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INTEGRAL FORM OF AFFINE KAC MOODY ALGEBRAS

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"Au milieu de l'hiver, j'apprenais enfin qu'il y avait en moi un été invincible."
Albert Camus

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Introduction

We use the following notations: $\mathbb{N}=\{n\in\mathbb{Z}\mid n\geq 0\}$, $\mathbb{Z}_+=\{n\in\mathbb{Z}\mid n>0\}$.

Let $X_n^{(k)}$ be an affine Kac-Moody algebra (see Chapter 2) and $\mathcal{U}$ its universal enveloping algebra. The aim of this work is to give a basis over $\mathbb{Z}$ of the $\mathbb{Z}$-subalgebra of $\mathcal{U}$ generated by the divided powers of the Drinfeld generators (see Definition 2.4), thus proving that this $\mathbb{Z}$-subalgebra is an integral form of $\mathcal{U}$. The integral forms for finite dimensional simple Lie algebras were first introduced by Chevalley in [2] for the study of the Chevalley groups and of their representation theory. The construction of the "divided power"-$\mathbb{Z}$-form for the simple finite dimensional Lie algebras is due to Kostant (see [9]); it has been generalized to the untwisted affine Kac-Moody algebras by Garland in [6] (see Section 2.3), as we shall quickly recall. Given a simple Lie algebra $\mathfrak{g}_0$ and the corresponding untwisted affine Kac-Moody algebra $\mathfrak{g}=\mathfrak{g}_0\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c$ provided with an (ordered) Chevalley basis, the $\mathbb{Z}$-subalgebra $\mathcal{U}_{\mathbb{Z}}$ of $\mathcal{U}=\mathcal{U}(\mathfrak{g})$ generated by the divided powers of the real root vectors is an integral form of $\mathcal{U}$; a $\mathbb{Z}$-basis of this integral form (hence its $\mathbb{Z}$-module structure) can be described by decomposing $\mathcal{U}_{\mathbb{Z}}$ as tensor product of its $\mathbb{Z}$-subalgebras relative respectively to the real root vectors ($\mathcal{U}_{\mathbb{Z}}^{re,+}$ and $\mathcal{U}_{\mathbb{Z}}^{re,-}$), to the imaginary root vectors ($\mathcal{U}_{\mathbb{Z}}^{im,+}$ and $\mathcal{U}_{\mathbb{Z}}^{im,-}$) and to the Cartan subalgebra ($\mathcal{U}_{\mathbb{Z}}^{\mathfrak{h}}$): $\mathcal{U}_{\mathbb{Z}}^{re,+}$ has a basis $B^{re,+}$ consisting of the (finite) ordered products of divided powers of the distinct positive real root vectors and $(\mathcal{U}_{\mathbb{Z}}^{re,-},B^{re,-})$ can be described in the same way:

$$B^{re,\pm}=\{x_{\pm\beta_1}^{(k_{\beta_1})}\cdot\ldots\cdot x_{\pm\beta_N}^{(k_{\beta_N})}\mid N\geq 0,\ \beta_1>\ldots>\beta_N>0 \text{ real roots},\ k_{\beta_j}>0\ \forall j\}.$$

Here a real root β of $\mathfrak{g}$ is said to be positive if there exists a positive root α of $\mathfrak{g}_0$ such that either $\beta=\alpha$ or $\beta-\alpha$ is imaginary; x_β is the Chevalley generator corresponding to the real root β. A basis $B^{\mathfrak{h}}$ of $\mathcal{U}_{\mathbb{Z}}^{\mathfrak{h}}$, which is commutative, consists of the products of the "binomials" of the (Chevalley) generators h_i ($i\in I$) of the Cartan subalgebra of $\mathfrak{g}$:

$$B^{\mathfrak{h}}=\Big\{\prod_i\binom{h_i}{k_i}\mid k_i\geq 0\forall\ i\Big\};$$

it is worth remarking that $\mathcal{U}_{\mathbb{Z}}^{\mathfrak{h}}$ is not an algebra of polynomials. $\mathcal{U}_{\mathbb{Z}}^{im,+}$ (and its symmetric $\mathcal{U}_{\mathbb{Z}}^{im,-}$) is commutative, too; as a $\mathbb{Z}$-module it is isomorphic to the tensor product of the $\mathcal{U}_{i,\mathbb{Z}}^{im,+}$'s (each factor corresponding to the i^{th} copy of $\mathcal{U}(\hat{\mathfrak{sl}}_2)$ inside $\mathcal{U}$), so that it is enough to describe it in the rank 1 case: the basis $B^{im,+}$ of $\mathcal{U}_{\mathbb{Z}}^{im,+}(\hat{\mathfrak{sl}}_2)$ provided by Garland can be described as a set of finite products of the elements $\Lambda_k(\xi(m))$ ($k\in\mathbb{N}$, $m>0$), where the $\Lambda_k(\xi(m))$'s ($k\geq -1, m>0$) are the elements of $\mathcal{U}^{im,+}=\mathbb{C}[h_r(=h\otimes t^r)\mid r>0]$ defined recursively (for all $m\neq 0$) by

$$\Lambda_{-1}(\xi(m))=1,\ \ k\Lambda_{k-1}(\xi(m))=\sum_{\substack{r\geq 0,s>0\\ r+s=k}}\Lambda_{r-1}(\xi(m))h_{ms}:$$

$$B^{im,+}=\left\{\prod_{m>0}\Lambda_{k_m-1}(\xi(m))\mid k_m\geq 0\ \forall m,\ \#\{m>0\mid k_m\neq 0\}<\infty\right\}.$$

It is not clear from this description that $\mathcal{U}_{\mathbb{Z}}^{im,+}$ and $\mathcal{U}_{\mathbb{Z}}^{im,-}$ are algebras of polynomials.
Thanks to the isomorphism of $\mathbb{Z}$-modules

$$\mathcal{U}_{\mathbb{Z}} \cong \mathcal{U}_{\mathbb{Z}}^{re,-} \otimes_{\mathbb{Z}} \mathcal{U}_{\mathbb{Z}}^{im,-} \otimes_{\mathbb{Z}} \mathcal{U}_{\mathbb{Z}}^{\mathfrak{h}} \otimes_{\mathbb{Z}} \mathcal{U}_{\mathbb{Z}}^{im,+} \otimes_{\mathbb{Z}} \mathcal{U}_{\mathbb{Z}}^{re,+}$$

a $\mathbb{Z}$-basis B of $\mathcal{U}_{\mathbb{Z}}$ is produced as multiplication of $\mathbb{Z}$-bases of these subalgebras:

$$B = B^{re,-} B^{im,-} B^{\mathfrak{h}} B^{im,+} B^{re,+}.$$

The same result has been proved for all the twisted affine Kac-Moody algebras by Mitzman in [11], where the author provides a deeper comprehension and a compact description of the commutation formulas by means of a drastic simplification of both the relations and their proofs. This goal is achieved remarking that the generating series of the elements involved in the basis can be expressed as suitable exponentials, observation that allows to apply very general tools of calculus, such as the well known properties

$$x \exp(y) = \exp(y) \exp([\cdot, y])(x)$$

if $\exp(y)$ and $\exp([\cdot, y])(x)$ are well defined, and

$$D(\exp(f)) = D(f) \exp(f)$$

if D is a derivation such that $[D(f), f] = 0$.
Here, too, it is not yet clear that $\mathcal{U}_{\mathbb{Z}}^{im,\pm}$ are algebras of polynomials.
However this property, namely In Fisher-Vasta's PhD thesis ([5] and see Section 2.3) is stated that $\mathcal{U}_{\mathbb{Z}}^{im,+} = \mathbb{Z}[\Lambda_{k-1} = \Lambda_{k-1}(\xi(1,1)) = p_{k,1} \mid k > 0]$, where the author describes the results of Garland for the untwisted case and of Mitzman for $A_2^{(2)}$ aiming at a better understanding of the commutation formulas. Yet the proof is missing: the theorem describing the integral form is based on observations which seem to forget some necessary commutations, those between $(x_r^+)^{(k)}$ and $(x_s^-)^{(l)}$ when $|r+s| > 1$; in [5] only the cases $r+s=0$ and $r+s=\pm 1$ are considered, the former producing the binomials appearing in $B^{\mathfrak{h}}$, the latter producing the elements $p_{n,1}$ (and their corresponding negative elements in $\mathcal{U}_{\mathbb{Z}}^{im,-}$).
Comparing the Kac-Moody presentation of the affine Kac-Moody algebras with its "Drinfeld" presentation as current algebra, one can notice a difference between the untwisted and twisted case, which is at the origin of our work. As in the simple finite dimensional case, also in the affine cases the generators of $\mathcal{U}_{\mathbb{Z}}$ described above are redundant: the $\mathbb{Z}$-subalgebra of $\mathcal{U}$ generated by $\{e_i^{(k)}, f_i^{(k)} \mid i \in I,\ k \in \mathbb{N}\}$, obviously contained in $\mathcal{U}_{\mathbb{Z}}$, is actually equal to $\mathcal{U}_{\mathbb{Z}}$.
On the other hand, the situation changes when we move to the Drinfeld presentation and study the $\mathbb{Z}$-subalgebra ${}^*\mathcal{U}_{\mathbb{Z}}$ of $\mathcal{U}$ generated by the divided powers of the Drinfeld generators $(x_{i,r}^{\pm})^{(k)}$: indeed, while in the untwisted case it is still true that $\mathcal{U}_{\mathbb{Z}} = {}^*\mathcal{U}_{\mathbb{Z}}$ and (also in the twisted case) it is always true that ${}^*\mathcal{U}_{\mathbb{Z}} \subseteq \mathcal{U}_{\mathbb{Z}}$, in general we get two different $\mathbb{Z}$-subalgebras of $\mathcal{U}$; more precisely ${}^*\mathcal{U}_{\mathbb{Z}} \subsetneq \mathcal{U}_{\mathbb{Z}}$ in case $A_{2n}^{(2)}$, that is when there exists a vertex i whose corresponding rank 1 subalgebra is not a copy of $\mathcal{U}(\hat{\mathfrak{sl}}_2)$ but is a copy of $\mathcal{U}(\hat{\mathfrak{sl}}_3{}^{\chi})$.
Thus in order to complete the description of ${}^*\mathcal{U}_{\mathbb{Z}}$ we need to study the case of $A_{2n}^{(2)}$.
In the present work we prove that the $\mathbb{Z}$-subalgebra generated by

$$\{(x_{i,r}^{+})^{(k)}, (x_{i,r}^{-})^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}\}$$

is an integral form of the enveloping algebra also in the case of $A_{2n}^{(2)}$, we exhibit a basis generalizing the one provided in [6] and in [11] and determine the commutation relations in a compact yet explicit formulation (see Theorems 5.44, 8.39 and Appendix 9.A). We use the same approach as Mitzman's, with a further simplification consisting in the remark that an element of the form

$G(u,v)=\exp(xu)\exp(yv)$ is characterized by two properties: $G(0,v)=\exp(yv)$ and $\frac{dG}{du}=xG$. Moreover, studying the rank 1 cases we prove that, both in the untwisted and in the twisted case, $\mathcal{U}_{\mathbb{Z}}^{im,+}$ and ${}^*\mathcal{U}_{\mathbb{Z}}^{im,+}$ are algebras of polynomials: as stated in [5], the generators of $\mathcal{U}_{\mathbb{Z}}^{im,+}$ are the elements Λ_k introduced in [6] and [11] (see Proposition 1.13 and Remark 4.12); the generators of ${}^*\mathcal{U}_{\mathbb{Z}}^{im,+}$ in the case $A_2^{(2)}$ are elements defined formally as the Λ_k's after a deformation of the h_r's (see Definition 5.12 and Remark 5.13): describing ${}^*\mathcal{U}_{\mathbb{Z}}^{im,+}(\hat{\mathfrak{sl}}_3{}^{\chi})$ (denoted by $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}$) has been the hard part of this work. In the higher rank the situation changes: it is no longer true that $\mathcal{U}_{\mathbb{Z}}^{im,+}$ is an algebra of polynomial in the $A_{2n}^{(2)}$ case if $n>1$ (see Remark 1.39).
We work over $\mathbb{Q}$ and dedicate a preliminary particular care to the description of some integral forms of $\mathbb{Q}[x_i \mid i\in I]$ and of their properties and relations when they appear in some non commutative situations, properties that will be repeatedly used for the computations in $\mathfrak{g}$: fixing the notations helps to understand the construction in the correct setting. With analogous care we discuss the symmetries arising in $\hat{\mathfrak{sl}}_2$ and $\hat{\mathfrak{sl}}_3{}^{\chi}$. We chose to recall also the case of $\mathfrak{sl}_2$ and to give in a few lines the proof of the theorem describing its divided power integral form in order to present in this easy context the tools that will be used in the more complicated affine cases.
The thesis is organized as follows.
Chapter 1 is devoted to review the description of some integral forms of the algebra of polynomials, polynomials over $\mathbb{Z}$, divided powers ,"binomials" are described in Section 1.1, symmetric functions (see [10]) are described in Section 1.2: they are introduced together with their generating series as exponentials of suitable series with null constant term, and their properties are rigorously stated, thus preparing to their use in the Lie algebra setting. We have inserted here, in Proposition 1.13, a result about the stability of the symmetric functions with integral coefficients under the homomorphism λ_m mapping x_i to x_i^m ($m>0$ fixed), which is almost trivial in the symmetric function context; it is a straightforward consequence of this observation that $\mathcal{U}_{\mathbb{Z}}^{im,+}$ is an algebra of polynomials and so is ${}^*\mathcal{U}_{\mathbb{Z}}^{im,+}$ in the rank-1 twisted case. We also provide a direct, elementary proof of this proposition (see Proposition 1.14). Section 1.3 is devoted to the description of a $\mathbb{Z}$-basis of $\mathbb{Z}^{(sym)}[h_r \mid r>0]$ alternative to that introduced in the Example 1.2. $\mathbb{Z}^{(sym)}[h_r \mid r>0]$ is the algebra of polynomials $\mathbb{Z}[\hat{h}_k \mid k>0]$, and as such it has a $\mathbb{Z}$-basis consisting of the monomials in the $\hat{h}_k$'s, which is the one considered in our work. In Section 1.4 we study a very special case of generalized (sym)-functor that depends on certain sequences $d:\mathbb{Z}_+\to\mathbb{Q}$, that is $\mathbb{Z}(\hat{h}_k^d \mid k\in\mathbb{Z}_+)=\mathbb{Z}^{(sym)}[h_rd_r \mid r>0]$. In particular we deal with two special case, that is $d(r)=\frac{1+(-1)^r}{2}$ and $d(r)=2^{r-1}$, these two sequences will play a crucial role in the study of the Integral form of $A_2^{(2)}$, when $A_2^{(2)}$ is seen has copy of the first node of the Dynkin Diagram of $A_{2n}^{(2)}$ (see Chapter 7). In Section 1.5 we collect some computations in non commutative situations that we shall systematically refer to in the following chapters.
In Chapter 2 we recall the information that frames this work. More precisely in Section 2.1 we recall general definition about Kac Moody algebras, in Section 2.2 we recall the loop presentation of the affine Kac-Moody algebras and in Section 2.3 we recall the results of Kostant, Garland and Mitzman about integral forms.
Chapter 3 deals with the case of $\mathfrak{sl}_2$: the one-page formulation and proof that we present (see Theorem 3.2) inspire the way we study $\hat{\mathfrak{sl}}_2$ and $\hat{\mathfrak{sl}}_3{}^{\chi}$, and offer an easy introduction to the strategy followed also in the harder affine cases: decomposing our $\mathbb{Z}$-algebra as a tensor product of commutative subalgebras; describing these commutative structures thanks to the examples introduced in Chapter 1; and gluing the pieces together applying the results of Section 1.5.
Even if the results of this section imply the commutation rules between $(x_r^+)^{(k)}$ and $(x_{-r}^+)^{(l)}$ ($r\in\mathbb{Z}$, $k,l\in\mathbb{N}$) in the enveloping algebra of $\hat{\mathfrak{sl}}_2$ (see Remark 4.13), it is worth remarking that Chapter 4 does not depend on Chapter 3, and can be read independently (see Remark 4.20).
In Chapter 4 we discuss the case of $\hat{\mathfrak{sl}}_2$. The first part of the Chapter is devoted to the choice of the notations in $\hat{\mathcal{U}}=\mathcal{U}(\hat{\mathfrak{sl}}_2)$; to the definition of its (commutative) subalgebras $\hat{\mathcal{U}}^{\pm}$ (corresponding to the real component of $\hat{\mathcal{U}}$), $\hat{\mathcal{U}}^{0,\pm}$ (corresponding to the imaginary component), $\hat{\mathcal{U}}^{0,0}$ (corresponding to the Cartan), of their integral forms $\hat{\mathcal{U}}_{\mathbb{Z}}^{\pm}$, $\hat{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$, $\hat{\mathcal{U}}_{\mathbb{Z}}^{0,0}$, and of the $\mathbb{Z}$-subalgebra

$\hat{\mathcal{U}}_{\mathbb{Z}}$ of $\hat{\mathcal{U}}$; and to a detailed reminder about the useful symmetries (automorphisms, antiautomorphisms, homomorphisms and triangular decomposition) thanks to which we can get rid of redundant computations. In the second part of the chapter the apparently tough computations involved in the commutation relations are reduced to four formulas whose proofs are contained in a few lines: Proposition 4.14, Proposition 4.15, Lemma 4.21, and Proposition 4.22, (together with Proposition 1.13) are all what is needed to show that $\hat{\mathcal{U}}_{\mathbb{Z}}$ is an integral form of $\hat{\mathcal{U}}$, to recognize that the imaginary (positive and negative) components $\hat{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$ of $\hat{\mathcal{U}}_{\mathbb{Z}}$ are the algebras of polynomials $\mathbb{Z}[\Lambda_k(\xi(\pm 1)) \mid k \geq 0] = \mathbb{Z}[\hat{h}_{\pm k} \mid k > 0]$, and to exhibit a $\mathbb{Z}$-basis of $\hat{\mathcal{U}}_{\mathbb{Z}}$ (see Theorem 4.25).

In Chapter 5 we present the case of $A_2^{(2)}$. As for $\hat{\mathfrak{sl}}_2$, we first highlight some general structures of $\mathcal{U}(\hat{\mathfrak{sl}}_3{}^{\chi})$ (that we denote here $\tilde{\mathcal{U}}$ in order to distinguish it from $\hat{\mathcal{U}} = \mathcal{U}(\hat{\mathfrak{sl}}_2)$): notations, subalgebras and symmetries. In order to study the $\mathbb{Z}$-subalgebra of $\tilde{\mathcal{U}}$ generated by the divide powers of the Drinfeld generators that we denote by $\tilde{\mathcal{U}}_{\mathbb{Z}}$, here we introduce the elements $\tilde{h}_k$ through the announced deformation of the formulas defining the elements $\hat{h}_k$'s (see Definition 5.12 and Remark 5.13). We also describe a $\mathbb{Q}[w]$-module structure on a Lie subalgebra L of $\hat{\mathfrak{sl}}_3{}^{\chi}$ (see Definitions 5.8 and 5.10), thanks to which we can further simplify the notations. In addition, in Remark 5.27 we recall the embeddings of $\hat{\mathcal{U}}$ inside $\tilde{\mathcal{U}}$ thanks to which a big part of the work can be translated from Chapter 4. The heart of the problem is thus reduced to the commutation of $\exp(x_0^+ u)$ with $\exp(x_1^- v)$ (which is technically more complicated than for $A_1^{(1)}$ since it is a product involving a higher number of factors) and to deducing from this formula the description of the imaginary part of the integral form as the algebra of the polynomials in the $\tilde{h}_k$'s. To the solution of this problem, which represents the central contribution of this work, we dedicate Subsection 5.2, where we concentrate, perform and explain the necessary computations.

In Chapter 6 we compare the Mitzman's integral form of the enveloping algebra of type $A_2^{(2)}$ with the one studied here, proving the inclusion stated above. We also show that our commutation relations imply Mitzman's Theorem, too.

In Chapter 7 we present two other integral forms of $A_2^{(2)}$ that we denote by $\bar{\mathcal{U}}_{\mathbb{Z}}$ and $\check{\mathcal{U}}_{\mathbb{Z}}$ in order to distinguish to $\tilde{\mathcal{U}}_{\mathbb{Z}}$. $\bar{\mathcal{U}}_{\mathbb{Z}}$ is generated by the divided powers of the Drinfeld generators $x_r^{\pm}$ and by the divide powers of the elements $\frac{1}{2}X_{2r+1}^{\pm}$, adapting certain straightening relations already studied in the case of $\tilde{\mathcal{U}}_{\mathbb{Z}}$ (see for example Lemma 7.10, Remarks 7.4, 7.5 and Proposition 7.6) we automatically deduce the structure of $\bar{\mathcal{U}}_{\mathbb{Z}} \cap \tilde{\mathcal{U}}^{\pm}$. The heart of the problem is thus reduced to describe $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,+} = \bar{\mathcal{U}}_{\mathbb{Z}} \cap \tilde{\mathcal{U}}^{0,+}$ and its symmetric ($\bar{\mathcal{U}}_{\mathbb{Z}}^{0,-}$). Here we introduce new elements that is: $\bar{h}_{2r}$ (see Definition 1.31), then thanks to Section 1.4 (Theorems 1.42 and 1.46), we can prove that is an integral form but not longer an algebra of polynomials. For this reason we decided to study $\check{\mathcal{U}}_{\mathbb{Z}}$, that is obtained by $\bar{\mathcal{U}}_{\mathbb{Z}}$ adding extra elements $\check{h}_r$ (see Definition 1.31) in order to have a polynomial structure in the imaginary components.

In Chapter 8 we present the case of $A_{2n}^{(2)}$. In Section 8.1 we introduce general definitions (see Definition 8.1), in particular we devote care to the description of the root system and the related group of automorphisms W_T generated by the τ_i (see Notation 8.4), also we highlight the presence of certain embeddings, namely a copy of $A_{2n-2}^{(2)}$ and $A_{n-1}^{(1)}$ (see Definition 8.9). Section 8.2 is devoted to the case of $A_4^{(2)}$. In the first part we devote ourselves to the study of positive real roots from which we see that the restriction of the integral form at the first node of the diagram turns out to be a copy of $\bar{\mathcal{U}}_{\mathbb{Z}}$ while the restriction at the second turns out to be a copy of $\hat{\mathcal{U}}_{\mathbb{Z}}$, at this point we reattach the various pieces of the decomposition and using the general relations from Chapter 1 it is possible to easily describe the structure of the integral form. In Section 8.3 we show inductively that the study of $A_4^{(2)}$ leads immediately to the case of $A_{2n}^{(2)}$ with $n > 2$.
At the end of the work some appendices are added for the sake of completeness.
In Appendix 9.A we collect all the straightening formulas: since not all of them are necessary to our proofs and in the previous sections we only computed those which were essential for our argument, we give here a complete explicit picture of the commutation relations.

Appendix 9.B As mentioned above, this algebra, that we are naturally interested in because it is isomorphic to the imaginary positive part of the integral form of the rank 1 Kac-Moody algebras, was not recognized by Garland and Mitzman as an algebra of polynomials: in this appendix the $\mathbb{Z}$-basis they introduce is studied from the point of view of the symmetric functions and thanks to this interpretation it is easily proved to generate freely the same $\mathbb{Z}$-submodule of $\mathbb{Q}[h_r \mid r > 0]$ as the monomials in the $\hat{h}_k$'s.
Finally, in order to help the reader to orientate in the notations and to find easily their definitions, we conclude the work with an index of symbols, collected in Appendix 9.C.

Chapter 1

Integral Forms

In this chapter we give the definition of integral form, fixing the notations used throughout the thesis. We first expose some simple commutative examples (polynomials, binomials, etc.), then we recall some well-known examples related to symmetric polynomials (deeply studied and systematically exposed in [10]), and finally we show some integral forms that emerge in a completely new way in the case of some affine algebras of twisted type. All these integral forms will play a central role in understanding in more detail the structure of a specific integral form in the case of enveloping algebras of finite and affine Kac-Moody algebras.

1.1 Generalities

Notation 1.1. *Let us define $V = \mathbb{Q} \otimes_{\mathbb{Z}} M$ where M a free $\mathbb{Z}$-module. We will denote by SV the symmetric algebra of V.*

Definition 1.2. *Let $\mathcal{U}$ be a $\mathbb{Q}$-algebra. An integral form of $\mathcal{U}$ is a $\mathbb{Z}$-algebra $\mathcal{U}_{\mathbb{Z}}$ such that*

1. *$\mathcal{U}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module;*
2. *$\mathcal{U} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{U}_{\mathbb{Z}}$*

In particular an integral form of $\mathcal{U}$ can be identified to a $\mathbb{Z}$-subalgebra of $\mathcal{U}$ and consequently a $\mathbb{Z}$-basis of an integral form of $\mathcal{U}$ is a $\mathbb{Q}$-basis of $\mathcal{U}$.

Example 1.3. *Clearly $\mathbb{Z}[x_i \mid i \in I]$ is an integral form of $\mathbb{Q}[x_i \mid i \in I]$. If $\{x_i \mid i \in I\}$ is a $\mathbb{Z}$-basis of M then $S_{\mathbb{Z}}M := \mathbb{Z}[x_i \mid i \in I]$ is an integral form of $SV = \mathbb{Q}[x_i \mid i \in I]$ and $S_{\mathbb{Z}}M \cap V = M$.*

By definition, every integral form of SV containing M contains $S_{\mathbb{Z}}M$, that is $S_{\mathbb{Z}}M$ is the least integral form of SV containing M.

Remark 1.4. *Let U be a unitary $\mathbb{Z}$-algebra and $f(u) \in U[[u]]$.*

1. *If $f(u) \in 1 + uU[[u]]$ then:*
   - i. *$f(u)$ is invertible in $U[[u]]$;*
   - ii. *the coefficients of $f(u)$, those of $f(-u)$ and those of $f(u)^{-1}$ generate the same $\mathbb{Z}$-subalgebra of U;*
2. *If $f(u) \in uU[[u]]$ then $\exp(f(u))$ is a well defined element of $1 + uU[[u]]$;*
3. *If $f(u) \in 1 + uU[[u]]$ then $\ln(f(u))$ is a well defined element of $uU[[u]]$;*

4. $\exp \circ \ln |_{1+uU[[u]]} = \mathrm{Id}$ *and* $\ln \circ \exp |_{uU[[u]]} = \mathrm{Id}$;

5. *If* $f(u) \in U[[u]]$ *then there exists a unique continuous algebra homomorphism* $\mathbb{Z}[[u]] \to U[[u]]$ *such that* $u \mapsto uf(u)$.

Definition 1.5. *Let a be an element of a unitary $\mathbb{Q}$-algebra U and $n \in \mathbb{N}$. The n-th divided powers of a is the element*

$$a^{(n)} = \frac{a^n}{n!}.$$

Notice that the generating series of the $a^{(n)}$'s is $\exp(au)$, that is

$$\sum_{n \geq 0} a^{(n)} u^n = \exp(au).$$

Definition 1.6. *Let $\{x_i\}_{i \in I}$ be a $\mathbb{Z}$-basis of M. The $\mathbb{Z}$-subalgebra $S^{(div)}M \subseteq SV$ generated by $\{x^{(k)}\}_{x \in M, k \in \mathbb{N}}$ containing M is called the* algebra of the divided powers of M.

It is well known that $S^{(div)}M$ satisfies the following properties

i) $S^{(div)}M \cap V = M$;

ii) $\{x_i^{(k)}\}_{i \in I, k \in \mathbb{N}}$ is a set of algebra-generators over $\mathbb{Z}$ of $S^{(div)}M$;

iii) the set $\{\mathbf{x}^{(\mathbf{k})} = \prod_{i \in I} x_i^{(k_i)} \mid \mathbf{k} : I \to \mathbb{N}$ is finitely supported$\}$ is a $\mathbb{Z}$-basis of $S^{(div)}M$;

iv) $S^{(div)}M$ is an integral form of SV.

Notation 1.7. *$S^{(div)}M$ is also denoted $\mathbb{Z}^{(div)}[x_i \mid i \in I]$.*

Setting $m(u) = \sum_{r \in \mathbb{N}} m_r u^r \in M[[u]]$, notice that if $m_0 = 0$ then

$$m(u)^{(k)} \in S^{(div)}M[[u]] \ \forall k \in \mathbb{N}$$

or equivalently

$$\exp(m(u)) \in S^{(div)}M[[u]].$$

The vice versa is obviously also true:

$$m(u) \in uV[[u]] \text{ and } \exp(m(u)) \in S^{(div)}M[[u]] \Leftrightarrow m(u) \in uM[[u]]. \tag{1.7.1}$$

Definition 1.8. *Let a be an element of a unitary $\mathbb{Q}$-algebra U. The binomials of a are the elements*

$$\binom{a}{k} = \frac{a(a-1) \cdot \cdots \cdot (a-k+1)}{k!} \quad (k \in \mathbb{N}).$$

Remark that the generating series of the $\binom{a}{k}$'s is $\sum_{k \geq 0} \binom{a}{k} u^k = \exp(a \ln(1+u))$.

Since $a \ln(1+u) \in uU[[u]]$, $\exp(a \ln(1+u))$ is a well defined element of $U[[u]]$ and it can and will be denoted as $(1+u)^a$; more explicitly

$$\sum_{k \in \mathbb{N}} \binom{a}{k} u^k = (1+u)^a = \exp\Big(\sum_{r>0} (-1)^{r-1} \frac{a}{r} u^r\Big).$$

It is clear from the definition of $(1+u)^a$ that if a and b are commuting elements of U then

$$(1+u)^{a+b} = (1+u)^a (1+u)^b.$$

It is also clear that the $\mathbb{Z}$-submodule of U generated by the coefficients of $(1+u)^{a+m}$ ($a \in U$, $m \in \mathbb{Z}$) depends only on a and not on m; it is actually a $\mathbb{Z}$-subalgebra of U: indeed for all $k, l \in \mathbb{N}$

$$\binom{a}{k}\binom{a-k}{l} = \binom{k+l}{k}\binom{a}{k+l}.$$

More precisely for each $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ the $\mathbb{Z}$-submodule of U generated by the $\binom{a+m}{k}$'s for $k = 0, ..., n$ ($a \in U$) depends only on a and n and not on m.

Finally notice that in $U[[u]]$ we have $\frac{\mathrm{d}}{\mathrm{d}u}(1+u)^a = a(1+u)^{a-1}$.

Definition 1.9. *Let $\{x_i\}_{i\in I}$ be a $\mathbb{Z}$-basis of M. The $\mathbb{Z}$-subalgebra $S^{(bin)}M \subseteq SV$ generated by $\{\binom{x}{k}\}_{x\in M, k\in\mathbb{N}}$ containing M is called the algebra of* binomials of M. *Then it is well known that:*

i) $\{\binom{x_i}{k}\}_{i\in I, k\in\mathbb{N}}$ *is a set of algebra-generators (over $\mathbb{Z}$) of $S^{(bin)}M$;*

ii) the set $\{\binom{\mathbf{x}}{\mathbf{k}} = \prod_{i\in I}\binom{x_i}{k_i}\} \mid \mathbf{k} : I \to \mathbb{N}$ *finitely supported} is a $\mathbb{Z}$-basis of $S^{(bin)}M$;*

iii) $S^{(bin)}M \cap V = M$;

iv) $S^{(bin)}M$ is an integral form of SV (called the algebra of binomials of M).

Notation 1.10. *$S^{(bin)}M$ is also denoted $\mathbb{Z}^{(bin)}[x_i \mid i \in I]$.*

1.2 Review of the symmetric functions

In this section we briefly recall the definition of the Symmetric Functions and some of their properties, for more details see [10] .

Let $n \in \mathbb{N}$. Consider the ring $\mathbb{Z}[x_1, ..., x_n]$ on n independent variables over $\mathbb{Z}$, then the symmetric group $\mathcal{S}_n$ acts on the variables permuting them and we set $\Lambda_n = \mathbb{Z}[x_1, ..., x_n]^{\mathcal{S}_n}$. It is well known that Λ_n is an integral form of $\mathbb{Q}[x_1, ..., x_n]^{\mathcal{S}_n}$ and that $\mathbb{Z}[x_1, ..., x_n]^{\mathcal{S}_n} = \mathbb{Z}[e_1^{[n]}, ..., e_n^{[n]}]$, where the (algebraically independent for $k = 1, ..., n$) elementary symmetric polynomials $e_k^{[n]}$'s are defined by

$$\prod_{i=1}^{n}(T - x_i) = \sum_{k\in\mathbb{N}}(-1)^k e_k^{[n]} T^{n-k}$$

and are homogeneous of degree k, that is $e_k^{[n]} \in \mathbb{Z}[x_1, ..., x_n]_k^{\mathcal{S}_n} \subseteq \mathbb{Q}[x_1, ..., x_n]_k^{\mathcal{S}_n}$.

It is also well known and trivial that for $n_1 \geq n_2$ the natural projection

$$\pi_{n_1,n_2} : \mathbb{Q}[x_1, ..., x_{n_1}] \to \mathbb{Q}[x_1, ..., x_{n_2}]$$

defined by

$$\pi_{n_1,n_2}(x_i) = \begin{cases} x_i & \text{if } i \leq n_2 \\ 0 & \text{otherwise} \end{cases}$$

is such that $\pi_{n_1,n_2}(e_k^{[n_1]}) = e_k^{[n_2]}$ for all $k \in \mathbb{N}$. Then

$$\bigoplus_{d\geq 0} \varprojlim \mathbb{Z}[x_1, ..., x_n]_d^{\mathcal{S}_n} = \mathbb{Z}[e_1, ..., e_k, ...] \ \ (e_k \text{ inverse limit of the } e_k^{[n]})$$

is an integral form of $\oplus_{d\geq 0} \varprojlim \mathbb{Q}[x_1, ..., x_n]_d^{\mathcal{S}_n}$, which is called the algebra of the symmetric functions.

Moreover the elements

$$p_r^{[n]} = \sum_{i=1}^{n} x_i^r \in \mathbb{Z}[x_1, ..., x_n]^{\mathcal{S}_n} \ (r > 0,\ n \in \mathbb{N})$$

and their inverse limits $p_r \in \mathbb{Z}[e_1, ..., e_k, ...]$ ($\pi_{n_1,n_2}(p_r^{[n_1]}) = p_r^{[n_2]}$ for all $r > 0$ and all $n_1 \geq n_2$) give another set of generators of the $\mathbb{Q}$-algebra of the symmetric functions: the p_r's are algebraically independent and

$$\bigoplus_{d\geq 0} \varprojlim \mathbb{Q}[x_1, ..., x_n]_d^{\mathcal{S}_n} = \mathbb{Q}[p_1, ..., p_r, ...].$$

Finally $\mathbb{Z}[e_1, ..., e_k, ...]$ is an integral form of $\mathbb{Q}[p_1, ..., p_r, ...]$ containing p_r for all $r > 0$ (more precisely a linear combination of the p_r's lies in $\mathbb{Z}[e_1, ..., e_k, ...]$ if and only if it has integral coefficients), the relation between the e_k's and the p_r's being given by:

$$\sum_{k\in\mathbb{N}} (-1)^k e_k u^k = \exp\Big(- \sum_{r>0} \frac{p_r}{r} u^r \Big).$$

In this context, to stress the dependence of the e_k's on the p_r's, we set $e_k = \hat{p}_k$, that is we fix the following notations:

$$\hat{p}(u) = \sum_{k\in\mathbb{N}} \hat{p}_k u^k = \exp\Big(\sum_{r>0} (-1)^{r-1} \frac{p_r}{r} u^r \Big)$$

and

$$\mathbb{Z}^{(sym)}[p_r \mid r > 0] = \mathbb{Z}[\hat{p}_k \mid k > 0] \subseteq \mathbb{Q}[p_r \mid r > 0].$$

Remark 1.11. *With the notations above, let $\varphi : \mathbb{Q}[p_1, ..., p_r, ...] \to U$ be an algebra-homomorphism and $a = \varphi(p_1)$:*

i) if $\varphi(p_r) = 0$ for $r > 1$ then $\varphi(\hat{p}_k) = a^{(k)}$ for all $k \in \mathbb{N}$;

ii) if $\varphi(p_r) = a$ for all $r > 0$ then $\varphi(\hat{p}_k) = \binom{a}{k}$ for all $k \in \mathbb{N}$.

Hence $\mathbb{Z}^{(sym)}$ is a generalization of both $\mathbb{Z}^{(div)}$ and $\mathbb{Z}^{(bin)}$.

Remark 1.12. *Let $\{p_r \mid r > 0\}$ be a $\mathbb{Z}$-basis of M. Then:*
i) as for the functors $S_{\mathbb{Z}}$, $S^{(div)}$ and $S^{(bin)}$, we have $\mathbb{Z}^{(sym)}[p_r \mid r > 0] \cap V = M$;
ii) unlike the functors $S_{\mathbb{Z}}$, $S^{(div)}$ and $S^{(bin)}$, $\mathbb{Z}^{(sym)}[p_r \mid r > 0]$ depends on $\{p_r \mid r > 0\}$ and not only on M, for instance

$$\mathbb{Z}^{(sym)}[-p_1, p_r \mid r > 1] \neq \mathbb{Z}^{(sym)}[p_r \mid r > 0]$$

(it is easy to check that these integral forms are different for example in degree 3);
iii) not all the sign changes of the p_r's produce different $\mathbb{Z}^{(sym)}$-forms of $\mathbb{Q}[p_r \mid r > 0]$:

$$\mathbb{Z}^{(sym)}[(-1)^r p_r \mid r > 0] = \mathbb{Z}^{(sym)}[-p_r \mid r > 0] = \mathbb{Z}^{(sym)}[p_r \mid r > 0]$$

since

$$\exp\left(\sum_{r>0} (-1)^{r-1} \frac{(-1)^r p_r}{r} u^r \right) = \exp\left(\sum_{r>0} (-1)^{r-1} \frac{p_r}{r} (-u)^r \right)$$

and

$$\exp\left(\sum_{r>0} (-1)^{r-1} \frac{-p_r}{r} u^r \right) = \exp\left(\sum_{r>0} (-1)^{r-1} \frac{p_r}{r} u^r \right)^{-1}$$

(see Remark 1.8,1),ii)).

In general it is not trivial to understand whether an element of $\mathbb{Q}[p_r \mid r > 0]$ belongs or not to $\mathbb{Z}^{(sym)}[p_r \mid r > 0]$; Proposition 1.13 gives an answer to this question, which is generalized in Proposition 1.18 (the examples in Remark 1.12, ii) and iii) can be obtained also as applications of Proposition 1.18).

Proposition 1.13. *Let us fix $m > 0$ and let $\lambda_m : \mathbb{Q}[p_r \mid r > 0] \to \mathbb{Q}[p_r \mid r > 0]$ be the algebra homomorphism defined by $\lambda_m(p_r) = p_{mr}$ for all $r > 0$.*

Then $\mathbb{Z}^{(sym)}[p_r \mid r > 0]$ $(= \mathbb{Z}[\hat{p}_k \mid k > 0])$ is λ_m-stable.

Proof. For $n \in \mathbb{N}$ let $\lambda_m^{[n]} : \mathbb{Q}[x_1, ..., x_n] \to \mathbb{Q}[x_1, ..., x_n]$ be the algebra homomorphism defined by $\lambda_m^{[n]}(x_i) = x_i^m$ for all $i = 1, ..., n$.

We obviously have that

$$\mathbb{Z}[x_1, ..., x_n] \text{ is } \lambda_m^{[n]}-\text{stable},$$

$$\mathbb{Q}[x_1, ..., x_n]_d \text{ is mapped to } \mathbb{Q}[x_1, ..., x_n]_{md} \ \forall d \geq 0,$$

$$\lambda_m^{[n]} \circ \sigma = \sigma \circ \lambda_m^{[n]} \ \forall n \in \mathbb{N},\ \sigma \in \mathcal{S}_n,$$

$$\pi_{n_1,n_2} \circ \lambda_m^{[n_1]} = \lambda_m^{[n_2]} \circ \pi_{n_1,n_2} \ \forall n_1 \geq n_2,$$

$$\lambda_m^{[n]}(p_r^{[n]}) = p_{mr}^{[n]} \ \forall n \in \mathbb{N},\ r > 0,$$

hence there exist the limits of the $\lambda_m^{[n]}\ |_{\mathbb{Q}[x_1,...,x_n]_d^{\mathcal{S}_n}}$'s: their direct sum over $d \geq 0$ stabilizes $\oplus_{d\geq 0} \lim \mathbb{Z}[x_1, ..., x_n]_d^{\mathcal{S}_n} = \mathbb{Z}[\hat{p}_k \mid k > 0]$ and is λ_m.

In particular $\lambda_m(\hat{p}_k) \in \mathbb{Z}[\hat{p}_l \mid l > 0]\ \forall k \in \mathbb{N}$.

□

We also propose a second, direct, proof of Proposition 1.13, which provides in addition an explicit expression of the $\lambda_m(\hat{p}_k)$'s in terms of the $\hat{p}_l$'s.

Proposition 1.14. *Let m and λ_m be as in Proposition 1.13 and $\omega \in \mathbb{C}$ a primitive m^{th} root of 1. Then*

$$\lambda_m(\hat{p}(-u^m)) = \prod_{j=0}^{m-1} \hat{p}(-\omega^j u) \in \mathbb{Z}[\hat{p}_k \mid k > 0][[u]].$$

Proof. The equality in the statement is an immediate consequence of

$$\sum_{j=0}^{m-1} \omega^{jr} = \begin{cases} m & \text{if } m \mid r \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$-\sum_{j=0}^{m-1} \sum_{r>0} \frac{p_r}{r} \omega^{jr} u^r = -\sum_{r>0} \frac{p_{mr}}{r} u^{mr} = \lambda_m \left(-\sum_{r>0} \frac{p_r}{r} (u^m)^r \right),$$

whose exponential is the claim.

Then for all $k > 0$

$$\lambda_m(\hat{p}_k) \in \mathbb{Q}[\hat{p}_l \mid l > 0] \cap \mathbb{Z}[\omega][\hat{p}_l \mid l > 0] = \mathbb{Z}[\hat{p}_l \mid l > 0]$$

since $\mathbb{Q} \cap \mathbb{Z}[\omega] = \mathbb{Z}$. □

In order to characterize the functions $a : \mathbb{Z}_+ \to \mathbb{Q}$ such that

$$\mathbb{Z}^{(sym)}[a_r p_r \mid r > 0] \subseteq \mathbb{Z}^{(sym)}[p_r \mid r > 0]$$

we introduce the Notation 1.15, where we rename the p_r's into h_r since in the affine Kac-Moody case the $\mathbb{Z}^{(sym)}$-construction describes the imaginary component of the integral form. Moreover from now on p_i will denote a positive prime number.

Notation 1.15. *Given $a : \mathbb{Z}_+ \to \mathbb{Q}$ set*

$$\sum_{k\geq 0} \hat{h}_k^{\{a\}} u^k = \hat{h}^{\{a\}}(u) = \exp\left(\sum_{r>0} (-1)^{r-1} \frac{a_r h_r}{r} u^r\right);$$

$\mathbb{1}$ denotes the function defined by

$$\mathbb{1}_r = 1 \textit{ for all } r \in \mathbb{Z}_+;$$

for all $m > 0$, $\mathbb{1}^{(m)}$ denotes the function defined by

$$\mathbb{1}_r^{(m)} = \begin{cases} m & \text{if } m \mid r \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\hat{h}^{\{\mathbb{1}\}}(u) = \hat{h}(u)$ (see the notation in Example 1.2) and $\hat{h}^{\{\mathbb{1}^{(m)}\}}(-u) = \lambda_m(\hat{h}(-u^m))$.

Remark 1.16. *Remark that $\hat{h}^{\{a+b\}}(u) = \hat{h}^{\{a\}}(u)\hat{h}^{\{b\}}(u)$ and that the function*

$$\begin{aligned} 1 + u\mathbb{Q}[[u]] &\to \mathbb{Q}[h_r \mid r > 0][[u]] \\ f(u) &\mapsto \hat{h}^{\{a\}}(u), \end{aligned}$$

where a is defined by $\ln(f(u)) = \sum_{r>0} (-1)^{r-1} \frac{a_r}{r} u^r$, preserves the multiplication. Of course $1 + u \mapsto \hat{h}(u)$ and $1 + u^m \mapsto \lambda_m(\hat{h}(u^m))$.

Recall 1.17. *The convolution product $*$ in the ring of the $\mathbb{Q}$-valued arithmetic functions*

$$\mathcal{A}r = \{f : \mathbb{Z}_+ \to \mathbb{Q}\}$$

is defined by

$$(f * g)(n) = \sum_{\substack{r,s: \\ rs=n}} f(r)g(s).$$

The Möbius function $\mu : \mathbb{Z}_+ \to \mathbb{Q}$ defined by

$$\mu\left(\prod_{i=1}^n p_i^{r_i}\right) = \begin{cases} (-1)^n & \text{if } r_i = 1\ \forall i \\ 0 & \text{otherwise} \end{cases}$$

is the inverse of $\mathbb{1}$ in the ring of the arithmetic functions, where $n \in \mathbb{N}$, the p_i's are distinct positive prime integers and $r_i \geq 1$ for all i.

Proposition 1.18. *Let $a : \mathbb{Z}_+ \to \mathbb{Q}$ be any function; then, with the notations fixed in 1.15,*

$$\hat{h}_k^{\{a\}} \in \mathbb{Z}[\hat{h}_l \mid l > 0]\ \ \forall k > 0 \Leftrightarrow n \mid (\mu * a)(n) \in \mathbb{Z}\ \ \forall n > 0.$$

Proof. Remark that $a = \mathbb{1} * \mu * a$, that is

$$\forall n > 0\ a_n = \sum_{m|n} (\mu * a)(m) = \sum_{m|n} \frac{(\mu * a)(m)}{m} m = \sum_{m>0} \frac{(\mu * a)(m)}{m} \mathbb{1}_n^{(m)},$$

which means

$$a = \sum_{m>0} \frac{(\mu * a)(m)}{m} \mathbb{1}^{(m)}.$$

Let $k_m = \frac{(\mu*a)(m)}{m}$ for all $m > 0$, choose $m_0 > 0$ such that $k_m \in \mathbb{Z}\ \forall m < m_0$ and set $a^{(0)} = \sum_{m<m_0} k_m \mathbb{1}^{(m)}$, $a' = a - a^{(0)}$, so that (see Remark 1.16)

$$\hat{h}^{\{a\}}(u) = \hat{h}^{\{a'\}}(u)\hat{h}^{\{a^{(0)}\}}(u),$$

and, by Proposition 1.13 (see also Notation 1.15),

$$\hat{h}^{\{a^{(0)}\}}(u) \in \mathbb{Z}[\hat{h}_k \mid k > 0][[u]].$$

It follows that

i) $\hat{h}^{\{a\}}(u) \in \mathbb{Z}[\hat{h}_k \mid k > 0][[u]] \Leftrightarrow \hat{h}^{\{a'\}}(u) \in \mathbb{Z}[\hat{h}_k \mid k > 0][[u]]$.

ii) $\forall n < m_0\ \hat{h}_n^{\{a'\}} = 0$, so that $\hat{h}_n^{\{a\}} = \hat{h}_n^{\{a^{(0)}\}} \in \mathbb{Z}[\hat{h}_k \mid k > 0]$;
in particular $\hat{h}^{\{a\}}(u) \in \mathbb{Z}[\hat{h}_k \mid k > 0][[u]]$ if $k_m \in \mathbb{Z}\ \forall m > 0$.

iii) $a'_{m_0} = (\mu * a)(m_0) = m_0 k_{m_0}$ so that $\hat{h}_{m_0}^{\{a'\}} = k_{m_0} h_{m_0}$, which belongs to $\mathbb{Z}[\hat{h}_k \mid k > 0]$ if and only if $k_{m_0} \in \mathbb{Z}$ (see Remark 1.12,i));
in particular $\hat{h}^{\{a\}}(u) \notin \mathbb{Z}[\hat{h}_k \mid k > 0][[u]]$ if $\exists m_0 \in \mathbb{Z}_+$ such that $k_{m_0} \notin \mathbb{Z}$. □

Proposition 1.19. *Let $a : \mathbb{Z}_+ \to \mathbb{Z}$ be a function satisfying the condition*

$$p^r \mid (a_{mp^r} - a_{mp^{r-1}})\ \ \forall p, m, r \in \mathbb{Z}_+ \ \text{ with } \ p \ \text{ prime and } (m, p) = 1.$$

Then $n \mid (\mu * a)(n)\ \forall n \in \mathbb{Z}_+$.

Proof. The condition $1 \mid (\mu * a)(1)$ is equivalent to the condition $a_1 \in \mathbb{Z}$.

For $n > 1$ remark that

$$n \mid (\mu * a)(n) \Leftrightarrow p^r \mid (\mu * a)(n)\ \ \forall p \text{ prime, } r > 0 \text{ such that } p^r \mid n.$$

Recall that if P is the set of the prime factors of n and $p \in P$ then

$$(\mu * a)(n) = \sum_{S \subseteq P} (-1)^{\#S} a_{\frac{n}{\prod_{q \in S} q}} =$$

$$= \sum_{S' \subseteq P \setminus \{p\}} (-1)^{\#S'} (a_{\frac{n}{\prod_{q \in S'} q}} - a_{\frac{n}{p \prod_{q \in S'} q}}).$$

The claim follows from the remark that $p^r \mid n$ if and only if $p^r \mid \frac{n}{\prod_{q \in S'} q}$. □

Remark 1.20. *The vice versa of Proposition 1.19 is trivially true, too, and is immediately proved applying 1.19 to the minimal $n > 0$ such that there exists $p \mid n$ and $r > 0$ ($p^r \mid n$, $n = mp^r$) not satisfying the hypothesis of the statement.*

Proposition 1.13 will play an important role in the study of the commutation relations in the enveloping algebra of $\hat{\mathfrak{sl}}_2$ (see Remarks 4.11,vi) and 4.19) and of $\hat{\mathfrak{sl}}_3{}^{\chi}$ (see Remark 5.16 and Proposition 5.19,iv)).

