}^2)$ with bounded gradient. On the other hand, we obtain conditions for existence and non existence of entire graphs which are related to a notion of critical mean curvature.

"There is no royal road to geometry"

— Euclid

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INTRODUCTION

A classical subject in Differential Geometry is the study of surfaces of constant mean curvature $H \in \mathbb{R}$ in the three-dimensional Euclidean space $\mathbb{R}^3$, sometimes denoted CMC surfaces or H-surfaces, that are critical points of the functional

$$\text{Area} - 2H \cdot \text{Volume}.$$

Of particular interest are surfaces with $H = 0$, which are known as minimal surfaces. This field remains very active nowadays, and constitutes a meeting point for a wide variety of techniques from different branches of mathematics such as for example Complex Analysis, Elliptic PDE Theory, Integrable Systems, Topology, Variational Calculus and so on.

Due to the role that Thurston geometries (that are particular cases of three-dimensional simply connected homogeneous manifolds[1]) play in the Poincaré conjecture solved by G. Perelman, the interest in extending the theory of minimal and CMC surfaces in these three-dimensional spaces has increased in the last twenty years. Indeed, despite some previous interesting works on the topic appeared in the late eighties, it is in the new millennium, after a series of pioneering works by U. Abresch and H. Rosenberg [AbrRos04, AbrRos05], and W. H. Meeks and H. Rosenberg [MeeRos05], that the study of minimal and CMC surfaces in homogeneous three-manifolds started to develop as a consistent unified theory. Today, the subject has grown rapidly in many different directions and already contains a large number of important contributions, but still presents several open problems.

It is important to notice that the simply connected homogeneous manifolds have been completely classified: except for the Riemannian products $\mathbb{S}^2(\kappa) \times \mathbb{R}$, where $\kappa > 0$, each one of them is isometric to a three-dimensional Lie group equipped with a left-invariant metric. A way to study $\mathbb{S}^2(\kappa) \times \mathbb{R}$ in a larger family of homogeneous spaces is by considering the two-parameter family $\mathbb{E}(\kappa, \tau)$, where κ and τ are real numbers. These spaces include Riemannian products $M^2(\kappa) \times \mathbb{R}$, where $M^2(\kappa)$ represents the simply connected surface with constant Gauss curvature $\kappa \in \mathbb{R}$, the Heisenberg space Nil_3,

1 Homogeneous means that the isometry group of the manifold acts transitively on the manifold.

the universal cover of the special linear group $SL_2(\mathbb{R})$ with a two-parameter family of left-invariant special metrics, and the Berger spheres $SU(2)$ (when $\kappa = 4\tau^2$ we get a sphere of constant sectional curvature). It is widely known that $\mathbb{E}(\kappa, \tau)$ carries a Riemannian submersion onto the surface $M^2(\kappa)$ with constant bundle curvature τ whose fibers are the integral curves of a unitary Killing vector field in $\mathbb{E}(\kappa, \tau)$. Both the Lie group structure and the $\mathbb{E}(\kappa, \tau)$ setting have been used to study the theory of CMC surfaces in homogeneous spaces, (see [MeePer12] and [DaHaMi09] for a comprehensive compilation of results).

The aim of this thesis is to extend to the wide class of three-manifolds with a Killing vector field (not necessarily unitary) some results about minimal surfaces that have been proved in some specific homogeneous manifolds. To do so, we only use the existence of the Killing vector field to give a description of the ambient manifold (see [LerMan17]). In this way, the results proved in this thesis will hold true in every homogeneous three-manifold and with respect to every Killing direction.

Killing submersions

The first chapter is dedicated to prove necessary and sufficient conditions that assure that a Riemannian (resp. Lorentzian) three-manifold $\mathbb{E}$ admitting a complete non-zero (resp. temporal) Killing vector field $\xi \in \mathfrak{X}(\mathbb{E})$ can be described as a Killing submersion. Moreover, we study global and local properties of Killing submersions.

Given a connected and oriented three-dimensional manifold $\mathbb{E}$, endowed with a Riemannian or Lorentzian metric $\langle \cdot, \cdot \rangle$, assume that it has a complete nowhere vanishing (and time-like when $(\mathbb{E}, \langle \cdot, \cdot \rangle)$ is Lorentzian) Killing vector field $\xi \in \mathfrak{X}(\mathbb{E})$, that is, ξ satisfies $\langle \nabla_X \xi, Y \rangle + \langle \nabla_Y \xi, X \rangle = 0$ for all $X, Y \in \mathfrak{X}(\mathbb{E})$ and the integral curves of ξ are defined for all $t \in \mathbb{R}$. We will denote by $G = \{\phi_t\}$ the one-parameter group of isometries of $\mathbb{E}$ associated to ξ, that are called *vertical translations*, and consider its natural smooth action on $\mathbb{E}$:

$$\begin{array}{rcl} G \times \mathbb{E} & \to & \mathbb{E}. \\ \phi_t \cdot p & \mapsto & \phi_t(p) \end{array}$$

When this action is free and proper, the orbit space $M = {}^{\mathbb{E}}/_{G}$ is well defined and it can be endowed with a unique Riemannian metric such that the quotient map $\pi \colon \mathbb{E} \to M$ is a Riemannian submersion, that is $d\pi_p$ is a linear isometry of the horizontal distribution $\ker(d\pi)^\perp \subset T\mathbb{E}$ for any $p \in \mathbb{E}$. Denoting by $\mathrm{Iso}(\mathbb{E})$ the isometry group of $\mathbb{E}$, we find necessary and sufficient conditions such that the action of $G \subseteq \mathrm{Iso}(\mathbb{E})$ is free and proper (see Theorems 1.4, 1.6 and 1.8). We start by noticing that since ξ is complete, G is either isomorphic to $\mathbb{R}$ or $\mathbb{S}^1$. Endowing $\mathrm{Iso}(\mathbb{E})$ with the compact-open topology, we will see that the properness of G will depend on its topological properties as a subgroup of $\mathrm{Iso}(\mathbb{E})$ and the results can be summarized as follows (see Corollary 1.9).

Theorem [Existence of Killing submersion]. *The action of* G *is proper if and only if* G *is closed in the compact-open topology. If* G *is closed, the action is free if and only if one of the following two conditions is satisfied:*

- G *is isomorphic to* $\mathbb{R}$*;*
- G *is isomorphic to* $\mathbb{S}^1$ *and the function of* $\mathbb{E}$ *describing the length of the fibers is proportional to the length of the Killing vector field* ξ*.*

In this case, we say that $\pi\colon \mathbb{E} \to M$ is a *Riemannian, or Lorentzian, Killing submersion*, depending on the causality of ξ.

In this setting, we can define the smooth function

$$\tau(p) = \frac{-1}{\mu(p)} \langle \nabla_{e_1}\xi, e_2\rangle_p ,$$

where $\{e_1, e_2, \xi_p/\|\xi_p\|\}$ is an oriented orthonormal basis of $T_p\mathbb{E}$ and $\mu(p) = \|\xi_p\|$. As it is shown in Remark 1.10 of Chapter 1, if μ is constant, τ is the function satisfying

$$\overline{\nabla}_X\xi = \tau X \times \xi,$$

where $\times$ is the cross product in $\mathbb{E}$. So, τ will be called *bundle curvature*, extending the definition given for the unitary Killing case [LeaRos09, SouVan12, EspDeO13, MerOrt14, Man14]. A direct computation shows that both the bundle curvature and the Killing length μ are constant along the fibers, so they induce functions in M that will be also denoted by $\tau, \mu \in \mathcal{C}^\infty(M)$. For every $p \in \mathbb{E}$, a tangent vector $v \in T_p\mathbb{E}$ will be called *vertical* when $v \in \ker(d\pi_p)$ and *horizontal* when $v \in \ker(d\pi_p)^\perp$.

In the Riemannian case, the Killing submersions have been completely classified in [LerMan17], where the authors proved that when $\mathbb{E}$ is simply connected, it is uniquely determined by the choice of M and $\tau, \mu \in \mathcal{C}^\infty(M)$, with $\mu > 0$. This result can be easily extended to the Lorentzian case, so along this essay we will denote $\mathbb{E} = \mathbb{E}(M, \tau, \mu, \epsilon)$ where $\epsilon = \pm 1$ denotes the causality of the vertical Killing vector field (see Section 1.3), proving that, fixing a Riemannian Surface M and the functions $\tau, \mu \in \mathcal{C}^\infty(M)$, $\mu > 0$,

- if M is simply connected, then there exist both a Riemannian and a Lorentzian Killing submersion $\pi\colon \mathbb{E} \to M$ with bundle curvature τ and Killing length μ, which is unique provided that $\mathbb{E}$ is simply connected and we write $\mathbb{E} = \mathbb{E}(M, \tau, \mu, \pm 1)$.
  - If M is topologically $\mathbb{R}^2$, then π is a trivial fibration, and we are able to get an explicit model for π. In particular, $\mathbb{E}$ is diffeomorphic to $\mathbb{R}^3$.
  - If M is topologically $\mathbb{S}^2$, then π admits a global section if and only if $\int_M \frac{\tau}{\mu} = 0$, and in that case $\mathbb{E}$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$. Otherwise π is topologically the Hopf fibration and $\mathbb{E}$ is diffeomorphic to $\mathbb{S}^3$.
- if $\pi\colon \mathbb{E} \to M$ is a Killing submersion and M and $\mathbb{E}$ are not simply connected, then there exists a Killing submersion $\tilde{\pi}\colon \tilde{\mathbb{E}} \to \tilde{M}$, being $\tilde{M}$ and $\tilde{\mathbb{E}}$ the universal coverings of M and $\mathbb{E}$, respectively, and a discrete group I of isometries on $\tilde{\mathbb{E}}$ preserving the Killing direction, such that I acts properly discontinuously on $\tilde{\mathbb{E}}$ and $\mathbb{E} = \tilde{\mathbb{E}}/I$.

We are also able to prove that when $\mathbb{E}$ has a Riemannian metric, the geodesic completeness of M implies the geodesic completeness of $\mathbb{E}$ (see Proposition 1.19) extending the result of the unitary case proved in [Man14]. This is not true in general in the Lorentzian case, as it is shown in Example 1.20, but it is true when μ is constant (see [AazRea23, Corollary 6.1]).

In this setting we deal with two types of surfaces depending on the fact that the angle function $\mathfrak{v} = \langle \xi, N \rangle$ is identically zero or nowhere vanishing:

- when $\mathfrak{v} = 0$, we have the *vertical cylinders*, which are always tangent to the Killing vector field ξ, and
- when $\mathfrak{v} \neq 0$ we have the *Killing multigraphs*, that are always transversal to the Killing vector field.

For what concerns a vertical cylinder, we prove that it projects to a curve in M and its mean curvature H is related to the geodesic curvature $\tilde{\kappa}_g$ computed in the conformal metric $\mu^2 ds_M^2$ of M. In particular, the mean curvature of the vertical cylinder $\Sigma = \pi^{-1}(\Gamma)$ with respect to a unit normal N satisfies

$$2H = \mu\, \widetilde{\kappa}_g,$$

where $\widetilde{\kappa}_g$ is the geodesic curvature of Γ with respect to the unit normal $\eta = \frac{1}{\mu}\pi_*(N)$ in the conformal metric $\mu^2 ds_M^2$ on M (see Proposition 1.21). We will use the prefix "μ-" to indicate that the corresponding term is computed with respect to the metric $\mu^2 ds_M^2$ in M. Notice that every surface immersed in any three-manifold and invariant by a continuous one-parameter group of isometries is locally a vertical cylinder for some Killing submersion structure. Specifically, we can give a geometric characterization of CMC surfaces invariant by a one-parameter group of isometries by studying the curves that generate them in the orbit space. In particular, such curves are characterized by their initial data (see Corollaries 1.22 and 1.23). This viewpoint also reveals the existence of minimal open book foliations of a neighborhood of any vertical fiber of any Killing submersion (with binding the fiber).

On the other hand, a Killing multigraph on $\mathbb{E}$ is locally a smooth section over an open subset $\Omega \subset M$, and can be seen as the graph of a function $u \in \mathcal{C}^\infty(\Omega)$. More precisely, if we prescribe a smooth zero section $F_0 : \Omega \to \mathbb{E}$, then such a graph can be parameterized as $F_u : \Omega \to \mathbb{E}$ with $F_u(p) = \phi_{u(p)}(F_0(p))$ for some $u \in \mathcal{C}^\infty(\Omega)$, where $\{\phi_t\}$ is the group of vertical translations. Considering $d \in \mathcal{C}^\infty(\mathbb{E})$ defined implicitly by $\phi_{d(q)}(F_0(\pi(q))) = q$, i.e., $d(q)$ is the signed distance along a fiber from the initial section to q, the mean curvature H of the graph of a function u with respect to the zero section F_0 satisfies the equation

$$H = \mathcal{Q}(u) = \frac{1}{2\mu}\operatorname{div}\left(\frac{\mu^2 Gu}{\sqrt{1+\epsilon\mu^2\|Gu\|^2}}\right),$$

where the divergence is computed in M, $Gu = \nabla u - \pi_*(\overline{\nabla} d)$ is the so called *generalized gradient* and ϵ denotes the causality of the Killing vector field (see Proposition 1.24).

The Dirichlet problem for the prescribed mean curvature equation

The second chapter, together with the Appendix, is dedicated to the study of the Dirichlet problem for the prescribed mean curvature equation with bounded boundary values in relatively compact domains in a Riemannian Killing submersion. We want to point out that, despite the fact that this result follows from [DajDel09, Theorem 1], our goal is to provide a complete proof of the theorem, detailing the techniques and giving a complete list of references of the results used to prove it.

The result we prove can be summarized as follows (see Theorem 2.1).

Theorem [Existence]. *Assume that* $\Omega \subset M$ *is a relatively compact domain and* $H \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$. *Assume also that* $\partial\Omega$ *is piecewise* $\mathcal{C}^1$ *and* $\mu\,\tilde{\kappa}_g(p) \geqslant 2H$ *for all* $p \in \partial\Omega \setminus E$, *where* $\tilde{\kappa}_g$ *is the* μ*-geodesic curvature of* $\partial\Omega$ *computed with respect to the normal pointing into* Ω *and* E *is the set of corner points of* $\partial\Omega$ *(that is, the points where* $\partial\Omega$ *is not* $\mathcal{C}^1$*). Assume also that* $f\colon \partial\Omega \to \mathbb{R}$ *is a piecewise continuous function and that, if* $H \neq 0$, Ω *is contained in a larger domain* $\tilde{\Omega}$ *such that*

- $\tilde{\Omega}$ *has* $\mathcal{C}^{2,\alpha}$ *boundary,*
- $\sup_{\Omega} |H| \leqslant \int_{\partial\tilde{\Omega}} \mu\tilde{\kappa}_g(\partial\tilde{\Omega})$ *and*
- $\mathrm{Ric}(\pi^{-1}(\tilde{\Omega})) \geqslant -\inf_{\partial\tilde{\Omega}}(\mu\tilde{\kappa}_g(\partial\tilde{\Omega}))^2$.

Hence, there exists a unique solution to the Dirichlet problem

$$P(\Omega, H, f) = \begin{cases} \frac{1}{2\mu}\operatorname{div}\left(\frac{\mu^2 Gu}{\sqrt{1+\mu^2\|Gu\|^2}}\right) = H & \text{in } \bar{\Omega}, \\ u = f & \text{in } \partial\Omega. \end{cases}$$

The strategy used to prove the result is the following.

- We prove a general Maximum Principle (see Proposition 2.3) for prescribed mean curvature graphs which guarantees the uniqueness.
- We prove a local existence result (see Theorem 2.11) using the classical Leray-Schauder's theory for quasilinear elliptic operator. Both the results of the Leray-Schauder theory (developed in the Appendix) and the estimates that are necessary to apply it have been explained in details.
- We use the Perron Process (see Section 2.4) to extend the local result to a larger class of domains.

In this chapter we also prove a removable singularity result (see Theorem 2.15) for graphs in arbitrarily Killing submersions proved by L. Bers [Ber55] for minimal graphs in $\mathbb{R}^3$, then by Finn [Finn65] for graphs of prescribed mean curvature in $\mathbb{R}^3$, by Nelli and Sa Earp [NelSaE96] for graphs of prescribed mean curvature in $\mathbb{H}^3$ and then extended to unitary Killing submersions by C. Leandro and H. Rosenberg [LeaRos09, Theorem 4.1]. We can adapt the technique used in [LeaRos09] since the function μ is continuous and hence bounded on relatively compact domains. This extension guarantees a removable singularity result, for example, in Sol_3 and for rotational multigraphs in $\mathbb{R}^3$. The general theorem can be stated as follows (see Theorem 2.15).

Theorem [Removable singularity]. *Let* $\Omega \subset M$, $p \in \Omega$ *and* $u\colon \Omega \setminus \{p\} \to \mathbb{R}$ *be a function whose Killing graph has prescribed mean curvature* $H \in C^{0,\alpha}(\bar{\Omega})$. *Then* u *extends smoothly to a solution at* p.

The Jenkins–Serrin Problem

In the third chapter we extend to general Riemannian Killing submersions the so called Jenkins–Serrin Theorem over relatively compact domains. This problem was firstly treated in Euclidean space $\mathbb{R}^3$ by Jenkins and Serrin [JenSer66, Thm. 3 and 4], who considered bounded domains $\Omega \subset \mathbb{R}^2$ with $\partial\Omega$ composed of straight segments and convex arcs. They found necessary and sufficient elementary conditions on the lengths of the sides of polygons inscribed in Ω that guarantee the existence of a minimal graph in $\mathbb{R}^3$ over Ω with prescribed values on the regular components of $\partial\Omega$, as well as its uniqueness (possibly up to vertical translations). They incorporated into the classical Dirichlet problem the possible asymptotic infinite values on some straight components of $\partial\Omega$. Over the years, analogous results over bounded domains have been proven in other Riemannian three-manifolds: in $\mathbb{H}^2 \times \mathbb{R}$ by Nelli and Rosenberg [NelRos02, NelRos07]; in $M \times \mathbb{R}$ by Pinheiro [Pin07] (geodesically convex domains), by Mazet, Rodriguez and Rosenberg [MaRoRo11] (general case), and Eichmair and Metzger [EicMet16] (under milder assumptions and also allowing closed geodesics as part of the boundary); in $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ by Younes [You10]; and in Sol_3 by Nguyen [Ngu14]. All these problems can be treated together by noticing that they deal with surfaces transverse to a Killing vector field. There are really few approaches to Dirichlet type problems with respect to non-Killing directions, (see for example [MeMiPe19]). There are also several works on the Jenkins–Serrin problem for positive constant mean curvature graphs (starting with the work of Spruck [Spr72], see also [HaRoSp09, FolMel11, EicMet16, KlaMen19]) as well as for graphs admitting infinite boundary values over unbounded domains in $M \times \mathbb{R}$, being M a Hadamard surface, and $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ (starting with the work of Collin and Rosenberg in $\mathbb{H}^2 \times \mathbb{R}$ [ColRos10]).

In our setting, we consider a relatively compact open connected domain $\Omega \subset M$ that will be called a *Jenkins–Serrin domain* if $\partial\Omega$ is piecewise regular and consists of μ-geodesic open arcs or simple closed μ-geodesics $A_1, \ldots, A_r$, $B_1, \ldots, B_s$ and μ-convex curves $C_1, \ldots, C_m$ with respect to the inner conormal to Ω. The finite set $E \subset \partial\Omega$ of intersections of all these curves will be called *corner set* of Ω. A Jenkins–Serrin domain $\Omega \subset M$ is said *admissible* if neither two of the A_i's nor two of the B_i's meet at a convex corner.

The *Jenkins–Serrin problem* consists in finding a minimal graph over Ω, with limit values $+\infty$ on each A_i and $-\infty$ on each B_i, and such that it extends continuously to $\Omega \cup (\cup_{i=1}^m C_i)$ with prescribed continuous values on each C_i with respect to a prescribed initial section F_0 defined on a neighborhood of Ω.

If Ω is a Jenkins–Serrin domain, we say that $\mathcal{P}$ is a μ-polygon (see Definition 3.4) inscribed in Ω if $\mathcal{P}$ is the union of disjoint curves $\Gamma_1 \cup \cdots \cup \Gamma_k$ satisfying the following conditions:

- $\mathcal{P}$ is the boundary of an open and connected subset of Ω;
- each Γ_j is either a closed μ-geodesic or a closed piecewise-regular curve with μ-geodesic components whose vertices are among the vertices of Ω.

For such an inscribed μ-polygon $\mathcal{P}$, we define

$$\alpha(\mathcal{P}) = \mathrm{Length}_\mu((\cup A_i) \cap \mathcal{P}), \qquad \beta(\mathcal{P}) = \mathrm{Length}_\mu((\cup B_i) \cap \mathcal{P}),$$
$$\gamma(\mathcal{P}) = \mathrm{Length}_\mu(\mathcal{P}),$$

and we can state the generalized Jenkins–Serrin Theorem as follows (see Theorem 3.5).

Theorem [Jenkins–Serrin]. *Let Ω be an admissible Jenkins–Serrin domain.*

(a) If the family $\{C_i\}$ is non-empty, then the Jenkins–Serrin problem in Ω has a solution if and only if the length condition

$$2\alpha(\mathcal{P}) < \gamma(\mathcal{P}) \quad \text{and} \quad 2\beta(\mathcal{P}) < \gamma(\mathcal{P})$$

holds for all inscribed μ-polygons $\mathcal{P} \subset \Omega$, in which case the solution is unique.

(b) If the family $\{C_i\}$ is empty, then the Jenkins–Serrin problem in Ω has a solution if and only if the length condition holds true for all inscribed μ-polygons $\mathcal{P} \neq \partial\Omega$ and $\alpha(\partial\Omega) = \beta(\partial\Omega)$. The solution is unique up to vertical translations.

In spite of the very diverse behaviors of Killing submersions, our main result shows that the necessary and sufficient conditions for the existence of a minimal graph over Ω are the very same as in the original Jenkins–Serrin result (using the μ-metric of M in the computation of lengths). It is quite satisfactory to realize that the original statement in $\mathbb{R}^3$ still applies with minor changes in this very general setting, but indeed our approach needs less assumptions. There are typically two conditions on a Jenkins–Serrin problem:

(C1) The value $+\infty$ or $-\infty$ is not assigned to two adjacent components of $\partial\Omega$ that meet at a convex corner.

(C2) If no continuous finite values are assigned, then the subsets of $\partial\Omega$ where $+\infty$ and $-\infty$ are assigned are both disconnected.

Condition (C1) is necessary for the existence of solutions and it is deduced by applying the so called Flux argument. Condition (C2) was used in Jenkins and Serrin's original argument and has been added to definition of "admissible domains" in the case of $M \times \mathbb{R}$ in [Pin07] or [MaRoR011] (but not in [EicMet16]). Note that (C2) is automatically satisfied in $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ or Sol_3, but it discards some configurations when the μ-metric has positive Gauss curvature as in the case of $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{S}^3$ (that fibers over $\mathbb{S}^2$ via the Hopf fibration and the μ-metric is round). Indeed, some symmetric configurations in $\mathbb{S}^2 \times \mathbb{R}$ show that (C2) is not strictly necessary, as pointed out in [MaRoR011, Remark. 3.5]. We give a counterexample that reveals a missing case in the proof of existence in [MaRoR011] in the general case of $M \times \mathbb{R}$ (see Example 3.6). This example suggests that condition (C2) cannot be dropped if one uses Jenkins and Serrin's approach.

Consequently, we have decided to extend the theory of divergence lines introduced by Mazet [Maz04] in $\mathbb{R}^3$ and developed in [MaRoR011] in $\mathbb{H}^2 \times \mathbb{R}$.

Besides simplifying some arguments in some of the cited papers, in this chapter there are several contributions that is worth highlighting:

1. Douglas criterion is commonly used to obtain a family of minimal annuli in the construction of Scherk barriers (as in [NelRos02]), but this is not possible in a general Killing submersion since minimal vertical cylinders are not necessarily area-minimizing. We use the Meeks-Yau solution of the Plateau problem to get minimal disks instead of annuli.

2. Contrary to the case of $\mathbb{H}^2 \times \mathbb{R}$, divergence lines might accumulate on $\overline{\Omega}$, but we will prove that they are actually properly embedded. We also

need to provide a new argument to prove that no divergence line ends at the interior of a boundary component because [MaRoRo11] uses the symmetries of $\mathbb{H}^2 \times \mathbb{R}$, which are not available in a Killing submersion.

3. We have to deal with the fact that there can be uncountably many divergence lines (again contrary to the case of $\mathbb{H}^2 \times \mathbb{R}$ in which this number is finite). We show that, up to a subsequence, they are disjoint and belong to finitely many nonempty isotopy classes (which can be understood rather well separately) and define different *divergence heights*. This settles a comment in [MaRoRo11, Remark. 4.5] and reveals that the number of relevant inscribed μ-polygons and convergence components is actually finite.

We must point out that most of the ideas developed in the study of divergence lines also apply (or can be adapted) to very general bounded or unbounded domains which are not of Jenkins–Serrin type.

Using this general Jenkins–Serrin Theorem, we have been able to produce new examples of minimal surfaces with boundary in $\mathbb{R}^3$ which are Jenkins–Serrin graphs with respect to rotations and accumulate on catenoids and planes (see Sections 3.5.2), as well as a complete Scherk type surface in Nil_3 which is neither embedded nor proper by the effect of the holonomy (see Sections 3.5.3).

Furthermore, using the solutions to the Jenkins–Serrin problem as barriers, we prove the existence of minimal graphs over certain unbounded domains of M with prescribed boundary values (see Section 3.5.1), extending the results of [RosSaE89, SaeTou00, SaeTou08, NeSaET017].

The Collin–Krust type estimates

In the fourth chapter we deal with the uniqueness of the Dirichlet problem for the minimal surface equation over unbounded domains of M. The pioneering work in this area was conducted by P. Collin and R. Krust [CoKu91]. Their research focused on the Dirichlet problem for the prescribed mean curvature equation in $\mathbb{R}^2$ over an unbounded domain $\Omega \subset \mathbb{R}^2$. The main theorem derived by Collin and Krust offers an asymptotic estimate of the difference between two solutions of this equation as these solutions approach infinity. This estimate serves as an essential tool in establishing the uniqueness of solutions and states that if $u, \tilde{u} \in C^2(\Omega)$ are such that $u_{|\partial\Omega} = \tilde{u}_{|\partial\Omega}$ and

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \operatorname{div}\left(\frac{\nabla \tilde{u}}{\sqrt{1+|\nabla \tilde{u}|^2}}\right),$$

then, denoting by $\Lambda(r) = \left\{(x,y) \in \Omega | \sqrt{x^2+y^2} = r\right\}$, $M(r) = \sup_{\Lambda(r)} |u - \tilde{u}|$ grows at least as $\log(r)$ for any Ω and at least linearly if $\Lambda(r)$ is uniformly bounded.

The result by Collin–Krust has been extended to unitary Killing submersions by C. Leandro and H. Rosenberg in [LeaRos09, Theorem 5.1], and improved in the specific case of minimal graphs in the three-dimensional Heisenberg group by J. M. Manzano and B. Nelli in [MaNe17, Theorem 7]. In all these results, the domain exhibits uniformly bounded or linear expansion, that is, there exists a positive constant C such that either

$$\limsup_{r\to\infty} \operatorname{Length}(\Lambda(r)) \leqslant C \quad \text{or} \quad \limsup_{r\to\infty} \tfrac{\operatorname{Length}(\Lambda(r))}{r} \leqslant C.$$

In Theorems 4.1 and 4.6, we provide a detailed description of the relationship between the growth of the vertical distance between two graphs with the same prescribed mean curvature and boundary values, and the rate of expansion of the domain where they are defined, without making any assumptions about the domain.

Theorem [Collin–Krust]. *Let $\Omega \subset M$ be an unbounded domain and assume that $p \in M$ is such that $\Omega \cap \operatorname{Cut}(p) = \emptyset$, where $\operatorname{Cut}(p)$ denotes the cut locus of $p \in M$. Assume also that $u, v \in \mathcal{C}^\infty(\Omega)$ satisfy $\mathcal{Q}(u) = \mathcal{Q}(v)$, $u > v$ in Ω and $u = v$ on $\partial\Omega$. Let*

$$M(r) = \sup_{\Lambda(r)} |u - v|, \qquad L(r) = \int_{\Lambda(r)} \mu^2 d\sigma \quad \textit{and} \quad g(r) = \int_{r_0}^{r} \tfrac{ds}{L(s)}$$

for some $r_0 > 0$. Then,

$$\liminf_{r\to\infty} \frac{M(r)}{g(r)} > 0.$$

When $\mathbb{E}$ can be described by the model $(\mathbb{R}^3, ds^2)$, where

$$ds^2 = \lambda(x,y)^2(dx^2+dy^2) + \mu^2(dz - \lambda(a dx + b dy))^2,$$

we prove the following theorem.

Theorem [Collin–Krust in local model]. *Let* $\Omega \subset M$ *be an unbounded domain ad assume that* $p \in M$ *is such that* $\Omega \cap \mathrm{Cut}(p) = \emptyset$. *Assume also that* $u \in \mathcal{C}^\infty(\Omega)$ *satisfy* $\mathcal{Q}(u) = H_0$, $u > 0$ *in* Ω *and* $u = 0$ *on* $\partial\Omega$. *Let*

$$M(r) = \sup_{\Lambda(r)} |u - v|, \quad L(r) = \int_{\Lambda(r)} \frac{2\mu^2 d\sigma}{\sqrt{1 + \mu^2(a^2 + b^2)}} \quad \text{and} \quad g(r) = \int_{r_0}^{r} \frac{ds}{L(s)},$$

for some $r_0 > 0$. *Then,*

$$\liminf_{r \to \infty} \frac{M(r)}{g(r)} > 0.$$

As a consequence of these Collin–Krust type estimates, we prove the uniqueness of solutions to the Dirichlet problem for the minimal surface equation with bounded boundary values in a domain contained in a strip of $\mathbb{R}^2$ in the Heisenberg group (see Theorem 4.10 and Corollary 4.11). This provides a positive answer to the two open questions posed in [NeSaET017]:

(a) Is the minimal solution with zero boundary value on a strip, unique?

(b) Let u be any minimal solution on a strip with boundary value f such that $|f| \leqslant M$ for some $M > 0$. Is $|u| \leqslant M$?

A Calabi-type correspondence

The last chapter is dedicated to extend a Calabi-type correspondence between spacelike graphs of prescribed mean curvature in Riemannian and Lorentzian Killing submersions. The starting point of this work relies on the fact that a minimal graph in the Euclidean space has divergence zero and can be transformed into a maximal (spacelike) graph in Lorentz–Minkowski space $\mathbb{L}^3$ by means of the Poincaré lemma. This clever trick is usually attributed to Calabi [Cal70], who used it to prove a Bernstein Theorem in the Lorentz-Minkowski space. In [Lee11], H. Lee has extended the Calabi duality to the case of homogeneous spaces with isometry group of dimension 4, obtaining a duality between graphs with constant mean curvature H in $\mathbb{E}(\kappa, \tau)$ and spacelike graphs with constant mean curvature τ in $\mathbb{L}(\kappa, H)$. The case of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ (i.e., the particular case $\kappa = 1$ and $\tau = H = 0$) was actually established earlier by Albujer and Alías [AlbAli09]. In [LeeMan19], the result was generalized to three-dimensional Killing submersions with unitary Killing vector field by prescribing non-necessarily constant mean and bundle curvature functions that are swapped by the duality. In this chapter, we move forward obtaining a duality under the presence of any Killing vector field with no zeros, not necessarily of constant length. This is possibly the most general scenario where the mean curvature of a surface immersed in a three-manifold still acquires a divergence type equation and there is a notion of bundle curvature that also admits a divergence type expression.

The main result of this chapter can be stated as follows (see Theorem 5.1).

Theorem [Conformal duality]. *Let M be a simply connected Riemannian surface and let $\tau, H, \mu \in \mathcal{C}^\infty(M)$ be arbitrary functions such that $\mu > 0$. There is a bijective correspondence between*

(a) entire graphs in $\mathbb{E}(M, \tau, \mu)$ with prescribed mean curvature H, and

(b) entire graphs in $\mathbb{L}(M, H, \mu^{-1})$ with prescribed mean curvature τ.

Assume that $\Sigma \subset \mathbb{E}(M, \tau, \mu)$ and $\widetilde{\Sigma} \subset \mathbb{L}(M, H, \mu^{-1})$ are such corresponding graphs.

1. *The graphs Σ and $\widetilde{\Sigma}$ determine each other up to vertical translations.*
2. *The corresponding angle functions $\mathfrak{v}, \widetilde{\mathfrak{v}} : M \to \mathbb{R}$ satisfy $\widetilde{\mathfrak{v}} = -\mathfrak{v}^{-1}$.*
3. *Denoting by $\pi : \mathbb{E}(M, \tau, \mu) \to M$ and $\widetilde{\pi} : \mathbb{L}(M, H, \mu^{-1}) \to M$ the involved Riemannian and Lorentzian Killing submersions, respectively, the diffeomorphism $\Phi : \Sigma \to \widetilde{\Sigma}$, such that $\widetilde{\pi} \circ \Phi = \pi$, is conformal with conformal factor*

$$\Phi^* ds^2_{\widetilde{\Sigma}} = \mu^{-2}\mathfrak{v}^2 ds^2_{\Sigma}.$$

Moreover, both families (a) *and* (b) *are empty if either $\int_M \frac{\tau}{\mu} \neq 0$ or $\int_M H\mu \neq 0$ and M is a topological sphere.*

As a first application of the duality, we will obtain entire spacelike graphs in Lorentz–Minkowski space $\mathbb{L}^3 = \mathbb{L}(\mathbb{R}^2, 0, 1)$ with bounded prescribed mean curvature $H \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that ∇H is also bounded. This is achieved by constructing the dual entire minimal graphs in $\mathbb{E}(\mathbb{R}^2, H, 1)$ using the theory of divergence lines, developed in the third chapter. In $\mathbb{E}(\mathbb{R}^2, H, 1)$, we discard the possible divergence lines by applying Mazet's halfspace theorem [Maz13] and it is precisely at this point where we use that H and ∇H are bounded.

In particular, we give a partial answer to a conjecture of [LeeMan19] that states that there are entire graphs in $\mathbb{L}^3$ with any prescribed mean curvature

$H \in \mathcal{C}^\infty(\mathbb{R}^2)$. We also prove this conjecture in Lorentzian warped products $\mathbb{L}(M, 0, \mu)$ in which M, μ and H are all invariant by rotations or translations with no assumptions on the growth of H. This means that our hypotheses are not sharp because there are entire spacelike graphs in Lorentz-Minkowski space $\mathbb{L}^3 = \mathbb{L}(\mathbb{R}^2, 0, 1)$ with (equivariant) unbounded H and unbounded ∇H.

The second application of the duality is about the non-existence of entire graphs. In particular, we prove that $\mathbb{E}(M, \tau, \mu)$ does not admit any entire graph with mean curvature satisfying $\inf_M |H| > \frac{1}{2}\mathrm{Ch}(M, \mu)$ and the dual statement that $\mathbb{L}(M, \tau, \mu^{-1})$ does not admit complete space-like surfaces (of any prescribed mean curvature) if $\inf_M |\tau| > \frac{1}{2}\mathrm{Ch}(M, \mu)$. Here, $\mathrm{Ch}(M, \mu)$ is a constant that we have named *Cheeger constant with density* μ,

$$\mathrm{Ch}(M, \mu) = \inf\left\{\frac{\int_{\partial D}\mu}{\int_D \mu} : D \subset M \text{ regular}\right\} \geqslant 0.$$

This result had already been proved in [LeeMan19] in the unitary case $\mu \equiv 1$, in which $\mathrm{Ch}(M, \mu)$ is the classical Cheeger constant. In the case of the homogeneous $\mathbb{E}(\kappa, \tau)$-spaces, the value $H_0 = \frac{1}{2}\mathrm{Ch}(M, \mu)$ is the so-called *critical mean curvature*. If $H \leqslant H_0$, then there are entire graphs with constant mean curvature H in $\mathbb{E}(\kappa, \tau)$ (and compact H-surfaces cannot exist because of the maximum principle); on the contrary, if $H > H_0$, then there are compact embedded surfaces with constant mean curvature H. This dichotomy plays a crucial role in the solution of the Hopf problem in homogeneous three-manifolds, see [AbrRos04, AbrRos05, MeMiPeRo21]. Motivated by this fact, we have investigated whether or not $H_0 = \frac{1}{2}\mathrm{Ch}(M, \mu)$ distinguishes the existence of entire graphs and compact surfaces in $\mathbb{E}(M, \tau, \mu)$. In the last theorem (see Theorem 5.10), we solve completely this problem in any rotationally invariant Riemannian warped product $\mathbb{E}(M, 0, \mu)$. Remarkably, we find that, depending on the metric of M and μ, there could be some specific values of $H > H_0$ that give rise to rotationally invariant non-entire complete graphs, which we call H-*cigars*. The existence of such surfaces contradicts the expected dichotomy which happens in the classical case. We also believe that the constant $\frac{1}{2}\mathrm{Ch}(M, \mu)$ is related to the critical mean curvature in all homogeneous three-manifolds for any of their (many) Killing submersion structures.

Part I

PRELIMINARIES ON KILLING SUBMERSIONS

1

KILLING SUBMERSIONS

Let $(\mathbb{E}, \langle\cdot,\cdot\rangle)$ be a Riemannian, or Lorentzian, connected and oriented three-dimensional manifold and assume that it has a complete non-zero Killing vector field $\xi \in \mathfrak{X}(\mathbb{E})$, that is, ξ satisfies $\langle\nabla_X \xi, Y\rangle + \langle\nabla_Y \xi, X\rangle = 0$ for any $X, Y \in \mathfrak{X}(\mathbb{E})$ and its integral curves extend for all $t \in \mathbb{R}$. Furthermore, we assume ξ to be timelike if $\mathbb{E}$ is Lorentzian. We will denote by $G = \{\phi_t\}$ the one-parameter group of isometries of $\mathbb{E}$ associated to ξ and consider its natural smooth action on $\mathbb{E}$:

$$\begin{aligned} G \times \mathbb{E} &\to \mathbb{E}. \\ \phi_t \cdot p &\mapsto \phi_t(p) \end{aligned}$$

Recall that an action is said to be *free* if the only element of G that fixes any point of $\mathbb{E}$ is the identity, and *proper* if the map

$$\begin{aligned} G \times \mathbb{E} &\to \mathbb{E} \times \mathbb{E} \\ \phi_t \cdot p &\mapsto (\phi_t(p), p) \end{aligned}$$

is proper, that is the inverse image of compact subsets is compact. A classical result in Differential Geometry (see [Lee03, Theorem 9.16]) assures that if G acts freely and properly on $\mathbb{E}$, then the orbit space $\mathbb{E}/G$ is a well-defined smooth surface M that can be endowed with a unique Riemannian metric with the property that the quotient map $\pi\colon \mathbb{E} \to M$ is a Riemannian submersion, that is $d\pi_p$ is a linear isometry of the horizontal distribution $\ker(d\pi)^\perp \subset T\mathbb{E}$ for any $p \in \mathbb{E}$.

Definition 1.1 . *If* G *acts freely and properly on* $\mathbb{E}$*, we call* $\pi\colon \mathbb{E} \to M$ *a Riemannian, or Lorentzian, Killing submersion depending on the fact that* $\mathbb{E}$ *is Riemannian or Lorentzian.*

Remark 1.2. We should notice that ξ is not unique under these conditions. Indeed, multiplying ξ by a non-zero real constant, we get another Killing vector field without zeroes generating the same integral curves and such that its associated group of isometries acts freely and properly onto $\mathbb{E}$.

Examples 1.3. We now see two cases in which the Killing submersion is not defined. In the first example, the action of G is not proper, while in the second it is not free.

1. Let $\mathbb{E}$ be the product space ${}^{\mathbb{R}^2}/_{\mathbb{Z}^2} \times \mathbb{R}$ endowed with the flat metric $ds^2 = dx^2 + dy^2 + dz^2$. Consider the Killing vector field $\xi \partial_x + \sqrt{2}\partial_y$. Its integral curves are dense in $\mathbb{E}$ and diffeomorphic to $\mathbb{R}$. In particular, the action cannot be proper.

2. Let $\mathbb{E}$ be the product space $\mathbb{R}^2 \times {}^{\mathbb{R}}/_{\mathbb{Z}}$ endowed with the flat metric $ds^2 = dx^2 + dy^2 + dz^2$. Consider the Killing vector field $\xi, -\pi y \partial_x + \pi x \partial_y + \partial_z$. Its associated group of isometries is defined by

$$\phi_t(x, y, z) = (x\cos(\pi t) - y\sin(\pi t),\ x\sin(\pi t) + y\cos(\pi t),\ z + t)$$

and it describes the helicoidal motion. A direct computation implies that $\phi_1(0,0,0) = (0,0,1)$ coincides with $(0,0,0)$ in the quotient by $\mathbb{Z}$, but $\phi_1(x,0,0) = (-x,0,0)$ for any $x \in \mathbb{R} \setminus \{0\}$. That is, the action is not free.

Denoting by $\mathrm{Iso}(\mathbb{E})$ the isometry group of $\mathbb{E}$, we need to give necessary and sufficient conditions such that the action of the one-parameter subgroup $G \subseteq \mathrm{Iso}(\mathbb{E})$ is free and proper. We start by noticing that since ξ is complete, G is either isomorphic to $\mathbb{R}$ or $\mathbb{S}^1$. Endowing $\mathrm{Iso}(\mathbb{E})$ with the compact-open topology, we will see that the properness of G will depend on its topological properties as a subgroup of $\mathrm{Iso}(\mathbb{E})$. In particular, we prove the following theorem.

Theorem 1.4 . *The following conditions are equivalent:*

1. G *acts properly on* $\mathbb{E}$.

2. G *is sequentially compact.*

3. G *is closed in the compact-open topology.*

Proof. The fact that (1) and (2) are equivalent is a classical result in Riemannian Geometry, see [Lee03, Proposition 9.13] for a proof. So, it is left to prove that (2) is equivalent to (3).

Let $\{\phi_n\}$ be a sequence in G such that $\phi_n \to \phi \in \mathrm{Iso}(\mathbb{E})$. Let $p \in \mathbb{E}$. Then, $\phi_n(p) \to \phi(p)$, and so, since G is sequentially compact, it follows that there exists a subsequence $\{\phi_{n_k}\}$ such that $\phi_{n_k} \to \tilde{\phi} \in G$. The uniqueness of limit implies $\phi = \tilde{\phi}$ and so G is closed.

Conversely, assume that G is closed and let $\{\phi_n\}$ be a sequence such that $\phi_n(p) \to q$, for some $p, q \in \mathbb{E}$. By [Hel62, Theorem 2.2, pag. 167], there exists a subsequence $\{\phi_{n_k}\}$ such that $\phi_{n_k}(p) \to \phi(p)$ for a $\phi \in \mathrm{Iso}(\mathbb{E})$. Since G is closed it follows that $\phi \in G$ and thus G is sequentially compact. $\square$

Remark 1.5. If $G \equiv \mathbb{S}^1$, the action is obviously proper. If $G \equiv \mathbb{R}$ is not closed in $\mathrm{Iso}(\mathbb{E})$ a result due to Lynge and Curras-Bosch (see [Lyn73, Proposition] and [Cur79, Theorem 2.1]) states that there exist $k \geqslant 2$ Killing vector fields $X_k \in \mathfrak{X}(\mathbb{E})$ that have compact orbits and such that $[X_i, X_j] = 0$ and $\xi = \sum a_k X_k$ with $a_k \in \mathbb{R}$. This guarantees that $\dim(\mathrm{Iso}(\mathbb{E})) \geqslant 2$ and there exists another one-parameter compact closed subgroup of $\mathrm{Iso}(\mathbb{E})$ that acts properly onto $\mathbb{E}$.

It remains to study the freeness of the action.

Theorem 1.6 . *If* $G \equiv \mathbb{R}$ *is closed in* $\mathrm{Iso}(\mathbb{E})$*, then the action of* G *onto* $\mathbb{E}$ *is free.*

Proof. Since $G \subset \mathrm{Iso}(\mathbb{E})$ is closed, for each $p \in \mathbb{E}$ the integral curve of ξ passing through p is closed, that is either the integral curve is compact or it is diffeomorphic to $\mathbb{R}$ and it is not dense in $\mathbb{E}$. If all the integral curves of ξ are diffeomorphic to $\mathbb{R}$, the statement is trivially satisfied. So, let us assume that there exists $p \in \mathbb{E}$ such that the integral curve of ξ passing through p is compact. Then, there exists $c \in \mathbb{R}$ and a sequence $\{\phi_k = \phi_{kc} \neq \mathrm{id}\} \subset G$ such that $\phi_k(p) = p$ for any $k \in \mathbb{N}$. It follows from [Hel62, Theorem 2.2, pag. 167] that there exists $\phi \in G$ such that $\{\phi_k\}$ subconverges to ϕ. Now, let $q \in \mathbb{E}$ be such that the integral curve of ξ passing through q is non-compact (it exists, since $G \equiv \mathbb{R}$). In particular, up to take a subsequence, $\phi_k(q)$ does not admit any convergent subsequence, providing us a contradiction. $\square$

Remark 1.7. Assuming G to be closed is a necessary condition. If we consider the manifold $\mathbb{E}$ of the Example 1.3 2 and $\xi = -\pi y \partial_x + \pi x \partial_y + a\partial_z$ with $a \in \mathbb{R}$ being an irrational number, we have an example of a Killing vector field such that the associated one-parameter group of isometries is diffeomorphic to $\mathbb{R}$, it is not close in $\mathrm{Iso}(\mathbb{E})$ and whose action onto $\mathbb{E}$ is not free.

Finally, Example 1.3 2 gives us a clue about which condition is necessary to prove the freeness of the action of a compact group of isometries. Indeed, it is easy to compute that $\|\xi_{(x,y,z)}\| = \sqrt{4+x^2+y^2}$, while the integral curve $\gamma_{(0,0)}$ of ξ passing through $(0,0,z)$ has length $\mathrm{Length}(\gamma_{(0,0)}) = 1$ and, choosing any $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, the integral curve $\gamma_{(x,y)}$ of ξ passing through (x,y,z) has length $\mathrm{Length}(\gamma_{(x,y)}) = 2\pi\sqrt{4+(x^2+y^2)}$. In particular, denoting by $\zeta(p)$ the length of the integral curve of ξ passing through p, we get $\frac{\zeta}{\mu} = 2\pi$, for any $(x,y) \neq (0,0)$, and $\frac{\zeta(0,0,z)}{\mu(0,0,z)} = 1$. That is, the quotient $c = \frac{\zeta}{\mu}$ is not well defined. Keeping this in mind, we can prove the following result.

Theorem 1.8 . *If G is compact, we define a piecewise continuous function ζ of $\mathbb{E}$ such that for any $p \in \mathbb{E}$, $\zeta(p)$ is the length of the integral curve of ξ passing through p, where ξ is the Killing vector field associated to G. Then the action of G onto $\mathbb{E}$ is free if and only if there exists $c \in \mathbb{R}_+$ such that $\zeta = c\mu$, where $\mu = \|\xi\|$. In particular, ζ has to be a smooth.*

Proof. If the action of G is free, [Lee03, Theorem 9.24] guarantees that all the fibers are diffeomorphic to G, in particular, all the fibers have finite length. Since $G \equiv \mathbb{S}^1$, there exists $c \in \mathbb{R}^+$ such that $G = \mathbb{R}/_{c\mathbb{Z}}$. For any $p \in \mathbb{E}$, the length of the fiber above p is equal to $c\mu(p)$, in particular, $\zeta = c\mu$.

So, suppose that $\zeta = c\mu$. In particular, reasoning as above we get $G = \mathbb{R}/_{c\mathbb{Z}}$. Suppose that for a point $p \in \mathbb{E}$ there exists a $t^* \in \mathbb{R}$ such that $\phi_{t^*}(p) = p$. The fact that $\phi_{t^*}(p) = p$ implies that $t^* = m\frac{\zeta(p)}{\mu(p)} = mc$, for some $m \in \mathbb{N}$. In particular, for any $q \in \mathbb{E}$, $\phi_{t^*}(q) = \phi_{mc}(q) = \phi_c(q) = q$, that is, ϕ_{t^*} fixes all the points of $\mathbb{E}$. $\square$

So we can resume all these results as follows.

Corollary 1.9 . *The action of G is proper if and only if G is closed in the compact-open topology. If G is closed, the action is free if and only if one of these two conditions are satisfied:*

- *G is isomorphic to $\mathbb{R}$;*
- *$G \equiv \mathbb{S}^1$ and the function of $\mathbb{E}$ describing the length of the fibers is proportional to the length of the Killing vector field ξ.*

1.1 Basic Riemannian and Lorentzian Properties

Let $(\mathbb{E}, \langle\cdot,\cdot\rangle)$ be a Riemannian, or Lorentzian, connected and oriented three-dimensional manifold and suppose that it admits a Riemannian, or Lorentzian, Killing submersion structure, that is, there exits a complete non-zero Killing vector field $\xi \in \mathfrak{X}(\mathbb{E})$ whose associated one-parameter group of isometries G acts freely and properly onto $\mathbb{E}$ (recall that we assume ξ to be temporal when $\mathbb{E}$ is Lorentzian). Hence, there exists a Riemannian submersion $\pi\colon \mathbb{E} \to M = {}^{\mathbb{E}}\!/_{G}$ onto a connected and oriented Riemannian surface (M, g) such that the fibers of π are the integral curves of ξ. For every $p \in \mathbb{E}$, a tangent vector $v \in T_p\mathbb{E}$ will be called *vertical* when $v \in \ker(d\pi_p)$ and *horizontal* when $v \in \ker(d\pi_p)^{\perp}$.

The Killing field ξ naturally define a 1-form α in $\mathbb{E}$ satisfying $\alpha(X) = \langle X, \xi\rangle$ and hence the curvature 2-form $\omega = \frac{1}{2}d\alpha$ such that $\omega(X,Y) = \left\langle \overline{\nabla}_X\xi, Y\right\rangle$ for all X, $Y \in \mathfrak{X}(\mathbb{E})$, being $\overline{\nabla}$ the Levi-Civita connection in $\mathbb{E}$. Since ξ is Killing, ω is skew-symmetric, so, it can be identified with the function $\tau \in \mathcal{C}^{\infty}(\mathbb{E})$, given by

$$\tau(p) = \frac{-1}{\|\xi_p\|^2}\,\omega(e_1, e_2), \quad p \in \mathbb{E}, \tag{1.1}$$

which depends neither on the oriented orthonormal basis $\left\{e_1, e_2, \frac{\xi}{\|\xi\|}\right\}$ of $T_p\mathbb{E}$ we choose nor on rescaling ξ by a constant factor. Furthermore, since $\phi_t \in G$ is an isometry satisfying $(\phi_t)_*\xi = \xi$ and $(\phi_t)_*\omega = \omega$, both the bundle curvature and the *Killing length* $\mu = \|\xi\| \in \mathcal{C}^{\infty}(\mathbb{E})$ are constant along the fibers of π. It follows that both τ and μ induce functions in M that will be also denoted by τ, $\mu \in \mathcal{C}^{\infty}(M)$.

Remark 1.10. Notice that the fact that ξ is a Killing vector field implies $\left\langle \overline{\nabla}_X\xi, X\right\rangle = 0$ and, when $\|\xi\|$ is constant, $\left\langle \overline{\nabla}_X\xi, \xi\right\rangle = \frac{1}{2}X\left(\langle\xi,\xi\rangle\right) = 0$, for all $X \in \mathfrak{X}(\mathbb{E})$, that is, τ satisfies the well-known identity $\overline{\nabla}_X\xi = \tau X \times \xi$, where $\times$ is the cross product in $\mathbb{E}$. Thus, τ will be called the *bundle curvature* of the Killing submersion, extending previous definitions in the unitary case.

Examples 1.11. Let us see some examples of Riemannian Killing submersions and their Lorentzian counterparts:

1. Let M be a Riemannian surface. Consider the warped product $M \times_{\mu} \mathbb{R}$ with one-dimensional fibers, that is the product manifold $M \times \mathbb{R}$ endowed with the metric $\pi_M^*(ds^2) + \mu^2\pi_{\mathbb{R}}^*$, where $\mu \in \mathcal{C}^{\infty}(M)$ is a pos-

itive function and π_M and $\pi_\mathbb{R}$ denote the usual projections. A simple computation implies that $\pi_M\colon M \times_\mu \mathbb{R} \to M$ is a Riemannian Killing submersion and $\tau \equiv 0$. Furthermore, if $\|\mu\|$ is constant, we get the Riemannian product space $M \times \mathbb{R}$. Likewise, if we endow $M \times \mathbb{R}$ with the Lorentzian metric $\pi_M^*(ds^2) - \mu^2\pi_\mathbb{R}^*$, we obtain a Lorentzian Killing submersion $\pi_M\colon M \times_\mu \mathbb{R} \to M$ with Killing length μ and bundle curvature $\tau \equiv 0$.

2. The Riemannian homogeneous spaces $\mathbb{E}(\kappa,\tau)$ can be described as Riemannian Killing submersions $\pi\colon \mathbb{E}(\kappa,\tau) \to M(\kappa)$ over the Riemannian surface $M(\kappa)$ of constant curvature κ, with unitary Killing length $\mu \equiv 1$ and constant bundle curvature τ. The same happens for the Lorentzian homogenous spaces $\mathbb{L}(\kappa,\tau)$; in this case $\pi\colon \mathbb{L}(\kappa,\tau) \to M(\kappa)$ is a Lorentzian Killing submersion with unitary Killing length. See [AbrRos05, Dan07, DaHaMi09, SouVan12, Man14] for details about $\mathbb{E}(\kappa,\tau)$-spaces and [Lee13, AazRea23] for details about $\mathbb{L}(\kappa,\tau)$-spaces.

3. In general, every homogeneous Riemannian manifold homeomorphic to $\mathbb{R}^3$ is isometric to the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ described as follows. Let A be a 2×2 real matrix and denote by $\left\{a_{ij}(z)\right\}_{ij} = e^{zA} = \sum_{k=0}^{\infty} \frac{z^k A^k}{k!}$ the exponential matrix. The semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is defined as $\mathbb{R}^3$ endowed with the Lie group structure
$$(p_1,z_1) \star (p_2,z_2) = (p_1 + e^{z_1 A}p_2, z_1+z_2), \quad (p_1,z_1),(p_2,z_2) \in \mathbb{R}^2 \times \mathbb{R},$$
and with the left-invariant metric
$$\tfrac{\alpha_{22}^2+\alpha_{21}^2}{(\alpha_{22}\alpha_{11}-\alpha_{12}\alpha_{21})^2}dx^2 + \tfrac{\alpha_{12}^2+\alpha_{11}^2}{(\alpha_{22}\alpha_{11}-\alpha_{12}\alpha_{21})^2}dy^2 - 2\tfrac{\alpha_{22}\alpha_{12}+\alpha_{21}\alpha_{11}}{(\alpha_{22}\alpha_{11}-\alpha_{12}\alpha_{21})^2}dxdy + dz^2$$
Furthermore, we know that ∂_x is a non-zero right-invariant vector field, so it is Killing (see [MeePer12]) and the Riemannian Killing submersion is the projection over the last two factors $\pi(x,y,z) = (y,z)$. Thus, we can manipulate the metric obtaining
$$\frac{1}{\alpha_{22}^2+\alpha_{21}^2}\,dy^2 + dz^2 + \frac{\alpha_{22}^2+\alpha_{21}^2}{(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21})^2}\left(dx - \frac{\alpha_{11}\alpha_{21}+\alpha_{12}\alpha_{22}}{\alpha_{22}^2+\alpha_{21}^2}\,dy\right)^2.$$
Using this metric, it is clear that
$$\mu = \|\partial_x\| = \frac{\sqrt{\alpha_{22}^2+\alpha_{21}^2}}{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}},$$

and, using Equation (1.5), we can also compute

$$2\tau = \frac{\alpha_{22}^2 + \alpha_{21}^2}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} \left(\frac{\alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22}}{\alpha_{22}^2 + \alpha_{21}^2} \right)_z .$$

4. If $\mathbb{E}$ is a Riemannian homogeneous three-manifold homeomorphic to $\mathbb{S}^3$, then $\mathbb{E}$ is isometric to the three-dimensional Lie group $SU(2)$ equipped with some left-invariant metric. We can identify $SU(2)$ with the group

$$\left(\mathbb{R}_1^4 = \left\{ (a, b, c, d) \in \mathbb{R}^4 : a^2 + b^2 + c^2 + d^2 = 1 \right\}, \star \right),$$

where

$$\begin{aligned} (a_1, b_1, c_1, d_1) \star (a_2, b_2, c_2, d_2) = \quad & (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2, \\ & a_2 b_1 + a_1 b_2 + c_1 d_2 - c_2 d_1, \\ & a_2 c_1 + a_1 c_2 + b_2 d_1 - b_1 d_2, \\ & b_1 c_2 - b_2 c_1 + a_2 d_1 + a_1 d_2) \end{aligned}$$

It is not difficult to see that $\xi \in \mathfrak{X}(\mathbb{E})$ defined such that

$$\xi_{(a,b,c,d)} = (-b, a, -d, c)$$

is a right-invariant vector field, that is, ξ is Killing for any left-invariant metric of $(\mathbb{R}_1^4, \star)$ and that the integral curve $\gamma_{(a,b,c,d)}(t)$ of ξ passing through $(a, b, c, d) \in \mathbb{R}_1^4$ is given by

$$(\cos(t)a - \sin(t)b, \sin(t)a + \cos(t)b, \cos(t)c - \sin(t)d, \sin(t)c + \cos(t)d) .$$

In particular, all the integral curves are compact and the one-parameter group of isometries of $\mathbb{E}$ associated to ξ is diffeomorphic to $\mathbb{S}^1$. Furthermore, noticing that $\gamma'_{(a,b,c,d)}(t) = \xi_{\gamma_{(a,b,c,d)}(t)}$, it follows that the length of $\gamma_{(a,b,c,d)}(t)$ is equal to $2\pi \|\xi_{(a,b,c,d)}\|$, that is, Theorem 1.8 is satisfied and there exists a Killing submersion structure. The Killing submersion defined by ξ is the Hopf fibration

$$\begin{array}{rccc} \pi_H : & \mathbb{R}_1^4 & \to & \mathbb{S}^2 \subset \mathbb{R}^3 \\ & (a, b, c, d) & \mapsto & \big(2(ad + bc), 2(bd - ac), a^2 + b^2 - c^2 - d^2\big) , \end{array}$$

where $\mathbb{S}^2$ is endowed with a Riemannian metric that makes π_H a Riemannian submersion.