Proposition 1.18 is based on and generalizes Proposition 1.13; it is a key tool in the study of the integral form in the case of $A_2^{(2)}$, see Corollary 5.41.

1.3 Garland's basis

Here we discuss the precise connection between the integral form $\mathbb{Z}^{(sym)}[h_r \mid r>0]$ of $\mathbb{Q}[h_r \mid r>0]$ and the homomorphisms λ_m's, namely we give another $\mathbb{Z}$-basis of $\mathbb{Z}^{(sym)}[h_r \mid r>0]$ (basis defined in terms of the elements $\lambda_m(\hat{h}_k)$'s and arising from Garland's and Mitzman's description of the integral form of the affine Kac-Moody algebras, discussed in Appendix 9.B).

Definition 1.21. *With the notations of Example 1.2 and Proposition 1.13 let us define the following elements and subsets in $\mathbb{Q}[h_r \mid r>0]$:*

i. $b_{\mathbf{k}} = \prod_{m>0} \lambda_m(\hat{h}_{k_m})$ where $\mathbf{k}: \mathbb{Z}_+ \to \mathbb{N}$ is finitely supported;

ii.

$$B_\lambda = \{b_{\mathbf{k}} \mid \mathbf{k}: \mathbb{Z}_+ \to \mathbb{N} \text{ is finitely supported}\};$$

iii. $\mathbb{Z}_\lambda[h_r \mid r>0] = \sum_{\mathbf{k}} \mathbb{Z} b_{\mathbf{k}}$ is the $\mathbb{Z}$-submodule of $\mathbb{Q}[h_r \mid r>0]$ generated by B_λ

We want to prove the following:

Theorem 1.22. *$\mathbb{Z}^{(sym)}[h_r \mid r>0]$ is a free $\mathbb{Z}$ module with basis B_λ. Equivalently:*

i. $\mathbb{Z}^{(sym)}[h_r \mid r>0] = \mathbb{Z}_\lambda[h_r \mid r>0]$,

ii. B_λ is linearly independent.

Remark 1.23. *Proposition 1.14 implies that $\mathbb{Z}^{(sym)}[h_r \mid r>0] \subseteq \mathbb{Z}_\lambda[h_r \mid r>0]$, so we are left to prove the reverse inclusion and the linear independence of B_λ, that we shall prove by comparing B_λ with a well known $\mathbb{Z}$-basis of this algebra.*

Remark 1.24. *Recall that $\mathbb{Z}[\hat{h}_k \mid k>0]$ is the algebra of the symmetric functions and that $\forall n \in \mathbb{N}$ the projection $\pi_n : \mathbb{Z}[\hat{h}_k \mid k>0] \to \mathbb{Z}[x_1,...,x_n]^{\mathcal{S}_n}$ induces an isomorphism $\mathbb{Z}[\hat{h}_1,...,\hat{h}_n] \cong \mathbb{Z}[x_1,...,x_n]^{\mathcal{S}_n}$ through which $\hat{h}_k$ corresponds to the k^{th} elementary symmetric polynomial $e_k^{[n]}$, and h_r corresponds to the sum of the r^{th}-powers $\sum_{i=1}^n x_i^r$ $\forall r>0$ (see Example 1.2).*

Then it is well known and obvious that:

i) $\forall \mathbf{k}: \mathbb{Z}_+ \to \mathbb{N}$ finitely supported $\exists!(\sigma x)_{\mathbf{k}} \in \mathbb{Z}[\hat{h}_k \mid k>0]$ such that

$$\pi_n((\sigma x)_{\mathbf{k}}) = \sum_{\substack{a_1,...,a_n \\ \#\{i \mid a_i=m\}=k_m \ \forall m>0}} \prod_{i=1}^n x_i^{a_i} \in \mathbb{Z}[x_1,...,x_n]^{\mathcal{S}_n} \ \ \forall n \in \mathbb{N};$$

ii) $\{(\sigma x)_{\mathbf{k}} \mid \mathbf{k}: \mathbb{Z}_+ \to \mathbb{N}$ finitely supported$\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[\hat{h}_k \mid k>0]$.

(It is the basis that in [10] is called {symmetric monomial functions} and is denoted by $\{m_\lambda \mid \lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq 0)\}$: $m_\lambda = (\sigma x)_{\mathbf{k}}$ where $\forall m>0$ $k_m = \#\{i \mid \lambda_i = m\}$).

Notation 1.25. *As in Remark 1.24, for all $\mathbf{k}: \mathbb{Z}_+ \to \mathbb{N}$ finitely supported let us denote by $(\sigma x)_{\mathbf{k}}$ the limit of the elements*

$$\sum_{\substack{a_1,...,a_n \\ \#\{i \mid a_i=m\}=k_m \ \forall m>0}} \prod_{i=1}^n x_i^{a_i} \ \ (n \in \mathbb{N}).$$

By abuse of notation, when $n \geq \sum_{m>0} k_m$ we shall write

$$(\sigma x)_{\mathbf{k}} = \sum_{\substack{a_1,...,a_n \\ \#\{i \mid a_i=m\}=k_m \ \forall m>0}} \prod_{i=1}^n x_i^{a_i},$$

which is justified because, under the hypothesis that $n \geq \sum_{m>0} k_m$, $\mathbf{k}$ is determined by the set $\{(a_1,...,a_n) \mid \#\{i=1,...,n \mid a_i = m\} = k_m \ \forall m>0\}$.

Definition 1.26. *$\forall n \in \mathbb{N}$ define $B_\lambda^{[n]}, B_x^{[n]}, \mathbb{Z}_\lambda^{[n]}, \mathbb{Z}_x^{[n]} \subseteq \mathbb{Q}[h_r \mid r > 0] = \mathbb{Q}[\hat{h}_k \mid k > 0]$ as follows:*

$$B_\lambda^{[n]} = \left\{ b_{\mathbf{k}} = \prod_{m>0} \lambda_m(\hat{h}_{k_m}) \in B_\lambda \mid \sum_{m>0} k_m \leq n \right\},$$

$$B_x^{[n]} = \left\{ (\sigma x)_{\mathbf{k}} \mid \sum_{m>0} k_m \leq n \right\},$$

$\mathbb{Z}_\lambda^{[n]}$ is the $\mathbb{Z}$-module generated by $B_\lambda^{[n]}$, $\mathbb{Z}_x^{[n]}$ is the $\mathbb{Z}$-module generated by $B_x^{[n]}$.

Remark 1.27. *By the very definition of $B_x^{[n]}$ we have that:*

i) $B_x^{[n]}$ is a basis of $\mathbb{Z}_x^{[n]} \subseteq \mathbb{Z}[\hat{h}_k \mid k > 0] = \sum_{n' \in \mathbb{N}} \mathbb{Z}_x^{[n']}$, see Remark 1.24, ii);

ii) $h \in \mathbb{Z}_x^{[n]}$ means that for all $N \geq n$ each monomial in the x_i's appearing in $\pi_N(h)$ with nonzero coefficient involves no more than n indeterminates x_i; hence in particular

$$h \in \mathbb{Z}_x^{[n]}, h' \in \mathbb{Z}_x^{[n']} \Rightarrow hh' \in \mathbb{Z}_x^{[n+n']}.$$

Lemma 1.28. *Let $n, n', n'' \in \mathbb{N}$ and $\mathbf{k}', \mathbf{k}'' : \mathbb{Z}_+ \to \mathbb{N}$ be such that $n' + n'' = n$, $\sum_{m>0} k'_m = n'$, $\sum_{m>0} k''_m = n''$. Then:*

i) $(\sigma x)_{\mathbf{k}'} \cdot (\sigma x)_{\mathbf{k}''} \in \mathbb{Z}(\sigma x)_{\mathbf{k}'+\mathbf{k}''} \oplus \mathbb{Z}_x^{[n-1]}$;

ii) if $k'_m k''_m = 0 \ \forall m > 0$ then $(\sigma x)_{\mathbf{k}'}(\sigma x)_{\mathbf{k}''} - (\sigma x)_{\mathbf{k}'+\mathbf{k}''} \in \mathbb{Z}_x^{[n-1]}$.

Proof. That $(\sigma x)_{\mathbf{k}'} \cdot (\sigma x)_{\mathbf{k}''}$ lies in $\mathbb{Z}_x^{[n]}$ follows from Remark 1.27,ii), so we just need to:

i) prove that if $\prod_{i=1}^n x_i^{a_i}$ with $a_i \neq 0 \ \forall i = 1, ..., n$ is the product of two monomials M' and M'' appearing with nonzero coefficient respectively in $(\sigma x)_{\mathbf{k}'}$ and in $(\sigma x)_{\mathbf{k}''}$ then $\#\{i \mid a_i = m\} = k'_m + k''_m$ for all $m > 0$;

ii) compute the coefficient of $(\sigma x)_{\mathbf{k}'+\mathbf{k}''}$ in the expression of $(\sigma x)_{\mathbf{k}'} \cdot (\sigma x)_{\mathbf{k}''}$ as a linear combination of the $(\sigma x)_{\mathbf{k}}$'s when $\forall m > 0$ k'_m and k''_m are not simultaneously non zero, and find that it is 1.

i) is obvious because the condition $a_i \neq 0 \ \forall i = 1, ..., n$ implies that the indeterminates involved in M' and those involved in M'' are disjoint sets.

For ii) it is enough to show that, under the further condition on k'_m and k''_m, the monomial $\prod_{i=1}^n x_i^{a_i}$ chosen in i) uniquely determines M' and M'' such that $\prod_{i=1}^n x_i^{a_i} = M'M''$: indeed

$$M' = \prod_{i:k'_{a_i} \neq 0} x_i^{a_i} \text{ and } M'' = \prod_{i:k''_{a_i} \neq 0} x_i^{a_i}.$$

□

Lemma 1.29. *Let $\mathbf{k} : \mathbb{Z}_+ \to \mathbb{N}$, $n \in \mathbb{N}$ be such that $\sum_{m>0} k_m = n$. Then:*

i) if $\exists m > 0$ such that $k_{m'} = 0$ for all $m' \neq m$ (equivalently $k_m = n$) we have

$$(\sigma x)_{\mathbf{k}} = \lambda_m(\hat{h}_n) = b_{\mathbf{k}} \in \mathbb{Z}_x^{[n]} \cap \mathbb{Z}_\lambda^{[n]};$$

ii) in general $b_{\mathbf{k}} - (\sigma x)_{\mathbf{k}} \in \mathbb{Z}_x^{[n-1]}$.

Proof. i) $\forall N \geq n$ we have

$$(\sigma x)_{\mathbf{k}} = \sum_{1 \leq i_1 < ... < i_n \leq N} x_{i_1}^m \cdot ... \cdot x_{i_n}^m = \lambda_m \left(\sum_{1 \leq i_1 < ... < i_n \leq N} x_{i_1} \cdot ... \cdot x_{i_n} \right) = \lambda_m(e_n^{[N]})$$

so that $(\sigma x)_{\mathbf{k}} = \lambda_m(\hat{h}_n)$.

ii) $b_{\mathbf{k}} = \prod_{m>0} \lambda_m(\hat{h}_{k_m}) = \prod_{m>0}(\sigma x)_{\mathbf{k}^{[m]}}$ where $k^{[m]}_{m'} = \delta_{m,m'}k_m$ $\forall m, m' > 0$; thanks to Lemma 1.28,ii) we have that $\prod_{m>0}(\sigma x)_{\mathbf{k}^{[m]}} - (\sigma x)_{\sum_m \mathbf{k}^{[m]}} \in \mathbb{Z}_x^{[n-1]}$; but $\sum_{m>0} \mathbf{k}^{[m]} = \mathbf{k}$ and the claim follows. □

Theorem 1.30. *B_λ is a $\mathbb{Z}$-basis of $\mathbb{Z}[\hat{h}_k \mid k > 0]$ (thus $\mathbb{Z}[\hat{h}_k \mid k > 0] = \mathbb{Z}_\lambda[h_r \mid r > 0]$).*

Proof. We prove by induction on n that $B_\lambda^{[n]}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}_x^{[n]} = \mathbb{Z}_\lambda^{[n]}$ $\forall n \in \mathbb{N}$, the case $n = 0$ being obvious.

Let $n > 0$: by the inductive hypothesis $B_\lambda^{[n-1]}$ and $B_x^{[n-1]}$ are both $\mathbb{Z}$-bases of $\mathbb{Z}_x^{[n-1]} = \mathbb{Z}_\lambda^{[n-1]}$; by definition $B_x^{[n]} \setminus B_x^{[n-1]}$ represents a $\mathbb{Z}$-basis of $\mathbb{Z}_x^{[n]}/\mathbb{Z}_x^{[n-1]}$ while $B_\lambda^{[n]} \setminus B_\lambda^{[n-1]}$ represents a set of generators of the $\mathbb{Z}$-module $\mathbb{Z}_\lambda^{[n]}/\mathbb{Z}_\lambda^{[n-1]}$.

Now Lemma 1.29,ii) implies that if $\sum_{m>0} k_m = n$ then $b_{\mathbf{k}}$ and $(\sigma x)_{\mathbf{k}}$ represent the same element in $\mathbb{Q}[\hat{h}_k \mid k > 0]/\mathbb{Z}_x^{[n-1]} = \mathbb{Q}[\hat{h}_k \mid k > 0]/\mathbb{Z}_\lambda^{[n-1]}$.

Hence $B_\lambda^{[n]} \setminus B_\lambda^{[n-1]}$ represents a $\mathbb{Z}$-basis of $\mathbb{Z}_x^{[n]}/\mathbb{Z}_x^{[n-1]} = \mathbb{Z}_x^{[n]}/\mathbb{Z}_\lambda^{[n-1]}$, that is $B_\lambda^{[n]}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}_x^{[n]}$; but $B_\lambda^{[n]}$ generates $\mathbb{Z}_\lambda^{[n]}$ and the claim follows. □

1.4 A "mixed symmetric" integral form

Given $a : \mathbb{Z}_+ \to \mathbb{Q}$ we have seen when $\hat{h}^{\{a\}}(u) \in \mathbb{Z}[\hat{h}_k \mid k > 0][[u]]$. But what happens if $\hat{h}^{\{a\}}(u) \notin \mathbb{Z}[\hat{h}_k \mid k > 0][[u]]$ and we consider the $\mathbb{Z}$-algebra generated by $\{\hat{h}_k, \hat{h}_k^{\{a\}}\}$? Is it still an integral form of $\mathbb{Q}[h_r \mid r > 0]$? Is it still an algebra of polynomials? Here we study a very particular case of this problem, that will play a crucial role in certain integral forms in the case of $A_2^{(2)}$ (see Chapter 7).

Definition 1.31. *Using the notations introduced in Notation 1.15, let us define the sequences $\frac{1}{2}\mathbb{1}^{(2)}, \frac{1}{2}\mathbb{1} : \mathbb{Z}_+ \to \mathbb{Q}$, more precisely*

$$\frac{1}{2}\mathbb{1}^{(2)}(r) = \begin{cases} 0 \text{ if } r \text{ is odd,} \\ 1 \text{ if } r \text{ is even.} \end{cases}$$

and

$$\frac{1}{2}\mathbb{1}(r) = \frac{1}{2} \text{ for all } r.$$

Let us set $\bar{h}(u) = \sum_{k\geq 0} \bar{h}_k u^k = \hat{h}^{\frac{1}{2}\mathbb{1}^{(2)}}(u)$ and $\check{h}(u) = \sum_{k\geq 0} \check{h}_k u^k = \hat{h}^{\frac{1}{2}\mathbb{1}}(u)$.

Remark 1.32. *$\bar{h}(u) \in \mathbb{Q}[h_{2r} \mid r > 0]$ and $\bar{h}_{2r+1} = 0$ $\forall r > 0$. More precisely*

$$\mathbb{Z}[\bar{h}_{2r} \mid k > 0] = \mathbb{Z}^{(sym)}[\frac{h_{2r}}{r} \mid r > 0]$$

and

$$\bar{h}(u^2) = \lambda_2(\hat{h}^{\frac{1}{2}}(u^2)) = \lambda_2(\check{h}(u^2)) = \check{h}(u)\check{h}(-u).$$

In particular $\hat{h}(u) \notin \mathbb{Z}[\bar{h}_{2r} \mid r > 0]$ (see Proposition 1.14).

Remark 1.33. *$\bar{h}(u) \notin \mathbb{Z}[\hat{h}_r \mid r > 0]$. Indeed the sequence $\frac{1}{2}\mathbb{1}^{(2)}$ does not satisfy condition of Proposition 1.19 (see Proposition 1.18).*

Remark 1.34. $\hat{h}(u), \bar{h}(u) \in \mathbb{Z}[\check{h}_k \mid k > 0] = \mathbb{Z}^{(sym)}[\frac{h_r}{2} \mid r > 0]$.

Definition 1.35. *Define* $\mathbb{Z}^{(mix)}[h_r \mid r > 0]$ *to be the* $\mathbb{Z}$*-subalgebra of* $\mathbb{Q}[h_r \mid r > 0]$ *generated by* $\{\hat{h}_r, \bar{h}_r \mid r > 0\}$.

Remark 1.36. *Of course* $\mathbb{Z}^{(mix)}[h_r \mid r > 0] \subseteq \mathbb{Z}[\check{h}_k \mid r > 0]$.

Remark 1.37. *Let* V *be the* $\mathbb{Q}$*-vector subspace of* $\mathbb{Q}[h_r \mid r > 0]$ *with basis* $\{h_r \mid r > 0\}$*. Then*

$$\mathbb{Z}^{(mix)}[h_r \mid r > 0] \cap V = \mathbb{Z}\langle h_{2r-1}, \frac{h_{2r}}{2} \mid r > 0\rangle.$$

Corollary 1.38. $\mathbb{Z}^{(mix)}[h_r \mid r > 0] \subsetneq \mathbb{Z}[\check{h}_r \mid r > 0]$*. Indeed* $\mathbb{Z}[\check{h}_r \mid r > 0] \cap V = \mathbb{Z}\langle \frac{h_r}{2} \mid r > 0\rangle$.

Remark 1.39. $\mathbb{Z}^{(mix)}[h_r \mid r > 0]$ *is a graded algebra with* $deg(h_r) = r$ *for all* $r > 0$*, that is*

$$\mathbb{Z}^{(mix)}[h_r \mid r > 0] = \bigoplus_{d \geq 0} \mathbb{Z}^{(mix)}[h_r \mid r > 0]_d,$$

and we have $\mathbb{Z}^{(mix)}[h_r \mid r > 0]_1 = \mathbb{Z}h_1$ *and*

$$\mathbb{Z}^{(mix)}[h_r \mid r > 0]_2 = \mathbb{Z}\langle h_1^2, \hat{h}_2 = \frac{1}{2}(h_1^2 + h_2), \bar{h}_2\rangle = \mathbb{Z}\langle \frac{1}{2}h_1^2, \frac{1}{2}h_2\rangle$$

which implies that h_1^2 *does not belongs to any* $\mathbb{Z}$*-basis of* $\mathbb{Z}^{(mix)}[h_r \mid r > 0]_2$*. Then* $\mathbb{Z}^{(mix)}[h_r \mid r > 0]$ *is not a polynomial algebra in homogeneous variables. In particular there does not exist a sequence* $a : \mathbb{Z}_+ \to \mathbb{Q}$ *such that* $\mathbb{Z}^{(mix)}[h_r \mid r > 0] = \mathbb{Z}[\hat{h}_k^{\{a\}} \mid k \geq 0]$.

We want tor prove that $\mathbb{Z}^{(mix)}[h_r \mid r > 0]$ is though an integral form of $\mathbb{Q}[h_r \mid r > 0]$, by exhibiting a λ-Garland type $\mathbb{Z}$-basis of $\mathbb{Z}^{(mix)}[h_r \mid r > 0]$. We shall also exhibit a polynomial-like basis of this $\mathbb{Z}$-algebra. In the following $k : \mathbb{Z}_+ \to \mathbb{N}$ will denote a finitely support function. Recall that

$$\{b_k = \prod_{m>0} \lambda_m(\hat{h}_{k_m})\}$$

is a basis of $\mathbb{Z}[\hat{h}_k \mid k > 0]$.

Definition 1.40. *Let us fix the following notation:*

$$b'_k = \prod_{m>0, m\ odd} \lambda_m(\hat{h}_{k_m}) \prod_{m>0, m\ even} \lambda_m(\check{h}_{k_m}),$$
$$B'_\lambda = \{b'_k \mid k : \mathbb{Z}_+ \to \mathbb{N} \text{ is finitely supported}\},$$
$$\mathbb{Z}'_\lambda[h_r \mid r > 0] = \mathbb{Z}\text{-linear span of } B'_\lambda.$$

Remark 1.41. *i.* $b'_k \in \mathbb{Z}^{(mix)}[h_r \mid r > 0]$,

ii. $\hat{h}_k, \bar{h}_k \in \mathbb{Z}'_\lambda[h_r \mid r > 0]$ $\forall k \geq 0$*: indeed* $\hat{h}_k = \lambda_1(\hat{h}_k)$ *and again* $\lambda_2(\check{h}_k) = \bar{h}_{2k}$.

Theorem 1.42. $\mathbb{Z}^{(mix)}[h_r \mid r > 0] = \mathbb{Z}'_\lambda[h_r \mid r > 0]$ *is and integral form of* $\mathbb{Q}[h_r \mid r > 0]$ *and* B'_λ *is* $\mathbb{Z}$*-basis of* $\mathbb{Z}^{(mix)}[h_r \mid r > 0]$.

Proof. Thanks to previous remark, in order to prove that $\mathbb{Z}^{(mix)}[h_r \mid r > 0] = \mathbb{Z}'_\lambda[h_r \mid r > 0]$ it is enough to show that $\mathbb{Z}'_\lambda[h_r \mid r > 0]$ is closed by multiplication. Notice that $\forall m > 0$ $\lambda_{2m}(\hat{h}(u)) \in \mathbb{Z}[\bar{h}_{2r} \mid r > 0][[u]]$ since $\hat{h}(u) \in \mathbb{Z}[\check{h}_k \mid k > 0]$ and $\lambda_2(\check{h}_k) = \bar{h}_k$. Then the fact that $\{b_k\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[\hat{h}_k \mid k > 0]$ implies the following facts, which imply the claim:

i. $\{\prod_{m>0, m \text{ is even}}\}\lambda_m(\check{h}_k) \mid k : \mathbb{Z} \to \mathbb{N}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[\bar{h}_{2k} \mid k > 0]$;

ii. $b_k = \prod_{m>0, m \text{ is odd}}\lambda_m(\hat{h}_k) \cdot b_k^{\text{even}}$ with $b_k^{\text{even}} \in \mathbb{Z}[\bar{h}_{2k} \mid k > 0]$.

iii. $b'_{k'}, b'_{k''} = \prod_{m>0, m \text{ is odd}} \lambda_m(\hat{h}_{k'_m})\lambda_m(\hat{h}_{k''_m}) \cdot \bar{b}'\bar{b}''$ with $\bar{b}', \bar{b}'' \in \mathbb{Z}[\bar{h}_{2k} \mid k > 0]$ is a $\mathbb{Z}$-linear combination of elements of the form $\prod_{m>0, m \text{ is odd}} \lambda_m(\hat{h}_{k_m})\bar{b}$ with $\bar{b} \in \mathbb{Z}[\bar{h}_{2k} \mid k > 0]$.

Finally it is obvious that the $\mathbb{Q}$-span of $\mathbb{Z}'_\lambda[h_r \mid r > 0]$ is $\mathbb{Q}[h_r \mid r > 0]$ and the linear independence of B'_λ now follows by dimension considerations:

$$\#\{b'_k \mid \deg(b'_k) = d\} = \#\{k : \mathbb{Z}_+ \to \mathbb{N} \mid \sum_{m>0} mk_m = d\} =$$
$$\#\{b_k \mid \deg(b_k) = d\} = \dim\mathbb{Q}[h_r \mid r > 0]_d.$$

□

Corollary 1.43. *$\mathbb{Z}^{(mix)}[h_r \mid r > 0]$ is a $\mathbb{Z}[\bar{h}_{2k} \mid k > 0]$-free module with basis*

$$\{\prod_{m>0} \lambda_{2m-1}(\hat{h}_{k_m}) \mid k : \mathbb{Z}_+ \to \mathbb{N} \textit{ is finitely supported.}\}$$

We now give also a polynomial-like $\mathbb{Z}$-basis of $\mathbb{Z}^{(mix)}[h_r \mid r > 0]$, before let us recall the following classical result:

Theorem 1.44 (Euler[4])**.** *The number of partitions of a positive integer n into distinct parts is equal to the number of partitions of n into odd parts.*

Proof. Let us denote by $D(n)$ and by $O(n)$ respectively the number of partitions of n into distinct parts and the number of partitions of n into odd parts, then it is immediate to see that:

$$\sum_{n\geq 0} D(n)x^n = \prod_{i\geq 1}(1 + x^i),$$
$$\sum_{n\geq 0} O(n)x^n = \prod_{i\geq 1} \frac{1}{1 - x^{2i-1}}.$$

The claim follows observing that

$$\prod_{i\geq 1}(1 + x^i) = \prod_{i\geq 1} \frac{1 - x^{2i}}{1 - x^i} = \prod_{i\geq 1} \frac{1}{1 - x^{2i-1}}.$$

□

Lemma 1.45. *The following identities hold in $\mathbb{Q}[h_r \mid r > 0][[u]]$:*

$$\lambda_2(\hat{h}(u^2)) = \hat{h}(u)\hat{h}(-u) = \bar{h}(u^2)^2, \tag{1.45.1}$$
$$\sum_{s=0}^{2r} \hat{h}_{2r-s}\hat{h}_s(-1)^s = \sum_{s=0}^{r} \bar{h}_{2r-2s}\bar{h}_{2s}. \tag{1.45.2}$$

Proof. Equation (1.45.1) follows directly from Definition 1.31 and Notation 1.15, Equation (1.45.2) follows from Equation (1.45.1) and Proposition 1.14. □

Theorem 1.46. *$\mathbb{Z}^{(mix)}[h_r \mid r > 0]$ is a $\mathbb{Z}[\bar{h}_{2r} \mid r > 0]$-free module with basis*

$$\{\prod_{k>0} \hat{h}_k^{\epsilon_i} \mid \epsilon : \mathbb{Z}_+ \to \{0,1\} \text{ is finitely supported}\}.$$

Equivalently

$$B_{q.pol} = \{\prod_{k>0} \hat{h}_k^{\epsilon_k} \prod_{k>0} \bar{h}_k^{d_k} \mid \epsilon : \mathbb{Z}_+ \to \{0,1\} \text{ and } d : \mathbb{Z}_+ \to \mathbb{N} \text{ are finitely supported}\}$$

is a $\mathbb{Z}$-basis of $\mathbb{Z}^{(mix)}[h_r \mid r > 0]$.

Proof. We prove that the $\mathbb{Z}[\bar{h}_{2r} \mid r > 0]$-span of $\{\prod_{k>0} \hat{h}_k^{\epsilon_k} \mid \epsilon \in \{0,1\}\}$ is stable by multiplication by the $\hat{h}_l$'s by induction on $N = \sum k\epsilon_k$. If $N = 0$ the claim is obvious, let us assume that $N > 0$ and the claim holds for all $\tilde{N} < N$. If $l > k$ or $l \neq k$ for all k such that $\epsilon_k = 1$ the claim is obvious. So suppose there exist a k such that $\epsilon_k = 1$ and $l = k$. Let us consider the monomial $\hat{h}_l^2 \hat{p}$ with $\hat{p} = \prod_{k \neq l} \hat{h}_k^{\epsilon_k}$ and $\deg(\hat{p}) = N - l$. Using relation (1.45.2) we have that

$$\hat{p}\hat{h}_l^2 = \hat{p}(2\sum_{j=1}^{l}(-1)^{j+1}\hat{h}_{l+j}\hat{h}_{l-j} + (-1)^l \sum_{j=0}^{l} \bar{h}_{2j}\bar{h}_{2l-2j}),$$

since the right summand is in the $\mathbb{Z}$-span of $B_{q.pol}$, let us focus on monomials of the form $\hat{p}\hat{h}_{l-j}\hat{h}_{l+j}$ for some $j \geq 1$. Since $\deg(\hat{p}) < N$, $\hat{h}_{l-j}\hat{p}$ is in the $\mathbb{Z}[\bar{h}_{2r} \mid r > 0]$-span of $\{\hat{h}_k^{\epsilon_k} \mid \sum k\epsilon_l \leq N - l + l - j = N - j\}$ so that by the induction hypothesis $\hat{h}_{l+j}\hat{h}_{l-j}\hat{p}$ lies in the $\mathbb{Z}[\bar{h}_{2r} \mid r > 0]$-span of $\{\hat{h}_k^{\epsilon_k} \mid \epsilon_k \in \{0,1\}\}$ We are left to prove that $B_{q.pol}$ is linearly independent. Let us observe that the elements of $B_{q.pol}$ of degree d are clearly indexed by the pairs of partitions (λ', λ'') such that $\lambda' \vdash n'$ consist only of not repeating integers, $\lambda'' \vdash n''$ consist of even integers and $n' + n'' = d$ on the other hand the elements of B'_λ of degree d are clearly indexed by the pairs of partitions $(\tilde{\lambda}', \lambda'')$ such that $\tilde{\lambda}' \vdash n'$ consist only of odd integers, $\lambda'' \vdash n''$ consist of even integers and $n' + n'' = d$. It follows from Euler's theorem (see Theorem 1.44) on partitions that these sets have the same cardinality. □

In this last part we still focus on $\mathbb{Z}^{(mix)}[h_r \mid r > 0]$.

Definition 1.47. *Let us consider the sequence $c : \mathbb{Z}_+ \to \mathbb{Q}$ defined by $c(r) = 2^{r-1}$ and set*

$$\hat{h}^{\{c\}}(u) = \sum_{k\geq 0} \hat{h}_k^{\{c\}} u^k.$$

More precisely: $\mathbb{Z}[\hat{h}_r^{\{c\}} \mid r > 0] = \mathbb{Z}^{(sym)}[2^{r-1}h_r \mid r > 0]$.

Remark 1.48. *We want to prove that $\mathbb{Z}[\hat{h}_r^{\{c\}} \mid r > 0] \subseteq \mathbb{Z}[\hat{h}_r, \bar{h}_{2r} \mid r > 0]$, from Proposition 1.19 that the claim follows if we show that there exist two sequences $a, b : \mathbb{Z}_+ \to \mathbb{Q}$ such that $\forall m, p, s > 0$ such that*

$$p \text{ is prime}, \qquad (1.48.1)$$
$$gcd(m, p) = 1,$$

we have

$$p^s | a_{mp^s} - a_{mp^{s-1}},$$
$$p^s | b_{mp^s} - b_{mp^{s-1}},$$

and

$$a_r = \begin{cases} 2^{r-1} \text{ if } r \text{ is odd,} \\ 2^{r-1} - b_{\frac{r}{2}} \text{ if } r \text{ is even} \end{cases}$$

In this case we have that: $\hat{h}^{\{a\}}(u) \in \mathbb{Z}[\hat{h}_r \mid r > 0][[u]]$, $\hat{h}^{\{\tilde{b}\}}(u) \in \mathbb{Z}[\hat{h}_r \mid r > 0][[u]]$, $\hat{h}^{\{c\}}(u) = \hat{h}^{\{a\}}\hat{h}^{\{\tilde{b}\}}(u)$ *where* $\tilde{b} = \{0, b_1, 0, b_2, \dots\}$.

Let now on $m, p, s \in \mathbb{Z}_+$ be such that they satisfy the conditions (1.48.1).

Lemma 1.49. $p^s | 2^{mp^s} - 2^{mp^{s-1}}$ *if and only if* $mp^s \neq 2$, *in particular*

$$\mathbb{Z}[\hat{h}_r^{\{c\}} \mid r > 0] \not\subseteq \mathbb{Z}[\hat{h}_r \mid r > 0] \tag{1.49.1}$$

.

Proof. Let $mp^s = 2$ then the claim holds in this case since $2 \not| 2^{2-1} - 2^{1-1}$, from which follows Relation (1.49.1). If $mp^s \neq 2$, let us observe that $2^{mp^s} - 2^{mp^{s-1}} = 2^{mp^{s-1}}(2^{mp^{s-1}(p-1)} - 1)$. The claim hold remarking that $2^s | 2^{mp^{s-1}-1}$ if $m > 1$ or $s > 1$ and $2^{mp^{s-1}(p-1)} \equiv 1 (\bmod p^s)$ if $s > 1$ and $p \neq 2$. □

Lemma 1.50. *Let* $a : 2\mathbb{Z}_+ - 1 \to \mathbb{Z}$, *let* mp^s *be odd and be such that* $p^s | a_{mp^s} - a_{mp^{s-1}}$, *then is possible to extend* $a : \mathbb{Z}_+ \to \mathbb{Z}$ *so that you have* $p^s | a_{mp^s} - a_{mp^{s-1}}$ *for all* $m, p, s > 0$.

Proof. We will prove that is possible to construct such succession on induction on N elements: $a_2, a_4, \dots, a_{2N}$, recalling that by hypothesis the claim holds if mp^s is odd. If $N = 1$ then of course $a_2 \equiv a_1 (\bmod 2)$ admits solutions. Let $N > 1$ and let be $\prod_{p=1}^r p^{s_p}$ its decomposition on prime factors, hence the following system of congruences is solvable by Chinese remainder theorem:

$$\begin{cases} a_{2N} \equiv a_N (\bmod 2^{s_2+1}), \\ a_{2N} \equiv a_{2\frac{N}{p}} (\bmod p^{s_p}) \text{ if } p | N. \end{cases}$$

□

Corollary 1.51. *From Remark 1.48 and Lemma 1.50 follows that exists* $a = (a_r)_{r>0}$ *such that* $a_r = 2^{r-1}$ *if* r *is odd and such that* $p^s | a_{mp^s} - a_{mp^{s-1}}$.

Proposition 1.52. *For all* $s \in \mathbb{N}$ *we have that* $\hat{h}_s^{\{c\}} \in \mathbb{Z}^{(mix)}[h_r \mid r > 0]$.

Proof. Let be a as in Corollary 1.51 and let us define $b_r = 2^{2r-1} - a_{2r}$. Then $\forall m, p, s > 0$ we have

$$b_{mp^s} - b_{mp^{s-1}} = 2^{mp^s} - 2^{mp^{s-1}} - (a_{2mp^s} - a_{2mp^{s-1}}).$$

Let us observe that by hypothesis $p^s | (a_{2mp^s} - a_{2mp^{s-1}})$ and $2^{mp^s} - 2^{mp^{s-1}} = 2^{mp^{s-1}-1}(2^{mp^{s-1}(p-1)} - 1)$, since $2 \leq 2mp^{s-1} - 1$ and $m^{p^{s-1}(p-1)} \equiv 1 (\bmod p^s)$, then for all m we have that $p^s | b_{mp^s} - b_{mp^{s-1}}$. The claim follows from Remark 1.48. □

1.5 Some non commutative cases

We start this section with a basic remark.

Remark 1.53. *i) Let U_1, U_2 be two $\mathbb{Q}$-algebras, with integral forms respectively $\tilde{U}_1$ and $\tilde{U}_2$. Then $\tilde{U}_1 \otimes_{\mathbb{Z}} \tilde{U}_2$ is an integral form of the $\mathbb{Q}$-algebra $U_1 \otimes_{\mathbb{Q}} U_2$.*

ii) Let U be an associative unitary $\mathbb{Q}$-algebra (not necessarily commutative) and $U_1, U_2 \subseteq U$ be two $\mathbb{Q}$-subalgebras such that $U \cong U_1 \otimes_{\mathbb{Q}} U_2$ as $\mathbb{Q}$-vector spaces. If $\tilde{U}_1, \tilde{U}_2$ are integral forms of U_1, U_2, then $\tilde{U}_1 \otimes_{\mathbb{Z}} \tilde{U}_2$ is an integral form of U if and only if $\tilde{U}_2\tilde{U}_1 \subseteq \tilde{U}_1\tilde{U}_2$.

Remark 1.53,ii) suggests that if we have a (linear) decomposition of an algebra U as an ordered tensor product of polynomial algebras U_i ($i = 1, ..., N$), that is we have a linear isomorphism

$$U \cong U_1 \otimes_{\mathbb{Q}} ... \otimes_{\mathbb{Q}} U_N,$$

then one can tackle the problem of finding an integral form of U by studying the commutation relations among the elements of some suitable integral forms of the U_i's.

Gluing together in a non commutative way the different integral forms of the algebras of polynomials discussed in Section 1 is the aim of this section, which collects the preliminary work of the work: the main results of the following sections are applications of the formulas found here.

Notation 1.54. *Let U be an associative $\mathbb{Q}$-algebra and $a \in U$.*

We denote by L_a and R_a respectively the left and right multiplication by a; of course $L_a - R_a = [a, \cdot] = -[\cdot, a]$.

Lemma 1.55. *Let U be an associative unitary $\mathbb{Q}$-algebra.*

Consider elements $a, b, c \in U[[u]]$. Then:

i) if $a, b \in uU[[u]]$ and $[a, b] = 0$ we have

$$\exp(a \pm b) = \exp(a)\exp(b)^{\pm 1};$$

ii) $[L_a, R_a] = 0$;

iii) if f is an algebra-homomorphism and $f(a) = a$ we have

$$[f, L_a] = [f, R_a] = 0;$$

iv) if $a \in uU[[u]]$ then $L_a, R_a \in End(U)[[u]]$ and we have

$$\exp(L_a) = L_{\exp(a)},\ \exp(R_a) = R_{\exp(a)},\ \exp(R_a) = L_{\exp(a)} \exp([\cdot, a]);$$

v) if $a, c \in uU[[u]]$ we have

$$ab = bc \Leftrightarrow \exp(a)b = b\exp(c);$$

vi) if $b \in uU[[u]]$ and $[b, c] = 0$ we have

$$[a, b] = c \Leftrightarrow a\exp(b) = \exp(b)(a + c);$$

vii) if $a, b, c \in uU[[u]]$ and $[a, c] = [b, c] = 0$ then

$$[a, b] = c \Leftrightarrow \exp(a)\exp(b) = \exp(b)\exp(a)\exp(c)$$

viii) if $a, b, c \in uU[[u]]$ and $[a, c] = [b, c] = 0$ then

$$[a, b] = c \Rightarrow \exp(a + b) = \exp(a)\exp(b)\exp(-c/2);$$

ix) if $[a, \frac{\mathrm{d}}{\mathrm{d}u}(a)] = 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}u}(\exp(a)) = \frac{\mathrm{d}}{\mathrm{d}u}(a)\exp(a) = \exp(a)\frac{\mathrm{d}}{\mathrm{d}u}(a).$$

x) if $a(u) = \sum_{r\in\mathbb{N}} a_r u^r$ ($a_r \in U\ \forall r \in \mathbb{N}$) and $\alpha \in U$ we have

$$\frac{\mathrm{d}}{\mathrm{d}u}a(u) = a(u)\alpha \Leftrightarrow a(u) = a_0 \exp(\alpha u)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}u}a(u) = \alpha a(u) \Leftrightarrow a(u) = \exp(\alpha u) a_0.$$

Proof. Statements v) and vi) are immediate consequence respectively of the fact that for all $n \in \mathbb{N}$:

v) $a^n b = bc^n$ (that is also $(\exp(a) - 1)^n b = b(\exp(c) - 1)^n$);

vi) $ab^{(n)} = b^{(n)}a + b^{(n-1)}c$ (that is also $a(\exp(b) - 1)^n = (\exp(b) - 1)^n a + n(\exp(b) - 1)^n c)$.

vii) follows from i), v) and vi).

viii) follows from vii):

$$(a+b)^{(n)} = \sum_{\substack{r,s,t:\\ r+s+2t=n}} \frac{(-1)^t}{2^t} a^{(r)} b^{(s)} c^{(t)}.$$

The other points are obvious. □

Proposition 1.56. *Let us fix $m \in \mathbb{Z}$ and consider the $\mathbb{Q}$-algebra structure on $U = \mathbb{Q}[x] \otimes_{\mathbb{Q}} \mathbb{Q}[h]$ given by $xh = (h-m)x$. Then $\mathbb{Z}^{(div)}[x] \otimes_{\mathbb{Z}} \mathbb{Z}^{(bin)}[h]$ and $\mathbb{Z}^{(bin)}[h] \otimes_{\mathbb{Z}} \mathbb{Z}^{(div)}[x]$ are integral forms of U: their images in U are closed under multiplication, and coincide. Indeed*

$$x^{(k)}\binom{h}{l} = \binom{h-mk}{l} x^{(k)}\ \forall k,l \in \mathbb{N} \tag{1.56.1}$$

or equivalently, with a notation that will be useful in the following,

$$\exp(xu)(1+v)^h = (1+v)^h \exp\left(\frac{xu}{(1+v)^m}\right).$$

Proof. The relation between x and h can be written as

$$xP(h) = P(h-m)x$$

and

$$x^{(k)}P(h) = P(h-mk)x^{(k)}$$

for all $P \in \mathbb{Q}[h]$ and for all $k > 0$. In particular it holds for $P(h) = \binom{h}{l}$, that is

$$x(1+v)^h = (1+v)^{h-m}x = (1+v)^h \frac{x}{(1+v)^m} \tag{1.56.2}$$

and

$$x^{(k)}(1+v)^h = (1+v)^h \left(\frac{x}{(1+v)^m}\right)^{(k)}. \tag{1.56.3}$$

The conclusion follows multiplying by u^k and summing over k. □

Proposition 1.57. *Let us fix $m \in \mathbb{Z}$ and consider the $\mathbb{Q}$-algebra structure on*

$$U = \mathbb{Q}[x] \otimes_{\mathbb{Q}} \mathbb{Q}[z] \otimes_{\mathbb{Q}} \mathbb{Q}[y]$$

defined by $[x,z] = [y,z] = 0$, $[x,y] = mz$.

Then $\mathbb{Z}^{(div)}[x] \otimes_{\mathbb{Z}} \mathbb{Z}^{(div)}[z] \otimes_{\mathbb{Z}} \mathbb{Z}^{(div)}[y]$ is an integral form of U.

Proof. Since z commutes with x and y we just have to straighten $y^{(r)}x^{(s)}$. Thus the claim is a straightforward consequence of Lemma 1.55,vii):

$$\exp(yu)\exp(xv) = \exp(xv)\exp(zuv)^{-m}\exp(yu).$$

□

Proposition 1.58. *Let us fix $m, l \in \mathbb{Z}$ and consider the $\mathbb{Q}$-algebra structure on $U = \mathbb{Q}[h_r \mid r < 0] \otimes_{\mathbb{Q}} \mathbb{Q}[h_0, c] \otimes_{\mathbb{Q}} \mathbb{Q}[h_r \mid r > 0]$ given by*

$$[c, h_r] = 0, \ [h_r, h_s] = \delta_{r+s,0} r(m + (-1)^r l)c \ \forall r, s \in \mathbb{Z}.$$

Then setting $(h_+)_r = h_r$ and $(h_-)_r = h_{-r}$ $\forall r > 0$, recalling the notation $\mathbb{Z}[\hat{h}_{\pm k} \mid k > 0] = \mathbb{Z}^{(sym)}[h_{\pm r} \mid r > 0]$ (see Example 1.2 and Formula 1.2) and defining $U_{\mathbb{Z}}$ to be the $\mathbb{Z}$-subalgebra of U generated by $U_{\mathbb{Z}}^{\pm} = \mathbb{Z}^{(sym)}[h_{\pm r} \mid r > 0]$ and $U_{\mathbb{Z}}^0 = \mathbb{Z}^{(bin)}[h_0, c]$, we have that

$$\hat{h}_+(u)\hat{h}_-(v) = \hat{h}_-(v)(1-uv)^{-mc}(1+uv)^{-lc}\hat{h}_+(u) \tag{1.58.1}$$

and $U_{\mathbb{Z}} = U_{\mathbb{Z}}^- U_{\mathbb{Z}}^0 U_{\mathbb{Z}}^+$, so that

$$U_{\mathbb{Z}} \cong \mathbb{Z}^{(sym)}[h_{-r} \mid r > 0] \otimes_{\mathbb{Z}} \mathbb{Z}^{(bin)}[h_0, c] \otimes_{\mathbb{Z}} \mathbb{Z}^{(sym)}[h_r \mid r > 0]$$

is an integral form of U.

Proof. Relation (1.58.1) follows from Lemma 1.55, vii) remarking that

$$\Big[\sum_{r>0}(-1)^{r-1}\frac{h_r}{r}u^r, \sum_{s>0}(-1)^{s-1}\frac{h_{-s}}{s}v^s\Big] = c\sum_{r>0}\frac{m+(-1)^r\,l}{r}u^r v^r =$$

$$= -mc\ln(1-uv) - lc\ln(1+uv).$$

Of course $U_{\mathbb{Z}}^0 U_{\mathbb{Z}}^- = U_{\mathbb{Z}}^- U_{\mathbb{Z}}^0$ is a $\mathbb{Z}$-subalgebra of U, $U_{\mathbb{Z}}^- U_{\mathbb{Z}}^0 U_{\mathbb{Z}}^+ \subseteq U_{\mathbb{Z}}$, $U_{\mathbb{Z}}$ is generated by $U_{\mathbb{Z}}^- U_{\mathbb{Z}}^0 U_{\mathbb{Z}}^+$ as $\mathbb{Z}$-algebra and $U_{\mathbb{Z}}^- U_{\mathbb{Z}}^0 U_{\mathbb{Z}}^+ \cong U_{\mathbb{Z}}^- \otimes_{\mathbb{Z}} U_{\mathbb{Z}}^0 \otimes_{\mathbb{Z}} U_{\mathbb{Z}}^+$ as $\mathbb{Z}$-modules.

Hence we need to prove that $U_{\mathbb{Z}}^- U_{\mathbb{Z}}^0 U_{\mathbb{Z}}^+$ is a $\mathbb{Z}$-subalgebra of U, or equivalently that it is closed under left multiplication by $U_{\mathbb{Z}}^+$ (because it is obviously closed under left multiplication by $U_{\mathbb{Z}}^- U_{\mathbb{Z}}^0$), which is a straightforward consequence of relation (1.58.1). □

Lemma 1.59. *Let U be a $\mathbb{Q}$-algebra, $T : U \to U$ an automorphism,*

$$f \in \sum_{r>0}\mathbb{Z}T^r u^r \subseteq End(U)[[u]] \subseteq End(U[[u]]),$$

$h \in uU[[u]]$ and $x \in U$ such that $T(h) = h$ and $[x, h] = f(x)$. Then

$$x\exp(h) = \exp(h)\cdot\exp(f)(x).$$

Proof. By Lemma 1.55,iv)

$$x\exp(h) = \exp(h)\exp([\cdot, h])(x),$$

so we have to prove that $\exp([\cdot, h])(x) = \exp(f)(x)$, or equivalently that $[\cdot, h]^n(x) = f^n(x)$ for all $n \in \mathbb{N}$.

If $n = 0, 1$ the claim is obvious; if $n > 1$, $f^{n-1}(x) = \sum_{r>0} a_r T^r u^r(x)$ with $a_r \in \mathbb{Z}$ for all $r > 0$, f commutes with T, and by the inductive hypothesis and Lemma 1.55,iii)

$$[\cdot, h]^n(x) = [f^{n-1}(x), h] = \left[\sum_{r>0} a_r T^r u^r(x), h\right] =$$

$$= \sum_{r>0} a_r u^r T^r([x, h]) = \sum a_r u^r T^r f(x) = f\sum a_r u^r T^r(x) = f(f^{n-1}(x)) = f^n(x).$$

□

Proposition 1.60. *Let us fix integers m_d's ($d > 0$) and consider elements $\{h_r,\ x_s \mid r > 0, s \in \mathbb{Z}\}$ in a $\mathbb{Q}$-algebra U such that*

$$[h_r, x_s] = \sum_{d|r} d m_d x_{r+s} \ \ \forall r > 0, s \in \mathbb{Z}.$$

Let T be an algebra automorphism of U such that

$$T(h_r) = h_r \text{ and } T(x_s) = x_{s-1} \ \forall r > 0, s \in \mathbb{Z}.$$

Then, recalling the notation $\mathbb{Z}[\hat{h}_k \mid k > 0] = \mathbb{Z}^{(sym)}[h_r \mid r > 0]$, we have that

$$x_r \hat{h}_+(u) = \hat{h}_+(u) \cdot \left(\prod_{d>0} (1 - (-T^{-1}u)^d)^{-m_d} \right) (x_r).$$

If moreover the subalgebras of U generated by $\{h_r \mid r > 0\}$ and $\{x_r \mid r \in \mathbb{Z}\}$ are isomorphic respectively to $\mathbb{Q}[h_r \mid r > 0]$ and $\mathbb{Q}[x_r \mid r \in \mathbb{Z}]$ and there is a $\mathbb{Q}$-linear isomorphism $U \cong \mathbb{Q}[h_r \mid r > 0] \otimes_{\mathbb{Q}} \mathbb{Q}[x_r \mid r \in \mathbb{Z}]$ then

$$\mathbb{Z}^{(sym)}[h_r \mid r > 0] \otimes_{\mathbb{Z}} \mathbb{Z}^{(div)}[x_r \mid r \in \mathbb{Z}]$$

is an integral form of U.