1.2 Local Structure

The goal of this section is to give a local canonical structure to study the Killing submersion $\pi\colon \mathbb{E} \to M$. To this end, we use the one-parameter group of vertical translations $\{\phi_t\}_{t\in\mathbb{R}}$ associated to ξ and the existence of local sections. Recall that, if $\pi\colon \mathbb{E} \to M$ is a Killing submersion whose fibers have infinite length, then there exists a global section, that is, there exists a map $F_0\colon M \to \mathbb{E}$ such that $\pi \circ F_0 = \mathrm{id}_M$ is the identity map (see [Ste51, Theorem 12.2]). The same is true if M is non-compact (see [GrHaVa76, Section VIII.5]). It is not restrictive to assume that the fibers have infinite length, since we can always pass to the universal cover.

Let $U \subset M$ be a simply connected neigborhood of $p \in M$ parameterized by $\varphi\colon (\Omega, ds^2_\Omega) \to U$, where $\Omega \subset \mathbb{R}^2$ is an open domain of the plane and $ds^2_\Omega = \lambda_1^2 dx^2 + \lambda_2^2 dy^2$ for some positive λ_1, $\lambda_2 \in \mathcal{C}^\infty(\Omega)$. Choosing the metric for M in this way, which is like using orthogonal coordinates and it extends the situation explained in [LerMan17], is helpful because it's simpler to get than the conformal option (check Example 1.11.2). Choosing a smooth section $F_0\colon U \to \mathbb{E}$ over U, we can consider the local diffeomorphism

$$\begin{array}{rccc} \psi\colon & \Omega\times\mathbb{R} & \to & \pi^{-1}(U) \\ & (x,y,t) & \mapsto & \phi_t(F_0(\varphi(x,y))) \end{array}$$

which makes the following diagram commutative

$$\begin{tikzcd} \Omega\times\mathbb{R} \arrow[r, "\psi"] \arrow[d, "\pi_1"] & \pi^{-1}(U) \arrow[d, "\pi"] \\ \Omega \arrow[r, "\varphi"] & U \end{tikzcd}$$

where $\pi_1\colon \Omega\times\mathbb{R} \to \Omega$ is the projection over the first factor. Now we can induce in $\Omega\times\mathbb{R} \subset \mathbb{R}^3$ the metric ds^2 that makes ψ an isometry, so that π_1 becomes a Killing submersion over (Ω, ds^2_Ω). To do so, we consider in (Ω, ds^2_Ω) the orthonormal frame $\left\{e_1 = \frac{1}{\lambda_1}\partial_x, e_2 = \frac{1}{\lambda_2}\partial_y\right\}$ which can be lifted via π_1 to the orthonormal frame $\{E_1, E_2\}$ of the horizontal distribution, which is orthogonal to $\xi = \partial_t$. Since π_1 is the canonical projection on the first two variables, there exist two functions a, $b \in \mathcal{C}^\infty(\Omega)$ such that

$$\begin{aligned} (E_1)_{(x,y,z)} &= \tfrac{1}{\lambda_1(x,y)}\,\partial_x + a(x,y)\partial_t, \\ (E_2)_{(x,y,z)} &= \tfrac{1}{\lambda_2(x,y)}\,\partial_y + b(x,y)\partial_t, \\ (E_3)_{(x,y,z)} &= \tfrac{1}{\mu(x,y)}\,\partial_t, \end{aligned} \tag{1.2}$$

define a positively oriented frame in $\Omega \times \mathbb{R}$. Notice that E_1 and E_2 are spacelike, whereas E_3 is spacelike in the Riemannian case and timelike in the Lorentzian case, and $\xi = \partial_t = \mu E_3$ is the Killing vector field. Therefore, the ambient metric in $\mathbb{E}$ can be written as

$$ds^2 = \lambda_1^2 dx^2 + \lambda_2^2 dy^2 + \epsilon\mu^2 \left(dt - \lambda_1 a dx - \lambda_2 b dy\right)^2, \tag{1.3}$$

where $\varepsilon = \pm 1$ depending on whether $\mathbb{E}$ is Riemannian or Lorentzian.

For any choice of a, $b \in \mathcal{C}^\infty(\Omega)$, Equation (1.3) defines a Riemannian (resp. Lorentzian) metric in $\Omega \times \mathbb{R}$ such that the projection π_1 is a Riemannian (resp. Lorentzian) submersion and ∂_t is a Killing vector field of length μ, constant along the fibers. In the next few lines, we will see that choosing a and b determines τ (see Equation (1.5)). Using the definition in (1.2), a simple computation implies that

$$\begin{aligned} [E_1, E_2] &= \tfrac{(\lambda_1)_y}{\lambda_1\lambda_2} E_1 - \tfrac{(\lambda_2)_x}{\lambda_1\lambda_2} E_2 + \tfrac{\mu}{\lambda_1\lambda_2}\left((\lambda_2 b)_x - (\lambda_1 a)_y\right) E_3, \\ [E_1, E_3] &= \tfrac{-\mu_x}{\lambda_1\mu} E_3, \qquad\qquad [E_2, E_3] = \tfrac{-\mu_y}{\lambda_2\mu} E_3. \end{aligned} \tag{1.4}$$

So, using (1.1) we deduce that

$$\begin{aligned} \tau &= -\tfrac{1}{\mu}\langle \nabla_{E_1}\partial_t, E_2\rangle = \langle \nabla_{E_1} E_2, E_3\rangle \\ &= \tfrac{1}{2}\langle [E_1, E_2], E_3\rangle = \tfrac{\epsilon\mu}{2\lambda_1\lambda_2}\left((\lambda_2 b)_x - (\lambda_1 a)_y\right) \\ &= \tfrac{\epsilon\mu}{2\lambda_1\lambda_2}\operatorname{div}_0\left(\lambda_2 b\partial_x - \lambda_1 a \partial_y\right), \end{aligned} \tag{1.5}$$

where div_0 is the divergence of the flat metric $dx^2 + dy^2$ in Ω.

Remark 1.12. If τ and μ are prescribed, there is a standard way of integrating (1.5) to obtain a and b. Assuming that $\Omega \subset \mathbb{R}^2$ is star-shaped with respect to the origin, the function

$$\mathbf{C}_{M,\tau,\mu}(x, y) = 2\int_0^1 s\,\frac{\tau(sx, sy)\,\lambda_1(sx, sy)\,\lambda_2(sx, sy)}{\mu(sx, sy)} ds \tag{1.6}$$

will be called the *Calabi potential*. It is straightforward to check that the following choice for a and b satisfies Equation (1.5):

$$a = \frac{-\epsilon y\, \mathbf{C}_{M,\tau,\mu}}{\lambda_1}, \qquad\qquad b = \frac{\epsilon x\, \mathbf{C}_{M,\tau,\mu}}{\lambda_2}. \tag{1.7}$$

Any other pair of functions $\widetilde{a}$ and $\widetilde{b}$ satisfying (1.5) produces another isometric metric which is nothing but a change of zero section. Indeed, equation (1.5) yields $\left((\lambda_2 b)_x - (\lambda_1 a)_y\right) = \left((\lambda_2\widetilde{b})_x - (\lambda_1\widetilde{a})_y\right)$, that is

$$\left(\lambda_2(b - \widetilde{b})\right)_x = \left(\lambda_1(a - \widetilde{a})\right)_y.$$

Since Ω is simply connected, Poincaré's lemma guarantees that there exists a function $d \in \mathcal{C}^\infty(\Omega)$ such that $\lambda_2(b-\widetilde{b}) = d_y$ and $\lambda_1(a-\widetilde{a}) = d_x$. If we denote by

$$d\widetilde{s}^2 = \lambda_1^2 dx^2 + \lambda_2^2 dy^2 + \epsilon\mu^2 \left(dt - \lambda_1 \widetilde{a} dx - \lambda_2 \widetilde{b} dy\right)^2,$$

the map

$$\begin{aligned} R\colon \ \left(\Omega \times \mathbb{R}, ds^2\right) &\to \left(\Omega \times \mathbb{R}, d\widetilde{s}^2\right) \\ (x, y, t) &\mapsto (x, y, t - d(x, y)) \end{aligned}$$

is an isometry that is equivalent to changing the zero section.

1.2.1 *The curvature tensor*

Our next goal is to compute the Riemann curvature tensor of the total space of a Killing submersion $\pi : \mathbb{E} \to M$ to understand its geometry. Since the computation is local, we will employ the coordinates we have just introduced, where $\mathbb{E}$ is (locally) identified with $\Omega \times \mathbb{R}$ for some $\Omega \subseteq \mathbb{R}^2$ with the metric in (1.3) for some positive functions $\lambda_1, \lambda_2, \mu \in \mathcal{C}^\infty(\Omega)$ and arbitrary functions $a, b \in \mathcal{C}^\infty(\Omega)$. Using (1.4), (1.5) and Koszul formula, we can write the Levi-Civita connection $\overline{\nabla}$ of $\mathbb{E}$ in the frame $\{E_1, E_2, E_3\}$ given by (1.2):

$$\begin{aligned}
&\overline{\nabla}_{E_1} E_1 = -\tfrac{(\lambda_1)_y}{\lambda_1\lambda_2} E_2, && \overline{\nabla}_{E_1} E_2 = \tfrac{(\lambda_1)_y}{\lambda_1\lambda_2} E_1 + \epsilon\tau E_3, && \overline{\nabla}_{E_1} E_3 = -\tau E_2,\\
&\overline{\nabla}_{E_2} E_1 = \tfrac{(\lambda_2)_x}{\lambda_1\lambda_2} E_2 - \epsilon\tau E_3, && \overline{\nabla}_{E_2} E_2 = -\tfrac{(\lambda_2)_x}{\lambda_1\lambda_2} E_1, && \overline{\nabla}_{E_2} E_3 = \tau E_1,\\
&\overline{\nabla}_{E_3} E_1 = -\tau E_2 + \tfrac{\mu_x}{\lambda_1\mu} E_3, && \overline{\nabla}_{E_3} E_2 = \tau E_1 + \tfrac{\mu_y}{\lambda_2\mu} E_3, && \overline{\nabla}_{E_3} E_3 = -\tfrac{\epsilon}{\mu}\overline{\nabla}\mu,
\end{aligned} \tag{1.8}$$

where $\overline{\nabla}\mu = \frac{\mu_x}{\lambda_1} E_1 + \frac{\mu_y}{\lambda_2} E_2$.

Therefore, we can work out $\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z$, the three-variable Riemann curvature tensor, over this frame to obtain

$$\begin{aligned}
R(E_1, E_2)E_1 &= -(K_M - 3\epsilon\tau^2)E_2 - \epsilon\langle T, E_1\rangle E_3,\\
R(E_1, E_2)E_2 &= (K_M - 3\epsilon\tau^2)E_1 - \epsilon\langle T, E_2\rangle E_3,\\
R(E_1, E_2)E_3 &= \langle T, E_1\rangle E_1 + \langle T, E_2\rangle E_2,\\
R(E_1, E_3)E_1 &= -\langle T, E_1\rangle E_2 - (\epsilon\tau^2 - a_{11})E_3,\\
R(E_1, E_3)E_2 &= \langle T, E_1\rangle E_1 + a_{12}E_3,\\
R(E_1, E_3)E_3 &= (\tau^2 - \epsilon a_{11})E_1 - \epsilon a_{12}E_2,\\
R(E_2, E_3)E_1 &= -\langle T, E_2\rangle E_2 + a_{21}E_3,\\
R(E_2, E_3)E_2 &= \langle T, E_2\rangle E_1 - (\epsilon\tau^2 - a_{22})E_3,\\
R(E_2, E_3)E_3 &= -\epsilon a_{21}E_1 + (\tau^2 - \epsilon a_{22})E_2,
\end{aligned}$$

where $T = \overline{\nabla}\tau + \frac{2\tau}{\mu}\overline{\nabla}\mu$ and $a_{ij} = \frac{1}{\mu}\overline{\mathrm{Hess}}(\mu)(E_i, E_j)$. Here, the Hessian is defined by $\overline{\mathrm{Hess}}(\mu)(X,Y) = X(Y(\mu)) - (\overline{\nabla}_X Y)(\mu)$ for all vector fields X and Y in $\mathbb{E}$. These coefficients a_{ij} are explicitly given by

$$a_{11} = \tfrac{1}{\mu}E_1(E_1(\mu)) + \tfrac{1}{\lambda_1\mu}E_2(\lambda_1)E_2(\mu), \quad a_{12} = \tfrac{1}{\mu}E_1(E_2(\mu)) - \tfrac{1}{\lambda_1\mu}E_2(\lambda_1)E_1(\mu),$$
$$a_{21} = \tfrac{1}{\mu}E_2(E_1(\mu)) - \tfrac{1}{\lambda_2\mu}E_1(\lambda_2)E_2(\mu), \quad a_{22} = \tfrac{1}{\mu}E_2(E_2(\mu)) + \tfrac{1}{\lambda_2\mu}E_1(\lambda_2)E_1(\mu).$$

Recall that $a_{12} = a_{21}$ by the symmetry of the Hessian. Also, in the above computations, we have introduced the Gauss curvature of M given by

$$K_M = \frac{(\lambda_1)_x(\lambda_2)_x\lambda_2^2 + (\lambda_1)_y(\lambda_2)_y\lambda_1^2}{\lambda_1^3\lambda_2^3} - \frac{(\lambda_2)_{xx}\lambda_2 + (\lambda_1)_{yy}\lambda_1}{\lambda_1^2\lambda_2^2},$$

which is computed using the classical formula for orthogonal coordinates

$$K = \frac{-1}{2\sqrt{EG}}\left(\left(\frac{E_y}{\sqrt{EG}}\right)_y + \left(\frac{G_x}{\sqrt{EG}}\right)_x\right)$$

where the first fundamental form is

$$E = \lambda_1^2, \qquad F = 0, \qquad G = \lambda_2^2.$$

The four-variable Riemann curvature tensor $\overline{R}(X,Y,Z,W) = \langle \overline{R}(X,Y)Z, W\rangle$ can be computed coordinate-freely as follows.

Proposition 1.13 . *If X, Y, Z, W are vector fields in $\mathbb{E}$, then*

$$\begin{aligned}\overline{R}(X,Y,Z,W) = &-\tau^2\langle X\times Y, Z\times W\rangle - (K_M - 4\epsilon\tau^2)\langle X\times Y, E_3\rangle\langle Z\times W, E_3\rangle \\ &+ \langle X\times Y, E_3\rangle\langle Z\times W, E_3\times T\rangle + \langle Z\times W, E_3\rangle\langle X\times Y, E_3\times T\rangle \\ &+ \tfrac{\epsilon}{\mu}\overline{\mathrm{Hess}}(\mu)((X\times Y)\times E_3, (Z\times W)\times E_3).\end{aligned}$$

In particular, the sectional curvature of a spacelike plane $\Pi \subset T_p\mathbb{E}$ is given by

$$\begin{aligned}\overline{K}(\Pi) = &\,\epsilon\tau^2 + (K_M - 4\epsilon\tau^2)\langle n, E_3\rangle^2 - 2\langle n, E_3\rangle\langle n\times E_3, T\rangle \\ &- \tfrac{\epsilon}{\mu}\overline{\mathrm{Hess}}(\mu)(n\times E_3, n\times E_3),\end{aligned}$$

where $n \in T_p\mathbb{E}$ is a unit normal to Π.

Proof. It suffices to check that both sides coincide on the frame $\{E_1, E_2, E_3\}$, which is a straightforward computation. It is important to notice first that $E_1 \times E_2 = \epsilon E_3$, $E_2 \times E_3 = E_1$ and $E_3 \times E_1 = E_2$ by definition of cross product. As for the sectional curvature, we choose an orthonormal basis $\{u, v\}$ of Π such that $u \times v = n$ and then compute $\overline{K}(\Pi) = \overline{R}(u, v, v, u)$ taking into account that $\langle n, n\rangle = \epsilon$. □

Remark 1.14. The structure of the expression for $\overline{R}(X, Y, Z, W)$ is meaningful. The first summand is the curvature of a space form of constant curvature since $\langle X \times Y, Z \times W \rangle = \langle X, Z \rangle \langle Y, W \rangle - \langle Y, Z \rangle \langle X, W \rangle$. The second summand shows up in homogeneous spaces $\mathbb{E}(\kappa, \tau)$ and $\mathbb{L}(\kappa, \tau)$ with four-dimensional isometry group for the standard Killing submersion over $M^2(\kappa)$. The next two summands appear in arbitrary Killing submersions with unitary Killing vector field (see also [Man14, Lem. 5.1]). The last summand containing the Hessian only appears if the Killing vector field has non-constant length.

1.3 Classification of Killing submersions

In this section we recall the classification result for Riemannian Killing submersions (see [LerMan17, Section 2]) and we extend them to Lorentzian Killing submersions. The arguments in Remark 1.12 imply a local classification result for Killing submersions when M is non-compact, which was proved in [LerMan17, Theorem 2.6]. Furthermore, when M is diffeomorphic to $\mathbb{S}^2$, we can use the same argument of [LerMan17, Theorem 2.9] to complete the classification of Killing submersions when M is simply connected, obtaining the following statement:

Theorem 1.15 . *Let M be a simply connected Riemannian surface, and let $\tau, \mu \in \mathcal{C}^\infty(M)$, $\mu > 0$. Then there exists a Killing submersion $\pi \colon \mathbb{E} \to M$ such that*

1. *the fibers of π have infinite length,*
2. *τ is the bundle curvature of π, and*
3. *μ is the length of a Killing field ξ whose integral curves are the fibers of π.*

Moreover, $\pi \colon \mathbb{E} \to M$ is unique in the sense that if $\pi_0 \colon \mathbb{E}' \to M$ is another Riemannian (resp. Lorentzian) Killing submersion satisfying conditions (1), (2) and (3) above, then there exists an isometry $T \colon \mathbb{E} \to \mathbb{E}'$ such that $\pi_0 \circ T = \pi$. Furthermore, when M is compact, we have that:

(a) *if $\int_M \frac{\tau}{\mu} = 0$, then the length of the fibers of π is infinite and π is isomorphic to*

$$\begin{aligned} \pi_1 \colon \quad (\mathbb{S}^2 \times \mathbb{R}, ds^2) &\to \mathbb{S}^2 \\ (p, t) &\mapsto p \end{aligned}$$

for some Riemannian (resp. Lorentzian) metric ds^2, with (temporal) Killing vector field $\xi_{(p,t)} = \partial_t$;

(b) *if* $\int_M \frac{\tau}{\mu} \neq 0$, *then the fibers of* π *are compact and* π *is isomorphic to the Hopf fibration*

$$\begin{array}{rccc} \pi_H\colon & (\mathbb{S}^3, ds^2) & \to & \mathbb{S}^2 \\ & (z,w) & \mapsto & (2zw, |z|^2 - |w|^2) \end{array}$$

for some Riemannian (resp. Lorentzian) metric ds^2 *in* $\mathbb{S}^3$ *with (temporal) Killing vector field* $\xi_{(z,w)} = (iz, iw)$. *Here* $\mathbb{S}^3$ *and* $\mathbb{S}^2$ *are the unit spheres in* $\mathbb{C}^2$ *and* $\mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$, *respectively.*

By means of this theorem, we can identify $\mathbb{E} = \mathbb{E}(M, \tau, \mu, \epsilon)$, where (M, τ, μ) is the triple defining the Killing submersion and $\epsilon = \pm$ describe the character of the Killing vector field, and consequently of $\mathbb{E}$. Sometimes we will denote by $\mathbb{E}(M, \tau, \mu) = \mathbb{E}(M, \tau, \mu, 1)$ and $\mathbb{L}(M, \tau, \mu) = \mathbb{E}(M, \tau, \mu, -1)$ to simplify the notation.

The proof of this theorem is omitted since it is quite technical and can be found in [Man13, Chapter 1.2] and [LerMan17, Section 2]. For completeness, we give a proof of the technical lemmas that are necessary to prove the theorem and whose proof differs in the Lorentzian case. The first lemma assures the existence of a unique horizontal lifting of any curve of M passing through a fixed point of $\mathbb{E}$.

Lemma 1.16 . *Given a piecewise* $\mathcal{C}^1$*-function* $\alpha\colon [a, b] \to M$ *and* $p_0 \in \mathbb{E}$ *such that* $\pi(p_0) = \alpha(a)$, *there exists a unique horizontal lifting* $\widetilde{\alpha}$ *of* α *such that* $\widetilde{\alpha}(a) = p_0$.

Proof. Consider a partition $a = t_0 < t_1 < \ldots < t_n = b$ such that the restriction to each segment $\alpha_{|[t_{i-1}, t_i]}$ is of class C^1. We refine this partition to ensure that $\alpha([t_{i-1}, t_i])$ lies completely within a specific chart (U_i, φ_i) of M, as it has been described in the previous section. Once we establish the existence and uniqueness for the lifting of each segment, it becomes evident that the statement will be proven.

Thus, without loss of generality, we can assume that the curve α itself lies within a chart (U, φ), and consequently, $\widetilde{\alpha}$ will be confined to $\pi^{-1}(U)$. This allows us to work within the chart on $\varphi(U) \times \mathbb{R}$, as described in the previous section. Writing α in coordinates as $\alpha(t) = (x(t), y(t)) \in \varphi(U)$, a horizontal lifting must have the form $\widetilde{\alpha}(t) = (x(t), y(t), z(t))$ for some function $z(t)$. Being horizontal is equivalent to satisfying $\langle\widetilde{\alpha}', \partial_t\rangle = 0$, which can be expressed as the differential equation

$$z' = \lambda_1 a x' + \lambda_2 b y' \tag{1.9}$$

where λ_1, λ_2, a and b are evaluated at (x, y). Since $\pi(p_0) = \alpha(a)$, we have that $p_0 = (x(a), y(a), z_0)$ in this parameterization for some $z_0 \in \mathbb{R}$. We deduce that there exists a unique $\mathcal{C}^1$-function $z(t)$ satisfying the equation (1.9) with the initial condition $z(a) = z_0$. Hence, the lifting exists and is unique. □

The second result extends [LerMan17, Proposition 2.8] to the Lorentzian case and it describes how, fixed a closed curve $\alpha \subset M$, its orientation and the bundle curvature of $\mathbb{E}$ affect its horizontal lifting.

Proposition 1.17 . *Let $\pi\colon \mathbb{E} \to M$ be a Killing submersion whose fibers have infinite length, and let $\alpha\colon [a, b] \to M$ be a simple $\mathcal{C}^1$-curve bounding an orientable relatively compact open set $O \subset M$. Assume that α is oriented such that the interior of O lies on the left side of α. Given a horizontal lift $\widetilde{\alpha}$ of α, there exists a unique $d \in \mathbb{R}$ such that $\phi_d(\widetilde{\alpha}(a)) = \widetilde{\alpha}(b)$ and it satisfies*

$$\int_O \frac{2\tau}{\mu} = d.$$

Proof. Consider an atlas of M. If the trace of α is not contained in one of the charts of the atlas, we can find a triangulation of the open set Ω. This triangulation consists of a finite number of piecewise regular triangles, denoted as T_n, each of which lies within an open set of the atlas. With this triangulation, it becomes possible to express α as a finite sum of the boundaries of these triangles, while following a consistent orientation that ensures shared edges are traversed twice but in opposite directions. So, without loss of generality we can assume that O is contained in one chart (U, φ) of the atlas and work with the parameterization defined in the previous section. Then, using Equation (1.5) and the Stokes Theorem, we get

$$\begin{aligned}\int_O \frac{2\epsilon\tau}{\mu} &= \int_{\varphi^{-1}(O)} \frac{2\epsilon\tau}{\mu}\lambda_1\lambda_2 dxdy = \int_{\varphi^{-1}(O)} \mathrm{div}_0\left(\lambda_2 b\partial_x - \lambda_1 a\partial_y\right) dxdy \\ &= \int_{\partial\varphi^{-1}(O)} \langle\lambda_2 b\partial_x - \lambda_1 a\partial_y, \eta\rangle,\end{aligned}$$

where η is the unit exterior conormal to $\partial\varphi^{-1}(O)$. If we write $\alpha = (x, y)$ and $\widetilde{\alpha} = (x, y, z)$ and assume that α is parameterized by arc length (i.e., $(x'\lambda_1)^2 + (y'\lambda_2)^2 = 1$), then $\eta = y'\partial_x - x'\partial_y$, and using (1.9), we can write

$$\int_O \frac{2\tau}{\mu} = \int_a^b \lambda_1 a x' + \lambda_2 b y' = \int_a^b z' = z(b) - z(a),$$

and complete the proof. □

Remark 1.18. We can establish a classification result by relaxing the topological assumptions, specifically, by considering cases where neither $\mathbb{E}$ nor M need to be simply connected. More precisely, when we assume that $\pi\colon \mathbb{E} \to M$ is a Killing submersion over an arbitrary orientable surface M with bundle curvature τ and Killing length μ, it can be shown that π can be treated as a quotient of a Killing submersion over simply connected surfaces by a subgroup of $\mathrm{Iso}(\mathbb{E})$ acting properly and discontinuously on $\mathbb{E}$, and consisting of isometries that preserve ξ, which have been classified in previous results. However, it is important to note that uniqueness is not guaranteed in this context. Due to the broader scope of this thesis, we refrain from delving into the details of this result, which can be found in [LerMan17, Section 2.3].

1.4 Geodesics and completeness

Consider a Killing submersion $\pi\colon \mathbb{E} \to M$, a curve $\alpha\colon [a, b] \to M$ and its horizontal lifting $\widetilde{\alpha}\colon [a, b] \to \mathbb{E}$, which is unique if we fix $\widetilde{\alpha}(a)$ over the fiber of $\alpha(a)$. Given two vector fields $X, Y \in \mathfrak{X}(M)$ and their horizontal lifting $\overline{X}, \overline{Y} \in \mathfrak{X}(\mathbb{E})$, it holds that

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla_X Y} + \left[\overline{X}, \overline{Y}\right]^{\nu}, \tag{1.10}$$

where ∇ and $\overline{\nabla}$ denote the Levi-Civita connections on M and $\mathbb{E}$, respectively, $\overline{\nabla_X Y}$ is the horizontal lifting of $\nabla_X Y$, and $\left[\overline{X}, \overline{Y}\right]^{\nu}$ is the vertical part of $\left[\overline{X}, \overline{Y}\right]$. Applying (1.10) to compute $\overline{\nabla}_{\widetilde{\alpha}'}\widetilde{\alpha}'$, it follows that $\overline{\nabla}_{\widetilde{\alpha}'}\widetilde{\alpha}'$ is the horizontal lifting of $\nabla_{\alpha'}\alpha'$. In particular, the horizontal lifting of a geodesic of M is a geodesic of $\mathbb{E}$. Furthermore, (1.8) implies that, if $p \in M$ is a critical point of μ, then the fiber $\pi^{-1}(p)$ above p is a geodesic of $\mathbb{E}$.

We now give a local description of the remaining geodesics. If $\gamma \subset \mathbb{E}$ is a geodesic, then $\langle\dot{\gamma}, \xi\rangle$ is constant. Indeed, taking the derivative,

$$\frac{d}{dt}\langle\dot{\gamma}, \xi\rangle = \left\langle\overline{\nabla}_{\dot{\gamma}}\dot{\gamma}, \xi\right\rangle + \left\langle\overline{\nabla}_{\dot{\gamma}}\xi, \dot{\gamma}\right\rangle = 0.$$

The first element in the right-hand side of the equation is 0 because γ is a geodesic, while the second one is 0 because ξ is Killing.

When $\epsilon = -1$, a curve $\alpha \subset \mathbb{E}$ is said to be *spacelike* when $\langle \alpha', \alpha' \rangle > 0$, *lightlike* when $\langle \alpha', \alpha' \rangle = 0$, *timelike* when $\langle \alpha', \alpha' \rangle < 0$.

Let $\alpha\colon (-\varepsilon, \varepsilon) \to M$ be a smooth curve and $\widetilde{\alpha}\colon (-\varepsilon, \varepsilon) \to \mathbb{E}$ be its horizontal lifting. For any fixed constant $\omega \in \mathbb{R}$ (with $|\omega| < \mu(\alpha(0))$ if $\mathbb{E}$ is Riemannian) assume that $\|\alpha'(t)\| = c - \frac{\epsilon\omega^2}{\mu^2(\alpha(t))}$, where $c = 1$ when $\mathbb{E}$ is Riemannian and $c = \{-1, 0, 1\}$, depending on the causal character of the geodesic that we are going to describe, when $\mathbb{E}$ is Lorentzian. We can consider the smooth curve

$$\gamma\colon (-\varepsilon, \varepsilon) \to \mathbb{E}, \qquad \gamma(t) = \phi_{f(t)}(\widetilde{\alpha}(t)),$$

where $f(t) = \int \frac{\omega dt}{\mu^2(\alpha(t))}$. The chain rule allows us to compute

$$\dot{\gamma}(t) = \tfrac{\omega}{\mu^2}\xi_{\widetilde{\alpha}'(t)} + \widetilde{\alpha}'(t). \tag{1.11}$$

In particular, we have that

$$\|\dot{\gamma}\|^2 = \tfrac{\epsilon\omega^2}{\mu^2} + \|\widetilde{\alpha}'\|^2_{\mathbb{E}} = \tfrac{\epsilon\omega^2}{\mu^2} + \|\alpha'\|^2_{M} = \tfrac{\epsilon\omega^2}{\mu^2} + c - \tfrac{\epsilon\omega^2}{\mu^2} = c$$

and, since $\widetilde{\alpha}'$ is horizontal, it follows

$$\langle \dot{\gamma}, \xi \rangle = \left\langle \tfrac{\omega}{\mu^2}\xi, \xi \right\rangle = \omega \langle E_3, E_3 \rangle = \epsilon\omega,$$

so γ will be our candidate to be a geodesic.

Using (1.11), it is easy to compute $\overline{\nabla}_{\dot{\gamma}}\dot{\gamma}$.

$$\begin{aligned}
\overline{\nabla}_{\dot{\gamma}}\dot{\gamma} =& \overline{\nabla}_{\frac{\omega}{\mu^2}\xi}\tfrac{\omega}{\mu^2}\xi + \overline{\nabla}_{\frac{c}{\mu^2}\xi}\widetilde{\alpha}' + \overline{\nabla}_{\widetilde{\alpha}'}\tfrac{\omega}{\mu^2}\xi + \overline{\nabla}_{\widetilde{\alpha}'}\widetilde{\alpha}' \\
=& \tfrac{\omega^2}{\mu^2}\overline{\nabla}_{E_3}E_3 + \tfrac{\omega}{\mu^2}\left(\overline{\nabla}_{\xi}\widetilde{\alpha}' + \overline{\nabla}_{\widetilde{\alpha}'}\xi\right) + \left\langle \widetilde{\alpha}', \overline{\nabla}\tfrac{\omega}{\mu^2} \right\rangle \xi + \overline{\nabla}_{\widetilde{\alpha}'}\widetilde{\alpha}' \\
=& -\tfrac{\epsilon\omega^2}{\mu^3}\overline{\nabla}\mu + \tfrac{\omega}{\mu^2}\left(\overline{\nabla}_{\xi}\widetilde{\alpha}' + \overline{\nabla}_{\widetilde{\alpha}'}\xi\right) + \left\langle \widetilde{\alpha}', \overline{\nabla}\tfrac{\omega}{\mu^2} \right\rangle \xi + \overline{\nabla}_{\widetilde{\alpha}'}\widetilde{\alpha}'
\end{aligned} \tag{1.12}$$

We first notice that $\left\langle \overline{\nabla}_{\dot{\gamma}}\dot{\gamma}, \xi \right\rangle = \dot{\gamma}\left(\langle \dot{\gamma}, \xi \rangle\right) - \left\langle \dot{\gamma}, \overline{\nabla}_{\dot{\gamma}}\xi \right\rangle = \dot{\gamma}(\omega) = 0$. We also notice that

$$\begin{aligned}
\left\langle \overline{\nabla}_{\xi}\widetilde{\alpha}', \widetilde{\alpha}' \right\rangle &= \tfrac{1}{2}\xi\left(\langle \widetilde{\alpha}', \widetilde{\alpha}' \rangle\right) = \xi\left(c - \tfrac{\epsilon\omega^2}{\mu^2}\right) = 0, \\
\left\langle \overline{\nabla}_{\widetilde{\alpha}'}\xi, \widetilde{\alpha}' \right\rangle &= 0, \\
\left\langle \overline{\nabla}_{\widetilde{\alpha}'}\widetilde{\alpha}', \widetilde{\alpha}' \right\rangle &= \tfrac{1}{2}\langle \widetilde{\alpha}', \widetilde{\alpha}' \rangle = \tfrac{1}{2}\widetilde{\alpha}'\left(c - \tfrac{\epsilon\omega^2}{\mu^2}\right) \\
&= -\tfrac{\epsilon\omega^2}{2}\left\langle \widetilde{\alpha}', \overline{\nabla}\tfrac{1}{\mu^2} \right\rangle = \tfrac{\epsilon\omega^2}{\mu^3}\left\langle \widetilde{\alpha}', \overline{\nabla}\mu \right\rangle,
\end{aligned}$$

that implies $\left\langle \overline{\nabla}_{\dot\gamma}\dot\gamma, \widetilde{\alpha}' \right\rangle = 0$.

Now consider J the $\frac{\pi}{2}$−rotation in TM such that, if $J\widetilde{\alpha}'$ is the horizontal lifting of $J\alpha'$, then $\{\widetilde{\alpha}', J\widetilde{\alpha}', \xi\}$ is an oriented orthogonal basis of $T_\gamma\mathbb{E}$. Thus, γ is a geodesic if and only if $\left\langle \overline{\nabla}_{\dot\gamma}\dot\gamma, J\widetilde{\alpha}' \right\rangle = 0$.

Notice that $\left\langle \overline{\nabla}_{\widetilde{\alpha}'}\widetilde{\alpha}', J\widetilde{\alpha}' \right\rangle = \langle \nabla_{\alpha'}\alpha', J\alpha' \rangle = \kappa_g \|\alpha'\|^3$ and, using the local frame (1.2) and the Levi-Civita connection in (1.8), a straightforward computation implies

$$\begin{aligned} \left\langle \overline{\nabla}_{\widetilde{\alpha}'}\xi, J\widetilde{\alpha}' \right\rangle &= - \left\langle \alpha', \nabla\mu \right\rangle \tau \|\alpha'\|^2, \\ \left\langle \overline{\nabla}_{\xi}\widetilde{\alpha}', J\widetilde{\alpha}' \right\rangle &= - \mu\tau\|\alpha'\|^2. \end{aligned}$$

Thus, γ is a geodesic in $\mathbb{E}$ if and only if the geodesic curvature of α in M satisfies the following equation:

$$\kappa_g = \frac{\left(\frac{\epsilon\omega^2}{\mu^3}\langle \nabla\mu, J\alpha'\rangle + \frac{\omega\tau}{\mu}\left(1 + \mu\langle \alpha', \nabla\mu\rangle\right)\left(c - \frac{\epsilon\omega^2}{\mu^2}\right)\right)}{\left(c - \frac{\epsilon\omega^2}{\mu^2}\right)^{-\frac{3}{2}}}. \tag{1.13}$$

Consider now a local conformal chart $\varphi\colon (\Omega \subset \mathbb{R}^2, ds^2_\Omega) \to U \subset M$, such that $\alpha(0) \in U$, let $ds^2_\Omega = \lambda(x,y)^2(dx^2 + dy^2)$ (choosing a conformal chart instead of orthogonal coordinate simplifies the computation and gives a simpler description of the geodesics) and in Ω we identify α with the coordinates $(x(t), y(t)) = \varphi^{-1} \circ \alpha(t)$. Then, there must exist a smooth function θ such that $x' = \sqrt{c - \frac{\epsilon\omega^2}{\mu^2}}\frac{\cos(\theta)}{\lambda}$ and $y' = \sqrt{c - \frac{\epsilon\omega^2}{\mu^2}}\frac{\sin(\theta)}{\lambda}$. The geodesic curvature of α with respect to $J\alpha' = -y'\partial_x + x'\partial_y$ is given by

$$\kappa_g = \theta'\left(c - \frac{\epsilon\omega^2}{\mu^2}\right)^{-1} + \frac{\lambda_y \sin(\theta) - \lambda_x\cos(\theta)}{\lambda^2}.$$

Now, equation (1.13) becomes the first-order ODEs system

$$\begin{cases} x' = & \sqrt{c - \frac{\epsilon\omega^2}{\mu^2}}\frac{\cos(\theta)}{\lambda}, \\ y' = & \sqrt{c - \frac{\epsilon\omega^2}{\mu^2}}\frac{\sin(\theta)}{\lambda}, \\ \theta' = & \left(c - \frac{\epsilon\omega^2}{\mu^2}\right)\left(\frac{\sin(\theta)\lambda_y - \cos(\theta)\lambda_x}{\lambda^2} + \frac{\epsilon\omega^2(\sin(\theta)\mu_x - \cos(\theta)\mu_y)}{c\mu^2 - \epsilon\omega^2\mu}\right) \\ & + \frac{\sqrt{c\mu^2 - \epsilon\omega^2}\tau}{\mu\left(\mu + \sqrt{c\mu^2-\epsilon\omega^2}(\sin(\theta)\mu_y + \cos(\theta)\mu_x)\right)}. \end{cases}$$

The general theory of ODEs guarantees the existence of a unique smooth solution in a neighborhood of the origin when prescribing $x(0)$, $y(0)$, $\theta(0)$.

Once we have a description of the geodesics, the next step is to give necessary and sufficient conditions which guarantee that $\mathbb{E}$ is geodesically complete. In the next proposition we give a necessary and sufficient condition that guarantees the completeness of a Riemannian manifold admitting a Killing submersion structure.

Proposition 1.19 . *Let* $\pi\colon \mathbb{E} \to M$ *be a Riemannian Killing submersion. Then* $\mathbb{E}$ *is complete if and only if* M *is complete.*

Proof. Recall that the horizontal lifting of geodesics in M are geodesics in $\mathbb{E}$. Therefore, if M is not complete, then $\mathbb{E}$ cannot be complete either. To prove that the hypotheses is sufficient we consider an arbitrary Cauchy sequence $\{p_n\}_n$ in $\mathbb{E}$ and prove that it is convergent. We consider the sequence $\{q_n = \pi(p_n)\}_n \subset M$. For any point $p \in \mathbb{E}$ and any tangent vector field $v \in T_p\mathbb{E}$, $\langle v, v\rangle_{\mathbb{E}} \geqslant \langle d\pi(v), d\pi(v)\rangle_M$. Then, for any curve $\alpha \subset \mathbb{E}$, $\mathrm{Length}_{\mathbb{E}}(\alpha) \geqslant \mathrm{Length}_M(\pi(\alpha))$. It follows that $\{q_n\}_n$ is a Cauchy sequence in M and, since M is complete, $\{q_n\}_n$ converges to a point $q \in M$. In particular, we can assume that $\{q_n\}_n$ is contained in a compact and simply connected subset $K \subset M$ Let $F_0\colon K \to \mathbb{E}$ be a local section, then, for any n, there exists $t_n \in \mathbb{R}$ such that $p_n = \phi_{t_n}(q_n)$. Denoting by $c = \min_K \mu$, then, for any $p \in \pi^{-1}(K)$ and any vector field $v \in T_p\mathbb{E}$, we have $\langle v, v\rangle_{\mathbb{E}} \geqslant c \left\langle d\pi^{\perp}(v), d\pi\perp(v)\right\rangle_{\mathbb{R}}$. This implies that, for any $i, j \in \mathbb{N}$ $\|p_i - p_j\|_{\mathbb{E}} \geqslant c|t_i - t_j|$, that is, $\{t_n\}_n$ is a Cauchy sequence in $(\mathbb{R}, g_{euc})$. Since $(\mathbb{R}, g_{euc})$ is complete, we can assume that there exist $a, b \in \mathbb{R}$ such that $t_n \in [a, b]$ for any n. It follows that $\{p_n\}_n$ is contained in the compact subset of $\pi^{-1}(K)$ delimited by $\phi_a(F_0)$ and $\phi_b(F_0)$. Hence, $\{p_n\}_n$ is a Cauchy sequence in a compact domain, that is convergent. This implies that $\mathbb{E}$ is complete and concludes the proof. □

When $\epsilon = -1$ and $c = 0, +1$, we have that $\|\alpha'\|$ restricted to U is greater then a positive constant, hence the solution can be extended as long as α is contained in U, so if M is complete and we take an atlas consisting of conformal parameterizations compatible with the orientation, then α extends to the whole real line. On the contrary, it could append that timelike geodesics are not complete, regardless of the completeness of M, as it is shown in the next example.

Example 1.20. Let us consider two Lorentzian three-manifolds:

1. The Anti-deSitter space as a Lorentzian $\mathbb{L}(\kappa,\tau)$-space (M_1, g_1):
$$M_1 = \mathbb{E}(\mathbb{H}^2(-1), 1, 1, -1) = \mathbb{L}(-4, 1) = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\},$$
$$g_1 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2} - \left(dz - \frac{ydx - xdy}{1 - x^2 - y^2}\right)^2.$$

2. The Anti-deSitter space as a Lorentzian warped product (M_2, g_2):
$$M_2 = \mathbb{E}(\mathbb{H}^2(-1), 0, \tfrac{1}{x}, -1) = \{(x, y, z) \in \mathbb{R}^3 : x > 0\},$$
$$g_2 = \frac{dx^2 + dy^2 - dz^2}{x^2}.$$

A direct computation applying Proposition 1.13 implies that both $(M1_,g_1)$ and (M_2, g_2) have constant sectional curvature -1. This means that these spaces must be at least locally isometric.

Equation (1.13) implies that the geodesics of (M_1, g_1) project onto curves of constant sectional curvature of $\mathbb{H}^2(-1)$ that are parameterized by arc length, so the completeness of $\mathbb{H}^2(-1)$ implies that (M_1, g_1) is geodesically complete. To prove that (M_2, g_2) is not geodesically complete we consider the surjective map:
$$\begin{array}{rccc} F\colon & (M_1, g_1) & \to & (M_2, g_2) \\ & (x, y, z) & \mapsto & \left(\frac{\sqrt{1-x^2-y^2}}{f(x,y,z)}, \frac{x\cos(z)-y\sin(z)}{f(x,y,z)}, \frac{\sin(z)}{f(x,y,z)}\right), \end{array} \tag{1.14}$$
where $f(x, y, z) = (1 - y)\cos(z) - x\sin(z)$. This is a local isometry that maps the region of M'_1 contained between the two helicoids (see Figure 1) into the all M_2. In particular, in (M_2, g_2), the geodesics that are the integral curves of the unitary Killing vector field are not complete.

Partial results about completeness of Lorentzian Killing submersion can be found in [RomSan94, Proposition 2.1] and [AazRea23, Corollary 6.1].

1.5 Surfaces in Killing Submersions

Let Σ be an orientable surface immersed in $\mathbb{E}$ and denote by N a smooth unit normal vector field along Σ. This defines the function $\mathfrak{v} = \langle N, \xi\rangle$, known as the *angle function* of the surface. Assuming that $\mathfrak{v}$ is identically zero or never vanishes gives rise to two distinguished families of surfaces in $\mathbb{E}$:

- If $\mathfrak{v} \equiv 0$, then Σ is everywhere vertical, so there exists a curve $\Gamma \subset M$ such that $\Sigma = \pi^{-1}(\Gamma)$ and Σ is called the vertical cylinder over Γ.

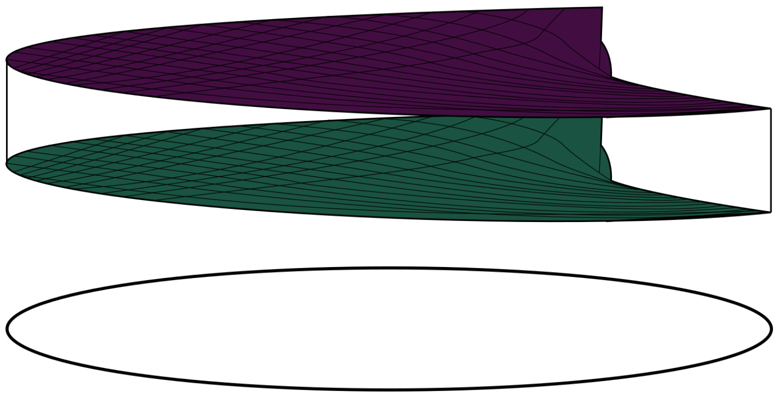

Figure 1: The domain of (M_1, g_1) mapped by F into (M_2, g_2).

- If $\mathfrak{v}$ has no zeroes, then Σ is everywhere transversal to the Killing vector field, and it is called a vertical multigraph. Note that Σ is a graph if and only if additionally $\pi_{|\Sigma}\colon \Sigma \to M$ is injective.

1.5.1 *Vertical Cylinders*

Consider a unit-speed parameterization $\gamma\colon [a, b] \to \Gamma \subset M$ and assume that $\Sigma = \pi^{-1}(\Gamma)$. We will call Σ *vertical cylinder* or *Killing cylinder* over Γ.

Consider the orthonormal frame $\{X, E_3 = \frac{1}{\mu}\xi\}$ in Σ, where X is a horizontal vector field on Σ that projects to γ'. The first fundamental form in the frame $\{X, E_3\}$ is given by the matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}$$

while the second fundamental form is given by

$$\sigma \equiv \begin{pmatrix} \langle \overline{\nabla}_X X, N\rangle & \langle \overline{\nabla}_X E_3, N\rangle \\ \langle \overline{\nabla}_{E_3} X, N\rangle & \langle \overline{\nabla}_{E_3} E_3, N\rangle \end{pmatrix} = \begin{pmatrix} \kappa_g & \tau \\ \tau & \langle -\frac{\epsilon}{\mu}\nabla\mu, \eta\rangle \end{pmatrix},$$

where κ_g is the geodesic curvature of γ in M with respect to the unit normal $\eta = \pi_* N$ to γ in M and ∇ denotes the gradient in M. This follows from (1.8) using that X and N are horizontal. In particular, the mean curvature of Σ is given by

$$2H = \kappa_g - \langle \eta, \tfrac{1}{\mu}\nabla\mu\rangle. \tag{1.15}$$

We can get rid of the term $\langle \eta, \frac{1}{\mu}\nabla\mu\rangle$ by considering a conformal factor in M.

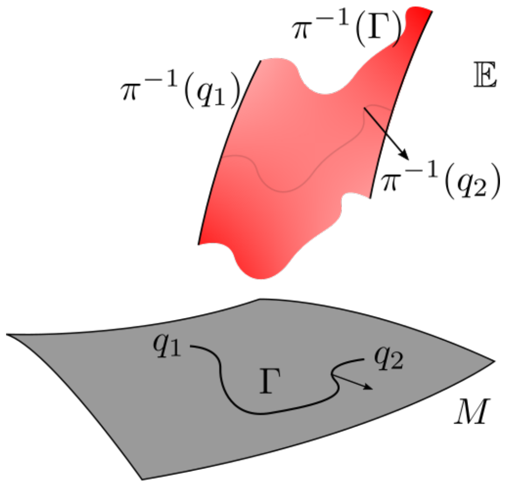

Figure 2: Vertical cylinder above the curve Γ.

Proposition 1.21 . *Let* $\pi : \mathbb{E} \to M$ *be a Killing submersion and let* $\Gamma \subset M$ *be a regular curve. The mean curvature of the vertical cylinder* $\Sigma = \pi^{-1}(\Gamma)$ *with respect to a unit normal* N *satisfies*

$$2H = \mu\,\widetilde{\kappa}_g,$$

where $\widetilde{\kappa}_g$ *is the geodesic curvature of* Γ *with respect to the unit normal* $\eta = \frac{1}{\mu}\pi_*(N)$ *in the conformal metric* $\mu^2 ds^2_M$ *on* M.

Proof. Since the computation is local, we can assume that M is a disk of $\mathbb{R}^2$ endowed with the metric $ds^2_\lambda = \lambda^2(dx^2 + dy^2)$ for some conformal factor λ in the usual coordinates (x, y). The Levi-Civita connection of ds^2_λ is given by

$$\begin{aligned} \nabla_{\partial_x}\partial_x &= \tfrac{\lambda_x}{\lambda}\partial_x - \tfrac{\lambda_y}{\lambda}\partial_y, & \nabla_{\partial_x}\partial_y &= \tfrac{\lambda_y}{\lambda}\partial_x + \tfrac{\lambda_x}{\lambda}\partial_y, \\ \nabla_{\partial_y}\partial_x &= \tfrac{\lambda_y}{\lambda}\partial_x + \tfrac{\lambda_x}{\lambda}\partial_y, & \nabla_{\partial_y}\partial_y &= -\tfrac{\lambda_x}{\lambda}\partial_x + \tfrac{\lambda_y}{\lambda}\partial_y. \end{aligned} \tag{1.16}$$

Given the curve $\gamma = (x, y)$ that parameterizes Γ, after swapping x and y if necessary, we can assume that the frame $\{\partial_x, \partial_y\}$ is oriented so that

$$\gamma' = x'\partial_x + y'\partial_y, \qquad \eta = \frac{-y'\partial_x + x'\partial_y}{\lambda((x')^2 + (y')^2)^{1/2}}.$$

On the one hand, taking into account (1.16), the geodesic curvature κ_g of γ (with respect to ds^2_λ and the unit normal η) can be computed as

$$\kappa_g = \frac{\langle \nabla_{\gamma'}\gamma', \eta\rangle}{|\gamma'|^2} = \frac{x'y'' - x''y'}{\lambda((x')^2 + (y')^2)^{3/2}} + \frac{\lambda_x y' - \lambda_y x'}{\lambda^2((x')^2 + (y')^2)^{1/2}}, \tag{1.17}$$

where we have used that

$$\nabla_{\gamma'}\gamma' = \left(x'' + \frac{\lambda_x}{\lambda}\left((x')^2-(y')^2\right)+2\frac{\lambda_y}{\lambda}x'y'\right)\partial_x \\ +\left(y''-\frac{\lambda_y}{\lambda}\left((x')^2-(y')^2\right)+2\frac{\lambda_x}{\lambda}x'y'\right)\partial_y.$$

On the other hand, we can also work out $\nabla\mu = \frac{1}{\lambda^2}(\mu_x\partial_x+\mu_y\partial_y)$ and hence

$$\langle\eta,\tfrac{1}{\mu}\nabla\mu\rangle = \frac{\langle J\gamma',\frac{1}{\mu}\nabla\mu\rangle}{|\gamma'|} = \frac{-\mu_x y'+\mu_y x'}{\mu\lambda((x')^2+(y')^2)^{1/2}}. \tag{1.18}$$

Plugging (1.17) and (1.18) into (1.15), we finally get

$$2H = \kappa_g - \langle\eta,\tfrac{1}{\mu}\nabla\mu\rangle = \frac{x'y''-x''y'}{\lambda((x')^2+(y')^2)^{3/2}} + \frac{(\lambda\mu)_x y'-(\lambda\mu)_y x'}{\lambda^2\mu((x')^2+(y')^2)^{1/2}}. \tag{1.19}$$

Observe that $\widetilde{\kappa}_g$, the curvature of γ with respect to the metric $\mu^2 ds^2_\lambda = ds^2_{\lambda\mu}$ can be computed by substituting λ with $\mu\lambda$ in (1.17), so it easily follows that the right-hand side in (1.19) is nothing but $\mu\,\widetilde{\kappa}_g$. □

In the sequel we will use the prefix 'μ-' to indicate that the corresponding term is computed with respect to the metric $\mu^2 ds^2_M$ in M. For instance, Proposition 1.21 implies that $\Sigma = \pi^{-1}(\Gamma)$ is minimal if and only if Γ is a μ-geodesic, and Σ is mean convex with respect to N if and only if Γ is μ-convex with respect to $\eta = \frac{1}{\mu}\pi_* N$.

The classification of H-surfaces invariant by any 1-parameter group of isometries in three-dimensional Killing submersions can be reduced by this argument to a problem for curves in the orbit space, which plays the role of base of the submersion. Since the local existence and uniqueness of curves with prescribed geodesic curvature is guaranteed (in an arbitrary surface) when some initial conditions have been fixed, we can also classify invariant H-surfaces by means of initial conditions.

Corollary 1.22 . *Let E be a three-manifold with a Killing vector field ξ, and fix $H \in \mathbb{R}$. Given $q \in E$ with $\xi_q \neq 0$, let $\{v, n, \xi_q/\|\xi_q\|\}$ be an orthonormal basis of T_qE.*

(1) There exists an H-surface invariant under the action of ξ passing through q, tangent to v with unit normal N such that $N_q = n$.

(2) Any two surfaces satisfying item (1) coincide in a neighborhood of q.

It is also interesting to notice that radial μ-geodesics at some point $p \in M$ produce an open book decomposition of a neighborhood of p, so the corresponding cylinders produce an open book decomposition by minimal surfaces of a neighborhood of $\pi^{-1}(\{p\})$.

Corollary 1.23 . *Let $\pi : \mathbb{E} \to M$ be a Killing submersion and let $p \in M$. Given an open neighborhood V of the origin in T_pM where the μ-exponential map is one-to-one, there exists an open book decomposition of $\pi^{-1}(O)$, where O is the μ-exponential image of V, by minimal cylinders with binding the fiber $\pi^{-1}(\{p\})$.*

1.5.2 *Killing Graphs*

A (Killing) graph in a Killing submersion $\pi : \mathbb{E} \to M$ is a smooth section over an open subset $U \subset M$. If we prescribe a smooth zero section $F_0 : U \to \mathbb{E}$, then such a graph can be parameterized as $F_u : U \to \mathbb{E}$ with $F_u(p) = \phi_{u(p)}(F_0(p))$ for some $u \in \mathcal{C}^\infty(U)$, where $\{\phi_t\}$ is the group of vertical translations. In the sequel, we will assume that the fibers of π have infinite length, which implies the existence of global smooth sections, see [LerMan17]. This assumption is not restrictive since, if the fiber are compact, we can work on a covering space of $\pi^{-1}(U)$.

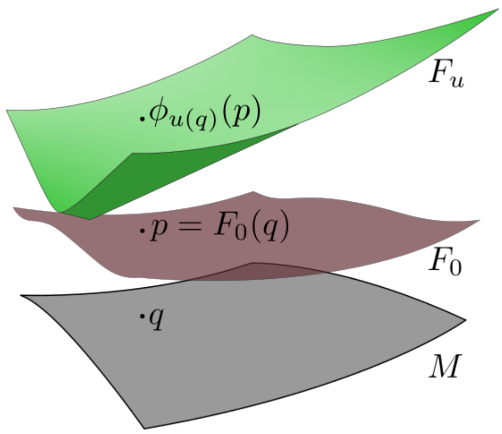

Figure 3: Killing graph of the function u with respect to the section F_0.

Given $u \in \mathcal{C}^\infty(U)$, we will denote by Σ_u the graph spanned by F_u, which will be assumed *spacelike*, i.e., the restriction of the metric of $\mathbb{E}$ is positive definite. Following the ideas in [LerMan17], we will consider the functions $\overline{u} \in \mathcal{C}^\infty(\mathbb{E})$ defined by $\overline{u} = u \circ \pi$ and $d \in \mathcal{C}^\infty(\mathbb{E})$ defined implicitly by $\phi_{d(q)}(F_0(\pi(q))) = q$, i.e., $d(q)$ is the signed Killing distance along a fiber from the initial section to q. Therefore, the upward pointing unit normal to Σ_u can be expressed as $N = \epsilon\overline{\nabla}(d-\overline{u})/\|\overline{\nabla}(d-\overline{u})\|_{\mathbb{E}}$, where $\overline{\nabla}$ and $\|\cdot\|_{\mathbb{E}}$ stand for the gradient and norm in $\mathbb{E}$, respectively.

Note that $\epsilon\langle\overline{\nabla}d,\xi\rangle = \xi(d) = 1$ by definition of d and $\langle\overline{\nabla}\overline{u},\xi\rangle = 0$ since $\overline{u}$ is constant along the fibers of π. Therefore, we can decompose in vertical and horizontal components $\overline{\nabla}(d-\overline{u}) = \frac{\epsilon}{\mu^2}\xi + (\overline{\nabla}(d-\overline{u}))^h$. It follows from the orthogonality of the vertical and horizontal components that

$$\|\overline{\nabla}(d-\overline{u})\|_{\mathbb{E}}^2 = \tfrac{\epsilon}{\mu^2} + \|(\overline{\nabla}(d-\overline{u}))^h\|_{\mathbb{E}}^2 = \tfrac{\epsilon}{\mu^2} + \|\nabla u - Z\|^2, \tag{1.20}$$

where $Z = \pi_*(\overline{\nabla}d)$ is a vector field on $U \subset M$ not depending on u. Here, ∇ and $\|\cdot\|$ denote the gradient and norm in M, respectively. We also define $Gu = \nabla u - Z$, usually known as the *generalized gradient* of u, see [LerMan17]. Observe that $\overline{\nabla}(d-\overline{u})$ is timelike in the Lorentzian case ($\epsilon = -1$), which amounts to saying that the right-hand side in (1.20) is negative, i.e., the spacelike condition is equivalent to $1+\epsilon\mu^2\|Gu\|^2 > 0$. This also means that we have to add a factor ϵ before taking square roots to get rid of the square in the left-hand side of (1.20). Consequently, the angle function $\mathfrak{v} = \langle N,\xi\rangle$ of Σ_u can be computed as

$$\mathfrak{v} = \frac{\epsilon\langle\overline{\nabla}(d-\overline{u}),\xi\rangle}{\|\overline{\nabla}(d-\overline{u})\|} = \frac{\epsilon\mu}{\sqrt{1+\epsilon\mu^2\|Gu\|^2}}. \tag{1.21}$$

Note that $0 < \mathfrak{v} \leqslant \mu$ if $\epsilon = 1$, whereas $\mathfrak{v} \leqslant -\mu$ if $\epsilon = -1$. Since Σ_u is a section of π, the projection $\pi|_{\Sigma_u} : \Sigma_u \to U$ is a diffeomorphism and the area element of Σ_u over U can be computed as the Jacobian of $\pi|_{\Sigma_u}$.

Let $\{\bar{v}_1,\bar{v}_2\}$ be an orthonormal basis of $T_q\Sigma_u$ at some $q \in \Sigma_u$ such that $\bar{v}_1$ is horizontal, and let $h \in T_q\mathbb{E}$ be an orizontal unit vector such that $\{\bar{v}_1,h\}$ is also orthonormal. Since ξ, N, $\bar{v}_2$ and h are coplanar (all of them are orthogonal to $\bar{v}_1$), we can easily express $N = \epsilon\frac{\mathfrak{v}}{\mu^2}\xi \pm \frac{1}{\mu}\sqrt{\epsilon(\mu^2-\mathfrak{v}^2)}h$ and then work out the orthogonal vector $v_2 = \frac{1}{\mu^2}\sqrt{\epsilon(\mu^2-\mathfrak{v}^2)}\xi \mp \frac{\mathfrak{v}}{\mu}h$, where the signs depend on the choice of h (it is determined up to the sign). Since π is a Riemannian submersion, we deduce that $\{\bar{v}_1,\bar{v}_2\}$ projects to an orthogonal basis $\{d\pi_q(\bar{v}_1), d\pi_q(\bar{v}_2)\}$ such that $\|d\pi_q(\bar{v}_1)\| = 1$ and $\|d\pi_q(\bar{v}_2)\| = \frac{|\mathfrak{v}|}{\mu}$. This implies that

$$|\operatorname{Jac}(\pi|_{\Sigma_u})| = \frac{|\mathfrak{v}|}{\mu} = \frac{1}{\sqrt{1+\epsilon\mu^2\|Gu\|^2}}. \tag{1.22}$$

For each relatively compact subdomain $\Omega \subset \bar{\Omega} \subset U$, a direct change of variables using (1.22) yields the desired area element:

$$\operatorname{area}(\Sigma_u \cap \pi^{-1}(\Omega)) = \int_\Omega \sqrt{1+\epsilon\mu^2\|Gu\|^2}. \tag{1.23}$$

Proposition 1.24 . *The mean curvature of a Killing graph parameterized by a function* $u \in \mathcal{C}^2(U)$ *under the above assumptions is given by*

$$2\mathcal{Q}(u) = \frac{1}{\mu}\operatorname{div}\left(\frac{\mu^2 Gu}{\sqrt{1+\epsilon\mu^2\|Gu\|^2}}\right), \tag{1.24}$$

where the divergence is computed in M.

Proof. Let $f \in \mathcal{C}_0^\infty(U)$ be a smooth function that vanishes outside a relatively compact open subset $\Omega \subset \bar{\Omega} \subset U$, and consider the functional $A_f(t) = \operatorname{area}(\Sigma_{u+tf} \cap \pi^{-1}(\Omega))$. It follows from (1.23) and the divergence theorem that

$$\begin{aligned} A_f'(0) &= \int_\Omega \frac{d}{dt}\bigg|_{t=0} \sqrt{1+\epsilon\mu^2\|G(u+tf)\|^2} \\ &= \int_\Omega \frac{d}{dt}\bigg|_{t=0} \sqrt{1+\epsilon\mu^2\|Gu+t\nabla f\|^2} \\ &= \int_\Omega \frac{\epsilon\mu^2\langle Gu, \nabla f\rangle}{\sqrt{1+\epsilon\mu^2\|Gu\|^2}} = -\int_\Omega \epsilon f \operatorname{div}\left(\frac{\mu^2 Gu}{\sqrt{1+\epsilon\mu^2\|Gu\|^2}}\right). \end{aligned} \tag{1.25}$$

Moreover, since the associated variational field of this graphical variation is just ξ, it is well known (e.g., see [BarOli93, Lem. 3.1]) that in both the Riemannian and Lorentzian cases, the first variation of the area functional is also given by

$$\begin{aligned} A_f'(0) &= -\int_{\Sigma_u\cap\pi^{-1}(\Omega)} 2\mathcal{Q}(u)\langle N, \xi\rangle = -\int_{\Sigma_u\cap\pi^{-1}(\Omega)} \frac{2\mathcal{Q}(u)\epsilon\mu f}{\sqrt{1+\epsilon\mu^2\|Gu\|^2}} \\ &= -\int_\Omega 2\mathcal{Q}(u)\epsilon\mu f. \end{aligned} \tag{1.26}$$

Since (1.25) and (1.26) must agree for all compactly supported functions $f \in \mathcal{C}_0^\infty(U)$, the formula in the statement follows readily. □

Remark 1.25. Another way to compute the mean curvature of a spacelike Killing graph is following the idea in [LerMan17, Lemma 3.1]. Notice first that

$$N_{\Sigma_u} = \frac{1}{\sqrt{1+\epsilon\mu^2\|Gu\|^2}}\left(E_3 - \epsilon\sum_{i=1}^{2}\langle\overline{\nabla}(\bar{u}-d), E_i\rangle\mu\, E_i\right), \tag{1.27}$$

that is, $\pi_* N = -\frac{\epsilon\mu G u}{\sqrt{1+\epsilon\mu^2\|Gu\|^2}}$. Indeed, a direct computation implies

$$\begin{aligned}
2\mathcal{Q}(u) =& -\operatorname{div}_{\mathbb{E}}(\epsilon N) = -\sum_{i=1}^{2}\langle \overline{\nabla}_{E_i}\epsilon N, E_i\rangle - \epsilon\langle \overline{\nabla}_{E_3}\epsilon N, E_3\rangle \\
=& -\operatorname{div}_M(\epsilon\,\pi_* N) - \tfrac{1}{\mu}\langle \epsilon N, \overline{\nabla}\mu\rangle_{\mathbb{E}} \\
=& \operatorname{div}_M\left(\frac{\mu G u}{\sqrt{1+\epsilon\mu^2\|Gu\|^2}}\right) + \frac{1}{\mu}\left\langle \frac{\mu G u}{\sqrt{1+\epsilon\mu^2\|Gu\|^2}}, \nabla\mu\right\rangle_M \\
=& \frac{1}{\mu}\operatorname{div}\left(\frac{\mu^2 G u}{\sqrt{1+\epsilon\mu^2\|Gu\|^2}}\right).
\end{aligned}$$

If we denote by $W_u^2 = 1+\epsilon\mu^2\|Gu\|^2$, manipulating the third line of the previous equation, we can define the mean curvature operator

$$2\mathcal{Q}(u) = \frac{\mu^2}{W_u^3}\sum_{i,j=1}^{2} A_{ij}\langle \nabla_{e_i} G u, e_j\rangle + \frac{1+W_u^2}{W_u^3}\langle Gu, \nabla\mu\rangle, \tag{1.28}$$

where the matrix (A_{ij}) is equal to

$$\begin{pmatrix} \frac{W_u^2}{\mu^2} - \langle Gu, e_1\rangle^2 & -\langle Gu, e_1\rangle\langle Gu, e_2\rangle \\ -\langle Gu, e_1\rangle\langle Gu, e_2\rangle & \frac{W_u^2}{\mu^2} - \langle Gu, e_2\rangle^2 \end{pmatrix}. \tag{1.29}$$

Furthermore,

$$\begin{aligned}
\langle \nabla_{e_i} G u, e_j\rangle =& \langle \nabla_{e_i}\pi_*\overline{\nabla}(\bar{u}-d), e_j\rangle \\
=& \langle \pi_*\left(\overline{\nabla}_{E_i}\overline{\nabla}(\bar{u}-d) - \overline{\nabla}_{E_i}\langle \overline{\nabla}(\bar{u}-d), E_3\rangle E_3\right), e_j\rangle \\
=& \left\langle \pi_*\left(\sum_{k=1}^{3}\langle \overline{\nabla}_{E_i}\overline{\nabla}(\bar{u}-d) - \overline{\nabla}_{E_i}E_3(\bar{u}-d)E_3, E_k\rangle E_k\right), e_j\right\rangle \\
=& \left\langle \sum_{k=1}^{3}\langle \overline{\nabla}_{E_i}\overline{\nabla}(\bar{u}-d) + \overline{\nabla}_{E_i}\tfrac{1}{\mu}E_3, E_k\rangle\,\pi_*(E_k), e_j\right\rangle \\
=& \langle \overline{\nabla}_{E_i}\overline{\nabla}(\bar{u}-d) + \overline{\nabla}_{E_i}\tfrac{1}{\mu}E_3, E_j\rangle \\
=& \langle \overline{\nabla}_{E_i}\overline{\nabla}\bar{u}, E_j\rangle + \langle \overline{\nabla}_{E_i}\overline{\nabla}d, E_j\rangle - \tfrac{1}{2\mu}\langle [E_i, E_j], E_3\rangle \\
=& \operatorname{Hess}_u(e_i, e_j) + d_{ij} - \gamma_{ij},
\end{aligned} \tag{1.30}$$

where $d_{ij} = \langle \overline{\nabla}_{E_i}\overline{\nabla}d, E_j\rangle \in \mathcal{C}^\infty(M)$ and

$$(\gamma_{ij}) = \begin{pmatrix} 0 & \frac{\tau}{2\mu} \\ -\frac{\tau}{2\mu} & 0 \end{pmatrix}.$$

Hence, the principal part of the mean curvature operator is given by the matrix $A = (A_{ij})$. Its eigenvalues are $\frac{1}{\mu^2}$ and $\frac{W_u^2}{\mu^2}$, in particular, $\mathcal{Q}$ is elliptic with respect to u (see Definition A.1).

1.5.2.1 *The mean curvature of a Killing graph in local coordinates*

We will now describe how to compute the mean curvature of a graph over an open subset $U \subset M$ in coordinates. We can choose the zero section $F_0 : U \to \mathbb{E}$ as $F_0(x, y) = (x, y, 0)$, so a graph parameterized by $u \in \mathcal{C}^\infty(U)$ can be expressed as $F_u(x, y) = (x, y, u(x, y))$. This also gives rise to the distance along vertical fibers $d(x, y, z) = z$. Taking into account (1.2), we can work out the gradient

$$\overline{\nabla} d = E_1(z)E_1 + E_2(z)E_2 + \epsilon E_3(z)E_3 = aE_1 + bE_2 + \frac{\epsilon}{\mu}E_3,$$

so that $Z = \pi_*(\overline{\nabla} d) = ae_1 + be_2$ and (1.5) yields

$$\operatorname{div}(JZ) = \operatorname{div}(-be_1 + ae_2) = \frac{-1}{\lambda_1\lambda_2}\left((\lambda_2 b)_x - (\lambda_1 a)_y\right) = \frac{-2\epsilon\tau}{\mu}, \qquad (1.31)$$

so that Z encodes information about the bundle curvature. Note also that

$$Gu = \alpha e_1 + \beta e_2, \qquad \text{where } \alpha = \frac{u_x}{\lambda_1} - a \text{ and } \beta = \frac{u_y}{\lambda_2} - b. \qquad (1.32)$$

Denoting by $\omega = \sqrt{1 + \epsilon\mu^2\|Gu\|^2} = \sqrt{1 + \epsilon\mu^2(\alpha^2 + \beta^2)}$ the area element we found in (1.23), it is easy to see that the upward-pointing normal to the Killing graph of u is given by

$$N = -\epsilon\frac{\mu\alpha}{\omega}E_1 - \epsilon\frac{\mu\beta}{\omega}E_2 + \frac{1}{\omega}E_3.$$

Notice that the spacelike condition in the Lorentzian case ($\epsilon = -1$) can be written as $\alpha^2 + \beta^2 < \mu^{-2}$.

Therefore, the equation for the mean curvature given by Proposition 1.24 can be written in coordinates as

$$2H = \frac{1}{\mu\lambda_1\lambda_2}\left[\frac{\partial}{\partial x}\left(\mu^2\frac{\lambda_2\alpha}{\omega}\right) + \frac{\partial}{\partial y}\left(\mu^2\frac{\lambda_1\beta}{\omega}\right)\right]. \qquad (1.33)$$

The standard frame $\{\partial_x, \partial_y\}$ in M can be lifted via π to the tangent frame $\{X = \lambda_1(E_1 + \mu\alpha E_3), Y = \lambda_2(E_2 + \mu\beta E_3)\}$ in Σ_u, whence

$$\langle X, X\rangle = \lambda_1^2(1 + \epsilon\mu^2\alpha^2), \quad \langle X, Y\rangle = \epsilon\lambda_1\lambda_2\mu^2\alpha\beta, \quad \langle Y, Y\rangle = \lambda_2^2(1 + \epsilon\mu^2\beta^2).$$

Therefore, $\pi|_{\Sigma_u} : \Sigma_u \to U$ induces the following Riemannian metric in $U \subset M$:

$$\lambda_1^2(1 + \epsilon\mu^2\alpha^2)dx^2 + 2\epsilon\lambda_1\lambda_2\mu^2\alpha\beta dxdy + \lambda_2^2(1 + \epsilon\mu^2\beta^2)dy^2. \qquad (1.34)$$

2 DIRICHLET PROBLEM FOR THE PRESCRIBED MEAN CURVATURE EQUATION

In this chapter we deal with the Dirichlet problem for the prescribed mean curvature equation over a relatively compact domain $\Omega \subsetneq M$ in a Riemannian Killing submersion $\pi\colon \mathbb{E} \to M$. Since Ω is compact, up to passing to the universal cover, we can assume without loss of generality that the fibers have infinite length, so a smooth zero section $F_0\colon \bar{\Omega} \to \pi^{-1}(\bar{\Omega})$ is always defined (see Section 1.2). Let $H \in \mathcal{C}^\infty(\bar{\Omega})$ and let f be a sufficiently regular function on $\partial\Omega$. The aim is to provide sufficient conditions of Ω, H and f that guarantee the existence and the uniqueness of a solution to the following Dirichlet problem:

$$P(\Omega, H, f) = \begin{cases} \mathcal{Q}(u) = \frac{1}{2\mu}\operatorname{div}\left(\frac{\mu^2 Gu}{\sqrt{1+\mu^2\|Gu\|^2}}\right) = H & \text{in } \bar{\Omega}, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

In particular, we will prove the following theorem.

Theorem 2.1 . *Assume that $\Omega \subset M$ is a relatively compact domain such that $\partial\Omega$ is piecewise $\mathcal{C}^1$ and $\mu\,\tilde{\kappa}_g(p) \geqslant 2H$ for all $p \in \partial\Omega \setminus E$, where $\tilde{\kappa}_g$ is the μ-geodesic curvature of $\partial\Omega$ computed with respect to the normal pointing into Ω and E is the set of corner points of $\partial\Omega$ (that is, the points where $\partial\Omega$ is not $\mathcal{C}^1$). Assume also that $f\colon \partial\Omega \to \mathbb{R}$ is a piecewise continuous function and that, if $H \neq 0$, Ω is contained in a larger domain $\tilde{\Omega}$ such that*

- *$\tilde{\Omega}$ has $\mathcal{C}^{2,\alpha}$ boundary,*
- *$\sup_\Omega |H| \leqslant \int_{\partial\tilde{\Omega}} \mu\tilde{\kappa}_g(\partial\tilde{\Omega})$ and*
- *$\mathrm{Ric}(\pi^{-1}(\tilde{\Omega})) \geqslant -\inf_{\partial\tilde{\Omega}}(\mu\tilde{\kappa}_g(\partial\tilde{\Omega}))^2$.*

Hence, there exists a unique solution to the problem $P(\Omega, H, f)$.

To provide all the details of the proof, we will follow the following general strategy:

- We prove a general Maximum Principle for prescribed mean curvature graphs which guarantees the uniqueness.
- We detail the proof of a local existence result ([DajDel09, Theorem 1]) using the classical theory of Leray-Schauder for quasilinear elliptic operator described in Appendix A.
- We use the Perron Process to extend the local result to the domains in the hypotheses of Theorem 2.1.
- We also prove a Removable Singularity Theorem.

2.1 A GENERAL MAXIMUM PRINCIPLE

The Maximum Principle is the key tool to prove the uniqueness of solutions. Remark 1.25 shows that the mean curvature operator $\mathcal{Q}$ is a quasilinear and elliptic operator. In particular, we can apply [GilTru01, Theorem 10.2] to guarantee the uniqueness of solution to $P(\Omega, H, f)$ when f is continuous. The aim of this section is to extend this result to the setting of Theorem 2.1.

We start by extending to general Killing submersions a result that was firstly proved for minimal graphs in $\mathbb{R}^3$ by Finn [Finn65] and Jenkins–Serrin [JenSer66], and later on generalized to many other ambient spaces including unit Killing submersions [LeaRos09].

Lemma 2.2 . *For any $u, v \in \mathcal{C}^1(\Omega)$, let N_u and N_v be the upward-pointing unit normal vector fields to the Killing graphs Σ_u and Σ_v, respectively. Then*

$$\left\langle \frac{Gu}{W_u} - \frac{Gv}{W_v}, Gu - Gv \right\rangle = \frac{1}{2\mu^2}(W_u + W_v)\|N_u - N_v\|^2 \geqslant 0.$$

Equality holds at some point $p \in M$ if and only if $\nabla u(p) = \nabla v(p)$.

Proof. On the one hand, we can write

$$\begin{aligned} \left\langle \frac{Gu}{W_u} - \frac{Gv}{W_v}, Gu - Gv \right\rangle &= \frac{\|Gu\|^2}{W_u} - \langle Gu, Gv\rangle \left(\frac{1}{W_u} + \frac{1}{W_v}\right) + \frac{\|Gv\|^2}{W_v} \\ &= \frac{W_u^2 - 1}{\mu^2 W_u} - \langle Gu, Gv\rangle \frac{W_u + W_v}{W_u W_v} + \frac{W_v^2 - 1}{\mu^2 W_v} \\ &= \mu^{-2}(W_u + W_v)\left(1 - \mu^2 \frac{\langle Gu, Gv\rangle}{W_u W_v} - \frac{1}{W_u W_v}\right). \end{aligned} \tag{2.2}$$

On the other hand, since $\pi_*(N_u) = \frac{\mu\, Gu}{W_u}$ and $\langle N_u, \frac{\xi}{\mu}\rangle = \frac{1}{W_u}$, we can decompose $N_u - N_v$ in horizontal and vertical components and compute

$$\begin{aligned} \|N_u - N_v\|^2 &= \left\| \frac{\mu\, Gu}{W_u} - \frac{\mu\, Gv}{W_v} \right\|^2 + \left(\frac{1}{W_u} - \frac{1}{W_v} \right)^2 \\ &= \frac{W_u^2 - 1}{W_u^2} - 2\mu^2 \frac{\langle Gu, Gv\rangle}{W_u W_v} + \frac{W_v^2 - 1}{W_v^2} + \left(\frac{1}{W_u^2} - \frac{2}{W_u W_v} + \frac{1}{W_v^2} \right) \\ &= 2\left(1 - \mu^2 \frac{\langle Gu, Gv\rangle}{W_u W_v} - \frac{1}{W_u W_v} \right). \end{aligned} \tag{2.3}$$

Plugging (2.3) into (2.2), we get the identity in the statement. Finally observe that $W_u + W_v > 0$ and $\|N_u - N_v\|^2 = 0$ if and only if $\nabla u = \nabla v$. $\square$

We can prove the following Maximum Principle.

Proposition 2.3 (Maximum Principle). *Let Ω be a relatively compact open subset of M with piecewise regular boundary. Let $u, v \in \mathcal{C}^\infty(\Omega)$ be functions that extend continuously to $\bar{\Omega} \setminus C$, where $C \subset \partial\Omega$ is the finite set of non-continuity points of $u_{|\partial\Omega}$ and $v_{|\partial\Omega}$. If*

i) $\mathcal{Q}(u) \geqslant \mathcal{Q}(v)$ in Ω and

ii) $u \leqslant v$ on $\partial\Omega \setminus C$,

then $u \leqslant v$ in Ω.

Proof. Reasoning by contradiction, consider $w = u - v$ and assume that $U = \{p \in \Omega : w(p) > 0\}$ is not empty. By adding a small enough positive constant to v so the condition $U \neq \emptyset$ is preserved, we can assume, without loss of generality, that ∇w does not vanish along ∂U and $u < v$ on $\partial\Omega \setminus V$. Therefore, ∂U is a family $\{C_\alpha\}$ of regular curves without intersection points. The Maximum Principle ([GilTru01, Theomem 10.2]) prevents the existence of any connected component of U whose boundary is contained in the interior of Ω. Moreover, the conditions $u < v$ on $\partial\Omega \setminus V$ and $\nabla w \neq 0$ on ∂U ensure that each C_α starts and ends in the vertex set $V \subset \partial\Omega$.

Given $\varepsilon > 0$, we will denote by U_ε the set of points of U which are not in the geodesic balls of radius ε with centers in V. For $\varepsilon > 0$ small enough, the discussion in the previous paragraph allows us to write $\partial U_\varepsilon = \Gamma_\varepsilon^1 \cup \Gamma_\varepsilon^2$, where $\Gamma_\varepsilon^1 \subset \partial U$ consists of finitely many curves and Γ_ε^2 is constituted by arcs of geodesic circles centered at the points of V.

Since the functions u and v satisfy $\mathcal{Q}(u) \geqslant \mathcal{Q}(v)$ in Ω, we get from Proposition 1.24 that $\operatorname{div}\frac{\mu^2\, Gu}{W_u} \geqslant \operatorname{div}\frac{\mu^2\, Gv}{W_v}$ in Ω. The divergence theorem yields

$$0 \leqslant \int_{U_\varepsilon} \operatorname{div}\left(\frac{\mu^2\, Gu}{W_u} - \frac{\mu^2\, Gv}{W_v}\right) = \int_{\partial U_\varepsilon} \mu^2 \left\langle \frac{Gu}{W_u} - \frac{Gv}{W_v}, \eta \right\rangle, \tag{2.4}$$

where η is the outer unit conormal vector field to U_ε along its boundary. On the other hand, Lemma 2.2 guarantees that

$$\left\langle \frac{Gu}{W_u} - \frac{Gv}{W_v}, \nabla w \right\rangle = \frac{1}{2\mu^2}(W_u + W_v)\|N_u - N_v\| > 0 \quad \text{on } \Gamma_\varepsilon^1, \tag{2.5}$$

where N_u and N_v stand for the downward unit vector fields, normal to F_u and F_v, respectively. The last strict inequality holds because $\nabla w \neq 0$ along Γ_ε^1. Nevertheless, since $w = 0$ in Γ_ε^1 and $w > 0$ in U_ε, the vector ∇w is a negative multiple of η along Γ_ε^1. Hence, the functions

$$\alpha_i(\varepsilon) = \int_{\Gamma_\varepsilon^i} \mu^2 \left\langle \frac{Gu}{W_u} - \frac{Gv}{W_v}, \eta \right\rangle, \qquad i \in \{1, 2\}, \tag{2.6}$$

satisfy $\lim_{\varepsilon\to 0} \alpha_1(\varepsilon) < 0$ by Equation (2.5), whereas $\lim_{\varepsilon\to 0} \alpha_2(\varepsilon) = 0$ since the integrand in $\alpha_2(\varepsilon)$ is bounded by Cauchy–Schwarz inequality and the length of Γ_ε^2 tends to zero as $\varepsilon \to 0$. Consequently, $\alpha_1(\varepsilon) + \alpha_2(\varepsilon) < 0$ for some small ε, contradicting the fact that $\alpha_1(\varepsilon) + \alpha_2(\varepsilon) \geqslant 0$ by Equation (2.4). □

2.2 Gradient estimates, local existence and convergence results

The Leray-Schauder existence theorem (Theorem A.9) reduces the solvability of the Dirichlet problem $P(\Omega, H, f)$ to find apriori $\mathcal{C}^{1,\alpha}$-estimates of the solutions of a related family of problems, for some $\alpha \in (0, 1)$. Since the mean curvature operator is of divergence form, as a consequence of Theorem A.2, it is sufficient to produce apriori $\mathcal{C}^1$-estimates. We follow the following general strategy:

- We estimate $\sup_\Omega |u|$ in terms of the boundary data f (see Proposition 2.7);
- We estimate $\sup_{\partial\Omega} |\nabla u|$ in terms of $\sup_\Omega |u|$ (see Proposition 2.8);
- We estimate $\sup_\Omega |\nabla u|$ in terms of $\sup_{\partial\Omega} |\nabla u|$ (see Proposition 2.9).

In the rest of this Section, we detail the proofs in [DajDel09], working in the following local setting. Recall that since $\Omega \subset M$ is relatively compact, we can assume that the fibers have infinite length. Fixed a zero section $F_0 \colon \bar{\Omega} \to \mathbb{E}$, we denote by $\Sigma_0 = F_0(\bar{\Omega})$ the surface transversal to the flow lines. We can consider the parameterization of $\pi^{-1}(\bar{\Omega})$ where Σ_0 is the set of initial values:

$$\begin{array}{rccc} \Psi\colon & \Sigma_0 \times \mathbb{R} & \to & \mathbb{E} \\ & (p,t) & \mapsto & \phi_t(p) \end{array}.$$

Notice that, in this setting, the Killing distance function (see Section 1.5.2) is simply $d = t$. We use the horizontal distance function $\delta = \mathrm{dist}_M(\cdot, \partial\Omega)$ defined as follows. Let $\Gamma \in M$ be a $\mathcal{C}^{2,\alpha}$ curve and define the horizontal distance function $\delta = \mathrm{dist}_M(\cdot, \Gamma) \in \mathcal{C}^{2,\alpha}(\pi^{-1}(\Omega_0))$, where $\Omega_0 \subset M$ is the largest set of points of $M \setminus \Gamma$ that can be joined to Γ by a unique minimizing geodesic orthogonal to Γ, so δ is well defined in Ω_0. In Ω_0 we consider the oriented orthonormal frame $\{e_1, e_2\}$ such that $e_1 = \nabla\delta$. In particular, e_2 will be the unitary tangent to the curves that are the level set of δ and e_1 will be their normal. Let E_i be the horizontal lifting of e_i in $\mathfrak{X}(\pi^{-1}(\Omega_0))$, so $\{E_1, E_2, E_3\}$ is an oriented orthonormal frame of $\pi^{-1}\Omega_0$. Notice that, by definition, $E_1 = \overline{\nabla}\bar{\delta}$ where $\bar{\delta} = \delta \circ \pi \in \mathcal{C}^{2,\alpha}(\pi^{-1}(\Omega_0))$.

Before proving the $\mathcal{C}^0$-estimate, we need to prove a couple of properties of the horizontal distance function. The first one we prove extends the result in [DaHiDe08, Lemma 5] and it is related to the ambient Ricci tensor in the direction ν, defined by

$$\mathrm{Ric}_{\mathbb{E}}(\nu) = \sum_{i=1}^{3} \langle \bar{R}(E_i, \nu)\nu, E_i \rangle,$$

where $\bar{R}$ is the curvature tensor in $\mathbb{E}$ defined in Section 1.2.1.

Lemma 2.4 . *Assume that the Ricci curvature satisfies*

$$\mathrm{Ric}_{|\mathbb{E}} \geqslant -\inf_{\partial\Omega} \left(\mu\, \tilde{\kappa}_g(\partial\Omega)\right)^2,$$

where $\tilde{\kappa}_g(\partial\Omega)$ is the μ-geodesic curvature of $\partial\Omega$. Let $y_0 \in \partial\Omega$ be the closest point to a given point $x_0 \in \partial\Omega_\varepsilon = \{q \in \Omega \mid \delta(q) = \varepsilon\}$, where $\varepsilon > 0$ is sufficiently small. If $H(\pi^{-1}(\partial\Omega)) > 0$, then, we have

$$H(\pi^{-1}(\partial\Omega_\varepsilon)))_{|\pi^{-1}(x_0)} \geqslant H(\pi^{-1}(\partial\Omega)))_{|\pi^{-1}(y_0)}$$

where we are comparing the mean curvature of $\pi^{-1}(\partial\Omega_\varepsilon)$ along the fiber $\pi^{-1}(x_0)$ with the mean curvature of $\pi^{-1}(\partial\Omega)$ along the fiber $\pi^{-1}(y_0)$.

Proof. Denote by A_δ the Weingarten operator of $K_\delta = \pi^{-1}(\partial\Omega_\delta))$. Since E_1 is a unit speed vector whose trajectories are geodesics, computing the derivative of the mean curvature of K_δ with respect to δ in $K_\delta = K_\varepsilon$ we get

$$\begin{aligned}\frac{d}{d\delta}_{|\delta=\varepsilon} 2H(K_\delta) =& E_1(\operatorname{tr} A_\delta) = E_1\left(\sum_{i=2}^{3}\langle -\nabla_{E_i}E_1, E_i\rangle\right)\\ =& -\sum_{i=2}^{3}\left(\langle \nabla_{E_1}\nabla_{E_i}E_1, E_i\rangle + \langle \nabla_{E_i}E_1, \nabla_{E_1}E_i\rangle\right)\\ =& \mathrm{Ric}_{\mathbb{E}}(E_1) + \sum_{i=2}^{3}\left(-\langle \nabla_{E_i}\nabla_{E_1}E_1, E_i\rangle + \langle \nabla_{[E_i,E_1]}E_1, E_i\rangle\right)\\ &+\sum_{i=2}^{3}\left(-\langle \nabla_{E_i}E_1, [E_i,E_1]\rangle + \langle \nabla_{E_i}E_1, \nabla_{E_i}E_1\rangle\right)\end{aligned}$$

Now, since E_1 is unitary and its integral curves are geodesics, it follows that $\langle \nabla_{E_i}\nabla_{E_1}E_1, E_i\rangle = 0$. Furthermore, using the Weingarten operator we get that

$$\langle \nabla_{[E_i,E_1]}E_1, E_i\rangle = -\langle A_\delta[E_i,E_1], E_i\rangle = -\langle [E_i,E_1], A_\delta E_i\rangle = \langle [E_i,E_1], \nabla_{E_i}E_1\rangle$$

and $\langle \nabla_{E_i}E_1, \nabla_{E_i}E_1\rangle = \langle A_\delta E_i, A_\delta E_i\rangle = \langle A_\delta^2 E_i, E_i\rangle$. In particular, we get that

$$\frac{d}{d\delta}_{|\delta=\varepsilon} 2H(K_\delta) = \mathrm{Ric}_{\mathbb{E}}(E_1) + \operatorname{tr}(A_\delta^2) \geqslant \mathrm{Ric}_{\mathbb{E}}(E_1) + 2H(K_\varepsilon)^2.$$

Let p be a fixed point of the fiber $\pi^{-1}(y_0)$ and denote by $\gamma(d) = \exp_p(dE_1)$ the horizontal geodesic normal to $\pi^{-1}(\partial\Omega))$ in p. From our hypotheses on $\mathrm{Ric}_{\mathbb{E}}$, using (1.15), we have that the function defined by

$$s(d) = H(K_d)_{|\exp_p(dE_1)} - H(\pi^{-1}(\partial\Omega))_{\pi^{-1}(y_0)}$$

satisfies

$$\begin{aligned} s'(d) &\geqslant H^2(K_d)_{|\exp_p(dE_2)} - \inf_\Gamma H^2(\pi^{-1}(\partial\Omega))\\ &\geqslant H^2(K_d)_{|\exp_p(dE_1)} - H^2(\pi^{-1}(\partial\Omega))_{|\pi^{-1}(y_0)}\\ &= \left(H(K_d)_{|\exp_p(dE_2)} + H(\pi^{-1}(\partial\Omega))_{|\pi^{-1}(y_0)}\right) s(d).\end{aligned}$$

Since $H(\pi^{-1}(\partial\Omega)) > 0$, it follows that there exists a constant $c > 0$, such that $s'(d) \geqslant c\, s(d)$ for d in some interval $[0, d_0 > 0]$. Then $H(K_d))_{|\exp_p(dE_1)}$ does not decrease when d increases. This concludes the proof of the lemma. □

The second lemma we prove describes how the mean curvature of a vertical cylinder can be computed as the laplacian of the horizontal distance from a fixed curve $\Gamma \subset M$ (see [DajDel09, Equation (12)]).

Lemma 2.5 . *For $\varepsilon \geqslant 0$, denote by $K_\varepsilon = \pi^{-1}(\Gamma_\varepsilon)$ the vertical cylinder above the curve $\Gamma_\varepsilon = \{q \in \Omega_0 \mid \delta(q) = \varepsilon\}$. Hence,*

$$(\overline{\Delta}\bar{\delta})_{|\varepsilon} = -2H(K_\varepsilon).$$

Proof. A direct computation gives

$$\begin{aligned}(\overline{\Delta}\bar{\delta})_{|\varepsilon} &= \sum_{i=1}^{3} \langle \overline{\nabla}_{E_i} \overline{\nabla}\bar{\delta}, E_i \rangle_{K_\varepsilon} = \sum_{i=1}^{3} \langle \overline{\nabla}_{E_i} E_1, E_i \rangle_{K_\varepsilon} \\ &= \langle \overline{\nabla}_{E_2} E_1, E_2 \rangle_{K_\varepsilon} + \langle \overline{\nabla}_{E_3} E_1, E_3 \rangle_{K_\varepsilon} \\ &= -\langle \overline{\nabla}_{E_2} E_2, E_1 \rangle_{K_\varepsilon} - \langle \overline{\nabla}_{E_3} E_3, E_1 \rangle_{K_\varepsilon}.\end{aligned}$$

Noticing that $(E_1)_{K_\varepsilon}$ (resp. $(E_2)_{K_\varepsilon}$) is the unit tangent (resp. normal) to K_ε, it follows that $(\overline{\Delta}\bar{\delta})_{|\varepsilon}$ is equal to the trace of the Weingarten operator of K_ε, that is, $(\overline{\Delta}\bar{\delta})_{|\varepsilon} = -2H(K_\varepsilon)$. □

Remark 2.6. From the proof, using Equation (1.10), it follows that

$$-2H = \overline{\Delta}\bar{\delta} = \Delta\delta - \langle \overline{\nabla}_{E_3} E_3, E_1 \rangle = -\kappa_g + \tfrac{1}{\mu}\langle \nabla\delta, \nabla\mu \rangle,$$

obtaining Equation(1.15).

Now we have all the ingredients to build the analytic barriers that allow us to prove a $\mathcal{C}^0$-estimate (see [DajDel09, Lemma 4]).

Proposition 2.7 . *Let $\Omega \subset M$ be a domain with compact closure and $\mathcal{C}^{2,\alpha}$-boundary. Suppose that $\partial\Omega$ is μ-convex and $\mathrm{Ric}_{\mathbb{E}} \geqslant -\inf_{\partial\Omega} (\mu\, \tilde{\kappa}_g(\partial\Omega))^2$, where $\tilde{\kappa}_g(\partial\Omega)$ is the μ-geodesic curvature of $\partial\Omega$. Let $H \in \mathcal{C}^{\alpha}(\bar{\Omega})$ and $f \in \mathcal{C}^{2,\alpha}(\partial\Omega)$ be given functions. If*

$$\sup_{\Omega} |H| \leqslant \inf_{\partial\Omega} \mu\, \tilde{\kappa}_g(\partial\Omega),$$

then there exists a constant $C = C(\Omega, H)$ such that

$$\sup_{\Omega} |u| \leqslant C + \sup_{\Omega} |f|$$

for any $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfying $\mathcal{Q}(u) = H$ and $u|_{\partial\Omega} = f$.

Proof. To prove a $\mathcal{C}^0$-estimate for the solution u of the Dirichlet problem (2.1), we follow the ideas in [GilTru01, Chapter 10] and construct an upper barrier

$$\varphi(x) = \sup_{\partial\Omega} f + h(\delta(x))$$

for u, where $\delta(x) = \mathrm{dist}_M(x, \partial\Omega)$ is the horizontal distance function defined in Section 1.5.1 and $h \in \mathcal{C}^\infty(\mathbb{R})$ will be chosen later. A lower barrier can be constructed in a similar way.

Since we are looking for an upper barrier we want to estimate from above

$$2\mathcal{Q}(u) = \frac{\mu^2}{W_u^3} \sum_{i,j=1}^{2} A_{i,j} \langle \nabla_{e_i} Gu, e_j \rangle + \frac{1+W_u^2}{W_u^3} \langle Gu, \nabla\mu \rangle, \tag{2.7}$$

defined in (1.28), we start by noticing that

$$\nabla\varphi = h'\nabla\delta, \qquad \mathrm{Hess}_\varphi(e_i, e_j) = h' \,\mathrm{Hess}_\delta(e_i, e_j) + h''\langle \nabla\delta, e_i\rangle\langle\nabla\delta, e_j\rangle,$$

$$\langle \nabla_{e_i} G\varphi, e_j \rangle = \mathrm{Hess}_\varphi(e_i, e_j) - \overline{\mathrm{Hess}}_t(E_i, E_j) + \tfrac{1}{\mu}\langle \overline{\nabla}_{E_i} E_j, E_3\rangle.$$

Since $A_{ij} = \frac{W_\varphi^2}{\mu^2}\delta_{ij} - \langle G\varphi, e_i\rangle\langle G\varphi, e_j\rangle$, where δ_{ij} is the Dirac's delta, it follows that

$$\begin{aligned}
\sum_{i,j=1}^{2} A_{ij}\langle \nabla_{e_i} G\varphi, e_j\rangle =& \frac{W_\varphi^2}{\mu^2}\left(\mathrm{Trace}\left(\mathrm{Hess}_\varphi\right) - \overline{\mathrm{Hess}}_t(E_1, E_1) - \overline{\mathrm{Hess}}_t(E_2, E_2)\right) \\
& - \sum_{i,j=1}^{2} \langle G\varphi, e_i\rangle\langle G\varphi, e_j\rangle\langle \nabla_{e_i} G\varphi, e_j\rangle \\
=& \frac{W_\varphi^2}{\mu^2}\left(h'' + h'\Delta\delta - \sum_{i=1}^{2}\overline{\mathrm{Hess}}_t(E_i, E_i)\right) \\
& - (h' - E_1(t))^2(h'' - \overline{\mathrm{Hess}}_t(E_1, E_1)) \\
& + (E_1(t))^2(h'\kappa_g + \overline{\mathrm{Hess}}_t(E_2, E_2)) \\
& - 2E_2(t)(h' - E_1(t))\overline{\mathrm{Hess}}_t(E_1, E_2).
\end{aligned}$$

Hence

$$\begin{aligned}
&\textstyle\sum_{i,j=1}^{2} A_{ij}\langle \nabla_{e_i} G\varphi, e_j\rangle = \\
&\quad h''\left(\tfrac{W_\varphi^2}{\mu^2} - (h')^2 + 2h'E_1(t) - (E_1(t))^2\right) - h'\tfrac{W_\varphi^2}{\mu^2}\Delta\delta + P_1(h'),
\end{aligned} \tag{2.8}$$

where P_1 is a polynomial of degree two in h'.

A direct computation implies that

$$\begin{aligned}\frac{W_\varphi^2}{\mu^2} &= \frac{1}{\mu^2}+\|\nabla\varphi-\pi_*\overline{\nabla}t\|^2\\ &= \frac{1}{\mu^2}+\|\nabla\varphi\|^2-2\langle\nabla\varphi,\pi_*\overline{\nabla}t\rangle+\|\pi_*\overline{\nabla}t\|^2\\ &= \frac{1}{\mu^2}+(h')^2\|\nabla\delta\|^2-2h'\langle\nabla\delta,\pi_*\overline{\nabla}t\rangle+\|\pi_*\overline{\nabla}t\|^2\\ &= (h')^2-2h'\langle\overline{\nabla}\bar{\delta},\overline{\nabla}t\rangle+\|\overline{\nabla}t\|^2\\ &= (h')^2-2h'E_1(t)+(E_1(t))^2+\frac{1}{\mu^2},\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\left\langle G\varphi,\frac{1}{\mu}\nabla\mu\right\rangle &= -\langle\overline{\nabla}\varphi,\overline{\nabla}_{E_3}E_3\rangle+\langle\overline{\nabla}t,\overline{\nabla}_{E_3}E_3\rangle\\ &= -h'\langle E_1,\overline{\nabla}_{E_3}E_3\rangle+\langle\overline{\nabla}t,\overline{\nabla}_{E_3}E_3\rangle.\end{aligned} \tag{2.10}$$

Finally, Lemma 2.5 implies that

$$\Delta\delta_{|\varepsilon}=\overline{\Delta}\bar{\delta}_{|\varepsilon}+\langle E_1,\overline{\nabla}_{E_3}E_3\rangle=-\mu\tilde{\kappa}_g(\partial\Omega_\varepsilon)+\langle E_1,\overline{\nabla}_{E_3}E_3\rangle, \tag{2.11}$$

and putting (2.8), (2.9) and (2.10) in (2.7), we get that

$$\frac{W_\varphi^3}{\mu^3}2\mathcal{Q}(\varphi)=\left(\frac{1}{\mu^2}+(E_2(t))^2\right)h''-\frac{1}{\mu^2}\left(\langle E_1,\overline{\nabla}_{E_3}E_3\rangle+2H\right)h'+P_2(h'),$$

where P_2 is again a polynomial of degree two in h'.

To define φ, we choose the test function

$$h=\frac{e^{CA}}{C}\left(1-e^{-C\delta}\right),$$

where $A>\operatorname{diam}(\bar{\Omega})$ and $C>0$ is a constant to be chosen later. Then,

$$h'=e^{C(A-\delta)}\quad\text{and}\quad h''=-Ch'.$$

Hence,

$$\mathcal{Q}(\varphi)\leqslant-(C+\langle E_1,\overline{\nabla}_{E_3}E_3\rangle)\frac{\mu h'}{W_\varphi^3}-\frac{\mu h'}{W_\varphi}2H+\frac{\mu^3P_2(h')}{W_\varphi^3}.$$

Observe that $W_\varphi^2\geqslant 1$. Moreover, as $C\to\infty$, we have that $\mu/W_\varphi\to 0$ and

$$\frac{\mu h'}{W\varphi}=\frac{h'}{\sqrt{(h')^2-2h'E_1(t)+(E_1(t))^2+\frac{1}{\mu^2}}}\to 1.$$

Furthermore, since $P_2(h')$ is a polynomial of degree two in h', it follows that

$$\frac{\mu^3P_2(h')}{W_\varphi^3}\to 0\quad\text{as}\quad C\to\infty.$$

Choose $C \gg 0$ such that, in particular, $C + \langle E_1, \overline{\nabla}_{E_3} E_3 \rangle > 0$. Since we are assuming $\sup_\Omega |H| \leqslant \inf\limits_{\partial\Omega} \mu\tilde{\kappa}_g(\partial\Omega)$ by hypotheses, as a consequence of Lemma 2.4 and Equation (1.15), we obtain

$$\mathcal{Q}(\varphi) < -|H| \leqslant H.$$

Thus, one has at points of Ω_0 that

$$\mathcal{Q}(\varphi) < \mathcal{Q}(u) = H, \qquad \varphi|_{\partial\Omega} \geqslant u|_{\partial\Omega}.$$

It remains to prove that $\varphi \geqslant u$ on $\bar{\Omega}$. By contradiction, assume that there exist points for which the continuous function $u^* := u - \varphi$ satisfies $u^* > 0$. Hence, $m := u^*(q) > 0$ at a maximum point $q \in \bar{\Omega}$ of u^*. Choose a minimizing geodesic γ joining q to $\partial\Omega$ for which the distance $\delta_q = \delta(q, \partial\Omega)$ is attained. Thus, $\gamma(t) = \exp_{q_0}(te_1)$, $0 \leqslant t \leqslant \delta_q$, starts from a point $q_0 \in \partial\Omega$ with unit speed e_1. Since γ is minimizing, we have $\delta(\gamma(t)) = t$ and the function φ restricted to γ is differentiable with $\varphi'(\gamma(t)) = e^{C(A-t)}$. Since the maximum of u^* restricted to γ occurs at $t = \delta_q$, i.e., at the point q, one has that

$$u'(\gamma(\delta_q)) - \varphi'(\gamma(\delta_q)) = (u^*)'(\gamma(\delta_q)) \geqslant 0.$$

This implies that

$$\langle \nabla u(q), \gamma'(\delta_q) \rangle \geqslant \varphi'(\gamma(\delta_q)) = e^{C(A-\delta_q)} > 0.$$

In particular, $\nabla u(q) \neq 0$, and Hence, the level curve

$$S = \{x \in \Omega \cap B_r(q) : u(x) = u(q)\}$$

is regular for a sufficiently small radius r. Along S we have

$$u^*(q_1) + \varphi(q_1) = u^*(q) + \varphi(q) \geqslant u^*(q_1) + \varphi(q),$$

and since φ is an increasing function of δ we have $\delta(q_1) \geqslant \dot{(y)} = \delta_q$. From this we conclude that the points in S are at a distance at least δ_q from $\partial\Omega$. Since S is of class C^2, it satisfies the interior sphere condition [Barb09, Theorem 1.0.9]: there exists a small ball $B_\varepsilon(q_2)$ touching S at q contained in the side to which $\nabla u(q)$ and $\gamma'(\delta_q)$ point. Thus, the points of $B_\varepsilon(q_2)$ satisfy $u(q_1) \geqslant u(q)$, and hence

$$\varphi(q_1) + m \geqslant u(q_1) \geqslant u(q) = \varphi(q) + m, \quad \text{for any } q_1 \in B_\varepsilon(q_2),$$

where in the first inequality we used the definition of m. Again because φ is an increasing function of δ_q, we have $\delta(q_1) \geqslant \delta_q$ on $B_\varepsilon(q_2)$ and therefore this ball is contained in the interior of Ω far away from $\partial\Omega$. This allows us to extend the geodesic γ through $B_\varepsilon(q_2)$. We claim that the center q_2 of the ball is contained in this extension. Otherwise, the broken line consisting of γ and of the radius in $B_\varepsilon(q_2)$ from q_2 to q has length smaller than *a* minimizing geodesic joining q_2 to $q_0 \in \partial\Omega$ (for a suitable small ε such a geodesic must cross the level curve S at a point $q_1 \neq q$ at distance to $\partial\Omega$ greater than δ_q). Thus, if there exists at least two distinct minimizing geodesics joining q to $\partial\Omega$, then the point q_2 is contained in the extension of both geodesics after its intersection at q. Choosing ε sufficiently small, we see that this configuration is not possible (the construction we made above applies to both geodesics). This contradiction implies that the maximum point q belongs to Ω_0. However, in this case, $u^*(q) \leqslant 0$, this gives a contradiction. We conclude that $u \leqslant \varphi$ in all$\bar{\Omega}$. In particular,

$$\sup_{\Omega} |u| \leqslant \sup_{\partial\Omega} f + \frac{e^{CA}}{C}\left(1 - e^{-C\,\mathrm{diam}(\bar{\Omega})}\right),$$

where A and C are sufficiently large constants depending on Ω, H and the ambient metric. □

The proof of the boundary gradient estimate is similar to the estimate in Proposition 2.7. It relies on the existence of upper and lower barriers in a tubular neighborhood Ω_ε of $\partial\Omega$. This barriers are build by deforming a certain $\mathcal{C}^{2,\alpha}$-extension of f in Ω_ε (see [DajDel09, Lemma 5]).

Proposition 2.8 . *Assume that* $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *satisfies* $\mathcal{Q}(u) = H$ *and* $u|_{\partial\Omega} = f$. *If* $|u|$ *is bounded in* $\bar{\Omega}$*, then*

$$\sup_{\partial\Omega} |\nabla u| \leqslant C$$

by a constant that depends on $\sup_\Omega |u|$.

Proof. Denote by $\phi \in \mathcal{C}^{2,\alpha}(\Omega_\varepsilon)$ an extension of f such that, at points of $\partial\Omega$, it holds

$$\langle \overline{\nabla}\phi, E_1 \rangle < \langle \overline{\nabla} t, E_1 \rangle,$$

and given $h(\delta) = C_1 \ln(1 + C_2\delta)$, for some positive constants C_1, C_2, denote by $w = h(\delta)$. We will show that with this choice for ϕ and h, the function $w + \phi$

is the upper barrier we are looking for. An analogous construction will give a lower barrier. The ellipticity of the mean curvature operator implies

$$2\mathcal{Q}(w+\phi)= \sum_{i,j=1}^{2} a^{ij}(x,\nabla w+\nabla\phi)\langle\nabla_{e_i}(Gw+G\phi),e_j\rangle + b(x,\nabla w+\nabla\phi)$$
$$\leqslant a^{ij}\langle\nabla_{e_i}Gw,e_j\rangle + \frac{1}{W}\|\phi\|_{2,\alpha}+b, \tag{2.12}$$

where $\|\cdot\|_{2,\alpha}$ is the $\mathcal{C}^{2,\alpha}$-norm,

$$a_{ij} := \frac{\mu^3 A_{ij}}{W^3_{w+\phi}} = \frac{\mu}{W_{w+\phi}}\delta_{ij} - \frac{\mu^3}{W^3_{w+\phi}}\langle G(w+\phi),e_i\rangle\langle G(w+\phi),e_j\rangle, \tag{2.13}$$

with δ_{ij} being the Dirac's delta, and

$$b = \frac{(1+W^2_{w+\phi})}{W^3_{w+\phi}}(\psi'\langle\nabla\mu,\nabla\delta\rangle + \langle\nabla\mu,G\phi\rangle - \langle\nabla\mu,\pi_*(\overline{\nabla}t)\rangle)$$

since $\pi_*(\overline{\nabla}_{E_3}E_3) = -\frac{1}{\mu}\nabla\mu$ and $G(w+\phi) = G\phi + h'\nabla\delta - \pi_*(\overline{\nabla}t)$.

In what follows, we denote by $P_j(h')$, for $j \geqslant 1$, polynomials in h' of at most degree two whose coefficients are smooth functions on Ω. As in Equation (2.9), a simple computation implies that

$$\frac{W^2_{w+\phi}}{\mu^2} = \tfrac{1}{\mu^2} + (h')^2 - 2h'\langle\overline{\nabla}\bar{\delta},\overline{\nabla}\phi-\overline{\nabla}t\rangle + \|\pi_*(\overline{\nabla}\phi-\overline{\nabla}t)\|^2,$$

from which follows that

$$\sum_{i,j=1}^{2}\left(\tfrac{W^2_{w+\phi}}{\mu^2}\delta_{ij} - \langle Gw,e_i\rangle\langle Gw,e_j\rangle\right)\langle\nabla_{e_i}Gw,e_j\rangle =$$
$$(|\overline{\nabla}t-\overline{\nabla}\phi|^2 - \langle\overline{\nabla}\bar{\delta},\overline{\nabla}t-\overline{\nabla}\phi\rangle^2)h'' + \tfrac{W^2_{w+\phi}}{\mu^2}\Delta\delta h' + P_1(h').$$

Moreover, a direct computation implies that

$$\sum_{i,j=1}^{2}\langle Gw,e_i\rangle\langle G\phi,e_j\rangle\langle\nabla_{e_i}Gw,e_j\rangle =$$
$$h''\left(\langle\overline{\nabla}\bar{\delta},\overline{\nabla}t\rangle\langle\overline{\nabla}\bar{\delta},\overline{\nabla}t-\overline{\nabla}\phi\rangle - h'\langle\overline{\nabla}\bar{\delta},\overline{\nabla}t-\overline{\nabla}\phi\rangle\right) + P_2(h')$$

and

$$\sum_{i,j=1}^{2}\langle G\phi,e_i\rangle\langle G\phi,e_j\rangle\langle\nabla_{e_i}Gw,e_j\rangle =$$
$$h''(e_1(\phi))^2 + h'\Delta\delta(e_2(\phi))^2 + \sum_{i,j=1}^{2} e_i(\phi)e_j(\phi)\overline{\mathrm{Hess}}_t(E_i,E_j) + P_3(h').$$

By the choice of $h \in \mathcal{C}^\infty(\mathbb{R})$, we have

$$h' = \frac{C_1 C_2}{1 + C_2\delta} \quad \text{and} \quad h'' = -\frac{1}{C_1}(h')^2.$$

Then using $\Delta\delta_{|\varepsilon} = -\mu\tilde{\kappa}_g(\partial\Omega_\varepsilon) + \langle E_1, \overline{\nabla}_{E_3}E_3\rangle$, we obtain

$$\sum_{i,j=1}^{2}\left(\frac{W^2_{w+\phi}}{\mu^2}\delta_{ij} - \langle Gw, e_i\rangle\langle Gw, e_j\rangle\right)\langle\nabla_{e_i}Gw, e_j\rangle = \\ -h'(\mu\tilde{\kappa}_g(\partial\Omega) - \langle E_1, \overline{\nabla}_{E_3}E_3\rangle)\frac{W^2_{w+\phi}}{\mu^2} + P_4(h'),$$

$$\sum_{i,j=1}^{2}\langle Gw, e_i\rangle\langle G\phi, e_j\rangle\langle\nabla_{e_i}Gw, e_j\rangle = -h'h''\langle\overline{\nabla}\bar{\delta}, \overline{\nabla}t - \overline{\nabla}\phi\rangle + P_5(h')$$

and

$$\sum_{i,j=1}^{2}\langle G\phi, e_i\rangle\langle G\phi, e_j\rangle\langle\nabla_{e_i}Gw, e_j\rangle = P_6(h').$$

We now conclude from (2.12) that

$$\begin{aligned}\frac{W^3_{w+\phi}}{\mu^3}(\mathcal{Q}(w+\phi) - 2H) \leqslant\ & -h'(\mu\tilde{\kappa}_g(\partial\Omega) - \langle E_1, \overline{\nabla}_{E_3}E_3\rangle)\frac{W^2_{w+\phi}}{\mu^2} \\ & -\frac{2}{C_1}(h')^3\langle\overline{\nabla}\bar{\delta}, \overline{\nabla}t - \overline{\nabla}\phi\rangle \\ & +(b - 2H)\frac{W^3_{w+\phi}}{\mu^3} + P_7(h').\end{aligned}$$

From the expressions above for b and $\frac{W^2_{w+\phi}}{\mu^2}$ it follows that

$$b\frac{W^3_{w+\phi}}{\mu^3} + h'\langle E_1, \overline{\nabla}_{E_3}E_3\rangle\frac{W^3_{w+\phi}}{\mu^3} = P_8(h').$$

Hence, we obtain

$$\frac{W^3_{w+\phi}}{\mu^3}(2\mathcal{Q}(w+\phi) - 2H) \leqslant -(2H + \mu\tilde{\kappa}_g(\partial\Omega) + \frac{2}{C_1}\langle\overline{\nabla}\bar{\delta}, \overline{\nabla}t - \overline{\nabla}\phi\rangle)(h')^3 + P_9(h').$$

We choose C_1 in such a way that $C_1 \to 0$ as $C_2 \to \infty$, namely,

$$C_1 = \frac{C}{\ln(1 + C_2)}$$

for some constant $C > 0$ to be chosen later. As $C_2 \to \infty$ we have that

$$h'(0) = \frac{C_2 C}{\ln(1 + C_2)} \to +\infty.$$

It also holds that $\mu h'/W_{w+\phi} \to 1$ as $C_2 \to \infty$. Thus, at points of $\partial\Omega$ the last inequality becomes

$$\frac{W^3_{w+\phi}}{\mu^3}(2\mathcal{Q}(w+\phi)-2H) \leqslant \\ -(2H+\mu\tilde{\kappa}_g(\partial\Omega)+\frac{2}{C_1}\langle\overline{\nabla}t-\overline{\nabla}\phi, E_1\rangle)(h')^3+P_9(h').$$

Since ϕ is such that $\langle\overline{\nabla}\phi, E_1\rangle < \langle\overline{\nabla}t, E_1\rangle$, choosing C_2 large enough and assuming that $\mu\tilde{\kappa}_g(\partial\Omega)+2H \geqslant 0$, on a small tubular neighborhood Ω_ε of $\partial\Omega$ we can assure that $2\mathcal{Q}(w+\phi)-2H<0$. Furthermore, notice that $(w+\phi)|_{\partial\Omega} = \phi|_{\partial\Omega}$. So, choosing C and C_2 large enough we also have that $w+\phi \geqslant u|_{\partial\Omega_\varepsilon}+\phi$ and this concludes the proof. □

We now discuss and detail the proof of the interior gradient estimates given in [DajDel09, Lemma 6], which use the classical ideas of Korevaar [Kor86].