Proof. This is an application of Lemma 1.59: let $h = \sum_{r>0} (-1)^{r-1} \frac{h_r}{r} u^r$; then

$$[x_0, h] = \sum_{r>0} \frac{(-1)^r}{r} u^r \sum_{d|r} d m_d T^{-r}(x_0) =$$

$$= \sum_{d>0} \sum_{s>0} \frac{(-1)^{ds}}{s} m_d T^{-ds} u^{ds}(x_0) = f(x_0)$$

where

$$f = -\sum_{d>0} m_d \ln\Big(1 - (-1)^d T^{-d} u^d\Big).$$

Then

$$x_0 \hat{h}_+(u) = \hat{h}_+(u) \cdot \exp(f)(x_0) = \hat{h}_+(u) \cdot \left(\prod_{d>0} (1 - (-T^{-1}u)^d)^{-m_d} \right)(x_0),$$

and the analogous statement for x_r follows applying T^{-r}.

Remark that $\prod_{d>0} (1 - (-T^{-1}u)^d)^{-m_d} = \sum_{r\geq 0} a_r T^{-r} u^r$ with $a_r \in \mathbb{Z}\ \forall r \in \mathbb{N}$; the hypothesis on the commutativity of the subalgebra generated by the x_r's implies that $(\sum_{r\geq 0} a_r x_r u^r)^{(k)}$ lies in the subalgebra of U generated by the divided powers $\{x_r^{(k)} \mid r \in \mathbb{Z}, k \geq 0\}$, which allows to conclude the proof thanks to the last hypotheses on the structure of U. □

Remark 1.61. *Proposition 1.60 implies Proposition 1.56: indeed when $m_1 = m$, $m_d = 0\ \forall d > 1$ we have a projection $h_r \mapsto h, x_r \mapsto x$, which maps $\exp(x_0 u)$ to $\exp(xu)$, $\hat{h}(u)$ to $(1+u)^h$ and T to the identity.*

Remark 1.62. *Proposition 1.60 implies Proposition 1.56: indeed when $m_1 = m$, $m_d = 0\ \forall d > 1$ we have a projection $h_r \mapsto h, x_r \mapsto x$, which maps $\exp(x_0 u)$ to $\exp(xu)$, $\hat{h}(u)$ to $(1+u)^h$ and T to the identity.*

Chapter 2

Kac-Moody algebras

In this part we will recall general notions about Kac-Moody algebras, in particular those of finite and affine type. We systematically refer to [7] and [8]. As announced in the Introduction , Section 2.1 is devoted recall definition of affine and finite Kac-Mooody algebras and Section 2.2 recall the loop construction of affine algebras and Section 2.3 is devoted to recall the results of Kostant, Garland and Mitzman on the integral forms.

2.1 Definition of finite and affine Kac-Moody Algebras

Fix $n \in \mathbb{N}_{>0}$ and set $I = \{1, \dots, n\}$

Definition 2.1. *A generalized Cartan Matrix is a $n \times n$ matrix $A = (a_{i,j})_{i,j \in I}$ with integral entries such that*

$$\begin{aligned} &a_{i,i} = 2 \\ &a_{i,j} \leq 0 \textit{ if } i \neq j \\ &a_{i,j} = 0 \Leftrightarrow a_{j,i} = 0 \end{aligned}$$

A is said to be decomposable if there exists a nonempty proper subset $\tilde{I} \subseteq I$ such that $a_{i,j} = 0$ whenever $i \in \tilde{I}$ and $j \notin \tilde{I}$, A is indecomposable if it is not decomposable. From now on we shall assume that A is indecomposable. We say that A is of finite type if all the principal minors of A are positive, of affine type if the proper principal minors of A are positive and A has determinant 0, and of indefinite type otherwise.

Finite and affine Cartan matrices are symmetrizable, that is, there exist a Diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$ such that DA is symmetric, moreover the diagonal entries d_i's can be chosen to be coprime positive integers, this condition determines them uniquely. It is a classical result that finite Cartan matrices classify simple Lie algebras of finite dimension.

Given a generalized Cartan Matrix A one can construct the associated Kac Moody algebra $\mathfrak{g}(A)$ as following.

Definition 2.2. *The Kac-Moody Algebra $\mathfrak{g}(A)$ associated to A is the Lie Algebra generated by the*

$$\{e_i, f_i, h_i \mid i \in I\}$$

with relations:

$$[e_i, f_j] = \delta_{i,j} h_i,$$
$$[h_i, e_j] = a_{i,j} e_j,$$
$$[h_i, f_j] = -a_{i,j} f_j,$$
$$(\,ade_i)^{1-a_{i,j}}(e_j) = 0 \text{ if } i \neq j,$$
$$(\,adf_i)^{1-a_{i,j}}(f_j) = 0 \text{ if } i \neq j.$$

Definition 2.3. *To A is associated its Dynkin diagram Γ, that is an oriented graph with vertices labeled by I and i-th vertex is connected to j-th vertex with* $\max\{|a_{i,j}|, |a_{j,i}|\}$ *edges with an arrow pointing from i to j if $|a_{i,j}| < |a_{j,i}|$.*
An automorphism χ of Γ is a permutation of its nodes such that $a_{i,j} = a_{\chi(i),\chi(j)}$, let us denote by k the order of χ. It is immediate to see that, if Γ is of finite type, then $k \in \{1,2,3\}$.

2.2 Loop construction

Consider a finite dimensional simple Lie algebra $\mathfrak{g}$, with Cartan Matrix $A_0 = (a_{i,j})_{i,j\in I}$, Dynkin Diagram Γ. And let χ be an automorphism of Γ, then χ induces an automorphism on $\mathfrak{g}$ defined on the generators by $\chi \cdot e_i = e_{\chi(i)}, \chi \cdot f_i = f_{\chi(i)}$ and its eigenvalues are $e^{\frac{2\pi i r}{k}}$ for $r = 0, \ldots, k$. Consider the decomposition of $\mathfrak{g}$ into eigenspaces

$$\mathfrak{g} = \bigoplus_{r=0}^{k-1} \mathfrak{g}^r$$

where $\mathfrak{g}^r$ is the eigenspace relative to the eigenvalue $e^{\frac{2\pi i r}{k}}$, of course it is a $\mathbb{Z}/k\mathbb{Z}$-grading that is $[\mathfrak{g}^r, \mathfrak{g}^s] \subseteq \mathfrak{g}^{r+s}\ \forall r, s \in \mathbb{Z}/k\mathbb{Z}$, in particular $\mathfrak{g}^0$ is a Lie subalgebra of $\mathfrak{g}$ (and $\mathfrak{g}_0$ is itself a finite dimensional simple Lie algebra) and $\mathfrak{g}^r$ is a $\mathfrak{g}^0$-module. The $\mathbb{Z}/k\mathbb{Z}$-grading induced by χ allows to construct the χ-Loop algebra of $\mathfrak{g}$, that is

$$\mathcal{L}^{\chi}(\mathfrak{g}) = \bigoplus_{r\in\mathbb{Z}} \mathfrak{g}^r \otimes \mathbb{C}[t^r].$$

The affine Kac-Moody algebras is a non trivial central extension of $\mathcal{L}^{\chi}(\mathfrak{g})$ via the Killing form, that is

$$\hat{\mathfrak{g}}^{\chi} = \mathcal{L}^{\chi}(\mathfrak{g}) \oplus \mathbb{C}c.$$

It was proven by Kac (see [8]) that the Affine Cartan matrices classify the affine Kac-Moody algebras, in particular they are said to be untwisted if $k = 1$, otherwise are said to be twisted.
Let I, $A = (a_{i,j})_{i\in I}$ and $\hat{\Gamma}$ be respectively the set of indices, the generalized Cartan matrix and the Dynkin diagram of $\hat{\mathfrak{g}}^{\chi}$ and let I_0, A_0 and Γ_0 be respectively the set of indices, the Cartan Matrix and the Dynkin Diagram of $\mathfrak{g}^0$. It is possible to identify I with $\{0, 1, \ldots, n\}$ and I_0 with $\{1, \ldots, n\}$, so that $I = I_0 \cup \{0\}$. It is possible to identify A_0 with $(a_{i,j})_{i,j\in I_0}$, and $\hat{\Gamma}$ can be construct by adding a vertex, labeled by 0, to Γ_0.
The affine Kac-Moody algebras are of type

$$A_n^{(k)}, B_{n+2}^{(1)}, C_{n+1}^{(1)}, D_{n+3}^{(k)}, E_6^{(k)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)}, D_4^{(3)}$$

for $k = 1, 2$ and $n \geq 1$.

$\hat{\mathfrak{g}}^{\chi}$ admits a presentation by generators and relations, that is:

Definition 2.4. *$\hat{\mathfrak{g}}^\chi$ is the Lie algebra generated by $\{x^+_{i,r}, x^-_{i,r}, h_{i,r}, c \mid i \in I_0, \tilde{d}_i|r \in \mathbb{Z}\}$ with relations*

$$\begin{aligned}
&[c,\cdot] = 0\\
&[h_{i,r}, h_{j,s}] = r\delta_{r+s,0}\frac{a_{i,j;r}}{d_j}Dc\\
&[x^+_{i,r}, x^-_{j,r}] = \delta_{i,j}(h_{i,r+s} + r\delta_{r+s,0}\frac{Dc}{d_j});\\
&[h_{i,r}, x^\pm_{j,s}] = \pm a_{i,j;r}x^\pm_{j,r+s};\\
&[x^\pm_{i,r}, x^\pm_{i,s}] = 0 \text{ if } (\hat{\mathfrak{g}}^\chi, d_i) \neq (A^{(2)}_{2n}, 1) \text{ or } r+s \text{ is even};\\
&[x^\pm_{i,r}, x^\pm_{i,s}] + [x^\pm_{i,r+1}, x^\pm_{i,s-1}] = 0 \text{ if } (\hat{\mathfrak{g}}^\chi, d_i) = (A^{(2)}_{2n}, 1) \text{ and } r+s \text{ is odd};\\
&[x^\pm_{1,r}, [x^\pm_{1,s}, x^\pm_{1,t}]] = 0\\
&(adx^\pm_{i,r})^{1-a_{i,j}}(x^\pm_{j,s}) = 0 \text{ if } i \neq j.
\end{aligned}$$

where $D = \max\{d_i \mid i \in I_0\}$,

$$\begin{aligned}
&a_{i,j;r} = 2(2+(-1)^r) \text{ if } i=j, d_i = 1 \text{ and } \hat{\mathfrak{g}}^\chi = A^{(2)}_{2n},\\
&a_{i,j;r} = a_{i,j} \text{ otherwise.}
\end{aligned}$$

and

$$\tilde{d}_i = \begin{cases} 1 \text{ if } k = 1 \text{ or } \hat{\mathfrak{g}}^\chi = A^{(2)}_{2n} \\ d_i \text{ otherwise,} \end{cases}$$

Remark 2.5. *Remark that this presentation implies that*

$$\begin{aligned}
x^\pm_r &\mapsto x^\pm_{i,r},\\
h_r &\mapsto h_{i,r},\\
c &\mapsto \frac{d}{d_j}c;
\end{aligned}$$

defines an embedding

$$\begin{aligned}
\varphi_i : A^{(1)}_1 \hookrightarrow \hat{\mathfrak{g}}^\chi \text{ if } (\hat{\mathfrak{g}}^\chi, d_i) \neq (A^{(2)}_{2n}, 1),\\
\varphi_i : A^{(2)}_2 \hookrightarrow \hat{\mathfrak{g}}^\chi \text{ if } (\hat{\mathfrak{g}}^\chi, d_i) = (A^{(2)}_{2n}, 1).
\end{aligned}$$

Remark 2.6. *The isomorphism between the two presentations $\hat{\mathfrak{g}}^\chi$ identifies*

$$\begin{aligned}
e_i &\leftrightarrow x^+_{i,0},\\
f_i &\leftrightarrow x^-_{i,0},\\
h_i &\leftrightarrow h_{i,0},
\end{aligned}$$

($i \in I_0$).

Definition 2.7. *Let us denote by Q and Q_0 the root lattice of respectively $\mathfrak{g}$ and $\mathfrak{g}^0$, that is $Q = \bigoplus_{i\in I}\mathbb{Z}\alpha_i$ and $Q_0 = \bigoplus_{i\in I_0}\mathbb{Z}\alpha_i$. A is symmetrizable, then DA induces a symmetric bilinear form on Q, that is $(\ ,\ )$. Since A is affine then DA has kernel of dimension one generated by an element $\delta \in Q$, moreover $\theta = \delta - \alpha_0 \in Q_0$ hence $Q = Q_0 \oplus \mathbb{Z}\delta$ and $(\ ,\ )|_{(Q_0,Q_0)} > 0$. Moreover A induces $Q \to \mathfrak{h}^* : \alpha_j(h_i) = a_{j,i}$.*

Definition 2.8. *The Weyl group W of $\mathfrak{g}$ is the subgroup of $Aut(Q)$ generated by the elements $\sigma_i(\alpha_j) = \alpha_j - a_{i,j}\alpha_i$ for all $i \in I$, let us denote by W_0 the subgroup of W generated by σ_i for $i \in I_0$. Remark that $\sigma_i(\delta) = \delta, \forall i \in I_0$.*

Remark 2.9. *W preserves $(\cdot,\cdot)$, that is $(w(\alpha), w(\beta)) = (\alpha, \beta)$ for all $\alpha, \beta \in Q$.*

Remark 2.10. *$\mathfrak{g}$ is Q-graded: $deg(e_i) = \alpha_i = -deg(f_i)$ and $deg(h_i) = 0$, hence $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in Q\setminus\{0\}} \mathfrak{g}_\alpha)$, equivalently: $deg(x^{\pm}_{i,r}) = \pm\alpha_i + r\delta$, $deg(h_{i,r}) = r\delta$ and $deg(c) = 0$.*

Definition 2.11. *Let us define the set of roots Φ of $\mathfrak{g}$ by*

$$\Phi = \{\alpha \in Q \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}.$$

The positive and negative roots are respectively $\Phi_+ = \Phi \cap \sum_{i\in I} \mathbb{N}\alpha_i$ and $\Phi_- = \Phi_+$, moreover $\Phi = \Phi_- \cup \Phi_+$. Let us denote by $\Phi_0 = W_0 \cdot \{\alpha_i \mid i \in I_0\}$.

Remark 2.12. *Let us remark that ade_i and adf_i) are nilpotent endomorphisms $\forall i \in I$, $adx^{\pm}_{i,r}$ are nilpotent endomorphisms $\forall i \in I_0$.*

Definition 2.13. *For all $i \in I$, let us define the following automorphisms of $\mathfrak{g}$:*

$$\tau_i = \exp(ade_i)\exp(-adf_i)\exp(ade_i),$$

and for all $i \in I_0$, let us define the following automorphisms of $\mathfrak{g}$:

$$\tau_{i,r} = \exp\left(adx^+_{i,r}\right)\exp\left(-adx^-_{i,r}\right)\exp\left(adx^+_{i,r}\right).$$

Of course if $i \in I_0$ we have that $\tau_i = \tau_{i,0}$. Denote by W_T the group generated by $\{\tau_i \mid i \in I\}$, obviously $\tau_{i,r} = T_i^{-r}\tau_i T_i^r$, for all $i \in I_0$ and for all $r \in \mathbb{Z}$.

Remark 2.14. *It is well known that $\tau_i(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma_i(\alpha)}$, for all $i \in I$ and $\forall\alpha \in Q$. In particular Φ is W-stable.*

Remark 2.15. *We have that $dim\mathfrak{g}_\alpha < \infty$ for all $\alpha \in \Phi$. There exist a unique element θ of Φ_0 such that $\theta - \alpha \in \sum_{i\in I} \mathbb{N}\alpha_i$ $\forall\alpha \in \Phi_0$, θ is called the highest root of Φ_0. There exist a unique element θ_s of Φ_0 such that $\theta_s - \alpha \in \sum_{i\in I} \mathbb{N}\alpha_i$ $\forall\alpha \in \Phi$ such that $(\alpha, \alpha) = 2$, θ_s is called the highest short root of Φ_0.*

The root system Φ of $\mathfrak{g}$ decompose into two parts, that is $\Phi = \Phi^{re} \cup \Phi^{im}$, where $\Phi^{re} = W \cdot \{\alpha_i \mid i \in I\}$ and $\Phi^{im} = \{m\delta \mid m \in \mathbb{Z}\}$, whose elements are called respectively real and imaginary roots. It is possible to describe Φ^{re} in terms of Φ_0, that is

$$\Phi^{re} = \begin{cases} \{\alpha + m\delta \mid \alpha \in \Phi_0,\ m \in \mathbb{Z}\} \text{ if } k = 1 \\ \{\alpha + m\delta \mid \alpha \in \Phi_0,\ m \in \mathbb{Z}\} \cup \{2\alpha + (2m+1)\delta \mid d_\alpha = 1,\ m \in \mathbb{Z}\} \text{ if } \hat{\mathfrak{g}}^\chi = A^{(2)}_{2n} \\ \{\alpha + md_\alpha\delta \mid \alpha \in \Phi_0,\ m \in \mathbb{Z}\} \text{ otherwise.} \end{cases}$$

where $d_\alpha = \frac{(\alpha,\alpha)}{2}$.

The weight lattice $\hat{P} \subseteq \mathbb{R} \otimes \mathbb{Z}Q_0$ is $\hat{P} = \bigoplus_{i\in I_0} \mathbb{Z}\lambda'_i$ where $\lambda'_i \in \mathbb{R} \otimes \mathbb{Z}Q_0$ is defined by $(\lambda'_i, \alpha_j) = \delta_{i,j}$ for all $i, j \in I_0$, Q_0 naturally embeds in $\hat{P}$. It is worth introducing another important sublattice P of $\hat{P}$ as $P = \bigoplus_{i\in I_0} \mathbb{Z}\lambda_i$, where $\lambda_i = \tilde{d}_i\lambda'_i$ for all $i \in I$; obviously $P \subseteq \hat{P}$. As subgroups of $Aut(Q)$ we have $W \leq \hat{P} \rtimes W_0 = \hat{W}$, in particular the equality holds if $\mathfrak{g} = A^{(2)}_{2n}$. $\hat{W}$ is called the extended Weyl group of $\mathfrak{g}$ and we have also $\hat{W} = W \rtimes \tau$, where $\tau = Aut(\Gamma) \cap \hat{W}$. $\hat{P}$ has a realisation as a group of transformations of Q as follows: define $t : \hat{P} \to \mathrm{Hom}(Q, Q)$ by setting $t_x(\alpha) = \alpha - (x, \alpha)\delta$, for all $x \in \hat{P}$.

2.3 Kostant, Garland and Mitzman integral form

Let $\mathfrak{g}$ be a finite or affine algebra with set of indices I.Let us denote by $\mathcal{U}$ its universal enveloping algebra and by $\mathcal{U}_{\mathbb{Z}}$ the $\mathbb{Z}$-subalgebra of $\mathcal{U}$ generated by $\{e_i^{(r)}, f_i^{(r)} \mid i \in I, r \in \mathbb{N}\}$. The study of $\mathcal{U}_{\mathbb{Z}}$ was begun by Kostant in the 1950s in the case where $\mathfrak{g}$ is finite and later extended to the related case by Garland in the 1970s and Mitzman in the 1980s. The investigation of $\mathcal{U}_{\mathbb{Z}}$ passes in all cases by the introduction of a Chevalley basis for the algebra $\mathfrak{g}$. If $\mathfrak{g}$ is finite we have the following result due to Kostant.

Notation 2.16. *In the following theorems where we speak about "the algebra of divided powers in the positive and negative roots vectors" we mean the $\mathbb{Z}$-subalgebra generated by $\{e_i^{(r)|i\in I, r\in\mathbb{N}}\}$ which is a free $\mathbb{Z}$-module with basis the ordered monomials in the x_α's.*

Theorem 2.17. *$\mathcal{U}_{\mathbb{Z}}$ is an integral form of $\mathcal{U}$, more precisely:*

$$\mathcal{U}_{\mathbb{Z}} \cong \mathcal{U}_{\mathbb{Z}}^{+} \otimes \mathcal{U}_{\mathbb{Z}}^{\mathfrak{h}} \otimes \mathcal{U}_{\mathbb{Z}}^{-}$$

where $\mathcal{U}_{\mathbb{Z}}^{+}$ and $\mathcal{U}_{\mathbb{Z}}^{-}$ are the algebras of divided powers respectively in the positive and negative root vectors, $\mathcal{U}_{\mathbb{Z}}^{\mathfrak{h}} = \mathbb{Z}^{(bin)}[h_i \mid i \in I]$ is the algebras of binomials in the h_i.

Let now on $\mathfrak{g}$ be affine.

Definition 2.18. *The Garland Λ-imaginary root vectors are the elements of $\mathcal{U}^{im,\pm}$: $\Lambda_k(\xi(i,m))$ are the elements of defined recursively for $k \geq -1, \tilde{d}_i|m, \pm m > 0, i \in I$ by*

$$\Lambda_{-1}(\xi(i,m)) = 1, \;\; k\Lambda_{k-1}(\xi(i,m)) = \sum_{\substack{r\geq 0, s>0 \\ r+s=k}} \Lambda_{r-1}(\xi(i,m)) e_{i,ms}.$$

then $\mathcal{U}_{\mathbb{Z}}^{im,\pm}$ is the $\mathbb{Z}$-algebra whose basis consisting in the following sets:

$$B^{im,\pm} = \left\{ \prod_{m>0} \Lambda_{k_m-1}(\xi(i,m)) \mid k_m \geq 0 \;\forall m,\; \#\{\pm m > 0 \mid k_m \neq 0\} < \infty, i \in I \right\}.$$

Theorem 2.19. *$\mathcal{U}_{\mathbb{Z}}$ is an integral form of $\mathcal{U}$, more precisely:*

$$\mathcal{U}_{\mathbb{Z}} \cong \mathcal{U}_{\mathbb{Z}}^{+} \otimes \mathcal{U}_{\mathbb{Z}}^{im,+} \otimes \mathcal{U}_{\mathbb{Z}}^{\mathfrak{h}} \otimes \mathcal{U}_{\mathbb{Z}}^{im,+} \otimes \mathcal{U}_{\mathbb{Z}}^{-}$$

where $\mathcal{U}_{\mathbb{Z}}^{+}$ and $\mathcal{U}_{\mathbb{Z}}^{-}$ are the algebras of divided powers respectively in the e_α with $\alpha \in \Phi^{re,+}$ and $\alpha \in \Phi^{re,-}$, $\mathcal{U}_{\mathbb{Z}}^{\mathfrak{h}}$ is an algebras of binomials in the h_i for $i \in I_0$. $\mathcal{U}_{\mathbb{Z}}^{im,\pm}$ is described in definition 2.18.

Even though it was stated in the literature (see [3] for example), it is not clear from this description that $\mathcal{U}_{\mathbb{Z}}^{im,+}$ and $\mathcal{U}_{\mathbb{Z}}^{im,-}$ are algebras of polynomials, hence we decide to fill this gap giving the proof of this fact (see 9.B and Proposition 1.14).

The question that arises naturally at this point is what is the relationship between the studied integral form generated the divided powers of the Chevalley generators (i.e., the one studied by Mitzman and Garland) and the analogous $\mathbb{Z}$-algebra generated by the divided powers of the $x_{i,r}^{+}$ and the $x_{i,r}^{-}$. As we shall see these coincide outside the case $A_{2n}^{(2)}$, instead in the latter case the integral form results smaller, we will prove it in Chapter 6.

Chapter 3

Integral form of A_1

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. The results about $\mathfrak{g}$ and the $\mathbb{Z}$-basis of the integral form $\mathcal{U}_{\mathbb{Z}}(\mathfrak{g})$ of its enveloping algebra $\mathcal{U}(\mathfrak{g})$ are well known (see [9] and [12]). Here we recall the description of $\mathcal{U}_{\mathbb{Z}}(\mathfrak{g})$ in terms of the non-commutative generalizations described in Section 1.5, with the notations of the commutative examples given in Chapter 1.

The proof expressed in this language has the advantage to be easily generalized to the affine case.

3.1 The integral form of $\mathfrak{sl}_2$ (A_1)

Definition 3.1. $\mathfrak{sl}_2$ *(respectively $\mathcal{U}(\mathfrak{sl}_2)$) is the Lie algebra (respectively the associative algebra) over $\mathbb{Q}$ generated by $\{e, f, h\}$ with relations*

$$[h, e] = 2e,\ [h, f] = -2f,\ [e, f] = h.$$

$\mathcal{U}_{\mathbb{Z}}(\mathfrak{sl}_2)$ is the $\mathbb{Z}$-subalgebra of $\mathcal{U}(\mathfrak{sl}_2)$ generated by $\{e^{(k)}, f^{(k)} \mid k \in \mathbb{N}\}$.

Theorem 3.2. *Let $\mathcal{U}^+, \mathcal{U}^-, \mathcal{U}^0$ denote the $\mathbb{Q}$-subalgebras of $\mathcal{U}(\mathfrak{sl}_2)$ generated respectively by e, by f, by h.*

Then $\mathcal{U}^+ \cong \mathbb{Q}[e], \mathcal{U}^- \cong \mathbb{Q}[f], \mathcal{U}^0 \cong \mathbb{Q}[h]$ and $\mathcal{U}(\mathfrak{sl}_2) \cong \mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+$; moreover

$$\mathcal{U}_{\mathbb{Z}}(\mathfrak{sl}_2) \cong \mathbb{Z}^{(div)}[f] \otimes_{\mathbb{Z}} \mathbb{Z}^{(bin)}[h] \otimes_{\mathbb{Z}} \mathbb{Z}^{(div)}[e]$$

is an integral form of $\mathcal{U}(\mathfrak{sl}_2)$.

Proof. Thanks to Proposition 1.56, we just have to study the commutation between $e^{(k)}$ and $f^{(l)}$ for $k, l \in \mathbb{N}$.

Let us recall the commutation relation

$$e \exp(fu) = \exp(fu)(e + hu - fu^2) \tag{3.2.1}$$

which is a direct application of Lemma 1.55,iv) and of the relations $[e, f] = h$, $[h, f] = -2f$ and $[f, f] = 0$.

We want to prove that in $\mathcal{U}(\mathfrak{sl}_2)[[u, v]]$

$$\exp(eu)\exp(fv) = \exp\Big(\frac{fv}{1+uv}\Big)(1+uv)^h \exp\Big(\frac{eu}{1+uv}\Big). \tag{3.2.2}$$

Let $F(u) = \exp\Big(\frac{fv}{1+uv}\Big)(1+uv)^h \exp\Big(\frac{eu}{1+uv}\Big)$.

It is obvious that $F(0)=\exp(fv)$; hence by Lemma 1.55,x) our claim is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}u}F(u)=eF(u).$$

To obtain this result we derive remarking Lemma 1.55,ix) and then apply the relations (1.56.2) and (3.2.1):

$$\frac{\mathrm{d}}{\mathrm{d}u}F(u)=$$

$$=\exp\Big(\frac{fv}{1+uv}\Big)(1+uv)^h\frac{e}{(1+uv)^2}\exp\Big(\frac{eu}{1+uv}\Big)+$$

$$+\exp\Big(\frac{fv}{1+uv}\Big)\Big(\frac{hv}{1+uv}-\frac{fv^2}{(1+uv)^2}\Big)(1+uv)^h\exp\Big(\frac{eu}{1+uv}\Big)=$$

$$=\exp\Big(\frac{fv}{1+uv}\Big)\Big(e+\frac{hv}{1+uv}-\frac{fv^2}{(1+uv)^2}\Big)(1+uv)^h\exp\Big(\frac{eu}{1+uv}\Big)=$$

$$=eF(u).$$

Remarking that

$$\frac{xu}{1+uv}\in\mathbb{Z}[x][[u,v]],\ \text{hence}\ \Big(\frac{xu}{1+uv}\Big)^{(k)}\in\mathbb{Z}^{(div)}[x][[u,v]]\ \forall k\in\mathbb{N},$$

it follows that the right hand side of 3.2 is an integral form of $\mathcal{U}(\mathfrak{sl}_2)$ (containing $\mathcal{U}_{\mathbb{Z}}(\mathfrak{sl}_2)$).

Finally remark that inverting the exponentials on the right hand side, the relation (3.2.2) gives an expression of $(1+uv)^h$ in terms of the divided powers of e and f, so that $\mathbb{Z}^{(bin)}[h]\subseteq\mathcal{U}_{\mathbb{Z}}(\mathfrak{sl}_2)$, which completes the proof.

□

Chapter 4

The integral form of $\hat{\mathfrak{sl}}_2$ ($A_1^{(1)}$)

The results about $\hat{\mathfrak{sl}}_2$ and the integral form $\hat{\mathcal{U}}_{\mathbb{Z}}$ of its enveloping algebra $\hat{\mathcal{U}}$ are due to Garland (see [6]). Here we simplify the description of the imaginary positive component of $\hat{\mathcal{U}}_{\mathbb{Z}}$ proving that it is an algebra of polynomials over $\mathbb{Z}$ and give a compact and complete proof of the assertion that the set given in Theorem 4.25 is actually a $\mathbb{Z}$-basis of $\hat{\mathcal{U}}_{\mathbb{Z}}$. This proof has the advantage, following [11], to reduce the long and complicated commutation formulas to compact, simply readable and easily proved ones. It is evident from this approach that the results for $\hat{\mathfrak{sl}}_2$ are generalizations of those for $\mathfrak{sl}_2$, so that the commutation formulas arise naturally recalling the homomorphism

$$ev : \hat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbb{Q}[t^{\pm 1}] \oplus \mathbb{Q}c \to \mathfrak{sl}_2 \otimes \mathbb{Q}[t^{\pm 1}] \to \mathfrak{sl}_2 \tag{4.0.1}$$

induced by the evaluation of t at 1.

On the other hand these results and the strategy for their proof will be shown to be in turn generalizable to $\hat{\mathfrak{sl}}_3{}^{\chi}$.

As announced in the Introduction, the proof of Theorem 4.25 is based on a few results: Proposition 4.14, Proposition 4.15, Lemma 4.21, and Proposition 4.22.

4.1 From A_1 to $A_1^{(1)}$

Definition 4.1. *$\hat{\mathfrak{sl}}_2$ (respectively $\hat{\mathcal{U}}$) is the Lie algebra (respectively the associative algebra) over $\mathbb{Q}$ generated by $\{x_r^+, x_r^-, h_r, c \mid r \in \mathbb{Z}\}$ with relations*

$$c \text{ is central,}$$

$$[h_r, h_s] = 2r\delta_{r+s,0}c, \quad [h_r, x_s^{\pm}] = \pm 2x_{r+s}^{\pm}$$

$$[x_r^+, x_s^+] = 0 = [x_r^-, x_s^-],$$

$$[x_r^+, x_s^-] = h_{r+s} + r\delta_{r+s,0}c.$$

Notice that $\{x_r^+, x_r^- \mid r \in \mathbb{Z}\}$ generates $\hat{\mathcal{U}}$.

$\hat{\mathcal{U}}^+$, $\hat{\mathcal{U}}^-$, $\hat{\mathcal{U}}^0$ are the subalgebras of $\hat{\mathcal{U}}$ generated respectively by $\{x_r^+ \mid r \in \mathbb{Z}\}$, $\{x_r^- \mid r \in \mathbb{Z}\}$, $\{c, h_r \mid r \in \mathbb{Z}\}$.

$\hat{\mathcal{U}}^{0,+}$, $\hat{\mathcal{U}}^{0,-}$, $\hat{\mathcal{U}}^{\mathfrak{h}}$, are the subalgebras of $\hat{\mathcal{U}}$ (of $\hat{\mathcal{U}}^0$) generated respectively by $\{h_r \mid r > 0\}$, $\{h_r \mid r < 0\}$, $\{c, h_0\}$.

Remark 4.2. *$\hat{\mathcal{U}}^+, \hat{\mathcal{U}}^-$ are (commutative) algebras of polynomials:*

$$\hat{\mathcal{U}}^+ \cong \mathbb{Q}[x_r^+ \mid r \in \mathbb{Z}], \quad \hat{\mathcal{U}}^- \cong \mathbb{Q}[x_r^- \mid r \in \mathbb{Z}];$$

$\hat{\mathcal{U}}^0$ is not commutative: $[h_r, h_{-r}] = 2rc$;

$\hat{\mathcal{U}}^{0,+}, \hat{\mathcal{U}}^{0,-}, \hat{\mathcal{U}}^{\mathfrak{h}}$, are (commutative) algebras of polynomials:

$$\hat{\mathcal{U}}^{0,+} \cong \mathbb{Q}[h_r \mid r > 0],\ \ \hat{\mathcal{U}}^{0,-} \cong \mathbb{Q}[h_r \mid r < 0],\ \ \hat{\mathcal{U}}^{\mathfrak{h}} \cong \mathbb{Q}[c, h_0];$$

Moreover we have the following "triangular" decompositions:

$$\hat{\mathcal{U}} \cong \hat{\mathcal{U}}^- \otimes \hat{\mathcal{U}}^0 \otimes \hat{\mathcal{U}}^+,$$

$$\hat{\mathcal{U}}^0 \cong \hat{\mathcal{U}}^{0,-} \otimes \hat{\mathcal{U}}^{\mathfrak{h}} \otimes \hat{\mathcal{U}}^{0,+}.$$

Remark that the images in $\hat{\mathcal{U}}$ of $\hat{\mathcal{U}}^- \otimes \hat{\mathcal{U}}^0$ and $\hat{\mathcal{U}}^0 \otimes \hat{\mathcal{U}}^+$ are subalgebras of $\hat{\mathcal{U}}$ and the images of $\hat{\mathcal{U}}^{0,-} \otimes \hat{\mathcal{U}}^{\mathfrak{h}}$ and $\hat{\mathcal{U}}^{\mathfrak{h}} \otimes \hat{\mathcal{U}}^{0,+}$ are commutative subalgebras of $\hat{\mathcal{U}}^0$.

Definition 4.3. *$\hat{\mathcal{U}}$ is endowed with the following anti/auto/homo/morphisms:*

σ is the antiautomorphism defined on the generators by:

$$x_r^+ \mapsto x_r^+, \quad x_r^- \mapsto x_r^-, \qquad (\Rightarrow h_r \mapsto -h_r, \quad c \mapsto -c);$$

Ω is the antiautomorphism defined on the generators by:

$$x_r^+ \mapsto x_{-r}^-, \quad x_r^- \mapsto x_{-r}^+, \quad (\Rightarrow h_r \mapsto h_{-r}, \quad c \mapsto c);$$

T is the automorphism defined on the generators by:

$$x_r^+ \mapsto x_{r-1}^+, \quad x_r^- \mapsto x_{r+1}^-, \quad (\Rightarrow h_r \mapsto h_r - \delta_{r,0}c, \quad c \mapsto c);$$

for all $m \in \mathbb{Z}$, λ_m is the homomorphism defined on the generators by:

$$x_r^+ \mapsto x_{mr}^+, \quad x_r^- \mapsto x_{mr}^-, \quad (\Rightarrow h_r \mapsto h_{mr}, \quad c \mapsto mc).$$

Remark 4.4. *$\sigma^2 = \mathrm{id}_{\hat{\mathcal{U}}}$, $\Omega^2 = \mathrm{id}_{\hat{\mathcal{U}}}$, T is invertible of infinite order; $\lambda_{-1}^2 = \lambda_1 = \mathrm{id}_{\hat{\mathcal{U}}}$; λ_m is not invertible if $m \neq \pm 1$; $\lambda_0 = ev$ (through the identification $< x_0^+, x_0^-, h_0 > \cong < e, f, h >$).*

Remark 4.5. *$\sigma\Omega = \Omega\sigma$, $\sigma T = T\sigma$, $\sigma\lambda_m = \lambda_m\sigma$ for all $m \in \mathbb{Z}$; $\Omega T = T\Omega$, $\Omega\lambda_m = \lambda_m\Omega$ for all $m \in \mathbb{Z}$; $\lambda_m T^{\pm 1} = T^{\pm m}\lambda_m$ for all $m \in \mathbb{Z}$; $\lambda_m\lambda_n = \lambda_{mn}$, for all $m, n \in \mathbb{Z}$.*

Remark 4.6. *$\sigma|_{\hat{\mathcal{U}}^\pm} = \mathrm{id}_{\hat{\mathcal{U}}^\pm}$, $\sigma(\hat{\mathcal{U}}^{0,\pm}) = \hat{\mathcal{U}}^{0,\pm}$, $\sigma(\hat{\mathcal{U}}^{\mathfrak{h}}) = \hat{\mathcal{U}}^{\mathfrak{h}}$.*

$\Omega(\hat{\mathcal{U}}^\pm) = \hat{\mathcal{U}}^\mp$, $\Omega(\hat{\mathcal{U}}^{0,\pm}) = \hat{\mathcal{U}}^{0,\mp}$, $\Omega|_{\hat{\mathcal{U}}^{\mathfrak{h}}} = \mathrm{id}_{\hat{\mathcal{U}}^{\mathfrak{h}}}$.

$T(\hat{\mathcal{U}}^\pm) = \hat{\mathcal{U}}^\pm$, $T|_{\hat{\mathcal{U}}^{0,\pm}} = \mathrm{id}_{\hat{\mathcal{U}}^{0,\pm}}$, $T(\hat{\mathcal{U}}^{\mathfrak{h}}) = \hat{\mathcal{U}}^{\mathfrak{h}}$.

For all $m \in \mathbb{Z}$ $\lambda_m(\hat{\mathcal{U}}^\pm) \subseteq \hat{\mathcal{U}}^\pm$, $\lambda_m(\hat{\mathcal{U}}^0) \subseteq \hat{\mathcal{U}}^0$, $\lambda_m(\hat{\mathcal{U}}^{\mathfrak{h}}) \subseteq \hat{\mathcal{U}}^{\mathfrak{h}}$,

$$\lambda_m(\hat{\mathcal{U}}^{0,\pm}) \subseteq \begin{cases} \hat{\mathcal{U}}^{0,\pm} & \text{if } m > 0 \\ \hat{\mathcal{U}}^{0,\mp} & \text{if } m < 0 \\ \hat{\mathcal{U}}^{\mathfrak{h}} & \text{if } m = 0. \end{cases}$$

Definition 4.7. *Here we define some $\mathbb{Z}$-subalgebras of $\hat{\mathcal{U}}$:*

$\hat{\mathcal{U}}_{\mathbb{Z}}$ is the $\mathbb{Z}$-subalgebra of $\hat{\mathcal{U}}$ generated by $\{(x_r^+)^{(k)}, (x_r^-)^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}\}$;

$\hat{\mathcal{U}}_{\mathbb{Z}}^\pm = \mathbb{Z}^{(div)}[x_r^\pm \mid r \in \mathbb{Z}]$;

$\hat{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} = \mathbb{Z}^{(bin)}[h_0, c]$;

$\hat{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} = \mathbb{Z}^{(sym)}[h_{\pm r} \mid r > 0]$;

$\hat{\mathcal{U}}_{\mathbb{Z}}^0$ is the $\mathbb{Z}$-subalgebra of $\hat{\mathcal{U}}$ generated by $\hat{\mathcal{U}}_{\mathbb{Z}}^{0,-}$, $\hat{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ and $\hat{\mathcal{U}}_{\mathbb{Z}}^{0,+}$.

The notations are those of Section 1.

We want to prove the following:

Theorem 4.8. *$\hat{\mathcal{U}}^0_{\mathbb{Z}} = \hat{\mathcal{U}}^{0,-}_{\mathbb{Z}}\hat{\mathcal{U}}^{\mathfrak{h}}_{\mathbb{Z}}\hat{\mathcal{U}}^{0,+}_{\mathbb{Z}}$: it is an integral form of $\hat{\mathcal{U}}^0$; $\hat{\mathcal{U}}_{\mathbb{Z}} = \hat{\mathcal{U}}^-_{\mathbb{Z}}\hat{\mathcal{U}}^0_{\mathbb{Z}}\hat{\mathcal{U}}^+_{\mathbb{Z}}$: it is an integral form of $\hat{\mathcal{U}}$.*

As in the case of $\mathfrak{sl}_2$, working in $\hat{\mathcal{U}}[[u]]$ (see the notation below) simplifies enormously the proofs and gives a deeper insight to the question.

Notation 4.9. *We shall consider the following elements in $\hat{\mathcal{U}}[[u]]$:*

$$x^+(u) = \sum_{r\geq 0} x^+_r u^r = \sum_{r\geq 0} T^{-r}u^r(x^+_0),$$

$$x^-(u) = \sum_{r\geq 0} x^-_{r+1} u^r = \sum_{r\geq 0} T^{r}u^r(x^-_1),$$

$$h_\pm(u) = \sum_{r\geq 1}(-1)^{r-1}\frac{h_{\pm r}}{r}u^r,$$

$$\hat{h}_\pm(u) = \exp(h_\pm(u)) = \sum_{r\geq 0}\hat{h}_{\pm r}u^r.$$

Remark 4.10. *Notice that $ev\circ T = ev$ and*

$$ev(x^+(-u)) = ev\left(\frac{1}{1+T^{-1}u}x^+_0\right) = \frac{e}{1+u},$$

$$ev(x^-(-u)) = ev\left(\frac{T}{1+Tu}x^-_0\right) = \frac{f}{1+u},$$

$$ev(h_\pm(u)) = h\ln(1+u),$$

$$ev(\hat{h}_\pm(u)) = (1+u)^h.$$

Remark 4.11. *Here we list some obvious remarks.*

i) $\hat{\mathcal{U}}^\pm_{\mathbb{Z}} \subseteq \hat{\mathcal{U}}_{\mathbb{Z}}\cap\hat{\mathcal{U}}^\pm$ and $\hat{\mathcal{U}}_{\mathbb{Z}}$ is the $\mathbb{Z}$-subalgebra of $\hat{\mathcal{U}}$ generated by $\hat{\mathcal{U}}^+_{\mathbb{Z}}\cup\hat{\mathcal{U}}^-_{\mathbb{Z}}$;

ii) $\hat{\mathcal{U}}^\pm_{\mathbb{Z}}$, $\hat{\mathcal{U}}^{\mathfrak{h}}_{\mathbb{Z}}$, $\hat{\mathcal{U}}^{0,\pm}_{\mathbb{Z}}$ and $\hat{\mathcal{U}}^{0,\pm}_{\mathbb{Z}}\hat{\mathcal{U}}^{\mathfrak{h}}_{\mathbb{Z}} = \hat{\mathcal{U}}^{\mathfrak{h}}_{\mathbb{Z}}\hat{\mathcal{U}}^{0,\pm}_{\mathbb{Z}}$ are integral forms respectively of $\hat{\mathcal{U}}^\pm$, $\hat{\mathcal{U}}^{\mathfrak{h}}$, $\hat{\mathcal{U}}^{0,\pm}$ and $\hat{\mathcal{U}}^{0,\pm}\hat{\mathcal{U}}^{\mathfrak{h}} = \hat{\mathcal{U}}^{\mathfrak{h}}\hat{\mathcal{U}}^{0,\pm}$;

iii) $\hat{\mathcal{U}}_{\mathbb{Z}}$ and $\hat{\mathcal{U}}^{\mathfrak{h}}_{\mathbb{Z}}$ are stable under σ, Ω, $T^{\pm1}$, λ_m for all $m\in\mathbb{Z}$;

iv) $\hat{\mathcal{U}}^\pm_{\mathbb{Z}}$ is stable under σ, $T^{\pm1}$, λ_m for all $m\in\mathbb{Z}$ and $\Omega(\hat{\mathcal{U}}^\pm_{\mathbb{Z}}) = \hat{\mathcal{U}}^\mp_{\mathbb{Z}}$;

v) $\hat{\mathcal{U}}^{0,\pm}_{\mathbb{Z}}$ is stable under σ, $T^{\pm1}$ and $\Omega(\hat{\mathcal{U}}^{0,\pm}_{\mathbb{Z}}) = \lambda_{-1}(\hat{\mathcal{U}}^{0,\pm}_{\mathbb{Z}}) = \hat{\mathcal{U}}^{0,\mp}_{\mathbb{Z}}$: more precisely

$$\sigma(\hat{h}_\pm(u)) = \hat{h}_\pm(u)^{-1},\ \Omega(\hat{h}_\pm(u)) = \lambda_{-1}(\hat{h}_\pm(u)) = \hat{h}_\mp(u),\ T^{\pm1}(\hat{h}_\pm(u)) = \hat{h}_\pm(u);$$

vi) for $m\in\mathbb{Z}$

$$\lambda_m(\hat{\mathcal{U}}^{0,\pm}_{\mathbb{Z}}) \subseteq \begin{cases}\hat{\mathcal{U}}^{0,\pm}_{\mathbb{Z}} & \text{if } m>0\\ \hat{\mathcal{U}}^{0,\mp}_{\mathbb{Z}} & \text{if } m<0\\ \hat{\mathcal{U}}^{\mathfrak{h}}_{\mathbb{Z}} & \text{if } m=0,\end{cases}$$

thanks to v), to Proposition 1.13 and Remark 4.10.

Remark 4.12. *The elements $\hat{h}_k$'s with $k>0$ generate the same $\mathbb{Z}$-subalgebra of $\hat{\mathcal{U}}$ as the elements Λ_k's ($k\geq 0$) defined in [6].*

Indeed let

$$\sum_{n\geq 0} p_n u^n = P(u) = \hat{h}(-u)^{-1};$$

then Remarks 1.8,1,ii) and 1.12,iii) imply that $\mathbb{Z}[\hat{h}_k \mid k > 0] = \mathbb{Z}[p_n \mid n > 0]$*; but*

$$\frac{\mathrm{d}}{\mathrm{d}u}P(u) = P(u)\sum_{r>0} h_r u^{r-1},$$

that is

$$p_0 = 1, \ \ p_n = \frac{1}{n}\sum_{r=1}^{n} h_r p_{n-r} \ \forall n > 0,$$

hence $p_n = \Lambda_{n-1}\ \forall n \geq 0$.

On the other hand applying λ_m *we get*

$$\lambda_m(p_0) = 1, \ \ \lambda_m(p_n) = \frac{1}{n}\sum_{r=1}^{n} h_{rm}\lambda_m(p_{n-r}),$$

so that $\lambda_m(p_n) = \lambda_m(\Lambda_{n-1}) = \Lambda_{n-1}(\xi(m))$ *(see [6]).*

Remark 4.13. *Remark that for all* $r \in \mathbb{Z}$ *the subalgebra of* $\hat{\mathfrak{sl}}_2$ *generated by*

$$\{x_r^+, x_{-r}^-, h_0 + rc\}$$

maps isomorphically onto $\mathfrak{sl}_2$ *through the evaluation homomorphism ev (see* (4.0.1)*). On the other hand for each* $r \in \mathbb{Z}$ *there is an injection* $\mathcal{U}(\mathfrak{sl}_2) \to \hat{\mathcal{U}}$*:*

$$e \mapsto x_r^+, \ f \mapsto x_{-r}^-, \ h \mapsto h_0 + rc.$$

In particular Theorem 3.2, implies that the elements $\binom{h_0+rc}{k}$ *belong to* $\hat{\mathcal{U}}_{\mathbb{Z}}$ *for all* $r \in \mathbb{Z}, k \in \mathbb{N}$ *(thus, remarking that the elements* $\binom{c}{k}$*'s are central and the Example 1.9, we get that* $\hat{\mathcal{U}}_{\mathbb{Z}}^h \subseteq \hat{\mathcal{U}}_{\mathbb{Z}}$*) and Proposition 1.56 implies that* $\hat{\mathcal{U}}_{\mathbb{Z}}^h\hat{\mathcal{U}}_{\mathbb{Z}}^+$ *and* $\hat{\mathcal{U}}_{\mathbb{Z}}^-\hat{\mathcal{U}}_{\mathbb{Z}}^h$ *are integral forms respectively of* $\hat{\mathcal{U}}^h\hat{\mathcal{U}}^+$ *and* $\hat{\mathcal{U}}^-\hat{\mathcal{U}}^h$*.*

Proposition 4.14. *The following identity holds in* $\hat{\mathcal{U}}[[u,v]]$*:*

$$\hat{h}_+(u)\hat{h}_-(v) = \hat{h}_-(v)(1-uv)^{-2c}\hat{h}_+(u).$$

$\hat{\mathcal{U}}_{\mathbb{Z}}^0 = \hat{\mathcal{U}}_{\mathbb{Z}}^{0,-}\hat{\mathcal{U}}_{\mathbb{Z}}^h\hat{\mathcal{U}}_{\mathbb{Z}}^{0,+}$*: it is an integral form of* $\hat{\mathcal{U}}^0$*.*

Proof. Since $[h_r, h_s] = 2r\delta_{r+s,0}c$, the claim is Proposition 1.58 with $m=2$, $l=0$. □

Proposition 4.15. *The following identity holds in* $\hat{\mathcal{U}}[[u]]$*:*

$$x_0^+\hat{h}_+(u) = \hat{h}_+(u)(1+T^{-1}u)^{-2}(x_0^+). \tag{4.15.1}$$

Hence for all $k \in \mathbb{N}$

$$(x_0^+)^{(k)}\hat{h}_+(u) = \hat{h}_+(u)((1+T^{-1}u)^{-2}(x_0^+))^{(k)} \in \hat{\mathcal{U}}_{\mathbb{Z}}^{0,+}\hat{\mathcal{U}}_{\mathbb{Z}}^+[[u]]. \tag{4.15.2}$$

Proof. The claim follows from Proposition 1.60 with $m_1 = 2$, $m_d = 0\ \forall d > 1$ and from 1.1. □

Remark 4.16. *The relation* (4.15.1) *can be written as*

$$x_0^+\hat{h}_+(u) = \hat{h}_+(u)\frac{\mathrm{d}}{\mathrm{d}u}(ux^+(-u)).$$

Indeed

$$(1+T^{-1}u)^{-2}(x_0^+) = \sum_{r\in\mathbb{N}}(-1)^r(r+1)x_r^+u^r = \frac{\mathrm{d}}{\mathrm{d}u}(ux^+(-u)).$$

Remark 4.17. *Remark that the relation* (4.15.2) *is the affine version of*

$$e^{(k)}(1+u)^h = (1+u)^h\left(\frac{e}{(1+u)^2}\right)^{(k)} \tag{4.17.1}$$

(see (1.56.3)*); indeed ev maps* (4.15.2) *to* (4.17.1).