Proposition 2.9 . *Assume that* $u \in C^3(\Omega)\cap C^1(\bar{\Omega})$ *satisfies* $\mathcal{Q}(u) = H$ *and* $u|_{\partial\Omega} = f$. *If* u *is bounded in* Ω *and* $|\nabla u|$ *is bounded on* $\partial\Omega$, *then* $|\nabla u|$ *is bounded in* Ω *by a constant that depends only on* $\sup_\Omega |u|$ *and* $\sup_{\partial\Omega}|\nabla u|$.

Proof. Since $\bar{\Omega}$ is compact and $Gu = \nabla u - \pi_*(\overline{\nabla}d)$ with $d \in \mathcal{C}^\infty(\mathbb{E})$, then $|\nabla u|$ is bounded if and only if $\|Gu\|$ is. So, suppose that the maximum of $\|Gu\|$ is attained at the interior point $q_0 \in \Omega$, where we may assume that $\|Gu\| \neq 0$ without loss of generality. Consider a geodesic ball $B \subset \Omega$ centered at q_0 with small radius $\rho \leqslant 1$ so that $|Gu| \geqslant C$ at points of $\bar{B}$ for some positive constant C. Without loss of generality, we may assume after a vertical translation that $u_{|B} < 0$.

We are going to work in the model of $\pi^{-1}(B)$ described in Section 1.2. Let $\eta(q,t) \in \mathcal{C}^\infty(\bar{B}\times\mathbb{R})$ be a non-negative function that vanishes on $\partial B\times\mathbb{R}$ and define $\bar{\Sigma}$ as the normal geodesic graph over Σ defined by

$$\bar{p} = \exp_p \varepsilon\eta(p)N(p)$$

where $p \in \Sigma$ is parametrized by $(q, u(q))$. Recall that N given in (1.27) was fixed to be pointing upwards. If $\varepsilon > 0$ is sufficiently small, we may describe $\bar{\Sigma}$ as a Killing graph of some function $\bar{u}$ defined in $\bar{\Omega}$. We denote by q_1 the point in Ω that maximizes $baru-u, e_1$. It is clear that $q_1 \in B$ and that, for $i = 1,2$, $\langle\nabla\bar{u}-\nabla u, e_1\rangle = 0$ at this point. By (1.27), the tangent planes to both graphs have the same slope with respect to fiber $\pi^{-1}(q_1)$ of ξ.

We claim that

$$H_{\bar{u}}(y) \leqslant H_u(y) \tag{2.14}$$

where H_u and $H_{\bar{u}}$ denote the mean curvature of Σ and $\bar{\Sigma}$, respectively. In fact, we can translate Σ upward in the vertical direction until the points $(q_1, u(q_1)) \in \Sigma$ and $(q_1, \bar{u}(q_1)) \in \bar{\Sigma}$ coincide, obtaining a tangency point for both graphs. Moreover, by the choice of q_1, it is clear that the translated copy of Σ is above $\bar{\Sigma}$ locally around the point. Thus, the inequality (2.14) is consequence of Proposition 2.3. In analytical terms, it is sufficient to notice that, by construction, $u = \bar{u}$ at ∂B and $u \leqslant \bar{u}$ in $\bar{B}$ and, since $H_{\bar{u}} = \mathcal{Q}(\bar{u}) \in \mathcal{C}^\infty(\Omega)$ and $H_u = \mathcal{Q}(u) \in \mathcal{C}^\infty(\Omega)$, Proposition 2.3 assures that $\bar{u} \leqslant u$ in B. Thus, this contradiction shows that (2.14) holds.

It is a well-known fact that since the variation of Σ we consider is along the normal direction, then the mean curvature may be expanded as

$$2H_{\bar{u}}(\bar{q}) = 2H_u(q) + \varepsilon J\eta + O(\varepsilon^2), \tag{2.15}$$

where $q, \bar{q} \in B$ are such that $\bar{u}(\bar{q}) = \exp_{(q,u(q))} \varepsilon\eta(q, u(q))N(q, u(q))$ and

$$J = \Delta_\Sigma + |A|^2 + \mathrm{Ric}_{\mathbb{E}}(N)$$

is the Jacobi operator produced by the linearization of the mean curvature equation (see [BarDoCEsc88]). Here, Δ_Σ is the Laplace-Beltrami operator induced on Σ and $|A|$ denotes the norm of its second fundamental form.

Let $q_2 \in B$ be such that $\bar{q}_2 = q_1$, that is, q_2 is such that

$$\bar{u}(q_1) = \exp_{(q_2,u(q_2))} \varepsilon\eta(q_2, u(q_2))N(q_2, u(q_2)).$$

Putting (2.15) in (2.14), it follows that

$$\varepsilon J\eta + O(\varepsilon^2) = 2(H_{\bar{u}}(q_1) - H_u(q_2)) \leqslant 2(H_u(q_1) - H_u(q_2)). \tag{2.16}$$

On the other hand, denoting by $\bar{H}_u \in \mathcal{C}^\infty(\mathbb{E})$ the extension of H_u in $\mathbb{E}$ that is constant along the fibers of π, Taylor's expansion of

$$\bar{H}_u(\varepsilon) = \exp_{(q_2,u(q_2))} \varepsilon\eta(q_2, u(q_2))N(q_2, u(q_2))$$

in $(q_1, u(q_1))$ gives

$$H_u(q_2) = \bar{H}_u(q_2, u(q_2)) = \bar{H}_u(q_1, u(q_1)) + \varepsilon\eta\ell(q_1) + O(\varepsilon^2),$$

where

$$\ell(q_1) = \sum_{i=1}^{3} \langle E_i, N\rangle_{|(q_1, u(q_1))}(E_i(\bar{H}_u))(q_1, u(q_1))$$
$$= \sum_{i=1}^{2} \langle E_i, N\rangle_{|(q_1, u(q_1))}(e_i(H_u))(q_1),$$

that is, ℓ is constant along the fibers and induces a smooth function in Ω, that we denote ℓ by an abuse of notation and then

$$H_u(q_1) - H_u(q_2) = -\varepsilon\eta\ell(q_1) + O(\varepsilon^2). \tag{2.17}$$

Thus, from (2.16) and (2.17) we get at q_1 that

$$\Delta_\Sigma \eta + \left(|A|^2 + \mathrm{Ric}(N, N) + 2\ell\right)\eta \leqslant O(\varepsilon).$$

Therefore,

$$\Delta_\Sigma \eta - M\eta \leqslant O(\varepsilon) \tag{2.18}$$

for some constant $M > 0$ which does not depend on η, but only on B.

In what follows we proceed as in [Kor86], choosing $\eta = g(\theta(q, t))$ for some real function g to be chosen later and a function θ defined so that $\Delta_\Sigma \eta$ is large for sufficiently large $|Gu|$. Since ϵ is chosen small, then (2.18) will give a contradiction. Observe that C being large implies that the tangent planes to Σ near $(q_1, u(q_1))$ are very steep. That a tangent plane to Σ is almost vertical means the tangential component $\nabla_\Sigma \theta$ of the gradient of θ is approximately θ_t. Then, we define

$$\theta(q, t) = \max\left\{0\,,\, Kt + \rho^2 - r^2\right\}$$

for some small constant $K > 0$, where $r(q) = \mathrm{dist}_M(q_0, q)$ is the geodesic distance measured from the center q_0 of B. We have that $0 \leqslant \theta \leqslant \rho$. Since we are assuming height estimates for Σ, we may choose K sufficiently small in such a way that $\theta > 0$ in a neighborhood of $(q_1, u(q_1))$ in $B \times \mathbb{R}^-$. We restrict ourselves to points where θ is differentiable. There,

$$\theta_t = K > 0.$$

Since $\eta = g \circ \theta$, we can compute

$$\Delta_\Sigma \eta = g'' \|\nabla_\Sigma \theta\|^2 + g' \Delta_\Sigma \theta, \tag{2.19}$$

an equations (2.18) and (2.19) give

$$g''\|\nabla_\Sigma\theta\|^2 + g'\Delta_\Sigma\theta - Mg \leqslant O(\varepsilon). \tag{2.20}$$

By hypotheses, the tangent plane of Σ at $(q_1, u(q_1))$ is not horizontal. (Otherwise, we obtain from (1.27) that $Gu(q_1) = 0$.) Let e be the unit vector that gives the steepest ascent direction in the tangent plane of Σ at $(q_1, u(q_1))$, namely

$$e = \frac{\mu}{W_u\|Gu\|}\left(\|Gu\|^2 E_3 + \tfrac{1}{\mu}(E_1(u)E_1 + E_2(u)E_2)\right).$$

Denoting by $\overline{\nabla}\theta$ the ambient gradient of θ and using that $\rho \leqslant 1$, we have

$$\begin{aligned}\langle\nabla_\Sigma\theta, e\rangle =&\langle\overline{\nabla}\theta, e\rangle = \frac{1}{W_u}\left(K\|Gu\| + \frac{E_1(u)E_1(\theta) + E_2(u)E_2(\theta)}{\|Gu\|}\right)\\ \geqslant&\frac{1}{W_u}(K\|Gu\| - \hat{C}K - 2),\end{aligned}$$

where $\hat{C} > 0$ is a constant independent of u that satisfies

$$\frac{E_i(u)}{\|Gu\|}E_i(\theta) = \frac{E_i(u)}{\|Gu\|}(KE_i(t) - 2re_i(r)) \geqslant -2 - \hat{C}K.$$

Since K and $\hat{C}$ are independent of u and the parameter s, we may assume that $\|Gu\| > 2/K + \hat{C}$, and conclude that

$$\|\nabla_\Sigma\theta\| > 0.$$

Finally, for $C_1 > 0$ large we choose

$$g(\theta) = e^{C_1\theta} - 1.$$

It is easily seen that this choice leads to a contradiction with (2.20). We conclude that $\|Gu\|$ and therefore $|\nabla u|$ is bounded by some constant which does not depend on u. □

The local existence theorem we are going to prove is a consequence of Theorem A.9. Beside gradient estimates, in order to apply Theorem A.9, we also need to prove the existence of a minimal solution with zero boundary value, which is the same as the existence of a minimal local section above U.

Lemma 2.10 . *Let $\pi : \mathbb{E} \to M$ a Killing submersion whose fibers have infinite length. If $U \subset M$ is open and relatively compact, then there is a minimal section over U.*

Proof. If M is compact, since the fibers of π have infinite length, then π admits a global smooth section [Ste51, Thm. 12.2], so $\int_M \frac{\tau}{\mu} = 0$ by [LerMan17, Prop. 3.3]. Hence, there is a minimal section over all M by [LerMan17, Thm. 3.6] and we are done. This means that we can assume M is not compact in what follows. Therefore, there is an increasing sequence of open subsets $G_n \subset M$ such that $\cup_{n\in\mathbb{N}} G_n = M$ and the boundary of each G_n consists of finitely many smooth Jordan curves. Since $\overline{U}$ is compact, there will be some $n_0 \in \mathbb{N}$ such that $G = G_{n_0}$ contains $\overline{U}$.

Let $\gamma_1, \dots, \gamma_r : [0,1] \to M$ be the boundary components of G. Each γ_k can be lifted to a horizontal curve $\widehat{\gamma}_k : [0,1] \to \mathbb{E}$ and let $d_k \in \mathbb{R}$ be the difference of heights of its endpoints, i.e., $\widehat{\gamma}_k(1) = \phi_{d_k}(\widehat{\gamma}_k(0))$. Let us attach smoothly a disk D_k to $\overline{G}$ such that $\partial D_k = \gamma_k$ and extend smoothly the Riemannian metric of G to $\overline{G} \cup D_k$. Let us also extend smoothly τ and μ to $\overline{G} \cup D_k$ in such a way that $\int_{D_k} \frac{\tau}{\mu} = 2d_k$. By uniqueness of Killing submersions [LerMan17], this implies that the total space $D_k \times \mathbb{R}$ of the Killing submersion over D_k with bundle curvature τ and Killing length μ can be glued smoothly with $\pi^{-1}(\overline{G})$ along $\pi^{-1}(\gamma_k)$, by just making a horizontal geodesic on $\partial D_k \times \mathbb{R}$ coincide with $\widehat{\gamma}_k$. After repeating this for all boundary components of G, we find a Killing submersion $\pi' : \mathbb{E}' \to M'$ whose fibers have infinite length, $M' = \overline{G} \cup D_1 \cup \ldots \cup D_r$ is compact, and induces on $(\pi')^{-1}(G)$ the same Riemannian metric as in $\pi^{-1}(G)$. The problem is therefore reduced to the compact case. □

Hence, without loss of generality, we can assume that the zero section F_0 is minimal. Then we can prove the following existence theorem.

Theorem 2.11 . *Let $\Omega \subset M$ be a domain with compact closure and $\mathcal{C}^{2,\alpha}$-boundary. If $H \neq 0$, suppose that $\partial\Omega$ is μ-convex and $\mathrm{Ric}(\pi^{-1}(\bar{\Omega})) \geqslant -\inf_{\partial\Omega} (\mu\, \tilde{\kappa}_g(\partial\Omega))^2$, where $\tilde{\kappa}_g(\partial\Omega)$ is the μ-geodesic curvature of $\partial\Omega$ Let $H \in \mathcal{C}^{\alpha}(\bar{\Omega})$ and $f \in \mathcal{C}^{2,\alpha}(\partial\Omega)$ be given functions. If*

$$\sup_{\Omega} |H| \leqslant \inf_{\partial\Omega} \mu\, \tilde{\kappa}_g(\partial\Omega),$$

then there exists a unique function $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ satisfying $u_{|\partial\Omega} = f$ whose Killing graph Σ has prescribed mean curvature H.

Proof. For $\sigma \in (0,1)$, consider the family of Dirichlet problems

$$P_\sigma(\Omega, H, f) = \begin{cases} \mathcal{Q}(u) = \sigma H & \text{in } \bar{\Omega}, \\ u = \sigma f & \text{on } \partial\Omega. \end{cases}$$

Lemma 2.10 and Proposition 2.3 imply that $P_0(\Omega, H, f) = P(\Omega, 0, 0)$ admits a unique solution $u \equiv 0$. Furthermore, under the hypotheses of theorem, Propositions 2.7, 2.8, 2.9 and Theorem A.2 implies that there exist $\beta \in (0,1)$ and $M > 0$ such that every $u \in \mathcal{C}^{2,\beta}(\bar{\Omega})$ solutions of $P_\sigma(\Omega, H, f)$, satisfies $\|u\|_{\mathcal{C}^{1,\beta}} < M$. Hence, Theorem A.9 implies that there exists a solution for $P_1(\Omega, H, f) = P(\Omega, H, f)$. □

Remark 2.12. Notice that $\mu\, \tilde{\kappa}_g(\partial\Omega)$ is just the mean curvature of the vertical cylinder above $\partial\Omega$. In particular, the condition $\sup_\Omega |H| \leqslant \inf_{\partial\Omega} \mu\, \tilde{\kappa}_g(\partial\Omega)$ is the natural extension of the classical convexity condition of the boundary. This condition can be avoided in the minimal case just by using the solution of [MeeYau82a, Theorem 1] as barriers to obtain $\mathcal{C}^0$-estimates and boundary gradient estimates.

As a consequence of the gradient estimates, the Arzela-Ascoli Theorem implies the following convergence result.

Theorem 2.13 (Compactness). *Let Ω be an open domain of M and $\{u_n\}$ be a $\mathcal{C}^0$-uniformly bounded sequence of smooth solutions of the Dirichlet problem for prescribed mean curvature equation in Ω. Then, there exists a subsequence of $\{u_n\}$ converging (in the $\mathcal{C}^k$-topology on compact subsets for all $k \in \mathbb{N}$) to a solution of the prescribed mean curvature equation in Ω.*

Remark 2.14. As a consequence of the Compactness Theorem, we can relax the hypotheses on boundary values of the Dirichlet problem. In particular, we can assume f to be piecewise continuous and prove the result in the following way. Let $\{\hat{f}_n\}$ (resp. $\{\tilde{f}_n\}$) an increasing (resp. decreasing) sequence of $\mathcal{C}^{2,\alpha}(\partial\Omega)$ functions converging to f and denote by $\hat{u}_n$ (resp. $\tilde{u}_n$) the solution of the Dirichlet problem $P(\Omega, \hat{f}_n)$ (resp. $P(\Omega, \tilde{f}_n)$). The Maximum Principle implies that $\{\hat{u}_n\}$ (resp. $\{\tilde{u}_n\}$) is an increasing (resp. decreasing) sequence of graphs having mean curvature H. In particular, in $\bar{\Omega}$ we have

$$\hat{u}_n < \hat{u}_{n+1} < \tilde{u}_{n+1} < \tilde{u}_n$$

for any $n \in \mathbb{N}$. Hence, applying the compactness Theorem and the Maximum Principle, we have that $u = \lim_{n\to\infty} \hat{u}_n = \lim_{n\to\infty} \tilde{u}_n$ is the solution of $P(\Omega, H, f)$.

2.3 Removable Singularity Theorem

In this Section we prove a removable singularity result firstly proved by L. Bers [Ber55] for minimal graphs, then by Finn [Finn65] for graphs of prescribed mean curvature in $\mathbb{R}^3$, by Nelli and Sa Earp[NelSaE96] for graphs of prescribed mean curvature in the hyperbolic space and then extended in unitary Killing submersions by C. Leandro and H. Rosenberg [LeaRos09, Theorem 4.1]. The same technique used in [LeaRos09] can be applied since the function μ has an upper bound in the domains of M where we are working. This extension guarantees a removable singularity result, for example, in Sol and for rotational graphs in $\mathbb{R}^3$.

Theorem 2.15 . *Let* $u\colon \Omega \setminus \{p\} \to \mathbb{R}$, $\Omega \subset M$, *be a function whose Killing graph has prescribed mean curvature* $H \in C^{0,\alpha}(\bar{\Omega})$. *Then* u *extends smoothly to a solution at* p.

Proof. For any $R > 0$, denote by $B_R(p)$ the μ-geodesic ball geodesic of radius R centered in $p \in M$. If R is sufficiently small, hypotheses of Theorem 2.11 are satisfied and then there exists a smooth function v defined on $B_R(p)$ satisfying the following Dirichlet problem:

$$\begin{cases} \operatorname{div}\left(\frac{\mu^2 Gv}{W_v}\right) = 2\mu H, & \text{in } B_R(p); \\ v = u, & \text{on } \partial B_R(p). \end{cases}$$

Fix a positive constant C and define the Lipschitz function

$$\varphi = \begin{cases} u - v, & \text{if } \mid u - v \mid < C; \\ C, & \text{if } \mid u - v \mid \geqslant C. \end{cases}$$

By definition, φ satisfies $\nabla\varphi = \nabla u - \nabla v = Gu - Gv$ in the set $\mid u - v \mid < C$ and $\nabla\varphi = 0$ in its complement.

For $0 < r < R$, let $A(r, R) = B_R(p) \setminus B_r(p)$ and denote by $m = \max\limits_{B_R(p)} \mu$. Hence,

$$\begin{aligned} \int_{\partial A(r,R)} \varphi\mu \left\langle \frac{\mu Gu}{W_u} - \frac{\mu Gv}{W_v}, \mathfrak{v} \right\rangle = & \int_{\partial B_r(p)} \varphi\mu \left\langle \frac{\mu Gu}{W_u} - \frac{\mu Gv}{W_v}, \mathfrak{v} \right\rangle \\ & + \int_{\partial B_R(p)} \varphi\mu \left\langle \frac{\mu Gu}{W_u} - \frac{\mu Gv}{W_v}, \mathfrak{v} \right\rangle \\ \leqslant & \int_{\partial B_r(p)} Cm = Cm \operatorname{Length}(\partial B_r(p)). \end{aligned}$$

Since the Killing graphs of u and v have the same mean curvature, we have that, when $| u - v | \geqslant C$, $\operatorname{div}\left[\varphi\left(\frac{\mu^2 Gu}{W_u} - \frac{\mu^2 Gv}{W_v}\right)\right] = 0$ and, when $| u - v | < C$,

$$\begin{aligned}
\operatorname{div}\left[\varphi\left(\tfrac{\mu^2 Gu}{W_u} - \tfrac{\mu^2 Gv}{W_v}\right)\right] &= \left\langle \nabla\varphi, \tfrac{\mu^2 Gu}{W_u} - \tfrac{\mu^2 Gv}{W_v}\right\rangle + \varphi \operatorname{div}\left(\tfrac{\mu^2 Gu}{W_u} - \tfrac{\mu^2 Gv}{W_v}\right) \\
&= \left\langle \nabla\varphi, \tfrac{\mu^2 Gu}{W_u} - \tfrac{\mu^2 Gv}{W_v}\right\rangle \\
&= \left\langle \nabla u - \nabla v, \tfrac{\mu^2 Gu}{W_u} - \tfrac{\mu^2 Gv}{W_v}\right\rangle \\
&= \left\langle Gu - Gv, \tfrac{\mu^2 Gu}{W_u} - \tfrac{\mu^2 Gv}{W_v}\right\rangle \\
&= \tfrac{W_u + W_v}{2} |N_u - N_v|^2_{\mathbb{E}} \leqslant |N_u - N_v|^2_{\mathbb{E}},
\end{aligned}$$

where the last equality follows by Lemma 2.2. By Stokes Theorem, we have

$$\begin{aligned}
\int_{A(r,R)} \operatorname{div}\left[\varphi\left(\frac{\mu^2 Gu}{W_u} - \frac{\mu^2 Gv}{W_v}\right)\right] &= \int_{\partial A(r,R)} \varphi \left\langle \frac{\mu^2 Gu}{W_u} - \frac{\mu^2 Gv}{W_v}, \mathfrak{v}\right\rangle \\
&= \int_{\partial A(r,R)} \varphi\mu \left\langle \frac{\mu Gu}{W_u} - \frac{\mu Gv}{W_v}, \mathfrak{v}\right\rangle \\
&\leqslant Cm \operatorname{Length}(\partial B_r(p)). \qquad (2.21)
\end{aligned}$$

Thus, it follows that

$$0 \leqslant \int_{A(r,R)\cap\{|u-v|<C\}} |N_u - N_v|^2_{\mathbb{E}} \leqslant Cm \operatorname{Length}(\partial B_r(p)).$$

Since m does not depend on r, as r decreases to zero we get that $N_u = N_v$ on the set $|u - v| < C$. Hence, $Gu = Gv$ in the set $| u - v | < C$. Since C was arbitrary, we have that $Gu = Gv$ in $A(0, R)$ and $u = v$ in $B_R(p) \setminus \{p\}$. Thus $u = v$ in $B_R(p)$. □

2.4 Perron Process

This method is rather well known (e.g. it was applied originally by Jenkins and Serrin [JenSer66] in the non-convex case with re-entrant corners), so we will just sketch it here in the Killing-submersion setting, for the sake of completeness. We will essentially follow Sa Earp and Toubiana's approach [SaeTou00, SaeTou08], see also [NeSaETo17, Ngu14]. Our goal is to solve the Dirichlet problem $P(\Omega, H, f)$ defined in (2.1), where f and H are continuous functions.

Given $u \in \mathcal{C}^0(\Omega)$ and $U \subset \Omega$ a small closed μ-convex disk, we will denote by $\tilde{u}_U$ the unique solution of $P(U, H, u_{|\partial U})$, with the same values as u on ∂U, which exists by Theorem 2.11. We also define $M_{U,u} \in \mathcal{C}^0(\bar{\Omega})$ as

$$M_{U,u}(p) = \begin{cases} u(p), & \text{if } p \in \bar{\Omega} \setminus U, \\ \tilde{u}_U(p), & \text{if } p \in U. \end{cases}$$

We say that $u \in \mathcal{C}^0(\bar{\Omega})$ is a *subsolution* (resp. *supersolution*) for $P(\Omega, H, f)$ if for any small closed disk $U \subset \Omega$, we have $u \leqslant M_{U,u}$ (resp. $u \geqslant M_{U,u}$), and $u|_{\partial\Omega} \leqslant f$ (resp. $u|_{\partial\Omega} \geqslant f$). Due to the ellipticity of the mean curvature equation, it easily follows that $u \in \mathcal{C}^2(\Omega)$ is a subsolution (resp. supersolution) if and only if $\mathcal{Q}(u) \geqslant H$ (resp. $\mathcal{Q}(u) \leqslant H$). Consequently, a solution $u \in \mathcal{C}^2(\Omega)$ of $P(\Omega, H, f)$ is both a subsolution and a supersolution. This fact will be used later to obtain subsolutions.

We also need to recall the notion of barrier.

Definition 2.16 . *We say that* $p_0 \in \partial\Omega$ *admits an* upper barrier *(resp.* lower barrier*) for* $P(\Omega, H, f)$ *if for any constant* $M_0 > 0$ *and any* $k \in \mathbb{N}$*, there exist an open neighborhood* V_k *of* p_0 *in* M *and a function* ω_k^+ *(resp.* ω_k^-*) of class* $\mathcal{C}^2(V_k \cap \Omega) \cap \mathcal{C}^0(\overline{V_k \cap \Omega})$ *such that*

i. $\omega_k^+ \geqslant f$ *(resp.* $\omega_k^- \leqslant f$*) on* $\partial\Omega \cap V_k$*,*

ii. $\omega_k^+ \geqslant M_0$ *(resp.* $\omega_k^- \leqslant -M_0$*) on* $\Omega \cap \partial V_k$*,*

iii. $\mathcal{Q}(\omega_k^+) \leqslant H$ *(resp.* $\mathcal{Q}(\omega_k^-) \geqslant H$*) in* $\Omega \cap V_k$*,*

iv. $\lim_{k\to\infty} \omega_k^+(p_0) = f(p_0)$ *(resp.* $\lim_{k\to\infty} \omega_k^-(p_0) = f(p_0)$*).*

This is motivated by the following result (see [Ngu14, Proposition 3.13] and [SaeTou10, Section 4] for the proof of this result in Sol_3 and $\mathbb{H}^n \times \mathbb{R}$).

Lemma 2.17 (Perron Process). *Let* $\Omega \subset M$ *be an open domain with piecewise regular boundary and assume that* $f : \partial\Omega \to \mathbb{R}$ *is continuous on each component of* $\partial\Omega$ *and has left and right limits at each vertex of* Ω*. Assume that* $P(\Omega, H, f)$ *has a supersolution* ϕ *and let* $\mathcal{S}_\phi$ *the set of subsolutions* φ *of* $P(\Omega, H, f)$ *such that* $\varphi \leqslant \phi$*.*

1. *If* $\mathcal{S}_\phi \neq \emptyset$*, then the function* $u(p) = \sup\{v(p) : v \in \mathcal{S}_\phi\}$ *is of class* $\mathcal{C}^2(\Omega)$ *and satisfies the equation* $\mathcal{Q}(u) = H$ *in* Ω*.*

2. *If Ω is bounded and $\partial\Omega$ admits upper and lower barriers at some regular point $p_0 \in \partial\Omega$ for the problem $P(\Omega, H, f)$, then the above solution u extends continuously at p_0 by setting $u(p_0) = f(p_0)$.*

Proof. Notice first that $M_U(\varphi) \in \mathcal{S}_\phi$ for any $\varphi \in \mathcal{S}_\phi$. Indeed, $M_U(\varphi) = \varphi < \phi$ on ∂U, and the Maximum Principle implies that $M_U(\varphi) < \phi$.

To show that u is in $\mathcal{C}^2(\Omega)$ and satisfies the minimal surface equation, consider any point $q \in \Omega$. Since $u(q)$ is defined as a supremum, we consider a sequence $\{v_n\} \subset \mathcal{S}_\phi$ satisfying $v_n(q) \to u(q)$ as $n \to +\infty$.

For each $n > 0$, let $u_n(p) = \sup\{v_1(p), \ldots, v_n(p)\}$, for any $p \in \bar{\Omega}$. Let $U \subset \Omega$ be a μ-convex neighborhood of q. By construction, it follows that $M_U(u_n)(q) \to u(q)$ as $n \to +\infty$. Furthermore, since $M_U(u_n) \geqslant M_U(u_m)$ on ∂U for any $n > m$, the Maximum Principle implies that $M_U(u_n)$ is an increasing sequence of solutions of $\mathcal{Q}(M_U(u_n)) = H$ in U, bounded above by ϕ. Hence, the Compactness Theorem implies that a subsequence of $M_U(u_n)$, which we call $M_U(u_n)$ by an abuse of notation, converges to a $\mathcal{C}^2$ function $\bar{u}$ on $\mathrm{Int}(U)$ satisfying $\mathcal{Q}(\bar{u}) = H$. We need to prove that $\bar{u}(p) = u(p)$ for any $p \in \mathrm{Int}(U)$.

To do so, fix a point $p_1 \in \mathrm{Int}(U)$ and consider a sequence $\{\tilde{v}_n\} \subset \mathcal{S}_\phi$ such that $\tilde{v}_n(p_1) \to u(p_1)$ as $n \to +\infty$. Using an argument similar to the one above, set $\tilde{u}_n = \sup\{\tilde{v}_n, M_U(u_n)\}$, and we have that $\{\tilde{u}_n\}$ is an increasing sequence, and thus, $\{M_U(\tilde{u}_n)\}$ is an increasing sequence of solutions to the minimal surface equation bounded from above. So the Compactness Theorem implies that a subsequence of $\{M_U(\tilde{u}_n)\}$, denoted by $\{M_U(\tilde{u}_n)\}$ without loss of generality, converges to $\tilde{u} \in \mathcal{C}^2(U)$ such that $H(\tilde{u}) = 0$. By construction, it follows that $M_U(u_n) \leqslant \tilde{u}_n \leqslant M_U(\tilde{u}_n)$ in U, and that $\tilde{u}_n(p_1) \leqslant M_U(\tilde{u}_n)(p_1) \leqslant u(p_1)$. Since $\bar{u} \leqslant \tilde{u}$ in $\mathrm{Int}(U)$ and $\bar{u}(q) = \tilde{u}(q)$, the Maximum Principle implies that $\bar{u} = \tilde{u}$ on $\mathrm{Int}(U)$, in particular, $\bar{u}(p_1) = u(p_1)$. As this is true for any $p_1 \in \mathrm{Int}(U)$, we can conclude that $\bar{u} = u$ on $\mathrm{Int}(U)$, and this concludes the proof of (1).

Next, let $p_0 \in \partial\Omega$ be a regular point admitting upper and lower barriers. Choose $M_0 > \sup_{\Omega \cap \partial V_k} \phi$ for all $k \in \mathbb{N}$. Then, $\omega_k^+(p) \geqslant \varphi(p)$ for every $\phi \in \mathcal{S}_\phi$, $k \in \mathbb{N}$, and $p \in \Omega \cap V_k$. Furthermore, for every $k \in \mathbb{N}$, we have that $\omega_k^- < \phi$ on $\bar{\Omega}$, and $M_U(\omega_k^-) \geqslant \omega_k^-$, meaning that $\omega_k^- \in \mathcal{S}_\phi$. Therefore, $\omega_k^- < u$. It follows that

$$\omega_k^-(p) - f(p_0) \leqslant u(p) - f(p_0) \leqslant \omega_k^+(p) - f(p_0)$$

for every $k \in \mathbb{N}$ and $q \in \Omega \cap V_k$. When p converges to p_0, and k diverges to $+\infty$, we get that $u(p)$ converges to $f(p_0)$ as desired. $\square$

End of the proof of Theorem 2.1. Perron Process is the key tool to prove Theorem 2.1 using the following argument. Notice that the existence of a solution to $P(\Omega, H, 0)$ is guaranteed by Lemma 2.10 for $H = 0$ and Theorem 2.11 when $H \neq 0$, and it allows us to assume that the zero section F_0 has mean curvature H. This implies that the first item of the Perron Process is satisfied. Furthermore, the Maximum Principle implies that the solution u of $P(\Omega, H, f)$ satisfies

$$\min f \leqslant u \leqslant \max f$$

and then we can build upper (resp. lower) barriers by considering μ-convex subdomains $D \subset \Omega$, such that $\partial D = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \Omega$ and $\Gamma_2 \subset \partial\Omega$ are sufficiently small continuous curves, and solving the Dirichlet problem

$$\begin{cases} \mathcal{Q}(u) = H & \text{in } D; \\ u = f & \text{on } \Gamma_2; \\ u = \max f \,(\text{resp. } \min f) & \text{on } \Gamma_1. \end{cases} \tag{2.22}$$

$\square$

Part II

MAIN RESULTS

3

THE GENERALIZED JENKINS–SERRIN THEOREM

In this chapter we deal with the so called Jenkins–Serrin problem, that is a Dirichlet problem with possible infinite boundary values. In particular, given a Riemannian Killing submersion $\pi\colon \mathbb{E} \to M$, we give necessary and sufficient conditions to solve the Dirichlet problem for the minimal surface equation in $\mathbb{E}$ over a relatively compact domain $\Omega \subset M$, with possible infinite boundary values on some arcs of $\partial\Omega$. In this chapter, as in Section 1.2, we will assume that the fibers of π have infinite length unless differently specified, which is a natural assumption for the Jenkins–Serrin problem, and we assume that the zero section F_0 we work with is minimal (see Lemma 2.10).

The first thing to understand is the properties of the arcs along which the minimal graph can diverge. In [RoSoTo10, Theorem 3.3], Rosenberg, Souam and Toubiana proved the following result.

Lemma 3.1 . *Let Σ be a graph of constant mean curvature H over $\bar{\Omega}$ given by $u \in \mathcal{C}^\infty(\Omega)$ with respect to F_0. Assume that $\gamma \subset \partial\Omega$ is a regular open arc such that $\lim\{u(p_n)\} = \pm\infty$ for all sequences $\{p_n\}$ of points in Ω converging to any $p \in \gamma$. Then $\pi^{-1}(\gamma)$ has mean curvature $\pm 2H$ and the angle function of Σ goes to 0 along any sequence approaching a point of γ.*

In particular, a minimal graph $u \in \mathcal{C}^\infty(\Omega)$ can diverge approaching a curve $\gamma \subset \partial\Omega$ only if γ is a μ-geodesic. The domain Ω is allowed to have simple closed μ-geodesics as boundary components with no vertices. This makes us consider the following problem:

Definition 3.2 . *A relatively compact open connected domain $\Omega \subset M$ will be called a* Jenkins–Serrin domain *if $\partial\Omega$ is piecewise regular and consists of μ-geodesic open arcs or simple closed μ-geodesics $A_1, \dots, A_r, B_1, \dots, B_s$ and μ-convex curves $C_1, \dots, C_m$ with respect to the inner conormal to Ω. The finite set $E \subset \partial\Omega$ of intersections of all these curves will be called the* vertex set *of Ω.*

The *Jenkins–Serrin problem* consists in finding a minimal graph over Ω, with limit values $+\infty$ on each A_i and $-\infty$ on each B_i, and such that it extends continuously to $\Omega \cup (\cup_{i=1}^{m} C_i)$ with prescribed continuous values f_i on each C_i with respect to a prescribed initial section F_0 defined on a neighborhood of Ω, i.e. the Jenkins–Serrin problem consists in finding a solution to the following Dirichlet problem:

$$P_{JS}(\Omega, f_i) = \begin{cases} \mathcal{Q}(u) = 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \cup A_i, \\ u = -\infty & \text{on } \cup B_i, \\ u = f_i & \text{on } C_i. \end{cases}$$

Note that all arcs are assumed to not contain their endpoints because a possible solution to the Jenkins–Serrin problem is not actually defined (as a function) at the vertices of the domain Ω in general, where discontinuities may occur.

An extra admissibility condition for Jenkins–Serrin domains is needed.

Definition 3.3 . *A Jenkins–Serrin domain $\Omega \subset M$ is said* admissible *if neither two of the A_i's nor two of the B_i's meet at a convex corner.*

The admissibility condition is a necessary condition as it will be shown in Proposition 3.9. If there are no C_i components, Jenkins and Serrin [JenSer66] use the fact that neither $\cup A_i$ nor $\cup B_i$ can be connected in $\mathbb{R}^2$. This condition has been required in the case of $M \times \mathbb{R}$ (see [Pin07] or [MaRoR011]) to prove the result using the same technique of Jenkins and Serrin, but it is not necessary (as it is shown in [MaRoR011, Remark 3.5]). Our approach allows us to drop this extra hypotheses for the admissibility of the domain. Eichmair and Metzger [EicMet16], using a different argument, also do not require this additional hypotheses in the case of product spaces $M \times \mathbb{R}$.

Definition 3.4 . *Let Ω be a Jenkins–Serrin domain. We will say that $\mathcal{P}$ is a μ-**polygon** inscribed in Ω if $\mathcal{P}$ is the union of disjoint curves $\Gamma_1 \cup \cdots \cup \Gamma_k$ satisfying the following conditions (see Figure 4):*

- *$\mathcal{P}$ is the boundary of an open and connected subset of Ω;*

- *each Γ_j is either a closed μ-geodesic or a closed piecewise-regular curve with μ-geodesic components whose vertices are among the vertices of Ω.*

For such an inscribed μ-polygon $\mathcal{P}$, define

$$\alpha(\mathcal{P}) = \mathrm{Length}_\mu((\cup A_i) \cap \mathcal{P}), \qquad \gamma(\mathcal{P}) = \mathrm{Length}_\mu(\mathcal{P}),$$
$$\beta(\mathcal{P}) = \mathrm{Length}_\mu((\cup B_i) \cap \mathcal{P}).$$

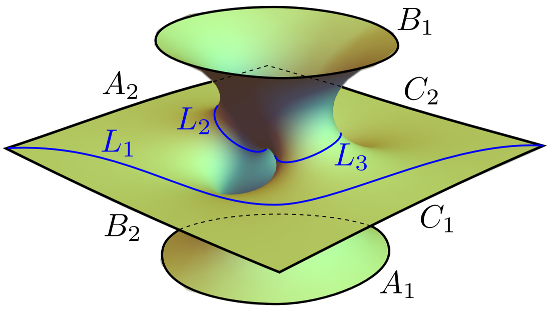

Figure 4: A Jenkins–Serrin problem with six μ-geodesic boundary components over a domain with the topology of a Costa surface. Here, a possible inscribed μ-polygon is $\mathcal{P} = \cup_{i=1}^4 \Gamma_i$ with $\Gamma_1 = L_1 \cup A_2 \cup C_2$, $\Gamma_2 = L_2$, $\Gamma_3 = L_3$ and $\Gamma_4 = A_1$.

Now we have all the ingredients to state the main theorem of this chapter.

Theorem 3.5 . *Let Ω be an admissible Jenkins–Serrin domain.*

(a) If the family $\{C_i\}$ is non-empty, then the Jenkins–Serrin problem in Ω has a solution if and only if

$$2\alpha(\mathcal{P}) < \gamma(\mathcal{P}) \quad \text{and} \quad 2\beta(\mathcal{P}) < \gamma(\mathcal{P}) \tag{3.1}$$

for all inscribed μ-polygons $\mathcal{P} \subset \Omega$, in which case the solution is unique.

(b) If the family $\{C_i\}$ is empty, then the Jenkins–Serrin problem in Ω has a solution if and only if (3.1) holds true for all inscribed μ-polygons $\mathcal{P} \neq \partial\Omega$ and $\alpha(\partial\Omega) = \beta(\partial\Omega)$. The solution is unique up to vertical translations.

The conditions in the statement about inscribed polygons will be called the *JS-conditions* for short. In the rest of this section, we will introduce the flux to prove that these JS-conditions are necessary (Proposition 3.9) as well as the uniqueness (Theorem 3.26). Finally, the existence of solutions will be proved by the method of divergence lines.

3.1 The Flux Argument

Let $\Omega \subset M$ be any domain. As shown by Proposition 1.24, $u \in \mathcal{C}^\infty(\Omega)$ satisfies the minimal surface equation if and only if $\operatorname{div}(X_u) = 0$, where $X_u = \mu^2 Gu/W_u$. This zero-divergence equation leads naturally to the definition of a flux for minimal graphs across curves of Ω.

Definition 3.6 . *Let $\Gamma \subset \Omega$ be a piecewise regular curve. The* flux *of $u \in \mathcal{C}^\infty(\Omega)$ across Γ with respect to a unit normal vector field η to Γ in M is defined as*

$$\operatorname{Flux}(u, \Gamma) = \int_\Gamma \langle X_u, \eta \rangle.$$

Since $\|X_u\| \leqslant \mu$ is bounded in Ω, the flux of u is well defined. This definition depends on the choice of the unit normal vector field, but the absolute value $|\operatorname{Flux}(u, \Gamma)|$ does not. The divergence theorem ensures that the flux across a curve enclosing a domain vanishes, so $|\operatorname{Flux}(u, \Gamma)| = |\operatorname{Flux}(u, \Gamma')|$ for two piecewise regular curves Γ and Γ' which are homotopic with respect to their common endpoints. Note also that Cauchy–Schwarz inequality yields the upper bound

$$|\operatorname{Flux}(u, \Gamma)| \leqslant \int_\Gamma \mu = \operatorname{Length}_\mu(\Gamma).$$

This last term denotes the μ-length of Γ, i.e., the length of Γ with respect to the conformal metric $\mu^2 ds_M^2$.

If X_u extends continuously to a regular curve $\Gamma \subset \partial\Omega$, then the flux across Γ can be defined similarly. Next lemma discusses the two different scenarios in which this idea has been typically applied.

Lemma 3.7 . *Let u be a solution to the minimal surface equation over Ω.*

1. *If u has limit value $\pm\infty$ along a μ-geodesic arc $A \subset \partial\Omega$, then* $\operatorname{Flux}(u, A) = \pm \operatorname{Length}_\mu(A)$ *with respect to the outer conormal to Ω along A.*

2. *If u extends continuously to $\Omega \cup C$, where $C \subset \partial\Omega$ is a μ-convex curve (with respect to the inner conormal), then* $|\operatorname{Flux}(u, C)| < \operatorname{Length}_\mu(C)$.

Proof. The equality $|\operatorname{Flux}(u, A)| = \operatorname{Length}_\mu(A)$ easily follows from the fact that if $u \to \pm\infty$ along A, then the tangent planes converge uniformly to vertical planes by Lemma 3.1. This means that ∇u is not bounded when approaching A, whence X_u is asymptotically equivalent to $\mu\, Gu/\|Gu\|$ or $\mu\nabla u/\|\nabla u\|$, where the norm is computed with respect to ds_M^2. Consequently, X_u can be extended continuously to $\Omega \cup A$ as $X_u = \pm\mu \cdot \eta$ on A, where the sign is positive if $u \to +\infty$ or negative if $u \to -\infty$ and η is the outer conormal to Ω along A.

In order to prove item (2), we will suppose without loss of generality that Ω is itself μ-convex and u has continuous values in $\partial\Omega$, because the argument is local. Let C' be a proper open subset of $\partial\Omega$. Theorem 2.1 guarantees the existence of $v \in \mathcal{C}^\infty(\Omega)$ satisfying the minimal graph equation such that $v = u - a$ in C' and $v = u$ in $\partial\Omega \setminus C'$. Since $u - v$ is not constant, Lemma 2.2 gives

$$\int_\Omega \langle \nabla u - \nabla v, X_u - X_v \rangle > 0.$$

Since $\operatorname{div}((u - v)(X_u - X_v)) = \langle \nabla u - \nabla v, X_u - X_v \rangle$, divergence theorem yields

$$0 < \int_{\partial\Omega} (u - v)\langle X_u - X_v, \eta \rangle = a \int_{C'} \langle X_u - X_v, \eta \rangle.$$

Letting $a = \pm 1$ and using the inequality $|\operatorname{Flux}(v, C')| \leqslant \operatorname{Length}_\mu(C')$, we obtain that $|\operatorname{Flux}(u, C')| < \operatorname{Length}_\mu(C')$, whence $|\operatorname{Flux}(u, C)| < \operatorname{Length}_\mu(C)$. □

Remark 3.8. Observe that with a slight modification it is possible to prove that given a domain Ω such that $A \subset \partial\Omega$ is a μ-geodesic arc and a sequence of solution to the minimal surface equation $\{u_n\}$, then:

(i) if $\{u_n\}$ diverges uniformly to $\pm\infty$ on compact subsets of Ω and remains uniformly bounded in compact subsets of A, then

$$\lim_{n \to \infty} \operatorname{Flux}(u_n, A) = \mp \operatorname{Length}_\mu(A);$$

(ii) if $\{u_n\}$ diverges uniformly to $\pm\infty$ on compact subsets of A and remains uniformly bounded in compact subsets of Ω, then

$$\lim_{n \to \infty} \operatorname{Flux}(u_n, A) = \pm \operatorname{Length}_\mu(A).$$

See [NelRos02, Lemma 1] for a detailed proof.

In the next proposition, we use the Flux Argument to show that the admissibility of the domain and the JS-conditions given by Theorem 3.5 are necessary.

Proposition 3.9 . *Consider a Jenkins–Serrin problem over some domain $\Omega \subset M$.*

1. *If two μ-geodesic components of $\partial\Omega$ meet at a convex corner and are both assigned the same value $+\infty$ or $-\infty$, then the problem has no solutions.*

2. *If the problem has a solution, then the JS-conditions are satisfied.*

Proof. Assume by contradiction that the problem with two adjacent sides A_1 and A_2 meeting at a convex corner p has a solution u. Let $p_1 \in A_1$ and $p_2 \in A_2$ be sufficiently close to p so that the minimazing μ-geodesic between p and p_1 (resp. p_2) is contained in A_1 (resp. A_2) and such that the μ-geodesic arcs joining p_1, p_2 is contained in Ω and it realizes the μ-distance between these two points. The flux of u across the boundary of the triangle of vertices p, p_1, p_2 is zero, which implies that $\mathrm{Length}_\mu(p_1p_2) > \mathrm{Length}_\mu(pp_1) + \mathrm{Length}_\mu(pp_2)$ by Lemma 3.7. This is in contradiction with the triangle inequality for the μ-metric.

As for item (2), let $\mathcal{P}$ be an inscribed μ-polygon which is the boundary of an open connected subset $\Omega_0 \subset \Omega$. The flux of a solution u across $\mathcal{P}$ gives

$$\mathrm{Flux}(u, (\cup A_i) \cap \mathcal{P}) + \mathrm{Flux}(u, (\cup B_i) \cap \mathcal{P}) + \mathrm{Flux}(u, \mathcal{P} \smallsetminus [(\cup A_i) \cup (\cup B_i)]) = 0, \tag{3.2}$$

with respect to the outer conormal to Ω_0 along $\mathcal{P}$. The first two summands in (3.2) add up to $\alpha(\mathcal{P}) - \beta(\mathcal{P})$, whereas the third one is, in absolute value, less than $\gamma(\mathcal{P}) - \alpha(P) - \beta(\mathcal{P})$ by Lemma 3.7. This gives the inequality

$$\gamma(\mathcal{P}) - \alpha(P) - \beta(\mathcal{P}) > |\,\mathrm{Flux}(u, \mathcal{P} \smallsetminus [(\cup A_i) \cup (\cup B_i)])| = |\alpha(\mathcal{P}) - \beta(\mathcal{P})|,$$

and it easily follows that $2\alpha(\mathcal{P}) < \gamma(\mathcal{P})$ and $2\beta(\mathcal{P}) < \gamma(\mathcal{P})$. However, this is true unless $\mathcal{P} = \partial\Omega$ and there are no C_i components, in which case the third summand in Equation (3.2) is identically zero, whence $\alpha(\mathcal{P}) = \beta(\mathcal{P})$. □

3.2 The Divergence-lines technique

In order to prove the existence of solution a to the Jenkins–Serrin problem, we will consider the possible limits of a sequence of graphs (not necessarily monotone), a context in which the theory of divergence lines plays an important role [Mazo4, MaRoRo11]. Recall that $\pi : \mathbb{E} \to M$ is a Killing submersion whose fibers have infinite length, $\Omega \subset M$ is a relatively compact domain, and we are considering Killing graphs with respect to a fixed zero minimal section F_0 defined on a neighborhood of $\overline{\Omega}$.

Let $\{u_n\}$ be a sequence of minimal graphs in Ω. For each $p \in \Omega$, define the translated minimal graph $\Sigma_n(p) \subset \mathbb{E}$ as the graph of $u_n - u_n(p)$. Observe that $\Sigma_n(p)$ contains the point $q = F_0(p)$ for all $n \in \mathbb{N}$ and has uniformly bounded curvature in a solid vertical cylinder of axis $\pi^{-1}(p)$ whose radius does not depend on n by Lemma 3.11. Since $\pi^{-1}(\Omega(\delta))$ has bounded geometry, standard convergence arguments show that a subsequence of $\Sigma_n(p)$ converges (locally nearby q) in the $\mathcal{C}^k$-topology on compact subsets for all $k \geqslant 0$ to a minimal surface Σ_∞ that contains q. In particular, the angle functions $\mathfrak{v}_n$ of $\Sigma_n(p)$ converge to the angle function $\mathfrak{v}_\infty$ of Σ_∞, whence $\mathfrak{v}_\infty \geqslant 0$. Since $\mathfrak{v}_\infty$ lies in the kernel of the Jacobi operator of Σ_∞, it satisfies a Maximum Principle (see [MePeRo08, Ass. 2.2]) so that either $\mathfrak{v}_\infty$ is identically zero or $\mathfrak{v}_\infty$ never vanishes.

- If the generalized gradients Gu_n are bounded at p, then any convergent subsequence of $\Sigma_n(p)$ actually converges to a minimal graph over a metric ball $D_M(p, R)$. By Proposition 2.9 and Theorem 2.13, the radius R can be chosen depending only on $d(p, \partial\Omega)$ and on the value of $\|Gu_n\|$ at p.

- If the generalized gradients Gu_n (and hence the usual gradients ∇u_n) are not bounded at p, up to a subsequence, we can assume that

  $$\mathfrak{v}_n(p) = (\mu^{-2} + \|Gu_n(p)\|^2)^{-1/2} \to 0 = \mathfrak{v}_\infty(p).$$

  This yields $\mathfrak{v}_\infty \equiv 0$ so we can produce a limit surface Σ_∞ which is part of a vertical cylinder over a μ-geodesic arc through p. Let L be the maximal extension of this μ-geodesic arc inside Ω. A standard diagonal argument says that there is a further subsequence $\Sigma_{\sigma(n)}(p)$ which converges uniformly to $\pi^{-1}(L)$ in the $\mathcal{C}^k$-topology on compact subsets for all $k \geqslant 0$ (see [MaRoRo11, Lemma 4.3]) and the unit normals of the sequence become horizontal along L. In the Killing-submersion setting, this means that

  $$\mathfrak{v}_{\sigma(n)} \longrightarrow 0, \qquad \text{and} \qquad \eta_{\sigma(n)} \longrightarrow \pm\eta_L, \tag{3.3}$$

  where $\eta_n = \nabla u_n / \|\nabla u_n\|$ and η_L is a unit normal to L in the metric ds^2_M (not in the μ-metric). Actually, to this end and for the arguments hereafter, we could have defined $\eta_n = Gu_n/\|Gu_n\|$ equivalently.

Definition 3.10 . *A μ-geodesic $L \subset \Omega$ is called a* ***divergence line*** *of a sequence of minimal graphs u_n over Ω if L is maximal (i.e., it is not a proper subset of another*

μ-geodesic $L' \subset \Omega$) and the graphs of $u_n - u_n(p)$ converge uniformly to $\pi^{-1}(L)$ on compact subsets for some (and hence for all) $p \in L$.

Before proving the properties of the divergence lines, it is convenient to prove a couple of results about convergence of minimal graphs in Killing submersions. We will consider a Killing submersion $\pi\colon \mathbb{E} \to M$ whose fibers have infinite length and a relatively compact domain $\Omega \subset M$ with piecewise smooth boundary, so there exists $\delta > 0$ such that the set $\Omega(\delta) \subset M$ consisting of the points at distance less than δ from Ω is also relatively compact. From Proposition 1.13, it follows that the sectional curvature of $\pi^{-1}(\Omega(\delta)) \subset \mathbb{E}$ is bounded by a constant $\Lambda > 0$ depending only on upper bounds for the Gauss curvature of M, τ and μ (and their first and second derivatives) on $\Omega(\delta)$. This is a key ingredient for the existence of gradient and curvature estimates.

A minimal graph is always stable because its angle function, which lies in the kernel of its Jacobi operator (also known as stability operator), has no zeros (see [LerMan17]). Stability implies curvature estimates, as proved by Schoen [Sch83, Theorem 3] and Rosenberg, Souam and Toubiana [RoSoTo10, Theorem 2.5]. We will rewrite the latter in terms of distance in the base.

Lemma 3.11 . *There exists C depending only on $\delta^2\Lambda$ such that the norm of the shape operator of any minimal graph Σ over Ω satisfies*

$$|A(q)| \leqslant \frac{C}{\min\{d_\Sigma(q,\partial\Sigma),\frac{\pi}{2\Lambda},\delta\}} \leqslant \frac{C}{\min\{d_M(\pi(q),\partial\Omega),\frac{\pi}{2\Lambda},\delta\}},\quad \textit{for all } q\in\Sigma,$$

where d_Σ and d_M are the distance functions in Σ and M, respectively.

Proof. There is no loss of generality if we assume that $G = \pi^{-1}(\Omega)$ is relatively compact in $\mathbb{E}$ after considering the Riemannian quotient of $\mathbb{E}$ by any vertical translation. Note that the graphical condition (and hence stability) is not affected by this quotient. This also implies that $\pi^{-1}(\Omega(\delta))$ is relatively compact. We will prove that $G(\delta)$, the set of points of $\mathbb{E}$ at distance from G less than δ, coincides with $\pi^{-1}(\Omega(\delta))$, so the statement follows directly from [RoSoTo10, Theorem 2.5].

Given $q \in \pi^{-1}(\Omega(\delta))$, there is some curve γ in M joining $\pi(q)$ and some $x \in \Omega$ whose length is less than δ. Denote by $\widehat{\gamma}$ the horizontal lift of γ (with respect to π) starting at q. Since the submersion is Riemannian, $\widehat{\gamma}$ has the same length as γ, and joins q and some $q' \in \pi^{-1}(x) \subset G$. This proves the inclusion

$\pi^{-1}(\Omega(\delta)) \subset G(\delta)$. To prove the other inclusion, let $q \in G(\delta)$ and $q' \in G$ such that $d(q', q) < \delta$. The minimum distance from p to the fiber $\pi^{-1}(\pi(q'))$ is realized by a geodesic $\widehat{\gamma}$ of $\mathbb{E}$ which is orthogonal to the fiber $\pi^{-1}(\pi(q'))$ at its endpoint. However, this implies that $\widehat{\gamma}$ is everywhere horizontal since the product $\langle \widehat{\gamma}', \xi \rangle$ is constant along $\widehat{\gamma}$ ($\widehat{\gamma}$ is a geodesic and ξ is Killing). This means that $\gamma = \pi \circ \widehat{\gamma}$ is a curve in M joining $\pi(q)$ and $\pi(q') \in \pi(G) = \Omega$ and the length of γ is equal to the length of $\widehat{\gamma}$, so it is less than δ. In particular, $q \in \pi^{-1}(\Omega(\delta))$. □

A consequence of these curvature estimates is the Uniform Graph Lemma, which we will state for graphs in Killing submersions. Since graphs admit curvature estimates only depending on Λ, δ and the distance to the boundary (see Lemma 3.11), we can rewrite [RoSoTo10, Prop. 4.3] in a more convenient way. Indeed, in the proof given in [RoSoTo10] it is shown that such a graph has uniformly bounded Euclidean second fundamental form in harmonic coordinates, so we can also use [PerRos02, Lemma 4.1.1] to ensure that the growth of the graph is under control as stated in item (2) of the following lemma:

Lemma 3.12 ([RoSoTo10, Prop. 4.3]). *Let Σ be a minimal graph over Ω and let $q \in \Sigma$. There exist constants $a, \rho, \rho_0 > 0$ (depending on δ, Λ, $d(q, \partial\Sigma)$ and on a positive lower bound for the injectivity radius of Ω) and an open neighborhood $U_q \subset \mathbb{E}$ of q that can be parametrized by harmonic coordinates such that:*

1. *A subset $\Sigma_q \subset \Sigma \cap U_q$ containing q is an Euclidean graph (in the harmonic coordinates) over the disk of $D(0, \rho) \subset T_q\Sigma$ of Euclidean radius ρ.*
2. *If $f \in \mathcal{C}^\infty(D(0, \rho))$ is the function that defines the Euclidean graph, then $|f(v)| \leqslant a|v|^2$ for all $v \in D(0, \rho)$, where $|\cdot|$ is the Euclidean norm in $T_q\Sigma$.*
3. *The subset Σ_q contains the geodesic disk $B_\Sigma(q, \rho_0)$.*

In what follows we will assume that Ω is a Jenkins–Serrin domain as in Definition 3.2, though most properties can be easily adapted to general bounded or unbounded domains (see for example the proof of Theorem 5.5). A divergence line can be a closed μ-geodesic or an open μ-geodesic arc of finite or infinite length. Observe that nulhomotopic divergence lines cannot exist for the Maximum Principle. As a matter of fact, μ-geodesics in an arbitrary surface have self-intersections or accumulation points but next result shows that this is not possible for divergence lines. It is worth mentioning that in other more specific cases in the literature (e.g., in $\mathbb{H}^2 \times \mathbb{R}$ [MaRoRo11]), this discussion is not pertinent because μ-geodesics are properly embedded automatically.

Lemma 3.13 . *Each divergence line of a sequence of minimal graphs in Ω is properly embedded in $\overline{\Omega}$. In particular, such a line is either a closed μ-geodesic or an open μ-geodesic arc with finite μ-length connecting two points of $\partial\Omega$.*

Proof. First, if a divergence line L has a self-intersection at $p \in \Omega$, we find a contradiction. Consider a compact subset $K \subset \pi^{-1}(L)$ that contains some $q \in \pi^{-1}(p)$ in the interior. Given a translated subsequence $\Sigma_{\sigma(n)}$ that uniformly converges to K, their unit normals also converge uniformly to the normal of K at q. Since the self-intersection of L is transverse (because L is a μ-geodesic), this contradicts the uniqueness of limit of $\nabla u_{\sigma(n)}/\|\nabla u_{\sigma(n)}\|$ as stated in (3.3).

Now we shall assume that L accumulates on some $p \in \Omega$ and find a contradiction again. Let $\Sigma_{\sigma(n)}$ be a translated subsequence that uniformly converges to $\pi^{-1}(L)$ on compact subsets and let U_q be a neighborhood of $q = F_0(p)$ where harmonic coordinates exist (see Lemma 3.12). Accumulation at p gives a sequence $p_k \in L$ converging to p and disjoint closed subarcs $L_k \subset L \cap \pi(U_q)$ of fixed length centered at $p_k \in L_k$ that converge to a limit μ-geodesic arc L_∞ through p. We can also assume that a μ-geodesic arc $\Gamma \subset \Omega$ orthogonal to L_∞ at p intersects all the arcs L_k transversely. For each $k \in \mathbb{N}$, consider $K_m = \cup_{k=1}^{m} \cup_{t\in[-1,1]} \phi_t(F_0(L_k))$, which is a compact subset of $\pi^{-1}(L)$. Since $\Sigma_{\sigma(n)}$ converges uniformly to K_m for any fixed m, the curve $\Sigma_{\sigma(n)} \cap \pi^{-1}(\Gamma)$ must go up and down many times, so there must be local maxima $q_k \in \pi^{-1}(\Gamma) \cap \Sigma_{\sigma(n)}$ of the height of this curve over F_0 (indeed as many as desired by making m and n large enough), see Figure 5. Since q is bounded away from the boundaries $\partial\Sigma_n$ (uniformly on n), we can translate vertically so that each q_k lies in U_q and the uniform graph lemma 3.12 implies that an intrinsic ball of $\Sigma_{\sigma(n)}$ centered at q_k of uniform radius is an Euclidean graph in the harmonic coordinates in U_q (that also contains all the points $F_0(p_k)$). This is clearly a contradiction when m and n are large because these uniform graphs cannot go up and down in arbitrarily narrow vertical strips. Note that they are also vertical graphs (not only graphs in the Euclidean sense over the tangent plane).

A similar argument discards the possibility that L accumulates at some $p \in \partial\Omega$. In this case, p belongs to the μ-geodesic arc $L_\infty \subset \partial\Omega$ so the maxima q_k in the above paragraph are bounded away from $\partial\Sigma_{\sigma(n)}$ (the arcs L_k con-

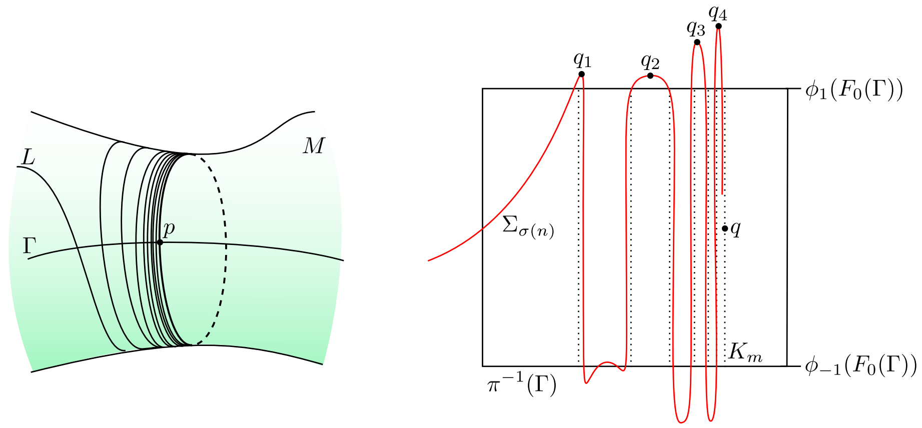

Figure 5: A divergence line that accumulates at some $p \in \overline{\Omega}$ (left) and the profile curve in the intersection $\pi^{-1}(\Gamma) \cap \Sigma_{\sigma(n)}$ (right). The dotted vertical lines represent the compact set K_m.

verging to L_∞ have fixed length). The contradiction arises again when m is large because the uniform graph lemma implies that the Euclidean graphs in harmonic coordinates on U_q must escape $\pi^{-1}(\Omega)$ by item (3) of Lemma 3.12 if $\pi(q_k)$ is close enough to $\partial\Omega$ (which is always the case for m large). $\square$

We show next that a divergence line cannot end at the interior of a component of $\partial\Omega$ where uniformly continuous boundary values have been prescribed. The proof of a similar result in $\mathbb{H}^2 \times \mathbb{R}$ [MaRoRo11, Prop. 4.8] strongly relies on reflections about horizontal geodesics, so we will need a different argument giving a slightly more general result. We want to point out that Lemma 3.14 applies if all the u_n have continuous fixed values at C but also when the value $\pm n$ is assigned to u_n on C, which is the case of the sequence (3.7) leading to the solution of the Jenkins–Serrin problem.

Lemma 3.14 . *Let $\{u_n\}$ be a sequence of minimal graphs over Ω and let $C \subset \partial\Omega$ be an open μ-convex arc (possibly μ-geodesic). If each u_n can be extended continuously to $\Omega \cup C$ and $\{u_n|_C - u_n(p)\}$ converges uniformly on C to a continuous function $f : C \to \mathbb{R}$ for some $p \in C$, then no divergence line of $\{u_n\}$ ends at p.*

Proof. Assume by contradiction that a divergence line L ends at p. Since L is μ-geodesic and C is μ-convex, their intersection at p is transverse. We will

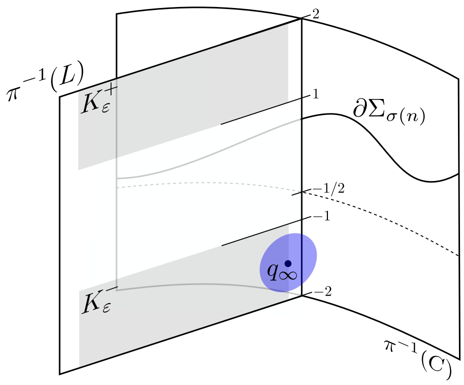

Figure 6: The compact sets $K_\varepsilon^\pm$ and the disk (in blue) that supports the graph which leaves the domain in the proof of Lemma 3.14. The dashed line represents the uniform lower bound for $\partial\Sigma_{\sigma(n)}$.

parametrize L as $\gamma : [0, \ell] \to L$ with unit speed and $\gamma(0) = p$, and define for $0 < \varepsilon < \frac{\ell}{2}$ the compact sets

$$K_\varepsilon^- = \bigcup_{t\in[-2,-1]} \phi_t(F_0(\gamma([\varepsilon, \ell-\varepsilon]))), \qquad K_\varepsilon^+ = \bigcup_{t\in[1,2]} \phi_t(F_0(\gamma([\varepsilon, \ell-\varepsilon]))).$$

Let v_n be a subsequence of $u_n - u_n(p_0)$, where $p_0 \in L$, that uniformly converges to $\pi^{-1}(L)$ on compact subsets, in particular on $K_\varepsilon^+ \cup K_\varepsilon^-$. Since each v_n is continuous on $\Omega \cup C$, we can consider a further subsequence to assume without loss of generality that $v_n(p) > 0$ for all n (the case $v_n(p) \leqslant 0$ for all n is similar). Choose a subarc $C' \subset C$ containing p such that $v_n|_{C'} \geqslant -\frac{1}{2}$, which does not depend on n by the uniform continuity on C given by the statement. If Σ_n denotes the graph of v_n, we can find a sequence $q_n \in \Sigma_n$ approaching $q_\infty = \phi_{-3/2}(\gamma(\epsilon)) \in K_\epsilon^-$, see Figure 6. This sequence verifies that $d_\Sigma(q_n, \partial\Sigma_n)$ is uniformly bounded away from zero because $v_n \geqslant -\frac{1}{2}$ on C' and the normal to Σ_n at q_n also approaches the normal to $\pi^{-1}(L)$ at q_∞.

If we start the above argument with ε small enough so that q_∞ lies in a prescribed harmonic coordinate chart centered at $\phi_{-3/2}(F_0(p))$, the uniform graph lemma 3.12 implies that Σ_n is an Euclidean graph over an almost vertical plane transverse to $\pi^{-1}(C)$, so Σ_n escapes $\pi^{-1}(\Omega)$ for small ε (the uniform radius does not depend on ε), which is the desired contradiction. In this argument, we have used item (2) in Lemma 3.12 strongly, since it implies that the bend of the graphs (in harmonic coordinates) is uniformly bounded. □

Assume that the divergence lines of a sequence of minimal graphs $\{u_n\}$ are disjoint and denote by $\mathcal{D}$ the union of all such lines. We can find a subsequence $\{u_{\sigma(n)}\}$ such that items (A)-(C) below hold (this was proved in $\mathbb{H}^2\times\mathbb{R}$, see [MaRoRo11, Prop. 4.4, Lemma 4.6, Rmk. 4.7] and the proof extends literally to the general case of Killing submersions). Let Ω_1 be a connected component of $\Omega\smallsetminus\mathcal{D}$ and let $p_1\in\Omega_1$.

(A) The translated sequence $u_{\sigma(n)}-u_{\sigma(n)}(p_1)$ converges uniformly on compact subsets of Ω_1 to a minimal graph u^1_∞ over Ω_1.

(B) If $L\subset\partial\Omega_1$ is a divergence line of $\{u_{\sigma(n)}\}$ and η_L is the outer unit conormal to Ω_1 along L, then $\eta_{\sigma(n)}\to\pm\eta_L$ and $u_{\sigma(n)}-u_{\sigma(n)}(p_1)\to\pm\infty$ uniformly on compact subsets of L (the sign $\pm$ is the same for both limits). The flux of $u_{\sigma(n)}$ in Ω_1 along L with respect to η_L gives (with the same choice of sign)

$$\lim_{n\to\infty}\mathrm{Flux}(u_{\sigma(n)},L)=\mathrm{Flux}(u^1_\infty,L)=\pm\,\mathrm{Length}_\mu(L).$$

(C) If Ω_2 is an adjacent component of $\Omega\smallsetminus\mathcal{D}$ such that $L\subset\partial\Omega_1\cap\partial\Omega_2$, then $\{u_{\sigma(n)}-u_{\sigma(n)}(p_1)\}$ diverges uniformly to $\pm\infty$ on compact subsets of Ω_2 provided that $\eta_{\sigma(n)}\to\pm\eta_L$ along L (the sign $\pm$ is the same for both limits).

In particular, two adjacent connected components of $\Omega\smallsetminus\mathcal{D}$ (at both sides of an isolated line of divergence L) cannot coincide. This topological obstruction discards some possible configurations of divergence lines.

It is important to notice that some divergence lines can disappear after passing to a subsequence but no new ones are created. Our next goal is to refine a sequence of minimal graphs so that the divergence lines are disjoint, whence they enjoy the above properties (A)-(C). In $\mathbb{H}^2\times\mathbb{R}$, this is not difficult since there are finitely many vertices and any two vertices are joined by a unique geodesic. However, over a relatively compact Jenkins–Serrin domain in general Killing submersions there might be an uncountable infinite number of divergence lines (see Remark 3.15) so we need again a new approach. We will have to deal with the two new situations depicted in Figure 7:

- Infinitely many closed disjoint μ-geodesics (as in the case of parallel circles in a round cylinder).
- Infinitely many disjoint μ-geodesics joining two fixed vertices (as in the case of meridians joining the north and south poles of the round sphere).

Remark 3.15. The above two situations can actually occur (we will prove later that this is not the case if the JS-conditions are satisfied). In $\mathbb{E} = \mathbb{S}^2 \times \mathbb{R}$ with $\tau \equiv 0$ and $\mu \equiv 1$, choose Ω as a wedge of $\mathbb{S}^2$ bounded by two meridians and let u_n take the value n and $-n$ on these meridians, then u_n spans a screw-motion invariant helicoid in $\mathbb{S}^2 \times \mathbb{R}$. Therefore, the limit of $\{u_n\}$ is a foliation of $\Omega \times \mathbb{R}$ by vertical cylinders, i.e., all geodesics of Ω joining the its vertices are divergence lines of $\{u_n\}$.

Likewise, in $\mathbb{E} = (\mathbb{S}^1 \times \mathbb{R}) \times \mathbb{R}$ with $\tau \equiv 0$ and $\mu \equiv 1$, consider the relatively compact domain $\Omega = \mathbb{S}^1 \times (-1, 1)$ and let u_n take the values $\pm n$ on $\mathbb{S}^1 \times \{\pm 1\}$. Then u_n spans a graph over Ω which is totally geodesic: it is a plane in the Euclidean space $\mathbb{R}^2 \times \mathbb{R}$, the universal cover of $\mathbb{E}$. These planes converge to vertical planes everywhere (i.e., tangent to the second factor $\mathbb{R}$), so the divergence lines of $\{u_n\}$ are the closed geodesics $\mathbb{S}^1 \times \{t_0\}$ with $-1 < t_0 < 1$.

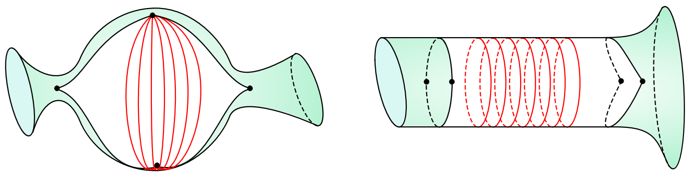

Figure 7: Two Jenkins–Serrin quadrilaterals containing open subsets isometric to part of a sphere (left) or a cylinder (right) so they have uncountably many potential divergence lines.

We will group the divergence lines in isotopy classes of closed μ-geodesics or open μ-geodesic arcs (with respect to their common endpoints in the latter case, i.e., the vertices remain fixed under the isotopy). Observe that, given such a class $\mathcal{J}$ and disjoint $L_1, L_2 \in \mathcal{J}$, the closed curve $\overline{L}_1 \cup \overline{L}_2$ is the boundary of a topological annulus (resp. disk) contained in Ω if $\mathcal{J}$ consists of closed curves (resp. open arcs).

Definition 3.16 . *Assume that all divergence lines are disjoint and their union is $\mathcal{D}$. The connected components of $\Omega \setminus \mathcal{D}$ will be called* convergence components.

Given a isotopy class of divergence lines $\mathcal{J}$ and $L_1, L_2 \in \mathcal{J}$, we will denote by $R(L_1, L_2) \subset \Omega$ the open disk or annulus with boundary $\overline{L}_1 \cup \overline{L}_2$. We will call isotopy region *the disk or annulus $R_{\mathcal{J}} = \cup_{L_1, L_2 \in \mathcal{J}} R(L_1, L_2)$.*

The closure of the divergence set (proved next in Lemma 3.17) will play a crucial role. Recall that a μ-geodesic L is a limit of μ-geodesics L_n if there is a sequence $p_n \in L_n$ converging to some $p \in L$ such that the unit tangent vectors to L_n at p_n converge to an unit tangent vector to L at p. This convergence is uniform in compact subsets (of the common arc-length parameter of these μ-geodesics) due to the smooth dependence of μ-geodesics on their initial conditions.

Lemma 3.17 . *Let $\{u_n\}$ be a sequence of minimal graphs over Ω.*

1. *Any limit of divergence lines of $\{u_n\}$ is either a μ-geodesic component of $\partial\Omega$ or again a divergence line of $\{u_n\}$.*

2. *Each isotopy class of divergence lines not isotopic to any μ-geodesic component of $\partial\Omega$ is closed (with respect to the convergence of μ-geodesics).*

Proof. Let L_n be a convergent sequence of divergence lines not converging to a component of $\partial\Omega$, so there exist $p_n \in L_n$ converging to some $p_\infty \in \Omega$ with unit tangent vectors ν_n to L_n at p_n that converge to a unit vector ν_∞ at p_∞, and let L_∞ be μ-geodesic through p_∞ with unit tangent vector ν_∞. Observe that $\pi^{-1}(L_n)$ converge as minimal surfaces to $\pi^{-1}(L_\infty)$ in the $\mathcal{C}^k$-topology on compact subsets for all k. Since L_n is a divergence line, denoting by $\mathfrak{v}_n$ the angle function of the graph of u_n in $u_n(p_n)$, we get that $\mathfrak{v}_n \to 0$ for $n \to \infty$. Assuming by contradiction that L_∞ is not a divergence line, we get that a subsequence $\{u_n - u_n(p_\infty)\}$ converges in a neighborhood U of p_∞ and, in particular, there exists a constant $C > 0$ such that $\mathfrak{v}_n > C$ for any n sufficiently large.

Item (2) follows readily from item (1) since a limit of simple closed μ-geodesics in some isotopy class is a simple closed μ-geodesic in the same isotopy class. □

Next, we will prove that a sequence of minimal graphs over a Jenkins–Serrin domain can be refined so that the divergence lines become disjoint and can be grouped into finitely many isotopy classes, and each isotopy class $\mathcal{I}$ defines the exclusive region $R_\mathcal{I}$ (see Definition 3.16) containing the (possibly uncountably many) divergence lines of $\mathcal{I}$ but no other lines in other isotopy classes. Furthermore, the lines of $\mathcal{I}$ separate countably many regions whose *divergence heights* are linearly ordered, that is, we only go up (or down) whenever we go through $R_\mathcal{I}$ transversely to the lines of $\mathcal{I}$. It was suggested in [MaRoRo11, Rmk. 4.5] that disjoint divergence lines can be obtained even in the uncountable case, so next result settles this question.

Proposition 3.18 . *Given a sequence of minimal graphs $\{u_n\}$ over a Jenkins–Serrin domain Ω, there is a subsequence $\{u_{\sigma(n)}\}$ whose divergence lines are pairwise disjoint, whence it has finitely many nonempty isotopy classes of divergence lines.*

Let $\mathcal{I}$ be one of such isotopy classes with at least two elements and assume that no μ-geodesic component of $\partial\Omega$ is isotopic to the elements of $\mathcal{I}$.

1. *There is a linear order $\prec$ in $\mathcal{I}$ such that $L_1 \prec L \prec L_2$ if and only if $L \subset R(L_1, L_2)$.*

2. *The ordered set $(\mathcal{I}, \prec)$ has maximum and minimum elements $L_+, L_- \in \mathcal{I}$.*

3. *All the curves of $\mathcal{I}$ have the same μ-length.*

4. *The order $\prec$ can be choosen uniquely (and we will do so) by assuming that $\eta_{\sigma(n)}$ converges to the unit inner conormal η_{L_-} to $R_{\mathcal{I}} = R(L_-, L_+)$ along L_-.*

5. *If $L \in \mathcal{I}$ is different from L_-, the normalized gradients $\eta_{\sigma(n)}$ converge to the outer unit conormal η_L to $R(L_-, L)$ along L.*

6. *Denote by $\mathcal{D}$ the union of all divergence lines of $\{u_{\sigma(n)}\}$. There are unique distinct convergence components $\Omega_\pm \subset \Omega \setminus R_{\mathcal{I}}$ with $L_\pm \subset \partial\Omega_\pm$.*
   - *(a) Given $p \in \Omega_-$ (resp. $p \in \Omega_+$), the sequence $\{u_{\sigma(n)} - u_{\sigma(n)}(p)\}$ diverges uniformly to $+\infty$ (resp. $-\infty$) on compact subsets of Ω_+ (resp. Ω_-).*
   - *(b) If $\Omega_0 = R(L_1, L_2) \subset R_{\mathcal{I}}$ is a convergence component with $L_1 \prec L_2$ and $p \in \Omega_0$, then $\{u_{\sigma(n)} - u_{\sigma(n)}(p)\}$ diverges uniformly to $+\infty$ (resp. $-\infty$) on compact subsets of $\overline{R(L_2, L_+)} \cup \Omega_+$ (resp. $\Omega_- \cup \overline{R(L_-, L_1)}$).*

Proof. Let $\{p_m\}$ be a countable and dense subset of Ω and $\mathcal{D}_1$ be the union of all divergence lines of $\{u_n\}$, which is a relatively closed subset of Ω. Let $L_1 \subset \mathcal{D}_1$ be a divergence line closest to p_1 (possible not unique), and let $\{u_n^1\}$ be a subsequence of $\{u_n\}$ such that the graphs of $u_n^1 - u_n^1(p)$ converge uniformly on compact subsets to $\pi^{-1}(L_1)$ for a fixed $p \in L_1$. Note that all divergence lines of $\{u_n^1\}$ (other than L_1) do not intersect L_1 because the normalized gradients η_n^1 converge along L_1 to an unit normal to L_1, and any other μ-geodesic intersects L_1 transversely.

By induction, suppose that we have a subsequence $\{u_n^{k-1}\}$ and let $\mathcal{D}_k$ its (relatively closed) set of divergence lines. Consider a divergence line $L_k \subset \mathcal{D}_k$ closest to p_k and define $\{u_n^k\}$ as a subsequence of $\{u_n^{k-1}\}$ such that the graphs of $u_n^k - u_n^k(p)$ converge uniformly on compact subsets to $\pi^{-1}(L_k)$ for a fixed $p \in$

L_k. As in the above argument, this leads to divergence lines $L_1, \dots, L_k$ which are pairwise disjoint (after removing possible repetitions) and also disjoint with any other divergence line of $\{u_n^k\}$. We will consider the diagonal sequence $u_{\sigma(n)} = u_n^n$ and the sequence of pairwise disjoint divergence lines $\{L_m\}$ we have constructed in this way: each L_m is disjoint with any other divergence line of $\{u_{\sigma(n)}\}$. Observe that the limits of elements of $\{L_m\}$ are disjoint too (if two limits intersect, then there must be sufficiently close elements of $\{L_m\}$ that also intersect since the convergence of μ-geodesics is uniform on compact subsets and their intersections are always transverse).

If $\{L_m\}$ is either finite (after removing repetitions) or contains all divergence lines of $u_{\sigma(n)}$, then we are done since this means that all divergence lines are disjoint. So, we will assume that L is a divergence line not in the sequence and prove that it is a limit of elements of $\{L_m\}$, which also proves that all divergence lines are disjoint. No point of L can be at positive distance from $\cup L_m$ (otherwise, as the p_m are dense, another divergence line different from any of the L_m should have been chosen in the process). Therefore, there is a sequence of points $x_k \in L_{m_k}$ converging to some $x_\infty \in L$. If ν_k is an unit normal to L_{m_k} at x_k, then up to its sign it must converge to an unit normal to L at x_∞ (otherwise, some of the L_m would intersect L). Therefore, L is a limit of elements of $\{L_m\}$ and we are done.

Since Ω has finite topology, it is diffeomorphic to a surface of finite genus minus some points given by its boundary curves. It is well known that such a surface cannot have infinite non-homotopic disjoint closed curves (e.g., see [Are15, Prop. 2.3.3]); also, it is clear that there cannot be infinitely disjoint non-isotopic arcs joining vertices of Ω. Since we have already shown that divergence lines are disjoint, the number of nonempty isotopy classes is finite. Recall that there are no nullhomotopic divergence lines by the Maximum Principle.

Let $\mathcal{I}$ be an isotopy class as in the statement, and let us prove items (1)–(6):

1. Given fixed distinct $L_1, L_2 \in \mathcal{I}$, we set $L_1 \prec L_2$. Then, there are three possible scenarios for another $L \in \mathcal{I}$, namely $L \in R(L_1, L_2)$, $L_1 \in R(L, L_2)$ or $L_2 \in R(L, L_1)$, in which case we set $L_1 \prec L \prec L_2$ or $L \prec L_1 \prec L_2$ or $L_1 \prec L_2 \prec L$, respectively. Given another $L' \in \mathcal{I}$, we can use the same argument to compare L and L', which easily leads to a total order in $\mathcal{I}$.

2. The region $R_{\mathcal{I}} = \cup_{L_1, L_2 \in \mathcal{I}} R(L_1, L_2)$ is nonempty because $\mathcal{I}$ contains at least two elements, so $R_{\mathcal{I}}$ is again a topological disk (resp. annulus) if $\mathcal{I}$ con-

sists of arcs (resp. closed curves). Given $p \in \partial R_{\mathcal{J}}$ not a vertex of Ω, there is a (possibly constant) sequence $\{L_n\}$ in $\mathcal{J}$ which accumulates at p, and hence a limit μ-geodesic L through p. Since no component of $\partial\Omega$ is isotopic to the elements of $\mathcal{J}$ by hypotheses, Lemma 3.17 ensures that $L \in \mathcal{J}$. This divergence line L must also be either a maximum or a minimum of $\prec$ by construction. Since $\partial R_{\mathcal{J}}$ cannot consist of just one divergence line by the Maximum Principle, we infer the existence of both the maximum L_+ and the minimum L_-, whence $R_{\mathcal{J}} = R(L_-, L_+)$.

(3)–(5) Given distinct $L_1, L_2 \in \mathcal{J}$, the divergence theorem on $R(L_1, L_2)$ gives

$$0 = \int_{L_1 \cup L_2} \langle X_{u_{\sigma(n)}}, \eta\rangle = \mathrm{Flux}(u_{\sigma(n)}, L_1) + \mathrm{Flux}(u_{\sigma(n)}, L_2), \tag{3.4}$$

where the flux is computed with respect to the outer unit conormal η to $R(L_1, L_2)$ along its boundary. Taking limits in (3.4) as $n \to \infty$, we get that $0 = \pm\,\mathrm{Length}_\mu(L_1) \pm \mathrm{Length}_\mu(L_2)$, where the signs depend on whether $\eta_{\sigma(n)}$ converges to η or $-\eta$. Clearly, both signs must be different so the result of the sum is zero, which proves item (3). In the case of $R(L_-, L_+)$, this means that $\eta_{\sigma(n)}$ converges to the inner conormal to $R_{\mathcal{J}}$ along L_- and to the outer conormal to $R_{\mathcal{J}}$ along L_+ up to reversing the order, so we have item (4). The very same argument proves item (5).

(6) Assume by contradiction that there is no such component Ω_+ or Ω_-. This means that there is a sequence of divergence lines outside $R_{\mathcal{J}}$ that accumulate at some $p \in \partial R_{\mathcal{J}}$. Since there are only finitely many nonempty isotopy classes of them, we can assume that they all belong to the same class, but this is clearly a contradiction since isotopy classes are closed and hence $\partial R_{\mathcal{J}}$ would intersect an element of an isotopy class other than $\mathcal{J}$. This proves the existence of the components $\Omega_\pm$ given in the statement.

Subitems (a) and (b) can be essentially proved in the same way and reflect the idea that all convergence components in $R_{\mathcal{J}}$ lie at different levels, which are linearly ordered, and this also applies to the adjacent ones $\Omega_\pm$. We will only consider the case $p \in \Omega_-$ as in item (a), since other cases are analogous. We first recall that $\{u_{\sigma(n)} - u_{\sigma(n)}(p)\}$ diverges uniformly to $+\infty$ on compact subsets of L_- because $L_- \subset \partial\Omega_-$. Assume by contradiction that $\{u_{\sigma(n)}(p_0) - u_{\sigma(n)}(p)\}$ remains bounded from above for some $p_0 \in R(L_-, L_+) \cup L_+ \cup \Omega_+$ (after possibly taking a further subsequence). Let $\gamma : [0, 1] \to \Omega$ be a regular curve joining p and p_0 meeting transversely the elements of $\mathcal{J}$ following the order given by $\prec$. The value of $u_{\sigma(n)}(\gamma(t)) - u_{\sigma(n)}(p)$ becomes arbitrarily high as $n \to \infty$ for points $\gamma(t) \in L_-$ and then remains bounded from above at $\gamma(1) = p_0$. This means that the graph of $u_{\sigma(n)}$ contains arbitrarily vertical directions that must subconverge to part of a cylinder over a divergence line (thus some $L \in \mathcal{J}$). However, since the value of the graph decreases from an arbitrarily high value as we cross L, the normalized gradient $\eta_{\sigma(n)}$ must converge to the inner conormal to $R(L_-, L)$, in contradiction to item (5). □

Remark 3.19. An interesting fact that may help understand the nature of the subsequence $\{u_{\sigma(n)}\}$ given by Proposition 3.18 is that all its divergence lines are not *removable*, in the sense that any further subsequence of $\{u_{\sigma(n)}\}$ has the same set of divergence lines. This is a consequence of the diagonal argument in the proof.

Under the JS-conditions, there will not be divergence lines in the isotopy class of a boundary component of type A_i or B_i (Lemma 3.27). However, most of the ideas of Proposition 3.18 can be adapted easily in the case that there is such a μ-geodesic $\Gamma \subset \partial\Omega$ (recall that the sides of Ω are not divergence lines, which must be interior to Ω by definition). We can extend the order $\prec$ to $\mathcal{J} \cup \{\Gamma\}$ and Γ acts as a maximum or minimum, in which case, one of the domains Ω_+ or Ω_- is not defined.

Also, an isotopy class $\mathcal{J}$ with just one element is not a problem since it can be understood using the above items (A)-(C) as in [MaRoRo11]. Because of the following corollaries, the structure of the divergence set is as depicted in Figure 8.

Corollary 3.20 . *Under the assumptions of Proposition 3.18, any connected component of $\Omega \setminus \mathcal{D}$ is either an inscribed μ-polygon or its boundary consists of strictly μ-convex arcs or closed curves $C_i \subset \partial\Omega$, μ-geodesics contained in $\mathcal{D}$ and, possibly, some μ-geodesic contained in $\partial\Omega$.*

Proof. A component $\Omega_0 \subset \Omega \setminus \mathcal{D}$ is bounded by disjoint μ-geodesic lines that can be either arcs joining two vertices of Ω or closed curves. We only need to prove that the number of such lines is finite, so assume by contradiction that it is not. Since there are finitely-many isotopy classes of divergence lines and finitely-many isotopy classes of the sides of $\partial\Omega$, we conclude that there are at

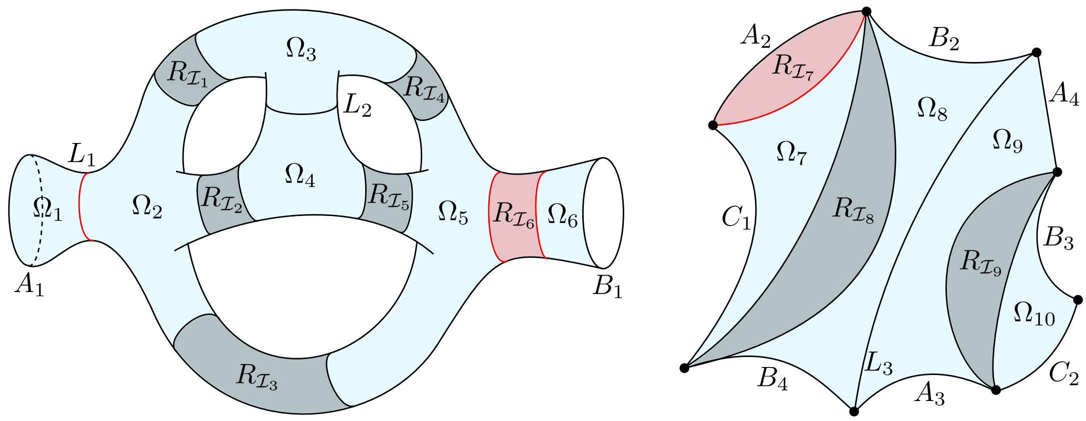

Figure 8: Two possible configurations of the divergence set in a surface of genus three (left) and in a genus zero octagon (right) with some isotopy classes with at least two elements (contained in the dark regions) plus several isolated divergence lines which are unique in their isotopy classes. Note that L_1, $R_{\mathcal{I}_6}$ and $R_{\mathcal{I}_7}$ (in red color) cannot exist under the JS-conditions by Lemma 3.27.

least three (infinitely many, indeed) isotopic distinct divergence lines L_1, L_2, L_3 that can be assumed to satisfy $L_1 \prec L_2 \prec L_3$. This is clearly a contradiction because the only two possible connected components of $\Omega \smallsetminus \mathcal{D}$ with $L_1 \cup L_3$ in their boundary are $R(L_1, L_3)$ and $\Omega \smallsetminus \overline{R(L_1, L_3)}$, none of which has L_2 as part of the boundary. $\square$

Corollary 3.21 . *Under the assumptions of Proposition 3.18,* $(\Omega \smallsetminus \mathcal{D}) \smallsetminus \cup_{\mathcal{I}} R_{\mathcal{I}}$ *has finitely many connected components.*

Proof. If m is the number of nonempty isotopy classes of divergence lines, there are at most $2m$ lines that can act as boundary components of the connected components of $(\Omega \smallsetminus \mathcal{D}) \smallsetminus \cup_{\mathcal{I}} R_{\mathcal{I}}$. Since each of these lines can only be in the boundary of two such connected components, we conclude that the number is finite. $\square$

3.3 A local Scherk-type surface and other barriers

We would like to obtain Scherk-type minimal surfaces on *small* μ-geodesic triangles $T \subset M$ that will serve as local barriers in our Jenkins–Serrin constructions. We will denote by p_1, p_2, p_3 the vertices of T and by ℓ_1, ℓ_2 and ℓ_3 the corresponding opposite geodesic sides. Corollary 1.23 yields the existence of a relatively compact open neighborhood U_i of p_i, where there is an open book decomposition by μ-geodesics with binding at p_i. We will say that T is *small* whenever $T \subset U$, where $U = U_1 \cap U_2 \cap U_3$, and all interior angles of T are at most π (notice that such a triangle T exists around any point $p \in M$ as long as we choose p_1, p_2, p_3 in a totally μ-convex neighborhood of p).

Proposition 3.22 . *There exists a minimal graph over* T *with zero value (with respect to* F_0*) on* $\ell_2 \cup \ell_3$ *and asymptotic value* $\pm\infty$ *on* ℓ_1*. Moreover, the tangent planes of* Σ *become vertical when approaching any interior point of* ℓ_1*.*

Proof. Assume that the boundary value on ℓ_1 is $+\infty$, since the case of $-\infty$ is analogous. For any n, the existence of a minimal solution u_n on T with value 0 on $\ell_2 \cup \ell_3$, and value n on ℓ_1 is guaranteed by Theorem 2.1. By Proposition 2.3, the sequence $\{u_n\}$ is non-decreasing and positive. Hence, to show that the limit $u = \lim_{n\to\infty} u_n$ exists, it is sufficient to prove that $\{u_n\}$ is uniformly bounded on any compact subset $K \subset \overline{T} \setminus \ell_1$ and then apply Theorem 2.13. Lemma 3.1 implies the last assertion of the statement.

Denote by Σ_n the graph of u_n. We will avoid the customary use of Douglas criterion by building a sequence of minimal disks $\{D_k\}$ such that

1. D_k is above Σ_n for all n and k, and
2. the family of the horizontal projections $\{\pi(D_k)\}$ exhausts T as $k \to \infty$.

The existence of D_k guarantees that $\{u_n\}$ is uniformly bounded on each compact subset $K \subset T$ since property (2) implies that $K \subset \pi(D_k)$ for some k.

The sequence D_k will be obtained inductively, but we need to set some notation first. For each $\varepsilon > 0$, let $\widetilde{T}$ be the μ-geodesic triangle with vertices p_1, $\tilde{p}_2$ and $\tilde{p}_3$, such that $\tilde{p}_2$ and $\tilde{p}_3$ belong to the μ-geodesic containing ℓ_1 at distance ε from p_2 and p_3, respectively. We will denote by $\tilde{\ell}_1$, $\tilde{\ell}_2$ and $\tilde{\ell}_3$ the sides of $\widetilde{T}$ opposite to p_1, $\tilde{p}_2$ and $\tilde{p}_3$, respectively (see Figure 9, left). We will assume that ε is small enough such that $\widetilde{T} \subset U$. To avoid a cumbersome notation, and only throughout this proof, we will consider the usual trivialization $F : U \times \mathbb{R} \to \pi^{-1}(U)$ given by $F(p,t) = \phi_t(F_0(p))$, where $\{\phi_t\}_{t\in\mathbb{R}}$ is the 1-parameter group of isometries associated to ξ. We will work on $U \times \mathbb{R}$ with the pullback metric by F in the sequel, so the minimality of $F_0|_U$ means that $U \times \{t_0\}$ is minimal for all $t_0 \in \mathbb{R}$.

Let $M_1 = \widetilde{T} \times [0,1]$ be the smooth compact three-manifold with boundary $\widetilde{T} \times \{0,1\} \cup (\ell_1 \cup \tilde{\ell}_2 \cup \tilde{\ell}_3) \times [0,1]$. Since ∂M_1 consists of five minimal smooth

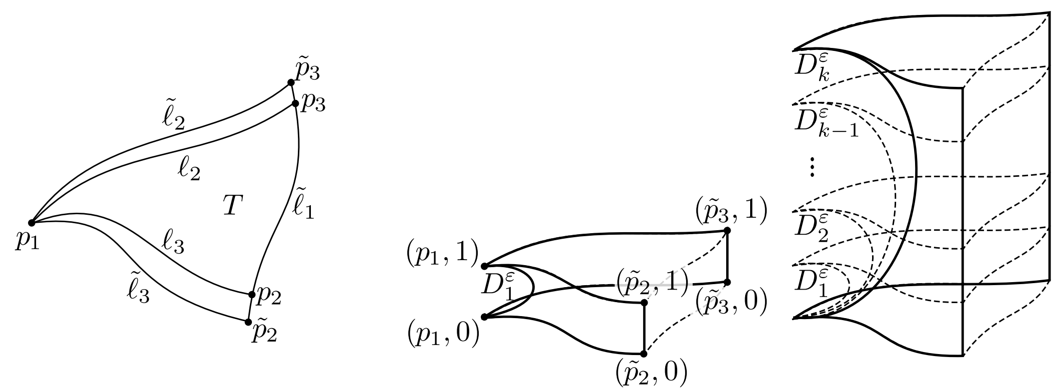

Figure 9: The μ-geodesic triangles T and $\widetilde{T}$ in the proof of Proposition 3.22, the initial minimal disk $D_1^\varepsilon \subset \widetilde{T} \times \mathbb{R}$ and the sequence of disjoint minimal disks D_k^ε constructed by recurrence.

pieces meeting at angles less than π, [MeeYau82a, Theorem 1] gives an (area-minimizing) smooth minimal disk D_1^ε with boundary

$$(\tilde{\ell}_2 \cup \tilde{\ell}_3) \times \{0, 1\} \cup (\{\tilde{p}_2, \tilde{p}_3\} \times [0, 1]),$$

that divides $T \times \mathbb{R}$ in two simply connected components (see Figure 9, center). The closure of the component whose boundary does not contain $\{p_1\} \times [0, 1]$ will be denoted by M_1^+.

Given $k \geqslant 2$, we define by recurrence $M_k = M_{k-1}^+ \cap (\widetilde{T} \times [0, k])$ so that

$$\partial M_k = (\widetilde{T} \times \{0, k\}) \cup D_{k-1}^\varepsilon \cup (\tilde{\ell}_1 \times [0, k]) \cup ((\tilde{\ell}_2 \cup \tilde{\ell}_3) \times [k-1, k]).$$

Again, by [MeeYau82a, Theorem 1], we find a minimal surface $D_k^\varepsilon \subset M_k$ with boundary $(\tilde{\ell}_2 \cup \tilde{\ell}_3) \times \{0, k\} \cup (\{\tilde{p}_2, \tilde{p}_3\} \times [0, k])$. We also define M_k^+ as the closure of the component of $(\widetilde{T} \times \mathbb{R}) \setminus D_k^\varepsilon$ whose boundary does not contain $\{p_1\} \times [0, k]$. Notice also that D_k^ε and D_{k-1}^ε do not have interior contact points (and they are not tangent at any point of their common boundary) by the Maximum Principle since D_{k-1}^ε acts as a barrier in the construction of D_k^ε (see Figure 9, right). Since D_k^ε is above Σ_n for all n, k and $\varepsilon > 0$ by the Maximum Principle and $\partial D_k^\varepsilon \cap \partial \Sigma_n = \{(p_1, 0)\}$, we define $D_k = \lim_{\varepsilon \to 0} D_k^\varepsilon$ and conclude that it lies above Σ_n. In particular, property (1) holds true and Σ_n is contained in $\cap_{k \in \mathbb{N}} M_k^+$ for all n. In this way, using the sequence $\{D_k\}$ and $T \times \{0\}$ as upper and lower barriers, we can assure that the limit of of the sequence $\{u_n\}$ is zero at $\ell_2 \cup \ell_3$.

As for property (2), observe that $\pi(D_k) \subset \pi(D_{k+1})$ for all k, so we we will reason by contradiction assuming that $\cup_{k \in \mathbb{N}} \pi(D_k)$ is not all T. Translate vertically each D_k so that it now lies in $T \times [\frac{-k}{2}, \frac{k}{2}]$. These translated disks are area-minimizing (in particular, stable) in $U \times \mathbb{R}$, which has bounded geometry. Let $\gamma \subset T \times \{0\}$ be a geodesic connecting $(p_1, 0)$ with $\ell_1 \times \{0\}$, since the sequence D_k is ordered (in the sense described in the previous paragraph), and $\cup_{k \in \mathbb{N}} \pi(D_k)$ is not all T, we can find an accumulation point $q_0 \in T \times \mathbb{R}$ for the ordered sequence $q_k = D_k \cup \gamma$. All in all, standard convergence arguments yield the existence of a stable minimal surface $D_\infty \subset T \times \mathbb{R}$ with boundary $\{p_1, p_2\} \times \mathbb{R}$. Since $q_0 \in D_\infty$, we conclude that D_∞ cannot be the vertical cylinder $\ell_1 \times \mathbb{R}$.

Consider the open-book decomposition of T with binding $\pi^{-1}(\tilde{p}_3)$ given by Corollary 1.23. Since D_∞ lies in $T \times \mathbb{R}$ and $\tilde{p}_3$ is outside T, we can find a leaf P of this open-book decomposition such that P and D_∞ are asymptotically tangent[1] and D_∞ lies in one of the components of $(T \times \mathbb{R}) \setminus P$ (note that there cannot be interior tangency points of D_∞ and P by Maximum Principle). Let $\{q_n\}$ be a sequence of points in D_∞ whose distance to P converges to zero, and let D_∞^n be the vertical translation of D_∞ that sends q_n to a point at height zero (in particular, $\mathrm{dist}_M(\pi(q_n), \pi(P))$ converges to 0 and hence $\lim_{n \to \infty}(\pi(q_n), 0) = (\pi(q_0), 0) \in P$). Again, we can take the limit of an ordered subsequence of D_∞^n as $n \to \infty$ and produce a minimal surface D_∞^∞ containing $(\pi(q_0), 0) \in P$ and lying in the closure of the same component of $(T \times \mathbb{R}) \setminus P$, so in this case D_∞^∞ does coincide with P by the Maximum Principle. Let $K \subset P$ be a compact domain such that $\pi(K) \setminus T \neq \emptyset$, so the convergence ensures that there exist domains K_n in Σ_n such that K_n converges uniformly to K. This says that there are points in Σ_n that project outside T, which is a contradiction. $\square$

Remark 3.23. Under the same assumptions, if p and p' are two points in ℓ_1 and γ is a μ-convex curve in T joining p and p', then the same argument in the proof of Proposition 3.22 yields the existence of a minimal graph u over Ω' such that $u|_\gamma = g$ and $u|_{\ell_1} = \pm\infty$ for any bounded function $g \in \mathcal{C}^0(\gamma)$, where Ω' is the relatively compact subdomain demarcated by ℓ_1 and γ.

We can use these Scherk type surfaces to analyze the boundary behavior of a sequence of minimal graphs that converges in the interior of the domain. This is a key step in the proof of Theorem 3.5 (see also [JenSer66, Lemma 7] and the Boundary Value Lemma in [ColRos10], whose proofs use different barriers).

1 Two surfaces S_1 and S_2 are asymptotically tangent when there are a sequence of points $\{p_n^1\} \in S_1$ and $\{p_n^2\} \in S_2$ such that $\mathrm{dist}(p_n^1, p_n^2)$ converges to 0.

Proposition 3.24 . *Let $\{u_n\}$ be a sequence of minimal graphs in a domain $\Omega \subset M$. Assume that there is a μ-convex arc $C \subset \partial\Omega$ such that each u_n can be extended continuously to $\Omega \cup C$. If u_n converges uniformly on compact subsets of Ω to a minimal graph u and $\{u_n|_C\}$ converges uniformly to a function f on C, then*

(a) $\{u_n\}$ *is uniformly bounded on a neighborhood of each* $p_0 \in C$*,*

(b) u *extends continuously to* $\Omega \cup C$ *by setting* $u|_C = f$*.*

Proof. In order to prove item (a), let us distinguish two cases:

1. If C is strictly μ-convex at p_0, then take two Scherk graphs over a small triangle T with values $\pm\infty$ along a side tangent to $\partial\Omega$ at p_0 and values $f(p_0) \pm 1$ on the other two sides (as in [ColRos10, Fig. 2], see also [JenSer66, Lemma 7]). It is clear that if T is small enough, all the u_n lie in between these two Scherk barriers.

2. Assume that C is a μ-geodesic arc with p_0 in the interior (we restrict C if necessary). Take $p_1 \in C$ close enough to p_0 such that $B_M(p_1, r)$ lies in a totally μ-convex neighborhood of p_1 that contains p_0. Let C_θ be the radial μ-geodesics through p_1 parametrized by the angle they make with $C_0 = C$. Let $d = d_{(M,\mu^2 ds^2)}(p_0, p_1)$ and, for a small $0 < \rho < d$, consider the μ-geodesic segment

$$C_{\theta,\rho} = \{p \in C_\theta : |d_{(M,\mu^2 ds^2)}(p_1, p) - d| < \rho\}$$

and the vertical region

$$Q_{\theta,\rho} = \cup_{t\in(0,2\rho)} \phi_t(C_{\theta,\rho}) \subset \pi^{-1}(C_\theta),$$

see Figure 10. For a fixed ρ, $\partial Q_{\theta,\rho}$ converges to $\partial Q_{0,\rho}$ as $\theta \to 0$ so we can find (not necessarily minimal) annuli $\Sigma_{\theta,\rho}$ of arbitrarily small area with boundary $\partial Q_{\theta,\rho} \cup \partial Q_{0,\rho}$ by making θ small enough. Since $\partial Q_{0,\rho}$ remains fixed, Douglas criterion ensures the existence of a minimal annulus S with boundary the two quadrilaterals $\partial Q_{0,\rho} \cup \partial Q_{\theta,\rho}$ for a small enough $\theta > 0$. Since u_n converges uniformly to f along $C_{0,\rho} \subset C$ and converges uniformly to u on the compact subset $C_{\rho,\theta} \subset \Omega$, a vertical translation of S provides the desired uniform estimate for u_n (from above and from below) on $\pi(S) \cup C_{0,\rho}$, which is a neighborhood of p_0 in $\Omega \cup C$.

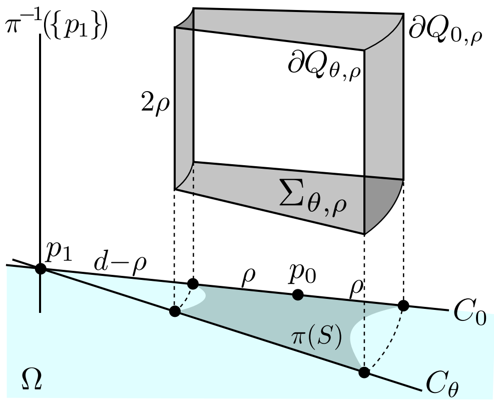

Figure 10: The minimal annulus $\Sigma_{\theta,\rho}$ has less area than any two disks with boundary $\partial Q_{\theta,\rho} \cup \partial Q_{0,\rho}$ for small enough θ. The shaded region in $\pi(S) \subset \Omega$ is the domain where the local barriers apply.

As for item (b), consider the barriers $\omega_k^{\pm}$ at p_0 given in Remark 2.14. Since f is continuous and $\{u_n\}$ converges uniformly to f, there exists $r > 0$ such that

$$|f(p) - f(p_0)| \leqslant \frac{1}{k} \quad \text{and} \quad |u_n(p) - f(p_0)| \leqslant \frac{1}{k}$$

for all $n \in \mathbb{N}$ and $p \in \partial\Omega$ with $d_{(M,\mu^2 ds^2)}(p, p_0) < r$. We can choose the triangle T that defines $\omega_k^{\pm}$ sufficiently small such that $d_{(M,\mu^2 ds^2)}(p, p_0) < r$ for all $p \in T \cap \Omega$. Item (a) allows us to assume that T is again small enough so $\{u_n\}$ is uniformly bounded on $\overline{T \cap \Omega}$. This means that we can choose the constant M_0 (see the definition of barrier in Section 2.4) large enough such that for all $n, k \in \mathbb{N}$ and $p \in T \cap \Omega$ we have

$$\omega_k^-(p) \leqslant u_n(p) \leqslant \omega_k^+(p). \tag{3.5}$$

This inequality holds in the boundary of $T \cap \Omega$ and extends to the interior by the Maximum Principle. Letting $n \to \infty$, the same inequality (3.5) holds for u at any interior point $p \in T \cap \Omega$. Finally, noticing that

$$f(p_0) - \tfrac{1}{k} = \lim_{p \to p_0} \omega_k^-(p) \leqslant \liminf_{p \to p_0} u(p) \leqslant \limsup_{p \to p_0} u(p) \lim_{p \to p_0} \omega_k^+(p) = f(p_0) + \tfrac{1}{k},$$

we get that $\lim_{p \to p_0} u(p) = f(p_0)$ by letting $k \to \infty$ so we get item (b). Here, we are using that M_0 is fixed in the process, whence ω_k^+ (resp. ω_k^-) is a decreasing (resp. increasing) sequence of functions. □

Proposition 3.25 . *Let $\{u_n\}$ be a sequence of minimal graphs in a domain $\Omega \subset M$. Assume that there is a μ-geodesic arc $A \subset \partial\Omega$ such that each u_n can be extended* *continuously to* $\Omega \cup A$. *If* u_n *converges uniformly on compact subsets of* Ω *to a minimal graph* u *and* $\{u_n|_A\}$ *diverges uniformly to* $\pm\infty$, *then* u *also diverges to* $\pm\infty$ *as we approach* A.

Proof. Assume that $\{u_n|_A\}$ diverges to $+\infty$ (the case of $-\infty$ is similar) and let $p_0 \in A$. The same argument as in case (2) of the proof of Proposition 3.24 implies that there is a neighborhood V of p_0 in $\Omega \cup A$ where $\{u_n\}$ is uniformly bounded from below, say there is some $a \in \mathbb{R}$ such that $u_n(p) \geqslant a$ for all $p \in V$ and $n \in \mathbb{N}$. Let $A' \subset A \cap V$ be a subarc centered at p_0 small enough so that there is a μ-convex curve $\Gamma \subset \Omega \cap V$ joining the endpoints of A' and $A' \cup \Gamma$ is the boundary of a topological disk D. Let v_m be the minimal graph over D with boundary values m on A' and a on Γ, which exists by Theorem 2.1. By the Maximum Principle, it follows that $u_n \geqslant v_m$ on D for n large enough, whence $u \geqslant v_m$ on D for all m. Since v_m is an increasing sequence of functions that take arbitrarily large values, we deduce that $\lim_{p \to p_0} u(p) = +\infty$. $\square$

3.4 Proof of the Jenkins–Serrin Theorem

We start by proving a uniqueness result that extends Proposition 2.3 to allow infinite values. It is stated as needed for proving Theorem 3.5, but it is worth observing that item (a) still holds true under milder assumptions with the same proof (e.g., if u extends continuously or has asymptotic value $-\infty$ on a μ-geodesic arc on which v has asymptotic value $+\infty$).

Theorem 3.26 (Uniqueness). *Let* $\Omega \subset M$ *be a Jenkins–Serrin domain. Suppose that* $u, v \in \mathcal{C}^\infty(\Omega)$ *define minimal graphs (with respect to a given initial section* F_0*) that extend continuously to* $\Omega \cup (\cup C_i)$ *and they both tend to* $+\infty$ *on each* A_i *and to* $-\infty$ *on each* B_j.

(a) If $\cup C_i \neq \emptyset$ *and* $u \leqslant v$ *in* $\cup C_i$, *then* $u \leqslant v$ *in* Ω.

(b) If $\cup C_i = \emptyset$, *then* $u = v + c$ *for some* $c \in \mathbb{R}$.

Proof. Let $w = u - v$ and assume that $U = \{p \in \Omega : w(p) > 0\} \neq \emptyset$ to reach a contradiction, which proves item (a), but also item (b) if we previously add a negative constant to v so that $\{p \in \Omega : w(p) \leqslant 0\}$ is not empty either. We will use and extend the notation and arguments of Proposition 2.3. By adding a small positive constant to v to assume that $\nabla w \neq 0$ along ∂U and $w < 0$ on

$\cup C_i$. Since u and v define minimal graphs, the divergence theorem guarantees that

$$\mathrm{Flux}(u, \partial U_\varepsilon) - \mathrm{Flux}(v, \partial U_\varepsilon) = \int_{\partial U_\varepsilon} \langle X_u - X_v, \eta \rangle = 0, \tag{3.6}$$

where η denotes the outer conormal to U_ϵ along its boundary. We can decompose $\partial U_\varepsilon = \Gamma_\varepsilon^1 \cup \Gamma_\varepsilon^2 \cup \Gamma_\varepsilon^3$, where $\Gamma_\varepsilon^1 \subset U \subset \mathrm{int}(\Omega)$, Γ_ε^2 lies in the boundary of the geodesic balls of radius ε, and $\Gamma_\varepsilon^3 \subset (\cup A_i) \cup (\cup B_i)$. Note that the component Γ_ε^3 did not appear in Proposition 2.3 because there were no infinite values in there. The assumption that $\nabla w \neq 0$ along ∂U implies that Γ_ε^1, Γ_ε^2 and Γ_ε^3 are away from $(\cup C_i) \cup V$ and consist of finitely many regular curves.

Consider the functions $\alpha_i(\varepsilon)$ defined as in (2.6) now for $i \in \{1, 2, 3\}$. By the same reasons as in Proposition 2.3, we have that $\lim_{\varepsilon \to 0} \alpha_1(\varepsilon) < 0$ and $\lim_{\varepsilon \to 0} \alpha_2(\varepsilon) = 0$. However, since $\mathrm{Flux}(u, \Gamma_\varepsilon^3) = \mathrm{Flux}(v, \Gamma_\varepsilon^3)$ by Lemma 3.7, we find that $\alpha_3(\varepsilon) = 0$ for all $\varepsilon > 0$, i.e., Γ_ε^3 does not contribute to the flux. This contradicts the fact that $\alpha_1(\varepsilon) + \alpha_2(\varepsilon) + \alpha_3(\varepsilon) = 0$ that follows from Equation (3.6). □

Consider a Jenkins–Serrin problem as in Definition 3.2. Theorem 2.1 yields the existence of minimal graphs $u_n \in \mathcal{C}^\infty(\Omega)$ with the following boundary conditions:

$$u_n = \begin{cases} n & \text{on } \cup A_i, \\ -n & \text{on } \cup B_i, \\ f_{i,n} & \text{on } C_i, \end{cases} \tag{3.7}$$

where $f_{i,n} = \min\{\max\{f_i, -n\}, n\}$ is the truncated continuous function f_i prescribed on the side C_i. Assume henceforth that the JS-conditions in Theorem 3.5 are satisfied and there is at least one side A_i or B_i (otherwise the JS-conditions become trivial and Theorem 3.5 follows from Theorem 2.1).

We will find a subsequence of $\{u_n\}$ that converges uniformly on compact sets of all Ω (i.e., without divergence lines) and achieves the desired boundary values. By Proposition 3.18, we can start with a subsequence $\{u_{\sigma(n)}\}$ such that all divergence lines are disjoint and can be grouped in finitely-many isotopy classes of either closed μ-geodesics or μ-geodesic arcs joining a pairs of vertices of Ω (see also Lemmas 3.13 and 3.14). We will denote by $\mathcal{D}$ the union of all divergence lines of $\{u_{\sigma(n)}\}$.

The next lemma completes the picture of the divergence set given by Proposition 3.18 by additionally assuming the JS conditions. Note that the flux limits still hold without such conditions (e.g., see [MaRoRo11, Lemma 3.6]).

Lemma 3.27 . *Under the JS-conditions no divergence lines of* $\{u_n\}$ *can either accumulate or be isotopic to any* A_i *or* B_i*.*

Proof. We will reason for one of the components A_i (the reasoning is completely analogous for a component B_i). We will first prove that divergence lines of $\{u_n\}$ cannot accumulate at A_i. By contraction, if they happen to accumulate, then A_i is a limit of divergence lines $\{L_n\}$ in the same isotopy class as A_i (isotopy classes of μ-geodesics are closed). Since all the L_n have the same μ-length by Proposition 3.18, so does A_i as a limit μ-geodesic. Therefore, $\mathcal{P} = \overline{A}_i \cup \overline{L}_1$ is an inscribed μ-polygon with $\gamma(\mathcal{P}) = 2\alpha(\mathcal{P})$, which is not possible by the JS-conditions.

Finally, assume by contradiction that there is a divergence line L isotopic to A_i, so the open region $R(A_i, L) \subset \Omega$ is either a disk or an annulus depending on whether A_i is an open arc or a closed curve. The very same argument as in item (3) of Proposition 3.18 implies that $\mathrm{Length}_\mu(A_i) = \mathrm{Length}_\mu(L)$. This in turn implies that the inscribed μ-polygon $\mathcal{P} = \partial R(A_i, L)$ verifies $\gamma(\mathcal{P}) = 2\alpha(\mathcal{P})$, which is the desired contradiction. □

We prove next that, under the JS-conditions, any subsequence of $\{u_n\}$ can be further refined to get rid of all divergence lines.