Corollary 4.18. *$\hat{\mathcal{U}}_{\mathbb{Z}}^{+}\hat{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} \subseteq \hat{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}\hat{\mathcal{U}}_{\mathbb{Z}}^{+}$ and $\hat{\mathcal{U}}_{\mathbb{Z}}^{\pm}\hat{\mathcal{U}}_{\mathbb{Z}}^{0} = \hat{\mathcal{U}}_{\mathbb{Z}}^{0}\hat{\mathcal{U}}_{\mathbb{Z}}^{\pm}$. Then $\hat{\mathcal{U}}_{\mathbb{Z}}^{0}\hat{\mathcal{U}}_{\mathbb{Z}}^{+}$ and $\hat{\mathcal{U}}_{\mathbb{Z}}^{-}\hat{\mathcal{U}}_{\mathbb{Z}}^{0}$ are integral forms respectively of $\hat{\mathcal{U}}^{0}\hat{\mathcal{U}}^{+}$ and $\hat{\mathcal{U}}^{-}\hat{\mathcal{U}}^{0}$.*

Proof. Applying T^{-r} to (4.15.2), we find that $(x_r^+)^{(k)}\hat{h}_+(u) \subseteq \hat{h}_+(u)\hat{\mathcal{U}}_{\mathbb{Z}}^{+}[[u]]$ $\forall r \in \mathbb{Z}, k \in \mathbb{N}$, hence $\hat{\mathcal{U}}_{\mathbb{Z}}^{+}\hat{h}_+(u) \subseteq \hat{h}_+(u)\hat{\mathcal{U}}_{\mathbb{Z}}^{+}[[u]]$ and $\hat{\mathcal{U}}_{\mathbb{Z}}^{+}\hat{\mathcal{U}}_{\mathbb{Z}}^{0,+} \subseteq \hat{\mathcal{U}}_{\mathbb{Z}}^{0,+}\hat{\mathcal{U}}_{\mathbb{Z}}^{+}$. From this, applying λ_{-1} we get $\hat{\mathcal{U}}_{\mathbb{Z}}^{+}\hat{\mathcal{U}}_{\mathbb{Z}}^{0,-} \subseteq \hat{\mathcal{U}}_{\mathbb{Z}}^{0,-}\hat{\mathcal{U}}_{\mathbb{Z}}^{+}$, hence $\hat{\mathcal{U}}_{\mathbb{Z}}^{+}\hat{\mathcal{U}}_{\mathbb{Z}}^{0} \subseteq \hat{\mathcal{U}}_{\mathbb{Z}}^{0}\hat{\mathcal{U}}_{\mathbb{Z}}^{+}$ thanks to Remark 4.13. Finally applying Ω we obtain that $\hat{\mathcal{U}}_{\mathbb{Z}}^{0}\hat{\mathcal{U}}_{\mathbb{Z}}^{-} \subseteq \hat{\mathcal{U}}_{\mathbb{Z}}^{-}\hat{\mathcal{U}}_{\mathbb{Z}}^{0}$ and applying σ we get the reverse inclusions. □

We are now left to prove that $\hat{\mathcal{U}}_{\mathbb{Z}}^{+}\hat{\mathcal{U}}_{\mathbb{Z}}^{-} \subseteq \hat{\mathcal{U}}_{\mathbb{Z}}^{-}\hat{\mathcal{U}}_{\mathbb{Z}}^{0}\hat{\mathcal{U}}_{\mathbb{Z}}^{+}$ and that $\hat{\mathcal{U}}_{\mathbb{Z}}^{0} \subseteq \hat{\mathcal{U}}_{\mathbb{Z}}$.

To this aim we study the commutation relations between $(x_r^+)^{(k)}$ and $(x_s^-)^{(l)}$ or equivalently between $\exp(x_r^+u)$ and $\exp(x_s^-v)$.

Remark 4.19. *Theorem 3.2 and Remark 4.13 imply that $\exp(x_r^+u)\exp(x_{-r}^-v) \in \hat{\mathcal{U}}_{\mathbb{Z}}^{-}\hat{\mathcal{U}}_{\mathbb{Z}}^{0}\hat{\mathcal{U}}_{\mathbb{Z}}^{+}[[u,v]]$ for all $r \in \mathbb{Z}$.*

In order to prove a similar result for $\exp(x_r^+u)\exp(x_s^-v)$ when $r+s \neq 0$ remark that in general

$$\exp(x_r^+u)\exp(x_s^-v) = T^{-r}\lambda_{r+s}(\exp(x_0^+u)\exp(x_1^-v)),$$

so that Remark 4.11,iv),v),vi) allows us to reduce to the case $r=0$, $s=1$.

This case will turn out to be enough also to prove that $\hat{\mathcal{U}}_{\mathbb{Z}}^{0} \subseteq \hat{\mathcal{U}}_{\mathbb{Z}}$.

Remark 4.20. *In the study of the commutation relations in $\hat{\mathcal{U}}_{\mathbb{Z}}$ remark that*

$$ev(\exp(x_0^+u)\exp(x_1^-v)) = \exp(eu)\exp(fv)$$

and that straightening $\exp(x_0^+u)\exp(x_1^-v)$ through the triangular decomposition $\hat{\mathcal{U}} \cong \hat{\mathcal{U}}^{-}\otimes\hat{\mathcal{U}}^{0}\otimes\hat{\mathcal{U}}^{+}$ we get an element of $\hat{\mathcal{U}}[[u,v]]$ whose coefficients involve $x_{r+1}^-, h_{r+1}, x_r^+$ with $r \geq 0$ and whose image through ev is

$$\exp\left(\frac{fv}{1+uv}\right)(1+uv)^h\exp\left(\frac{eu}{1+uv}\right)$$

(see Remark 4.10).

Vice versa once we have such an expression for $\exp(x_0^+u)\exp(x_1^-v)$ applying $T^{-r}\lambda_{r+s}$ we can deduce from it the relation (3.2.2) *and the expression for $\exp(x_r^+u)\exp(x_s^-v)$ for all $r,s \in \mathbb{Z}$ (also in the case $r+s=0$).*

Remark that

$$\exp(vx^-(-uv))\hat{h}_+(uv)\exp(ux^+(-uv))$$

is an element of $\hat{\mathcal{U}}[[u,v]]$ which has the required properties (see Remark 4.10) and belongs to $\hat{\mathcal{U}}_{\mathbb{Z}}^{-}\hat{\mathcal{U}}_{\mathbb{Z}}^{0}\hat{\mathcal{U}}_{\mathbb{Z}}^{+}[[u,v]]$.

Our aim is to prove that

$$\exp(x_0^+u)\exp(x_1^-v) = \exp(vx^-(-uv))\hat{h}_+(uv)\exp(ux^+(-uv)).$$

Lemma 4.21. *In $\hat{\mathcal{U}}[[u,v]]$ we have*

$$x_0^+\exp(vx^-(-uv)) = \exp(vx^-(-uv))\left(x_0^+ + \frac{\mathrm{d}h_+(uv)}{\mathrm{d}u} + \frac{\mathrm{d}vx^-(-uv)}{\mathrm{d}u}\right).$$

Proof. The claim follows from Lemma 1.55,iv) remarking that

$$[x_0^+, vx^-(-uv)] = v\sum_{r\in\mathbb{N}} h_{r+1}(-uv)^r = \frac{\mathrm{d}}{\mathrm{d}u}\sum_{r\in\mathbb{N}}\frac{h_{r+1}}{r+1}(-1)^r(uv)^{r+1} = \frac{\mathrm{d}h_+(uv)}{\mathrm{d}u},$$

$$\Big[\frac{\mathrm{d}h_+(uv)}{\mathrm{d}u}, vx^-(-uv)\Big] = -2v^2\sum_{r,s\in\mathbb{N}} x^+_{r+s+2}(-uv)^{r+s} =$$

$$= -2v^2\sum_{r\in\mathbb{N}}(r+1)x^-_{r+2}(-uv)^r = 2\frac{\mathrm{d}vx^-(-uv)}{\mathrm{d}u}$$

and

$$\Big[\frac{\mathrm{d}vx^-(-uv)}{\mathrm{d}u}, vx^-(-uv)\Big] = 0.$$

□

Proposition 4.22. *In $\hat{\mathcal{U}}[[u,v]]$ we have*

$$\exp(x_0^+u)\exp(x_1^-v) = \exp(vx^-(-uv))\hat{h}_+(uv)\exp(ux^+(-uv)).$$

Proof. Let $F(u) = \exp(vx^-(-uv))\hat{h}_+(uv)\exp(ux^+(-uv))$. It is clear that $F(0) = \exp(x_1^-v)$, so that thanks to Lemma 1.55,x) it is enough to prove that

$$\frac{\mathrm{d}}{\mathrm{d}u}F(u) = x_0^+F(u).$$

Remark that, thanks to the derivation rules (Lemma 1.55,ix)), to Proposition 4.15, and to Lemma 4.21, we have:

$$\frac{\mathrm{d}}{\mathrm{d}u}F(u) = \exp(vx^-(-uv))\hat{h}_+(uv)\frac{\mathrm{d}}{\mathrm{d}u}(ux^+(-uv))\exp\big(ux^+(-uv)\big)+$$

$$+\exp(vx^-(-uv))\Big(\frac{\mathrm{d}}{\mathrm{d}u}h_+(uv) + \frac{\mathrm{d}}{\mathrm{d}u}(vx^-(-uv))\Big)\hat{h}_+(uv)\exp(ux^+(-uv)) =$$

$$= \exp(vx^-(-uv))\Big(x_0^+ + \frac{\mathrm{d}(h_+(uv)+vx^-(-uv))}{\mathrm{d}u}\Big)\hat{h}_+(uv)\exp(ux^+(-uv)) =$$

$$= x_0^+\exp(vx^-(-uv))\hat{h}_+(uv)\exp(ux^+(-uv)) = x_0^+F(u).$$

□

Corollary 4.23. $\hat{\mathcal{U}}^0_{\mathbb{Z}} \subseteq \hat{\mathcal{U}}_{\mathbb{Z}}$.

Proof. That $\hat{\mathcal{U}}^{0,+}_{\mathbb{Z}} \subseteq \hat{\mathcal{U}}_{\mathbb{Z}}$ is a consequence of Proposition 4.22 inverting the exponentials (see the proof Theorem 3.2), which implies also (applying Ω) that $\hat{\mathcal{U}}^{0,-}_{\mathbb{Z}} \subseteq \hat{\mathcal{U}}_{\mathbb{Z}}$; the claim then follows thanks to Remark 4.13. □

Proposition 4.24. *$\hat{\mathcal{U}}^-_{\mathbb{Z}}\hat{\mathcal{U}}^0_{\mathbb{Z}}\hat{\mathcal{U}}^+_{\mathbb{Z}}$ is a $\mathbb{Z}$-subalgebra of $\hat{\mathcal{U}}$ (hence $\hat{\mathcal{U}}_{\mathbb{Z}} = \hat{\mathcal{U}}^-_{\mathbb{Z}}\hat{\mathcal{U}}^0_{\mathbb{Z}}\hat{\mathcal{U}}^+_{\mathbb{Z}}$).*

Proof. We want to prove that $\hat{\mathcal{U}}^-_{\mathbb{Z}}\hat{\mathcal{U}}^0_{\mathbb{Z}}\hat{\mathcal{U}}^+_{\mathbb{Z}}$ (which is obviously a $\hat{\mathcal{U}}^-_{\mathbb{Z}}$-module and, by Corollary 4.18, a $\hat{\mathcal{U}}^0_{\mathbb{Z}}$-module) is also a $\hat{\mathcal{U}}^+_{\mathbb{Z}}$-module, or equivalently that $\hat{\mathcal{U}}^+_{\mathbb{Z}}\hat{\mathcal{U}}^-_{\mathbb{Z}} \subseteq \hat{\mathcal{U}}^-_{\mathbb{Z}}\hat{\mathcal{U}}^0_{\mathbb{Z}}\hat{\mathcal{U}}^+_{\mathbb{Z}}$.

By Proposition 4.22 together with Remark 4.19, relation (3.2.2) and Remark 4.13 we have that $y_+y_- \in \hat{\mathcal{U}}^-_{\mathbb{Z}}\hat{\mathcal{U}}^0_{\mathbb{Z}}\hat{\mathcal{U}}^+_{\mathbb{Z}}$ in the particular case when $y_+ = (x_r^+)^{(k)}$ and $y_- = (x_s^-)^{(l)}$, thus we just need to perform the correct induction to deal with the general $y_\pm \in \hat{\mathcal{U}}^\pm_{\mathbb{Z}}$.

Remark that setting

$$deg(x_r^\pm) = \pm 1, \;\; deg(h_r) = deg(c) = 0$$

induces a $\mathbb{Z}$-gradation on $\hat{\mathcal{U}}$ (since the relations defining $\hat{\mathcal{U}}$ are homogeneous) and on $\hat{\mathcal{U}}_{\mathbb{Z}}$ (since its generators are homogeneous), which is preserved by σ, $T^{\pm1}$ and λ_m $\forall m \in \mathbb{Z}$; in particular it induces $\mathbb{N}$-gradations

$$\hat{\mathcal{U}}^{\pm} = \bigoplus_{k\in\mathbb{N}} \hat{\mathcal{U}}^{\pm}_{\pm k}, \qquad \hat{\mathcal{U}}^{\pm}_{\mathbb{Z}} = \bigoplus_{k\in\mathbb{N}} \hat{\mathcal{U}}^{\pm}_{\mathbb{Z},\pm k}$$

with the properties that

$$\Omega(\hat{\mathcal{U}}^{\pm}_{\mathbb{Z},\pm k}) = \hat{\mathcal{U}}^{\mp}_{\mathbb{Z},\mp k},$$

$$\hat{\mathcal{U}}^{+}_{\mathbb{Z},k} = \sum_{\substack{n\in\mathbb{N}\\ k_1+\dots+k_n=k}} \mathbb{Z}(x^+_{r_1})^{(k_1)}\cdot\ldots\cdot(x^+_{r_n})^{(k_n)} = \sum_{r\in\mathbb{Z}} \mathbb{Z}(x^+_r)^{(k)} + \sum_{\substack{k_1,k_2>0\\ k_1+k_2=k}} \hat{\mathcal{U}}^{+}_{\mathbb{Z},k_1}\hat{\mathcal{U}}^{+}_{\mathbb{Z},k_2},$$

$$\hat{\mathcal{U}}^{+}_{\mathbb{Z},k}\hat{\mathcal{U}}^{0}_{\mathbb{Z}} = \hat{\mathcal{U}}^{0}_{\mathbb{Z}}\hat{\mathcal{U}}^{+}_{\mathbb{Z},k} \ \text{(because } \hat{\mathcal{U}}_k\hat{\mathcal{U}}^0 = \hat{\mathcal{U}}^0\hat{\mathcal{U}}_k \text{ and } \hat{\mathcal{U}}^{+}_{\mathbb{Z}}\hat{\mathcal{U}}^{0}_{\mathbb{Z}} = \hat{\mathcal{U}}^{0}_{\mathbb{Z}}\hat{\mathcal{U}}^{+}_{\mathbb{Z}})$$

and thanks to Definition 4.1 and Remark 4.2

$$[\hat{\mathcal{U}}^{+}_{k}, \hat{\mathcal{U}}^{-}_{-l}] \subseteq \sum_{m>0} \hat{\mathcal{U}}^{-}_{-l+m}\hat{\mathcal{U}}^{0}\hat{\mathcal{U}}^{+}_{k-m} \ \ \forall k,l \in \mathbb{N}.$$

We want to prove that

$$\hat{\mathcal{U}}^{+}_{\mathbb{Z},k}\hat{\mathcal{U}}^{-}_{\mathbb{Z},-l} \subseteq \sum_{m\geq 0} \hat{\mathcal{U}}^{-}_{\mathbb{Z},-l+m}\hat{\mathcal{U}}^{0}_{\mathbb{Z}}\hat{\mathcal{U}}^{+}_{\mathbb{Z},k-m} \ \ \forall k,l \in \mathbb{N}, \tag{4.24.1}$$

the claim being obvious for $k=0$ or $l=0$.

Suppose $k \neq 0$, $l \neq 0$ and the claim true for all $(\tilde{k},\tilde{l}) \neq (k,l)$ with $\tilde{k} \leq k$ and $\tilde{l} \leq l$. Then:

a) Proposition 4.22 together with Remark 4.19, relation (3.2.2) and Remark 4.13 imply that

$$(x^+_r)^{(k)}(x^-_s)^{(l)} \in \sum_{m\geq 0} \hat{\mathcal{U}}^{-}_{\mathbb{Z},-l+m}\hat{\mathcal{U}}^{0}_{\mathbb{Z}}\hat{\mathcal{U}}^{+}_{\mathbb{Z},k-m} \ \ \forall r,s \in \mathbb{Z};$$

b) if $k_1,k_2 > 0$ are such that $k_1+k_2=k$ or $l_1,l_2>0$ are such that $l_1+l_2=l$, then

$$\hat{\mathcal{U}}^{+}_{\mathbb{Z},k_1}\hat{\mathcal{U}}^{+}_{\mathbb{Z},k_2}\hat{\mathcal{U}}^{-}_{\mathbb{Z},-l} \subseteq \sum_{m_2\geq 0} \hat{\mathcal{U}}^{+}_{\mathbb{Z},k_1}\hat{\mathcal{U}}^{-}_{\mathbb{Z},-l+m_2}\hat{\mathcal{U}}^{0}_{\mathbb{Z}}\hat{\mathcal{U}}^{+}_{\mathbb{Z},k_2-m_2} \subseteq$$

$$\subseteq \sum_{m_1,m_2\geq 0} \hat{\mathcal{U}}^{-}_{\mathbb{Z},-l+m_2+m_1}\hat{\mathcal{U}}^{0}_{\mathbb{Z}}\hat{\mathcal{U}}^{+}_{\mathbb{Z},k_1-m_1}\hat{\mathcal{U}}^{0}_{\mathbb{Z}}\hat{\mathcal{U}}^{+}_{\mathbb{Z},k_2-m_2} =$$

$$= \sum_{m_1,m_2\geq 0} \hat{\mathcal{U}}^{-}_{\mathbb{Z},-l+m_2+m_1}\hat{\mathcal{U}}^{0}_{\mathbb{Z}}\hat{\mathcal{U}}^{+}_{\mathbb{Z},k_1-m_1}\hat{\mathcal{U}}^{+}_{\mathbb{Z},k_2-m_2} \subseteq \sum_{m\geq 0} \hat{\mathcal{U}}^{-}_{\mathbb{Z},-l+m}\hat{\mathcal{U}}^{0}_{\mathbb{Z}}\hat{\mathcal{U}}^{+}_{\mathbb{Z},k-m}$$

and symmetrically applying Ω

$$\hat{\mathcal{U}}^{+}_{\mathbb{Z},k}\hat{\mathcal{U}}^{-}_{\mathbb{Z},-l_1}\hat{\mathcal{U}}^{-}_{\mathbb{Z},-l_2} = \Omega(\hat{\mathcal{U}}^{+}_{\mathbb{Z},l_2}\hat{\mathcal{U}}^{+}_{\mathbb{Z},l_1}\hat{\mathcal{U}}^{-}_{\mathbb{Z},-k}) \subseteq$$

$$\subseteq \Omega(\sum_{m\geq 0} \hat{\mathcal{U}}^{-}_{\mathbb{Z},-k+m}\hat{\mathcal{U}}^{0}_{\mathbb{Z}}\hat{\mathcal{U}}^{+}_{\mathbb{Z},l-m}) = \sum_{m\geq 0} \hat{\mathcal{U}}^{-}_{\mathbb{Z},-l+m}\hat{\mathcal{U}}^{0}_{\mathbb{Z}}\hat{\mathcal{U}}^{+}_{\mathbb{Z},k-m}.$$

(4.24.1) follows from a) and b). □

We have thus proved Theorem 4.8, summarized in the following:

Theorem 4.25. *The $\mathbb{Z}$-subalgebra $\hat{\mathcal{U}}_{\mathbb{Z}}$ of $\hat{\mathcal{U}}$ generated by*

$$\{(x^{\pm}_r)^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}\}$$

is an integral form of $\hat{\mathcal{U}}$.

More precisely

$$\hat{\mathcal{U}}_{\mathbb{Z}} \cong \hat{\mathcal{U}}_{\mathbb{Z}}^{-} \otimes \hat{\mathcal{U}}_{\mathbb{Z}}^{0} \otimes \hat{\mathcal{U}}_{\mathbb{Z}}^{+} \cong \hat{\mathcal{U}}_{\mathbb{Z}}^{-} \otimes \hat{\mathcal{U}}_{\mathbb{Z}}^{0,-} \otimes \hat{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} \otimes \hat{\mathcal{U}}_{\mathbb{Z}}^{0,+} \otimes \hat{\mathcal{U}}_{\mathbb{Z}}^{+}$$

and a $\mathbb{Z}$-basis of $\hat{\mathcal{U}}_{\mathbb{Z}}$ is given by the product

$$\hat{B}^{-}\hat{B}^{0,-}\hat{B}^{\mathfrak{h}}\hat{B}^{0,+}\hat{B}^{+}$$

where $\hat{B}^{\pm}$, $\hat{B}^{0,\pm}$ and $\hat{B}^{\mathfrak{h}}$ are the $\mathbb{Z}$-bases respectively of $\hat{\mathcal{U}}_{\mathbb{Z}}^{\pm}$, $\hat{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$ and $\hat{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ given as follows:

$$\hat{B}^{\pm} = \left\{ (\mathbf{x}^{\pm})^{(\mathbf{k})} = \prod_{r\in\mathbb{Z}} (x_r^{\pm})^{(k_r)} \;\middle|\; \mathbf{k} : \mathbb{Z} \to \mathbb{N} \text{ is finitely supported} \right\}$$

$$\hat{B}^{0,\pm} = \left\{ \hat{\mathbf{h}}_{\pm}^{\mathbf{k}} = \prod_{l\in\mathbb{Z}_+} \hat{h}_{\pm l}^{k_l} \;\middle|\; \mathbf{k} : \mathbb{Z}_+ \to \mathbb{N} \text{ is finitely supported} \right\}$$

$$\hat{B}^{\mathfrak{h}} = \left\{ \binom{h_0}{k}\binom{c}{\tilde{k}} \;\middle|\; k, \tilde{k} \in \mathbb{N} \right\}.$$

Remark that $\hat{B}^{\pm} = B^{re,\pm}$ and that exhibiting the basis $\hat{B}^{0,\pm}$ proves that $\hat{\mathcal{U}}_{\mathbb{Z}}^{im,\pm} = \hat{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$ is an algebra of polynomials (see the Introduction).

Chapter 5

The integral form of $\hat{\mathfrak{sl}_3}^{\chi}(A_2^{(2)})$

In this chapter we describe the integral form $\tilde{\mathcal{U}}_{\mathbb{Z}}$ of the enveloping algebra $\tilde{\mathcal{U}}$ of the Kac-Moody algebra of type $A_2^{(2)}$ generated by the divided powers of the Drinfeld generators $x_r^{\pm}$; unlike the untwisted case, this integral form is strictly smaller than the one (studied in [11] by Mitzman) generated by the divided powers of the Chevalley generators e_0, e_1, f_0, f_1 (see Chapter 6).

However, the construction of a $\mathbb{Z}$-basis of $\tilde{\mathcal{U}}_{\mathbb{Z}}$ follows the idea of the analogous construction in the case $A_1^{(1)}$, seen in the previous section; this method allows us to overcome the technical difficulties arising in case $A_2^{(2)}$ - difficulties which seem otherwise overwhelming.

The commutation relations needed to our aim can be partially deduced from the case $A_1^{(1)}$: indeed, underlining some embeddings of $\hat{\mathfrak{sl}}_2$ into $\hat{\mathfrak{sl}_3}^{\chi}$ (see Remark 5.27), the commutation relations in $\hat{\mathcal{U}}$ can be directly translated into a class of commutation relations in $\tilde{\mathcal{U}}$ (see Corollary 5.28, Proposition 5.29 and the Appendix 9.A for more details).

Yet, there are some differences between $A_1^{(1)}$ and $A_2^{(2)}$.

First of all, the real (positive and negative) components of $\tilde{\mathcal{U}}$ are no more commutative (this is well known: it happens in all the affine cases different from $A_1^{(1)}$, as well as in all the finite cases different from A_1), hence the study of their integral form requires some - easy - additional observations (see Lemma 5.22).

The non commutativity of the real components of $\tilde{\mathcal{U}}$ makes the general commutation formula between the exponentials of positive and negative Drinfeld generators technically more complicated to compute and express than in the case of $\hat{\mathfrak{sl}}_2$; nevertheless, general and explicit compact formulas can be given in this case, too, always thanks to the exponential notation. As already seen, the simplification provided by the exponential approach lies essentially on Lemma 1.55,iv), which allows to perform the computations in $\tilde{\mathcal{U}}$ reducing to much simpler computations in $\hat{\mathfrak{sl}_3}^{\chi}$, and even, thanks to the symmetries highlighted in Definition 5.4, in the Lie subalgebra $L = \hat{\mathfrak{sl}_3}^{\chi} \cap (\mathfrak{s}l_3 \otimes \mathbb{Q}[t]) \subseteq \hat{\mathfrak{sl}_3}^{\chi}$ (see Definition 5.8). Recognizing a $\mathbb{Q}[w]$-module structure on each direct summand of $L = L^- \oplus L^0 \oplus L^+$ and unifying them in a $\mathbb{Q}[w]$-module structure on L (see Definition 5.10) provides a further simplification in the notations: one could have done the same construction for $\hat{\mathfrak{sl}}_2$, but we have the feeling that in the case of $\hat{\mathfrak{sl}}_2$ it would be unnecessary and that on the other hand it is useful to present both formulations. All this is dealt with in Section 5.1.

The most remarkable difference with respect to $A_1^{(1)}$ on one hand and to Mitzman's integral form on the other hand lies in the description of the generators of the imaginary (positive and negative) components; it can be surprising that they are not what one could expect: $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+} \neq$

$\mathbb{Z}^{(sym)}[h_r \mid r > 0]$. More precisely (see Remark 5.13 and Theorem 5.44)

$$\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+} \not\subseteq \mathbb{Z}^{(sym)}[h_r \mid r > 0] \text{ and } \mathbb{Z}^{(sym)}[h_r \mid r > 0] \not\subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+};$$

as we shall show, we need to somehow "deform" the h_r's (by changing some of their signs) to get a basis of $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ by the (sym)-construction (see Definition 5.12, Example 1.2 and Remark 1.12). To this we dedicate Section 5.2.

Notice that in order to prove that $\tilde{\mathcal{U}}_{\mathbb{Z}}$ is an integral form of $\tilde{\mathcal{U}}$ and that B is a $\mathbb{Z}$-basis of $\tilde{\mathcal{U}}_{\mathbb{Z}}$ (Theorem 5.44) it is not necessary to find explicitly all the commutation formulas between the basis elements. In any case, for completeness, we shall collect them in the Appendix 9.A.

5.1 From $A_1^{(1)}$ to $A_2^{(2)}$

Definition 5.1. $\hat{\mathfrak{sl}}_3^{\chi}$ *(respectively* $\tilde{\mathcal{U}}$*) is the Lie algebra (respectively the associative algebra) over* $\mathbb{Q}$ *generated by* $\{c, h_r, x_r^{\pm}, X_{2r+1}^{\pm} \mid r \in \mathbb{Z}\}$ *with relations*

$$c \text{ is central}$$

$$[h_r, h_s] = \delta_{r+s,0} 2r(2 + (-1)^{r-1})c$$

$$[h_r, x_s^{\pm}] = \pm 2(2 + (-1)^{r-1})x_{r+s}^{\pm}$$

$$(s \text{ odd}) \qquad [h_r, X_s^{\pm}] = \begin{cases} \pm 4X_{r+s}^{\pm} & \text{if } 2 \mid r \\ 0 & \text{if } 2 \nmid r \end{cases}$$

$$[x_r^{\pm}, x_s^{\pm}] = \begin{cases} 0 & \text{if } 2 \mid r+s \\ \pm(-1)^s X_{r+s}^{\pm} & \text{if } 2 \nmid r+s \end{cases}$$

$$[x_r^{\pm}, X_s^{\pm}] = [X_r^{\pm}, X_s^{\pm}] = 0$$

$$[x_r^{+}, x_s^{-}] = h_{r+s} + \delta_{r+s,0} rc$$

$$(s \text{ odd}) \qquad [x_r^{\pm}, X_s^{\mp}] = \pm(-1)^r 4x_{r+s}^{\mp}$$

$$(r, s \text{ odd}) \qquad [X_r^{+}, X_s^{-}] = 8h_{r+s} + 4\delta_{r+s,0} rc$$

Notice that $\{x_r^+, x_r^- \mid r \in \mathbb{Z}\}$ generates $\tilde{\mathcal{U}}$.

Moreover $\{c, h_r, x_r^{\pm}, X_{2r+1}^{\pm} \mid r \in \mathbb{Z}\}$ is a basis of $\hat{\mathfrak{sl}}_3^{\chi}$; hence the ordered monomials in these elements (with respect to any total ordering of the basis) is a PBW-basis of $\tilde{\mathcal{U}}$.

$\tilde{\mathcal{U}}^+, \tilde{\mathcal{U}}^-, \tilde{\mathcal{U}}^0$ are the subalgebras of $\tilde{\mathcal{U}}$ generated respectively by

$$\{x_r^+ \mid r \in \mathbb{Z}\},\ \{x_r^- \mid r \in \mathbb{Z}\},\ \{c, h_r \mid r \in \mathbb{Z}\}.$$

$\tilde{\mathcal{U}}^{\pm,0}, \tilde{\mathcal{U}}^{\pm,1}$ and $\tilde{\mathcal{U}}^{\pm,c}$ are the subalgebras of $\tilde{\mathcal{U}}^{\pm}$ generated respectively by

$$\{x_r^{\pm} \mid r \equiv 0\ (mod\ 2)\},\ \{x_r^{\pm} \mid r \equiv 1\ (mod\ 2)\} \text{ and } \{X_{2r+1}^{\pm} \mid r \in \mathbb{Z}\}.$$

$\tilde{\mathcal{U}}^{0,+}, \tilde{\mathcal{U}}^{0,-}, \tilde{\mathcal{U}}^{\mathfrak{h}}$, are the subalgebras of $\tilde{\mathcal{U}}$ (of $\tilde{\mathcal{U}}^0$) generated respectively by

$$\{h_r \mid r > 0\},\ \{h_r \mid r < 0\},\ \{c, h_0\}.$$

Remark 5.2. *Recalling that the root system of $\hat{\mathfrak{sl}}_3^{\chi}$ is*

$$(\pm\alpha+\mathbb{Z}\delta)\cup(\pm 2\alpha+(1+2\mathbb{Z})\delta)\cup(\mathbb{Z}\setminus\{0\})\delta$$

notice that $\{h_0,c\}$ is a basis of the Cartan subalgebra and

$$x_r^{\pm}\in\mathfrak{g}_{r\delta\pm\alpha},\ X_{2r+1}^{\pm}\in\mathfrak{g}_{(2r+1)\delta\pm 2\alpha},\ h_r\in\mathfrak{g}_{r\delta}$$

(see [8]).

The following remark is a consequence of trivial applications of the PBW-Theorem to different subalgebras of $\hat{\mathfrak{sl}}_3^{\chi}$.

Remark 5.3. *$\tilde{\mathcal{U}}^+$ and $\tilde{\mathcal{U}}^-$ are not commutative: $[x_0^+,x_1^+]=-X_1^+$ and $[x_0^-,x_1^-]=X_1^-$.*

$\tilde{\mathcal{U}}^{\pm,0}$, $\tilde{\mathcal{U}}^{\pm,1}$ and $\tilde{\mathcal{U}}^{\pm,c}$ are (commutative) algebras of polynomials:

$$\tilde{\mathcal{U}}^{+,0}\cong\mathbb{Q}[x_{2r}^+\mid r\in\mathbb{Z}],\ \tilde{\mathcal{U}}^{+,1}\cong\mathbb{Q}[x_{2r+1}^+\mid r\in\mathbb{Z}],\ \tilde{\mathcal{U}}^{+,c}\cong\mathbb{Q}[X_{2r+1}^+\mid r\in\mathbb{Z}],$$

$$\tilde{\mathcal{U}}^{-,0}\cong\mathbb{Q}[x_{2r}^-\mid r\in\mathbb{Z}],\ \tilde{\mathcal{U}}^{-,1}\cong\mathbb{Q}[x_{2r+1}^-\mid r\in\mathbb{Z}],\ \tilde{\mathcal{U}}^{-,c}\cong\mathbb{Q}[X_{2r+1}^-\mid r\in\mathbb{Z}].$$

We have the following "triangular" decompositions of $\tilde{\mathcal{U}}^{\pm}$:

$$\tilde{\mathcal{U}}^{\pm}\cong\tilde{\mathcal{U}}^{\pm,0}\otimes\tilde{\mathcal{U}}^{\pm,c}\otimes\tilde{\mathcal{U}}^{\pm,1}\cong\tilde{\mathcal{U}}^{\pm,1}\otimes\tilde{\mathcal{U}}^{\pm,c}\otimes\tilde{\mathcal{U}}^{\pm,0}$$

Remark that $\tilde{\mathcal{U}}^{\pm,c}$ is central in $\tilde{\mathcal{U}}^{\pm}$, so that the images in $\tilde{\mathcal{U}}^{\pm}$ of $\tilde{\mathcal{U}}^{\pm,0}\otimes\tilde{\mathcal{U}}^{\pm,c}$ and $\tilde{\mathcal{U}}^{\pm,1}\otimes\tilde{\mathcal{U}}^{\pm,c}$ are commutative subalgebras of $\tilde{\mathcal{U}}$.

$\tilde{\mathcal{U}}^0$ is not commutative: $[h_r,h_{-r}]\neq 0$ if $r\neq 0$;

$\tilde{\mathcal{U}}^{0,+}$, $\tilde{\mathcal{U}}^{0,-}$, $\tilde{\mathcal{U}}^{\mathfrak{h}}$, are (commutative) algebras of polynomials:

$$\tilde{\mathcal{U}}^{0,+}\cong\mathbb{Q}[h_r\mid r>0],\ \tilde{\mathcal{U}}^{0,-}\cong\mathbb{Q}[h_r\mid r<0],\ \tilde{\mathcal{U}}^{\mathfrak{h}}\cong\mathbb{Q}[c,h_0];$$

Moreover we have the following triangular decomposition of $\tilde{\mathcal{U}}^0$:

$$\tilde{\mathcal{U}}^0\cong\tilde{\mathcal{U}}^{0,-}\otimes\tilde{\mathcal{U}}^{\mathfrak{h}}\otimes\tilde{\mathcal{U}}^{0,+}\cong\tilde{\mathcal{U}}^{0,+}\otimes\tilde{\mathcal{U}}^{\mathfrak{h}}\otimes\tilde{\mathcal{U}}^{0,-}.$$

Remark that $\tilde{\mathcal{U}}^{\mathfrak{h}}$ is central in $\tilde{\mathcal{U}}^0$, so that the images in $\tilde{\mathcal{U}}^0$ of $\tilde{\mathcal{U}}^{0,-}\otimes\tilde{\mathcal{U}}^{\mathfrak{h}}$ and $\tilde{\mathcal{U}}^{\mathfrak{h}}\otimes\tilde{\mathcal{U}}^{0,+}$ are commutative subalgebras of $\tilde{\mathcal{U}}$.

Finally remark the triangular decomposition of $\tilde{\mathcal{U}}$:

$$\tilde{\mathcal{U}}\cong\tilde{\mathcal{U}}^-\otimes\tilde{\mathcal{U}}^0\otimes\tilde{\mathcal{U}}^+\cong\tilde{\mathcal{U}}^+\otimes\tilde{\mathcal{U}}^0\otimes\tilde{\mathcal{U}}^-,$$

and observe that the images of $\tilde{\mathcal{U}}^-\otimes\tilde{\mathcal{U}}^0$ and $\tilde{\mathcal{U}}^0\otimes\tilde{\mathcal{U}}^+$ are subalgebras of $\tilde{\mathcal{U}}$.

Definition 5.4. *$\hat{\mathfrak{sl}}_3^{\chi}$ and $\tilde{\mathcal{U}}$ are endowed with the following anti/auto/homo/morphisms:*

σ is the antiautomorphism defined on the generators by:

$$x_r^+\mapsto x_r^+,\ \ x_r^-\mapsto x_r^-,\ \ (\Rightarrow X_r^{\pm}\mapsto -X_r^{\pm},\ \ h_r\mapsto -h_r,\ \ c\mapsto -c);$$

Ω is the antiautomorphism defined on the generators by:

$$x_r^+\mapsto x_{-r}^-,\ \ x_r^-\mapsto x_{-r}^+,\ \ (\Rightarrow X_r^{\pm}\mapsto X_{-r}^{\mp},\ \ h_r\mapsto h_{-r},\ \ c\mapsto c);$$

T is the automorphism defined on the generators by:

$$x_r^+\mapsto x_{r-1}^+,\ \ x_r^-\mapsto x_{r+1}^-,\ \ (\Rightarrow X_r^{\pm}\mapsto -X_{r\mp 2}^{\pm},\ \ h_r\mapsto h_r-\delta_{r,0}c,\ \ c\mapsto c);$$

for all odd integer $m\in\mathbb{Z}$, λ_m is the homomorphism defined on the generators by:

$$x_r^+\mapsto x_{mr}^+,\ \ x_r^-\mapsto x_{mr}^-,\ \ (\Rightarrow X_r^{\pm}\mapsto X_{mr}^{\pm},\ \ h_r\mapsto h_{mr},\ \ c\mapsto mc).$$

Remark that if m is even λ_m is not defined on $\tilde{\mathcal{U}}$, but it is still defined on $\tilde{\mathcal{U}}^{0,+}=\mathbb{Q}[h_r\mid r>0]$.

Remark 5.5. *$\sigma^2 = \mathrm{id}_{\tilde{\mathcal{U}}}$, $\Omega^2 = \mathrm{id}_{\tilde{\mathcal{U}}}$, T is invertible of infinite order;*
$\lambda_{-1}^2 = \lambda_1 = \mathrm{id}_{\tilde{\mathcal{U}}}$; λ_m is not invertible if $m \neq \pm 1$.

Remark 5.6. *$\sigma\Omega = \Omega\sigma$, $\sigma T = T\sigma$, $\Omega T = T\Omega$. Moreover for all m, n odd we have $\sigma\lambda_m = \lambda_m\sigma$, $\Omega\lambda_m = \lambda_m\Omega$, $\lambda_m T^{\pm 1} = T^{\pm m}\lambda_m$, $\lambda_m\lambda_n = \lambda_{mn}$.*

Remark 5.7. *$\sigma|_{\tilde{\mathcal{U}}^{\pm,0}} = \mathrm{id}_{\tilde{\mathcal{U}}^{\pm,0}}$, $\sigma|_{\tilde{\mathcal{U}}^{\pm,1}} = \mathrm{id}_{\tilde{\mathcal{U}}^{\pm,1}}$, $\sigma(\tilde{\mathcal{U}}^{\pm,c}) = \tilde{\mathcal{U}}^{\pm,c}$, $\sigma(\tilde{\mathcal{U}}^{0,\pm}) = \tilde{\mathcal{U}}^{0,\pm}$, $\sigma(\tilde{\mathcal{U}}^{\mathfrak{h}}) = \tilde{\mathcal{U}}^{\mathfrak{h}}$.*
$\Omega(\tilde{\mathcal{U}}^{\pm,0}) = \tilde{\mathcal{U}}^{\mp,0}$, $\Omega(\tilde{\mathcal{U}}^{\pm,1}) = \tilde{\mathcal{U}}^{\mp,1}$, $\Omega(\tilde{\mathcal{U}}^{\pm,c}) = \tilde{\mathcal{U}}^{\mp,c}$, $\Omega(\tilde{\mathcal{U}}^{0,\pm}) = \tilde{\mathcal{U}}^{0,\mp}$, $\Omega|_{\tilde{\mathcal{U}}^{\mathfrak{h}}} = \mathrm{id}_{\tilde{\mathcal{U}}^{\mathfrak{h}}}$.
$T(\tilde{\mathcal{U}}^{\pm,0}) = \tilde{\mathcal{U}}^{\pm,1}$, $T(\tilde{\mathcal{U}}^{\pm,1}) = \tilde{\mathcal{U}}^{\pm,0}$, $T(\tilde{\mathcal{U}}^{\pm,c}) = \tilde{\mathcal{U}}^{\pm,c}$, $T|_{\tilde{\mathcal{U}}^{0,\pm}} = \mathrm{id}_{\tilde{\mathcal{U}}^{0,\pm}}$, $T(\tilde{\mathcal{U}}^{\mathfrak{h}}) = \tilde{\mathcal{U}}^{\mathfrak{h}}$.
For all odd $m \in \mathbb{Z}$:
$\lambda_m(\tilde{\mathcal{U}}^{\pm,0}) \subseteq \tilde{\mathcal{U}}^{\pm,0}$, $\lambda_m(\tilde{\mathcal{U}}^{\pm,1}) \subseteq \tilde{\mathcal{U}}^{\pm,1}$, $\lambda_m(\tilde{\mathcal{U}}^{\pm,c}) \subseteq \tilde{\mathcal{U}}^{\pm,c}$, $\lambda_m(\tilde{\mathcal{U}}^{\mathfrak{h}}) \subseteq \tilde{\mathcal{U}}^{\mathfrak{h}}$,

$$\lambda_m(\tilde{\mathcal{U}}^{0,\pm}) \subseteq \begin{cases} \tilde{\mathcal{U}}^{0,\pm} & \text{if } m > 0 \\ \tilde{\mathcal{U}}^{0,\mp} & \text{if } m < 0. \end{cases}$$

Definition 5.8. *L, $L^{\pm}$, L^0, $L^{\pm,0}$, $L^{\pm,1}$, $L^{\pm,c}$ are the Lie-subalgebras of $\hat{\mathfrak{sl}}_3^{\chi}$ generated by:*

$$L : \{x_r^+, x_r^- \mid r \geq 0\},$$
$$L^+ : \{x_r^+ \mid r \geq 0\},\ \ L^- : \{x_r^- \mid r \geq 0\},\ \ L^0 : \{h_r \mid r \geq 0\},$$
$$L^{+,0} : \{x_{2r}^+ \mid r \geq 0\},\ \ L^{+,1} : \{x_{2r+1}^+ \mid r \geq 0\},\ \ L^{+,c} : \{X_{2r+1}^+ \mid r \geq 0\}.$$
$$L^{-,0} : \{x_{2r}^- \mid r \geq 0\},\ \ L^{-,1} : \{x_{2r+1}^- \mid r \geq 0\},\ \ L^{-,c} : \{X_{2r+1}^- \mid r \geq 0\}.$$

Remark 5.9. *L^0, $L^{\pm,0}$, $L^{\pm,1}$ and $L^{\pm,c}$ are commutative Lie-algebras; for these subalgebras of L the Lie-generators given in Definition 5.8 are bases over $\mathbb{Q}$.*
Moreover we have $\mathbb{Q}$-vector space decompositions

$$L = L^- \oplus L^0 \oplus L^+, \quad L^+ = L^{+,0} \oplus L^{+,1} \oplus L^{+,c}, \quad L^- = L^{-,0} \oplus L^{-,1} \oplus L^{-,c}.$$

Finally remark that L^+ is T^{-1}-stable and that L^- is T-stable; more in detail $T^{\mp 1}(L^{\pm,0}) = L^{\pm,1}$, $T^{\mp 1}(L^{\pm,1}) \subseteq L^{\pm,0}$ (so that $L^{\pm,0}$ and $L^{\pm,1}$ are $T^{\mp 2}$-stable); $L^{\pm,c}$ is $T^{\mp 1}$-stable.

Definition 5.10. *L is endowed with the $\mathbb{Q}[w]$-module structure defined by $w|_{L^-} = T|_{L^-}$, $w|_{L^+} = T^{-1}|_{L^+}$, $w.h_r = h_{r+1}$ $\forall r \in \mathbb{N}$. Explicitly w acts on $L^{\pm}$ as follows: $w.x_r^{\pm} = x_{r+1}^{\pm}$, $w.X_{2r+1}^{\pm} = -X_{2r+3}^{\pm}$ $\forall r \geq 0$.*

Lemma 5.11. *Let $\xi_1(w), \xi_2(w) \in \mathbb{Q}[w][[u,v]]$. Then:*

i)
$$[\xi_1(w^2).x_0^{\pm}, \xi_2(w^2).x_1^{\pm}] = \mp(\xi_1\xi_2)(-w).X_1^{\pm};$$

ii)
$$[\xi_1(w).x_0^+, \xi_2(w).x_0^-] = (\xi_1\xi_2)(w).h_0;$$

iii)
$$[\xi_1(w).x_0^+, \xi_2(w).X_1^-] = 4\xi_1(-w)\xi_2(-w^2).x_1^-;$$

iv)
$$[\xi_1(w).h_0, \xi_2(w).x_0^{\pm}] = \pm(4\xi_1(w) - 2\xi_1(-w))\xi_2(w).x_0^{\pm}.$$

Proof. The assertions are just a translation of the defining relations of $\tilde{\mathcal{U}}$:

$$[x_{2r}^{\pm}, x_{2s+1}^{\pm}],\ \ [x_r^+, x_s^-],\ \ [x_r^+, X_{2s+1}^-],\ \ [h_r, x_s^{\pm}].$$

For iv), remark that

$$2(2 + (-1)^{r-1})w^r = 4w^r - 2(-w)^r.$$

□

Definition 5.12. *Here we define some $\mathbb{Z}$-subalgebras of $\tilde{\mathcal{U}}$:*

$\tilde{\mathcal{U}}_{\mathbb{Z}}$ *is the $\mathbb{Z}$-subalgebra of $\tilde{\mathcal{U}}$ generated by* $\{(x_r^+)^{(k)}, (x_r^-)^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}\}$;

$\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ *and* $\tilde{\mathcal{U}}_{\mathbb{Z}}^-$ *are the $\mathbb{Z}$-subalgebras of $\tilde{\mathcal{U}}$ (and of $\tilde{\mathcal{U}}_{\mathbb{Z}}$) generated respectively by* $\{(x_r^+)^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}\}$, *and* $\{(x_r^-)^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}\}$;

$\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,0} = \mathbb{Z}^{(div)}[x_{2r}^{\pm} \mid r \in \mathbb{Z}]$;

$\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,1} = \mathbb{Z}^{(div)}[x_{2r+1}^{\pm} \mid r \in \mathbb{Z}]$;

$\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,c} = \mathbb{Z}^{(div)}[X_{2r+1}^{\pm} \mid r \in \mathbb{Z}]$;

$\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} = \mathbb{Z}^{(bin)}[h_0, c]$;

$\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} = \mathbb{Z}^{(sym)}[\varepsilon_r h_{\pm r} \mid r > 0]$ *with* $\varepsilon_r = \begin{cases} 1 & \text{if } 4 \nmid r \\ -1 & \text{if } 4 \mid r \end{cases}$;

$\tilde{\mathcal{U}}_{\mathbb{Z}}^{0}$ *is the $\mathbb{Z}$-subalgebra of $\tilde{\mathcal{U}}$ generated by* $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ *and* $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}$.

The notations are those of Section 1.

In particular remark the definition of $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$ (where the ε_r's represent the necessary "deformation" announced in the Introduction of this section, and discussed in details in Proposition 1.18) and introduce the notation

$$\mathbb{Z}[\tilde{h}_k \mid \pm k > 0] = \mathbb{Z}^{(sym)}[\varepsilon_r h_{\pm r} \mid r > 0]$$

where

$$\tilde{h}_{\pm}(u) = \sum_{k\in\mathbb{N}} \tilde{h}_{\pm k} u^k = \exp\Big(\sum_{r>0} (-1)^{r-1} \frac{\varepsilon_r h_{\pm r}}{r} u^r\Big).$$

Remark 5.13. *It is worth underlining that $\tilde{h}_+(u) \neq \hat{h}_+(u)$, where*

$$\mathbb{Z}[\hat{h}_k \mid k > 0] = \mathbb{Z}^{(sym)}[h_r \mid r > 0],$$

that is

$$\hat{h}_+(u) = \sum_{k\in\mathbb{N}} \hat{h}_k u^k = \exp\Big(\sum_{r>0} (-1)^{r-1} \frac{h_r}{r} u^r\Big).$$

More precisely the $\mathbb{Z}$-subalgebras generated respectively by $\{\hat{h}_k \mid k > 0\}$ and $\{\tilde{h}_k \mid k > 0\}$ are different and not included in each other: indeed $\tilde{h}_1 = \hat{h}_1$, $\tilde{h}_2 = \hat{h}_2$, $\tilde{h}_3 = \hat{h}_3$ but $\hat{h}_4 \notin \mathbb{Z}[\tilde{h}_k \mid k > 0]$ and $\tilde{h}_4 \notin \mathbb{Z}[\hat{h}_k \mid k > 0]$ (see Propositions 1.18 and 1.19 and Remark 1.20).