Lemma 3.28 . *For each* $p \in \Omega$*, any subsequence of* $\{u_n\}$ *has a further subsequence* $\{u_{\sigma(n)}\}$ *such that* $\{u_{\sigma(n)} - u_{\sigma(n)}(p)\}$ *uniformly converges on compact subsets of* Ω *to a solution of the minimal surface equation.*

Proof. Take a subsequence $\{u_{\sigma(n)}\}$ of the original subsequence having only disjoint divergence lines using Proposition 3.18, and assume by contradiction that the union of all divergence lines $\mathcal{D}$ is nonempty. Note that $\mathcal{D} \neq \Omega$ because, by Lemmas 3.17 and 3.27, isotopy classes of μ-geodesics are closed and $\mathcal{D} = \Omega$ would imply the existence of divergence lines isotopic to some A_i or B_i. Therefore, $\Omega \setminus \mathcal{D} \neq \emptyset$ and Lemma 3.27 ensures the existence of a convergence component $\Omega_1 \subset \Omega \setminus \mathcal{D}$ whose boundary contains one of the sides B_i (we can argue similarly if we assume that it contains one of the sides A_i); in particular, Ω_1 is disjoint with any of the regions $R_{\mathcal{J}}$. Note that $\{u_{\sigma(n)} - u_{\sigma(n)}(p_1)\}$ converges uniformly on compact subsets of Ω_1 for a fixed $p_1 \in \Omega_1$ and $\partial\Omega_1$ intersects an inscribed μ-polygon by Corollary 3.20. If there is a divergence line $L_1 \subset \partial\Omega_1$ such that $\{u_{\sigma(n)} - u_{\sigma(n)}(p_1)\}$ diverges to $+\infty$ along L_1, then the normalized gradients $\eta_{\sigma(n)}$ converge to the outer conormal to Ω_1 along L_1. We can define a new convergence component Ω_2 as follows:

1. If $L_1 = L_-$ and $\Omega_1 = \Omega_-$ for some isotopy class of divergence lines $\mathcal{J}$ with at least two elements, then define $\Omega_2 = \Omega_+$ (with the notation of Proposition 3.18). Hence, by item (6) of that Proposition, $\{u_{\sigma(n)} - u_{\sigma(n)}(p_2)\}$ converges uniformly on compact subsets of Ω_2 for any $p_2 \in \Omega_2$ and diverges to $-\infty$ on $\Omega_1 \cup \overline{R}_{\mathcal{J}}$.

2. Otherwise, L_1 is unique in its isotopy class so there is an adjacent component $\Omega_2 \subset \Omega \setminus \mathcal{D}$ such that $L_1 \subset \partial\Omega_1 \cap \partial\Omega_2$. By the already discussed results in [MaRoRo11], it follows that $\{u_{\sigma(n)} - u_{\sigma(n)}(p_2)\}$ converges uniformly to a minimal graph on compact subsets of Ω_2 for any $p_2 \in \Omega_2$ and diverges to $-\infty$ on $\Omega_1 \cup L_1$.

Either way, we found a component Ω_2 at a higher level than Ω_1. This process can be repeated to produce a sequence of convergence components $\Omega_1, \Omega_2, \ldots$ which are pairwise disjoint and disjoint with any of the regions $R_{\mathcal{J}}$. There is a finite number of such convergence components by Corollary 3.21, so the aforesaid process must end after finitely many steps. This means that we can find a component $\Omega_k \subset \Omega \setminus \mathcal{D}$ and $p_k \in \Omega_k$ such that $\{u_{\sigma(n)} - u_{\sigma(n)}(p_k)\}$ goes to $-\infty$ along all the divergence lines in $\partial\Omega_k$. If $\partial\Omega_k \cap (\cup C_i) = \emptyset$, then $\partial\Omega_k$ is an inscribed μ-polygon $\mathcal{P}_k$ and $\partial\Omega_k \cap (\cup A_i) \neq \emptyset$. The Flux Argument then implies that $2\alpha(\mathcal{P}_k) = \gamma(\mathcal{P}_k)$, contradicting the JS-conditions. Otherwise, if there exists a strictly μ-convex $C_k \subset \partial\Omega_k$, we work in the subdomain $\tilde{\Omega}_k \subset \Omega_k$ bounded by the μ-geodesics contained in $\partial\Omega_k$ and a μ-geodesic $\tilde{C}_k \subset \Omega_k$ isotopic to C_k and call $\mathcal{P}_k = \partial\tilde{\Omega}_k$. The divergence theorem on $\tilde{\Omega}_k$ with respect to its outer unit conormal η yields

$$0 = \int_{(\cup A_i)\cap\mathcal{P}_k} \langle X_{u_n}, \eta\rangle + \int_{(\cup B_i)\cap\mathcal{P}_k} \langle X_{u_n}, \eta\rangle + \int_{\mathcal{P}_k \setminus \partial\Omega} \langle X_{u_n}, \eta\rangle. \qquad (3.8)$$

Notice that this computation can be done indistinctly for u_n or $u_n - u_n(p_k)$ since they differ in a vertical translation. The first summand in (3.8) is bounded by $\alpha(\mathcal{P}_k)$ in absolute value, whereas the second and the third summands converge to $-\beta(\mathcal{P})$ and $\alpha(\mathcal{P}_k) + \beta(\mathcal{P}_k) - \gamma(\mathcal{P}_k)$, respectively, as $n \to \infty$ (by the same argument as in the proof of Lemma 3.7). In particular, the limit of (3.8) as $n \to \infty$ gives $\gamma(\mathcal{P}_k) - 2\alpha(\mathcal{P}_k) \leqslant 0$, which is not possible by the JS-conditions, whence $\mathcal{D} = \emptyset$. □

Now we have all ingredients to finish the proof of the main theorem.

Proof of the existence in Theorem 3.5. Fix $p \in \Omega$ and let $\{u_{\sigma(n)} - u_n(p)\}$ be the subsequence given by Lemma 3.28. We will assume first that $u_n(p)$ is bounded, so $\{u_{\sigma(n)}(p)\} \to a \in \mathbb{R}$ (up to a subsequence). Thus, it easily follows that $\{u_{\sigma(n)} - a\}$, or equivalently $\{u_{\sigma(n)}\}$, converges uniformly on compact subsets of Ω to a minimal graph u. Propositions 3.24 and 3.25 ensure that u achieves the desired boundary values along the components of $\partial\Omega$.

Now suppose that $\{u_n(p)\}$ is unbounded and also that $\{u_{\sigma(n)}(p)\} \to +\infty$ up to considering a further subsequence (the case $\{u_{\sigma(n)}(p)\} \to -\infty$ follows similarly). Let u be the limit of $\{u_{\sigma(n)} - u_{\sigma(n)}(p)\}$, which has the correct asymptotic value $-\infty$ on $\cup B_i$ by Proposition 3.25. Let $a_n = n - u_n(p)$ be the value that each graph $u_n - u_n(p)$ takes on $\cup A_i$, which is a sequence of positive numbers by the Maximum Principle (F_0 is a minimal section). Notice that we need that $a_n \to +\infty$ in order to get the desired solution. We will distinguish two cases:

1. If $\cup C_i \neq \emptyset$, then u takes the value $-\infty$ on each C_i. If $\{a_{\sigma(n)}\}$ is bounded, we can pass to a subsequence such that $\{a_{\sigma(n)}\} \to a \in \mathbb{R}$ so that u takes the constant value a on $\cup A_i$; otherwise, we can pass to a subsequence such that $\{a_{\sigma(n)}\} \to +\infty$ increasingly and u takes the value $+\infty$ on $\cup A_i$. Either way, by computing the flux of u across $\partial\Omega$ we get $2\alpha(\partial\Omega) \geqslant \gamma(\partial\Omega)$ which is not compatible with the JS-conditions.

2. If $\cup C_i = \emptyset$, then we apply a similar argument. If $\{a_{\sigma(n)}\}$ is not bounded, we get the desired solution with the correct boundary values. If $\{a_{\sigma(n)}\}$ is bounded, we find a graph u in all Ω with constant value on each A_i and $-\infty$ value on each B_i. This leads to $\alpha(\partial\Omega) > \beta(\partial\Omega)$, which is a contradiction. □

Remark 3.29. Assume the JS-conditions hold. If $\cup C_i \neq \emptyset$, the above argument shows that any subsequence of $\{u_n\}$ has a further subsequence that converges uniformly on compact subsets of Ω to a solution of the Jenkins–Serrin problem. Since the solution is unique by Proposition 2.3, this easily implies that the original sequence $\{u_n\}$ given by (3.7) converges itself to the solution. If $\cup C_i = \emptyset$, the same is true for $\{u_n - u_n(p)\}$ for any prescribed $p \in \Omega$ (no need of subsequences).

Remark 3.30. Our approach also gives information if the JS-conditions do not hold or there are two adjacent arcs of type A_i or B_i by analysing the behavior of $\{u_n\}$. There are two possible scenarios:

(a) Every subsequence of $\{u_n\}$ has divergence lines. In particular, we can find a subsequence of $\{u_n\}$ where these divergence lines are disjoint and hence they are grouped in isotopy classes that behave as in Proposition 3.18 and its corollaries (see Figure 8).

(b) There is a subsequence of $\{u_n\}$ without divergence lines, in which case we produce a minimal graph over all Ω with different boundary values. The rectangle of $\mathbb{R}^2$ in Figure 11 (left) cannot have divergence lines by symmetry and uniqueness of solution, so $u_n(p) \to +\infty$ at any point p of the rectangle. This means that $\{u_n - u_n(p)\}$ converges uniformly on compact subsets of the rectangle but we have performed an infinite translation downwards, so that the prescribed boundary values 0 become $-\infty$ whilst the values $+\infty$ become 0.

We also point out that divergence lines cannot end at convex corners where two of the C_i with finite values meet (we can use the small Scherk graphs as barriers at such a corner). However, there do exist examples in which divergence lines actually end on reentrant corners where two curves of type C_i meet. The example in $\mathbb{R}^2$ given in Figure 11 (right) cannot converge after bounded or unbounded translations because the JS-conditions are not satisfied. It is not difficult to see that the divergence lines are those in dashed line that end at the concave vertex.

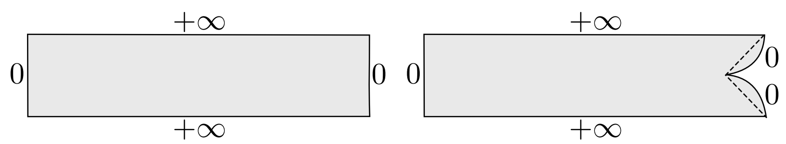

Figure 11: On the left, an Jenkins–Serrin problem which has a solution if we change the boundary values. On the right, the dashed segments show up as divergence lines of the sequence $\{u_n\}$.

3.5 Applications

3.5.1 *Minimal surfaces over unbounded domains*

Here we deal with the Dirichlet problem for minimal Killing graphs over unbounded domains of M. The result we are going to prove is inspired by

[NeSaET017, Theorem 4.3] and use the solutions of the Jenkins–Serrin problem as barriers in order to be able to apply the Compactness Theorem.

First of all, we need to define in which kind of unbounded domains we are going to work.

Definition 3.31 . *For* $p \in M$ *and* $\alpha \in (0, 2\pi)$ *let* $\tilde{W}$ *be a wedge of angle* α *in* T_pM. *Then, if* $\exp_p \colon \tilde{W} \to M$ *is a diffeomorphism, we say that* $W = \exp_p(\tilde{W})$ *is a* μ-wedge *of angle* α *and origin* p.

Definition 3.32 . *Let* $\gamma_1, \gamma_2 \in M$ *be two complete non-intersecting curves, both diffeomorphic to* $\mathbb{R}$, *such that* $\gamma_1 \cup \gamma_2$ *is the boundary of a connected domain* $S \subset M$. *We will say S is a (convex)* μ*-strip if the* μ*-geodesic curvature of* $\gamma_1 \cup \gamma_2$ *with respect to the inner normal pointing S is non-negative.*

We can now state the existence result for the Dirichlet problem as follows:

Theorem 3.33 . *Let* $\Omega \subset M$ *be an unbounded* μ*-convex domain contained either in a* μ*-wedge* W *of angle* $\alpha < \pi$ *or in a* μ*-strip* S *such that the* μ*-metric of* M *restricted to* W *or* S *is asymptotically flat. Let* φ *be a function on* $\partial\Omega$ *continuous except at a discrete and closed set* $U \subset \partial\Omega$ *of points where* φ *has finite left and right limits. Then there exists a minimal extension of* φ *over* $\bar{\Omega}$.

Proof. The argument is inspired by the proof of [NeSaET017, Theorem 4.3] and relies on the Compactness Theorem and on the existence of local barriers, that is guaranteed by the existence of solutions to the Jenkins–Serrin problem. In particular, the goal is to construct a sequence $\{u_n\}$ of minimal graph that will converge to the solution.

We study separately the cases $\Omega \subseteq W$ and $\Omega \subseteq S$.

Case 1: $\Omega \subseteq W$. Let $p \in M$ be the vertex of the μ-wedge W containing Ω and $\gamma_1(t)$ and $\gamma_2(t)$ be the two half μ-geodesics parametrized by arc length such that $\gamma_1(0) = \gamma_2(0) = p$ and $\gamma_1(\mathbb{R}_+) \cup \gamma_2(\mathbb{R}_+) = \partial W$, that is $\gamma_1(t) = \exp_p(\nu_1 t)$ and $\gamma_2(t) = \exp_p(\nu_2 t)$ for two directions $\nu_1, \nu_2 \in \mathbb{S}^1$. For all $n \in \mathbb{N}$ sufficiently large, let $r_n = \gamma_1(n)$ and $s_n = \gamma_2(n)$ and let $\gamma_{r_n}^{s_n} \subset W$ be a μ-geodesic that joins r_n and s_n such that, denoting by T_n the μ-geodesic triangle with vertices p, r_n and s_n with $\gamma_{r_n}^{s_n} \subset \partial T_n$, any possible other μ-geodesic connecting r_n and s_n does not lie in the interior of T_n.

We denote by a_n (resp. b_n) the point of $\Omega \cap \gamma_{r_n}^{s_n}$ closest to r_n (resp. s_n) and by Γ_n the μ-geodesic closest to p joining a_n and b_n (notice that Γ_n and $\gamma_{r_n}^{s_n}$ could be distinct). Finally, we call Ω_n the domain bounded by $\partial\Omega \cap T_n$ and Γ_n (see Figure 12).

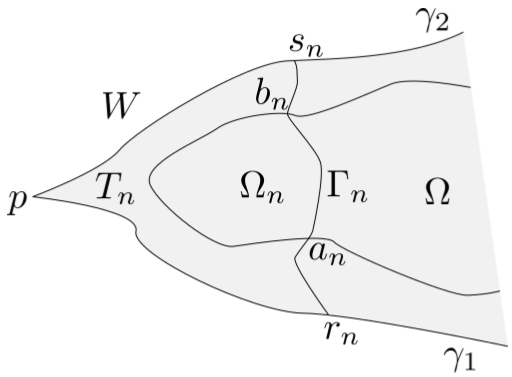

Figure 12: Sequence of domains in the μ-wedge

Since U is discrete, we can assume that φ is continuous at a_n and b_n. Notice that, by construction, Theorem 3.5 implies that in each $\Omega_n \neq \emptyset$ we can find a minimal graph $\omega_n^\pm$ such that $\omega_n^\pm = \varphi$ in $\partial\Omega \cap T_n$ and diverges to $\pm\infty$ approaching Γ_n. Now we build a sequence of solutions as in [NeSaET017, Theorem 4.3]. On $\partial\Omega_n$, we consider a piecewise continuous function φ_n such that it is continuous on Γ_n, with values between $\varphi(a_n)$ and $\varphi(b_n)$ and

$$\varphi_n(q) = \begin{cases} \varphi(q) & \text{if } q \in \partial\Omega_n \setminus \Gamma_n; \\ \varphi(a_n) & \text{if } q = a_n; \\ \varphi(b_n) & \text{if } q = b_n. \end{cases}$$

As Ω_n is bounded and μ-convex and φ_n is piecewise continuous, Theorem 3.5 guarantees the existence of a minimal extension u_n of φ_n on Ω_n. We recall that for any discontinuity point $q \in \partial\Omega$, the boundary of the graph of each u_n contains part of the fiber above q with endpoints the left and the right limit of φ at q. Moreover, there are no other points of the closure of the graph of u_n on the vertical geodesic passing through the discontinuity points. Let n_0 be the smaller natural number such that $\Omega_{n_0} \neq \emptyset$. The Maximum Principle implies that $\omega_{n_0}^+ \geqslant u_m \geqslant \omega_{n_0}^-$ for all $m > n_0$. Then, using the Compactness Theorem, we can take a subsequence $\{u_{n_0,m}\}_m$ converging to a function $\tilde{u}_{n_0}$ in Ω_{n_0}. For any $n > n_0$, using ω_n^+ and ω_n^- as barriers, we can solve the problem in Ω_n taking, by induction, a subsequence $\{u_{n,m}\}_m$ of $\{u_{n-1,m}\}_m$ converging to the function $\tilde{u}_n$. By construction, $\tilde{u}_m = \tilde{u}_n$ in Ω_n for any $m > n$, that is, $\tilde{u}_m$ is the analytic extension of $\tilde{u}_n$ in Ω_m. Thus, $u = \lim_{n\to\infty} \tilde{u}_u$ will be the solution that we are looking for.

Case 2: $\Omega \subseteq S$. Let $\gamma_1(t)$ and $\gamma_2(t)$ be the μ-convex curves parametrized by arc length such that $\partial S = \gamma_1(\mathbb{R}) \cup \gamma_2(\mathbb{R})$. For any $n > 0$ we call $\eta_n^l \subset S$ (resp. $\eta_n^r \subset S$) the μ-geodesic that minimizes the distance between $\gamma_1(-n)$ and $\gamma_2(-n)$ (resp. $\gamma_1(n)$ and $\gamma_2(n)$) and denote by Q_n the quadrilateral domain bounded by $\gamma_1([-n,n]) \cup \gamma_1([-n,n]) \cup \eta_n^l \cup \eta_n^r$. Let a_n (resp. d_n) be the point in η_n^l closest to γ_1 (resp. γ_2) and b_n (resp. c_n) be the point in η_n^r closest to γ_1 (resp. γ_2). We denote by Γ_n^l (resp. Γ_n^r) the μ-geodesic closest to η_n^l (resp. η_n^l) joining a_n and d_n (resp. b_n and c_n) and call Ω_n the domain bounded by $\Gamma_n = \Gamma_n^l \cup \Gamma_n^r$ and $\partial\Omega \cap Q_n$ (see Figure 13).

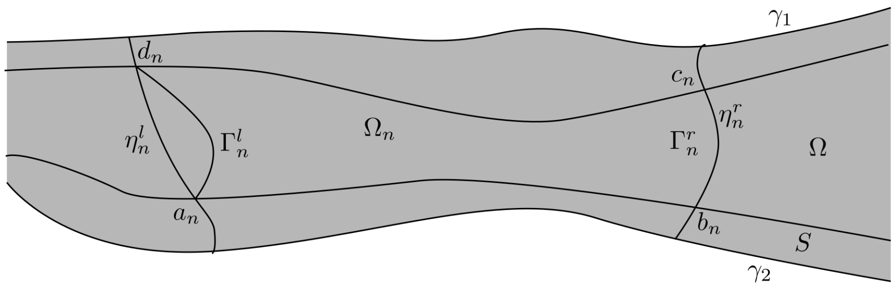

Figure 13: Sequence of domains in the μ-strip

Since the μ-metric in S is asymptotically flat, there exists $n_0 \in \mathbb{N}$ such that Ω_n satisfies the hypotheses of Theorem 3.5 for any $n > n_0$. Therefore, there exist functions $\omega_n^\pm$ satisfying the minimal surface equation such that $\omega_n^\pm = \varphi$ in $\partial\Omega \cap Q_n$ and diverging to $\pm\infty$ as we approach Γ_n.

Now we can proceed as in the case of the wedge: since U is discrete, without loss of generality, we can suppose that φ is continuous at a_n, b_n, c_n, d_n. On the boundary of Ω_n, we consider a piecewise continuous function φ_n, continuous on Γ_n^l, with values between $\varphi(a_n)$ and $\varphi(d_n)$, and on Γ_n^r, with values between $\varphi(b_n)$ and $\varphi(c_n)$, such that

$$\varphi_n(q) = \begin{cases} \varphi(q) & \text{if } q \in \partial\Omega \cap Q_n; \\ \varphi(a_n) & \text{if } q = a_n; \\ \varphi(b_n) & \text{if } q = b_n; \\ \varphi(c_n) & \text{if } q = c_n; \\ \varphi(d_n) & \text{if } q = d_n. \end{cases}$$

For any n sufficiently large, we denote by u_n the solution to the Dirichlet problem for minimal surface equation in Ω_n such that $u_n = \varphi_n$ in $\partial\Omega_n$. By construction, for any $n, m \in \mathbb{N}$ with $m > n$, the Maximum Principle implies that $\omega_n^+ \geqslant u_m \geqslant \omega_n^-$ in Ω_n. From this point on we can use the same argument as in Case 1 to conclude the proof. □

Remark 3.34. Notice that, without assuming that the μ-metric of M restricted to W (resp. S) is asymptotically flat, the sequence of μ-geodesic triangles T_n (resp μ-quadrilaterals Q_n) satisfying the hypotheses of Theorem 3.5 may not cover the μ-wedge (resp. the μ-strip).

Remark 3.35. As in [NeSaET017, Remark 4.4 (C)], if we assume F_0 to be minimal, when the boundary value φ is bounded above (respectively below) by a constant M, then the solution given by our proof is also bounded above (respectively below) by the same constant M. Furthermore, if φ is bounded both above and below, a global barrier is given by a vertical translation of F_0 and the solution that we find is bounded.

In general we can not say much about the uniqueness of solutions in μ-wedges and μ-strips. In the forthcoming section, we will establish a Maximum Principle at infinity (see Theorems 4.1 and 4.6), which represents the initial step towards demonstrating the uniqueness of the solution of the Dirichlet problem over unbounded domains.

3.5.2 *New minimal surfaces in the Euclidean space*

Let ℓ be the z-axis in $\mathbb{R}^3$, so that $\mathbb{R}^3 \setminus \ell$ can be seen as a Killing submersion with the Killing vector field $\xi = y\partial_x - x\partial_y$ generated by rotations about ℓ. The affine planes of $\mathbb{R}^3$ containing ℓ are everywhere orthogonal to ξ, so the horizontal distribution associated to this Killing submersion is integrable. The metric in the orbit space $M = \{(x,z) \in \mathbb{R}^2 : x > 0\}$ that makes the projection $\pi : \mathbb{R}^3 \setminus \ell \to M$ Riemannian is the Euclidean one, and we also infer that $\tau(x,z) = 0$ and $\mu(x,z) = x$ on M. The Killing submersion is completely determined in this way by also taking into account that $\mathbb{R}^3 - \ell$ is not simply connected: it is the quotient by a vertical translation of the simply connected space $\mathbb{E}(M,\tau,\mu)$ that fibers over M with bundle curvature τ and Killing length μ. Recall that *vertical* in $\mathbb{E}(M,\tau,\mu)$ is not the same as vertical in $\mathbb{R}^3$. Consequently, if $\Omega \subset M$ is an admissible Jenkins–Serrin domain, an eventual solution Σ of the Jenkins–Serrin problem in $\mathbb{E}(M,\mu,\tau)$ that diverges in $\alpha \subset \partial\Omega$ will be embedded around α (since it is a Killing graph), but not properly embedded since it accumulates at $\pi^{-1}(\alpha)$. Also, the vertices of Ω always give rise to self-intersections of the boundary of the graph.

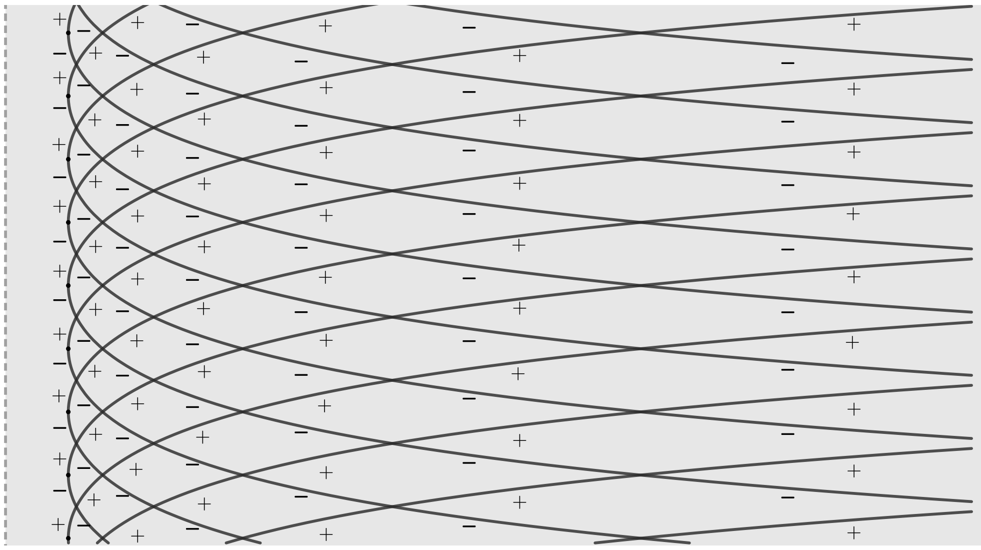

Figure 14: Tessellation of a vertical halfplane by catenaries that produce catenoids of $\mathbb{R}^3$ by rotation about the axis (in dashed line).

Rotational minimal surfaces in $\mathbb{R}^3$ are catenoids and planes, from where we infer that μ-geodesics in M are catenaries (with respect to the z-axis) and straight lines (orthogonal to the z-axis). This gives the following μ-geodesics depending on two parameters $a, b \in \mathbb{R}$:

- $\alpha_a(t) = (t, a)$, which is defined for $t > 0$ and hence noncomplete;
- $\beta_{a,b}(t) = (a\cosh(t/a), t + b)$ with $a > 0$.

For fixed $b \in \mathbb{R}$ and $a, c > 0$, the curves $\{\beta_{a,b+kc}\}_{k\in\mathbb{Z}}$ produce a tiling of M as shown in Figure 14. Each μ-geodesic $\beta_{a,b+kc}(t)$ is marked with the values $+\infty$ if $t > 0$ and $-\infty$ if $t < 0$. This produces a Jenkins–Serrin problem in each tile, which satisfies the JS-conditions since each tile is a μ-quadrilateral symmetric with respect to the horizontal line passing through two of its vertices. The solution viewed in $\mathbb{R}^3 \smallsetminus \ell$ is an embedded graph in the rotational direction that accumulates on closed subsets of four catenoids and whose boundary consists of four circumferences.

It is natural to ask if there exist values of a, b, c such that the solutions on two tiles that are opposite by a vertex continue analytically each other. This is not trivial since there is no Schwarz reflection across circumferences of $\mathbb{R}^3$.

We also remark that constructions in the same spirit can be also done with respect to screw motions in $\mathbb{R}^3$ by taking advantage of the symmetric configuration of the μ-geodesics. The same applies to the rest of $\mathbb{E}(\kappa,\tau)$-spaces.

3.5.3 *Scherk-like minimal surfaces in Heisenberg group and Berger spheres*

Consider the unit square $\Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$ and assign values $\pm\infty$ to opposite sides of Ω, as in the classical Scherk graph of $\mathbb{R}^3$. This gives rise trivially to a minimal graph in Nil_3 over Ω with these assigned values. The resulting surface Σ_0 has boundary four vertical lines projecting to the vertices of Ω, so it can be extended to a complete minimal surface by successive axial symmetries about its boundary components.

The minimal set of such axial symmetries that needed to go back to the tile Ω consists of the symmetries about $(0,0)$, $(0,-1)$, $(1,-1)$ and $(1,0)$, as shown in Figure 15. The axial symmetry of Nil_3 about the vertical axis $\{x = x_0, y = y_0\}$ reads

$$R_{(x_0,y_0)}(x,y,z) = (2x_0 - x, 2y_0 - y, z + 2\tau(y_0 x - x_0 y)),$$

and it follows that $R_{(1,0)} \circ R_{(1,-1)} \circ R_{(0,-1)} \circ R_{(0,0)}$ is the vertical translation $(x,y,z) \mapsto (x,y,z+4\tau)$. This means that on each shaded tile of the infinite chessboard, one finds infinitely many copies of Σ_0 evenly distributed at vertical distance 4τ from the neighboring ones. Therefore, a Scherk-like surface in Nil_3 is neither proper nor embedded.

Similar constructions can be done in other $\mathbb{E}(\kappa,\tau)$-spaces by taking tessellations of $\mathbb{H}^2(\kappa)$ by regular $2m$-gons such that $2k$ of them meet at each vertex (such a tessellation exists if and only if $\frac{1}{m}+\frac{1}{k}\leqslant 1$). The above construction in Nil_3 can be mimicked to get Scherk surfaces in $\mathbb{H}^2(\kappa)\times\mathbb{R}$ or $\widetilde{\mathrm{SL}}_2(\mathbb{R})$; in the latter case, we will find the same holonomy problem as in Nil_3, so the resulting complete surface is invariant by a vertical translation and it is neither embedded nor proper.

There is a special case which is worth mentioning, namely considering a *beach ball* tessellation of $\mathbb{S}^2(\kappa)$ consisting of $2m$ sectors (or 2-gons) whose sides are split in two arcs by adding the midpoints. Each sector becomes a quadrilateral in this way in which we can solve a Jenkins–Serrin problem (in $\mathbb{S}^2(\kappa)\times\mathbb{R}$ or in a Berger sphere) by prescribing alternating boundary values $\pm\infty$, see

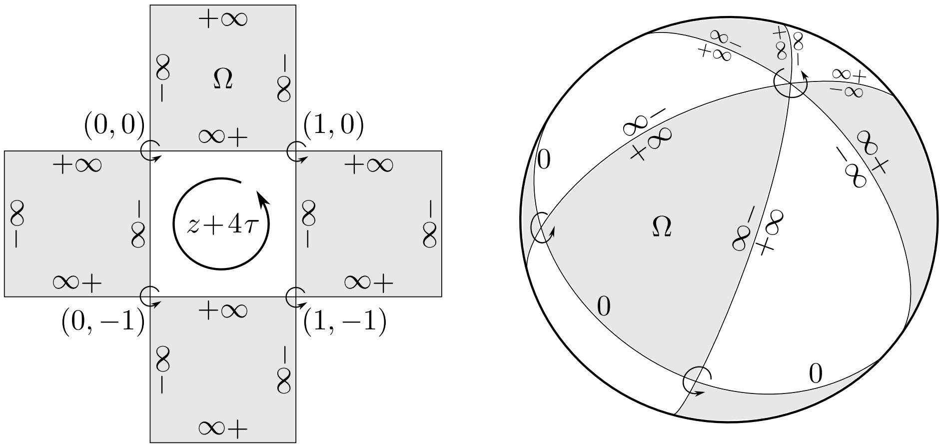

Figure 15: Fundamental domains of a Scherk-like surface in Nil_3 and the effect of the holonomy (left). Beach ball tessellation of $\mathbb{S}^2$ that leads to another complete surface in $\mathbb{S}^2 \times \mathbb{R}$ or in Berger spheres. The values 0 actually mean *horizontal geodesic*.

Figure 15 (right). The solution Σ_0 exists if $m \geqslant 2$ and has an additional axial symmetry that has been marked as zero in the Figure (this equator spans a horizontal geodesic $\Gamma \subset \Sigma_0$).

- If $\tau = 0$, then Σ_0 is completed by successive axial symmetries about the vertical geodesics projecting to Γ, since this process also provides an extension of Σ_0 beyond the geodesics projecting to the poles. All in all, we obtain a complete surface that consists of $4m$ copies of Σ, each of them projecting to one of the triangles (shaded or not) in Figure 15. This surface is properly immersed in $\mathbb{S}^2(\kappa) \times \mathbb{R}$ with $2m$ annular ends asymptotic to vertical planes, since it takes $+\infty$ (resp. $-\infty$) values along m great circles.

- If $\tau \neq 0$, then the same ideas still apply, though we need $8m$ copies of Σ_0. The holonomy makes the horizontal geodesic Γ project two-to-one to a great circle of $\mathbb{S}^2(\kappa)$, so that the complete surface projects two-to-one onto the interior of each of the triangles in Figure 15 and we have $4m$ annular ends.

There are other possible configurations that lead to interesting minimal surfaces consisting of finitely many isometric copies of a solution to a Jenkins–Serrin problem. It is likely that all these surfaces have finite total curvature in $\mathbb{S}^2(\kappa) \times \mathbb{R}$ or in a Berger sphere but this is an open question.

3.6 Some topological observations

In the above examples, we have seen that the condition that fibers have infinite length is not actually necessary for practical purposes, since one can work in the universal cover and then pass to the quotient. There are other three scenarios that is worth mentioning.

First, it is not necessary that the domain $\Omega \subset M$ is embedded. Assume that $\Omega' \subset M'$ is a relatively compact domain on some simply connected Riemannian surface M' and let $\psi : M' \to M$ be an isometric immersion. Then we can consider the Killing submersion $\pi' : \mathbb{E}' \to M'$ with bundle curvature and Killing length the pullback of τ and μ by φ, respectively. Since ψ lifts to an isometric immersion $\Psi : \mathbb{E}' \to \mathbb{E}$ such that $\pi \circ \Psi = \psi \circ \pi'$, the solution of a Jenkins–Serrin problem over Ω' can be mapped by Ψ to a solution of a Jenkins–Serrin problem over the (possibly not embedded) domain $\psi(\Omega) \subset M$.

Second, an extremal case in our Jenkins–Serrin problem is the construction of minimal annuli over annular domains bounded by two closed geodesics in M. For instance, in Figure 16, we have a rotational unduloid M, where we assume that $\mu \equiv 1$ and τ is arbitrary. Also, A_1, A_2, B_1, B_2, B_3 are closed embedded μ-geodesics corresponding to maximal or minimal radii. In the following problems, we prescribe $+\infty$ (resp. $-\infty$) values in the components A_i (resp. B_i) when they lie in the boundary of the domains under consideration.

- In the domain bounded by A_1 and B_1, the Jenkins–Serrin problem has solution, because any possible closed simple μ-geodesic has μ-length larger than the (common) μ-length of A_1 and B_1 (they minimize lengths in their isotopy class).
- In the domain bounded by A_2 and B_2, there is no solution because the inscribed polygon $A_2 \cup B_1$ does not satisfy the JS-conditions.
- In the domain bounded by A_1 and B_3 there is no solution either, because the inscribed polygon $A_1 \cup B_1$ does not satisfy the JS-conditions.

Third and last, we would like to point out some issue related to the condition which is assumed in $M \times \mathbb{R}$ in order to adapt the original ideas by Jenkins and Serrin (see the Fourth case in the proof of [MaRoRo11, Theorem 3.3]), that is:

(C) If no continuous finite values are assigned, then the subsets of $\partial\Omega$ where $+\infty$ and $-\infty$ are assigned are both disconnected.

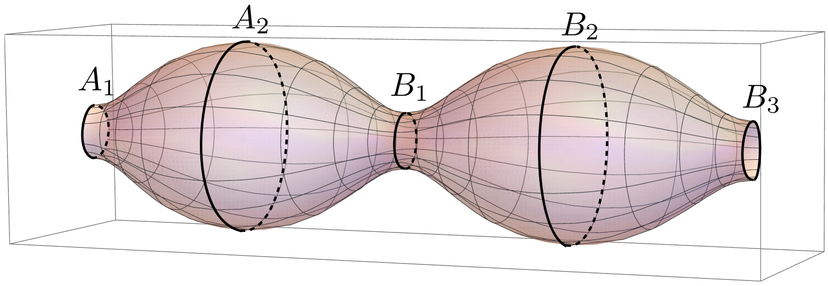

Figure 16: Unduloid-like domains for Jenkins–Serrin problems.

In the case $\cup C_i = \emptyset$, take the sequence ν_n with values 0 at the A_i and n at the B_i, and define the sets $E_c = \{p \in \Omega : \nu_n(p) > c\}$ and $F_c = \{p \in \Omega : \nu_n(p) < c\}$, which are disconnected when c or $n - c$ are close enough to zero by condition (C). The classical approach defines μ_n as the infimum of $c \in (0, n)$ such that F_c is connected and claims that E_{μ_n} and F_{μ_n} are both disconnected.

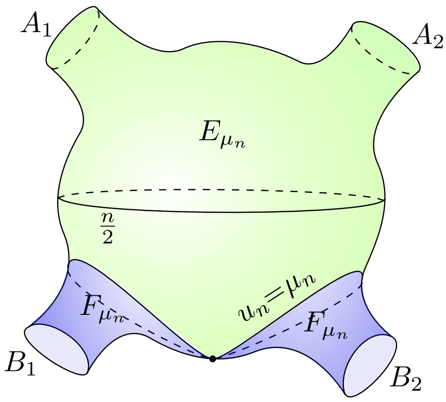

Figure 17: The set E_{μ_n} (green) is connected.

To see that this is not true in general, consider a sphere in which we add four necks with boundary geodesics A_1, A_2, B_1, B_2 disposed symmetrically, as shown in Figure 17. By uniqueness, the solution of the Jenkins–Serrin problem given by Theorem 3.5 has a symmetry with respect to a horizontal geodesic. Note that a similar example can be produced by removing four small polygons (with reentrant corners) in the round sphere $\mathbb{S}^2$.

The aforesaid symmetry implies that ν_n has value $\frac{n}{2}$ along the symmetry curve (as shown in the figure), so it takes values larger (resp. smaller) than $\frac{n}{2}$ on the upper (resp. lower) half of the surface. At the first instant that the purple set F_c gets disconnected, the green set E_c is still connected.

With the divergence lines approach we avoid this problem removing the hypotheses (C).

4

GENERALIZED COLLIN–KRUST ESTIMATES

In this chapter we deal with the uniqueness of solutions to the Dirichlet problem for minimal surfaces equation over unbounded domains of M in a Killing submersion $\pi\colon \mathbb{E} \to M$. In particular, we prove a Maximum Principle at infinity known as Collin–Krust Theorem. The original result of Collin and Krust (see [CoKu91]) estimates the growth of the difference between two minimal graphs having the same boundary values and it can be stated as follows:

Theorem [Collin–Krust, 1991]. *Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain and let $u, \tilde{u} \in C^2(\Omega)$ be such that $u_{|\partial\Omega} = \tilde{u}_{|\partial\Omega}$ and*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \operatorname{div}\left(\frac{\nabla \tilde{u}}{\sqrt{1+|\nabla \tilde{u}|^2}}\right).$$

Denote $\Lambda(r) = \Omega \cap \left\{(x,y) \in \mathbb{R}^2 : x^2+y^2 = r\right\}$ and $M(r) = \sup_{\Lambda(r)} |u-\tilde{u}|$. Hence

$$\liminf_{r\to\infty} \frac{M(r)}{\ln r} > 0.$$

Furthermore, if the length of $\Lambda(r)$ is uniformly bounded, then $\liminf_{r\to\infty} \frac{M(r)}{r} > 0$.

This result has been extended to unitary Killing submersions by C. Leandro and H. Rosenberg in [LeaRos09, Theorem 5.1], and improved in the specific case of minimal graphs in the three-dimensional Heisenberg group by J. M. Manzano and B. Nelli in [MaNe17, Theorem 7]. In all these results, the expansion of the domain is either uniformly bounded or linear, that is, there exists a positive constant C such that either

$$\limsup_{r\to\infty} \operatorname{Length}(\Lambda(r)) \leqslant C \quad \text{or} \quad \limsup_{r\to\infty} \frac{\operatorname{Length}(\Lambda(r))}{r} \leqslant C.$$

In what follows, we extend this result to the Riemannian Killing submersions, providing a detailed description of the relationship between the growth of the vertical distance between two graphs with the same prescribed mean curvature and boundary values, and the expansion of the domain where they are defined, without making any assumptions about the domain. We will see that in this general setting both the Killing length μ (see Theorem 4.1) and the bundle curvature τ and the mean curvature H (see Theorem 4.6) will have an important role.

Let $\Omega \subset M$ be an unbounded domain and assume that there exists a point $p \in M$ such that $\Omega \cap \mathrm{Cut}(p) = \emptyset$. This assumption assures that the distance function from p, $\mathrm{dist}_M(p, \cdot)$, is differentiable in $\Omega \setminus \{p\}$ and this will allow the use of the co-area formula. We denote by $B_p(r)$ the geodesic ball in M centered at p of radius r and for any $r > r_0$ such that $\Omega(r) = B_p(r) \cap \Omega \neq \emptyset$ we call $\Lambda(r) = \partial B_p(r) \cap \Omega$. The first result we prove provides an estimate of the growth of the difference of the disjoint Killing graphs of the functions u and v defined over an unbounded domain $\Omega \subset M$, having the same prescribed mean curvature and boundary values. The theorem introduces three key functions: $M(r)$, $L(r)$, and $g(r)$;

- $M(r)$ measures the maximum of the difference between u and v over the region $\Lambda(r)$.
- $L(r)$ is defined as $\int_{\Lambda(r)} \mu^2$, where μ is the Killing length of the Killing submersion. This integral reflects the expansion rate of the domain Ω with density μ.
- $g(r)$ is defined as $\int_{r_0}^{r} \frac{ds}{L(s)}$ and measures the growth rate of $M(r)$.

Theorem 4.1 states that if the function $g(r)$ tends to infinity as r approaches infinity, the maximum difference between u and v over $\Lambda(r)$ grows at a rate that is at least comparable to the growth rate of $g(r)$.

In Theorem 4.6, we employ the idea presented in [MaNe17, Theorem 7] to improve the estimate of Theorem 4.1 when one of the two surfaces is known. Specifically, we consider one of the graphs as a fixed zero section of the Killing submersion. By doing so, we establish that the vertical growth of any Killing graph with zero boundary values and the same prescribed mean curvature H_0 as that of the zero section depends on the function $L(r) = \int_{\Lambda(r)} \frac{2\mu^2}{\sqrt{1+\mu^2(a^2+b^2)}}$. Here, the smooth functions a and b defined in the domain Ω carry information regarding the bundle curvature τ, as expressed in Equation (1.5), and the mean curvature of the zero section, as expressed in Equation (1.33).

Theorem 4.1 . *Let $\Omega \subset M$ be an unbounded domain and assume that $p \in M$ is such that $\Omega \cap \mathrm{Cut}(p) = \emptyset$. Assume also that $u, v \in \mathcal{C}^\infty(\Omega)$ satisfy $\mathcal{Q}(u) = \mathcal{Q}(v)$, $u > v$ in Ω and $u = v$ in $\partial\Omega$. Let*

$$M(r) = \sup_{\Lambda(r)} |u - v|, \qquad L(r) = \int_{\Lambda(r)} \mu^2 \quad \textit{and} \quad g(r) = \int_{r_0}^{r} \frac{ds}{L(s)}$$

for some $r_0 > 0$. Then,

$$\liminf_{r\to\infty} \frac{M(r)}{g(r)} > 0.$$

Proof. Denote by $\rho(r) = \int_{\Omega(r)} \left| \frac{\mu G u}{W_u} - \frac{\mu G v}{W_v} \right|^2$. The fact that $u - v > 0$ in Ω implies that there exists $r_0 > 0$ such that $\rho(r_0) > 0$. Let us define $\eta(r) = \int_{\Lambda(r)} \mu \left| \frac{\mu G u}{W_u} - \frac{\mu G v}{W_v} \right|$, for all $r \geqslant r_0$. Using Lemma 2.2, the divergence theorem, and the fact that $|N_u - N_v| \geqslant \left| \frac{\mu G u}{W_u} - \frac{\mu G v}{W_v} \right|$, we can estimate for all $r \geqslant r_0$

$$\begin{aligned}
M(r)\eta(r) &\geqslant \int_{\Lambda(r)} (u-v)\mu \left| \frac{\mu G u}{W_u} - \frac{\mu G v}{W_v} \right| = \int_{\partial\Omega(r)} (u-v)\mu \left| \frac{\mu G u}{W_u} - \frac{\mu G v}{W_v} \right| \\
&\geqslant \int_{\partial\Omega(r)} (u-v) \left\langle \frac{\mu^2 G u}{W_u} - \frac{\mu^2 G v}{W_v}, \chi \right\rangle \\
&= \int_{\Omega(r)} \operatorname{div} \left((u-v) \left(\frac{\mu^2 G u}{W_u} - \frac{\mu^2 G v}{W_v} \right) \right) \\
&= \int_{\Omega(r)} \left\langle \nabla u - \nabla v, \frac{\mu^2 G u}{W_u} - \frac{\mu^2 G v}{W_v} \right\rangle \\
&= \int_{\Omega(r)} \frac{W_u + W_v}{2} |N_u - N_v|^2 \\
&= \rho(r_0) + \int_{\Omega(r)\setminus\Omega(r_0)} \frac{W_u + W_v}{2} |N_u - N_v|^2 \\
&\overset{(1)}{\geqslant} \rho(r_0) + \int_{r_0}^{r} \left(\int_{\Lambda(s)} \left| \frac{\mu G u}{W_u} - \frac{\mu G v}{W_v} \right|^2 \right) ds \\
&\overset{(2)}{\geqslant} \rho(r_0) + \int_{r_0}^{r} \frac{\eta^2(s)}{L(s)} ds,
\end{aligned} \tag{4.1}$$

where χ denotes a unit co-normal vector field to $\Omega(r)$ along its boundary. In (4.1), inequality (1) is a consequence of the co-area formula with respect to the Riemannian distance and (2) follows from the Cauchy-Schwarz inequality. Since $g(r) = \int_{r_0}^{r} \frac{ds}{L(s)}$, we get that

$$M(r)\eta(r) \geqslant \rho(r_0) + \int_{r_0}^{r} g'(s)\eta^2(s) ds \tag{4.2}$$

for all $r \geqslant r_0$.

The Maximum Principle implies that the function $r \mapsto M(r)$ does not decrease. Given $r_1 > r_0$, let us write $a = M(r_1)$, so $a\eta(r) \geqslant M(r)\eta(r)$ for all $r_0 < r < r_1$. Hence, η satisfies the integral inequality

$$\eta(r) \geqslant \frac{\rho(r_0)}{a} + \frac{1}{a} \int_{r_0}^{r} g'(s)\eta^2(s) ds.$$

Let us define the function $\zeta\colon [r_0, R) \to \mathbb{R}$ as

$$\zeta(r) = \frac{a\rho(r_0)}{2a^2 - \rho(r_0)[g(r) - g(r_0)]}, \tag{4.3}$$

where $R > r_1$ is defined as $R = g^{-1}\left(\frac{2a^2}{\rho(r_0)} + g(r_0)\right)$ if $\left(\frac{2a^2}{\rho(r_0)} + g(r_0)\right) \in \operatorname{Im}(g)$ and $R = +\infty$ otherwise. Observe that

$$\zeta'(r) = \frac{a\rho(r_0)g'(r)}{(2a^2 - \rho(r_0)(g(r) - g(r_0)))^2} = \tfrac{1}{a}\zeta(r)^2 g'(r),$$

whence

$$\zeta(r) = \frac{\rho(r_0)}{2a} + \frac{1}{a}\int_{r_0}^{r} \zeta(s)^2 g'(s)\,ds.$$

Thus, a simple comparison yields $\eta \geqslant \zeta$ for all $r_0 \leqslant r \leqslant r_2 \leqslant R$, so that

$$\begin{aligned} r_1 \leqslant r_2 &\iff g(r_1) \leqslant g(r_2) \leqslant g(r_0) + \tfrac{2a^2}{\rho(r_0)} \\ &\iff 2a^2 \geqslant \rho(r_0)\,(g(r_1) - g(r_0)) \\ &\iff a \geqslant \sqrt{\tfrac{\rho(r_0)}{2}\,[g(r_1) - g(r_0)]}, \text{ for all } r_1 > r_0. \end{aligned} \tag{4.4}$$

We claim that the function η is bounded away from zero at infinity. Note that, for $r > r_0$,

$$\begin{aligned} \eta(r) &\geqslant \left| \int_{\Lambda(r)} \left\langle \tfrac{\mu^2 Gu}{W_u} - \tfrac{\mu^2 Gv}{W_v}, \chi \right\rangle \right| \\ &\geqslant \left| \int_{\partial\Omega(r)} \left\langle \tfrac{\mu^2 Gu}{W_u} - \tfrac{\mu^2 Gv}{W_v}, \chi \right\rangle - \int_{\partial\Omega(r)\setminus\Lambda(r)} \left\langle \tfrac{\mu^2 Gu}{W_u} - \tfrac{\mu^2 Gv}{W_v}, \chi \right\rangle \right|. \end{aligned} \tag{4.5}$$

The first integral of the right hand side of (4.5) vanishes by Stokes Theorem. So the result follows by proving that $\int_\Gamma \left\langle \frac{\mu^2 Gu}{W_u} - \frac{\mu^2 Gv}{W_v}, \chi \right\rangle$ has constant sign on any arc Γ contained on $\partial\Omega$, as in [MaNe17]. Notice that $Gu - Gv = \nabla u - \nabla v \neq 0$ along $\partial\Omega$, except at isolated points, because $u - v \geqslant 0$ in Ω by assumption. In particular, $Gu - Gv$ is oriented toward Ω, where it is not zero. Hence, $Gu - Gv$ can be used to orient $\partial\Omega$. Then, if $\mu^2 \left\langle \frac{Gu}{W_u} - \frac{Gv}{W_v}, Gu - Gv \right\rangle$ has constant sign along $\partial\Omega$, the same holds for $\left\langle \mu^2 \frac{Gu}{W_u} - \mu^2 \frac{Gv}{W_v}, \chi \right\rangle$. By Lemma 2.2,

$$\mu^2 \left\langle \frac{Gu}{W_u} - \frac{Gv}{W_v}, Gu - Gv \right\rangle = \frac{1}{2}(W_u + W_v)\,|N_u - N_v|^2$$

is positive at any point where $Gu - Gv$ is not zero. Then there exists a constant n such that $\eta(r) \geqslant \int_\Gamma \mu^2 \left\langle \frac{Gu}{W_u} - \frac{Gv}{W_v}, \chi \right\rangle \geqslant n > 0$, which proves the claim.

For any $r_2 > r_0$, we deduce that

$$\begin{aligned} \rho(r_2) &= \int_{\Omega(r_2)} \left| \tfrac{\mu Gu}{W_u} - \tfrac{\mu Gv}{W_v} \right|^2 \geqslant \int_{r_0}^{r_2} \left(\int_{\Lambda(s)} \left| \tfrac{\mu Gu}{W_u} - \tfrac{\mu Gv}{W_v} \right|^2 \right) ds \\ &\geqslant \int_{r_0}^{r_2} \tfrac{\eta^2(s)}{L(s)}\,ds \geqslant n^2 \int_{r_0}^{r_2} g'(s)\,ds \geqslant n^2\,[g(r_2) - g(r_0)]\,. \end{aligned} \tag{4.6}$$

Observe that $g(r_0) \leqslant \frac{g(r_1)-g(r_0)}{2} \leqslant g(r_1)$, so there is $r_2 \in [r_0, r_1]$ such that $g(r_2) = \frac{g(r_1)-g(r_0)}{2}$. Applying (4.4) to r_2 instead of r_0 we get

$$\begin{aligned} M(r_1) &\geqslant \sqrt{\tfrac{\rho(r_2)}{2}\left[g(r_1)-g(r_2)\right]} \geqslant \tfrac{n}{\sqrt{2}}\sqrt{\left[g(r_1)-g(r_2)\right]\left[g(r_2)-g(r_0)\right]} \\ &= \tfrac{n}{2\sqrt{2}}\left[g(r_1)-g(r_0)\right] \end{aligned} \tag{4.7}$$

for all $r_1 > r_0$. Finally, this means that

$$\liminf_{r\to\infty} \frac{M(r)}{g(r)} \geqslant \liminf_{r\to\infty} \left(\tfrac{n}{2\sqrt{2}}\left(1-\tfrac{g(r_0)}{g(r)}\right)\right) > 0$$

and this concludes the proof. □

Remark 4.2. Up to lose some information, we can take

$$g(r) = \int_{r_0}^{r} \frac{ds}{T(s)^2\,\mathrm{Length}(\Lambda(s))},$$

where $T(r) = \sup_{\Lambda(r)} \mu$, to simplify the computation. Hence, if there exists a constant $C > 0$ such that $\mu_{|\Omega} \leqslant C$, then the growth function $g(r)$ will depend only on how the domain expands, that is, $g'(r) \geqslant \frac{1}{C^2\,\mathrm{Length}(\Lambda(r))}$.

In the next example, when μ is bounded, we find a sharper bound on the growth of a domain Ω which guarantees a divergent Collin-Krust type estimate. Such domains exist, for instance, in $\mathbb{H}^2$ (see Figure 18).

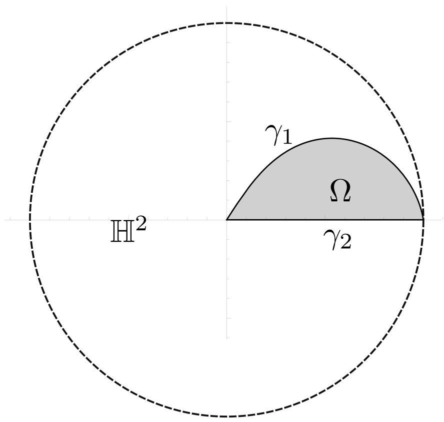

Figure 18: A domain in $\mathbb{H}^2$ whose expansion is equal to $(r+1)\log(r+1)$.

Using the Poincaré's disk model, that is,

$$\mathbb{H}^2 = \left(\left\{(x,y) \in \mathbb{R}^2 | x^2+y^2 < 1\right\}, \frac{4}{(1-x^2-y^2)^2}(dx^2+dy^2)\right),$$

we can consider the convex domain Ω bounded by bounded by $\gamma_1 \cup \gamma_2$, where $\gamma_1, \gamma_2 \colon [0, +\infty) \to \mathbb{H}^2$ are such that

$$\gamma_1(t) = \left\{\tanh\left(\tfrac{t}{2}\right)\cos\left(\tfrac{(t+1)\log(t+1)}{\sinh(t)}\right), \tanh\left(\tfrac{t}{2}\right)\sin\left(\tfrac{(t+1)\log(t+1)}{\sinh(t)}\right)\right\},$$
$$\gamma_2(t) = \left\{\tanh\left(\tfrac{t}{2}\right), 0\right\},$$

which has expansion rate function $L(r) = (r+1)\log(r+1)$.

Example 4.3. It is not difficult to prove that $g(x)$ does not diverge whenever $f(x) = \frac{1}{g'(x)} \geqslant cx(\log x)^{b+1}$ for some $b, c > 0$, since

$$\int \frac{1}{cx(\log x)^{b+1}}\,dx = -\frac{1}{bc\,(\log x)^b} + C,$$

which is bounded above. Nevertheless, we can build a sequence of monotone functions $\{f_n(x)\}_n$ such that, for all $n \geqslant 0$,

$$\lim_{x \to +\infty} \frac{f_{n+1}(x)}{f_n(x)} = +\infty,$$

and $\int \frac{dx}{f_n(x)}$ diverges for $x \to +\infty$, with $f_0(x) = x$.

Define a sequence of function as follows:

$$\begin{cases} a_0(x) = x; \\ a_i(x) = \log(a_{i-1}(x)) \qquad \text{for } i \geqslant 1. \end{cases}$$

Now we define $F_n(x) = \prod_{i=0}^{n} a_i(x)$ and a sequence of *translation terms*

$$\begin{cases} x_t^0 = 0, \\ x_t^i = e^{x_t^{i-1}}, \qquad \text{for all } i \geqslant 1. \end{cases}$$

Finally, we can define

$$f_n(x) = F_n(x + x_t^n). \tag{4.8}$$

Hence, we get that

$$g_n(x - x_t^{n+1}) = \int \frac{1}{f_n(x - x_t^{n+1})}\,dx = \begin{cases} \tilde{a}_0(x) = \int \frac{1}{a_0(x)}\,dx & \text{for } n = 0, \\ \tilde{a}_n(x) = \log(\tilde{a}_{n-1}) & \text{for } n \geqslant 1, \end{cases}$$

which diverges. Notice that the faster the domain expands, the smaller the growth function will be.

We now present an example of two graphs with identical boundary values and mean curvature, while ensuring their vertical distance remains bounded. In particular, given the function describing the expansion of Ω, we are going to find for which choice of $\mu(r)$ the grow rate function $g(r)$, such that $g'(r) = \frac{1}{L(r)}$, does not diverge for $r \to +\infty$ and then build a non-divergent minimal graph.