Notice that we are considering the algebra involution of $\mathbb{Q}[h_r \mid r > 0]$ defined by $h_r \mapsto \varepsilon_r h_r \,\forall r > 0$ through which (using Notation 1.15) $\hat{h}^{\{a\}}(u)$ is mapped to $\hat{h}^{\{\varepsilon a\}}(u)$; in particular $\tilde{h}(u) = \hat{h}^{\{\varepsilon\}}(u)$ so that $\hat{h}^{\{a\}}(u) \subseteq \mathbb{Z}[\hat{h}_k \mid k > 0][[u]]$ if and only if $\hat{h}^{\{\varepsilon a\}}(u) \subseteq \mathbb{Z}[\tilde{h}_k \mid k > 0][[u]]$.

Remark 5.14. *Let $\xi(w) \in \mathbb{Q}[w][[u]]$; the elements*

$$\exp\Big(\xi(w^2).x_0^{\pm}\Big), \ \exp\Big(\xi(w^2).x_1^{\pm}\Big) \ \text{ and } \ \exp\big(\xi(w).X_1^{\pm}\big)$$

lie respectively in $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,0}[[u]]$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,1}[[u]]$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,c}[[u]]$ if and only if $\xi(w)$ has integral coefficients, that is if and only if $\xi(w) \in \mathbb{Z}[w][[u]]$ (see Definitions 1.6 and 5.10).

Remark also that (see Remark 1.16)

$$\hat{h}_+(u) = \exp(\ln(1 + wu).h_0),$$

while

$$\tilde{h}_+(u) = \exp\bigg(\Big(\ln(1 + uw) - \frac{1}{2}\ln\Big(1 - u^4 w^4\Big)\Big).h_0\bigg).$$

Before entering the study of the integral forms just introduced, we still dwell on the comparison between $\tilde{h}_+(u)$ and $\hat{h}_+(u)$, proving Lemma 5.16, that will be useful later.

Lemma 5.15. *For all $m \in \mathbb{Z} \setminus \{0\}$ we have*

$$(1+m^2u)^{\frac{1}{m}} \in 1 + mu\mathbb{Z}[[u]].$$

Proof. $(1+\sum_{r>0} a_r u^r)^m = 1+m^2u$ implies

$$1+m^2u = 1+m\sum_{r>0} a_r u^r + \sum_{k>1}\binom{m}{k}\big(\sum_{r>0} a_r u^r\big)^k.$$

Let us prove by induction on s that $a_s \in m\mathbb{Z}$:

if $s=1$ we have that $ma_1 = m^2$;

if $s>1$ the coefficient c_s of u^s in $\sum_{k>1}\binom{m}{k}\big(\sum_{r>0} a_r u^r\big)^k$ is a combination with integral coefficients of products of the a_t's with $t<s$, which are all multiple of m. Then, since $k \geq 2$, $m^2 \mid c_s$. But $ma_s + c_s = 0$, thus $m \mid a_s$. □

Lemma 5.16. *Let us consider the integral forms $\mathbb{Z}[\hat{h}_k \mid k>0]$ and $\mathbb{Z}[\tilde{h}_k \mid k>0]$ of $\mathbb{Q}[h_r \mid r>0]$ (see Example 1.2, notation 1.2, Definiton 5.12 and Remark 5.13); for all $m>0$ recall the $\mathbb{Q}$-algebra homomorphism λ_m of $\mathbb{Q}[h_r \mid r>0]$ (see Proposition 1.13) and define the analogous homomorphism $\tilde{\lambda}_m$ mapping each $\varepsilon_r h_r$ to $\varepsilon_{mr} h_{mr}$ (of course $\mathbb{Z}[\tilde{h}_k \mid k>0]$ is $\tilde{\lambda}_m$-stable $\forall m>0$).*

We have that:

i) if m is odd then $\tilde{\lambda}_m = \lambda_m$; in particular $\mathbb{Z}[\tilde{h}_k \mid k>0]$ is λ_m-stable;

ii) $\lambda_2(\hat{h}_k) \in \mathbb{Z}[\tilde{h}_l \mid l>0]$ for all $k>0$;

iii) $\hat{h}_+(4u)^{\frac{1}{2}} \in \mathbb{Z}[\tilde{h}_k \mid k>0][[u]]$;

Proof. i) If m is odd then $4 \mid mr \Leftrightarrow 4 \mid r$, hence $\varepsilon_{mr} = \varepsilon_r$ $\forall r>0$ and the claim follows from Proposition 1.13

ii) By Proposition 1.13 we know that $\mathbb{Z}[\tilde{h}_k \mid k>0]$ is $\tilde{\lambda}_2$-stable; but

$$\tilde{\lambda}_2(\tilde{h}_+(u^2)) = \exp\sum_{r>0}(-1)^{r-1}\frac{\varepsilon_{2r}h_{2r}}{r}u^{2r} = \exp\sum_{r>0}\frac{h_{2r}}{r}u^{2r} = \lambda_2(\hat{h}_+(-u^2))^{-1};$$

equivalently

$$\lambda_2(\hat{h}_+(u^2)) = \tilde{\lambda}_2(\tilde{h}_+(-u^2))^{-1}$$

which implies the claim.

iii) Remark that

$$\hat{h}_+(u)\tilde{h}_+(u)^{-1} = \exp\left(-\sum_{r>0}\frac{2h_{4r}}{4r}u^{4r}\right) = \tilde{\lambda}_4(\tilde{h}_+(-u^4))^{-\frac{1}{2}};$$

then

$$\hat{h}_+(4u)^{\frac{1}{2}} = \tilde{h}_+(4u)^{\frac{1}{2}}\tilde{\lambda}_4(\tilde{h}_+(-4^4u^4))^{-\frac{1}{4}}.$$

Since $\tilde{h}_+(4u) \in 1+4u\mathbb{Z}[\tilde{h}_k \mid k>0][[u]]$ and $\tilde{\lambda}_4(\tilde{h}_+(4^4u^4)) \in 1+4^4u\mathbb{Z}[\tilde{h}_k \mid k>0][[u]]$ we deduce from Lemma 5.15 and Remark 1.8,5) that

$$\tilde{h}_+(4u)^{\frac{1}{2}},\ \ \tilde{\lambda}_4(\tilde{h}_+(4^4u^4))^{\frac{1}{4}} \in \mathbb{Z}[\tilde{h}_k \mid k>0][[u]],$$

which implies the claim. □

Remark 5.17. *It is obvious that $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,0}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,1}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,c}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ are integral forms respectively of $\tilde{\mathcal{U}}^{\pm,0}$, $\tilde{\mathcal{U}}^{\pm,1}$, $\tilde{\mathcal{U}}^{\pm,c}$, $\tilde{\mathcal{U}}^{0,\pm}$ and $\tilde{\mathcal{U}}^{\mathfrak{h}}$.*

Hence by the commutativity properties we also have that $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,0}\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,c}$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,c}$ are integral forms respectively of $\tilde{\mathcal{U}}^{\pm,0}\tilde{\mathcal{U}}^{\pm,c}$ and $\tilde{\mathcal{U}}^{\pm,c}\tilde{\mathcal{U}}^{\pm,1}$.

Analogously $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-}\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ are integral forms respectively of $\tilde{\mathcal{U}}^{\mathfrak{h}}\tilde{\mathcal{U}}^{0,+}$ and $\tilde{\mathcal{U}}^{0,-}\tilde{\mathcal{U}}^{\mathfrak{h}}$.

We want to prove the following

Theorem 5.18. *1) $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0} = \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-}\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}$, so that $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0}$ is an integral form of $\tilde{\mathcal{U}}^{0}$;*

2) $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm} = \tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,0}$, so that $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+}$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{-}$ are integral forms respectively of $\tilde{\mathcal{U}}^{+}$ and $\tilde{\mathcal{U}}^{-}$;

3) $\tilde{\mathcal{U}}_{\mathbb{Z}} = \tilde{\mathcal{U}}_{\mathbb{Z}}^{-}\tilde{\mathcal{U}}_{\mathbb{Z}}^{0}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+}$, so that $\tilde{\mathcal{U}}_{\mathbb{Z}}$ is an integral form of $\tilde{\mathcal{U}}$.

It is useful to evidentiate the behaviour of the $\mathbb{Z}$-subalgebras introduced above under the symmetries of $\tilde{\mathcal{U}}$.

Proposition 5.19. *The following stability properties under the action of σ, Ω, $T^{\pm 1}$ and λ_m ($m \in \mathbb{Z}$ odd) hold:*

i) $\tilde{\mathcal{U}}_{\mathbb{Z}}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+}$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{-}$ are σ-stable, $T^{\pm 1}$-stable, λ_m-stable.

$\tilde{\mathcal{U}}_{\mathbb{Z}}$ is also Ω-stable, while $\Omega(\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm}) = \tilde{\mathcal{U}}_{\mathbb{Z}}^{\mp}$.

ii) $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}$ are σ-stable, $T^{\pm 2}$-stable, λ_m-stable.

$\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}$ is also $T^{\pm 1}$-stable, while $T^{\pm 1}(\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}) = \tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}$.

$\Omega(\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}) = \tilde{\mathcal{U}}_{\mathbb{Z}}^{-,0}$, $\Omega(\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}) = \tilde{\mathcal{U}}_{\mathbb{Z}}^{-,1}$ and $\Omega(\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}) = \tilde{\mathcal{U}}_{\mathbb{Z}}^{-,c}$.

iii) $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-}$ are σ-stable and $T^{\pm 1}$-stable.

$\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ is also Ω-stable and λ_m-stable; $\Omega(\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}) = \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\mp}$; $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$ is λ_m-stable if $m > 0$, while $\lambda_m(\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}) \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\mp}$ if $m < 0$.

iv) $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0}$ is σ-stable, Ω-stable, $T^{\pm 1}$-stable, λ_m-stable.

Proof. The only non-trivial assertion is the claim that $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ is λ_m-stable when $m > 0$, which was proved in Lemma 5.16,i).

The assertion about $\lambda_m(\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\pm})$ in the general case follows using that

$$\Omega(\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}) = \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\mp} = \lambda_{-1}(\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}), \ \ \lambda_m\Omega = \Omega\lambda_m \ \text{ and } \ \lambda_{-m} = \lambda_{-1}\lambda_m.$$

Remark that

$$\sigma(\tilde{h}_{\pm}(u)) = \tilde{h}_{\pm}(u)^{-1},\ \Omega(\tilde{h}_{\pm}(u)) = \lambda_{-1}(\tilde{h}_{\pm}(u)) = \tilde{h}_{\mp}(u),\ T^{\pm 1}(\tilde{h}_{\pm}(u)) = \tilde{h}_{\pm}(u).$$

□

Remark 5.20. *The stability properties described in Proposition 5.19 imply that:*

i) $\sigma(\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-}\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}) = \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-}$; in particular

$$\tilde{\mathcal{U}}_{\mathbb{Z}}^{0} = \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-}\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+} \Leftrightarrow \tilde{\mathcal{U}}_{\mathbb{Z}}^{0} = \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-}.$$

ii) $T^{\pm 1}(\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}) = \tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}$ is $T^{\pm 2}$-stable and λ_m-stable ($m \in \mathbb{Z}$ odd); in particular:

$$\tilde{\mathcal{U}}_{\mathbb{Z}}^{+} = \tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0} \Leftrightarrow \tilde{\mathcal{U}}_{\mathbb{Z}}^{+} = \tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}.$$

iii) $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+}$ is $T^{\pm 1}$-stable and λ_{-1}-stable, and $\Omega(\tilde{\mathcal{U}}_{\mathbb{Z}}^{0}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+}) = \tilde{\mathcal{U}}_{\mathbb{Z}}^{-}\tilde{\mathcal{U}}_{\mathbb{Z}}^{0}$; in particular it is enough to prove that $(x_0^{+})^{(k)}\tilde{h}_{+}(u) \in \tilde{h}_{+}(u)\tilde{\mathcal{U}}_{\mathbb{Z}}^{+}[[u]]$ $\forall k \geq 0$ in order to show that

$$(x_r^{+})^{(k)}\tilde{h}_{\pm}(u) \in \tilde{h}_{\pm}(u)\tilde{\mathcal{U}}_{\mathbb{Z}}^{+}[[u]], \tilde{h}_{\pm}(u)(x_r^{-})^{(k)} \in \tilde{\mathcal{U}}_{\mathbb{Z}}^{-}[[u]]\tilde{h}_{\pm}(u) \ \ \forall r \in Z,\ k \in \mathbb{N},$$

or equivalently that $\tilde{\mathcal{U}}_{\mathbb{Z}}^+\tilde{\mathcal{U}}_{\mathbb{Z}}^0 \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ *and* $\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^- \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0$.

iv) $\tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ *is* $T^{\pm 1}$*-stable and* λ_m*-stable* ($m \in \mathbb{Z}$ *odd*)*; in particular if one shows that* $(x_0^+)^{(k)}(x_1^-)^{(l)} \in \tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ *it follows that* $\forall r,s \in \mathbb{Z}$ *such that* $2 \nmid (r+s)$

$$(x_r^+)^{(k)}(x_s^-)^{(l)} = T^{-r}\lambda_{r+s}((x_0^+)^{(k)}(x_1^-)^{(l)}) \in \tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+.$$

Proposition 5.21. *The following identities hold in* $\tilde{\mathcal{U}}$*:*

$$\hat{h}_+(u)\hat{h}_-(v) = \hat{h}_-(v)(1-uv)^{-4c}(1+uv)^{2c}\hat{h}_+(u)$$

and

$$\tilde{h}_+(u)\tilde{h}_-(v) = \tilde{h}_-(v)(1-uv)^{-4c}(1+uv)^{2c}\tilde{h}_+(u).$$

In particular $\tilde{\mathcal{U}}_{\mathbb{Z}}^0 = \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-}\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ *and* $\tilde{\mathcal{U}}_{\mathbb{Z}}^0$ *is an integral form of* $\tilde{\mathcal{U}}^0$.

Proof. Since $[h_r,h_s] = [\varepsilon_r h_r, \varepsilon_s h_s] = \delta_{r+s,0}2r(2+(-1)^{r-1})c$, the claim is Proposition 1.58 with $m=4, l=-2$. □

Lemma 5.22. *The following identity holds in* $\tilde{\mathcal{U}}$ *for all* $r,s \in \mathbb{Z}$*:*

$$\exp(x_{2r}^+u)\exp(x_{2s+1}^+v) = \exp(x_{2s+1}^+v)\exp(-X_{2r+2s+1}^+uv)\exp(x_{2r}^+u).$$

Proof. The claim is an immediate consequence of Lemma 1.55,vii), thanks to the relation $[x_{2r}^+, x_{2s+1}^+] = -X_{2r+2s+1}^+$. □

Corollary 5.23. $\tilde{\mathcal{U}}_{\mathbb{Z}}^+ = \tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}$*; then* $\tilde{\mathcal{U}}_{\mathbb{Z}}^\pm$ *is an integral form of* $\tilde{\mathcal{U}}^\pm$.

More in detail $\tilde{\mathcal{U}}_{\mathbb{Z}}^\pm = \tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,0}\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,1} = \tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,0}$.

Proof. From Lemma 5.22 we deduce that:

i) $(X_{2r+1}^+)^{(k)} \in \tilde{\mathcal{U}}_{\mathbb{Z}}^+$ $\forall k \in \mathbb{N}, r \in \mathbb{Z}$; this implies that

$$\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^+ \text{ and } \tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^+.$$

ii) $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}$, hence $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}$ is stable by left multiplication by $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}$, hence by $\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ (which is generated by $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}$).

Since $1 \in \tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}$, we deduce $\tilde{\mathcal{U}}_{\mathbb{Z}}^+ \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}$, and the claim follows applying Ω and T (see Proposition 5.19,i) and ii)).

□

Proposition 5.24. $\tilde{\mathcal{U}}_{\mathbb{Z}}^+\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\tilde{\mathcal{U}}_{\mathbb{Z}}^+$*; more precisely*

$$(x_r^+)^{(k)}\binom{h_0}{l} = \binom{h_0-2k}{l}(x_r^+)^{(k)} \quad \forall r \in \mathbb{Z},\ k,l \in \mathbb{N}.$$

Proof. The claim follows by immediate application of (1.56.1). □

Proposition 5.25. *In* $\tilde{\mathcal{U}}$ *the following holds:*

i) $x_0^+\tilde{h}_+(u) = \tilde{h}_+(u)(1-uT^{-1})^6(1-u^2T^{-2})^{-3}(1+u^2T^{-2})(x_0^+)$*;*

ii) $(x_0^+)^{(k)}\tilde{h}_+(u) \in \tilde{h}_+(u)\tilde{\mathcal{U}}_{\mathbb{Z}}^+[[u]]$ $\forall k \in \mathbb{N}$*;*

iii) $\tilde{\mathcal{U}}_{\mathbb{Z}}^+\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}\tilde{\mathcal{U}}_{\mathbb{Z}}^+$.

Proof. i) We have that $[\varepsilon_r h_r, x_0^+] = \varepsilon_r 2(2+(-1)^{r-1})x_r^+$ and

$$\varepsilon_r 2(2+(-1)^{r-1}) = \begin{cases} 6 & \text{if } 2 \nmid r \\ 2 = 6-4 & \text{if } 2 \mid r \text{ and } 4 \nmid r \\ -2 = 6-4-4 & \text{if } 4 \nmid r, \end{cases}$$

hence Proposition 1.60 applies, with $m_1 = 6$, $m_2 = -2$, $m_4 = -1$ and implies that

$$x_0^+ \tilde{h}_+(u) = \tilde{h}_+(u)(1+uT^{-1})^{-6}(1-u^2T^{-2})^2(1-u^4T^{-4})(x_0^+) =$$

$$= \tilde{h}_+(u)(1-uT^{-1})^6(1-u^2T^{-2})^{-3}(1+u^2T^{-2})(x_0^+).$$

ii) Let us underline that $(1-u^2)^{-3}(1+u^2) \in \mathbb{Z}[[u^2]]$, hence from the coefficients of $(1-u)^6$ it can be deduced that

$$(1-u)^6(1-u^2)^{-3}(1+u^2) \in \mathbb{Z}[[u^2]] + 2u\mathbb{Z}[[u^2]]$$

and

$$x_0^+ \tilde{h}_+(u) = \tilde{h}_+(u) \sum_{r \geq 0} a_r x_r^+ u^r \ \text{ with } \ a_r \in \mathbb{Z}\ \forall r \geq 0 \text{ and } 2 \mid a_r\ \forall r \text{ odd}.$$

If we define $y_0 = \sum_{r\geq 0} a_{2r} x_{2r}^+ u^{2r}$, $y_1 = \frac{1}{2}\sum_{r \geq 0} a_{2r+1} x_{2r+1}^+ u^{2r+1}$ we have that, thanks to Lemma 1.55,v) and viii)

$$\exp(x_0^+ v)\tilde{h}_+(u) = \tilde{h}_+(u)\exp((y_0+2y_1)v) =$$

$$= \tilde{h}_+(u)\exp(2y_1 v)\exp\Big([y_0, y_1]v^2\Big)\exp(y_0 v) \in \tilde{h}_+(u)\tilde{\mathcal{U}}_{\mathbb{Z}}^+[[u,v]],$$

thanks to Remark 5.14, from which the claim follows.

iii) From the $T^{\pm 1}$-stability of $\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ and the fact that $T^{\pm 1}|_{\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}} = id$ we deduce that for all $r \in \mathbb{Z}$, $k \in \mathbb{N}$

$$(x_r^+)^{(k)} \tilde{h}_+(u) \in \tilde{h}_+(u)\tilde{\mathcal{U}}_{\mathbb{Z}}^+[[u]].$$

The claim follows recalling that the $(x_r^+)^{(k)}$'s generate $\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ and the $\tilde{h}_k$'s generate $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}$. □

Corollary 5.26. *$\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm}\tilde{\mathcal{U}}_{\mathbb{Z}}^0 = \tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm}$. In particular $\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0$ are subalgebras of $\tilde{\mathcal{U}}_{\mathbb{Z}}$.*

Proof. $\tilde{\mathcal{U}}_{\mathbb{Z}}^+\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ (see Proposition 5.24) and $\tilde{\mathcal{U}}_{\mathbb{Z}}^+\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ (see Proposition 5.25,iii)); moreover

$$\tilde{\mathcal{U}}_{\mathbb{Z}}^+\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-} = \lambda_{-1}(\tilde{\mathcal{U}}_{\mathbb{Z}}^+\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}) \subseteq \lambda_{-1}(\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}\tilde{\mathcal{U}}_{\mathbb{Z}}^+) = \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-}\tilde{\mathcal{U}}_{\mathbb{Z}}^+.$$

Hence $\tilde{\mathcal{U}}_{\mathbb{Z}}^+\tilde{\mathcal{U}}_{\mathbb{Z}}^0 \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+$.

Applying σ we get the reverse inclusion and applying Ω we obtain the claim for $\tilde{\mathcal{U}}_{\mathbb{Z}}^-$. □

Now that we have described $\tilde{\mathcal{U}}_{\mathbb{Z}}^0, \tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm}$ and the $\mathbb{Z}$-subalgebras generated by $\tilde{\mathcal{U}}_{\mathbb{Z}}^0$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ (respectively by $\tilde{\mathcal{U}}_{\mathbb{Z}}^0$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^-$), in order to show that $\tilde{\mathcal{U}}_{\mathbb{Z}} = \tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ it remains to prove that

$$\tilde{\mathcal{U}}_{\mathbb{Z}}^0 \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}} \ \text{ and } \ \tilde{\mathcal{U}}_{\mathbb{Z}}^+\tilde{\mathcal{U}}_{\mathbb{Z}}^- \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+.$$

Before attaching this problem in its generality it is worth evidentiating the existence of some copies of $\hat{\mathfrak{sl}}_2$ inside $\hat{\mathfrak{sl}}_3^{\chi}$, hence of embeddings $\hat{\mathcal{U}} \hookrightarrow \tilde{\mathcal{U}}$, that induce some useful commutation relations in $\tilde{\mathcal{U}}$.

Remark 5.27. *The* $\mathbb{Q}$*-linear maps* $f, F : \hat{\mathfrak{sl}}_2 \to \hat{\mathfrak{sl}}_3{}^\chi$ *defined by*

$$f: \qquad x_r^\pm \mapsto x_{2r}^\pm, \;\; h_r \mapsto h_{2r}, \;\; c \mapsto 2c$$

$$F: \qquad x_r^\pm \mapsto \frac{X_{2r\mp1}^\pm}{4}, \;\; h_r \mapsto \frac{h_{2r}}{2} - \delta_{r,0}\frac{c}{4}, \;\; c \mapsto \frac{c}{2}$$

are Lie-algebra homomorphisms, obviously injective, inducing embeddings $f, F : \hat{\mathcal{U}} \hookrightarrow \tilde{\mathcal{U}}$.

Corollary 5.28. $f(\hat{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}) \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}$.

Proof. Since $f(\hat{\mathcal{U}}_{\mathbb{Z}}^\pm) \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,0} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}$ we have that f maps $\hat{\mathcal{U}}_{\mathbb{Z}}$ (which is generated by $\hat{\mathcal{U}}_{\mathbb{Z}}^+$ and $\hat{\mathcal{U}}_{\mathbb{Z}}^-$) into $\tilde{\mathcal{U}}_{\mathbb{Z}}$; in particular $f(\hat{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}) \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}$. But (recalling Example 1.9)

$$f(\hat{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}) = f(\mathbb{Z}^{(bin)}[h_0, c]) = \mathbb{Z}^{(bin)}[h_0, 2c],$$

thus $\mathbb{Z}^{(bin)}[h_0, 2c] \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}$. Since $\tilde{\mathcal{U}}_{\mathbb{Z}}$ is T-stable and $T(h_0) = h_0 - c$ we also have $\mathbb{Z}^{(bin)}[h_0 - c] \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}$, so that

$$f(\hat{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}) = \mathbb{Z}^{(bin)}[h_0, 2c] \subseteq \mathbb{Z}^{(bin)}[h_0, c] = \mathbb{Z}^{(bin)}[h_0, h_0 - c] \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}$$

which is the claim because $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} = \mathbb{Z}^{(bin)}[h_0, c]$. □

Proposition 5.29. $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}\tilde{\mathcal{U}}_{\mathbb{Z}}^{-,0} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ *and* $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{-,1} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+$.

Proof. $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}\tilde{\mathcal{U}}_{\mathbb{Z}}^{-,0} = f(\hat{\mathcal{U}}_{\mathbb{Z}}^+\hat{\mathcal{U}}_{\mathbb{Z}}^-) \subseteq f(\hat{\mathcal{U}}_{\mathbb{Z}}^-\hat{\mathcal{U}}_{\mathbb{Z}}^0\hat{\mathcal{U}}_{\mathbb{Z}}^+) = \tilde{\mathcal{U}}_{\mathbb{Z}}^{-,0} f(\hat{\mathcal{U}}_{\mathbb{Z}}^0)\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}$: we want to prove that $f(\hat{\mathcal{U}}_{\mathbb{Z}}^0) = f(\hat{\mathcal{U}}_{\mathbb{Z}}^{0,-}\hat{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\hat{\mathcal{U}}_{\mathbb{Z}}^{0,+}) \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^0$.

By Corollary 5.28 $f(\hat{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}) \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$.

On the other hand

$$f(\hat{\mathcal{U}}_{\mathbb{Z}}^{0,+}) = f(\mathbb{Z}^{(sym)}[h_r \mid r > 0]) = \mathbb{Z}^{(sym)}[h_{2r} \mid r > 0] = \lambda_2(\mathbb{Z}[\hat{h}_k \mid k > 0]),$$

hence $f(\hat{\mathcal{U}}_{\mathbb{Z}}^{0,+}) \subseteq \mathbb{Z}[\tilde{h}_k \mid k > 0] = \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ thanks to Lemma 5.16,ii).

Finally remark that $f\Omega = \Omega f$, thus $f(\hat{\mathcal{U}}_{\mathbb{Z}}^{0,-}) = f\Omega(\hat{\mathcal{U}}_{\mathbb{Z}}^{0,+}) \subseteq \Omega\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-}$ (see Proposition 5.19,iii)).

It follows that $f(\hat{\mathcal{U}}_{\mathbb{Z}}^0) \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^0$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}\tilde{\mathcal{U}}_{\mathbb{Z}}^{-,0} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+$.

The assertion for $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,1}$ follows applying T, see Proposition 5.19,i),ii) and iv). □

5.2 $\exp(x_0^+u)\exp(x_1^-v)$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}$: here comes the hard work

We shall deal with the commutation between $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{-,1}$ following the strategy already proposed for $\hat{\mathcal{U}}_{\mathbb{Z}}$ and recalling Remark 5.20,iv): finding an explicit expression involving suitable exponentials for

$$\exp(x_0^+u)\exp(x_1^-v) \in \tilde{\mathcal{U}}^{-,1}\tilde{\mathcal{U}}^{-,c}\tilde{\mathcal{U}}^{-,0}\tilde{\mathcal{U}}^{0,+}\tilde{\mathcal{U}}^{+,1}\tilde{\mathcal{U}}^{+,c}\tilde{\mathcal{U}}^{+,0}[[u,v]]$$

and proving that all its coefficients lie in

$$\tilde{\mathcal{U}}_{\mathbb{Z}}^{-,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{-,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{-,0}\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+.$$

Since here there are more factors involved, the computation is more complicated than in the case of $\hat{\mathfrak{sl}}_2$ and the simplification provided by this approach is even more evident. On the other hand it is not immediately clear from the commutation formula that our element belongs to $\tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+$, or better: the factors relative to the (negative, resp. positive) real root vectors will be evidently elements of $\tilde{\mathcal{U}}_{\mathbb{Z}}^-$, resp. $\tilde{\mathcal{U}}_{\mathbb{Z}}^+$, while proving that the null part lies indeed in $\tilde{\mathcal{U}}_{\mathbb{Z}}^0$ is not evident at all and will require a deeper inspection (see Remark 5.39, Lemma 5.40 and Corollary 5.41).

As we shall see, in order to complete the proof that $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}\subseteq\tilde{\mathcal{U}}_{\mathbb{Z}}$ (see Proposition 5.43), it is useful to compute also $\exp\left(x_0^+u\right)\exp\left(X_1^-v\right)$. The two computations ($\exp\left(x_0^+u\right)\exp(yv)$ with $y=x_1^-$ or $y=X_1^-$) are essentially the same and will be performed together (see the considerations from Remark 5.30 to Lemma 5.34, of which the Propositions 5.35 and 5.36 are straightforward applications); even though $\exp\left(x_0^+u\right)\exp\left(x_1^-v\right)$ presents more symmetries than $\exp\left(x_0^+u\right)\exp\left(X_1^-v\right)$ (see Remark 5.32,iii)), its interpretation will require more work, since it is not evident the connection with $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}$, as just mentioned.

Remark 5.30. *Let $G=G(u,v)\in\tilde{\mathcal{U}}[[u,v]]$ and $y\in L^-$ (see Definition 5.8); then*

$$G(u,v)=\exp\left(x_0^+u\right)\exp(yv)$$

if and only if the following two conditions hold (see Lemma 1.55,x)):

a) $G(0,v)=\exp(yv)$;

b) $\frac{d}{du}G(u,v)=x_0^+G(u,v)$.

Notation 5.31. *In the following (recalling Definition 5.10) G^-, G^0, G^+ will denote elements of $\tilde{\mathcal{U}}[[u,v]]$ of the form*

$$G^-=\exp(\alpha_-)\exp(\beta_-)\exp(\gamma_-),$$
$$G^+=\exp(\gamma_+)\exp(\beta_+)\exp(\alpha_+),$$
$$G^0=\exp(\eta)$$

with

$$\alpha_-\in v\mathbb{Q}[w^2][[u,v]].x_1^-,\ \beta_-\in v\mathbb{Q}[w][[u,v]].X_1^-,\ \gamma_-\in v\mathbb{Q}[w^2][[u,v]].x_0^-,$$
$$\alpha_+\in u\mathbb{Q}[w^2][[u,v]].x_0^+,\ \beta_+\in u\mathbb{Q}[w][[u,v]].X_1^+,\ \gamma_+\in u\mathbb{Q}[w^2][[u,v]].x_1^+,$$
$$\eta\in uvw\mathbb{Q}[w][[u,v]].h_0.$$

$G(u,v)$ will denote the element $G(u,v)=G=G^-G^0G^+$.

Remark 5.32. *Let $G=G^-G^0G^+\in\tilde{\mathcal{U}}[[u,v]]$ be as in Notation 5.31. Then:*

i) Of course

$$\frac{dG}{du}=\frac{dG^-}{du}G^0G^++G^-\frac{dG^0}{du}G^++G^-G^0\frac{dG^+}{du}$$

where, considering the commutativity properties, we have that

$$\frac{dG^-}{du}=\exp(\alpha_-)\exp(\beta_-)\frac{d(\alpha_-+\beta_-+\gamma_-)}{du}\exp(\gamma_-),$$
$$\frac{dG^+}{du}=\exp(\gamma_+)\frac{d(\alpha_++\beta_++\gamma_+)}{du}\exp(\beta_+)\exp(\alpha_+),$$
$$\frac{dG^0}{du}=\frac{d\eta}{du}G^0.$$

ii) If moreover $G=\exp\left(x_0^+u\right)\exp(yv)$ with $y\in L^-$, the property b) of Remark 5.30 translates into

$$x_0^+G=\exp(\alpha_-)\exp(\beta_-)\frac{d(\alpha_-+\beta_-+\gamma_-)}{du}\exp(\gamma_-)G^0G^++$$

$$+G^-\frac{d\eta}{du}G^0G^+ + G^-G^0\exp(\gamma_+)\frac{d(\alpha_+ + \beta_+ + \gamma_+)}{du}\exp(\beta_+)\exp(\alpha_+).$$

iii) If in addition to condition ii) we also have $y = x_1^-$, *then* $T\lambda_{-1}\Omega(G(u,v)) = G(v,u)$; *hence*

$$G^-(u,v) = T\lambda_{-1}\Omega(G^+)(v,u),$$

$$\alpha_-(u,v) = T\lambda_{-1}\Omega(\alpha_+)(v,u),$$
$$\beta_-(u,v) = T\lambda_{-1}\Omega(\beta_+)(v,u),$$
$$\gamma_-(u,v) = T\lambda_{-1}\Omega(\gamma_+)(v,u),$$
$$\eta(u,v) = \eta(v,u).$$

Observe that $T\lambda_{-1}\Omega(X_{2r+1}^+) = -X_{2r+3}^- \ \forall r \in \mathbb{Z}$.

The following lemma is based on Lemma 1.55,iv) and on the defining relations of $\tilde{\mathcal{U}}$ (Definition 5.1).

Lemma 5.33. *With the notations fixed in 5.31 we have that:*

i)
$$x_0^+\exp(\alpha_-) =$$

$$= \exp(\alpha_-)\left(x_0^+ + [x_0^+,\alpha_-] + \frac{1}{2}[[x_0^+,\alpha_-],\alpha_-] + \frac{1}{6}[[[x_0^+,\alpha_-],\alpha_-],\alpha_-]\right);$$

ii)
$$x_0^+\exp(\alpha_-)\exp(\beta_-) = \exp(\alpha_-)\exp(\beta_-)\cdot$$

$$\cdot\left(x_0^+ + [x_0^+,\alpha_-] + \frac{1}{2}[[x_0^+,\alpha_-],\alpha_-] + \frac{1}{6}[[[x_0^+,\alpha_-],\alpha_-],\alpha_-] + [x_0^+,\beta_-]\right);$$

iii)
$$(x_0^+ + [x_0^+,\alpha_-])\exp(\gamma_-) =$$

$$= \exp(\gamma_-)\left(x_0^+ + [x_0^+,\alpha_-] + [x_0^+,\gamma_-]\right) +$$
$$+\left([[x_0^+,\alpha_-],\gamma_-] + \frac{1}{2}[[x_0^+,\gamma_-],\gamma_-] - \frac{1}{2}[[[x_0^+,\alpha_-],\gamma_-],\gamma_-]\right)\exp(\gamma_-);$$

iv) $x_0^+\exp(\eta) = \exp(\eta)(y_0 + y_1)$ *with*

$$y_0 \in \mathbb{Q}[w^2][[u,v]].x_0^+, \ \ y_1 \in w\mathbb{Q}[w^2][[u,v]].x_0^+;$$

v)
$$(y_0 + y_1)\exp(\gamma_+) = \exp(\gamma_+)(y_0 + y_1 + [y_0,\gamma_+]).$$

vi) In conclusion

$$x_0^+G = \frac{dG}{du}$$

if and only if the following relations hold:

$$\frac{d\alpha_-}{du} = [x_0^+,\beta_-] + [[x_0^+,\alpha_-],\gamma_-]$$

$$\frac{d\beta_-}{du} = \frac{1}{6}[[[x_0^+,\alpha_-],\alpha_-],\alpha_-] - \frac{1}{2}[[[x_0^+,\alpha_-],\gamma_-],\gamma_-]$$

$$\frac{d\gamma_-}{du} = \frac{1}{2}[[x_0^+, \alpha_-], \alpha_-] + \frac{1}{2}[[x_0^+, \gamma_-], \gamma_-]$$
$$\frac{d\eta}{du} = [x_0^+, \gamma_-] + [x_0^+, \alpha_-]$$
$$\frac{d\alpha_+}{du} = y_0$$
$$\frac{d\beta_+}{du} = [y_0, \gamma_+]$$
$$\frac{d\gamma_+}{du} = y_1.$$

Remark that $\frac{d\alpha_+}{du} = y_0$ *and* $\frac{d\gamma_+}{du} = y_1$ *is equivalent to* $\frac{d(\alpha_+ + \gamma_+)}{du} = y_0 + y_1$.

Proof. i)-v) are straightforward repeated applications of Lemma 1.55,iv) remarking that:

i) and ii): $[[[x_0^+, \alpha_-], \alpha_-], \alpha_-] \in \tilde{\mathcal{U}}^{-,c}[[u,v]]$, hence it commutes with both α_- and β_- (which are in $\tilde{\mathcal{U}}^-[[u,v]]$);

ii): $\beta_- \in \tilde{\mathcal{U}}^{-,c}[[u,v]]$, hence it commutes also with $[[x_0^+, \alpha_-], \alpha_-]$ and $[x_0^+, \beta_-]$ (which belong to $\tilde{\mathcal{U}}^-[[u,v]]$) and with $[x_0^+, \alpha_-]$ (because $[h_{2r+1}, \tilde{\mathcal{U}}^{-,c}] = 0\ \forall r \in \mathbb{Z}$);

iii): $[[x_0^+, \gamma_-], \gamma_-]$ and $[[[x_0^+, \alpha_-], \gamma_-], \gamma_-]$ belong respectively to $\tilde{\mathcal{U}}^{-,0}[[u,v]]$ and $\tilde{\mathcal{U}}^{-,c}[[u,v]]$, so that they commute with $\gamma_- \in \tilde{\mathcal{U}}^{-,0}[[u,v]]$; the claim follows from the identities

$$(x_0^+ + [x_0^+, \alpha_-])\exp(\gamma_-) = \exp(\gamma_-) \cdot \Big(x_0^+ + [x_0^+, \alpha_-] +$$
$$+ [x_0^+, \gamma_-] + [[x_0^+, \alpha_-], \gamma_-] + \frac{1}{2}[[x_0^+, \gamma_-], \gamma_-] + \frac{1}{2}[[[x_0^+, \alpha_-], \gamma_-], \gamma_-]\Big)$$

and

$$\exp(\gamma_-)[[x_0^+, \alpha_-], \gamma_-] = ([[x_0^+, \alpha_-], \gamma_-] - [[[x_0^+, \alpha_-], \gamma_-], \gamma_-])\exp(\gamma_-);$$

iv): Lemma 1.59 implies that $\exp(\eta)^{-1} x_0^+ \exp(\eta) \in \mathbb{Q}[w][[u,v]].x_0^+$;

v): $\gamma_+ \in \tilde{\mathcal{U}}^{+,1}[[u,v]]$ commutes with both $y_1 \in \tilde{\mathcal{U}}^{+,1}[[u,v]]$ and $[y_0, \gamma_+] \in \tilde{\mathcal{U}}^{+,c}[[u,v]]$.

Point vi) is a consequence of points i)-v) and Remark 5.32,i).

□

Lemma 5.34. *By abuse of notation let* $\alpha_\pm$, $\beta_\pm$, $\gamma_\pm$ *and* η *(see Notation 5.31 and Lemma 5.33,iv)) denote also the elements of* $\mathbb{Q}[w][[u,v]]$ *such that*

$$\alpha_+ = \alpha_+(w^2).x_0^+, \ \ \beta_+ = \beta_+(w).X_1^+, \ \ \gamma_+ = \gamma_+(w^2).x_1^+,$$
$$\alpha_- = \alpha_-(w^2).x_1^-, \ \ \beta_- = \beta_-(w).X_1^-, \ \ \gamma_- = \gamma_-(w^2).x_0^-,$$
$$\eta = \eta(w).h_0.$$

Then the relations of Lemma 5.33,vi) can be written as:

$$\frac{d\alpha_-(w^2)}{du} = 4\beta_-(-w^2) - 6\alpha_-(w^2)\gamma_-(w^2),$$
$$\frac{d\beta_-(w)}{du} = \alpha_-(-w)(w\alpha_-^2(-w) - 3\gamma_-^2(-w)),$$
$$\frac{d\gamma_-(w^2)}{du} = -3w^2\alpha_-^2(w^2) - \gamma_-^2(w^2),$$
$$\frac{d\eta(w)}{du} = w\alpha_-(w^2) + \gamma_-(w^2),$$

$$\frac{d(\alpha_+(w^2)+w\gamma_+(w^2))}{du} = \exp(-4\eta(w)+2\eta(-w)),$$
$$\frac{d\beta_+(w)}{du} = -\frac{d\alpha_+(-w)}{du}\gamma_+(-w).$$

Proof. The claim is obtained using Lemma 5.11. Indeed
$\frac{d\alpha_-}{du} = \frac{d\alpha_-(w^2)}{du}.x_1^-$ and $[x_0^+,\beta_-]+[[x_0^+,\alpha_-],\gamma_-] =$

$$= [x_0^+,\beta_-(w).X_1^-]+[[x_0^+,\alpha_-(w^2).x_1^-],\gamma_-(w^2).x_0^-] = (\text{by } (iii))$$
$$= 4\beta_-(-w^2).x_1^- + [[x_0^+, w\alpha_-(w^2).x_0^-],\gamma_-(w^2).x_0^-] = (\text{by } (ii))$$
$$= 4\beta_-(-w^2).x_1^- + [w\alpha_-(w^2).h_0,\gamma_-(w^2).x_0^-] = (\text{by } (iv))$$
$$= 4\beta_-(-w^2).x_1^- - (4w\alpha_-(w^2)+2w\alpha_-(w^2))\gamma_-(w^2).x_0^- =$$
$$= (4\beta_-(-w^2) - 6\alpha_-(w^2)\gamma_-(w^2)).x_1^-;$$

$\frac{d\beta_-}{du} = \frac{d\beta_-(w)}{du}.X_1^-$ and $\frac{1}{6}[[[x_0^+,\alpha_-],\alpha_-],\alpha_-] - \frac{1}{2}[[[x_0^+,\alpha_-],\gamma_-],\gamma_-] =$

$$= \frac{1}{6}[[[x_0^+, w\alpha_-(w^2).x_0^-], w\alpha_-(w^2).x_0^-],\alpha_-(w^2).x_1^-]+$$
$$-\frac{1}{2}[[[x_0^+, w\alpha_-(w^2).x_0^-],\gamma_-(w^2).x_0^-],\gamma_-(w^2).x_0^-] = (\text{by } (ii))$$
$$= \frac{1}{6}[[w\alpha_-(w^2).h_0, w\alpha_-(w^2).x_0^-],\alpha_-(w^2).x_1^-]+$$
$$-\frac{1}{2}[[w\alpha_-(w^2).h_0,\gamma_-(w^2).x_0^-],\gamma_-(w^2).x_0^-] = (\text{by } (iv))$$
$$= -[w^2\alpha_-^2(w^2).x_0^-,\alpha_-(w^2).x_1^-] + 3[w\alpha_-(w^2)\gamma_-(w^2).x_0^-,\gamma_-(w^2).x_0^-] =$$
$$= (w\alpha_-^3(-w) - 3\alpha_-(-w)\gamma_-^2(-w)).X_1^-;$$

$\frac{d\gamma_-}{du} = \frac{d\gamma_-(w^2)}{du}.x_0^-$ and $\frac{1}{2}[[x_0^+,\alpha_-],\alpha_-]+\frac{1}{2}[[x_0^+,\gamma_-],\gamma_-] =$

$$= \frac{1}{2}[[x_0^+, w\alpha_-(w^2).x_0^-], w\alpha_-(w^2).x_0^-] + \frac{1}{2}[[x_0^+,\gamma_-(w^2).x_0^-],\gamma_-(w^2).x_0^-] = (\text{by } (ii))$$
$$= \frac{1}{2}[w\alpha_-(w^2).h_0, w\alpha_-(w^2).x_0^-] + \frac{1}{2}[\gamma_-(w^2).h_0,\gamma_-(w^2).x_0^-] = (\text{by } (iv))$$
$$= -\frac{1}{2}[6w\alpha_-(w^2)w\alpha_-(w^2).x_0^- - \frac{1}{2}2\gamma_-^2(w^2).x_0^- =$$
$$= -3w^2\alpha_-^2(w^2) - \gamma_-^2(w^2)).x_0^-;$$

$\frac{d\eta}{du} = \frac{d\eta(w)}{du}.h_0$ and $[x_0^+,\gamma_-]+[x_0^+,\alpha_-] =$

$$[x_0^+,\gamma_-(w^2).x_0^-]+[x_0^+, w\alpha_-(w^2).x_0^-] = (\text{by}$$
$$(\gamma_-(w^2)+w\alpha_-(w^2)).h_0;$$

$\frac{d(\alpha_++\gamma_+)}{du} = \frac{d(\alpha_+(w^2)+w\gamma_+(w^2))}{du}.x_0^+$ and $y_0+y_1 =$

$$= \exp(-\eta)x_0^+\exp(\eta) =$$
$$= \exp(-\eta(w).h_0)x_0^+\exp(\eta(w).h_0) = (\text{by } (iv) \text{ and Lemma 1.59})$$

$$= \exp(-4\eta(w) + 2\eta(-w)).x_0^+;$$

$\frac{d\beta_+}{du} = \frac{d\beta_+(w)}{du}.X_1^+$ and $[y_0, \gamma_+] = \left[\frac{d\alpha_+}{du}, \gamma_+\right] =$

$$= \left[\frac{d\alpha_+(w^2)}{du}.x_0^+, \gamma_+(w^2).x_1^+\right] = \text{(by } (i))$$

$$= -\frac{d\alpha_+(-w)}{du}\gamma_+(-w).X_1^+.$$

□

Proposition 5.35.

$$\exp(x_0^+ u)\exp(X_1^- v) =$$
$$= \exp(\alpha_-)\exp(\beta_-)\exp(\gamma_-)\exp(\eta)\exp(\gamma_+)\exp(\beta_+)\exp(\alpha_+)$$

where, with the notations of Lemma 5.34,

$$\alpha_-(w) = \frac{4uv}{1-4^2wu^4v^2}, \quad \alpha_+(w) = \frac{u}{1-4^2wu^4v^2},$$

$$\beta_-(w) = \frac{(1+3\cdot 4^2wu^4v^2)v}{(1+4^2wu^4v^2)^2}, \quad \beta_+(w) = \frac{(1-4^2wu^4v^2)u^4v}{(1+4^2wu^4v^2)^2},$$

$$\gamma_-(w) = \frac{-4^2wu^3v^2}{1-4^2wu^4v^2}, \quad \gamma_+(w) = \frac{-4u^3v}{1-4^2wu^4v^2},$$

$$\eta(w) = \frac{1}{2}\ln\Big(1+4wu^2v\Big).$$

In particular:

i) $(x_0^+)^{(k)}(X_1^-)^{(l)} \in \tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ *for all* $k, l \in \mathbb{N}$;

ii) $\hat{h}_+(4u)^{\frac{1}{2}} \in \tilde{\mathcal{U}}_{\mathbb{Z}}[[u]]$.

Proof. We use the notation fixed in 5.31.

It is obvious that $G(0,v) = \exp(X_1^- v)$, so that the condition a) of Remark 5.30 is fulfilled, and we need to verify condition b), following Lemmas 5.33,vi) and 5.34.

Remark that

$$\frac{d\eta(w)}{du} = \frac{4wuv}{1+4wu^2v} = \frac{4wuv(1-4wu^2v)}{1-4^2w^2u^4v^2} = w\alpha_-(w^2) + \gamma_-(w^2)$$

and

$$\exp(-4\eta(w) + 2\eta(-w)) = \frac{1-4wu^2v}{(1+4wu^2v)^2},$$

$$\alpha_+(w^2) + w\gamma_+(w^2) = \frac{u(1-4wu^2v)}{1-4^2w^2u^4v^2} = \frac{u}{1+4wu^2v},$$

so that

$$\frac{d(\alpha_+(w^2) + w\gamma_+(w^2))}{du} = \frac{1+4wu^2v-8wu^2v}{(1+4wu^2v)^2} = \exp(-4\eta(w) + 2\eta(-w)).$$

Now let us recall that $\forall n, m \in \mathbb{N}$ and $\forall a$ not depending on u

$$\frac{d}{du}\frac{u^n}{(1-au^4)^m} = \frac{nu^{n-1} + (4m-n)au^{n+3}}{(1-au^4)^{m+1}},$$

hence, fixing $a = 4^2 w^2 v^2$, we get

$$\frac{d\alpha_-(w^2)}{du} = \frac{4v(1+3au^4)}{(1-au^4)^2},$$
$$\frac{d\beta_-(-w^2)}{du} = \frac{-4au^3v(1+3au^4)}{(1-au^4)^3},$$
$$\frac{d\gamma_-(w^2)}{du} = \frac{-a(3u^2+au^6)}{(1-au^4)^2},$$
$$\frac{d\alpha_+(w^2)}{du} = \frac{1+3au^4}{(1-au^4)^2},$$
$$\frac{d\beta_+(-w^2)}{du} = \frac{4vu^3(1+3au^4)}{(1-au^4)^3};$$

remark that $w \mapsto -w^2$ induces an injective algebra endomorphism of $\mathbb{Q}[w][[u]]$ commuting with $\frac{d}{du}$, which allowed us to use the same $a = a(w,u)$ in the computations involving $\beta_\pm$.

The relations to prove are then equivalent to the following:

$$4v(1+3au^4) = 4(1-3au^4)v + 6\cdot 4uv\cdot au^3,$$
$$-4au^3v(1+3au^4) = 4uv(-w^2 4^2u^2v^2 - 3a^2u^6),$$
$$-a(3u^2+au^6) = -3w^2\cdot 4^2u^2v^2 - a^2u^6,$$
$$4u^3v(1+3au^4) = (1+3au^4)4u^3v,$$

which are easily verified.

Then, since $\alpha_\pm$, $\beta_\pm$, $\gamma_\pm$ have integral coefficients, i) follows from Remark 5.14 and Lemma 5.16,iii).

ii) follows at once from the above considerations, inverting the exponentials: indeed, recalling Remark 5.13 and Notation 5.31 we have

$$\exp(\eta) = \exp(\eta(w).h_0) = \hat{h}_+(4u^2v)^{1/2} = (G^-)^{-1}\exp\big(x_0^+u\big)\exp\big(X_1^-v\big)(G^+)^{-1}$$

which belongs to $\tilde{\mathcal{U}}_{\mathbb{Z}}[[u,v]]$ because so do all the factors.

□

Proposition 5.36.

$$\exp\big(x_0^+u\big)\exp\big(x_1^-v\big) =$$
$$= \exp(\alpha_-)\exp(\beta_-)\exp(\gamma_-)\exp(\eta)\exp(\gamma_+)\exp(\beta_+)\exp(\alpha_+)$$

where, with the notations of Lemma 5.34,

$$\alpha_+(w) = \frac{(1+wu^2v^2)u}{1-6wu^2v^2+w^2u^4v^4}, \quad \alpha_-(w) = \frac{(1+wu^2v^2)v}{1-6wu^2v^2+w^2u^4v^4},$$
$$\beta_+(w) = \frac{(1-4wu^2v^2-w^2u^4v^4)u^3v}{(1+6wu^2v^2+w^2u^4v^4)^2}, \quad \beta_-(w) = \frac{(1-4wu^2v^2-w^2u^4v^4)wuv^3}{(1+6wu^2v^2+w^2u^4v^4)^2},$$
$$\gamma_+(w) = \frac{(-3+wu^2v^2)u^2v}{1-6wu^2v^2+w^2u^4v^4}, \quad \gamma_-(w) = \frac{(-3+wu^2v^2)wuv^2}{1-6wu^2v^2+w^2u^4v^4},$$
$$\eta(w) = \frac{1}{2}\ln\Big(1+2wuv-w^2u^2v^2\Big).$$

Proof. We use the notations fixed in 5.31.

It is obvious that $G(0,v) = \exp(x_1^- v)$, so that the condition a) of Remark 5.30 is fulfilled, and we need to verify condition b), following Lemma 5.34.

First of all remark that

$$1 - 6t^2 + t^4 = (1 + 2t - t^2)(1 - 2t - t^2)$$

and that

$$1 + t^2 + (-3 + t^2)t = 1 - 3t + t^2 + t^3 = (1 - t)(1 - 2t - t^2);$$

thus, replacing t by wuv, we get

$$\alpha_+(w^2) + w\gamma_+(w^2) = \frac{(1 - wuv)u}{1 + 2wuv - w^2u^2v^2}$$

and

$$w\alpha_-(w^2) + \gamma_-(w^2) = \frac{(1 - wuv)wv}{1 + 2wuv - w^2u^2v^2}.$$

Hence the relations of Lemma 5.34 involving η are easily proved:

$$\frac{d\eta(w)}{du} = \frac{(1 - wuv)wv}{1 + 2wuv - w^2u^2v^2} = w\alpha_-(w^2) + \gamma_-(w^2)$$

and

$$\exp(-4\eta(w) + 2\eta(-w)) = \frac{1 - 2wuv - w^2u^2v^2}{(1 + 2wuv - w^2u^2v^2)^2}$$

while, on the other hand,

$$\frac{d}{dt}\frac{t - t^2}{1 + 2t - t^2} = \frac{1 - 2t - t^2}{(1 + 2t - t^2)^2}$$

so that

$$\frac{d}{du}(\alpha_+(w^2) + w\gamma_+(w^2)) = \frac{1 - 2wuv - w^2u^2v^2}{(1 + 2wuv - w^2u^2v^2)^2}$$

and

$$\exp(-4\eta(w) + 2\eta(-w)) = \frac{d}{du}(\alpha_+(w^2) + w\gamma_+(w^2)).$$

In order to prove the remaining relations remark that for all $n, m \in \mathbb{N}$

$$\frac{d}{dt}\frac{t^n}{(1 - 6t^2 + t^4)^m} = \frac{nt^{n-1} + 6(2m - n)t^{n+1} + (n - 4m)t^{n+3}}{(1 - 6t^2 + t^4)^{m+1}},$$

which helps to compute the derivative of $\alpha_\pm(w^2)$, $\beta_\pm(-w^2)$ (which is equivalent to computing that of $\beta_\pm(w)$, see the proof of Proposition 5.35) and $\gamma_-(w^2)$, fixing $t = wuv$ and recalling that $\frac{d}{du} = wv\frac{d}{dt}$:

$$\frac{d\alpha_-(w^2)}{du} = \frac{wv^2(14t - 4t^3 - 2t^5)}{(1 - 6t^2 + t^4)^2},$$

$$\frac{d\beta_-(-w^2)}{du} = \frac{w^2v^3(-1 - 30t^2 - 12t^4 + 14t^6 - 3t^8)}{(1 - 6t^2 + t^4)^3},$$

$$\frac{d\gamma_-(w^2)}{du} = \frac{w^2v^2(-3 - 15t^2 + 3t^4 - t^6)}{(1 - 6t^2 + t^4)^2},$$

$$\frac{d\alpha_+(w^2)}{du} = \frac{1 + 9t^2 - 9t^4 - t^6}{(1 - 6t^2 + t^4)^2},$$

$$\frac{d\beta_+(-w^2)}{du} = \frac{w^{-2}v^{-1}(3t^2+26t^4-36t^6+6t^8+t^{10})}{(1-6t^2+t^4)^3}.$$

The relations to prove are then equivalent to the following:

$$14t-4t^3-2t^5 = -4(1+4t^2-t^4)t-6(1+t^2)(-3+t^2)t,$$
$$-1-30t^2-12t^4+14t^6-3t^8 = (1+t^2)(-(1+t^2)^2-3(-3+t^2)^2t^2),$$
$$-3-15t^2+3t^4-t^6 = -3(1+t^2)^2-(-3+t^2)^2t^2,$$
$$3t^2+26t^4-36t^6+6t^8+t^{10} = -(1+9t^2-9t^4-t^6)(-3+t^2)t^2,$$

which are easily verified.

□

Remark 5.37. *Since $(1\pm 6t^2+t^4)^{-1}\in\mathbb{Z}[[t]]$ Proposition 5.36 implies that $(G^{\pm})^{\pm 1}\in\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm}[[u,v]]$ (see Notation 5.31). As in Proposition 5.35,ii), it also implies that $\exp(\eta)\in\tilde{\mathcal{U}}_{\mathbb{Z}}$. Then, in order to prove that*

$$(x_0^+)^{(k)}(x_1^-)^{(l)}\in\tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+,$$

we just need to show that $\exp(\eta)\in\tilde{\mathcal{U}}_{\mathbb{Z}}^0[[u,v]]$. This will imply that $\tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ is closed under multiplication, hence it is an integral form of $\tilde{\mathcal{U}}$, obviously containing $\tilde{\mathcal{U}}_{\mathbb{Z}}$.

In order to prove that $\tilde{\mathcal{U}}_{\mathbb{Z}}=\tilde{\mathcal{U}}_{\mathbb{Z}}^-\tilde{\mathcal{U}}_{\mathbb{Z}}^0\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ we need to show in addition that $\tilde{\mathcal{U}}_{\mathbb{Z}}^0\subseteq\tilde{\mathcal{U}}_{\mathbb{Z}}$.

The last part of this Chapter is devoted to prove that

$$\exp\left(\frac{1}{2}\ln\Big(1+2u-u^2\Big).h_0\right)\in\tilde{\mathcal{U}}_{\mathbb{Z}}^0[[u]]$$

(see Corollary 5.41) and that $\tilde{\mathcal{U}}_{\mathbb{Z}}^0\subseteq\tilde{\mathcal{U}}_{\mathbb{Z}}$ (see Proposition 5.43).

Notation 5.38. *In the following $d:\mathbb{Z}_+\to\mathbb{Q}$ denotes the function defined by*

$$\sum_{n>0}(-1)^{n-1}\frac{d_n}{n}u^n = \frac{1}{2}\ln\Big(1+2u-u^2\Big)$$

and $\tilde{d}=\varepsilon d$ (that is $\tilde{d}_n=\varepsilon_n d_n$ for all $n>0$, where ε_n has been defined in Definition 5.12).

Remark that with this notation we have $\exp(\eta)=\hat{h}_+^{\{d\}}(uv)$ (η as in Lemma 5.34 and Proposition 5.36, $\hat{h}_+^{\{d\}}(u)$ as in Notation 1.15, where we replace $\hat{h}^{\{d\}}(u)$ by $\hat{h}_+^{\{d\}}(u)$ in order to distinguish it from its symmetric $\hat{h}_-^{\{d\}}(u)=\Omega(\hat{h}_+^{\{d\}}(u))$).

Remark 5.39. *From $1+2u-u^2=(1+(1+\sqrt{2})u)(1+(1-\sqrt{2})u)$, we get that:*

i) for all $n\in\mathbb{Z}_+$ $d_n=\frac{1}{2}((1+\sqrt{2})^n+(1-\sqrt{2})^n)$; equivalently $\exists\delta_n\in\mathbb{Z}$ such that

$$\forall n\in\mathbb{Z}_+\quad (1+\sqrt{2})^n=d_n+\delta_n\sqrt{2}.$$

ii) d_n is odd for all $n\in\mathbb{Z}_+$; δ_n is odd if and only if n is odd.

iii) $\mathbb{Z}[\hat{h}_k^{\{d\}}\mid k>0]\not\subseteq\mathbb{Z}[\hat{h}_k\mid k>0]$ (indeed $(\mu*d)(4)=d_4-d_2=17-3=14$, which is not a multiple of 4, see Propositions 1.18 and 1.19).

iv) $\mathbb{Z}[\hat{h}_k^{\{d\}}\mid k>0]\subseteq\mathbb{Z}[\tilde{h}_k\mid k>0]$ if and only if $\mathbb{Z}[\hat{h}_k^{\{\tilde{d}\}}\mid k>0]\subseteq\mathbb{Z}[\hat{h}_k\mid k>0]$ (see Remark 5.13).

Lemma 5.40. *Let $p,m,r\in\mathbb{Z}_+$ be such that p is prime and $(m,p)=1$. Then*

$$\text{if } p^r=4\quad p^r=4\mid(d_{4m}+d_{2m}),$$
$$\text{if } p^r\neq 4\quad p^r\mid(d_{p^rm}-d_{p^{r-1}m}).$$

Proof. The claim is obvious for $p^r = 2$ since the d_n's are all odd.

In general if n is any positive integer it follows from Remark 5.39 that

$$d_{np} + \delta_{np}\sqrt{2} = (d_n + \delta_n\sqrt{2})^p.$$

If $p = 2$ this means that

$$d_{2n} = d_n^2 + 2\delta_n^2,$$
$$\delta_{2n} = 2d_n\delta_n,$$

hence by induction on r

$$2^r \mid \delta_{2^r m} \text{ and } 2^{r+1} \nmid \delta_{2^r m} \ \text{(recall that } \delta_m \text{ is odd since } m \text{ is odd)}$$

$$d_{2^r m} \equiv d_{2^{r-1}m}^2 \ (mod\ 2^{2r-1}),$$

from which it follows that

$$d_{2m} \equiv -1 \ (mod\ 4),$$
$$d_{2^r m} \equiv 1 \ (mod\ 2^{r+1}) \quad \text{if } r > 1:$$

indeed, since d_m and δ_m are odd,

$$d_{2m} \equiv_{(8)} 1 + 2 \equiv_{(4)} -1,$$

while if $r \geq 2$ then $2r - 1 \geq r + 1$ and by induction on r we get

$$d_{2^r m} \equiv d_{2^{r-1}m}^2 = (\pm 1 + 2^r k)^2 \equiv 1 \ (mod\ 2^{r+1}).$$

These last relations immediately imply the claim for $p = 2$.

Now let $p \neq 2$. Then

$$d_{pn} = \sum_{h\geq 0} \binom{p}{2h} 2^h d_n^{p-2h} \delta_n^{2h},$$

$$\delta_{pn} = \sum_{h\geq 0} \binom{p}{2h+1} 2^h d_n^{p-2h-1} \delta_n^{2h+1}.$$

Suppose that $d_n = d + p^{r-1}k$, $\delta_n = \delta + p^{r-1}k'$ with $k = k' = 0$ if $r = 1$. Then

$$d_{pn} \equiv \sum_{h\geq 0} \binom{p}{2h} 2^h d^{p-2h} \delta^{2h} \ (mod\ p^r)$$

$$\delta_{pn} \equiv \sum_{h\geq 0} \binom{p}{2h+1} 2^h d^{p-2h-1} \delta^{2h+1} \ (mod\ p^r)$$

The above relations allow us to prove by induction on $r > 0$ that if ζ_p is defined by the properties $\zeta_p \in \{\pm 1\}$, $\zeta_p \equiv_{(p)} 2^{\frac{p-1}{2}}$ then

$$d_{p^r m} \equiv d_{p^{r-1}m} \ (mod\ p^r) \quad \text{and} \quad \delta_{p^r m} \equiv \zeta_p \delta_{p^{r-1}m} \ (mod\ p^r):$$

indeed if $r = 1$

$$d_{pm} \equiv d_m^p \equiv d_m \ (mod\ p),$$

$$\delta_{pm} \equiv 2^{\frac{p-1}{2}} \delta_m^p \equiv \zeta_p \delta_m \ (mod\ p);$$

remark that $(d+p^{r-1}k)^p \equiv d^p (mod\ p^r)$ and if $0<h<p$, $p \mid \binom{p}{h}$ and $(d+p^{r-1}k)^h \equiv d^h (mod\ p^{r-1})$; then if $r>1$, using relations 5.2 and 5.2 with $d=d_{p^{r-2}m}$ and $\delta=\delta_{p^{r-2}m}$, we get then

$$d_{p^r m} \equiv_{(p^r)} \sum_{h\geq 0} \binom{p}{2h} 2^h d_{p^{r-2}m}^{p-2h} \delta_{p^{r-2}m}^{2h} = d_{p^{r-1}m},$$

$$\delta_{p^r m} \equiv_{(p^r)} \zeta_p \sum_{h\geq 0} \binom{p}{2h+1} 2^h d_{p^{r-2}m}^{p-2h-1} \delta_{p^{r-2}m}^{2h+1} = \zeta_p \delta_{p^{r-1}m}.$$

□

Corollary 5.41. *$\hat{h}_n^{\{d\}} \in \mathbb{Z}[\tilde{h}_k \mid k>0]$ for all $n>0$.*

In particular $(x_0^+)^{(k)}(x_1^-)^{(l)} \in \tilde{\mathcal{U}}_{\mathbb{Z}}^- \tilde{\mathcal{U}}_{\mathbb{Z}}^0 \tilde{\mathcal{U}}_{\mathbb{Z}}^+$ $\forall k,l \in \mathbb{N}$.

Proof. The claim follows from Propositions 1.18 and 1.19, Remark 5.39 and Lemma 5.40, remarking that if m is odd then

$$d_{4m}+d_{2m} = -(\tilde{d}_{4m}-\tilde{d}_{2m})$$

while if $(m,p)=1$ and $p^r \neq 4$ then

$$d_{p^r m}-d_{p^{r-1}m} = \pm(\tilde{d}_{p^r m}-\tilde{d}_{p^{r-1}m}).$$

Thus for all $n>0$ $\hat{h}_n^{\{\tilde{d}\}} \in \mathbb{Z}[\hat{h}_k \mid k>0]$ and $\hat{h}_n^{\{d\}} \in \mathbb{Z}[\tilde{h}_k \mid k>0]$. □

Corollary 5.42. *$\tilde{\mathcal{U}}_{\mathbb{Z}}^+ \tilde{\mathcal{U}}_{\mathbb{Z}}^- \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}^- \tilde{\mathcal{U}}_{\mathbb{Z}}^0 \tilde{\mathcal{U}}_{\mathbb{Z}}^+$; equivalently $\tilde{\mathcal{U}}_{\mathbb{Z}}^- \tilde{\mathcal{U}}_{\mathbb{Z}}^0 \tilde{\mathcal{U}}_{\mathbb{Z}}^+$ is an integral form of $\tilde{\mathcal{U}}$.*

Proof. The proof is identical to that of Proposition 4.24 replacing $\hat{\mathcal{U}}$ with $\tilde{\mathcal{U}}$, having care to remark that in this case, too,

$$(x_r^+)^{(k)}(x_s^-)^{(l)} \in \sum_{m\geq 0} \tilde{\mathcal{U}}_{\mathbb{Z},-l+m}^- \tilde{\mathcal{U}}_{\mathbb{Z}}^0 \tilde{\mathcal{U}}_{\mathbb{Z},k-m}^+ \quad \forall r,s \in \mathbb{Z},\ \forall k,l \in \mathbb{N}:$$

if $r+s$ is even this follows at once comparing Proposition 5.29 with the properties of the gradation, while if $r+s$ is odd it is true by Proposition 5.36 and Remark 5.20,iv). □

Proposition 5.43. *$\tilde{\mathcal{U}}_{\mathbb{Z}}^0 \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}$ and $\tilde{\mathcal{U}}_{\mathbb{Z}} = \tilde{\mathcal{U}}_{\mathbb{Z}}^- \tilde{\mathcal{U}}_{\mathbb{Z}}^0 \tilde{\mathcal{U}}_{\mathbb{Z}}^+$.*

Proof. Let $\mathcal{Z}$ be the $\mathbb{Z}$-subalgebra of $\mathbb{Q}[h_r \mid r>0]$ generated by the coefficients of $\hat{h}_+^{\{d\}}(u)$ and of $\hat{h}_+(4u)^{1/2}$. Remark that, by Propositions 5.35 and 5.36, $\mathcal{Z} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}$.

We have already proved that $\mathcal{Z} \subseteq \mathbb{Z}[\tilde{h}_k \mid k>0]$ (see Lemma 5.16,iii) and Corollary 5.41). Let us prove, by induction on j, that $\tilde{h}_j \in \mathcal{Z}$ for all $j>0$.

If $j=1$ the claim depends on the equality $\tilde{h}_1 = h_1 = \hat{h}_1^{\{d\}}$ (since $\varepsilon_1 = d_1 = 1$).

Let $j>1$ and suppose that $\tilde{h}_1,...,\tilde{h}_{j-1} \in \mathcal{Z}$.

We notice that if $a:\mathbb{Z}_+ \to \mathbb{Z}$ is such that $\hat{h}_j^{\{a\}} \in \mathcal{Z}$ then $a_j\tilde{h}_j \in \mathcal{Z}$: indeed it is always true that

$$\tilde{h}_j + (-1)^j \frac{\varepsilon_j h_j}{j} \in \mathbb{Q}[h_1,...,h_{j-1}]$$

and

$$\hat{h}_j^{\{a\}} + (-1)^j \frac{a_j h_j}{j} \in \mathbb{Q}[h_1,...,h_{j-1}]$$

from which we get that

$$\hat{h}_j^{\{a\}} - \varepsilon_j a_j \tilde{h}_j \in \mathbb{Q}[h_1,...,h_{j-1}];$$

but the condition $\hat{h}_j^{\{a\}} \in \mathcal{Z} \subseteq \mathbb{Z}[\tilde{h}_k \mid k > 0]$ and the inductive hypothesis $\mathbb{Z}[\tilde{h}_1, ..., \tilde{h}_{j-1}] \subseteq \mathcal{Z}$ imply that

$$\hat{h}_j^{\{a\}} - \varepsilon_j a_j \tilde{h}_j \in \mathbb{Q}[h_1, ..., h_{j-1}] \cap \mathbb{Z}[\tilde{h}_k \mid k > 0] = \mathbb{Z}[\tilde{h}_1, ..., \tilde{h}_{j-1}] \subseteq \mathcal{Z}$$

hence $a_j \tilde{h}_j \in \mathcal{Z}$.

This in particular holds for $a = d$ and for $\hat{h}^{\{a\}}(u) = \hat{h}_+(4u)^{\frac{1}{2}}$, hence

$$d_j \tilde{h}_j \in \mathcal{Z} \text{ and } 2^{2j-1}\tilde{h}_j \in \mathcal{Z}.$$

But $(d_j, 2^{2j-1}) = 1$ because d_j is odd, hence $\tilde{h}_j \in \mathcal{Z}$.

Then $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+} = \mathbb{Z}[\tilde{h}_k \mid k > 0] = \mathcal{Z} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}$ and, applying Ω, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-} \subseteq \tilde{\mathcal{U}}_{\mathbb{Z}}$. The claim follows recalling Corollary 5.28. □

We can now collect all the results obtained till now in the main theorem of this work (see Theorem 5.18).

Theorem 5.44. *The $\mathbb{Z}$-subalgebra $\tilde{\mathcal{U}}_{\mathbb{Z}}$ of $\tilde{\mathcal{U}}$ generated by*

$$\{(x_r^+)^{(k)}, (x_r^-)^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}\}$$

is an integral form of $\tilde{\mathcal{U}}$.

More precisely

$$\tilde{\mathcal{U}}_{\mathbb{Z}} \cong \tilde{\mathcal{U}}_{\mathbb{Z}}^{-,1} \otimes \tilde{\mathcal{U}}_{\mathbb{Z}}^{-,c} \otimes \tilde{\mathcal{U}}_{\mathbb{Z}}^{-,0} \otimes \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-} \otimes \tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} \otimes \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+} \otimes \tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1} \otimes \tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c} \otimes \tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}$$

and a $\mathbb{Z}$-basis of $\tilde{\mathcal{U}}_{\mathbb{Z}}$ is given by the product

$$B^{-,1}B^{-,c}B^{-,0}B^{0,-}B^{\mathfrak{h}}B^{0,+}B^{+,1}B^{+,c}B^{+,0}$$

where $B^{\pm,0}$, $B^{\pm,1}$, $B^{\pm,c}$, $B^{0,\pm}$ and $B^{\mathfrak{h}}$ are the $\mathbb{Z}$-bases respectively of $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,0}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,1}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,c}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ given as follows:

$$B^{\pm,0} = \Big\{(\mathbf{x}^{\pm,0})^{(\mathbf{k})} = \prod_{r\in\mathbb{Z}} (x_{2r}^{\pm})^{(k_r)} \mid \mathbf{k} : \mathbb{Z} \to \mathbb{N} \text{ is finitely supported}\Big\}$$

$$B^{\pm,1} = \Big\{(\mathbf{x}^{\pm,1})^{(\mathbf{k})} = \prod_{r\in\mathbb{Z}} (x_{2r+1}^{\pm})^{(k_r)} \mid \mathbf{k} : \mathbb{Z} \to \mathbb{N} \text{ is finitely supported}\Big\}$$

$$B^{\pm,c} = \Big\{(\mathbf{X}^{\pm})^{(\mathbf{k})} = \prod_{r\in\mathbb{Z}} (X_{2r+1}^{\pm})^{(k_r)} \mid \mathbf{k} : \mathbb{Z} \to \mathbb{N} \text{ is finitely supported}\Big\}$$

$$B^{0,\pm} = \Big\{\tilde{\mathbf{h}}_{\pm}^{\mathbf{k}} = \prod_{l\in\mathbb{N}} \tilde{h}_{\pm l}^{k_l} \mid \mathbf{k} : \mathbb{N} \to \mathbb{N} \text{ is finitely supported}\Big\}$$

$$B^{\mathfrak{h}} = \Big\{\binom{h_0}{k}\binom{c}{\tilde{k}} \mid k, \tilde{k} \in \mathbb{N}\Big\}.$$

Chapter 6

Comparison with the Mitzman integral form

In the present chapter we compare the integral form $\tilde{\mathcal{U}}_{\mathbb{Z}} = {}^*\mathcal{U}_{\mathbb{Z}}(\hat{\mathfrak{sl}}_3^{\chi})$ of $\tilde{\mathcal{U}}$ (described in Chapter 5) with the integral form $\mathcal{U}_{\mathbb{Z}}(\hat{\mathfrak{sl}}_3^{\chi})$ of the same algebra $\tilde{\mathcal{U}}$ introduced and studied by Mitzman in [11], that we denote here by $\mathcal{U}_{\mathbb{Z},M}$ and that is easily defined as the $\mathbb{Z}$-subalgebra of $\tilde{\mathcal{U}}$ generated by the divided powers of the Kac-Moody generators e_i, f_i ($i = 0, 1$): see also Remark 6.11.

More precisely:

Definition 6.1. *$\tilde{\mathcal{U}}$ is the enveloping algebra of the Kac-Moody algebra whose generalized Cartan matrix is* $A_2^{(2)} = (a_{i,j})_{i,j\in\{0,1\}} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$ *(see [8]): it has generators $\{e_i, f_i, h_i \mid i = 0, 1\}$ and relations*

$$[h_i, h_j] = 0,\ \ [h_i, e_j] = a_{i,j}e_j,\ \ [h_i, f_j] = -a_{i,j}f_j,\ \ [e_i, f_j] = \delta_{i,j}h_i\ \ (i, j \in \{0, 1\})$$

$$(\mathrm{ad}e_i)^{1-a_{i,j}}(e_j) = 0 = (\mathrm{ad}f_i)^{1-a_{i,j}}(f_j)\ \ (i \neq j \in \{0, 1\}).$$

Definition 6.2. *The Mitzman integral form $\mathcal{U}_{\mathbb{Z},M}$ of $\tilde{\mathcal{U}}$ is the $\mathbb{Z}$-subalgebra of $\tilde{\mathcal{U}}$ generated by $\{e_i^{(k)}, f_i^{(k)} \mid i = 0, 1,\ k \in \mathbb{N}\}$.*

Remark 6.3. *The Kac-Moody presentation of $\tilde{\mathcal{U}}$ (Definition 6.1) and its presentation given in Definition 5.1 are identified through the following isomorphism:*

$$e_1 \mapsto x_0^+,\ \ f_1 \mapsto x_0^-,\ \ h_1 \mapsto h_0,\ \ e_0 \mapsto \frac{1}{4}X_1^-,\ \ f_0 \mapsto \frac{1}{4}X_{-1}^+,\ \ h_0 \mapsto \frac{1}{4}c - \frac{1}{2}h_0.$$

Notation 6.4. *In order to avoid in the following any confusion and heavy notations, we set:*

$$y_{2r+1}^{\pm} = \frac{1}{4}X_{2r+1}^{\pm},\ \ \mathtt{k}_r = \frac{1}{2}h_r,\ \ \tilde{c} = \frac{1}{4}c$$

where the $X_{2r+1}^{\pm}$'s, the h_r's and c are those introduced in Definition 5.1: thus $e_0 = y_1^-$, $f_0 = y_{-1}^+$, while the Kac-Moody h_0 and h_1 appearing in Definition 6.1 are respectively $\tilde{c} - \mathtt{k}_0$ and $2\mathtt{k}_0$; moreover $\mathcal{U}_{\mathbb{Z},M}$ is the $\mathbb{Z}$-subalgebra of $\tilde{\mathcal{U}}$ generated by $\{(x_0^{\pm})^{(k)}, (y_{\mp 1}^{\pm})^{(k)} \mid k \in \mathbb{N}\}$.

Remark 6.5. *$\mathcal{U}_{\mathbb{Z},M}$ is Ω-stable, $\exp(\pm\mathrm{ad}e_i)$-stable and $\exp(\pm\mathrm{ad}f_i)$-stable. In particular $\mathcal{U}_{\mathbb{Z},M}$ is stable under the action of*

$$\tau_0 = \exp(\mathrm{ad}e_0)\exp(-\mathrm{ad}f_0)\exp(\mathrm{ad}e_0) = \exp\big(\mathrm{ad}y_1^-\big)\exp\big(-\mathrm{ad}y_{-1}^+\big)\exp\big(\mathrm{ad}y_1^-\big),$$

of

$$\tau_1 = \exp(\mathrm{ad}e_1)\exp(-\mathrm{ad}f_1)\exp(\mathrm{ad}e_1) = \exp\big(\mathrm{ad}x_0^+\big)\exp\big(-\mathrm{ad}x_0^-\big)\exp\big(\mathrm{ad}x_0^+\big)$$

and of their inverses (cfr. [7]).

Proof. The claim for Ω follows at once from the definitions; the remaining claims are an immediate consequence of the identity $(\mathrm{ad}a)^{(n)}(b) = \sum_{r+s=n}(-1)^s a^{(r)} b a^{(s)}$. □

Remark 6.6. *Recalling the embedding $F : \hat{\mathcal{U}} \to \tilde{\mathcal{U}}$ defined in Remark 5.27, Theorem 4.25 implies that the $\mathbb{Z}$-subalgebra of $\tilde{\mathcal{U}}$ generated by the divided powers of the $y^{\pm}_{2r+1}$'s is the tensor product of the $\mathbb{Z}$-subalgebras $\mathbb{Z}^{(div)}[y^{\pm}_{2r+1} \mid r \in \mathbb{Z}]$, $\mathbb{Z}^{(sym)}[\mathtt{k}_{\pm r} \mid r > 0]$, $\mathbb{Z}^{(bin)}[\mathtt{k}_0 - \tilde{c}, 2\tilde{c}]$.*

Mitzman completely described the integral form generated by the divided powers of the Kac-Moody generators in all the twisted cases; in case $A_2^{(2)}$ his result can be stated as follows, using our notations (see Examples 1.9 and 1.2, Definitions 1.6 1.21 and Notation 6.4):

Theorem 6.7. *$\mathcal{U}_{\mathbb{Z},M} \cong \mathcal{U}^-_{\mathbb{Z},M} \otimes_{\mathbb{Z}} \mathcal{U}^0_{\mathbb{Z},M} \otimes_{\mathbb{Z}} \mathcal{U}^+_{\mathbb{Z},M}$ where*

$$\mathcal{U}^{\pm}_{\mathbb{Z},M} \cong \mathbb{Z}^{(div)}[x^{\pm}_{2r} \mid r \in \mathbb{Z}] \otimes_{\mathbb{Z}} \mathbb{Z}^{(div)}[y^{\pm}_{2r+1} \mid r \in \mathbb{Z}] \otimes_{\mathbb{Z}} \mathbb{Z}^{(div)}[x^{\pm}_{2r+1} \mid r \in \mathbb{Z}] \cong$$

$$\cong \mathbb{Z}^{(div)}[x^{\pm}_{2r+1} \mid r \in \mathbb{Z}] \otimes_{\mathbb{Z}} \mathbb{Z}^{(div)}[y^{\pm}_{2r+1} \mid r \in \mathbb{Z}] \otimes_{\mathbb{Z}} \mathbb{Z}^{(div)}[x^{\pm}_{2r} \mid r \in \mathbb{Z}],$$

$$\mathcal{U}^0_{\mathbb{Z},M} \cong \mathbb{Z}_{\lambda}[\mathtt{k}_{-r} \mid r > 0] \otimes_{\mathbb{Z}} \mathbb{Z}^{(bin)}[2\mathtt{k}_0, \tilde{c} - \mathtt{k}_0] \otimes_{\mathbb{Z}} \mathbb{Z}_{\lambda}[\mathtt{k}_r \mid r > 0].$$

The isomorphisms are all induced by the product in $\tilde{\mathcal{U}}$.

Remark that $\mathbb{Z}^{(bin)}[2\mathtt{k}_0, \tilde{c} - \mathtt{k}_0] = \mathbb{Z}^{(bin)}[\mathtt{k}_0 - \tilde{c}, 2\tilde{c}]$ (see Example 1.9) and $\mathbb{Z}_{\lambda}[\mathtt{k}_r \mid r > 0] = \mathbb{Z}^{(sym)}[\mathtt{k}_r \mid r > 0]$ (see Theorem 1.30).

Remark 6.8. *As in the case of $\hat{\mathfrak{sl}}_2$ (see Remark 4.12) we can evidentiate the relation between the elements $\hat{\mathtt{k}}_k$'s with $k > 0$ and the elements $p_{n,1}$'s ($n > 0$) defined in [5] following Garland's Λ_k's.*

Setting

$$\sum_{n \geq 0} p_n u^n = P(u) = \hat{\mathtt{k}}(-u)^{-1}$$

we have on one hand $\mathbb{Z}[\hat{\mathtt{k}}_k \mid k > 0] = \mathbb{Z}[p_n \mid n > 0]$ and on the other hand

$$p_0 = 1, \;\; p_n = \frac{1}{n}\sum_{r=1}^{n} \mathtt{k}_r p_{n-r} \; \forall n > 0,$$

hence $p_n = p_{n,1}$ $\forall n \geq 0$ (see [5]) and $\mathbb{Z}[\hat{\mathtt{k}}_k \mid k > 0] = \mathbb{Z}[p_{n,1} \mid n > 0]$.

Corollary 6.9. *$\tilde{\mathcal{U}}_{\mathbb{Z}} \subsetneq \mathcal{U}_{\mathbb{Z},M}$.*

More precisely:

$$\mathbb{Z}^{(div)}[X^{\pm}_{2r+1} \mid r \in \mathbb{Z}] \subsetneq \mathbb{Z}^{(div)}[y^{\pm}_{2r+1} \mid r \in \mathbb{Z}],$$

so that $\tilde{\mathcal{U}}^+_{\mathbb{Z}} \subsetneq \mathcal{U}^+_{\mathbb{Z},M}$ and $\tilde{\mathcal{U}}^-_{\mathbb{Z}} \subsetneq \mathcal{U}^-_{\mathbb{Z},M}$;

$$\mathbb{Z}^{(bin)}[h_0, c] = \mathbb{Z}^{(bin)}[2\mathtt{k}_0, 4\tilde{c}] \subsetneq \mathbb{Z}^{(bin)}[2\mathtt{k}_0, \tilde{c} - \mathtt{k}_0]$$

and (see Definition 5.12)

$$\mathbb{Z}^{(sym)}[\varepsilon_r h_r \mid r > 0] \subsetneq \mathbb{Z}^{(sym)}[\mathtt{k}_r \mid r > 0]$$

(and similarly for the negative part of $\mathcal{U}^0_{\mathbb{Z},M}$), so that $\tilde{\mathcal{U}}^0_{\mathbb{Z}} \subsetneq \mathcal{U}^0_{\mathbb{Z},M}$.

Proof. For $\mathbb{Z}^{(div)}$ and $\mathbb{Z}^{(bin)}$ the claim is obvious. For $\mathbb{Z}^{(sym)}$ the inequality follows at once from the fact that $\mathtt{k}_1 = \frac{h_1}{2}$ does not belong to $\mathbb{Z}^{(sym)}[\varepsilon_r h_r \mid r > 0]$ while the inclusion follows from Propositions 1.18 and 1.19 remarking that for all $r > 0$ $\varepsilon_r h_r = 2\varepsilon_r \mathtt{k}_r$.

Then the assertion for $\tilde{\mathcal{U}}_{\mathbb{Z}}$ and $\mathcal{U}_{\mathbb{Z},M}$ follows from Theorems 5.44 and 6.7. □

Remark 6.10. *Theorem 6.7 can be deduced from the commutation formulas discussed in this work and collected in Appendix 9.A, thanks to the triangular decompositions (see Remark 5.3) and to the following observations:*

i) $\mathcal{U}^0_{\mathbb{Z},M}$ *is a* $\mathbb{Z}$*-subalgebra of* $\tilde{\mathcal{U}}$*:*

indeed, since the map $h_r \mapsto \mathtt{k}_r$, $c \mapsto \tilde{c}$ *defines an automorphism of* $\tilde{\mathcal{U}}^0$*, Proposition 5.21 implies that*

$$\hat{\mathtt{k}}_+(u)\hat{\mathtt{k}}_-(v) = \hat{\mathtt{k}}_-(v)(1-uv)^{-4\tilde{c}}(1+uv)^{2\tilde{c}}\hat{\mathtt{k}}_+(u).$$

ii) $\mathcal{U}^+_{\mathbb{Z},M}$ *and* $\mathcal{U}^-_{\mathbb{Z},M}$ *are* $\mathbb{Z}$*-subalgebras of* $\tilde{\mathcal{U}}$*:*

indeed the $[(x^+_{2r})^{(k)}, (x^+_{2s+1})^{(l)}]$*'s (the only non trivial commutators in* $\mathcal{U}^+_{\mathbb{Z},M}$*) lie in* $\tilde{\mathcal{U}}^+_{\mathbb{Z}} \subseteq \mathcal{U}^+_{\mathbb{Z},M}$*; on the other hand* $\mathcal{U}^-_{\mathbb{Z},M} = \Omega(\mathcal{U}^+_{\mathbb{Z},M})$*.*

iii) $\exp\left(\sum_{r>0} a_r x^+_r u^r\right) \in \mathcal{U}^+_{\mathbb{Z},M}[[u]]$ *if* $a_r \in \mathbb{Z}$ *for all* $r > 0$*:*

see Lemma 1.55,viii), condition (1.7.1) *and the relation* $[x^+_{2r}, x^+_{2s+1}] = -4y^+_{2r+2s+1}$*.*

iv) $\mathcal{U}^0_{\mathbb{Z},M}\mathcal{U}^+_{\mathbb{Z},M}$ *and* $\mathcal{U}^-_{\mathbb{Z},M}\mathcal{U}^0_{\mathbb{Z},M}$ *are* $\mathbb{Z}$*-subalgebras of* $\tilde{\mathcal{U}}$*:*

that $(y^+_{2r+1})^{(k)}\mathcal{U}^0_{\mathbb{Z},M} \subseteq \mathcal{U}^0_{\mathbb{Z},M}\mathcal{U}^+_{\mathbb{Z},M}$ *follows from Remark 6.6; moreover by Propositions 1.56 and 1.60 we get*

$$(x^+_r)^{(k)}\binom{\mathtt{k}_0-\tilde{c}}{l} = \binom{\mathtt{k}_0-\tilde{c}-k}{l}(x^+_r)^{(k)},$$

$$(x^+_r)^{(k)}\hat{\mathtt{k}}_+(u) = \hat{\mathtt{k}}_+(u)\left(\frac{1-uT^{-1}}{(1+uT^{-1})^2}x^+_r\right)^{(k)},$$

$$\lambda_{-1}(x^+_r) = x^+_{-r}, \;\; \lambda_{-1}(\hat{\mathtt{k}}_+(u)) = \hat{\mathtt{k}}_-(u).$$

On the other hand $\mathcal{U}^-_{\mathbb{Z},M}\mathcal{U}^0_{\mathbb{Z},M} = \Omega(\mathcal{U}^0_{\mathbb{Z},M}\mathcal{U}^+_{\mathbb{Z},M})$*.*

v) $\mathcal{U}^-_{\mathbb{Z},M}\mathcal{U}^0_{\mathbb{Z},M}\mathcal{U}^+_{\mathbb{Z},M}$ *is a* $\mathbb{Z}$*-subalgebra of* $\tilde{\mathcal{U}}$*:*

$$(x^+_r)^{(k)}(x^-_s)^{(l)} \in \tilde{\mathcal{U}}_{\mathbb{Z}} = \tilde{\mathcal{U}}^-_{\mathbb{Z}}\tilde{\mathcal{U}}^0_{\mathbb{Z}}\tilde{\mathcal{U}}^+_{\mathbb{Z}} \subseteq \mathcal{U}^-_{\mathbb{Z},M}\mathcal{U}^0_{\mathbb{Z},M}\mathcal{U}^+_{\mathbb{Z},M}$$

(see Theorem 5.44 and Corollary 6.9),

$$(y^+_{2r+1})^{(k)}(y^-_{2s+1})^{(l)} \in \mathcal{U}^-_{\mathbb{Z},M}\mathcal{U}^0_{\mathbb{Z},M}\mathcal{U}^+_{\mathbb{Z},M}$$

(see Remark 6.6), and

$$\exp(x^+_0 u)\exp(y^-_1 v) =$$

$$= \exp(\alpha_-)\exp(\beta_-)\exp(\gamma_-)\hat{\mathtt{k}}_+(u^2v)\exp(\gamma_+)\exp(\beta_+)\exp(\alpha_+) \tag{6.10.1}$$

where

$$\alpha_- = \frac{uv}{1-w^2u^4v^2}.x^-_1, \quad \alpha_+ = \frac{u}{1-w^2u^4v^2}.x^+_0,$$

$$\beta_- = \frac{(1+3\cdot wu^4v^2)v}{(1+wu^4v^2)^2}.y^-_1, \quad \beta_+ = \frac{(1-wu^4v^2)u^4v}{(1+wu^4v^2)^2}.y^+_1,$$

$$\gamma_- = \frac{-w^2u^3v^2}{1-w^2u^4v^2}.x^-_0, \quad \gamma_+ = \frac{-u^3v}{1-w^2u^4v^2}.x^+_1$$

(see Proposition 5.35 recalling Definition 5.10 and Remark 5.14), so that $(x^+_0)^{(k)}(y^-_1)^{(l)}$ *lies in* $\mathcal{U}^-_{\mathbb{Z},M}\mathcal{U}^0_{\mathbb{Z},M}\mathcal{U}^+_{\mathbb{Z},M}$ *for all* $k,l \geq 0$*; from this it follows that* $(x^+_r)^{(k)}(y^-_{2s+1})^{(l)}$ *and* $(y^+_{2s+1})^{(l)}(x^-_r)^{(k)}$ *lie in* $\mathcal{U}^-_{\mathbb{Z},M}\mathcal{U}^0_{\mathbb{Z},M}\mathcal{U}^+_{\mathbb{Z},M}$ *for all* $r,s \in \mathbb{Z}$, $k,l \geq 0$ *because* $\mathcal{U}^-_{\mathbb{Z},M}\mathcal{U}^0_{\mathbb{Z},M}\mathcal{U}^+_{\mathbb{Z},M}$ *is stable under* $T^{\pm 1}$, λ_m *(*$m \in \mathbb{Z}$ *odd) and* Ω*, and*

$$x^+_r = T^{-r}\lambda_{2r+2s+1}(x^+_0), \;\; y^-_{2s+1} = (-1)^r T^{-r}\lambda_{2r+2s+1}(y^-_1),$$

$$y^+_{2s+1}=\Omega(y^-_{-2s-1}),\ \ x^-_r=\Omega(x^+_{-r});$$

vi) $\mathcal{U}_{\mathbb{Z},M}\subseteq\mathcal{U}^-_{\mathbb{Z},M}\mathcal{U}^0_{\mathbb{Z},M}\mathcal{U}^+_{\mathbb{Z},M}$*:*
it follows from v) since $(x_0^{\pm})^{(k)}\in\mathbb{Z}^{(div)}[x^{\pm}_{2r}\mid r\in\mathbb{Z}]$ *and* $(y^{\pm}_{\mp1})^{(k)}\in\mathbb{Z}^{(div)}[y^{\pm}_{2r+1}\mid r\in\mathbb{Z}]$*.*
vii) $\mathcal{U}^{\pm}_{\mathbb{Z},M}\subseteq\mathcal{U}_{\mathbb{Z},M}$*:*
this follows from Remark 6.5, observing that

$$\tau_0(x^+_r)=(-1)^{r-1}x^-_{r+1},\ \ \tau_1(x^-_r)=x^+_r,\ \ \tau_1(y^-_{2r+1})=y^+_{2r+1},\ \ \tau_0(y^+_{2r+1})=-y^-_{2r+3}.$$

viii) $\mathcal{U}^0_{\mathbb{Z},M}\subseteq\mathcal{U}_{\mathbb{Z},M}$*:*
it follows from vii), relation (6.10.1) *and the stability under* Ω*.*
ix) $\mathcal{U}^-_{\mathbb{Z},M}\mathcal{U}^0_{\mathbb{Z},M}\mathcal{U}^+_{\mathbb{Z},M}\subseteq\mathcal{U}_{\mathbb{Z},M}$*:*
this is just vii) and viii) together.
Then $\mathcal{U}_{\mathbb{Z},M}=\mathcal{U}^-_{\mathbb{Z},M}\mathcal{U}^0_{\mathbb{Z},M}\mathcal{U}^+_{\mathbb{Z},M}$*, which is the claim.*

Remark 6.11. *As one can see from Remark 6.10,vii),*

$$\{x^{\pm}_r,y^{\pm}_{2r+1},\mathtt{k}_s,2\mathtt{k}_0,\tilde{c}-\mathtt{k}_0\mid r,s\in\mathbb{Z},s\neq0\}$$

is, up to signs, a Chevalley basis of $\hat{\mathfrak{sl}}_3^{\chi}$ *(see [11]).*

It is actually through these basis elements that Mitzman introduces, following [6], the integral form of $\tilde{\mathcal{U}}$*, as the* $\mathbb{Z}$*-subalgebra of* $\tilde{\mathcal{U}}$ *generated by*

$$\{(x^{\pm}_r)^{(k)},(y^{\pm}_{2r+1})^{(k)}\mid r\in\mathbb{Z},k\in\mathbb{N}\};$$

but this $\mathbb{Z}$*-subalgebra is precisely the algebra* $\mathcal{U}_{\mathbb{Z},M}$ *introduced in Definition 6.2: indeed it turns out to be generated over* $\mathbb{Z}$ *just by* $\{e_i^{(k)},f_i^{(k)}\mid i=0,1,\ k\geq0\}$*, that is by* $\{(x_0^{\pm})^{(k)},(y^{\pm}_{\mp1})^{(k)}\mid k\geq0\}$*, thanks to Remarks 6.5 and 6.10,vii).*

Chapter 7

Other integral forms of $A_2^{(2)}$

In this chapter we describe two other integral forms $\bar{\mathcal{U}}_{\mathbb{Z}}$ and $\check{\mathcal{U}}_{\mathbb{Z}}$ of the enveloping algebra $\tilde{\mathcal{U}} = \mathcal{U}(\hat{\mathfrak{sl}_3}^{\chi})$ of the Kac-Moody algebra of type $A_2^{(2)}$ (see Definitions 7.1 and 7.2), $\bar{\mathcal{U}}_{\mathbb{Z}}$ is generated by the divided powers of the Drinfeld generators $x_r^{\pm}$ and by the divide powers of the elements $\frac{1}{2}X_{2r+1}^{\pm}$, $\check{\mathcal{U}}_{\mathbb{Z}}$ is generated by adding extra elements $\check{h}_r$ to $\bar{\mathcal{U}}_{\mathbb{Z}}$ (see Definition 1.31). As we shall see later (see Chapter 8), if we consider the $\mathbb{Z}$-algebra generated by the divided powers of the positive Drinfeld generators $x_{i,r}^{+}$ ($i \in I$, $r \in \mathbb{Z}$) in the case of $A_{2n}^{(n)}$ for $n > 1$ then this algebra also contains the divided powers of the elements $\frac{1}{2}X_{1,2r+1}^{+}$, for this reason we are interested in the study of $\bar{\mathcal{U}}_{\mathbb{Z}}$. There are two remarkable differences between $\bar{\mathcal{U}}_{\mathbb{Z}}$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}$: the first, as previously announced, is the presence of the divided powers of the elements $\frac{1}{2}X_{2r+1}^{\pm}$. The second difference concerns the structure of the (positive and negative) imaginary component. In fact, in this case $\bar{\mathcal{U}}_{\mathbb{Z}} \cap \tilde{\mathcal{U}}^{0,+} \neq \mathbb{Z}^{sym}[\bar{h}_r \mid r > 0]$ is no longer an algebra of polynomials (see Remark 1.39 and Theorem 7.14), but we exhibit fora Garland-type $\mathbb{Z}$ basis (see the description if $\mathbb{Z}^{(mix)}[h_r \mid r > 0]$ in Definition 1.35). We shall also show that $\bar{\mathcal{U}}_{\mathbb{Z}}$ can be enlarged to another integral form $\check{\mathcal{U}}_{\mathbb{Z}}$ of $\tilde{\mathcal{U}}$ with the same positive part (that is $\check{\mathcal{U}}_{\mathbb{Z}} \cap \tilde{\mathcal{U}}^{+} = \bar{\mathcal{U}}_{\mathbb{Z}} \cap \tilde{\mathcal{U}}^{+} = \bar{\mathcal{U}}_{\mathbb{Z}}^{+}$) is the $\mathbb{Z}$-subalgebra of $\tilde{\mathcal{U}}$ generated by $\{(x_r^{+})^{(k)}, (X_{2r+1}^{+})^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}\}$) and such that $\check{\mathcal{U}}_{\mathbb{Z}} \cap \tilde{\mathcal{U}}^{0,+} = \check{\mathcal{U}}_{\mathbb{Z}}^{0,+} \supsetneq \bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ is an algebra of polynomials. $\check{\mathcal{U}}_{\mathbb{Z}}$ and $\bar{\mathcal{U}}_{\mathbb{Z}}$ will be introduced and studied together and the description of $\check{\mathcal{U}}_{\mathbb{Z}}$ will also avoid unnecessary computation in $\bar{\mathcal{U}}_{\mathbb{Z}}$.

The notations are those of Chapter 5.

Definition 7.1. *Let us define $\bar{\mathcal{U}}_{\mathbb{Z}}$ to be the $\mathbb{Z}$-subalgebra of $\tilde{\mathcal{U}}$ generated by*

$$\{(x_r^{+})^{(k)}, (x_r^{-})^{(k)}, (\frac{1}{2}X_{2r+1}^{+})^{(k)}, (\frac{1}{2}X_{2r+1}^{-})^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}\},$$

$\bar{\mathcal{U}}_{\mathbb{Z}}^{+}$ and $\bar{\mathcal{U}}_{\mathbb{Z}}^{-}$ be respectively the $\mathbb{Z}$-subalgebras of $\tilde{\mathcal{U}}^{+}$ and $\tilde{\mathcal{U}}^{-}$ generated respectively by

$$\{(x_r^{+})^{(k)}, (\frac{1}{2}X_{2r+1}^{+})^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}\},$$

and

$$\{(x_r^{-})^{(k)}, (\frac{1}{2}X_{2r+1}^{-})^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}\},$$

and let $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} = \bar{\mathcal{U}}_{\mathbb{Z}} \cap \tilde{\mathcal{U}}^{0,\pm}$, $\bar{\mathcal{U}}_{\mathbb{Z}}^{0} = \bar{\mathcal{U}}_{\mathbb{Z}} \cap \tilde{\mathcal{U}}^{0}$ and $\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} = \mathbb{Z}^{(bin)}[h_0, \frac{c}{4}]$.