Example 4.4. Let $\pi\colon \mathbb{E} \to M$ be a Killing submersion such that M is the euclidean plane, $\tau \equiv 0$ and μ is a smooth radial function. We can consider in $\mathbb{R}^2$ the polar coordinates,

$$(\mathbb{R}^2, g_{euc} = dx^2 + dy^2) \equiv \left(\mathbb{R}^+ \times [0, 2\pi), g_{pol} = dr^2 + r^2 d\theta\right).$$

We are looking for non-negative solutions u defined in the domain $\Omega = \{(r,\theta) \in \mathbb{R} \times [0, 2\pi] \mid r \geqslant 1\}$ such that $u(1,\theta) = 0$. It is not difficult to show that $u(r,\theta) = u(r)$ is a radial solution of the minimal surface equation in Ω if and only if $f(r) = \frac{\partial u}{\partial_r}(r)$ satisfies the following ODE:

$$\frac{\partial}{\partial r}\left(\frac{r\mu^2(r)f(r)}{\sqrt{1+\mu^2(r)f^2(r)}}\right) = 0. \tag{4.9}$$

If $\mu(r,\theta) = r$, then the ODE becomes

$$\frac{\partial}{\partial r}\left(\frac{r^3 f(r)}{\sqrt{1+r^2(r)f^2(r)}}\right) = 0$$

Now, the solution is easy to compute: for all $c > 1$,

$$f(r) = \frac{1}{\sqrt{cr^6 - r^2}}.$$

Hence, $u\colon \Omega \to \mathbb{R}$ is given by

$$u(r) = \int_1^r \frac{ds}{\sqrt{cs^6 - s^2}} = \frac{1}{2}\arctan\left(\sqrt{cr^4 - 1}\right) - \frac{1}{2}\arctan\left(\sqrt{c-1}\right).$$

First, notice that $u(r)$ is defined also for $c = 1$ (this is the solution whose tangent at the boundary is vertical). Notice also that for $c \to +\infty$, the solutions $u(r)$ converge to $u(r) \equiv 0$. Finally, fixed $c \geqslant 1$,

$$0 \leqslant \sup_{r>1} u(r) = \frac{\pi}{4} - \frac{1}{2}\arctan\left(\sqrt{c-1}\right) \leqslant \frac{\pi}{4}.$$

If $\mu(r,\theta) = \mu(r)$, the solution of (4.9) is the one-parameter family

$$u_c(r) = \pm \frac{1}{\sqrt{cr^2\mu(r)^4 - \mu(r)^2}} = \pm \frac{1}{r\mu(r)^2\sqrt{c - \frac{1}{r^2\mu(r)^2}}}, \tag{4.10}$$

depending on c. The comparison theorem for ODEs implies that, whenever $\mu(r)$ grows faster than $\log(r)^{\frac{a}{2}}$ with $a > 1$, then

$$\lim_{r\to+\infty} \int_{r_0}^{r} u_c(s)ds < +\infty.$$

Hence, we can find two distinct minimal Killing graphs with the same boundary and bounded distance.

Taking $\mu(r)$ such that $\mu(r)^2_{|r\geqslant 2} = \log(r)$ and using again the Comparison Theorem for ODEs, it is easy to see that the integral solution $u(r) = \int_{r_0}^{r} f_c(s)ds$ grows faster than $k\log(\log(r))$, for some constant $k > 0$, which diverges. The same argument applies by taking $\mu(r)$ such that $r\mu^2(r) = f_n(r)$ for some $n > 0$ where $\{f_n(x)\}_n$ is the sequence of function defined in (4.8). In particular, since $\mathrm{Length}(\Lambda(r)) = 2\pi r$, we obtain that the space can admit two Killing graphs with the same mean curvature, the same boundary values and bounded difference, only if $\mu(r)$ grows faster than $\log(r)^{\frac{b}{2}}$ with $b > 1$.

We can apply Theorem 4.1 to study when and how the difference between two Killing graphs in Sol_3 with the same mean curvature and the same boundary values diverges. We want to show that, unlike in $\mathbb{H}^2\times\mathbb{R}$, in Sol_3 there are some wedges where it makes sense to calculate a Collin–Krust type estimate.

Example 4.5. The homogeneous manifold Sol_3 is isometric to the warped product

$$\left(\left\{(x,y,z)\in\mathbb{R}^3 \mid y>0\right\}, \frac{dx^2+dy^2}{y^2} + y^2dz^2\right),$$

(see [Ngu14]). In this setting, a direct computation implies that the function $u(x,y) = 1 - 1/y$ defines a positive minimal graph in the unbounded domain of the hyperbolic plane $\{y > 1\}$ that has zero boundary values and bounded height. Hence, in general, we can not expect to have a Collin–Krust type estimate in any domain.

Using the Mobius transformations, it easy to see that Sol_3 is isometric to

$$\left(\mathbb{D}(1)\times\mathbb{R}, ds^2 = \frac{4(dx^2+dy^2)}{(1-(x^2+y^2))^2} + \left(\frac{1-x^2-y^2}{(x-1)^2+y^2}\right)^2 dz^2\right).$$

That is, Sol_3 is a Killing submersion over $\mathbb{H}^2$ (described with the Poincaré Disk Model) with $\tau \equiv 0$ and $\mu(x,y) = \frac{1-x^2-y^2}{(x-1)^2+y^2}$.

Let us define first in which kind of domain we want to compute the estimate. Let $\theta_1, \theta_2 \in (0,\pi)$ and for $t \in [0,1)$ define the geodesics $\gamma_1(t) = (t\cos\theta_1, t\sin\theta_1)$ and $\gamma_2(t) = (t\cos\theta_2, -t\sin\theta_2)$. We call (θ_1,θ_2)*-wedge* the domain in $\mathbb{D}(1)$ bounded by γ_1, γ_2 and such that its asymptotic boundary is $\gamma_3 = (\cos\phi, \sin\phi)$, with $\phi \in (\theta_1, 2\pi - \theta_2)$.

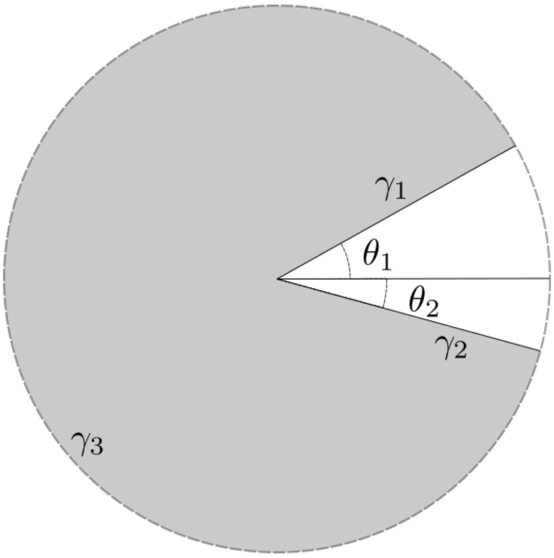

Figure 19: (θ_1,θ_2)-wedge.

Let $\Omega \subset \mathbb{H}^2$ be an unbounded domain contained in a (θ_1,θ_2)-wedge W and $\Lambda(\rho)$ be the boundary of the geodesic ball of geodesic radius ρ centered at the center of the Poincaré's disk contained in Ω. Thus, $\mathrm{Length}(\Lambda(\rho)) \leqslant [2\pi - (\theta_1+\theta_2)]\sinh(\rho)$ and

$$T(\rho) = \sup_{\Lambda(\rho)} \mu = \frac{1-\tanh(\rho)^2}{1+\tanh(\rho)^2 - 2\tanh(\rho)\cos(\theta)},$$

where $\theta = \min\{\theta_1,\theta_2\}$. As explained in Theorem 4.1, $g(\rho) = \int \frac{d\rho}{\int_{\Lambda(r)} \mu^2}$, hence

$$g'(\rho) \leqslant \left(T^2(\rho)\,\mathrm{Length}(\Lambda(\rho))\right)^{-1} = \frac{(1+\tanh(\rho)^2 - 2\tanh(\rho)\cos(\theta))^2}{[2\pi-(\theta_1+\theta_2)]\sinh(\rho)[1-\tanh(\rho)^2]^2}.$$

Integrating this inequality we have that

$$\begin{aligned} g(\rho) \leqslant \ & \frac{1}{2\pi-(\theta_1+\theta_2)}\left[\frac{1}{2}(3+\cos(2\theta)) + \frac{1}{6}(2+\cos(2\theta))\cosh(3\rho)\right. \\ & \left. + \log\left(\tanh\left(\frac{\rho}{2}\right)\right) - 2\cos(\theta)\sinh(\rho) - \frac{2}{3}\cos(\theta)\sinh(3\rho)\right] \end{aligned}$$

which diverges whenever $\theta > 0$.

Notice that, if Ω is an unbounded domain such that, for a (θ_1, θ_2)-wedge W, $\Omega \setminus W \neq \emptyset$ is compact and $u \in C^\infty(\Omega)$ describes a minimal Killing graph with bounded boundary values, then the positive (resp. negative) part of $u - \max_{\Omega \cap W} u$ (resp. $u + \min_{\Omega \cap W} u$) is a minimal Killing graph with zero boundary values over a non-compact domain contained in W and we can apply the previous estimate.

To prove the last Collin–Krust type result, we recall the local coordinates described in Chapter 1. We assume $\mathbb{E}$ to be locally isometric to $(\mathcal{U} \times \mathbb{R}, ds^2)$ where $\mathcal{U} \subseteq \mathbb{R}^2$ and $ds^2 = \lambda^2(dx^2 + dy^2) + \mu^2[dz - \lambda(a dx + b dy)]^2$ for some $\lambda, a, b \in C^\infty(\mathcal{U})$, with $\lambda > 0$, such that

$$\frac{2\tau}{\mu} = \frac{1}{\lambda^2}[(\lambda b)_x - (\lambda a)_y],$$

that is, the functions a and b are uniquely determined by the choice of the zero section F_0. Equation (1.33) shows that the functions a and b also describe the mean curvature H_0 of the section F_0. Furthermore, we can compute that the area element of $\{z = 0\}$ is exactly $\sqrt{1 + \mu^2(a^2 + b^2)}$. So, putting this information in (4.1), we can prove the following theorem.

Theorem 4.6 . *Let $\Omega \subset M$ be an unbounded domain ad assume that $p \in M$ is such that $\Omega \cap \mathrm{Cut}(p) = \emptyset$. Assume also that $u \in \mathcal{C}^\infty(\Omega)$ satisfy $\mathcal{Q}(u) = H_0$, $u > 0$ in Ω and $u = 0$ on $\partial\Omega$. Let*

$$M(r) = \sup_{\Lambda(r)} |u - v|, \quad L(r) = \int_{\Lambda(r)} \frac{2\mu^2}{\sqrt{1 + \mu^2(a^2 + b^2)}} \quad \textit{and} \quad g(r) = \int_{r_0}^{r} \frac{ds}{L(s)},$$

for some $r_0 > 0$. Then,

$$\liminf_{r \to \infty} \frac{M(r)}{g(r)} > 0.$$

Proof. We do a slight modification of the proof of Theorem 4.1, starting from Equation (4.1). Then, recalling that $W_u \geqslant 1$ and $W_0 = \sqrt{1 + \mu^2(a^2 + b^2)} \geqslant 1$ we get

$$\begin{aligned} M(r)\eta(r) &\geqslant \int_{\Omega(r)} \frac{W_u + W_0}{2} |N_u - N_0|^2 \\ &> \rho + \int_{\Omega(r) \setminus \Omega(r_0)} \frac{\sqrt{1 + \mu^2(a^2 + b^2)}}{2} \left| \frac{\mu G u}{W_u} - \frac{\mu Z}{W_0} \right|^2 \\ &\overset{(1)}{\geqslant} \rho + \int_{r_0}^{r} \left(\int_{\Lambda(s)} \frac{\sqrt{1 + \mu^2(a^2 + b^2)}}{2} \left| \frac{\mu G u}{W_u} - \frac{\mu Z}{W_0} \right|^2 \right) ds \\ &\overset{(2)}{\geqslant} \rho + \int_{r_0}^{r} \frac{\eta^2(s)}{L(s)} ds, \end{aligned} \tag{4.11}$$

where we have used the co-area formula in (1) and the Cauchy-Schwarz inequality in (2). From this point the argument is the same as the one in the proof of Theorem 4.1. □

To conclude this section, we show how Theorem 4.6 can be applied to the space $\mathbb{E}(-1,\tau)$, providing a better estimate than the one given in [LeaRos09, Theorem 5.1]. In the case of unbounded domains where $\Lambda(\rho)$ is uniformly bounded, the result of Leandro and Rosenberg [LeaRos09, Theorem 5.1] states that for every choice of τ and H, the distance between two surfaces which have the same mean curvature grows at least as ρ. We show that if $H = 1/2$, the distance between two graphs having the same boundary values grows at least as $e^{\frac{\rho}{2}}$. We also show that, if we consider exterior domains, for any choice of τ and $H \in [0, 1/2]$, the growth function $g(\rho)$ is not divergent.

Example 4.7. Consider for $\mathbb{E}(-1,\tau)$ the global model given by

$$\left(\mathbb{D}(1) \times \mathbb{R}, ds^2 = \lambda^2(dx^2 + dy^2) + [2\tau\lambda(y\,dx - x\,dy) + dz]^2\right),$$

where $\lambda = \frac{2}{1-(x^2+y^2)}$. In this model $a(x,y) = -2\tau y$ and $b(x,y) = 2\tau x$. If we call $r = \sqrt{x^2+y^2}$, the geodesic distance of a point $(x,y) \in \mathbb{D}(1)$ from the center of the disk is given by $\rho = 2\tanh^{-1}(r)$.

In [Pen12], Peñafiel shows that an entire rotationally invariant graph of constant mean curvature $H \in [0, 1/2]$ is

$$(\tanh(\rho/2)\cos\theta, \tanh(\rho/2)\sin\theta, u(\rho)),$$

where $u(\rho)$ satisfies

$$u'(\rho) = \frac{(2H\cosh(\rho) - 2H)\sqrt{1 + 4\tau^2\tanh^2(\rho/2)}}{\sqrt{\sinh^2(\rho) - (2H\cosh(\rho) - 2H)^2}}.$$

Hence,

$$\tfrac{\partial}{\partial r}u(\rho(r)) = u'(\rho(r))\frac{\partial\rho}{\partial r} = \frac{4Hr\sqrt{1+4\tau^2 r^2}}{(1-r^2)\sqrt{1-4H^2r^2}}.$$

Now, in order to have an estimate for the vertical distance between an H-graph Σ_H and the rotational entire H-graph P_H described by Peñafiel, we have to compute

$$\tilde{a}(x,y) = \frac{u_r(\sqrt{x^2+y^2})}{\lambda}\frac{\partial r}{\partial x} + a(x,y), \quad \tilde{b}(x,y) = \frac{u_r(\sqrt{x^2+y^2})}{\lambda}\frac{\partial r}{\partial y} + b(x,y).$$

An easy computation implies

$$\tilde{a}(x,y) = 2Hx\sqrt{\frac{1+4\tau^2(x^2+y^2)}{1-4H^2(x^2+y^2)}} - 2y\tau,$$

$$\tilde{b}(x,y) = 2Hy\sqrt{\frac{1+4\tau^2(x^2+y^2)}{1-4H^2(x^2+y^2)}} + 2x\tau.$$

Thus, defining

$$h(\rho) = (\tilde{a}^2+\tilde{b}^2)(\rho) = \frac{4(H^2+\tau^2)\tanh^2(\frac{\rho}{2})}{1-4H^2\tanh^2(\frac{\rho}{2})}$$

and $L(r) = \int_{\Lambda(r)} \frac{ds}{\sqrt{1+h(s)}} = \frac{2\,\mathrm{Length}(\Lambda(r))}{1+h(\rho)}$, we have $g(\rho) = \int \frac{\sqrt{1+h(\rho)}}{2\,\mathrm{Length}(\Lambda(r))} d\rho$.

Denoting by $\Omega \subset \mathbb{H}^2$ the domain bounded by $\pi(\Sigma_H \cap P_H)$, our result shows that, if $\mathrm{Length}(\Lambda(\rho))$ is uniformly bounded, then $\Sigma_H - P_H$ grows as

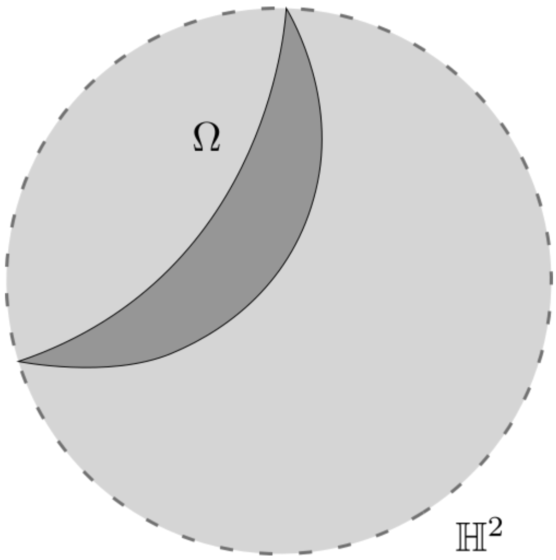

Figure 20: Domain in $\mathbb{H}^2$ such that $\Lambda(\rho)$ is uniformly bounded.

$$g(r) \simeq \begin{cases} \left(\frac{1}{2} + \sqrt{\frac{H^2+\tau^2}{1-4H^2}}\right) r + o(r) & \text{for } 0 \leqslant H < \frac{1}{2}; \\[2em] \frac{\sqrt{1+4\tau^2}}{2} e^{\frac{r}{2}} + o\left(e^{\frac{r}{2}}\right) & \text{for } H = \frac{1}{2}. \end{cases}$$

If $\Omega \subset \mathbb{H}^2$ is an exterior domain, then $\Lambda(r) \simeq 2\pi \sinh(r)$. Hence,

$$g'(r) = \frac{1}{L(r)} \simeq \begin{cases} \sqrt{\frac{1-H^2-\tau^2}{4\pi^2(1-4H^2)}} e^{-r} + o\left(e^{-r}\right) & \text{if } H, \tau \neq 0, \\ \sqrt{\frac{1+4\tau^2}{2\pi}} e^{-\frac{r}{2}} + o\left(e^{-\frac{r}{2}}\right) & \text{if } H = 1/2. \end{cases}$$

that is, $\lim_{r\to\infty} g(r)$ converges for any $0 \leqslant H \leqslant 1/2$ and for any $\tau \in \mathbb{R}$. The existence of bounded graphs over exterior domains, characterized by zero boundary values and a constant mean curvature H with respect to a rotational zero section of constant mean curvature H, has been established in various cases. Citti and Senni [CiSe12] demonstrated this existence for $H \in (0, 1/2)$ in $\mathbb{E}(-1, 0)$. Peñafiel [Pen12], focusing on surfaces invariant by rotation, proved the result for $H < 1/2$ in any $\mathbb{E}(-1, \tau)$. In the work of Elbert, Nelli and Sa Earp [ElNeSaE12], the case of $H = 1/2$ in $\mathbb{E}(-1, 0)$ was proven. However, the existence of a similar result for $H = 1/2$ and $\tau \neq 0$ remains unknown.

In all the examples we have seen, we were always able to find two different solutions whenever it was not possible to compute a generalized Collin-Krust estimate. So it seems natural to ask the following question:

- Assume that $\Omega \subset M$ is an unbounded domain such that for any choice of $p \in M$, with $\Omega \cap \mathrm{Cut}(p) = \emptyset$, the function $g(r) = \int_{r_0}^{r} \frac{ds}{L(s)}$ does not diverge to $+\infty$. Does a positive and bounded solution to the following Dirichlet problem
$$\begin{cases} \mathcal{Q}(u) = \mathcal{Q}(0) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$
exist?

4.1 A uniqueness result in a strip of the Heisenberg group

The three-dimensional Heisenberg group is a particular case of Killing submersion where the base M is $\mathbb{R}^2$ endowed with the Euclidean metric, $\mu \equiv 1$ and τ is constant. The classical model that describes $\mathrm{Nil}_3(\tau)$ is given by $\mathbb{R}^3$ endowed with the metric $ds^2 = dx^2 + dy^2 + [\tau(y dx - x dy) + dz]^2$. In this model the Riemannian submersion reads as $\pi(x, y, z) = (x, y)$ and the Killing vector field is $\xi = \partial_z$.

In the Heisenberg space, Cartier constructed non-zero graphs over a wedge $\Omega \subset \mathbb{R}^2$ (with the vertex at the origin and any angle less the π) with zero values on $\partial\Omega$, which shows that the solution of the Dirichlet problem in Ω is not unique [Car16, Corollary 3.8]. We will prove an uniqueness result for minimal graphs with bounded boundary values over domains contained in a strip. The analogous problem was studied in $\mathbb{R}^3$ by Collin and Krust [CoKu91, Theorem 1] and by Elbert and Rosenberg in the product space $M \times \mathbb{R}$, [ElRo08, Theorem 1.1].

The ambient isometries of this model are generated by the following maps (see [FiMePe99] for more details):

$$\begin{aligned}
\varphi_1^c(x,y,z) &= (x+c, y, z+c\tau y),\\
\varphi_2^c(x,y,z) &= (x, y+c, z-c\tau x),\\
\varphi_3^c(x,y,z) &= (x, y, z+c),\\
\varphi_4^\theta(x,y,z) &= (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z),\\
\varphi_5(x,y,z) &= (x, -y, -z).
\end{aligned}$$

For convenience, we will introduce the following notation. Let $\Omega \subseteq \mathbb{R}^2$ be an open subset and let $S \subset \mathrm{Nil}_3(\tau)$ be the graph of a function $u^S \in \mathcal{C}^2(U)$ where $U \subset \mathbb{R}^2$ is an open subset containing $\overline{\Omega}$. We call $P(S,\Omega)$ the following Dirichlet problem:

$$P(S,\Omega) : \begin{cases} \mathcal{Q}(u) = 0 & \text{in } \Omega, \\ u = u^S & \text{on } \partial\Omega. \end{cases}$$

Let Ω be a domain contained in a strip of $\mathbb{R}^2$. Without loss of generality, applying a rotation φ_4^θ, we can assume $\Omega \subseteq \Omega_a^b = \{(x,y) \in \mathbb{R}^2 \mid a < y < b\}$ for some $a, b \in \mathbb{R}$ such that $a < b$. Let $T \subset \mathrm{Nil}_3(\tau)$ be the entire minimal graph invariant by φ_1 given by $u^T(x,y) = \tau xy$.

Lemma 4.8 . *The only solution to* $P(T,\Omega)$ *is* $u^T_{|\Omega}$.

Proof. Let $c \in \mathbb{R}$ be such that $c > b - a$. Hence, Theorem 3.5 implies that in the rectangle R of vertices $A = (0,a)$, $B = (c,a)$, $C = (c,b)$ and $D = (0,b)$ there exists a unique minimal surface $\Sigma^\pm$, graph of the function $\omega^\pm$, intersecting the surface T above the sides $\overline{AB}$ and $\overline{CD}$ and diverging to $\pm\infty$ over $\overline{BC}$ and $\overline{DA}$. If u is any solution of $P(T,\Omega_a^b)$, it follows that $\omega^- \leqslant u \leqslant \omega^+$ on $\partial(\Omega_a^b \cap R)$, then the Maximum Principle implies that $\omega^- \leqslant u \leqslant \omega^+$ in $\Omega_a^b \cap R$ (see Figure 21). Since $\varphi_1^t(T) = T$ and φ_1^t is an isometry for all $t \in \mathbb{R}$, $\varphi_1^t(\Sigma^+)$ (resp. $\varphi_1^t(\Sigma^-)$) is above (resp. below) T for all t. It follows that there exists a positive constant M such that any solution $\tilde{u}$ of $P(T,\Omega)$ satisfies $|\tilde{u}(p) - u^T(p)| < M$ for all $p \in \Omega$. However, Theorem 4.1 implies that $|\tilde{u} - u^T|$ is not bounded, so we are done. $\square$

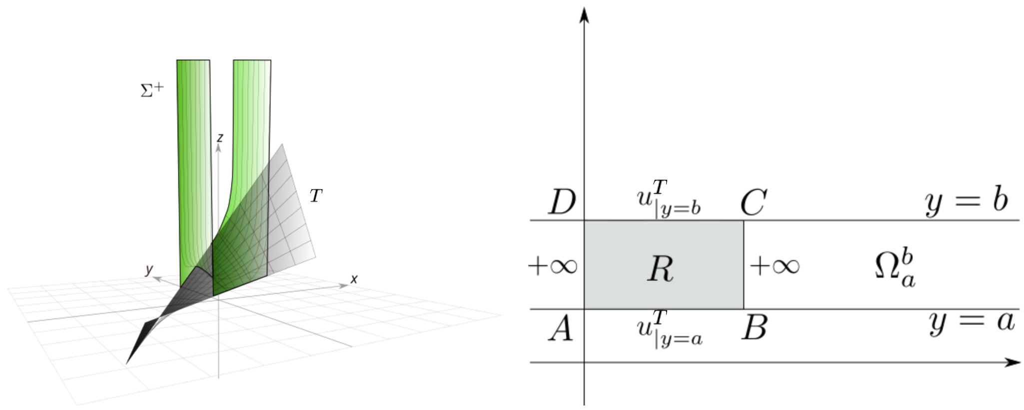

Figure 21: The upper barrier Σ^+.

Remark 4.9. Notice that, if Ω is a convex domain contained in a strip of $\mathbb{R}^2$, f is a piecewise continuous function over $\partial\Omega$, and there exists a positive constant C such that $|u^T - f| < C$, then [NeSaET017, Theorem 4.3] and Lemma 4.8 imply that there exists a unique minimal graph u over Ω with $u_{|\partial\Omega} = f$ and $|u - u^T| < C$.

Using Lemma 4.8, we can prove the following theorem, giving a positive answer to [NeSaET017, Question (a), p.17].

Theorem 4.10 . *The only minimal graph in* $\mathrm{Nil}_3(\tau)$ *over a strip of* $\mathbb{R}^2$ *with zero values on the boundary of the strip is the trivial one.*

Proof. After applying a rotation φ_4^θ for some θ, we consider the strip Ω_a^b parallel to the x-axis described above. For each $n \in \mathbb{N}$, denote by R_n the rectangle of vertices $A_n = (-n, a)$, $B_n = (n, a)$, $C_n = (n, b)$ and $D_n = (-n, b)$. If $n_0 > \frac{b-a}{2}$, Theorem 3.5 guarantees that for any $n > n_0$ there exists a unique solution $\omega_n^\pm$ to the Jenkins–Serrin problem in R_n that is zero on the sides parallel to the x-axis and diverges to $\pm\infty$ on the sides parallel to the y-axis. It is clear that, if u is a minimal solution in Ω_a^b with zero boundary values, then $\omega_n^+ > u > \omega_n^-$ for $n \in \mathbb{N}$. Furthermore, the Maximum Principle implies that $\omega_{n-1}^+ > \omega_n^+ > 0$ (resp. $\omega_{n-1}^- < \omega_n^- < 0$) in R_{n-1}, for any $n > n_0$. Thus, the Compactness Theorem implies that the limit $\omega^\pm = \lim\limits_{n\to\infty} \omega_n^\pm$ exist and we call $\Sigma^\pm$ their graphs.

If $\omega^+ \equiv 0 \equiv \omega^-$, then we are done. So, suppose for instance that $\omega^+ \neq 0$ in Ω_a^b (the same argument can be applied with slight modifications to ω^-).

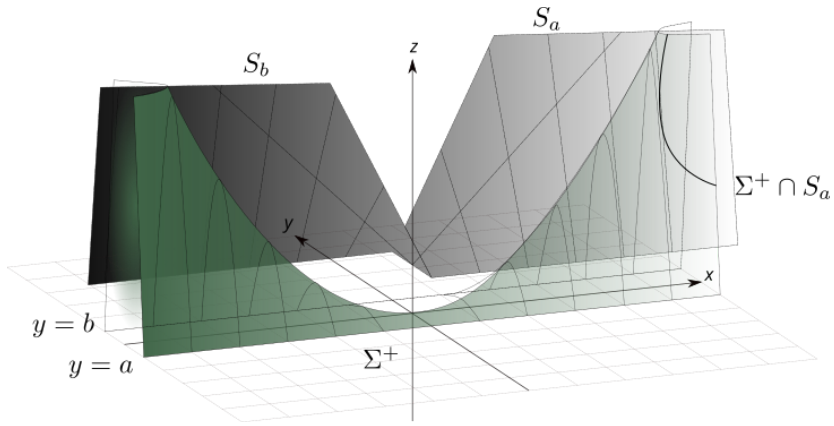

Figure 22: A contradiction on the growth of ω^+.

Theorem 4.6 implies that ω^+ has a quadratic height growth. We first study the asymptotic behaviour of ω^+ in $\Omega_a^b \cap \{x \geqslant 0\}$. Let

$$M = \sup_{y \in (a,b)} \omega^+_{n_0}(0, y)$$

and, for any $c \in \mathbb{R}$, denote by $S = \varphi_2^{c/2}(T)$ (that is, the graph of the function $u(x,y) = \tau x(y-c)$), and $S_c = \varphi_3^M(S)$ (that is, the graph of $u_c(x,y) = M + \tau x(y-c)$). Thus, by construction, $u_a \geqslant \omega^+$ on $\partial(\Omega_a^b \cap \{x \geqslant 0\})$. Since u_a has a linear height growth, it follows that

$$\Omega = \Big\{(x,y) \in \Omega_a^b \mid x \geqslant 0, u_a(x,y) < \omega^+(x,y)\Big\} \neq \emptyset.$$

Hence, $\omega^+_{|\Omega} \neq u_a$ is a solution of $P(S_a, \Omega)$ in contradiction with Lemma 4.8. To study the behaviour of ω^+ in $\Omega_a^b \cap \{x \geqslant 0\}$, we apply the same argument by replacing S_a with S_b. □

Once we have proved that the only minimal solution with constant boundary values on a strip is the constant one, we can give a positive answer to [NeSaET017, Question (b), p.17].

Corollary 4.11 . *Let Ω be a domain contained in a strip of $\mathbb{R}^2$ and $\phi\colon \partial\Omega \to \mathbb{R}$ be a piecewise $\mathcal{C}^2$-function and suppose that there exist two constants $m, M \in \mathbb{R}$ such that $m < \phi < M$. Hence the solution to the problem*

$$\begin{cases} \mathcal{Q}(u) = 0 & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

is unique and satisfies $m < u < M$.

Remark 4.12. It is possible to study the problem of uniqueness of minimal graphs in a strip of $\widetilde{\mathrm{Sl}}_2(\mathbb{R}) = \mathbb{E}(\mathbb{H}^2(-1), \tau, 1)$, that is isometric to $\left(\mathbb{D} \times \mathbb{R}, ds^2\right)$, where $\mathbb{D} = \left\{(x, y) \in \mathbb{R}^2 \mid \lambda(x, y) > 0\right\}$ and

$$ds^2 = \lambda^2(dx^2 + dy^2) + [2\tau\lambda(y\,dx - x\,dy) + dz]^2\,,$$

with $\lambda = \frac{2}{1-(x^2+y^2)}$. In this context φ_1 should be defined as the lifting of hyperbolic translation, so it makes sense to consider a strip invariant by a hyperbolic translation, i.e., the non-compact domain whose boundary is the union of two complete curves which are equidistant from a fixed geodesic. The minimal graphs invariant by φ_1 in $\widetilde{\mathrm{Sl}}_2(\mathbb{R})$ have bounded height (see for example [Cas22] or [Pen12]), so the arguments used for Nil_3 can be easily adapted to this case.

5 THE CONFORMAL CALABI–TYPE DUALITY

The classical Calabi duality [Cal70] provides a correspondence between minimal graphs in the Euclidean space $\mathbb{R}^3$ and maximal graphs in the Lorentz-Minkowski space $\mathbb{L}^3$ and it relies on the fact that the functions defining the graphs in both spaces satisfy a divergence-zero equation in $\mathbb{R}^2$. In general, in a simply connected base surface M, the Poincaré lemma says that a divergence-zero equation $\operatorname{div}(X) = 0$ implies the existence of a function f in M such that $X = J\nabla f$, where J is a $\frac{\pi}{2}$-rotation in the tangent bundle of M. In [Lee11], Lee managed to produce divergence zero equations when the mean curvature is constant and possibly not zero proving a correspondence between graphs with constant mean curvature H in $\mathbb{E}(\kappa, \tau)$ and space-like graphs with constant mean curvature τ in $\mathbb{L}(\kappa, H)$. The case of minimal graphs in $\mathbb{S}^2 \times \mathbb{R}$ (that is, $\tau = H = 0$ and $\kappa = 1$) was actually proved earlier by Albujer and Alías [AlbAli09]. In [LeeMan19], Lee and Manzano extend the result proved by Lee prescribing non-necessarily constant mean curvature and bundle curvature functions that are swapped by the duality. In particular, they proved a Calabi–type duality in unitary Killing submersions. The aim of this chapter is to extend this correspondence to the more general setting of non-unitary Killing submersions.

Using the natural notion of graph in $\mathbb{E}(M, \tau, \mu)$ and $\mathbb{L}(M, \tau, \mu)$ defined as a section of the submersion over M, described in Section 1.5.2, we prove a conformal duality between entire graphs in $\mathbb{E}(M, \tau, \mu) = \mathbb{E}(M, \tau, \mu, 1)$ with mean curvature H and entire space-like graphs in $\mathbb{L}(M, H, \mu^{-1}) = \mathbb{E}(M, H, \mu^{-1}, -1)$ with mean curvature τ. This is a very general result that covers all previously known cases since M is an arbitrary simply connected surface and $H, \tau, \mu \in \mathcal{C}^\infty(M)$ are also arbitrary (such that $\mu > 0$).

This yields a geometric connection between two apparently different theories which helps understand some geometric features. For instance, Fernández and Mira's classification [FerMir09] of entire minimal graphs in Heisenberg space $\mathrm{Nil}_3 = \mathbb{E}(0, \frac{1}{2})$ becomes transparent by considering the dual entire space-like graphs in $\mathbb{L}^3$ with constant mean curvature $\frac{1}{2}$, see [Man19]. Also, Manzano and Nelli [MaNe17] showed that gradient estimates for entire minimal graphs in Nil_3 are related to the Cheng and Yau's estimates [CheYau76] for the dual graphs in $\mathbb{L}^3$. In [LerMan17], the duality was used to show the existence of entire minimal graphs in Riemannian Killing submersions over compact surfaces using the existence results for prescribed mean curvature graphs in Lorentzian warped products obtained by Gerhardt [Ger83]. In [LeeMan19], the duality revealed that many Lorentzian Killing submersions do not admit any complete space-like surface by an extension of a classical argument of Heinz [Hei55] for constant mean curvature graphs in the Riemannian setting (see also Theorem 5.9).

As a first application of the duality, we will obtain entire space-like graphs in Lorentz–Minkowski space $\mathbb{L}^3 = \mathbb{L}(\mathbb{R}^2, 0, 1)$ with bounded prescribed mean curvature $H \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that ∇H is also bounded, see Theorem 5.5. This is achieved by constructing the dual entire minimal graphs in $\mathbb{E}(\mathbb{R}^2, H, 1)$ using the theory of divergence lines, developed by Mazet, Rosenberg and Rodríguez [MaRoRo11] and adapted to the case of Killing submersions in Section 3.2. In our proof, we have extended some of the results of Section 3.2 to take limits in three-manifolds whose geometry is not necessarily bounded by a diagonal argument with respect to an exhaustion by relatively compact domains. In $\mathbb{E}(\mathbb{R}^2, H, 1)$, we discard the possible divergence lines by applying Mazet's halfspace theorem [Maz13], and it is precisely at this point where we use that H and ∇H are bounded.

In particular, we give a partial answer to a conjecture stated in [LeeMan19] that there are entire graphs in $\mathbb{L}^3$ with any possible prescribed mean curvature $H \in \mathcal{C}^\infty(\mathbb{R}^2)$. We also prove this conjecture in Lorentzian warped products $\mathbb{E}(M, 0, \mu)$ in which M, μ and H are all invariant by rotations or translations with no assumptions on the growth of H, see Proposition 5.8. This means that our hypotheses in Theorem 5.5 are not sharp because there are entire space-like graphs in Minkowski space $\mathbb{L}^3 = \mathbb{L}(\mathbb{R}^2, 0, 1)$ with (equivariant) unbounded H and unbounded ∇H. In higher dimensions, this problem has been discussed in the literature as related to the Born–Infeld equation in which the mean curvature plays the role of the density of charge of an electrostatic physical system, and a solution is usually required to vanish at infinity (e.g., see [BoDaPo16, ByIkMaMa] and the references therein). In our approach, we are able to prescribe the normal at a given point of the base by means of a topological argument, see Lemma 5.4 and Remark 5.6.

The application of the duality is about the non-existence of entire graphs. In Theorem 5.9, we prove that $\mathbb{E}(M, \tau, \mu)$ does not admit any entire graph with

$\inf_M |H| > \frac{1}{2}\mathrm{Ch}(M, \mu)$ and the dual statement that $\mathbb{L}(M, \tau, \mu^{-1})$ does not admit complete space-like surfaces (of any mean curvature) if $\inf_M |\tau| > \frac{1}{2}\mathrm{Ch}(M, \mu)$. Here, $\mathrm{Ch}(M, \mu)$ is a constant that we have named *Cheeger constant with density* μ, see Equation (5.8). Theorem 5.9 had already been proved in [LeeMan19] in the unitary case $\mu \equiv 1$, in which $\mathrm{Ch}(M, \mu)$ is the classical Cheeger constant. In the case of the homogeneous $\mathbb{E}(\kappa, \tau)$-spaces, the value $H_0 = \frac{1}{2}\mathrm{Ch}(M, \mu)$ is the so-called *critical mean curvature*. If $H \leqslant H_0$, then there are entire graphs with constant mean curvature H in $\mathbb{E}(\kappa, \tau)$; on the contrary, if $H > H_0$, then there are compact surfaces with constant mean curvature H. This dichotomy plays a crucial role in the solution of the Hopf problem in homogeneous three-manifolds, see [MeMiPeRo021]. Motivated by this fact, we have investigated if $H_0 = \frac{1}{2}\mathrm{Ch}(M, \mu)$ distinguishes the existence of entire graphs and compact surfaces in $\mathbb{E}(M, \tau, \mu)$. In Theorem 5.10, we solve completely this problem in any rotationally invariant Riemannian warped product $\mathbb{E}(M, 0, \mu)$. Remarkably, we find that some specific values of $H > H_0$ give rise to rotationally invariant non-entire complete graphs, which we call H-*cigars*, see Figure 24. We also believe that the constant $\frac{1}{2}\mathrm{Ch}(M, \mu)$ is related to the critical mean curvature in all homogeneous three-manifolds for any of their (many) Killing submersion structures.

All the rotational examples we have obtained in $\mathbb{E}(M, 0, \mu)$ or $\mathbb{L}(M, 0, \mu)$ in the proofs of Proposition 5.8 and Theorem 5.10 have been constructed by means of the duality. It is hard to get a direct solution of the associated ODE since we recall that M, μ and H are arbitrary (rotationally symmetric) objects. It is also important to mention that Theorem 5.5 uses strongly the duality since we transform the prescribed mean curvature problem in the Lorentzian setting into a problem for minimal graphs in the Riemannian setting, where there are many more results that come in handy to analyze convergence.

Before stating and proving the main theorem of this chapter, we recall a couple of definitions about Killing graphs given in Section 1.5.2. In a Killing submersion $\mathbb{E}(M, \tau, \mu, \epsilon)$[1], the choice of the zero section F_0 naturally defines a vector field on M as $Z = \pi_*(\overline{\nabla} d)$, where $d \in \mathcal{C}^\infty(\mathbb{E})$ is the signed Killing distance from F_0 along the fibers of π. The vector field Z allows to define the

1 Here $\epsilon = \pm 1$ denote the causality of the vertical Killing vector field.

generalized gradient of a function $u \in \mathcal{C}^\infty(M)$ as $Gu = \nabla u - Z$ and compute the mean curvature of its graph as

$$2H\mu = \operatorname{div}\left(\frac{\mu^2 Gu}{\sqrt{1+\epsilon\mu^2\|Gu\|^2}}\right).$$

Furthermore, as explained in Remark 1.10, it carries informations about the bundle curvature:

$$\operatorname{div}(JZ) = \frac{2\epsilon\tau}{\mu},$$

where div is the divergence of M and J is a $\frac{\pi}{2}$-rotation in the tangent bundle of M. The main theorem of this chapter reads as follows.

Theorem 5.1 (Conformal duality). *Let M be a simply connected Riemannian surface and let $\tau, H, \mu \in \mathcal{C}^\infty(M)$ be arbitrary functions such that $\mu > 0$. There is a bijective correspondence between*

(a) entire graphs in $\mathbb{E}(M, \tau, \mu)$ with prescribed mean curvature H, and

(b) entire graphs in $\mathbb{L}(M, H, \mu^{-1})$ with prescribed mean curvature τ.

Assume that $\Sigma \subset \mathbb{E}(M, \tau, \mu)$ and $\widetilde{\Sigma} \subset \mathbb{L}(M, H, \mu^{-1})$ are such corresponding graphs.

1. *The graphs Σ and $\widetilde{\Sigma}$ determine each other up to vertical translations.*

2. *The corresponding angle functions $\mathfrak{v}, \widetilde{\mathfrak{v}} : M \to \mathbb{R}$ satisfy $\widetilde{\mathfrak{v}} = -\mathfrak{v}^{-1}$.*

3. *Denoting by $\pi : \mathbb{E}(M, \tau, \mu) \to M$ and $\widetilde{\pi} : \mathbb{L}(M, H, \mu^{-1}) \to M$ the involved Riemannian and Lorentzian Killing submersions, respectively, the diffeomorphism $\Phi : \Sigma \to \widetilde{\Sigma}$, such that $\widetilde{\pi} \circ \Phi = \pi$, is conformal with conformal factor*

$$\Phi^* ds^2_{\widetilde{\Sigma}} = \mu^{-2}\mathfrak{v}^2 ds^2_{\Sigma}.$$

Moreover, both families (a) *and* (b) *are empty if either $\int_M \frac{\tau}{\mu} d\sigma \neq 0$ or $\int_M H\mu d\sigma \neq 0$ and M is a topological sphere.*

Proof. If M is a topological sphere and $\int_M \frac{\tau}{\mu} \neq 0$, then the Killing submersion $\pi : \mathbb{E}(M, \tau, \mu) \to M$ is the Hopf fibration [LerMan17, Theorem 2.9], which admits no entire sections. Also, there is no entire graph with prescribed mean curvature τ in $\mathbb{L}(M, H, \mu^{-1})$, because such a graph would produce a smooth field X on M such that $\operatorname{div}(X) = \frac{\tau}{2\mu}$, whence $\int_M \frac{\tau}{\mu} = 0$ by the divergence theorem. This means that both families in (a) and (b) are empty if M is a topological sphere and $\int_M \frac{\tau}{\mu} \neq 0$. Analogously, both are empty if M is a topological sphere and $\int_M H\mu \neq 0$.

Therefore, we can assume that there are global sections $F_0 : M \to \mathbb{E}(M, \tau, \mu)$ and $\widetilde{F}_0 : M \to \mathbb{L}(M, H, \mu^{-1})$, see [LerMan17, Proposition 3.3]. These sections produce smooth vector fields $Z, \widetilde{Z} \in \mathfrak{X}(M)$ such that $\operatorname{div}(JZ) = \frac{-2\tau}{\mu}$ and $\operatorname{div}(J\widetilde{Z}) = 2H\mu$. Let $u \in \mathcal{C}^\infty(M)$ whose graph over the zero section F_0 has prescribed mean curvature H, that is,

$$2H\mu = \operatorname{div}\left(\frac{\mu^2 Gu}{\sqrt{1+\mu^2\|Gu\|^2}}\right) = \operatorname{div}(J\widetilde{Z}). \tag{5.1}$$

Since M is simply connected and (5.1) can be written as a divergence zero equation, the Poincaré lemma yields the existence of $v \in \mathcal{C}^\infty(M)$ such that

$$\frac{\mu^2 Gu}{\sqrt{1+\mu^2\|Gu\|^2}} - J\widetilde{Z} = -J\nabla v \;\Leftrightarrow\; \frac{\mu^2 Gu}{\sqrt{1+\mu^2\|Gu\|^2}} = -J\widetilde{G}v, \tag{5.2}$$

where $\widetilde{G}v = \nabla v - \widetilde{Z}$ is the generalized gradient in $\mathbb{L}(M, H, \mu^{-1})$. The function v is univocally determined up to an additive constant, which proves item (1) in the statement. Taking square norms in (5.2), we find that

$$\frac{\mu^4\|Gu\|^2}{1+\mu^2\|Gu\|^2} = \|\widetilde{G}v\|^2 \;\Leftrightarrow\; \frac{1}{1+\mu^2\|Gu\|^2} = 1 - \mu^{-2}\|\widetilde{G}v\|^2. \tag{5.3}$$

The right-hand side in (5.3) reveals that $1 - \mu^{-2}\|\widetilde{G}v\|^2 > 0$, whence the graph defined by v over the zero section $\widetilde{F}_0$ is space-like. Taking into account (1.21) and (5.3), we easily reach item (2). Also, we can plug (5.3) into (5.2) to get

$$\operatorname{div}\left(\frac{\mu^{-2}\widetilde{G}v}{\sqrt{1+\mu^{-2}\|\widetilde{G}v\|^2}}\right) = \operatorname{div}(JGu) = \operatorname{div}(J\nabla u) - \operatorname{div}(JZ) = \frac{2\tau}{\mu}, \tag{5.4}$$

so the graph defined by v has mean curvature τ in $\mathbb{L}(M, H, \mu^{-1})$. Likewise, we can obtain a graph in $\mathbb{E}(M, \tau, \mu)$ with mean curvature H starting with a space-like graph in $\mathbb{L}(M, H, \mu^{-1})$ with mean curvature τ, so the duality is a bijection.

It remains to check item (3) to finish the proof. It suffices to check that the metrics induced by π and $\widetilde{\pi}$ in M differ in the desired conformal factor. Since this property is local, we will work in coordinates using the background described in Chapter 1, where $M = (\Omega, \lambda_1^2 dx^2 + \lambda_2 dy^2)$ with $\Omega \subset \mathbb{R}^2$. Equation (1.32) says that we can express $Gu = \alpha e_1 + \beta e_2$ and $\widetilde{G}v = \widetilde{\alpha} e_1 + \widetilde{\beta} e_2$, where $\alpha = \frac{u_x}{\lambda_1} - a$, $\beta = \frac{u_y}{\lambda_2} - b$, $\widetilde{\alpha} = \frac{v_x}{\lambda_1} - \widetilde{a}$ and $\widetilde{\beta} = \frac{v_y}{\lambda_2} - \widetilde{b}$. If we consider the area elements

$$\omega = \sqrt{1 + \mu^2(\alpha^2 + \beta^2)}, \qquad \widetilde{\omega} = \sqrt{1 - \mu^{-2}(\widetilde{\alpha}^2 + \widetilde{\beta}^2)},$$

then (5.3) implies that $\omega\widetilde{\omega} = 1$, whence (5.2) can be written in two equivalent ways:

$$\left(\widetilde{\alpha}, \widetilde{\beta}\right) = \left(\frac{-\mu^2\beta}{\omega}, \frac{\mu^2\alpha}{\omega}\right) \Leftrightarrow (\alpha, \beta) = \left(\frac{\widetilde{\beta}}{\mu^2\widetilde{\omega}}, \frac{-\widetilde{\alpha}}{\mu^2\widetilde{\omega}}\right). \tag{5.5}$$

These *twin relations* allow us to compute

$$\begin{aligned}
\lambda_1^2\left(1 - \frac{\widetilde{\alpha}^2}{\mu^2}\right) &= \lambda_1^2\left(1 - \frac{\mu^2\beta^2}{\omega^2}\right) = \frac{\lambda_1^2(1 + \mu^2\alpha^2)}{\omega^2}, \\
-\lambda_1\lambda_2\frac{\widetilde{\alpha}\widetilde{\beta}}{\mu^2} &= \frac{\lambda_1\lambda_2\mu^2\alpha\beta}{\omega^2}, \\
\lambda_2^2\left(1 - \frac{\widetilde{\beta}^2}{\mu^2}\right) &= \lambda_2^2\left(1 - \frac{\mu^2\alpha^2}{\omega^2}\right) = \frac{\lambda_2^2(1 + \mu^2\beta^2)}{\omega^2}.
\end{aligned}$$

Taking into account the expression (1.34) for the induced metrics in M, we deduce that both metrics are conformal with conformal factor $\omega^{-2} = \mu^{-2}\mathfrak{v}^2$. □

Remark 5.2. In local coordinates, we only need to choose the functions a, b, $\widetilde{a}$ and $\widetilde{b}$ giving the desired bundle curvatures (which amounts to choosing the initial section). Once this is achieved, the twin relations (5.5) actually give a first-order ODE system

$$\left(\widetilde{\alpha}, \widetilde{\beta}\right) = \left(\frac{-\mu^2\beta}{\omega}, \frac{\mu^2\alpha}{\omega}\right) \Leftrightarrow \begin{cases} v_x = \lambda_1\widetilde{a} + \dfrac{-\lambda_1\mu(\frac{u_y}{\lambda_2} - b)}{\sqrt{\mu^{-2} + (\frac{u_x}{\lambda_1} - a)^2 + (\frac{u_y}{\lambda_2} - b)^2}}, \\ v_y = \lambda_2\widetilde{b} + \dfrac{\lambda_2\mu(\frac{u_x}{\lambda_1} - a)}{\sqrt{\mu^{-2} + (\frac{u_x}{\lambda_1} - a)^2 + (\frac{u_y}{\lambda_2} - b)^2}}. \end{cases} \tag{5.6}$$

Equivalently,

$$(\alpha, \beta) = \left(\frac{\widetilde{\beta}}{\mu^2\widetilde{\omega}}, \frac{-\widetilde{\alpha}}{\mu^2\widetilde{\omega}}\right) \Leftrightarrow \begin{cases} u_x = \lambda_1 a + \dfrac{\frac{\lambda_1}{\mu}(\frac{v_y}{\lambda_2} - \widetilde{b})}{\sqrt{\mu^2 - (\frac{v_x}{\lambda_1} - \widetilde{a})^2 - (\frac{v_y}{\lambda_2} - \widetilde{b})^2}}, \\ u_y = \lambda_2 b + \dfrac{-\frac{\lambda_2}{\mu}(\frac{v_x}{\lambda_1} - \widetilde{a})}{\sqrt{\mu^2 - (\frac{v_x}{\lambda_1} - \widetilde{a})^2 - (\frac{v_y}{\lambda_2} - \widetilde{b})^2}}. \end{cases} \tag{5.7}$$

The prescribed mean curvature H or τ equation in $\mathbb{E}(M, \tau, \mu)$ or $\mathbb{L}(M, H, \mu^{-1})$, respectively, can be now thought of as the compatibility conditions for these systems.

- The classical Calabi duality [Cal70] is recovered by Theorem 5.1 for $a = b = \widetilde{a} = \widetilde{b} = 0$ and $\mu = \lambda_1 = \lambda_2 = 1$ (so we get the flat base surface $M = \mathbb{R}^2$).
- The duality in homogeneous spaces with four-dimensional isometry group is recovered by Theorem 5.1 for $\lambda_1 = \lambda_2 = (1 + \frac{\kappa}{4}(x^2 + y^2))^{-1}$, $a = -\tau y$, $b = \tau x$, $\widetilde{a} = Hy$, $\widetilde{b} = -Hx$, and $\mu = 1$, see [Lee11, Cor. 2]. In this case, we have the base surface $M = \mathbb{M}^2(\kappa)$ (minus a point if $\kappa > 0$) as explained in Example 1.11.2.

5.1 Entire graphs of prescribed mean curvature in $\mathbb{L}^3$

Let $H \in \mathcal{C}^\infty(\mathbb{R}^2)$ be a smooth function. We would like to obtain an entire space-like graph $z = v(x, y)$ in $\mathbb{L}^3 = \mathbb{L}(\mathbb{R}^2, 0, 1)$ whose mean curvature at the point $(x, y, v(x, y))$ is precisely $H(x, y)$ for all $(x, y) \in \mathbb{R}^2$. By the duality in Theorem 5.1 (indeed, it suffices to apply the duality in the unitary case, see [LeeMan19]), this is equivalent to finding an entire minimal graph in $\mathbb{R}^3_H = \mathbb{E}(\mathbb{R}^2, H, 1)$. We will need a couple of lemmas to prove the existence of such an entire minimal graph, though we will need that both H and ∇H are bounded in order to apply the following halfspace theorem.

Lemma 5.3 . *If H and ∇H are bounded, then there is no properly immersed surfaces in a connected component of $\mathbb{R}^3_H - P$, where P is a vertical plane.*

Proof. Using the Calabi potential (see Remark 1.12), the manifold $\mathbb{R}^3_H$ can be modeled as $\mathbb{R}^3$ endowed with the Riemannian metric

$$dx^2 + dy^2 + (dz + y\,\mathbf{C}dx - x\,\mathbf{C}dy)^2, \qquad \text{where } \mathbf{C}(x, y) = 2\int_0^1 sH(xs, ys)ds,$$

where we can also assume (after an *a priori* rotation) that P is given by $y = d$ for some $d \in \mathbb{R}$. Consider the foliation by planes $P_t = \{(x, y, z) \in \mathbb{R}^3 : y = t\}$, in which $P_d = P$ and each P_t is flat and minimal since it projects onto a geodesic of $\mathbb{R}^2$. In particular, each leave P_t is a parabolic surface. Observe that

$E_1 = \partial_x - y\mathbf{C}\partial_z$ and $E_3 = \partial_z$ form an orthonormal tangent frame to all P_t in which we can compute the second fundamental form as

$$\sigma_t \equiv \begin{pmatrix} \sigma_t(E_1, E_1) & \sigma_t(E_1, E_3) \\ \sigma_t(E_3, E_3) & \sigma_t(E_3, E_3) \end{pmatrix} = \begin{pmatrix} 0 & H \\ H & 0 \end{pmatrix}.$$

This computation is essentially the same as in [LerMan17, p. 1361] taking into account that $\mu \equiv 1$ and P_t projects onto a geodesic. Therefore, $\|\sigma_t\|^2 = 2H^2$ is uniformly bounded not depending on t. Finally, consider the projection $\Phi_t : P_t \to P_0$ sending $(x, d+t, z)$ to (x, d, z). Its differential $d\Phi_t$ sends the orthonormal frame $\{E_1, E_3\}$ in P_t to the frame $\{E_1 - t\mathbf{C}E_3, E_3\}$ in P_0. However, since H is bounded, so is C and it trivially follows that Φ_t is a quasi-isometric projection into P_0 when t is close to d. Proposition 1.13 yields the following bound for the sectional curvature of $\mathbb{R}^3_H$:

$$|\overline{K}(\Pi)| = \left| H^2 - 4H^2\langle n, E_3\rangle^2 - 2\langle n, E_3\rangle\langle n \times E_3, \nabla H\rangle \right| \leqslant 3H^2 + 2\|\nabla H\|^2.$$

Since H and ∇H are bounded, the geometry of $\mathbb{R}^3_H$ is bounded. All the hypotheses of the halfspace theorem in [Maz13, Theorem 7] are met, so we deduce that there are no properly immersed surfaces in a connected component of $\mathbb{R}^3_H - P$. □

Lemma 5.4 . *For each $r > 0$, there is a minimal graph in $\mathbb{R}^3_H$ over $D_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}$ with angle function equal to 1 at $(0,0)$.*

Proof. Let $\mathbb{S}^2_+ = \{\varphi \in \mathbb{R}^3 : \varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 1,\ \varphi_3 > 0\}$ be the open upper halfsphere in $\mathbb{R}^3$. For each $\varphi \in \mathbb{S}^2_+$ with $(\varphi_1, \varphi_2) \neq (0,0)$, decompose $\partial D_r = S^+_\varphi \cup S^-_\varphi$, where

$$S^+_\varphi = \{(x,y) \in \partial D_r : \langle (x,y), (\varphi_1, \varphi_2)\rangle > 0\},$$
$$S^-_\varphi = \{(x,y) \in \partial D_r : \langle (x,y), (\varphi_1, \varphi_2)\rangle < 0\},$$

and consider the boundary data in ∂D_r that assigns a value $\pm(\varphi_3^{-2} - 1)$ to the component $S^\pm_\varphi$, see Figure 23. If $\varphi_1 = \varphi_2 = 0$, the value 0 is assigned to all ∂D_r. Let $\Sigma_\varphi \subset \mathbb{R}^3_H$ be the minimal graph over $\overline{D}_r$ that solves the Dirichlet problem for such boundary data. Note that such a minimal surface exists and is unique by Theorem 2.1. The uniqueness also guarantees that Σ_φ depends continuously on φ since the boundary data we have defined in turn depends continuously on φ. Additionally, we define $\Sigma_\varphi \subset \mathbb{R}^3_H$ as the minimal vertical plane with normal $\varphi_1\partial_x + \varphi_2\partial_y$ at the origin whenever $\varphi_1^2 + \varphi_2^2 = 1$ and $\varphi_3 = 0$. Recall that $\{\partial_x, \partial_y, \partial_z\}$ is an orthonormal basis of $\mathbb{R}^3_H$ at the origin in our model.

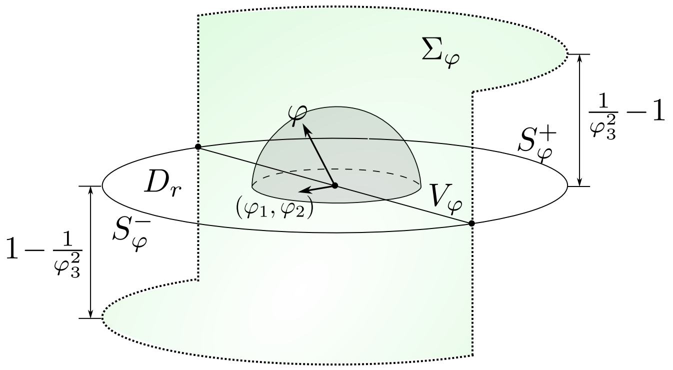

Figure 23: The green surface Σ_φ that solves the Dirichlet problem over D_r with boundary values $\pm(\varphi_3^{-2} - 1)$ on the half-circles $S^\pm_\varphi$.

This allows us to define a map $\eta : \overline{\mathbb{S}}^2_+ \to \overline{\mathbb{S}}^2_+$ in the closed upper hemisphere such that $\eta(\varphi) = \psi$ if the unit normal of Σ_φ is expressed as $\psi_1\partial_x + \psi_2\partial_y + \psi_3\partial_z$. This map is continuous on $\mathbb{S}^2_+$, the interior of the hemisphere, by the continuity of Σ_φ with respect to φ, but it is also continuous in the closure. To see this, for each $\varphi \in \mathbb{S}^2_+$ with $\varphi_3 \neq 1$, let V_φ be the diameter of D_r joining the endpoints of the arc S^+_φ and let $\Sigma^\pm_\varphi \subset \mathbb{R}^3_H$ be the minimal graph that solves the Jenkins–Serrin problem over the half-disk demarcated by $S^\pm_\varphi$ and V_φ with boundary values $\pm(\varphi_3^{-2} - 1)$ on $S^\pm_\varphi$ and $\mp\infty$ on V_φ, which exists by Theorem 3.5. By the Maximum Principle (see Theorem 3.26), the surface Σ_φ lies below Σ^+_φ and above Σ^-_φ as graphs.

Given $\varphi_0 \in \partial\mathbb{S}^2_+$, the radial limit of Σ_φ as $\varphi \to \varphi_0$ is the vertical plane $\Sigma_{\varphi_0} = V_\varphi \times \mathbb{R}$ because S^+_φ and S^-_φ sweep out the whole region outside this plane as $\varphi \to \varphi_0$ radially (V_φ does not change in the radial limit). This also means that $\eta(\varphi_0) = \varphi_0$. Since the radial limit of η is continuous in all $\partial\mathbb{S}^2_+$, we infer that $\eta : \overline{\mathbb{S}}^2_+ \to \overline{\mathbb{S}}^2_+$ is continuous. Since $\eta(\varphi) = \varphi$ for all $\varphi \in \partial\mathbb{S}^2_+$, an easy degree argument shows that η is onto, whence there is some $\varphi_0 \in \mathbb{S}^2_+$ such that $\eta(\varphi_0) = (0, 0, 1)$ so that Σ_{φ_0} is the desired minimal graph over D_r. □

Theorem 5.5 . *If* $H \in \mathcal{C}^\infty(\mathbb{R}^2)$ *is a bounded function such that* ∇H *is also bounded, then there is an entire space-like graph in* $\mathbb{L}^3$ *with prescribed mean curvature* H.

Proof. For each $n \in \mathbb{N}$, let u_n be the minimal graph in $\mathbb{R}^3_H$ over the disk $D_n \subset \mathbb{R}^2$ of Euclidean radius n passing through the origin with angle function 1 given by Lemma 5.4. This means that ∇u_n is bounded at the origin. If we fix some $r > 0$, the theory of divergence lines ensures that there is some domain $\Omega \subset D_r$ containing 0 and bounded by straight segments such that the translated graphs $u_n - u_n(0)$ subconverge to a minimal graph over Ω. We claim that there is a further subsequence such that these divergence lines have no intersection in D_r.

To prove the claim, we start with a disk $D_\rho \subset \Omega$ of maximal radius and assume that $\rho < r$ (there is nothing to prove if $\rho = r$), which means that we can find some divergence lines touching ∂D_ρ. Choose one, say L, and take a further subsequence of $u_n - u_n(0)$ such that L does not intersect any other divergence line inside D_r. In particular, the convergence domain of this subsequence is contained in the connected component Ω_L of $D - L$, which in turn contains the origin. Since there cannot be infinitely many disjoint tangent segments to ∂D_ρ connecting points of ∂D_r, we can proceed likewise with the rest of them to ensure that such disjoint divergence lines tangent to ∂D_ρ do not intersect any other divergence line of the refined subsequence. By successively enlarging the radius ρ within the intersection of the regions Ω_L for the lines we have met so far, we can possibly reach new divergence lines, in which case we apply the same reasoning. However, for any $\rho < r$, we can only meet a finite number of divergence lines touching D_ρ, which ensures that the process can be repeated until reaching $\rho = r$ in at most countably many steps. Although this potentially means a countable number of refinements of the original subsequence, a diagonal argument finishes the proof of the claim.

Just to make it clear and set the notation, we have proved that there is a limit minimal graph u^r_∞ (for some subsequence of $u_n - u_n(0)$) over a domain $\Omega_r \subset D_r$ such that $\partial\Omega_r \cap D_r$ is a union of (countably many) disjoint divergence lines. Note that u^r_∞ diverges to $\pm\infty$ along each component of $\partial\Omega_r \cap D_r$ as in [MaRoRo11, Lem. 4.6].

Given $r_1 > r$, we can use the above argument to find another minimal graph $u^{r_1}_\infty$ in another maximal subset $\Omega_{r_1} \subset D_{r_1}$, but the trick is to start with the subsequence of $u_n - u_n(0)$ that already converges to u^r_∞ in Ω_r instead of the original sequence. Doing so, we have that $\Omega_r \subset \Omega_{r_1}$ but some of the divergence lines that demarcate Ω_r might be lost. This actually occurs if the divergence segments in the boundary of Ω_r intersect outside D_r, since we can apply this reasoning for some $r_1 > r$ such that the intersection occurs in D_{r_1}, and then pass to a subsequence that eliminates one of the divergence lines. All in all, we can assume that there are at most two (parallel) divergence segments.

This process can be repeated for an increasing sequence of radii $r_n \to \infty$ to obtain minimal graphs $u_\infty^{r_n}$, each of them by further refining the subsequence of $u_n - u_n(0)$ that converges to the previous one. In particular, $u_\infty^{r_n}$ extends $u_\infty^{r_{n-1}}$ to a larger domain for all n. A diagonal argument yields a complete minimal graph u_∞ with angle function 1 at the origin over all the plane $\mathbb{R}^2$ or a halfplane or a strip, depending on whether there are 0, 1 or 2 divergence lines, respectively. (Recall that u_∞ tends to $\pm\infty$ uniformly on compact subsets of these lines.) If H and ∇H are bounded, the halfplane and the strip can be discarded by Lemma 5.3, so the dual space-like graph in $\mathbb{L}^3$ is entire and has prescribed mean curvature H. □

Remark 5.6. The same argument shows that there is always an entire minimal graph in $\mathbb{R}^3_H$ with prescribed unit normal at some fixed $p \in \mathbb{R}^2$.

To see this, note that the (upward-pointing) unit normal of the minimal graph in $\mathbb{R}^3_H$ is given by $N = -\frac{\alpha}{\omega}E_1 - \frac{\beta}{\omega}E_2 + \frac{1}{\omega}E_3$, so that we can prescribe α and β at p by choosing all the elements of the convergent sequence with this normal at p (the map η in Lemma 5.4 is a bijection). By the twin relations (5.5), this means that we can prescribe the (time-like) unit normal of the dual graph in $\mathbb{L}^3$ since it is given by $\widetilde{N} = \frac{\widetilde{\alpha}}{\widetilde{\omega}}\widetilde{E}_1 + \frac{\widetilde{\beta}}{\widetilde{\omega}}\widetilde{E}_2 + \frac{1}{\widetilde{\omega}}\widetilde{E}_3 = -\beta\widetilde{E}_1 + \alpha\widetilde{E}_2 + \omega\widetilde{E}_3$.

Remark 5.7. This technique for constructing complete minimal surfaces works in any Killing submersion $\mathbb{E}(M, \tau, \mu)$ provided that there is an exhaustion of M by disks whose boundaries are convex in the conformal metric $\mu^2 ds_M^2$. For instance, it works in a unit Killing submersion over a Hadamard surface or in $\mathrm{Sol}_3 = \mathbb{E}(\mathbb{H}^2, 0, x^2)$ (this is the metric given by Nguyen [Ngu14] in her solution to the Jenkins–Serrin problem).

The point is that, if it is not an entire graph, the domain of the constructed complete graph is bounded by disjoint geodesics (in the conformal metric $\mu^2 ds_M^2$, see Lemma 3.1). The hypotheses of Theorem 5.5 on H are just the hypotheses of Lemma 5.3, so an improved halfspace theorem with respect to vertical planes in Killing submersions would allow us to show the existence of solutions to the prescribed mean curvature equation under milder hypotheses.

As a matter of fact, motivated by the next result, we believe that all hypotheses on H and ∇H can be dropped. Indeed, we show that if both M, μ, τ and H are invariant with respect to a one-parameter group of isometries (that is, in the rotational and translational cases), we do not need any assumption on the boundedness of H and ∇H to guarantee the existence of a global minimal section.

Proposition 5.8 . *The Lorentzian warped product $\mathbb{L}(M, 0, \mu)$, where the base surface is $M = (\Omega, \lambda^2(dx^2 + dy^2))$, admits an entire space-like graph with prescribed mean curvature H under any of the following two assumptions:*

(a) $\Omega \subseteq \mathbb{R}^2$ is a disk centered at the origin with radius $0 < R \leqslant +\infty$ and $\lambda, \mu, H \in \mathcal{C}^\infty(\Omega)$ are radial functions (such that $\lambda, \mu > 0$).

(b) $\Omega \subseteq \mathbb{R}^2$ is a strip of width $0 < R \leqslant +\infty$ and $\lambda, \mu, H \in \mathcal{C}^\infty(\Omega)$ are functions invariant by translations along the strip (such that $\lambda, \mu > 0$).

Proof. In the rotational case, consider the Riemannian space $\mathbb{E}(M, H, \mu^{-1})$ modeled as $\Omega \times \mathbb{R}$ with the metric $\lambda^2(dx^2 + dy^2) + \mu^{-2}(dz + y\,\mathbf{C}\,dx - x\,\mathbf{C}\,dy)^2$, where $\mathbf{C}$ is the Calabi potential (see Remark 1.12). It is easy to check that the graph $z = 0$ is minimal in this model using Equation (1.33) and the fact that λ, H and μ (and hence $\mathbf{C}$) are radial functions. Therefore, the dual graph in $\mathbb{L}(M, 0, \mu)$ is an entire space-like graph with mean curvature function H.

In item (b) we will consider $\mathbb{E}(M, H, \mu^{-1})$ modeled as $\Omega \times \mathbb{R}$ with the metric

$$\lambda(x)^2(dx^2 + dy^2) + \frac{1}{\mu(x)^2}(dz - f(x)\,dy)^2, \qquad f(x) = 2\int \frac{H(x)\lambda(x)^2}{\mu(x)}dx.$$

This model is obtained by assuming that the strip runs in the direction of the x-axis and integrating (1.5) with $a \equiv 0$. Again, the graph $z = 0$ is minimal by Equation (1.33) and satisfies $\alpha \equiv 0$ and β depends only on the variable x. As in item (a), the dual graph in $\mathbb{L}(M, 0, \mu)$ is the desired entire space-like graph. □

5.2 Existence and non-existence of entire graphs

Given a Riemannian surface M and a positive function $\mu \in \mathcal{C}^\infty(M)$, we define the *Cheeger constant of* M *with density* μ as

$$\mathrm{Ch}(M, \mu) = \inf\left\{\frac{\int_{\partial D} \mu d\sigma}{\int_D \mu d\sigma} : D \subset M \text{ regular}\right\} \geqslant 0. \tag{5.8}$$

Here, an open subset $D \subset M$ is said *regular* if it is relatively compact and its boundary is piecewise smooth so the quotient in (5.8) makes sense. If M is compact, then $\mathrm{Ch}(M,\mu) = 0$ by choosing $D = M$ in (5.8). Note that $\mathrm{Ch}(M,\mu)$ remains invariant when changing μ into $a\mu$ for any positive constant a.

Theorem 5.9 . *Let M be a non-compact simply-connected surface and consider an arbitrary positive function $\mu \in \mathcal{C}^\infty(M)$.*

(a) Given $H \in \mathcal{C}^\infty(M)$ such that $\inf_M |H| > \frac{1}{2}\mathrm{Ch}(M,\mu)$, the space $\mathbb{E}(M,\tau,\mu)$ admits no entire graphs with mean curvature H for any $\tau \in \mathcal{C}^\infty(M)$.

(b) Given $\tau \in \mathcal{C}^\infty(M)$ such that $\inf_M |\tau| > \frac{1}{2}\mathrm{Ch}(M,\mu)$, the space $\mathbb{L}(M,\tau,\mu^{-1})$ admits neither complete space-like surfaces nor entire space-like graphs.

Proof. We will use a standard argument due to Heinz [Hei55] to get item (a). Let us argue by contradiction supposing that such an entire graph exists and it is given by $u \in \mathcal{C}^\infty(M)$ with respect to some initial section. Applying the divergence theorem to the mean curvature equation given by Proposition 1.24 over an open regular domain $D \subset M$ and Cauchy-Schwarz inequality, we get

$$\begin{aligned} 2H_0 \int_D \mu \, d\sigma &\leqslant \int_D \operatorname{div}\left(\frac{\mu^2\, Gu}{\sqrt{1+\mu^2\|Gu\|^2}}\right) d\sigma \\ &= \int_{\partial D} \mu \left\langle \frac{\mu\, Gu}{\sqrt{1+\mu^2\|Gu\|^2}}, \eta \right\rangle d\sigma < \int_{\partial D} \mu \, d\sigma, \end{aligned} \tag{5.9}$$

where η is an outer unit conormal to D along its boundary and $H_0 = \inf_M(H)$. The condition $\inf |H| > \frac{1}{2}\mathrm{Ch}(M,\mu) \geqslant 0$ implies that H has a sign. If $H_0 > 0$ (and hence $H > 0$), since (5.9) holds for all regular domains D, we find that

$$H_0 = \inf_M(H) = \inf_M |H| < \tfrac{1}{2}\mathrm{Ch}(M,\mu),$$

contradicting the hypotheses in the statement. Otherwise, we have $H_0 < 0$, so we change the sign of the normal in the above argument to get that $-2H_0 \int_D \mu \leqslant \int_{\partial D} \mu$, so $-H_0 = \inf_M |H| < \frac{1}{2}\mathrm{Ch}(M,\mu)$ and we get a contradiction again.

As for item (b), we will reason by contradiction again: if there is a complete space-like surface $\widetilde{\Sigma} \subset \mathbb{L}(M,\tau,\mu^{-1})$, then $\widetilde{\Sigma}$ would be an entire graph (the proof is the same as in [LeeMan19, Lem. 4.11] since the projection $\pi|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to M$ is distance non-decreasing) so its dual surface $\Sigma \subset \mathbb{E}^3(M,H,\mu)$ is an entire graph, where H denotes the mean curvature of $\widetilde{\Sigma}$. Now, τ becomes the mean curvature of Σ and verifies $\inf_M |\tau| > \frac{1}{2}\mathrm{Ch}(M,\mu)$, in contradiction with item (a). □

In $\mathbb{E}(\kappa,\tau)$-spaces, the Cheeger constant (with density $\mu \equiv 1$) is given by

$$\mathrm{Ch}(\mathbb{M}^2(\kappa), 1) = \begin{cases} \sqrt{-\kappa} & \text{if } \kappa \leqslant 0, \\ 0 & \text{if } \kappa \geqslant 0. \end{cases}$$

Consequently, the value $\frac{1}{2}\mathrm{Ch}(M,\mu)$ given by Theorem 5.9 is nothing but the *critical* mean curvature in $\mathbb{E}(\kappa,\tau)$-spaces. It is well known that this also reflects the dichotomy between the existence of entire H-graphs and the existence of compact H-surfaces (both types of surfaces cannot coexist by the Maximum Principle, with the exception of horizontal slices in $\mathbb{S}^2(\kappa)\times\mathbb{R}$).

We will show next that this dichotomy extends to rotationally invariant Riemannian warped products by a means of a tricky application of the duality. However, in this general case, we will find another type of surface that we will call H-*cigar* since it is a graph over a disk with asymptotic value $+\infty$ on the boundary of the disk, see Figure 24. It can be thought of as a half-sphere of infinite height.

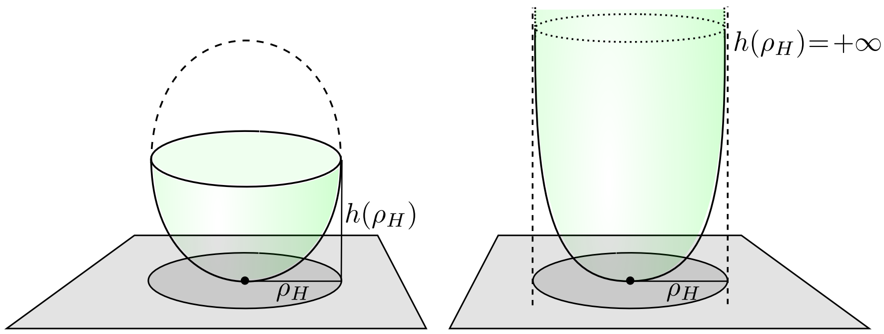

Figure 24: An H-halfsphere (left) and an H-cigar (right).

Theorem 5.10 . *Let* $M = (\Omega, \lambda^2(dx^2+dy^2))$ *be such that* $\Omega \subseteq \mathbb{R}^2$ *is a disk centered at the origin with radius* $R \in (0,+\infty]$ *and* $\lambda,\mu \in \mathcal{C}^\infty(\Omega)$ *are radial functions such that* $\lambda,\mu > 0$*, that is,* M *is a rotationally invariant Riemannian surface. Given a constant* $H \geqslant 0$*:*

(a) *If* $H > \frac{1}{2}\mathrm{Ch}(M,\mu)$*, then* $\mathbb{E}(M,0,\mu)$ *admits a smooth embedded rotationally invariant* H*-sphere or* H*-cigar.*

b *If* $H \leqslant \frac{1}{2}\mathrm{Ch}(M,\mu)$*, then* $\mathbb{E}(M,0,\mu)$ *admits a rotationally invariant entire* H*-graph.*

As a consequence, $\mathbb{E}(M,0,\mu)$ *does not admit compact* H*-surfaces for* $H \leqslant \frac{1}{2}\mathrm{Ch}(M,\mu)$*, and does not admit entire* H*-graphs with* $H > \frac{1}{2}\mathrm{Ch}(M,\mu)$ *either.*

Proof. In the sequel we will use the radial coordinates $\rho = (x^2+y^2)^{1/2}$ and write $\mu = \mu(\rho)$ and $\lambda = \lambda(\rho)$. Consider the Lorentzian space $\mathbb{L}^3(M,H,\mu^{-1})$, whose Calabi potential with respect to the conformal parameterization is also radial, given by $\mathbf{C}(\rho) = 2Hc(\rho)$, where

$$c : [0,R) \to \mathbb{R}, \qquad c(\rho) = \int_0^1 s\,\lambda(s\rho)^2\mu(s\rho)\,ds = \frac{1}{\rho^2}\int_0^\rho s\lambda(s)\mu(s)^2 ds \geqslant 0.$$

This means that $\mathbb{L}^3(M,H,\mu^{-1})$ is modeled as $\Omega \times \mathbb{R}$ with metric

$$\lambda^2(dx^2+dy^2) - \mu^{-2}\,(dz - y\,\mathbf{C}\,dx + x\,\mathbf{C}\,dy)^2\,,$$

which follows from Equation (1.3) and Remark 1.12 with $\widetilde{a} = \frac{2Hyc}{\lambda}$ and $\widetilde{b} = \frac{-2Hxc}{\lambda}$. Equation (1.33) easily implies that the graph $z = 0$ is maximal in $\mathbb{L}^3(M,H,\mu^{-1})$ and the space-like condition (1.20) for this graph reads

$$\mu(\rho) - \frac{2H\rho\,c(\rho)}{\lambda(\rho)} > 0. \tag{5.10}$$

Since (5.10) holds true for $\rho = 0$, it must still hold true in a neighborhood of 0, so there is some maximal radius $\rho_H \in (0,R]$ such that (5.10) is satisfied for $0 \leqslant \rho < \rho_H$. Theorem 5.1 gives a dual H-graph in $\mathbb{E}(M,0,\mu)$ over the disk of radius ρ_H. As $\mathbb{E}(M,0,\mu)$ has zero bundle curvature, we will choose $a = b = 0$ in (1.3) and model it as $\Omega \times \mathbb{R}$ with the metric $\lambda^2(dx^2+dy^2) + \mu^2 dz^2$. In this model we parametrize the aforesaid dual H-graph as $z = u(x,y)$ for some smooth function u on the disk of radius ρ_H. The twin relations (5.7) give the derivatives of u:

$$u_x = \frac{2Hxc}{\mu\sqrt{\mu^2 - \frac{4H^2c^2}{\lambda^2}(x^2+y^2)}}, \qquad u_y = \frac{2Hyc}{\mu\sqrt{\mu^2 - \frac{4H^2c^2}{\lambda^2}(x^2+y^2)}}, \tag{5.11}$$

whence $yu_x - xy_y = 0$ and u also defines a rotationally invariant surface in $\mathbb{E}(M,0,\mu)$. In particular, we can reparametrize the graph of u as

$$(\rho,\theta) \mapsto (\rho\sin(\theta), \rho\cos(\theta), h(\rho)) \in \Omega \times \mathbb{R} \equiv \mathbb{E}(M,0,\mu),$$

where $0 \leqslant \rho < \rho_H$ and $\theta \in \mathbb{R}$. The profile function h is given by

$$h(\rho) = \int_0^\rho \frac{2Hr\,c(r)\,dr}{\mu(r)\sqrt{\mu(r)^2 - \frac{4H^2r^2c(r)^2}{\lambda(r)^2}}} = \int_0^\rho \frac{g_1(r)\,dr}{\sqrt{1-g_2(r)^2}},$$

where $g_1(r) = \frac{2Hrc(r)}{\mu(r)^2}$ and $g_2(r) = \frac{2Hrc(r)}{\lambda(r)\mu(r)}$ are non-negative functions defined for all $r \in [0, R)$ which only vanish at $r = 0$. We will distinguish three cases:

Case 1. If $\rho_H = R$, then $z = 0$ is an entire maximal graph so the dual surface $z = u(x, y)$ is an entire rotationally invariant H-graph.

Case 2. Assume that $\rho_H < R$ and $g_2'(\rho_H) \neq 0$. Since $\rho \mapsto h(\rho)$ is increasing and the function $\varphi = g_2' g_1^{-1}$ is continuous and bounded away from zero in a neighborhood of ρ_H, it follows that $h(\rho_H) < +\infty$ if and only if

$$\int_0^{\rho_H} \frac{\varphi(r) g_1(r)\, dr}{\sqrt{1 - g_2(r)^2}} = \int_0^{\rho_H} \frac{g_2'(r)\, dr}{\sqrt{1 - g_2(r)^2}} < +\infty.$$

The last integral equals $\arcsin(g_2(\rho_H)) < +\infty$, so this argument shows that $h(\rho_H) < +\infty$ and the boundary of the graph $z = u(x, y)$ lies in the slice $\Omega \times \{h(\rho_H)\}$. The graph meets the slice orthogonally since $g_2(\rho_H) = 1$ by the maximality of ρ_H, whence the angle function of $z = u(x, y)$, given by $\mathfrak{v}(\rho) = \mu(\rho)\sqrt{1 - g_2(\rho)^2}$ tends to zero as $\rho \to \rho_H$. Moreover, the transformation $(x, y, z) \mapsto (x, y, 2h(\rho_H) - z)$ is an isometry in $\mathbb{E}(M, 0, \mu)$ keeping the (totally geodesic) slice $\Omega \times \{h(\rho_H)\}$ fixed, so the graph can be reflected about this slice to get an embedded H-sphere.

This H-sphere is of class $\mathcal{C}^1$. We will show that it is of class $\mathcal{C}^2$. The curve $\rho \mapsto (\rho, h(\rho))$ defines the profile curve of a halfsphere with $0 \leqslant \rho \leqslant \rho_H$. Since $h(\rho)$ is one-to-one, we can consider the reparametrization $t \mapsto (h^{-1}(t), t)$ with $0 \leqslant t \leqslant h(\rho_H)$. In the interval $(0, \rho_H)$, we get

$$(h^{-1})' \circ h = \frac{1}{h'} = \frac{\sqrt{1 - g_2^2}}{g_1},$$

$$(h^{-1})'' \circ h = -\frac{h''}{(h')^3} = \frac{\frac{g_1'(1-g_2^2) - g_2' g_1 g_2}{(1-g_2)^{3/2}}}{\frac{g_1^3}{(1-g_2)^{3/2}}} = \frac{g_1'(1 - g_2^2) - g_2' g_1 g_2}{g_1^3}.$$

This reveals that h^{-1} is of class $\mathcal{C}^2$ up to $h(\rho_H)$ with derivatives

$$(h^{-1})'(h(\rho_H)) = 0 \text{ and } (h^{-1})''(h(\rho_H)) = \frac{g_2'(\rho_H)}{g_1(\rho_H)^2}$$

(note that $g_2(\rho_H) = 1$ and $g_1(\rho_H) > 0$). In particular, the extension of the profile curve of the H-sphere is of class $\mathcal{C}^2$ after reflection.

To finish the proof that the H-sphere is smooth, observe that the profile curve $s \mapsto (x(s), 0, z(s))$ that spans the whole H-sphere by rotations about the z-axis satisfies a second-order ODE (e.g., see Proposition 1.21). The initial value problem for this equation has a unique $\mathcal{C}^2$-solution through the point $(\rho_H, h(\rho_H))$ with speed $(0,1)$, so it must coincide with the profile of the H-sphere. Since the aforesaid ODE has smooth coefficients, we conclude that our H-spheres are smooth.

Case 3. Finally, assume that $\rho_H < R$ and $g_2'(\rho_H) = 0$. Let Γ be the vertical cylinder of equation $x^2 + y^2 = \rho_H^2$, which has constant mean curvature

$$\begin{aligned} 2H_\Gamma &\overset{(1)}{=} \frac{(\lambda\mu)'(\rho_H)}{\lambda(\rho_H)^2\mu(\rho_H)} + \frac{1}{\rho_H\lambda(\rho_H)} \overset{(2)}{=} \frac{1}{\rho_H\lambda(\rho_H)} + \frac{c(\rho_H) + \rho_H c'(\rho_H)}{\rho_H\lambda(\rho_H)c(\rho_H)} \\ &\overset{(3)}{=} \frac{1}{\rho_H\lambda(\rho_H)} + \frac{\mu(\rho_H)\lambda^2(\rho_H) - c(\rho_H)}{\rho_H\lambda(\rho_H)c(\rho_H)} \\ &\overset{(4)}{=} \frac{1}{\rho_H\lambda(\rho_H)} + \frac{\mu(\rho_H)\lambda^2(\rho_H) - \frac{\lambda(\rho_H)\mu(\rho_H)}{2H\rho_H}}{\rho_H\lambda(\rho_H)\frac{\lambda(\rho_H)\mu(\rho_H)}{2H\rho_H}} = 2H. \end{aligned} \tag{5.12}$$

The equality (1) in (5.12) to compute the mean curvature of a vertical cylinder follows from Equation (1.15); (2) uses the condition $g_2'(\rho_H) = 0$, in which we solve for $(\lambda\mu)'(\rho_0)$; (3) uses the identity $\frac{d}{dr}(rc(r)) = \mu(r)\lambda(r)^2$, which in turn follows from (1.5) and the fact that the bundle curvature of $\mathbb{L}(M, H, \mu^{-1})$ is H (note that the Killing length is μ^{-1}); finally, (4) is a consequence of the fact that $g_2(\rho_H) = 1$ by the maximality of ρ_H. We will conclude that $h(\rho_H) = +\infty$ by contradiction. If $h(\rho_H) < +\infty$, then the H-graph $z = u(x, y)$ lies in the interior of the H-cylinder $x^2 + y^2 = \rho_H^2$. They are tangent along the boundary because $\mathfrak{v}(\rho_H) = \mu(\rho_H)\sqrt{1 - g_2(\rho_H)^2} = 0$ as in the above item. The boundary Maximum Principle for H-surfaces yields the desired contradiction.

Now observe that (5.10) implies that $H \mapsto \rho_H$ is a continuous and decreasing function of H. That means that there exists $H_0 \geqslant 0$ such that $z = u(x, y)$ defines an entire graph for $H \leqslant H_0$ and an H-halfsphere or an H-cigar for $H > H_0$ (depending on whether $g_2'(\rho_H)$ vanishes or not). Note that in the case $H = 0$, then $u \equiv 0$ is an entire minimal graph. Recall that entire H-graphs and H-spheres (or H-cigars) cannot coexist due to the Maximum Principle for H-surfaces. Hence, it remains to prove that $H_0 = \frac{1}{2}\mathrm{Ch}(M, \mu)$ and we will be done.

On the one hand, Theorem 5.9 yields non-existence of entire H-graphs for $H > \frac{1}{2}\mathrm{Ch}(M, \mu)$, so we deduce that $H_0 \leqslant \frac{1}{2}\mathrm{Ch}(M, \mu)$. On the other hand, let D_ρ be the disk of center 0 and Euclidean radius $0 < \rho < R$. By definition of Cheeger constant,

$$\mathrm{Ch}(M, \mu) \leqslant \frac{\int_{\partial D_\rho} \mu d\sigma}{\int_{D_\rho} \mu d\sigma} = \frac{\int_0^{2\pi} \rho\,\lambda(\rho)\mu(\rho)\,d\theta}{\int_0^{2\pi}\int_0^\rho r\,\lambda(r)^2\mu(r)\,dr\,d\theta} = \frac{\lambda(\rho)\mu(\rho)}{\rho\,c(\rho)} \tag{5.13}$$

for all $0 < \rho < R$, where we have used polar coordinates (r, θ). Given $0 \leqslant H < \frac{1}{2}\mathrm{Ch}(M, \mu)$, the estimate (5.13) implies that the causality condition (5.10) holds for all $0 < \rho < R$, so the above construction (Case 1) provides an entire H-graph for all $H < \frac{1}{2}\mathrm{Ch}(M, \mu)$. It follows that $H_0 = \frac{1}{2}\mathrm{Ch}(M, \mu)$. □

Remark 5.11. The H-cigars are tangent to a vertical cylinder at infinity of equation $x^2 + y^2 = \rho_H^2$. This vertical cylinder (which is homogeneous as a surface of $\mathbb{E}(M, 0, \mu)$) has the same constant mean curvature H as shown in the proof (Case 3). One can see the H-cigars as solutions to a Jenkins–Serrin problem for H-surfaces in $\mathbb{E}(M, 0, \mu)$ with just one boundary component.

Remark 5.12. The proof shows indirectly that the Cheeger constant can be obtained explicitly from the radial geometric data λ and μ as

$$\mathrm{Ch}(M, \mu) = \inf_{0<\rho<R} \frac{\rho\, \lambda(\rho)\mu(\rho)}{\int_0^\rho s\, \lambda(s)^2 \mu(s)\, ds}.$$

The inequality $\leqslant$ follows directly from the computations for disks D_ρ in the proof of Theorem 5.10. Assume by contradiction that a strict inequality holds. In that case, there exists $H > 0$ such that

$$\mathrm{Ch}(M, \mu) < 2H < \inf_{0<\rho<R} \frac{\rho\, \lambda(\rho)\mu(\rho)}{\int_0^\rho s\lambda(s)^2 \mu(s)\, ds}.$$

In particular, (5.10) is satisfied for all $0 < \rho < R$, so $z = 0$ in $\mathbb{L}(M, H, \mu^{-1})$ defines an entire space-like maximal graph, and its twin graph in $\mathbb{E}(M, 0, \mu)$ has constant mean curvature $H > \frac{1}{2}\mathrm{Ch}(M, \mu)$, in contradiction with item (a) of Theorem 5.9.

APPENDIX

A LERAY–SCHAUDER THEORY FOR QUASILINEAR ELLIPTIC EQUATIONS

In this appendix we deal with the existence of classical solutions to the Dirichlet problem

$$\begin{cases} Q[u] = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{A.1}$$

where Q is a second order quasilinear elliptic operator, f is a sufficiently regular function on $\partial\Omega$ and $\Omega \subset \mathbb{R}^n$, with $n \geqslant 2$, is a bounded domain. So, Q is an operator of the form

$$Q[u] = a^{ij}(x, u, \nabla u)\partial_{ij}u + b(x, u, \nabla u) \tag{A.2}$$

where $x = (x_1, \ldots, x_n) \in \Omega$ for some domain $\Omega \subset \mathbb{R}^n$. The function u is assumed to be $\mathcal{C}^2(\Omega)$, so the matrix $\left\{a^{ij}\right\}_{i,j=1}^{n}$ is symmetric. We assume that the coefficients $a^{ij}(x, z, p)$ and $b(x, z, p)$ of Q are defined for $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n = \mathcal{U}$ and we denote by $\lambda(x, z, p)$ and $\Lambda(x, z, p)$ the minimum and maximum eigenvalues of the coefficient matrix $\left\{a^{ij}(x, z, p)\right\}_{i,j}$ respectively.

Definition A.1 (Ellipticity). *Let* Q *be the operator defined by* (A.2). *We say that* Q *is elliptic in* Ω *if the coefficient matrix* $\left\{a^{ij}(x, z, p)\right\}_{i,j}$ *is positive definite for every* $(x, z, p) \in \mathcal{U}$, *that is,*

$$0 < \lambda(x, z, p)|\xi|^1 \leqslant a^{ij}(x, z, p)\xi_i\xi_j \leqslant \Lambda(x, z, p)|\xi|^2$$

for every $\xi \in \mathbb{R}^2 \setminus \{0\}$ *and every* $(x, z, p) \in \mathcal{U}$. *If* $u \in \mathcal{C}^1(\Omega)$ *and the matrix* $\left\{a^{ij}(x, u\nabla u)\right\}_{i,j}$ *is positive definite, we say* Q *is elliptic with respect to* u.

This theory was pioneered by Leray and Schauder in the 1930s: at its heart is the Leray–Schauder fixed point theorem which allows us to establish existence of solutions to PDEs from a priori estimates. The essence of the Leray–Schauder existence theorem is as follows: we embed the Dirichlet problem

(A.1) into a family of related problems of the same type, depending on a parameter $\sigma \in [0,1]$, say

$$\begin{cases} Q_\sigma[u] = 0 & \text{in } \Omega, \\ u = \sigma f & \text{on } \partial\Omega. \end{cases} \tag{A.3}$$

The theorem asserts that (A.1) has a solution if for some $\beta \in (0,1)$, there exists a positive constant M such that, for each σ, every solution u of (A.3) satisfies the bound

$$\|u\|_{\mathcal{C}^{1,\beta}(\bar{\Omega})} \leqslant M.$$

Thus, the problem has been reduced to estimating Hölder norms of solutions of second order quasilinear elliptic equations, assuming such solutions exist. In particular, noticing that

$$\begin{aligned} \|u\|_{\mathcal{C}^{1,\beta}(\bar{\Omega})} &= \sup_\Omega |u| + \sup_\Omega \sup_{|\gamma|=1} |D^\gamma u| + [Du]_{\beta,\Omega} \\ &\leqslant \sup_\Omega |u| + \sup_\Omega |\nabla u| + [Du]_{\beta,\Omega}, \end{aligned}$$

it will be sufficient to estimate $\sup_\Omega |u|$, $\sup_\Omega |\nabla u|$ and

$$[Du]_{\beta,\Omega} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|Du(x) - Du(y)\|}{|x-y|^\beta}.$$

For our purpose, we focus on the case where Q is of divergence form, that is,

$$Q[u] = \operatorname{div} A(x,u,\nabla u) + b(x,u,\nabla u), \tag{A.4}$$

where the vector function $A \in \mathcal{C}^1(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ and $b \in \mathcal{C}^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. Whenever we are in this case, it will be sufficient to just prove the first two estimates, since we can apply [GilTru01, Theorem 13.2] that states what follows.

Theorem A.2 . *Let $u \in \mathcal{C}^2(\bar{\Omega})$ satisfy $Q[u] = 0$ in Ω, where Q is elliptic in $\bar{\Omega}$ and it is of divergence form and let $f \in \mathcal{C}^2(\bar{\Omega})$. Then, if $\partial\Omega \in \mathcal{C}^2$ and $u = f$ on $\partial\Omega$, we have the estimate*

$$[Du]_{\alpha,\Omega} \leqslant C,$$

where $C = C(\sup_\Omega |u|, \sup_\Omega |\nabla u|, a^{ij}, b, \lambda, \Lambda)$ and $\alpha = \alpha(\Omega, \lambda, \Lambda, n) > 0$.

In the subsequent sections we provide an outline of the theory formulated by Leray and Schauder. Our focus here is to give a broad overview without delving deeply into proof intricacies. Each proof reference will be cited for those seeking more specific details.

A.1 The Leray-Schauder fixed point theorem

The Leray-Schauder existence theorem is based on a generalization of the classical result known as Brouwer Fixed Point Theorem:

Theorem A.3 (Brouwer Fixed Point). *Let* $T\colon B \to B$ *be a continuous map of the closed unit ball* $B \subset \mathbb{R}^n$ *into itself. Then* T *has a fixed point.*

Recalling that a compact map between two Banach spaces maps bounded sets to precompact sets, the Leray-Schauder fixed point theorem can be stated as follows (see [GilTru01, Theorem 11.6]).

Theorem A.4 (Leray-Schauder fixed point theorem). *Let* $\mathcal{B}$ *be a Banach space and* $T\colon \mathcal{B} \times [0,1] \to \mathcal{B}$ *be a compact map such that*

- $T(x,0) = 0$ *for each* $x \in \mathcal{B}$, *and*
- *there exists a constant* $M > 0$ *such that for each pair* $(x,\sigma) \in \mathcal{B} \times [0,1]$ *which satisfies* $x = T(x,\sigma)$, *we have*

$$\|x\| < M. \tag{A.5}$$

Then, the map

$$\begin{aligned} T_1\colon \mathcal{B} &\to \mathcal{B} \\ y &\mapsto T(y,1) \end{aligned}$$

has a fixed point.

The very first result needed to prove the Leray–Schauder fixed point theorem is a generalization of the Brouwer theorem to Banach spaces (see [GilTru01, Theorem 11.1]):

Theorem A.5 . *Let* K *be a compact convex set in a Banach space* $\mathcal{B}$ *and* $T\colon K \to K$ *be continuous. Then* T *has a fixed point.*

For a later purpose, this result can be extended as follows.

Corollary A.6 . *Let* K *be a closed convex set in a Banach space* $\mathcal{B}$ *and* $T\colon K \to K$ *be a continuous map such that* $T(K)$ *is precompact. Then* T *has a fixed point.*

Proof. We will find a compact convex subset $A \subseteq K$ such that $T(A) \subseteq A$. Then, the previous theorem implies that T has a fixed point in A, and hence K. Indeed, let A be the convex hull of $\overline{T(K)}$. Certainly A is convex, and since the convex hull of a compact set is itself compact, A is compact. Moreover, $A \subseteq K$ because $T(K) \subseteq K$ and K is closed, so $\overline{T(K)} \subseteq K$, but K is convex by assumption so $A \subseteq K$. Thus

$$T_{|A}\colon A \to T(A) \subseteq T(K) \subseteq \overline{T(K)} \subseteq A,$$

so T maps A into itself and we are done. □

Before proving Theorem A.4, we need the following lemma.

Lemma A.7 . *Let* $\mathcal{B}$ *be a Banach space with open unit ball* B. *Suppose* $T\colon \overline{B} \to \mathcal{B}$ *is a continuous map such that*

1. $T(B)$ *is precompact, and*
2. $T(\partial B) \subseteq B$.

Then T *has a fixed point.*

Proof. Define the map $T^*\colon \mathcal{B} \to \overline{B}$ such that

$$T^*(x) = \begin{cases} T(x), & \text{if } \|T(x)\| \leqslant 1; \\ \frac{T(x)}{\|T(x)\|}, & \text{if } \|T(x)\| \geqslant 1. \end{cases}$$

It is clear that T^* is continuous, and that T^* maps $\overline{B}$ into itself. Moreover, if $T(\overline{B})$ is precompact, then also $T^*(\overline{B})$ is precompact. Indeed,

$$T^*(\overline{B}) = I_1 \cup I_2,$$

where $I_1 = T\left(\left\{x \in \overline{B}\colon \|T(x)\| \leqslant 1\right\}\right)$ and $I_2 = \left\{\frac{T(x)}{\|T(x)\|}\colon x \in \overline{B},\ \|T(x)\| \geqslant 1\right\}$. Now $\overline{T^*(\overline{B})} \subseteq \overline{I_1} \cup \overline{I_2}$, and the former is closed, so to show compactness of

$\overline{T^*(\overline{B})}$, it is enough to show that $\overline{I_1} \cup \overline{I_2}$ is compact. As a finite union of compact sets is compact, we need only to show that $\overline{I_1}$ and $\overline{I_2}$ are compact. The first one is obviously compact since it is a closed subset of $\overline{T(\overline{B})}$, that is compact. To show that $\overline{I_2}$ is compact, let $\{p_i\}$ be a sequence in I_2. Two possible cases arise:

1. either infinitely many $p_i \in I_2$ or
2. there are only finitely many $p_i \in I_2$.

In the first case we can consider a subsequence, which we still denote $\{p_i\}$, such that each $p_i \in I_2$. Then for each i, there exists $x_i \in \overline{B}$ such that $\|T(x)\| \geqslant 1$ and $p_i = \frac{T(x_i)}{\|T(x_i)\|}$. So $T(x_i) \in T(\overline{B})$ and $T(\overline{B})$ is precompact, so there is a subsequence $T(x_{i_k})$ which converges to some $z \in \overline{T(\overline{B})}$, and moreover $\|z\| \geqslant 1$. So $p_{i_k} \to \frac{z}{\|z\|}$, and this limit is in $\overline{I_2}$, since $\overline{I_2}$ is closed.

In the second case, without loss of generality, we may assume $\{p_i\} \subset \partial I_2$. Now

$$\partial I_2 \subset \left\{ \frac{T(x)}{\|T(x)\|} : x \in \overline{B},\ \|T(x)\| = 1 \right\} \cup \left\{ \frac{T(x)}{\|T(x)\|} : x \in \partial B,\ \|T(x)\| \geqslant 1 \right\}.$$

But by assumption $T(\partial B) \subseteq B$, so that the rightmost set above is empty. So $\{p_i\} \subseteq \left\{ \frac{T(x)}{\|T(x)\|} : x \in \overline{B},\ \|T(x)\| = 1 \right\} \subseteq I_1 \subseteq \overline{I_1}$, which is compact by the above. So $\{p_i\}$ has a convergent subsequence, with limit $p \in \overline{I_1}$ say. But $p_i \in \partial I_2$ for each i and ∂I_2 is closed, so $p \in \partial I_2 \subset \overline{I_2}$. So in either case, $\{p_i\}$ has a convergent subsequence in $\overline{I_2}$, so $\overline{I_2}$ is compact, as desired.

So we conclude that $T^*(\overline{B})$ is precompact, so by Corollary A.6, T^* has a fixed point $x \in \overline{B}$. Since $T(\partial B) \subseteq B$, then $x \notin \partial B$ and so $x \in B$. Therefore, $\|T^*(x)\| = \|x\| < 1$, so by definition of T^*, we must have $\|T(x)\| < 1$, and hence $Tx = T^*x = x$, so that x is a fixed point for T. □

Finally, we have all the ingredients to prove Theorem A.4.

Proof of the Theorem A.4. Without loss of generality, we may assume $M = 1$. Otherwise just rescale the norm on $\mathcal{B}$ by a factor of $1/M$. For $0 < \epsilon < 1$, define $T^*_\epsilon \colon \overline{B} \to \mathcal{B}$ such that

$$T^*_\epsilon(x) = \begin{cases} T\left(\frac{x}{\|x\|}, \frac{1-\|x\|}{\epsilon}\right), & \text{if } 1-\epsilon \leqslant \|x\| \leqslant 1, \\ T\left(\frac{x}{1-\epsilon}, 1\right), & \text{if } \|x\| \leqslant 1-\epsilon, \end{cases}$$

where B denotes the open unit ball around 0 in $\mathcal{B}$. Certainly T^*_ϵ is continuous, and by compactness of T, similarly to the proof of the previous lemma, $T^*_\epsilon(\overline{B})$ is precompact. Moreover, since $\|x\| = 1$ for $x \in \partial B$, we have $T^*_\epsilon(x) = T\left(\frac{x}{\|x\|}, 0\right) = 0$ by hypotheses, so $T^*_\epsilon(\partial B) = \{0\} \subset B$. So we may apply the previous lemma to conclude that T^*_ϵ has a fixed point which we denote x_ϵ.

Now take $\epsilon = \frac{1}{k}$ for $k = 2, 3, \ldots$. So, $T^*_{1/k}$ has a fixed point $x_{1/k}$. Denote

$$\sigma_k \colon = \begin{cases} k(1 - \|x_{1/k}\|), & \text{if } 1 - \frac{1}{k} \leqslant \|x_{1/k}\| \leqslant 1, \\ 1, & \text{if } \|x_{1/k}\| < 1 - \frac{1}{k}. \end{cases}$$

Set $A = \left\{(x_{1/k}, \sigma_k) \colon k \geqslant 2\right\}$. By compactness of T, we may assume there is a subsequence of A, which we still denote $\left\{(x_{1/k}, \sigma_k)\right\}$, which converges to some $(x, \sigma) \in \mathcal{B} \times [0, 1]$.

Suppose $\sigma < 1$. Then for large enough k, $\sigma_k < 1$ so that $\|x_{1/k}\| \geqslant 1 - \frac{1}{k}$ (the inequality must be strict since otherwise $\sigma_k = 1$). So $\|x_{1/k}\| \to 1$, and so $\|x\| = 1$. But $\|x_{1/k}\| = 1$ implies that $x_{1/k} = T^*_{1/k}(x_{1/k}) = T\left(\frac{x_{1/k}}{\|x_{1/k}\|}, \sigma_k\right) \to T(x, \sigma)$ by continuity of T. So $x = T(x, \sigma)$ and $\|x\| = 1$, which contradicts (A.5). Hence $\sigma = 1$. Now, by continuity of T, we have $x_{1/k} = T^*_{1/k}(x_{1/k}) \to T(x, 1)$. But $x_{1/k} \to x$, so x is a fixed point of T_1, as required. □

A.2 The Leray–Schauder existence theorem

Throughout this section, Ω will denote a bounded set in $\mathbb{R}^n$ with boundary $\partial\Omega \in \mathcal{C}^{2,\alpha}$ and $f \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ is a given function. We assume the operator Q to be defined on $\mathcal{C}^2(\Omega)$ and the functions $a^{ij}, b \in \mathcal{C}^\alpha(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ for some $\alpha \in (0, 1)$. To solve the Dirichlet problem (A.1), we embed it in a family of problems (A.3), where $\sigma \in [0, 1]$ and

$$Q_\sigma[u] = a^{ij}(x, u, \nabla u; \sigma)\partial_{ij}u + b(x, u, \nabla u; \sigma), \tag{A.6}$$

satisfying the following assumptions:

1. $Q_1 = Q$,
2. $b(x, u, \nabla u; 0) = 0$ for each $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$,
3. Q_σ is elliptic in $\bar{\Omega}$ for each $\sigma \in [0, 1]$, and

4. $a^{ij}(\cdot;\sigma), b(\cdot;\sigma) \in \mathcal{C}^{\alpha}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ for each $\sigma \in [0,1]$, and the maps

$$a^{ij}(x,z,p;\cdot), b(x,z,p;\cdot)\colon [0,1] \to \mathcal{C}^{\alpha}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$$

are continuous.

We want to apply Theorem A.4, so we start choosing a Banach space and defining an operator T. Let $\beta \in (0,1)$, choose $\mathcal{B}$ to be the Banach space $\mathcal{C}^{1,\beta}(\bar{\Omega})$ and define the operator

$$\begin{array}{rccc} T\colon & \mathcal{C}^{1,\beta}(\bar{\Omega}) \times [0,1] & \to & \mathcal{C}^{2,\alpha\beta}(\bar{\Omega}) \subset \mathcal{C}^{1,\beta}(\bar{\Omega}) \\ & (v,\sigma) & \mapsto & u \end{array}, \tag{A.7}$$

where $u = T(v,\sigma)$ is the unique solution of the linear elliptic Dirichlet problem

$$\begin{cases} a^{ij}(x,v,\nabla v;\sigma)\partial_{ij}u + b(x,v,\nabla v;\sigma) = 0 & \text{in } \Omega, \\ \hfill u = \sigma f & \text{on } \partial\Omega. \end{cases} \tag{A.8}$$

Note that the existence of a unique $\mathcal{C}^{2,\alpha\beta}(\bar{\Omega})$ solution is guaranteed by the theory for linear, strictly elliptic operators. Indeed, $v \in \mathcal{C}^{1,\beta}(\bar{\Omega})$ implies that $\nabla v \in \mathcal{C}^{\beta}(\bar{\Omega})$, so that the coefficients $\tilde{a}^{ij}(x) = a^{ij}(x,v(x),\nabla v(x);\sigma)$ and $\tilde{b}(x) = b(x,v(x),\nabla v(x);\sigma)$ satisfy $\tilde{a}^{ij}, \tilde{b} \in \mathcal{C}^{\alpha\beta}(\bar{\Omega})$. Since $\alpha\beta < \alpha$, we have $\partial\Omega \in \mathcal{C}^{2,\alpha\beta}$ and $f \in \mathcal{C}^{2,\alpha\beta}(\bar{\Omega})$. Hence, applying the following theorem, whose proof can be found in [GilTru01, Theorem 6.14], we see that (A.8) has a unique solution in $\mathcal{C}^{2,\alpha\beta}(\bar{\Omega})$.

Theorem A.8 . *Let Ω be a $\mathcal{C}^{2,\alpha}$ domain in $\mathbb{R}^n$ and*

$$L[u] = a^{ij}\,\partial_{ij}u + b^i\,\partial_i u + c$$

be a strictly elliptic operator in Ω with $a^{ij}, b^i, c \in \mathcal{C}^{\alpha}(\bar{\Omega})$ and $c \leqslant 0$. Then, for any $f \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ and $h \in \mathcal{C}^{\alpha}(\bar{\Omega})$, the Dirichlet problem

$$\begin{cases} L[u] = h & \textit{in } \Omega, \\ \quad u = f & \textit{on } \partial\Omega, \end{cases}$$

admits a unique solution in $\mathcal{C}^{2,\alpha}(\bar{\Omega})$.

So the operator T defined in (A.7) is well-defined. From $Q_1 = Q$. listed above, solvability of (A.1) is equivalent to the existence of a fixed point $u \in \mathcal{C}^{1,\beta}(\bar{\Omega})$ for the map

$$\begin{array}{rccc} T_1\colon & \mathcal{C}^{1,\beta}(\bar{\Omega}) & \to & \mathcal{C}^{1,\beta}(\bar{\Omega}) \\ & v & \mapsto & T(v,1) \end{array}.$$

We are now ready to prove the Leray–Schauder existence theorem (see [GilTru01, Theorem 11.4]).

Theorem A.9 (Leray–Schauder existence theorem). *Let* $0 < \alpha < 1$. *Suppose that*

- $\Omega \subset \mathbb{R}^n$ *is a bounded domain with* $\partial\Omega \in \mathcal{C}^{2,\alpha}$*; and*
- $f \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$.

Let $\{Q_\sigma\colon \sigma \in [0,1]\}$ *be the family of operators defined in* (A.6)*, satisfying conditions 1.-4. above. Suppose that for some* $\beta \in (0,1)$ *there exists a constant* $M > 0$ *such that for every* $\sigma \in [0,1]$*, every* $\mathcal{C}^{2,\alpha}(\bar{\Omega})$ *solution of* u *of*

$$\begin{cases} Q_\sigma[u] = 0 & \text{in } \Omega, \\ u = \sigma f & \text{on } \partial\Omega \end{cases}$$

satisfies $\|u\|_{\mathcal{C}^{1,\beta}(\bar{\Omega})} < M$. *Then the Dirichlet problem*

$$\begin{cases} Q[u] = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

has a solution in $\mathcal{C}^{2,\alpha}(\bar{\Omega})$.

Proof. In view of the comments preceding the theorem, it is enough to show that the operator T defined in (A.7) satisfies the hypotheses of Theorem A.4. So, we have reduced the proof to checking properties of T. Since the bound in Theorem A.4 is assumed to hold in our hypotheses, we need only to check that

1. $T(v,0) = 0$ for each $v \in \mathcal{C}^{1,\beta}(\bar{\Omega})$;
2. T is compact and continuous.

The first property is easy to see. Indeed, let $v \in \mathcal{C}^{1,\beta}(\bar{\Omega})$. Condition 2 above ensures $b(x, v, \nabla v; 0) = 0$ and then $T(v, 0) = u \equiv 0$ is the unique solution of (A.8).

To show compactness of T, we first show that T maps bounded sets in $\mathcal{C}^{1,\beta}(\bar{\Omega}) \times [0, 1]$ to precompact sets in $\mathcal{C}^{1,\beta}$ and $\mathcal{C}^2$, and then use the latter to show that T is continuous.

We will first use the global Schauder estimates ([GilTru01, Theorem 6.6]) to show that T maps bounded sets to bounded sets. In particular, for $v \in \mathcal{C}^{1,\beta}(\bar{\Omega})$, applying the global Schauder estimate to $u = T(v, \sigma)$, we get that

$$\begin{aligned} |T(v,\sigma)|_{2,\alpha\beta,\Omega} &\leqslant C(|T(v,\sigma)|_{0,\Omega} + \sigma|f|_{2,\alpha\beta,\Omega} + |b(\cdot, v\nabla v;\sigma)|_{0,\alpha\beta,\Omega}) \\ &= C\left(\sup_{\Omega}|T(v,\sigma)| + \sigma\|f\|_{\mathcal{C}^{2,\alpha\beta}(\bar{\Omega})} + |b(\cdot, v\nabla v;\sigma)|_{0,\alpha\beta,\Omega}\right) \end{aligned}$$

where $C = C\left(n, \alpha\beta, \lambda, \sup\left(\|a^{ij}\|_{\mathcal{C}(\bar{\Omega})} + [a^{ij}]_{\alpha\beta,\Omega}\right), \Omega\right)$. The first term on the right-hand side is bounded in terms of the boundary data f by the maximum principle (for linear elliptic operators), see [GilTru01, Theorem 3.7]. Furthermore, the second term is bounded by hypotheses since $f \in \mathcal{C}^{2,\alpha}(\bar{\Omega}) \subset \mathcal{C}^{2,\alpha\beta}(\bar{\Omega})$. So using condition 4 for the third term, we see that T maps bounded sets in $\mathcal{C}^{1,\beta}(\bar{\Omega}) \times [0, 1]$ to bounded sets in $\mathcal{C}^{2,\alpha\beta}(\bar{\Omega})$. Finally, by the Arzela–Ascoli theorem, these bounded $\mathcal{C}^{2,\alpha\beta}(\bar{\Omega})$ sets are precompact in $\mathcal{C}^{1,\beta}(\bar{\Omega})$ and $\mathcal{C}^2(\bar{\Omega})$.

To prove continuity of T, we suppose $(v_n, \sigma_n) \to (v, \sigma)$ in $\mathcal{C}^{1,\beta}(\bar{\Omega})$, and show $T(v_n, \sigma_n) \to T(v, \sigma)$. Note that $\{(v_n, \sigma_n)\}_n$ is convergent for $n \to \infty$, hence bounded, so it follows from above that $\{T(v_n, \sigma_n)\}_n$ is precompact in $\mathcal{C}^2(\bar{\Omega})$. Thus every subsequence of $\{T(v_n, \sigma_n)\}_n$ has a convergent subsequence. We let $\{T(v_{n_k}, \sigma_{n_k})\}_k$ denote any such convergent subsequence, and let

$$u := \lim_{k\to\infty} T(v_{n_k}, \sigma_{n_k}).$$

Hence,

$$\begin{aligned} &a^{ij}(x, v, \nabla v; \sigma)\partial_{ij}u + b(x, v, \nabla v; \sigma) \\ =&\lim_{k\to\infty} a^{ij}(x, v_{n_k}, \nabla v_{n_k}; \sigma_{n_k})\partial_{ij}T(v_{n_k}, \sigma_{n_k}) + b(x, v_{n_k}, \nabla v_{n_k}; \sigma_{n_k}) \\ =&0, \end{aligned}$$

where we have used continuity of the coefficients (condition 4 above) for the first equality. Moreover, since $\sigma_{n_k} \to \sigma$, on $\partial\Omega$ we have $T(v_{n_k}, \sigma_{n_k}) = \sigma_{n_k} f \to \sigma f$, so that $u = \sigma f$ on $\partial\Omega$. Hence, by uniqueness of solutions to the Dirichlet problem (A.8), we have $u = T(v, \sigma)$. Since this holds for every such sequence $\{(v_{n_k}, \sigma_{n_k})\}$, we have that $T(v_n, \sigma_n) \to T(v, \sigma)$. □

Remark A.10. Note that the regularity of the solution to the quasi-linear Dirichlet problem is related to the regularity of the solution for the linear Dirichlet problem. Moreover, the regularity of the solution to the linear Dirichlet problem relies on the regularity of the coefficients of the operator. In particular, if these coefficients belong to $\mathcal{C}^k(\bar{\Omega})$, then the solution attains a level of smoothness of $\mathcal{C}^{k+2}(\bar{\Omega})$, as demonstrated in [GilTru01, Theorem 7.11, Theorem 8.10, and Corollary 8.11]. This same principle extends to the quasi-linear case.

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