Definition 7.2. *Let us define $\check{\mathcal{U}}_{\mathbb{Z}}$ by the $\mathbb{Z}$-algebra of $\tilde{\mathcal{U}}$ generated by*

$$\{(\frac{1}{2}X_{2r+1}^{+})^{(k)}, (\frac{1}{2}X_{2r+1}^{-})^{(k)}, (x_r^{+})^{(k)}, (x_r^{-})^{(k)}, \check{h}_s \mid r \in \mathbb{Z}, k \in \mathbb{N}, s \in \mathbb{Z}^{*}\},$$

$\check{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} = \mathbb{Z}^{(sym)}[\frac{1}{2}h_r \mid \pm r > 0] = \mathbb{Z}[\check{h}_r \mid \pm r > 0]$, $\check{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} = \mathbb{Z}^{(bin)}[h_0, \frac{c}{4}]$ *and* $\check{\mathcal{U}}_{\mathbb{Z}}^{0}$ *is the* $\mathbb{Z}$*-subalgebra of* $\tilde{\mathcal{U}}^0$ *generated by* $\check{\mathcal{U}}_{\mathbb{Z}}^{0,+}$, $\check{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ *and* $\check{\mathcal{U}}_{\mathbb{Z}}^{0,-}$ *(see Definition 1.31).*

Remark 7.3. $\bar{\mathcal{U}}_{\mathbb{Z}}^{+}$ *and* $\bar{\mathcal{U}}_{\mathbb{Z}}^{-}$ *are integral form of* $\tilde{\mathcal{U}}^{+}$ *and* $\tilde{\mathcal{U}}^{-}$ *respectively, a basis of* $\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}$*, that is* $\bar{B}^{\pm}$*, is given by the ordered monomials in the divided powers of the elements* $\{x_r^{\pm}, \frac{1}{2}X_{2r+1}^{\pm} \mid r \in \mathbb{Z}\}$*, it follows from Definition 5.12 and Corollary 5.23 observing that* $\frac{1}{2}X_{2r+1}^{\pm}$ *are central in* $\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}$*.*

Remark 7.4. *Of course* $\check{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ *and* $\check{\mathcal{U}}_{\mathbb{Z}}^{0,-}$ *are integral form of* $\tilde{\mathcal{U}}^{0,+}$ *and* $\tilde{\mathcal{U}}^{0,-}$*.*

Lemma 7.5. *The following identity holds in* $\tilde{\mathcal{U}}^0[[u,v]]$*:*

$$\check{h}^+(u)\check{h}^-(u) = \check{h}^-(u)(1-uv)^c(1+uv)^{-\frac{c}{2}}\check{h}^+(u), \tag{7.5.1}$$

in particular $\check{\mathcal{U}}_{\mathbb{Z}}^{0,-}\check{\mathcal{U}}_{\mathbb{Z}}^{0,+} \subseteq \check{\mathcal{U}}_{\mathbb{Z}}^{0}$ *and* $\check{\mathcal{U}}_{\mathbb{Z}}^{0}$ *is an integral form of* $\tilde{\mathcal{U}}^0$*.*

Proof. Since $[\frac{1}{2}h_r, \frac{1}{2}h_s] = \delta_{r+s,0}r(2+(-1)^{r-1})\frac{1}{2}c$, Equation (7.5.1) follows from Proposition 1.58 with $m=1$, $l=\frac{1}{2}$ by substituting $\frac{c}{2}$ in place of c.

□

Proposition 7.6. *The following relations hold in* $\tilde{\mathcal{U}}[[u]]$

$$x_0^+\check{h}^+(u) = \check{h}^+(u)(1-T^{-1}u)^{-1}(1-T^{-2}u^2)^{-3}(x_0^+) \tag{7.6.1}$$
$$X_1^+\check{h}^+(u) = \check{h}^+(u)(1-T^{-1}u^2)^{-1}(X_1^+) \tag{7.6.2}$$

hence for all $k \geq 0$

$$(x_0^+)^{(k)}\check{h}^+(u) = \check{h}^+(u)\big((1-T^{-1}u)^{-1}(1-T^{-2}u^2)^{-3}(x_0^+)\big)^{(k)} \in \check{\mathcal{U}}_{\mathbb{Z}}^{0,+}\bar{\mathcal{U}}_{\mathbb{Z}}^{+}[[u]] \tag{7.6.3}$$
$$(\frac{1}{2}X_1^+)^{(k)}\check{h}^+(u) = \check{h}^+(u)\big((1-T^{-1}u^2)^{-1}\frac{1}{2}(X_1^+)\big)^{(k)} \in \check{\mathcal{U}}_{\mathbb{Z}}^{0,+}\bar{\mathcal{U}}_{\mathbb{Z}}^{+}[[u]]. \tag{7.6.4}$$

in particular

$$\bar{\mathcal{U}}_{\mathbb{Z}}^{+}\check{\mathcal{U}}_{\mathbb{Z}}^{0,+} \subseteq \check{\mathcal{U}}_{\mathbb{Z}}^{0,+}\bar{\mathcal{U}}_{\mathbb{Z}}^{+}$$

Proof. Equations (7.6.1) and (7.6.2) follow from Proposition 1.60 respectively with $m_1 = 1, m_2 = 3$ and $m_d = 0$ if $d > 2$ and $m_2 = 1$ and $m_d = 0$ if $d > 2$. Equations (7.6.3) and (7.6.4) follow respectively by equation (7.6.1) and (7.6.2). From the $T^{\pm}$ stability of $\bar{\mathcal{U}}_{\mathbb{Z}}^{+}$ and the fact that $T|_{\tilde{\mathcal{U}}^{0,+}} = \mathrm{id}|_{\tilde{\mathcal{U}}^{0,+}}$ we deduce that for all $k \geq 0$ $(x_r^+)^{(k)}\check{h}^+(u) \subseteq \check{h}^+(u)\bar{\mathcal{U}}_{\mathbb{Z}}^{+}[[u]]$ and $(\frac{1}{2}X_{2r+1}^+)^{(k)}\check{h}^+(u) \subseteq \check{h}^+(u)\bar{\mathcal{U}}_{\mathbb{Z}}^{+}[[u]]$, the claim following recalling that the $\check{h}_r$ generate $\check{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ and the $(x_r^+)^{(k)}$ and the $(\frac{1}{2}X_r^+)^{(k)}$ generate $\bar{\mathcal{U}}_{\mathbb{Z}}^{+}$. □

Proposition 7.7. *The following identity holds in* $\tilde{\mathcal{U}}$*:*

$$(\frac{1}{2}X_{2r+1}^+)^{(k)}\binom{h_0}{l} = \binom{h_0-4k}{l}(\frac{1}{2}X_{2r+1}^+)^{(k)} \tag{7.7.1}$$

hence $\bar{\mathcal{U}}_{\mathbb{Z}}^{+}\check{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} \subseteq \check{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\check{\mathcal{U}}_{\mathbb{Z}}^{+}$*.*

Proof. Equation (7.7.1) follows from (9.2.2) by multiplying both side by $(\frac{1}{2})^k$. The claim follows by Proposition 5.24 and Equation (7.7.1). □

Corollary 7.8. $\bar{\mathcal{U}}^{\pm}_{\mathbb{Z}}\check{\mathcal{U}}^{0}_{\mathbb{Z}} = \check{\mathcal{U}}^{0}_{\mathbb{Z}}\bar{\mathcal{U}}^{\pm}_{\mathbb{Z}}$. *In particular* $\check{\mathcal{U}}^{0}_{\mathbb{Z}}\bar{\mathcal{U}}^{+}_{\mathbb{Z}}$ *and* $\bar{\mathcal{U}}^{-}_{\mathbb{Z}}\check{\mathcal{U}}^{0}_{\mathbb{Z}}$ *are integral form of* $\tilde{\mathcal{U}}^{0}\tilde{\mathcal{U}}^{+}$ *and* $\tilde{\mathcal{U}}^{-}\tilde{\mathcal{U}}^{0}$.

Proof. From Propositions 7.6 and 7.7 follow that $\bar{\mathcal{U}}^{+}_{\mathbb{Z}}\check{\mathcal{U}}^{0}_{\mathbb{Z}} \subseteq \check{\mathcal{U}}^{0}_{\mathbb{Z}}\bar{\mathcal{U}}^{+}_{\mathbb{Z}}$, then the proof is the same as Corollary 5.26. □

Remark 7.9. *It is worth underling that*

$$\tilde{\mathcal{U}}^{0,\pm}_{\mathbb{Z}} \subsetneq \check{\mathcal{U}}^{0,\pm}_{\mathbb{Z}}, \tag{7.9.1}$$

$$\tilde{\mathcal{U}}^{\mathfrak{h}}_{\mathbb{Z}} \subsetneq \check{\mathcal{U}}^{\mathfrak{h}}_{\mathbb{Z}}, \tag{7.9.2}$$

$$\tilde{\mathcal{U}}^{0}_{\mathbb{Z}} \subsetneq \check{\mathcal{U}}^{0}_{\mathbb{Z}}, \tag{7.9.3}$$

since by the very definition

$$\check{h}^{\pm}(u) = \hat{h}^{\pm}(u)^{\frac{1}{2}}$$

and

$$\tilde{h}^{\pm}(u) = \hat{h}^{\pm}(u)\lambda_4(\hat{h}^{\pm}(-u^4)^{-\frac{1}{2}}) = \check{h}^{\pm}(u)^2\lambda_4(\check{h}^{\pm}(-u^4)^{-1}),$$

hence Relation (7.9.1) *holds (see Remark 5.14), Relation* (7.9.2) *follows from Lemma 7.5, Relation* (7.9.3) *follows from Relations* (7.9.1) *and* (7.9.2).

Lemma 7.10. *The following identities hold in* $\tilde{\mathcal{U}}[[u,v]]$:

$$\exp(x_0^+u)\exp\left(\frac{1}{2}X_1^-v\right) = \tag{7.10.1}$$
$$\exp\left(\frac{2}{1-4T^2u^4v^2}x_0^-uv\right)\exp\left(\frac{-4T^2}{1-4T^2u^4v^2}x_1^-u^3v^2\right)\cdot$$
$$\cdot\exp\left(\frac{1-3\cdot 4Tu^4v^2}{(1-4T^1u^4v^2)^2}\frac{1}{2}X_1^-v\right)\hat{h}^+(2u^2v)^{\frac{1}{2}}\exp\left(\frac{1+4T^{-1}u^4v^2}{(1-4T^{-1}u^4v^2)^2}\frac{1}{2}X_1^+u^4v\right)\cdot$$
$$\cdot\exp\left(\frac{-2}{1-4T^{-2}u^4v^2}x_1^+u^3v\right)\exp\left(\frac{1}{1-4T^{-2}u^4v^2}x_1^+u\right);$$

$$\exp\left(\frac{1}{2}X^+_{2r+1}u\right)\exp\left(\frac{1}{2}X^-_{2s-1}v\right) = \tag{7.10.2}$$
$$\exp\left(\frac{1}{1+T^{s+r}uv}\frac{1}{2}X^-_{2s-1}v\right)\cdot\lambda_{2(r+s)}(\hat{h}^+((u^rv^s)^2)^{\frac{1}{2}})\cdot\exp\left(\frac{1}{1+uvT^{-s-r}}\frac{1}{2}X^+_{2r+1}u\right),\ \textit{if } r+s\neq 0;$$

$$\exp\left(\frac{1}{2}X^+_{2r+1}u\right)\exp\left(\frac{1}{2}X^-_{2s-1}v\right) = \tag{7.10.3}$$
$$\exp\left(\frac{1}{2}X^-_{2s-1}v\right)\cdot(1+4uv)^{(\frac{h_0}{2}+\frac{(2r+1)c}{4})}\cdot\exp\left(\frac{1}{2}X^+_{2r+1}u\right),\ \textit{if } r+s=0;$$

Proof. Equations (7.10.1) follows from (9.2.5) substituting $\frac{1}{2}v$ to v. Equation (7.10.2) follows from (9.2.4) substituting respectively $\frac{1}{2}u$ to u and $\frac{1}{2}v$ to v. Equation (7.10.3) follows by (9.2.3) substituting $\frac{1}{2}u$ and $\frac{1}{2}v$ respectively to u and v. □

Theorem 7.11. *The following relations hold in* $\tilde{\mathcal{U}}$

$$\bar{\mathcal{U}}^{+}_{\mathbb{Z}}\bar{\mathcal{U}}^{-}_{\mathbb{Z}} \subseteq \check{\mathcal{U}}_{\mathbb{Z}},$$
$$\bar{\mathcal{U}}_{\mathbb{Z}}\cap\bar{\mathcal{U}}^{0} \subseteq \check{\mathcal{U}}^{0}_{\mathbb{Z}}.$$

Proof. From relation (7.10.3) it follows that $\bar{\mathcal{U}}_{\mathbb{Z}}^{+}\bar{\mathcal{U}}_{\mathbb{Z}}^{-} \subseteq \bar{\mathcal{U}}_{\mathbb{Z}}^{-} \otimes \check{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} \otimes \bar{\mathcal{U}}_{\mathbb{Z}}^{+}$ since $(1+4u)^{\frac{h_0}{2}} \subseteq \mathbb{Z}^{(bin)}[h_0][[u]]$ (see Lemma 5.15). From Lemma 7.10 and Theorem 5.1 follows that

$$(\bar{\mathcal{U}}_{\mathbb{Z}}^{+}\bar{\mathcal{U}}_{\mathbb{Z}}^{-}) \cap \tilde{\mathcal{U}}^{0,+} = \mathbb{Z}(\lambda_{2s}(\check{h}_r), \lambda_{2r+1}(\hat{h}^{\{c\}}), \tilde{h}_r \mid r > 0) =$$
$$\mathbb{Z}(\lambda_s(\bar{h}_r), \lambda_{2r+1}(\hat{h}^{\{c\}}), \tilde{h}_r \mid r > 0) \subsetneq \mathbb{Z}[\check{h}_r \mid r > 0]$$

(see Remarks 7.9 and 1.34), hence from Equations (7.10.1), (7.10.2) and (7.10.3) follow that $\bar{\mathcal{U}}_{\mathbb{Z}}^{+}\bar{\mathcal{U}}_{\mathbb{Z}}^{-} \subseteq \check{\mathcal{U}}_{\mathbb{Z}}$. □

We can now collect all the results obtained till now in the main theorem of this section:

Theorem 7.12. *The $\mathbb{Z}$-subalgebra $\check{\mathcal{U}}_{\mathbb{Z}}$ of $\tilde{\mathcal{U}}$ generated by*

$$\{(\frac{1}{2}X_{2r+1}^{+})^{(k)}, (\frac{1}{2}X_{2r+1}^{-})^{(k)}, (x_r^{+})^{(k)}, (x_r^{-})^{(k)}, \check{h}_r \mid r \in \mathbb{Z}, k \in \mathbb{N}\}$$

is an integral form of $\tilde{\mathcal{U}}$. More precisely

$$\check{\mathcal{U}}_{\mathbb{Z}} \cong \bar{\mathcal{U}}_{\mathbb{Z}}^{-} \otimes \check{\mathcal{U}}_{\mathbb{Z}}^{0,+} \otimes \check{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} \otimes \check{\mathcal{U}}_{\mathbb{Z}}^{0,+} \otimes \bar{\mathcal{U}}_{\mathbb{Z}}^{+}$$

and a $\mathbb{Z}$-basis of $\bar{\mathcal{U}}_{\mathbb{Z}}$ is given by the product $\bar{B}^{-}\check{B}^{0}\bar{B}^{+}$, where $\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}$ and $\bar{B}^{\pm}$ are described in Remark 7.3, $B^{0,\pm}$ and $B^{\mathfrak{h}}$ are the $\mathbb{Z}$-bases respectively of $\check{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$ and $\check{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ given as follows:

$$\check{B}^{0,\pm} = \left\{\check{h}^{\pm\mathbf{k}} = \prod_{l \in \mathbb{N}} \check{h}_{\pm l}^{k_l} \mid \mathbf{k} : \mathbb{N} \to \mathbb{N} \text{ is finitely supported}\right\}$$

$$B^{\mathfrak{h}} = \left\{\binom{h_0}{k}\binom{\frac{c}{4}}{\tilde{k}} \mid k, \tilde{k} \in \mathbb{N}\right\}.$$

We can now concentrate on $\bar{\mathcal{U}}_{\mathbb{Z}}$

Remark 7.13. *It follows directly by Definitions 5.1, 7.2 and 7.1 that:*

$$\tilde{\mathcal{U}}_{\mathbb{Z}} \subseteq \bar{\mathcal{U}}_{\mathbb{Z}} \subseteq \check{\mathcal{U}}_{\mathbb{Z}}.$$

From Theorems 5.44 and 7.12 it follows that

$$\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm} \subsetneq \bar{\mathcal{U}}_{\mathbb{Z}} \cap \tilde{\mathcal{U}}^{\pm} = \bar{\mathcal{U}}_{\mathbb{Z}}^{\pm} = \check{\mathcal{U}}_{\mathbb{Z}} \cap \tilde{\mathcal{U}}^{\pm}.$$

From Theorem 5.44 and 7.11 and Lemma 7.10 that $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} \subseteq \check{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$ and $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$ is generated by the elements whose the generating series series are $\lambda_2(\check{h}^{+}(u^2))$, $\check{h}^{+}(2u)$ and $\tilde{h}^{+}(u)$, equivalently is generated by the elements whose the generating series series are $\bar{h}(u^2)$, $\hat{h}^{\{c\}}(u)$ and $\hat{h}^{\{\varepsilon\}}(u)$ (see Notation 1.15, Definitions 1.31, 1.47).

Theorem 7.14. *$\bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ and $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,-}$ are integral forms of respectively $\tilde{\mathcal{U}}^{0,+}$ and $\tilde{\mathcal{U}}^{0,-}$, with basis given by*

$$B_{q.pol}^{0,\pm} = \{\prod_{k>0} \hat{h}_{\pm k}^{\epsilon_k} \prod_{k>0} \bar{h}_{\pm k}^{d_k} \mid \epsilon : \mathbb{Z}_+ \to \{0,1\} \text{ and } d : \mathbb{Z}_+ \to \mathbb{N} \text{ are finitely supported}\}$$

or equivalently

$$B_{q.\lambda}^{0,\pm} = \{\prod_{m>0, m \text{ odd}} \lambda_m(\hat{h}_{k_m}) \prod_{m>0, m \text{ even}} \lambda_m(\check{h}_{k_m}), \mid k : \mathbb{Z}_+ \to \mathbb{N} \text{ is finitely supported}\}.$$

Proof. From Remark 7.13 follows that $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} = \mathbb{Z}(\tilde{h}_r, \hat{h}_r^{\{c\}}, \bar{h}_{2r} \mid r > 0)$. By the very definition

$$\tilde{h}^+(u) = \hat{h}^+(u)\lambda_4(h^+(u^4)^{-\frac{1}{2}}) = \hat{h}^+(u)\lambda_2(\bar{h}^+(u^2)^{-1}),$$

hence we can consider the $\hat{h}_r$s instead of $\tilde{h}_r$s, hence $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} = \mathbb{Z}(\hat{h}_r, \hat{h}_r^{(c)}, \bar{h}_{2r} \mid r > 0)$. From Proposition 1.52 follows that $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} = \mathbb{Z}(\hat{h}_r, \bar{h}_{2r} \mid r > 0)$. From Theorems 1.46 and 1.42 follows that $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ is an integral form of $\tilde{\mathcal{U}}^{0,+}$ and $B_{q.pol}^{0,\pm}$ and $B_{q.\lambda}^{0,\pm}$ are basis of $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}$. The claim for $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,-}$ follows by applying Ω. □

Remark 7.15. *From Definition 1.31 and Remark 1.32 we have the following relations:*

$$\hat{h}^+(u) = \check{h}^+(u)^2,$$
$$\bar{h}_1^+(u^2) = \check{h}^+(u)\check{h}^+(-u),$$

Proposition 7.16. *The following identities holds in $\bar{\mathcal{U}}[[u,v]]$:*

$$\hat{h}^+(u)\hat{h}^-(u) = \hat{h}^-(u)(1-uv)^{2c}(1+uv)^{-c}\hat{h}^+(u); \tag{7.16.1}$$
$$\bar{h}^+(u^2)\bar{h}^-(v^2) = \bar{h}^-(v^2)(1-(uv)^2)^{2c}(1-(uv)^2)^{-c}\bar{h}^+(u^2); \tag{7.16.2}$$
$$\hat{h}^+(u)\bar{h}^-(v^2) = \bar{h}^-(v^2)(1-(uv)^2)^{c}\hat{h}^+(u). \tag{7.16.3}$$

Proof. Equations (7.16.1), (7.16.2) and (7.16.3) follow from Equation (7.5.1) and Remark 7.15. □

Corollary 7.17. *$\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ and $\bar{\mathcal{U}}_{\mathbb{Z}}^0 = \bar{\mathcal{U}}_{\mathbb{Z}}^{0,-}\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ is an integral form of $\tilde{\mathcal{U}}^0$.*

Corollary 7.18. *$\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} = \bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} \subsetneq \check{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} \subsetneq \bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} \subsetneq \check{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$.*

Remark 7.19. *Let $\mathcal{U}_{\mathbb{Z},M}$ be the Mitzman integral form (see Chapter 6), we want to underline that $\mathcal{U}_{\mathbb{Z},M}$ strictly contains $\check{\mathcal{U}}_{\mathbb{Z}}$, more precisely we have the following relations:*

$$\mathcal{U}_{\mathbb{Z},M} \supsetneq \check{\mathcal{U}}_{\mathbb{Z}} \supsetneq \bar{\mathcal{U}}_{\mathbb{Z}} \supsetneq \tilde{\mathcal{U}}_{\mathbb{Z}},$$
$$\mathcal{U}_{\mathbb{Z},M}^{\pm} \supsetneq \check{\mathcal{U}}_{\mathbb{Z}}^{\pm} = \bar{\mathcal{U}}_{\mathbb{Z}}^{\pm} \supsetneq \tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm},$$
$$\mathcal{U}_{\mathbb{Z},M}^{0,\pm} = \check{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} \supsetneq \bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} \supsetneq \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\pm},$$
$$\mathcal{U}_{\mathbb{Z},M}^{\mathfrak{h}} \supsetneq \check{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} \supsetneq \bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} = \tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}.$$

We can now recollect the results regarding $\bar{\mathcal{U}}_{\mathbb{Z}}$ in the following:

Theorem 7.20. *The $\mathbb{Z}$-subalgebra $\bar{\mathcal{U}}_{\mathbb{Z}}$ of $\tilde{\mathcal{U}}$ generated by*

$$\{(x_r^+)^{(k)}, (x_r^-)^{(k)}, (\frac{1}{2}X_{2r+1}^+)^{(k)}, (\frac{1}{2}X_{2r+1}^-)^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}\}$$

is an integral form of $\tilde{\mathcal{U}}$.
More precisely

$$\bar{\mathcal{U}}_{\mathbb{Z}} \cong \bar{\mathcal{U}}_{\mathbb{Z}}^- \otimes \bar{\mathcal{U}}_{\mathbb{Z}}^0 \otimes \bar{\mathcal{U}}_{\mathbb{Z}}^+$$

and a $\mathbb{Z}$-basis of $\bar{\mathcal{U}}_{\mathbb{Z}}$ is given by $\bar{B}^-\bar{B}^0\bar{B}^+$, $\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}$ and $\bar{B}^{\pm}$ are described in Theorem 7.12.
$\bar{\mathcal{U}}_{\mathbb{Z}}^0$ is an integral form of $\tilde{\mathcal{U}}^0$, more precisely

$$\bar{\mathcal{U}}_{\mathbb{Z}}^0 \cong \bar{\mathcal{U}}_{\mathbb{Z}}^{0,-} \otimes \bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} \otimes \bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}.$$

$\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} = \tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ and $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$ are integral form of respectively and $\tilde{\mathcal{U}}^{\mathfrak{h}}$ and $\tilde{\mathcal{U}}^{0,\pm}$. A $\mathbb{Z}$-basis $\bar{\mathcal{U}}_{\mathbb{Z}}^0$ of is given by the product $B_{q.pol}^{0,-}\bar{B}^{\mathfrak{h}}B_{q.pol}^{0,+}$. $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ and $B^{\mathfrak{h}}$ is described in Theorem 5.44, $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$ and $B_{q.pol}^{0,\pm}$ are described in Theorem 7.14.

Chapter 8

Integral form of $A_{2n}^{(2)}$

8.1 Generalities

In this section we will use the results of the two previous chapters to describe the integral form of $A_{2n}^{(2)}$ $(n > 1)$. As wee shall see, in this case is not more true that the positive part of the $\mathbb{Z}$-subalgebra by $\{(x_{i,r}^{+})^{(k)}, (x_{i,r}^{-})^{(k)} \mid i \in I, r \in \mathbb{Z}, k \in \mathbb{N}\}$ is the $\mathbb{Z}$-subalgebra generated by $\mathbb{Z}$-subalgebra by $\{(x_{i,r}^{+})^{(k)} \mid i \in I, r \in \mathbb{Z}, k \in \mathbb{N}\}$, it turn out that we have to add the generators $(\frac{1}{2}X_{1,2r+1}^{+})^{(k)}$.

Definition 8.1. *Let $I = \{1, \dots, n\}$, then $A_{2n}^{(2)}$ (respectively $\tilde{\mathcal{U}}$) is the Lie algebra (respectively the associative algebra) over $\mathbb{Q}$ generated by $\{c, h_{i,r}, x_{i,r}^{\pm}, X_{1,2r+1}^{\pm} | r \in \mathbb{Z}, i \in I\}$ with relations:*

$$[c, \cdot] = 0;$$

$$[h_{i,r}, h_{j,s}] = r\delta_{r+s,0}a_{i,j;r}\frac{2c}{d_j}$$

$$[x_{i,r}^{+}, x_{j,r}^{-}] = \delta_{i,j}(h_{i,r+s} + r\delta_{r+s,0}\frac{2c}{d_j});$$

$$[h_{i,r}, x_{j,s}^{\pm}] = \pm a_{i,j;r}x_{j,r+s}^{\pm};$$

$$[x_{1,r}^{\pm}, x_{1,s}^{\pm}] = \begin{cases} \pm(-1)^s X_{1,r+s}^{\pm} \text{ if } r+s \text{ is odd} \\ 0 \text{ otherwise;} \end{cases}$$

$$[x_{1,r}^{\pm}, X_{1,2s+1}^{\pm}] = 0;$$

$$(adx_{i,r}^{\pm})^{1-a_{i,j}}(x_{j,s}^{\pm}) = 0 \text{ if } i \neq j;$$

$$[x_{i,r}^{\pm}, x_{i,s}^{\pm}] = 0 \text{ if } r+s \text{ is even or } i \neq 1; \tag{8.1.1}$$

$$[x_{1,r}^{+}, [x_{1,r}^{+}, x_{2,s}^{+}]] = -[x_{1,r+1}^{+}, [x_{1,r+1}^{+}, x_{2,s-2}^{+}]]; \tag{8.1.2}$$

where

$$A = (a_{i,j})_{i,j=1,\dots,n,0} = \begin{pmatrix} 2 & -2 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -1 & 2 & -2 \\ \vdots & \dots & 0 & -1 & 2 \end{pmatrix},$$

$$D = diag(d_1, d_2, \dots, d_n, d_0) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \dots & 0 \\ \vdots & 0 & 0 & 2 & 0 \\ & \dots & 0 & 0 & 4 \end{pmatrix}$$

and

$$a_{i,j;r} = \begin{cases} 2(2 + (-1)^{r-1}) \text{ if } (i,j) = (1,1); \\ a_{i,j} \text{ otherwise }. \end{cases}$$

Notice that $\{x_{i,r}^+, x_{i,r}^- | r \in \mathbb{Z}, i \in I\}$ generates $\tilde{\mathcal{U}}$.

Definition 8.2. *Let us define the following subalgebra of $\bar{\mathcal{U}}$:*
$\bar{\mathcal{U}}^+, \bar{\mathcal{U}}^-, \bar{\mathcal{U}}^0$ are the subalgebras of $\bar{\mathcal{U}}$ generated respectively by

$$\{x_{i,r}^+ \mid i \in I, r \in \mathbb{Z}\}, \ \{x_{i,r}^- \mid i \in I, r \in \mathbb{Z}\}, \ \{c, h_{i,r} \mid i \in I, r \in \mathbb{Z}\}.$$

Definition 8.3. *$A_{2n}^{(2)}$ and $\bar{\mathcal{U}}$ are endowed with the following anti/auto/homo/morphisms:*
σ is the antiautomorphism defined on the generators by:

$$\begin{aligned} &x_{i,r}^\pm \mapsto x_{i,r}^\pm, \\ &X_{1,r}^\pm \mapsto -X_{1,r}^\pm, \\ &h_{i,r} \mapsto h_{i,r}, \\ &c \mapsto -c; \end{aligned}$$

Ω is the antiautomorphism defined on the generators by:

$$\begin{aligned} &x_{i,r}^\pm \mapsto x_{i,-r}^\mp, \\ &X_{1,r}^\pm \mapsto X_{1,-r}^\mp, \\ &h_{i,r} \mapsto h_{i,-r}, \\ &c \mapsto c; \end{aligned}$$

T is the automorphism defined on the generators by:

$$\begin{aligned} &x_{i,r}^\pm \mapsto x_{i,r\mp 1}^\mp, \\ &X_{1,r}^\pm \mapsto -X_{1,r\mp 2}^\mp, \\ &h_{i,r} \mapsto h_{i,-r} - r\delta_{r,0}c, \\ &c \mapsto c. \end{aligned}$$

Notation 8.4. *Let us define the following sets*

$$\begin{aligned} &\Phi_{0,s}^+ = \{\alpha_i + \dots + \alpha_j \mid 1 \le i \le j \le n\}, \\ &\Phi_{0,m}^+ = \{2\alpha_1 + \dots + 2\alpha_i + \alpha_{i+1} + \dots + \alpha_j \mid 1 \le i < j \le n\}, \\ &\Phi_0^+ = \Phi_{0,s}^+ \cup \Phi_{0,m}^+. \end{aligned}$$

Recalling that the root system of $A_{2n}^{(2)}$ is $\Phi = \Phi^{re} \cup \Phi^{im}$ where $\Phi^{re} = \Phi^{re,+} \cup \Phi^{re,-}$, $\Phi^{re,+} = -\Phi^{re,-}$, $\Phi^{re,+} = \Phi_s^{re,+} \cup \Phi_m^{re,+} \cup \Phi_l^{re,+}$,

$$\begin{aligned} &\Phi_s^{re,+} = \{\alpha + r\delta \mid \alpha \in \Phi_{0,s}^+, r \in \mathbb{Z}\}, \\ &\Phi_m^{re,+} = \{\alpha + r\delta \mid \alpha \in \Phi_{0,m}^+, r \in \mathbb{Z}\} \\ &\Phi_l^{re,+} = \{2\alpha + (2r+1)\delta \mid \alpha \in \Phi_{0,s}^+, r \in \mathbb{Z}\}, \end{aligned}$$

in particular we have that:

$$\Phi_s^{re} = \Phi_s^{re,+} \cup \Phi_s^{re,-} = W \cdot \alpha_1,$$
$$\Phi_m^{re} = \Phi_m^{re,+} \cup \Phi_m^{re,-} = W \cdot \alpha_2,$$
$$\Phi_l^{re} = \Phi_l^{re,+} \cup \Phi_l^{re,-} = W \cdot \alpha_0,$$
$$\Phi^{re} = W \cdot \alpha_1 \cup W \cdot \alpha_2 \cup W \cdot \alpha_0.$$

$\Phi^{im} = \Phi^{im,+} \cup \Phi^{im,-}$, $\Phi^{im,+} = -\Phi^{im,-}$ *where*

$$\Phi^{im,\pm} = \{\pm m\delta \mid m \in \mathbb{Z}_{>0}\}.$$

Definition 8.5. *For all $\alpha \in W_0 \cdot \{\alpha_i \mid i \in I \cup \{0\}\}$, that is $\alpha = w(\alpha_i)$ for some $w \in W_0$, let us define $x_{\alpha,r} = \tau_{i_1} \dots \tau_{i_N}(x_{i,0}^+)$ if $w = \sigma_{i_1} \dots \sigma_{i_N}$.*

Remark 8.6. *$x_{\alpha,r}$ is defined up to sign. In particular the $\mathbb{Z}$-subalgebra generated by $\{x_{\alpha,r}^{(k)} \mid \alpha \in W \cdot \{\alpha_i \mid i \in I, r \in \mathbb{Z}, k \in \mathbb{N}\}\}$ is uniquely determined.*

Remark 8.7. *$\{x_\alpha \mid \alpha \in \Phi^{re}\}$ is the set of Chevalley generators used by Mitzman. In particular the $\mathbb{Z}$-subalgebra of $\tilde{\mathcal{U}}$ generated by $\{(e_i)^{(k)} \mid i \in I \cup \{0\}, r \in \mathbb{N}\}$ is a free $\mathbb{Z}$-module with basis the ordered monomials in the divided powers of the x_α's.*

Definition 8.8. *For all $i \in I \setminus \{1\}$, let us define the following elements:*

$$X_{i,2r+1}^{\pm} = \tau_i(X_{i-1,2r+1}^{\pm}).$$

Definition 8.9. *The following maps are Lie-algebra homomorphisms, obviously injective, inducing embeddings:*

$$\begin{aligned} \phi_1 : A_2^{(2)} &\to A_{2n}^{(2)} \\ x_r^{\pm} &\mapsto x_{1,r}^{\pm} \\ h_r &\mapsto h_{1,r} \\ c &\mapsto 2c. \end{aligned}$$

$$\begin{aligned} \phi_i : A_1^{(1)} &\to A_{2n}^{(2)} \text{ if } i \neq 1 \\ x_r^{\pm} &\mapsto x_{i,r}^{\pm} \\ h_r &\mapsto h_{i,r} \\ c &\mapsto c \end{aligned}$$

$$\begin{aligned} \bar{\psi} : A_{2(n-1)}^{(2)} &\to A_{2n}^{(2)} \\ x_{i,r}^{\pm} &\mapsto x_{i,r}^{\pm} \\ h_r &\mapsto h_{i,r} \\ c &\mapsto c \end{aligned} \tag{8.9.1}$$

$$\begin{aligned} \tilde{\psi} : A_{n-1}^{(1)} &\to A_{2n}^{(2)} \\ x_{i,r}^{\pm} &\mapsto x_{i+1,r}^{\pm} \\ h_{i,r} &\mapsto h_{i+1,r} \\ c &\mapsto c \end{aligned} \tag{8.9.2}$$

Definition 8.10. *Here we define some $\mathbb{Z}$-subalgebras of $\bar{\mathcal{U}}$:*
$\bar{\mathcal{U}}_{\mathbb{Z}}$ is the $\mathbb{Z}$-subalgebras of $\bar{\mathcal{U}}$ generated by

$$\{(x_{i,r}^{+})^{(k)}, (x_{i,r}^{-})^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}, i \in I\};$$

$\mathcal{U}_{\mathbb{Z}}^{+}$ and $\mathcal{U}_{\mathbb{Z}}^{-}$ are the $\mathbb{Z}$-subalgebras of $\bar{\mathcal{U}}$ respectively generated by

$$\{(x_{i,r}^{+})^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}, i \in I\},$$
$$\{(x_{i,r}^{-})^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}, i \in I\};$$

$\bar{\mathcal{U}}_{\mathbb{Z}}^{+}$ and $\bar{\mathcal{U}}_{\mathbb{Z}}^{-}$ are the $\mathbb{Z}$-subalgebras of $\bar{\mathcal{U}}$ respectively generated by

$$\{(x_{i,r}^{+})^{(k)}, (\frac{1}{2}X_{1,2r+1}^{+})^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}, i \in I\},$$
$$\{(x_{i,r}^{-})^{(k)}, (\frac{1}{2}X_{1,2r+1}^{-})^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}, i \in I\};$$

$\bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ and $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,-}$ are the $\mathbb{Z}$-subalgebras of $\bar{\mathcal{U}}$ respectively generated by

$$\{\hat{h}_{i,r}, \bar{h}_{1,r}, \mid r > 0, i \in I\},$$
$$\{\hat{h}_{i,r}, \bar{h}_{1,r}, \mid r < 0, i \in I\};$$

$\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} = \mathbb{Z}^{bin}[h_{i,0}, c \mid i \in I]$,
The notations are those of Chapter 1.

Remark 8.11. *Of course we have that $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} \subseteq \mathbb{Z}[\check{h}_{1,r}, \hat{h}_{1,r} \mid i \in I \setminus \{1\}, \pm r > 0]$. $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$ and $\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ are integral form of respectively $\bar{\mathcal{U}}^{0,\pm}$ and $\bar{\mathcal{U}}^{\mathfrak{h}}$.*

8.2 $A_4^{(2)}$

Let us fix in this section $n = 2$. In this part we want to study the algebra $\bar{\mathcal{U}}_{\mathbb{Z}}^{+}(A_4^{(2)})$, to do this we first want to study the straightening formulas within the algebra and express them in terms of the Lie bracket.

Lemma 8.12.

$$[x_{1,0}^{-}, [x_{1,0}^{+}, x_{2,r}^{+}]] = 2x_{2,r}^{+}; \tag{8.12.1}$$
$$[x_{1,0}^{-}, [x_{1,0}^{+}, [x_{1,0}^{+}, x_{2,r}^{+}]]] = 2[x_{1,0}^{+}, x_{2,r}^{+}]; \tag{8.12.2}$$
$$[x_{1,0}^{-}, [x_{1,0}^{-}, [x_{1,0}^{+}, [x_{1,0}^{+}, x_{2,r}^{+}]]]] = 4x_{2,r}^{+}; \tag{8.12.3}$$
$$[x_{2,0}^{-}, [x_{2,0}^{+}, x_{1,r}^{+}]] = x_{1,r}^{+}; \tag{8.12.4}$$
$$[h_{2,0}, X_{1,r}^{+}] = -2X_{1,r}^{+} \tag{8.12.5}$$
$$[x_{2,0}^{-}, [x_{2,0}^{+}, X_{1,r}^{+}]] = 2X_{1,r}^{+}. \tag{8.12.6}$$
$$[x_{2,0}^{-}, [x_{2,0}^{+}, [x_{2,0}^{+}, X_{1,r}^{+}]]] = 2[x_{2,0}^{+}, X_{1,r}^{+}] \tag{8.12.7}$$
$$[x_{2,0}^{-}, [x_{2,0}^{-}, [x_{2,0}^{+}, [x_{2,0}^{+}, X_{1,r}^{+}]]]] = 4X_{1,r}^{+}. \tag{8.12.8}$$

Proof. Prove of equations (8.12.1),(8.12.2),(8.12.3) and (8.12.4)

$$[x_{1,0}^-, [x_{1,0}^+, x_{2,r}^+]] = -[x_{2,r}^+, [x_{1,0}^-, x_{1,0}^+]] = [x_{2,r}^+, h_{1,0}] = -[h_{1,0}, x_{2,r}^+] = 2x_{2,r}^+.$$

$$[x_{1,0}^-, [x_{1,0}^+, [x_{1,0}^+, x_{2,r}^+]]] = -([[x_{1,0}^+, x_{2,r}^+], [x_{1,0}^-, x_{1,0}^+]] + [x_{1,0}^+, [[x_{1,0}^+, x_{2,r}^+], x_{1,0}^-]]) =$$
$$[[x_{1,0}^+, x_{2,r}^+], h_{1,0}] + [x_{1,0}^+, [x_{1,0}^-, [x_{1,0}^+, x_{2,r}^+]]] = -[h_{1,0}, [x_{1,0}^+, x_{2,r}^+]] + 2[x_{1,0}^+, x_{2,r}^+] =$$
$$[x_{2,r}^+, [h_{1,0}, x_{1,0}^+]] + [x_{1,0}^+, [x_{2,r}^+, h_{1,0}]] + 2[x_{1,0}^+, x_{2,r}^+] =$$
$$2[x_{2,r}^+, x_{1,0}^+] + 2[x_{1,0}^+, x_{2,r}^+] + 2[x_{1,0}^+, x_{2,r}^+] = 2[x_{1,0}^+, x_{2,r}^+].$$

$$[x_{1,0}^-, [x_{1,0}^-, [x_{1,0}^+, [x_{1,0}^+, x_{2,r}^+]]]] = 2[x_{1,0}^-, [x_{1,0}^+, x_{2,r}^+]] = 4x_{2,r}^+;$$

$$[x_{2,0}^-, [x_{2,0}^+, x_{1,r}^+]] = -[x_{1,r}^+, [x_{2,0}^-, x_{2,0}^+] = [x_{1,r}^+, h_{2,0}] = x_{1,r}^+.$$

Proof of equation (8.12.5), (8.12.6), (8.12.7) and (8.12.8)

$$[h_{2,0}, X_{1,r}^+] = [h_{2,0}, [x_{1,r}^+, x_{1,0}^+]] =$$
$$-[x_{1,0}^+, [h_{2,0}, x_{1,r}^+]] - [x_{1,r}^+, [x_{1,0}^+, h_{2,0}]] = -[x_{1,0}^+, x_{1,r}^+] + [x_{1,r}^+, x_{1,0}^+] = -2X_{1,r}^+$$

$$[x_{2,0}^-, [x_{2,0}^+, X_{1,r}^+]] = -[X_{1,r}^+, [x_{2,0}^-, x_{2,0}^+]] = [X_{1,r}^+, h_{2,0}] = 2X_{1,r}^+.$$

$$[x_{2,0}^-, [x_{2,0}^+, [x_{2,0}^+, X_{1,r}^+]]] = -[[x_{2,0}^+, X_{1,r}^+], [x_{2,0}^-, x_{2,0}^+]] - [x_{2,0}^+, [[x_{2,0}^+, X_{1,r}^+], x_{2,0}^-]] =$$
$$[[x_{2,0}^+, X_{1,r}^+], h_{2,0}]] + [x_{2,0}^+, [x_{2,0}^-, [x_{2,0}^+, X_{1,r}^+]]] = -[h_{2,0}, [x_{2,0}^+, X_{1,r}^+]] + 2[x_{2,0}^+, X_{1,r}^+] =$$
$$[X_{1,r}^+, [h_{2,0}, x_{2,0}^+]] + [x_{2,0}^+, [X_{1,r}^+, h_{2,0}]] + 2[x_{2,0}^+, X_{1,r}^+] =$$
$$2[X_{1,r}^+, x_{2,0}^+] + 2[x_{2,0}^+, X_{1,r}^+] + 2[x_{2,0}^+, X_{1,r}^+] = 2[x_{2,0}^+, X_{1,r}^+]$$

$$[x_{2,0}^-, [x_{2,0}^-, [x_{2,0}^+, [x_{2,0}^+, X_{1,r}^+]]]] = 2[x_{2,0}^-, [x_{2,0}^+, X_{1,r}^+]] = 4X_{1,r}^+$$

□

Lemma 8.13. *The following identities hold in $\tilde{\mathcal{U}}$:*

$$\tau_1(x_{2,r}^+) = \frac{1}{2}[x_{1,0}^+, [x_{1,0}^+, x_{2,r}^+]]. \tag{8.13.1}$$
$$\tau_2(x_{1,r}^+) = [x_{2,0}^+, x_{1,r}^+]; \tag{8.13.2}$$
$$\tau_2(X_{1,r}^+) = \frac{1}{2}[x_{2,0}^+, [x_{2,0}^+, X_{1,r}^+]]. \tag{8.13.3}$$

Proof. We use relations of Lemma 8.12.

Proof of Equation (8.13.1):

$$\tau_1(x^+_{2,r}) = \exp\left(\mathrm{ad}x^+_{1,0}\right)\exp\left(-\mathrm{ad}x^-_{1,0}\right)\exp\left(\mathrm{ad}x^+_{1,0}\right)(x^+_{2,r}) =$$
$$\exp\left(\mathrm{ad}x^+_{1,0}\right)\exp\left(-\mathrm{ad}x^-_{1,0}\right)(x^+_{2,r} + [x^+_{1,0}, x^+_{2,r}] + \frac{1}{2}[x^+_{1,0}, [x^+_{1,0}, x^+_{2,r}]]) =$$
$$\exp\left(\mathrm{ad}x^+_{1,0}\right)(x^+_{2,r} + [x^+_{1,0}, x^+_{2,r}] - [x^-_{1,0}, [x^+_{1,0}, x^+_{2,r}]] +$$
$$\frac{1}{2}[x^+_{1,0}, [x^+_{1,0}, x^+_{2,r}]] - \frac{1}{2}[x^-_{1,0}, [x^+_{1,0}, [x^+_{1,0}, x^+_{2,r}]]] + \frac{1}{4}[x^-_{1,0}, [x^-_{1,0}, [x^+_{1,0}, [x^+_{1,0}, x^+_{2,r}]]]]) =$$
$$\exp\left(\mathrm{ad}x^+_{1,0}\right)(x^+_{2,r} + [x^+_{1,0}, x^+_{2,r}] - 2x^+_{2,r} + \frac{1}{2}[x^+_{1,0}, [x^+_{1,0}, x^+_{2,r}]] - [x^+_{1,0}, x^+_{2,r}] + x^+_{2,r}) =$$
$$\exp\left(\mathrm{ad}x^+_{1,0}\right)(\frac{1}{2}[x^+_{1,0}, [x^+_{1,0}, x^+_{2,r}]]) =$$
$$\frac{1}{2}[x^+_{1,0}, [x^+_{1,0}, x^+_{2,r}]]$$

Proof of Equation (8.13.2):

$$\tau_2(x^+_{1,r}) = \exp\left(\mathrm{ad}x^+_{2,0}\right)\exp\left(-\mathrm{ad}x^-_{2,0}\right)\exp\left(\mathrm{ad}x^+_{2,0}\right)(x^+_{1,r}) =$$
$$\exp\left(\mathrm{ad}x^+_{2,0}\right)\exp\left(-\mathrm{ad}x^-_{2,0}\right)(x^+_{1,r} + [x^+_{2,0}, x^+_{1,r}]) =$$
$$\exp\left(\mathrm{ad}x^+_{2,0}\right)(x^+_{1,r} + [x^+_{2,0}, x^+_{1,r}] - [x^-_{2,0}, [x^+_{2,0}, x^+_{1,r}]]) =$$
$$\exp\left(\mathrm{ad}x^+_{2,0}\right)(x^+_{1,r} + [x^+_{2,0}, x^+_{1,r}] - x^+_{1,r}) =$$
$$\exp\left(\mathrm{ad}x^+_{2,0}\right)([x^+_{2,0}, x^+_{1,r}]) = [x^+_{2,0}, x^+_{1,r}]$$

Proof of Equation (8.13.3):

$$\tau_2(X^+_{1,r}) = \exp\left(\mathrm{ad}x^+_{2,0}\right)\exp\left(-\mathrm{ad}x^-_{2,0}\right)\exp\left(\mathrm{ad}x^+_{2,0}\right)(X^+_{1,r}) =$$
$$\exp\left(\mathrm{ad}x^+_{2,0}\right)(X^+_{1,r} + [x^+_{2,0}, X^+_{1,r}] - [x^-_{2,0}, [x^+_{2,0}, X^+_{1,r}]] +$$
$$\frac{1}{2}[x^+_{2,0}, [x^+_{2,0}, X^+_{1,r}]] - \frac{1}{2}[x^-_{2,0}, [x^+_{2,0}, [x^+_{2,0}, X^+_{1,r}]]] + \frac{1}{4}[x^-_{2,0}, [x^-_{2,0}, [x^+_{2,0}, [x^+_{2,0}, X^+_{1,r}]]]]) =$$
$$\exp\left(\mathrm{ad}x^+_{2,0}\right)(\frac{1}{2}[x^+_{2,0}, [x^+_{2,0}, X^+_{1,r}]]) =$$
$$\frac{1}{2}[x^+_{2,0}, [x^+_{2,0}, X^+_{1,r}]].$$

□

We will now use the τ_is to prove straightening formulas of the positive real root vectors.

Lemma 8.14. *The following identities hold in $\tilde{\mathcal{U}}^+[[u,v]]$*

$$i)\ \exp\left(x^+_{1,r}u\right)\exp\left(x^+_{2,s}v\right) = \exp\left(x^+_{2,s}v\right)\exp\left(x^+_{1,r}u\right)\exp\left(x^+_{\alpha_1+\alpha_2,r+s}uv\right)\exp\left((-1)^{r+1}x^+_{2\alpha_1+\alpha_2,2r+s}u^2v\right), \tag{8.14.1}$$

$$ii)\ \exp\left(x^+_{1,r}u\right)\exp\left(x^+_{\alpha_1+\alpha_2,r}v\right) = \exp\left(x^+_{\alpha_1+\alpha_2,r}v\right)\exp\left(2(-1)^r x^+_{2\alpha_1+\alpha_2,r+s}uv\right)\exp\left(x^+_{1,r}u\right), \tag{8.14.2}$$

$$iii)\ \exp\left(\frac{1}{2}X^+_{1,r}u\right)\exp\left(x^+_{2,r}v\right) = \exp\left(x^+_{2,s}v\right)\exp\left(\frac{1}{2}X^+_{1,r}u\right)\exp\left(2x^+_{2\alpha_1+\alpha_2,r+s}uv\right), \tag{8.14.3}$$

$$iv)\ \exp\left(x^+_{2,r}u\right)\exp\left(x^+_{2\alpha_1+\alpha_2,s}v\right) = \exp\left(x^+_{2\alpha_1+\alpha_2,r}v\right)\exp\left(-\frac{1}{2}X^+_{2,r+s}uv\right)\exp\left(x^+_{2,s}u\right),\ \textit{if } r+s \textit{ is odd} \tag{8.14.4}$$

Proof. Proof of Equation (8.14.1):
From Lemma 1.55,vi) follows that

$$\exp\left(x^+_{1,r}u\right)\exp\left(x^+_{2,s}v\right) = \exp\left(x^+_{2,s}v\right)\exp\left(x^+_{1,r}u + [x^+_{1,r}, x^+_{2,s}]uv\right) =$$
$$= \exp\left(x^+_{2,s}v\right)\exp\left(x^+_{1,r}u\right)\exp\left([x^+_{1,r}, x^+_{2,s}]uv\right)\exp\left(-\frac{1}{2}[x^+_{1,r}, [x^+_{1,r}, x^+_{2,s}]]u^2v\right))$$

where the last equality follows from lemma 1.55,viii).
Using Relations (8.1.1) and (8.1.2) follows that

$$[x^+_{1,r}, x^+_{2,s}] = -[x^+_{2,0}, x^+_{1,r+s}] = -x^+_{\alpha_1+\alpha_2,s+r}$$

and

$$-\frac{1}{2}[x^+_{1,r}, [x^+_{1,r}, x^+_{2,s}]] = (-1)^{r+1}([x^+_{1,0}, [x^+_{1,0}, x^+_{2,s+2r}]) = (-1)^{r+1}x^+_{2\alpha_1+\alpha_2,s+2r}$$

Proof of Equation (8.14.2).
From Lemma 1.55,iv) follows that

$$\exp\left(x^+_{1,r}u\right)\exp\left(x^+_{\alpha_1+\alpha_2,r}v\right),$$
$$= \exp\left(x^+_{\alpha_1+\alpha_2,r}v\right)\exp\left([x^+_{1,r}, x^+_{\alpha_1+\alpha_2,s}]uv\right)\exp\left(x^+_{1,r}u\right)$$

Using Relations (8.1.1) and (8.1.2) we have that

$$[x^+_{1,r}, x^+_{\alpha_1+\alpha_2,s}] = [[x^+_{1,r}, [x^+_{2,0}, x^+_{1,s}]] =$$
$$-[[x^+_{1,r}, [x^+_{1,s}, x^+_{2,0}]] = -[x^+_{1,r}, [x^+_{1,r}, x^+_{2,s-r}]] =$$
$$(-1)^r[x^+_{1,0}, [x^+_{1,0}, x^+_{2,s+r}]] = 2(-1)^r x^+_{2\alpha_1+\alpha_2,r+s}.$$

Proof of Equation (8.14.3).
From Lemma 1.55,vi) we get:

$$\exp\left(X^+_{1,r}u\right)\exp\left(x^+_{2,s}v\right) = \exp\left(x^+_{2,s}v\right)\exp\left(X^+_{1,r}u\right)\exp\left([X^+_{1,r}, x^+_{2,s}]uv\right)$$

the claim follows observing that:

$$\begin{aligned}&[X_{1,r}^+, x_{2,s}^+] = [[x_{1,r}^+, x_{1,0}^+], x_{2,s}^+] =\\&-[x_{2,s}^+, [x_{1,r}^+, x_{1,0}^+]] =\\&\big([x_{1,0}^+, [x_{2,s}^+, x_{1,r}^+]] + [x_{1,r}^+, [x_{1,0}^+, x_{2,s}^+]]\big) =\\&[x_{1,0}^+, [x_{1,0}^+, x_{2,s+r}^+]] - [x_{1,r}^+, [x_{1,r}^+, x_{2,s-r}^+]]\big) =\\&2x_{2\alpha_1+\alpha_2,s+r}^+ + (-1)^{r+1}[x_{1,0}^+, [x_{1,0}^+, x_{2,s+r}^+]]\big) =\\&4x_{2\alpha_1+\alpha_2,s+r}^+.\end{aligned}$$

Proof of Equation (8.14.4).
From Lemma 1.55,iv) follows that

$$\exp\left(x_{2,r}^+ u\right)\exp\left(x_{2\alpha_1+\alpha_2,s}^+ v\right) = \exp\left(x_{2\alpha_1+\alpha_2,r}^+ v\right)\exp\left([x_{2\alpha_1+\alpha_2,r}^+, x_{2,r}^+]uv\right)\exp\left(x_{2,s}^+ u\right)$$

hence the claim follows observing that:

$$\begin{aligned}&[x_{2\alpha_1+\alpha_2,r}^+, x_{2,s}^+] = \frac{1}{2}[[x_{1,0}^+, [x_{1,0}^+, x_{2,r}^+]], x_{2,s}^+] =\\&-\frac{1}{2}[[x_{2,s}^+, x_{1,0}^+], [x_{1,0}^+, x_{2,r}^+]] = -\frac{1}{2}X_{2,r+s}^+.\end{aligned}$$

□

Corollary 8.15. *$\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} \subseteq \bar{\mathcal{U}}_{\mathbb{Z}}$, more precisely :*

1. *$(x_{\alpha_1+\alpha_2,r}^+)^{(k)}$, $(x_{2\alpha_1+\alpha_2,r}^+)^{(k)}$ and $(\frac{1}{2}X_{2,2r+1}^+)^{(k)}$ belong to the $\mathbb{Z}$-subalgebra of $\bar{\mathcal{U}}$ generated by $(x_{i,r}^+)^{(k)}$, in particular they belong to $\bar{\mathcal{U}}_{\mathbb{Z}} \cap \bar{\mathcal{U}}^+$.*
2. *$(\frac{1}{2}X_{1,2r+1}^+)^{(k)} \in \bar{\mathcal{U}}_{\mathbb{Z}} \cap \bar{\mathcal{U}}^+$ even if it does not belong to the $\mathbb{Z}$-subalgebra generated by $(x_{i,r}^+)^{(k)}$.*

Proof. 1. From Lemma 8.14,i) it follows that

$$\exp\left(x_{\alpha_1+\alpha_2,r}^+ uv\right)\exp\left(x_{2\alpha_1+\alpha_2,r}^+ u^2 v\right) \in \mathbb{Z}((x_{i,r}^+)^{(k)} \mid i \in I, r \in \mathbb{Z}, k \in \mathbb{N})[[u,v]],$$

then considering the coefficients of $u^k v^k$ and of $u^{2k} v^k$ we get that

$$x_{\alpha_1+\alpha_2,r}^+, x_{2\alpha_1+\alpha_2,r}^+ u^2 v \in \mathbb{Z}((x_{i,r}^+)^{(k)} \mid i \in I, r \in \mathbb{Z}, k \in \mathbb{N}),$$

then Lemma 8.14,iv) implies that

$$(\frac{1}{2}X_{2,2r+1}^+)^{(k)} \in \mathbb{Z}((x_{i,r}^+)^{(k)} \mid i \in I, r \in \mathbb{Z}, k \in \mathbb{N});$$

2. $\bar{\mathcal{U}}_{\mathbb{Z}}$ is τ_2-invariant, hence

$$\mathcal{U}^+ \ni (\frac{1}{2}X_{1,2r+1}^+)^{(k)} = \tau_2(\frac{1}{2}X_{2,2r+1}^+)^{(k)} \in \bar{\mathcal{U}}_{\mathbb{Z}},$$

but $(\frac{1}{2}X_{1,2r+1}^+)^{(k)} \in \mathbb{Z}((x_{i,r}^+)^{(k)} \mid i \in I, r \in \mathbb{Z}, k \in \mathbb{N})$ (see Chapter 7).

□

Theorem 8.16. *$\bar{\mathcal{U}}_{\mathbb{Z}}^+ \subseteq \bar{\mathcal{U}}_{\mathbb{Z}} \cap \bar{\mathcal{U}}^+$ and $\bar{\mathcal{U}}_{\mathbb{Z}}^- \subseteq \bar{\mathcal{U}}_{\mathbb{Z}} \cap \bar{\mathcal{U}}^-$ are integral form of $\bar{\mathcal{U}}^+$ and $\bar{\mathcal{U}}^-$, a $\mathbb{Z}$-basis of $\bar{\mathcal{U}}_{\mathbb{Z}}^\pm$ is given by the ordered monomials of the set:*

$$\{(x_{\alpha,r}^\pm)^{(k)}, (\frac{1}{2}X_{i,2r+1}^\pm)^{(k)} \mid \alpha \in \Phi_0^+, i \in I, r \in \mathbb{Z}, k \in \mathbb{N}\}.$$

Proof. From Lemma 8.14 follows that the $\mathbb{Z}$-subalgebra of $\bar{\mathcal{U}}_{\mathbb{Z}}$ generated by $\{(x_{i,r}^+)^{(k)} \mid i \in I, r \in \mathbb{Z}, k \in \mathbb{N}\}$ has basis consisting in the ordered monomials in the set

$$\{(x_{\alpha,r}^+)^{(k)}, (\frac{1}{2}X_{2,2r+1}^+)^{(k)}, (X_{1,2r+1}^+)^{(k)} \mid \alpha \in \Phi_0^+, r \in \mathbb{Z}, k \in \mathbb{N}\},$$

moreover,

$$W_T \cdot \{(x_{\alpha,r}^+)^{(k)}, (\frac{1}{2}X_{2,2r+1}^+)^{(k)}, (X_{1,2r+1}^+)^{(k)} \mid \alpha \in \Phi_0^+, r \in \mathbb{Z}, k \in \mathbb{N}\} =$$
$$\{(x_{\alpha,r}^\pm)^{(k)}, (\frac{1}{2}X_{i,2r+1}^\pm)^{(k)} \mid \alpha \in \Phi_0^+, i \in I, r \in \mathbb{Z}, k \in \mathbb{N}\},$$

then the claim follows observing that

$$\tilde{\mathcal{U}}^+ \cap \{(x_{\alpha,r}^\pm)^{(k)}, (\frac{1}{2}X_{i,2r+1}^\pm)^{(k)} \mid \alpha \in \Phi_0^+, i \in I, r \in \mathbb{Z}, k \in \mathbb{N}\} =$$
$$\{(x_{\alpha,r}^+)^{(k)}, (\frac{1}{2}X_{i,2r+1}^+)^{(k)} \mid \alpha \in \Phi_0^+, i \in I, r \in \mathbb{Z}, k \in \mathbb{N}\}.$$

□

Proposition 8.17. *The following identities hold in $\bar{\mathcal{U}}[[u,v]]$:*

$$\check{h}_1^+(u)\hat{h}_2^-(v) = \hat{h}_2^-(v)(1-uv)^c\check{h}_1^+(u). \tag{8.17.1}$$

In particular $\bar{\mathcal{U}}_{\mathbb{Z}}^0 = \bar{\mathcal{U}}_{\mathbb{Z}}^{0,-}\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ and $\bar{\mathcal{U}}_{\mathbb{Z}}^0$ is an integral form of $\tilde{\mathcal{U}}^0$. $\check{\mathcal{U}}_{\mathbb{Z}}^0 = \check{\mathcal{U}}_{\mathbb{Z}}^{0,-}\check{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\check{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ and $\check{\mathcal{U}}_{\mathbb{Z}}^0$ is an integral form of $\tilde{\mathcal{U}}^0$.

Proof. Equation (8.17.1) follows from Propositions 1.58 with $m = 1$ and $l = 0$, hence $\check{\mathcal{U}}_{\mathbb{Z}}^0 = \check{\mathcal{U}}_{\mathbb{Z}}^{0,-}\check{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\check{\mathcal{U}}_{\mathbb{Z}}^{0,+}$. □

Corollary 8.18. $\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} = \bar{\mathcal{U}}^{\mathfrak{h}} \cap \bar{\mathcal{U}}_{\mathbb{Z}}$

Proof. The claim follows by Corollary 7.17, Proposition 8.17 and Definition 8.9. □

Corollary 8.19. $\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} = \bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}$

Proof. From Proposition 1.56 with $m = a_{i,j}$ we have that

$$(x_{i,r}^+)^{(k)}\binom{h_{0,j}}{l} = \binom{h_{0,j} - a_{i,j}}{l}(x_{i,r}^+)^{(k)},$$

from Proposition 1.56 with $m = a_{2,1}^2 = 4$ by multiplying both side for $(\frac{1}{2})^k$ we have that:

$$(\frac{1}{2}X_{1,2r+1}^+)^{(k)}\binom{h_{0,2}}{l} = \binom{h_{0,2} - 4k}{l}(\frac{1}{2}X_{1,2r+1}^+)^{(k)}.$$

Hence we have that

$$\bar{\mathcal{U}}_{\mathbb{Z}}^{+}\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} = \bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\bar{\mathcal{U}}_{\mathbb{Z}}^{+}, \tag{8.19.1}$$

remarking that the $(\frac{1}{2}X_{1,2r+1}^+)^{(k)}$'s and $(x_{i,r}^+)^{(k)}$s generate $\bar{\mathcal{U}}_{\mathbb{Z}}$, then by applying Ω to Relation (8.19.1) we get

$$\bar{\mathcal{U}}_{\mathbb{Z}}^{-}\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} = \bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\bar{\mathcal{U}}_{\mathbb{Z}}^{-}.$$

□

Proposition 8.20. *The following identities hold in $\bar{\mathcal{U}}[[u]]$*

$$x_{1,0}^+\hat{h}_2^+(u) = \hat{h}_2^+(u)(1+uT^{-1})(x_{1,0}^+), \tag{8.20.1}$$
$$x_{2,0}^+\check{h}_1^+(u) = \check{h}_1^+(u)(1+uT^{-1})(x_{2,0}^+) \tag{8.20.2}$$
$$\frac{1}{2}X_{1,1}^+\hat{h}_2^+(u) = \hat{h}_2^+(u)(1+Tu^2)(\frac{1}{2}X_{1,1}^+). \tag{8.20.3}$$

hence for all $k \in \mathbb{N}$

$$(x_{1,0}^+)^{(k)}\hat{h}_2^+(u) = \hat{h}_2^+(u)\big((1+uT^{-1})(x_{1,0}^+)\big)^{(k)} \in \bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}\bar{\mathcal{U}}_{\mathbb{Z}}^+, \tag{8.20.4}$$
$$(x_{2,0}^+)^{(k)}\check{h}_1^+(u) = \check{h}_1^+(u)((1+uT^{-1})(x_{2,0}^+))^{(k)} \in \check{\mathcal{U}}_{\mathbb{Z}}^{0,+}\bar{\mathcal{U}}_{\mathbb{Z}}^+ \tag{8.20.5}$$
$$(\frac{1}{2}X_{1,1}^+)^{(k)}\hat{h}_2^+(u) = \hat{h}_2^+(u)((1+Tu^2)(\frac{1}{2}X_{1,1}^+))^{(k)} \in \bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}\bar{\mathcal{U}}_{\mathbb{Z}}^+. \tag{8.20.6}$$

In particular $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm} = \bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}\bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}$, $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,-}\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm} = \bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}\bar{\mathcal{U}}_{\mathbb{Z}}^{0,-}$, $\check{\mathcal{U}}_{\mathbb{Z}}^{0,+}\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm} = \bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}\check{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ and $\check{\mathcal{U}}_{\mathbb{Z}}^{0,-}\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm} = \bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}\check{\mathcal{U}}_{\mathbb{Z}}^{0,-}$, moreover are integral form of respectively $\tilde{\mathcal{U}}^{\pm}\bar{\mathcal{U}}^{0,+}$ and $\tilde{\mathcal{U}}^{\pm}\bar{\mathcal{U}}^{0,-}$. $\check{\mathcal{U}}_{\mathbb{Z}}^{0}\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm} = \bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}\check{\mathcal{U}}_{\mathbb{Z}}^{0}$ and $\bar{\mathcal{U}}_{\mathbb{Z}}^{0}\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm} = \bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}\bar{\mathcal{U}}_{\mathbb{Z}}^{0}$ are integral form of $\tilde{\mathcal{U}}^{\pm}\bar{\mathcal{U}}^{0}$.

Proof. Equations (8.20.1) and (8.20.2) and follow from Proposition 1.60 with $m_1 = -1$ and $m_d = 0$ if $d > 1$, Equation (8.20.3) follows (8.20.1) and (8.20.2). Equations (8.20.4), (8.20.5) and (8.20.6) follow from (8.20.1) and (8.20.2) since $\bar{\mathcal{U}}_{\mathbb{Z}}^+$ is T-stable and $T|_{\bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}} = \mathrm{id}$. $\bar{\mathcal{U}}_{\mathbb{Z}}^+\bar{\mathcal{U}}_{\mathbb{Z}}^{0,+} = \bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}\bar{\mathcal{U}}_{\mathbb{Z}}^+$ and $\bar{\mathcal{U}}_{\mathbb{Z}}^+\check{\mathcal{U}}_{\mathbb{Z}}^{0,+} = \check{\mathcal{U}}_{\mathbb{Z}}^{0,+}\bar{\mathcal{U}}_{\mathbb{Z}}^+$ follow directly, the others follow by applying $\Omega \circ \sigma$ and λ_{-1}. The last Relation follows from previous relation and Corollary 8.19. □

Remark 8.21. *We know from Chapter 7 that $\hat{h}_1^+(u), \bar{h}_1^+(u) \in \bar{\mathcal{U}}_{\mathbb{Z}}$, hence $\bar{\mathcal{U}}_{\mathbb{Z}}^0 \subseteq \bar{\mathcal{U}}_{\mathbb{Z}}$.*

Theorem 8.22. *$\bar{\mathcal{U}}_{\mathbb{Z}}^+\bar{\mathcal{U}}_{\mathbb{Z}}^- \subseteq \bar{\mathcal{U}}_{\mathbb{Z}}^-\bar{\mathcal{U}}_{\mathbb{Z}}^0\bar{\mathcal{U}}_{\mathbb{Z}}^+$ is a $\mathbb{Z}$-subalgebra of $\bar{\mathcal{U}}$: it is an integral form of $\mathcal{U}(A_4^{(2)})$. Then:*

- *$\bar{\mathcal{U}}_{\mathbb{Z}}^+\bar{\mathcal{U}}_{\mathbb{Z}}^0\bar{\mathcal{U}}_{\mathbb{Z}}^- = \bar{\mathcal{U}}_{\mathbb{Z}}$ (hence $\bar{\mathcal{U}}_{\mathbb{Z}}$ is an integral form of $\bar{\mathcal{U}}$),*
- *$\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm} = \bar{\mathcal{U}}_{\mathbb{Z}} \cap \bar{\mathcal{U}}^{\pm}$;*
- *$\bar{\mathcal{U}}_{\mathbb{Z}}^{0} = \bar{\mathcal{U}}_{\mathbb{Z}} \cap \bar{\mathcal{U}}^{0}$;*
- *$\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} = \bar{\mathcal{U}}_{\mathbb{Z}} \cap \bar{\mathcal{U}}^{\mathfrak{h}}$;*
- *$\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} = \bar{\mathcal{U}}_{\mathbb{Z}} \cap \bar{\mathcal{U}}^{0,\pm}$;*

Theorem 8.23. *$\bar{\mathcal{U}}_{\mathbb{Z}}^+\check{\mathcal{U}}_{\mathbb{Z}}^0\bar{\mathcal{U}}_{\mathbb{Z}}^- \supsetneq \bar{\mathcal{U}}_{\mathbb{Z}}$ is an integral form of $\mathcal{U}(A_4^{(2)})$.*

8.3 $A_{2n}^{(2)}$

We want now prove that $\bar{\mathcal{U}}_{\mathbb{Z}}$ is an integral form of $\bar{\mathcal{U}}$. We will prove it by induction on n, we will systematically refer to the identifications defined in Definition 8.9. The claim for $n = 2$ is the Section 8.2.

Remark 8.24. *Recall that:*
The results for $A_4^{(2)}$ and $A_1^{(1)}$ shows that

$$\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}, \bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}, \bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} \subseteq \bar{\mathcal{U}},$$

in particular the results for $A_4^{(2)}$ *and* $A_{n-1}^{(1)}$ *imply that*

$$\bar{\mathcal{U}}_{\mathbb{Z}}^{+}\bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\bar{\mathcal{U}}_{\mathbb{Z}}^{0,-}\bar{\mathcal{U}}_{\mathbb{Z}}^{-}$$

is a $\mathbb{Z}$ *subalgebra of* $\bar{\mathcal{U}}_{\mathbb{Z}}$.

It follows from previous remark that to our aim we just need to study $\bar{\mathcal{U}}_{\mathbb{Z}}^{+}$ and to prove that it is the $\mathbb{Z}$ linear span of the monomials in

$$\{(x_{\alpha,r}^{+})^{(k)}, (\frac{1}{2}X_{i,2r+1}^{+})^{(k)} \mid r \in \mathbb{Z}, \alpha \in \Phi_0^{+}, i \in I\}.$$

Notation 8.25. *Let* $\mathbb{Z} < B_n^{+} >$ *be the* $\mathbb{Z}$ *linear span of the monomials in*

$$\{(x_{\alpha,r}^{+})^{(k)}, (\frac{1}{2}X_{i,2r+1}^{+})^{(k)} \mid r \in \mathbb{Z}, \alpha \in \Phi_0^{+}, i \in I\},$$

We want to prove that

$$\bar{\mathcal{U}}_{\mathbb{Z}}^{+} = \mathbb{Z} < B_n^{+} >,$$

we shall proceed by induction on n, the case of $n = 2$ is the previous section.

Remark 8.26. $\bar{\mathcal{U}}_{\mathbb{Z}}^{+}(A_{2(n-1)}^{(2)}), \mathcal{U}_{\mathbb{Z}}^{+}(A_{n-1}^{(1)}) \subseteq \mathbb{Z} < B_n^{+} >$

Remark 8.27.

$$\tau_n(B_{n-1}^{1}) \subseteq B_n^{+}.$$

Notation 8.28. *Let us set denote by* $\bar{\Phi}_0^{+}$ *and* $\tilde{\Phi}_0^{+}$ *denotes the sub-root system* Φ_0^{+} *of respectively* $A_{2n-2}^{(2)}$ *and* $A_{2n-1}^{(1)}$ *via the Identifications* (8.9.1) *and* (8.9.2).

Remark 8.29. *Let us observe that from Theorem 8.16 follows that*

$$\bar{\mathcal{U}}_{\mathbb{Z}}^{+} \subseteq \mathbb{Z}((x_{i,r}^{+})^{(k)}, (\frac{1}{2}X_{1,2r+1}^{+})^{(k)} \mid i \in I, r \in \mathbb{Z}, k \in \mathbb{N}).$$

We want to prove that $\bar{\mathcal{U}}_{\mathbb{Z}}^{+}$ *is an algebra of divided powers, whose basis is given by the ordered divided powers monomials in the elements:*

$$\{x_{\alpha,r}^{+}, \frac{1}{2}X_{i,2r+1}^{+} \mid i \in I, r \in \mathbb{Z}, \alpha \in \Phi_0^{+}\}. \tag{8.29.1}$$

By Identification (8.9.2) *and by induction hypothesis it follows that*

$$\mathbb{Z}^{(Div)}[x_{\alpha,r}^{+}, \frac{1}{2}X_{i,2r+1}^{+} \mid i \in I \setminus \{n\}, r \in \mathbb{Z}, \alpha \in \bar{\Phi}_0^{+}) \subseteq \bar{\mathcal{U}}_{\mathbb{Z}}^{+}.$$

By Identification (8.9.2) *we have that*

$$\mathbb{Z}^{(Div)}[x_{\alpha,r}^{+} \mid r \in \mathbb{Z}, \alpha \in \tilde{\Phi}_0^{+}] \subseteq \bar{\mathcal{U}}_{\mathbb{Z}}^{+}.$$

If we prove that $(x_{\alpha,r}^{+})^{(k)}$, $(\frac{1}{2}X_{n,2r+1}^{+})^{(k)}$ $\forall \alpha \in \Phi_0^{+} \setminus (\bar{\Phi}_0^{+} \cup \tilde{\Phi}_0^{+})$, *then follows that*

$$\mathbb{Z}((x_{\alpha,r}^{+})^{(k)}, (\frac{1}{2}X_{i,2r+1}^{+})^{(k)} \mid i \in I, r \in \mathbb{Z}, \alpha \in \Phi_0^{+}, k \in \mathbb{N}) \subseteq \bar{\mathcal{U}}_{\mathbb{Z}}^{+},$$

then Relation (8.29.1) *turns out to be equivalent to show that*

- $(x^+_{\alpha,r})^{(k)}(x^+_{\beta,s})^{(j)}$ $\forall\alpha \in \Phi^+_0 \setminus (\bar\Phi^+_0 \cup \tilde\Phi^+_0)$,$\forall\beta \in \Phi^+_0$, $\forall r,s \in \mathbb{Z}$ *and* $\forall k,j \in \mathbb{N}$,
- $(x^+_{\alpha,r})^{(k)}(x^+_{\beta,s})^{(j)}$ $\forall\alpha \in \tilde\Phi^+_0 \setminus (\bar\Phi^+_0)$, $\forall\beta \in \bar\Phi^+_0 \setminus (\tilde\Phi^+_0)$, $\forall r,s \in \mathbb{Z}$ *and* $\forall k,j \in \mathbb{N}$,
- $(\frac{1}{2}X^+_{i,2r+1})^{(k)}(x^+_{\beta,s})^{(j)}$, $\forall i \in I$, $\forall\beta \in \Phi^+_0 \setminus \bar\Phi^+_0$, $\forall r,s \in \mathbb{Z}$ *and for all* $k,j \in \mathbb{N}$,

Proposition 8.30. $(x^+_{\alpha,r})^{(j)}, (\frac{1}{2}X^\pm_{n,2r+1})^{(k)} \in \bar{\mathcal{U}}^+_\mathbb{Z}$ *for all* $\alpha \in \bar\Phi^+_0 \cup \tilde\Phi^+_0$ *and* $j,k \in \mathbb{N}$.

Proof. By induction hypothesis $(\frac{1}{2}X^\pm_{i,2r+1})^{(k)} \in \bar{\mathcal{U}}^+_\mathbb{Z}$ if $i \neq n$, let us observe that

$$\tau_n((\frac{1}{2}X^\pm_{n-1,2r+1})^{(k)}) = (\frac{1}{2}X^\pm_{n,2r+1})^{(k)} \in \bar{\mathcal{U}}^+_\mathbb{Z}.$$

By Notation 8.28 follows that

$$\tilde\Phi^+_0 \cup \bar\Phi^+_0 = \{\alpha_1 + \cdots + \alpha_n, 2\alpha_1 + \cdots + 2\alpha_j + \cdots + \alpha_{j+1} + \cdots + \alpha_n\}$$

$(x^+_{\alpha,r})^{(j)} \in \bar{\mathcal{U}}^+_\mathbb{Z}$

$$\begin{aligned}
&\tau_n(x^+_{\alpha_1+\cdots+\alpha_{n-1},r}) = x^+_{\alpha_1+\cdots+\alpha_n,r},\\
&\tau_n(x^+_{2\alpha_1+\cdots+2\alpha_j+\cdots+\alpha_{j+1}+\cdots+\alpha_{n-1},r}) = x^+_{2\alpha_1+\cdots+2\alpha_j+\cdots+\alpha_{j+1}+\cdots+\alpha_n,r} \text{ if } j \neq n-1,\\
&\tau_{n-1}\tau_n(x^+_{2\alpha_1+\cdots+2\alpha_{n-2}+\alpha_{n-1},r}) = x^+_{2\alpha_1+\cdots+2\alpha_{n-1}+\alpha_n,r}.
\end{aligned}$$

□

Lemma 8.31. $(\frac{1}{2}X^+_{i,2r+1})^{(k)}(x^+_{\beta,s})^{(j)}$, $\forall i \in I$, $\forall\beta \in \Phi^+_0 \setminus \bar\Phi^+_0$, $\forall r,s \in \mathbb{Z}$ *and for all* $k,j \in \mathbb{N}$

Proof. Let us observe that $[(\frac{1}{2}X^+_{i,2r+1})^{(k)}, (x^+_{\beta,s})^{(j)}] \neq 0$ only if $\beta = \alpha_{i+1} + \cdots + \alpha_n$. If $i \neq n-1$ the claim follows by applying τ_n since in this case $\tau_n((\frac{1}{2}X^+_{i,2r+1})^{(k)}) = (\frac{1}{2}X^+_{i,2r+1})^{(k)}$ and $\sigma_n(\alpha_{i+1} + \cdots + \alpha_n) = \alpha_{i+1} + \cdots + \alpha_{n-1}$. If $i = n-1$ let us observe that $\tau_n\tau_{n-1}((\frac{1}{2}X^+_{n-1,2r+1})^{(k)}) = \tau_n(\frac{1}{2}X^+_{n-2,2r+1})^{(k)}$ and $\sigma_n\sigma_{n-1}(\alpha_n) = \alpha_{n-1}$, hence the claim follows by Remark 8.29. □

Lemma 8.32. $(x^+_{\alpha,r})^{(k)}(x^+_{\beta,s})^{(j)}$ $\forall\alpha \in \Phi^+_0 \setminus (\bar\Phi^+_0 \cup \tilde\Phi^+_0)$,$\forall\beta \in \Phi^+_0$, $\forall r,s \in \mathbb{Z}$ *and* $\forall k,j \in \mathbb{N}$,

Proof. Let us observe that

$$\Phi^+_0 \setminus (\bar\Phi^+_0 \cup \tilde\Phi^+_0) = \{\alpha_1 + \cdots + \alpha_n, 2\alpha_1 + \cdots + 2\alpha_j + \alpha_{j+1} + \cdots + \alpha_n\}.$$

If $\alpha = \alpha_1 + \cdots + \alpha_n$ then $\alpha + \beta \in \Phi^{0,+}$ only if $\beta = \alpha_1 + \cdots + \alpha_k$

- if $k = n$ then $\sigma_n(\alpha) = \sigma_n(\beta) = \alpha_1 + \cdots + \alpha_{n-1} \in \bar\Phi^+_0$,
- if $k < n-1$ then $\sigma_n(\alpha) = \alpha_1 + \cdots + \alpha_{n-1}, \sigma_n(\beta) = \beta \in \bar\Phi^+_0$,
- if $k = n-1$ then $\sigma_{n-1}(\alpha) = (\alpha), \sigma_{n-1}(\beta) = \alpha_1 + \cdots + \alpha_{n-2}$ hence we can lead back to case $k < n-1$.

If $\alpha = 2\alpha_1 + \cdots + 2\alpha_j + \alpha_{j+1} + \cdots + \alpha_n$ then $\alpha + \beta \in \Phi^{0,+}$ only if $\beta = \alpha_{j+1} + \cdots + \alpha_k$, let us first observe that if $j \neq 1$ then

$$\sigma_{j-1}(\alpha) = 2\alpha_1 + \cdots + 2\alpha_{j-1} + \alpha_j + \cdots + \alpha_n$$

and

$$\sigma_{j-1}(\beta) = \alpha_{j-1} + \cdots + \alpha_k,$$

hence we can assume that $j < n-1$.

- if $k = n$ then $\sigma_n\sigma_{n-1}(\alpha) = 2\alpha_1 + \cdots + 2\alpha_{n-2} + \alpha_{n-1}$, $\sigma_n\sigma_{n-1}(\beta) = \alpha_{n-1} \in \bar{\Phi}_0^+$,
- if $k < n$ then $\sigma_n(\alpha) = 2\alpha_1 + \cdots + 2\alpha_j + \alpha_{j+1} + \cdots + \alpha_{n-1}, \sigma_n(\beta) = \alpha_{j+1} + \cdots + \alpha_{n-1} \in \bar{\Phi}_0^+$.

□

Lemma 8.33. $(x_{\alpha,r}^+)^{(k)}(x_{\beta,s}^+)^{(j)}$ $\forall\alpha \in \tilde{\Phi}_0^+ \setminus (\bar{\Phi}_0^+)$, $\forall\beta \in \bar{\Phi}_0^+ \setminus (\tilde{\Phi}_0^+)$, $\forall r, s \in \mathbb{Z}$ *and* $\forall k, j \in \mathbb{N}$.

Proof. If $\alpha \in \bar{\Phi}_0^+ \setminus (\tilde{\Phi}_0^+)$ and $\beta \in \tilde{\Phi}_0^+ \setminus (\bar{\Phi}_0^+)$ this implies that $\alpha = \alpha_1 + \cdots + \alpha_k$ or $\alpha = 2\alpha_1 + \cdots + 2\alpha_j + \alpha_{j+1} + \cdots + \alpha_{n-1}$ and $\beta = \alpha_k + \cdots + \alpha_n$.

If $\alpha = 2\alpha_1 + \cdots + 2\alpha_j + \alpha_{j+1} + \cdots + \alpha_{n-1}$ and $\beta = \alpha_{j+1} + \cdots + \alpha_n$, let us observe if $j \neq 1$ then

$$\sigma_{j-1}(\alpha) = 2\alpha_1 + \cdots + 2\alpha_{j-1} + \alpha_j + \cdots + \alpha_n$$

and

$$\sigma_{j-1}(\beta) = \alpha_{j-1} + \cdots + \alpha_k,$$

then we can assume that $j < n - 1$, if $k \neq n$, then $\sigma(\alpha) \in \bar{\Phi}_0^+$ and $\sigma(\beta) =\in \bar{\Phi}_0^+$ if $k \neq n - 1$. If $\alpha = \alpha_1 + \cdots + \alpha_k$ and $\alpha = \alpha_j + \cdots + \alpha_n$ then $k = j + 1$, if $j \neq n$ the claim follows directly by applying σ_n, if $j \neq n$ we can lead back to Lemma 8.32. □

Theorem 8.34. $\bar{\mathcal{U}}_{\mathbb{Z}}^+$ *and* $\bar{\mathcal{U}}_{\mathbb{Z}}^-$ *are integral form of* $\tilde{\mathcal{U}}^+$ *and* $\tilde{\mathcal{U}}^-$*, more precisely*

$$\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm} = \mathbb{Z}^{(div)}[x_{\alpha,r}^{\pm}, \frac{1}{2}X_{i,2r+1}^{\pm} \mid \alpha \in \Phi_0^+, i \in I, r \in \mathbb{Z}],$$

a $\mathbb{Z}$*-basis* $B^{\pm}$ *of* $\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}$ *a base is given by the divided powers of the elements of the set* $\{x_{\alpha,r}^{\pm}, \frac{1}{2}X_{i,2r+1}^{\pm}, \alpha \in \Phi_0^+, i \in I, r \in \mathbb{Z}\}$.

Proof. The claim follows from Remark 8.29, Proposition 8.30 and Lemmas 8.31, 8.32 and 8.33. □

Lemma 8.35.

$$\bar{\mathcal{U}}_{\mathbb{Z}}^{0,-}\bar{\mathcal{U}}_{\mathbb{Z}}^{0,+} = \bar{\mathcal{U}}_{\mathbb{Z}}^{0,+} \otimes \mathbb{Z}^{(bin)}[c] \otimes \bar{\mathcal{U}}_{\mathbb{Z}}^{0,-}$$

Proof. The claim follows from Theorems 4.25 and 7.14 and Definition 8.9 □

Lemma 8.36. $\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} = \bar{\mathcal{U}}_{\mathbb{Z}} \cap \tilde{\mathcal{U}}^{\mathfrak{h}} = \mathbb{Z}^{(bin)}[h_i, c \mid i \in I]$

Proof. The claim follows observing that $\bar{\mathcal{U}}_{\mathbb{Z}} \cap \tilde{\mathcal{U}}^{\mathfrak{h}} \cap \tilde{\mathcal{U}}_i = \mathbb{Z}^{(bin)}[h_i, c]$. □

Remark 8.37. $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$ *is an integral form of* $\tilde{\mathcal{U}}^{0,\pm}$*, more precisely:*

$$\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm} \cong \mathbb{Z}^{(mix)}[h_{1,r} \mid \pm r > 0] \otimes \mathbb{Z}[\hat{h}_{i,r} \mid i \in I, i \neq 1, \pm r > 0],$$

a basis is given by the product $B^{0,\pm} = B_{1,q.pol}^{0,\pm} \prod_{i=2}^{n} \hat{B}_i^{0,\pm}$*, where* $B_{1,q.pol}^{0,\pm} = \phi_1(B_{q.pol}^{0,\pm})$ *and* $\hat{B}_i^{0,\pm} = \phi_i(\hat{B}^{0,\pm})$ *if* $i > 1$ *(see Theorems 4.25 and 7.14 and Definition 8.9).*

Proposition 8.38.

$$\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}\bar{\mathcal{U}}_{\mathbb{Z}}^{0} = \bar{\mathcal{U}}_{\mathbb{Z}}^{0}\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}$$

Proof. Let us prove first that $\bar{\mathcal{U}}_{\mathbb{Z}}^{+}\bar{\mathcal{U}}_{\mathbb{Z}}^{0} \subseteq \bar{\mathcal{U}}_{\mathbb{Z}}^{0}\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}$, the claim is equivalent to show that for all $i, j \in I$ then $(x_{i,r}^+)^{(k)}(\bar{\mathcal{U}}_{\mathbb{Z}}^{0} \cap \bar{\mathcal{U}}_j) \subseteq \bar{\mathcal{U}}_{\mathbb{Z}}^{0}\bar{\mathcal{U}}_{\mathbb{Z}}^{+}$, if $|i - j| > 1$ the claim is obvious, if $i, j < n$ follows from Identification (8.9.1), if $i = n$ or $j = n$ follows from Identification (8.9.2). □

Theorem 8.39. *The $\mathbb{Z}$-subalgebra $\bar{\mathcal{U}}_{\mathbb{Z}}$ of $\tilde{\mathcal{U}}$ generated by*

$$\{(x_{i,r}^{+})^{(k)}, (x_{i,s}^{-})^{(k)} \mid r \in \mathbb{Z}, k \in \mathbb{N}, i \in I\}$$

is an integral form of $\tilde{\mathcal{U}}$. More precisely

$$\bar{\mathcal{U}}_{\mathbb{Z}} \cong \bar{\mathcal{U}}_{\mathbb{Z}}^{-} \otimes \bar{\mathcal{U}}_{\mathbb{Z}}^{0} \otimes \bar{\mathcal{U}}_{\mathbb{Z}}^{+},$$

a $\mathbb{Z}$-basis of $\bar{\mathcal{U}}_{\mathbb{Z}}$ is given by the product $B^{-}B^{0}B^{+}$ where $B^{\pm}$ and $\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}$ are described in Theorem 8.34 and

$$\bar{\mathcal{U}}_{\mathbb{Z}}^{0} \cong \bar{\mathcal{U}}_{\mathbb{Z}}^{0,-} \otimes \bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}} \otimes \bar{\mathcal{U}}_{\mathbb{Z}}^{0,+}.$$

a $\mathbb{Z}$-basis of $\bar{\mathcal{U}}_{\mathbb{Z}}^{0}$ is given by the product $B^{0} = B^{-}B^{0}B^{+}$ where $B^{\pm}$ and B^{0} are basis of respectively $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$ and $\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ are described respectively in Remark 8.37 in Lemma 8.36.

8.4 Conclusions

The study of the integral form of the affine Kac-Moody algebras from the point of view of the Drinfeld presentation, which differs from the one defined through the Kac-Moody presentation ([6] and [11]) in the case $A_{2n}^{(2)}$ as outlined above, is motivated by the interest in the representation theory over $\mathbb{Z}$, since for the affine Kac-Moody algebras the notion of highest weight vector with respect to the e_i's has been usefully replaced with that defined through the action of the $x_{i,r}^{+}$'s (see the works of Chari and Pressley [1] and [13]): in order to study what happens over the integers it is useful to work with an integral form defined in terms of the same $x_{i,r}^{+}$'s.
This work is also intended to be the preliminary classical step in the project of constructing and describing the *quantum* integral form for the twisted affine quantum algebras (with respect to the Drinfeld presentation). The commutation relations involved are extremely complicated and appear to be unworkable by hands without a deeper insight; we hope that a simplified approach can open a viable way to work in the quantum setting, this is the reason why (in the case of $A_{4}^{(2)}$) it has been shown that $\check{\mathcal{U}}_{\mathbb{Z}}$ is also an integral form.

Chapter 9

Appendices

9.A Straightening formulas of $A_2^{(2)}$

For the sake of completeness we collect here the commutation formulas of $A_2^{(2)}$, inserting also the formulas that we didn't need for the proof of Theorem 5.44.

Notation 9.1 and Remark 9.2 will help writing some of the following straightening relations and to understand the origin of some apparently mysterious terms.

Notation 9.1. *Given $p(t) \in \mathbb{Q}[[t]]$ let us define $p_+(t), p_-(t) \in \mathbb{Q}[[t^2]]$ and $p_0(t) \in \mathbb{Q}[[t]]$ by*

$$p(t) = p_+(t) + tp_-(t),\ p_0(t^2) = \frac{1}{2}p_+(t)p_-(t).$$

Remark that the maps $p(t) \mapsto p_+(t)$ and $p(t) \mapsto p_-(t)$ are homomorphisms of $\mathbb{Q}[[t^2]]$-modules while $q(t) \in \mathbb{Q}[[t^2]], \tilde{q}(t^2) = q(t) \Rightarrow (qp)_0(t) = \tilde{q}(t)^2 p_0(t)$.

Remark 9.2. *Given $p(t) \in \mathbb{Q}[[t]]$, Lemma 1.55, viii) implies that*

$$\exp\big(p(uw).x_0^+\big) =$$
$$= \exp\big(p_+(uw).x_0^+\big) \exp\Big(up_0(-u^2w).X_1^+\Big) \exp\big(up_-(uw).x_1^+\big) =$$
$$= \exp\big(up_-(uw).x_1^+\big) \exp\Big(-up_0(-u^2w).X_1^+\Big) \exp\big(p_+(uw).x_0^+\big).$$

We shall now list a complete set of **straightening formulas** in $\tilde{\mathcal{U}}_{\mathbb{Z}}$.

I) Zero commutations regarding $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$:

$$\binom{c}{k} \text{ is central in } \tilde{\mathcal{U}}_{\mathbb{Z}};$$

$$\binom{h_0}{k} \text{ is central in } \tilde{\mathcal{U}}_{\mathbb{Z}}^{0} : \left[\binom{h_0}{k}, \tilde{h}_l\right] = 0\ \forall k \geq 0,\ l \neq 0.$$

II) Relations in $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ (from which those in $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-}$ follow as well):

$$\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+} \text{ is commutative}:\ [\tilde{h}_k, \tilde{h}_l] = 0\ \forall k, l > 0;$$

$$\tilde{\lambda}_m(\tilde{h}_+(-u^m)) = \prod_{j=1}^{m} \tilde{h}_+(-\omega^j u)\ \forall m \in \mathbb{Z}_+$$

where ω is a primitive m^{th} root of 1 (see Proposition 1.14 and Remark 5.13), that is

$$\tilde\lambda_m(\tilde h_k)=(-1)^{(m-1)k}\sum_{\substack{(k_1,\dots,k_m):\\ k_1+\dots+k_m=mk}}\omega^{\sum_{j=1}^m jk_j}\tilde h_{k_1}\dots\tilde h_{k_m};$$

if m is odd

$$\lambda_m(\tilde h_k)=\tilde\lambda_m(\tilde h_k)\ \forall k\geq 0;$$

if m is even

$$\lambda_m(\hat h_+(u))=\tilde\lambda_m(\tilde h_+((-1)^{\frac{m}{2}}u)^{-1}).$$

In order to describe the dependence of $\hat h^{\{d\}}(u)$ on the $\tilde\lambda_m(\tilde h_k)$'s (where d is as defined in Remark 5.39) remark first that

$$\hat h_+(u)=\tilde h_+(u)\tilde\lambda_4(\tilde h_+(-u^4)^{-\frac12})=\tilde h_+(u)^{\frac12}\tilde h_+(-u)^{-\frac12}\tilde h_+(iu)^{-\frac12}\tilde h_+(-iu)^{-\frac12},$$

so that

$$\hat h_+^{\{d\}}(u)=\hat h_+((1+\sqrt2)u)^{\frac12}\hat h_+((1-\sqrt2)u)^{\frac12}=$$
$$=\tilde h_+((1+\sqrt2)u)^{\frac14}\tilde h_+((1-\sqrt2)u)^{\frac14}\tilde h_+(-(1+\sqrt2)u)^{-\frac14}\tilde h_+(-(1-\sqrt2)u)^{-\frac14}\cdot$$
$$\cdot(\tilde h_+((1+\sqrt2)iu)^{-\frac14}\tilde h_+((1-\sqrt2)iu)^{-\frac14}\tilde h_+(-(1+\sqrt2)iu)^{-\frac14}\tilde h_+(-(1-\sqrt2)iu)^{-\frac14}.$$

Now recall that through the involution $h_r\mapsto\varepsilon_r h_r\ \forall r>0$ (see Remark 5.13) $\hat h(u)$ corresponds to $\tilde h(u)$ and λ_m corresponds to $\tilde\lambda_m$, so that our problem is equivalent to describing

$$\begin{aligned}&\hat h_+((1+\sqrt2)u)\hat h_+((1-\sqrt2)u)\hat h_+(-(1+\sqrt2)u)^{-1}\hat h_+(-(1-\sqrt2)u)^{-1}\cdot\\ &\cdot\big(\hat h_+((1+\sqrt2)iu)\hat h_+((1-\sqrt2)iu)\hat h_+(-(1+\sqrt2)iu)\hat h_+(-(1-\sqrt2)iu)\big)^{-1}\end{aligned}\tag{9.2.1}$$

in terms of the $(\lambda_m(\hat h_k))^4$'s; since Remark 1.16 implies that (9.2.1) corresponds to

$$\frac{(1+2u-u^2)}{(1-2u-u^2)(1+6u^2+u^4)};$$

then we get

$$\hat h^{\{d\}}(u)=\prod_{m>0}\tilde\lambda_m(\tilde h_+(u^m))^{k_m}$$

where the k_m's are the integers defined by the identity

$$1+2u-u^2=(1-2u-u^2)(1+6u^2+u^4)\prod_{m>0}(1+u^m)^{4k_m}.$$

The corresponding relations in $\tilde{\mathcal U}_{\mathbb Z}^{0,-}$ are obtained applying Ω, that is just replacing $\tilde h_k$, $\tilde h_+(u)$ and $\hat h_+(u)$ with $\tilde h_{-k}$, $\tilde h_-(u)$ and $\hat h_-(u)$.

III) Other straightening relations in $\tilde{\mathcal U}_{\mathbb Z}^0$ (see Proposition 5.21):

$$\tilde h_+(u)\tilde h_-(v)=\tilde h_-(v)(1-uv)^{-4c}(1+uv)^{2c}\tilde h_+(u).$$

IV) Commuting elements and straightening relations in $\tilde{\mathcal U}_{\mathbb Z}^+$ (and in $\tilde{\mathcal U}_{\mathbb Z}^-$):

$$(X_{2r+1}^+)^{(k)}\text{ is central in }\tilde{\mathcal U}_{\mathbb Z}^+:$$

$$[(X_{2r+1}^+)^{(k)},(x_s^+)^{(l)}]=0=[(X_{2r+1}^+)^{(k)},(X_{2s+1}^+)^{(l)}]\ \forall r,s\in\mathbb Z,\ k,l\in\mathbb N;$$

$$\text{if } r+s \text{ is even } [(x_r^+)^{(k)},(x_s^+)^{(l)}]=0\ \forall k,l\in\mathbb N;$$

$$\text{if } r+s \text{ is odd } \exp(x_r^+u)\exp(x_s^+v) = \exp(x_s^+v)\exp((-1)^s X_{r+s}^+uv)\exp(x_r^+u)$$

(see Lemma 5.22).

All the relations in $\tilde{\mathcal{U}}_{\mathbb{Z}}^-$ are obtained from those in $\tilde{\mathcal{U}}_{\mathbb{Z}}^+$ applying the antiautomorphism Ω; in particular if $r+s$ is odd

$$\exp(x_r^-u)\exp(x_s^-v) = \exp(x_s^-v)\exp((-1)^r X_{r+s}^-uv)\exp(x_r^-u).$$

V) Straightening relations for $\tilde{\mathcal{U}}_{\mathbb{Z}}^+\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ (and for $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}\tilde{\mathcal{U}}_{\mathbb{Z}}^-$): $\forall r \in \mathbb{Z},\ k,l \in \mathbb{N}$

$$\begin{aligned}(x_r^+)^{(k)}\binom{h_0}{l} &= \binom{h_0-2k}{l}(x_r^+)^{(k)},\\ (X_{2r+1}^+)^{(k)}\binom{h_0}{l} &= \binom{h_0-4k}{l}(X_{2r+1}^+)^{(k)},\end{aligned} \tag{9.2.2}$$

and

$$\begin{aligned}\binom{h_0}{l}(x_r^-)^{(k)} &= (x_r^-)^{(k)}\binom{h_0-2k}{l},\\ \binom{h_0}{l}(X_{2r+1}^-)^{(k)} &= (X_{2r+1}^-)^{(k)}\binom{h_0-4k}{l}.\end{aligned}$$

VI) Straightening relations for $\tilde{\mathcal{U}}_{\mathbb{Z}}^+\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,+}$ (and for $\tilde{\mathcal{U}}_{\mathbb{Z}}^+\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,-}, \tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}\tilde{\mathcal{U}}_{\mathbb{Z}}^-$):

$$(X_{2r+1}^+)^{(k)}\tilde{h}_+(u) = \tilde{h}_+(u)\left((1-u^2T^{-1})^2X_{2r+1}^+\right)^{(k)}$$

(see Lemma 1.59) and

$$(x_r^+)^{(k)}\tilde{h}_+(u) = \tilde{h}_+(u)\left(\frac{(1-uT^{-1})^6(1+u^2T^{-2})}{(1-u^2T^{-2})^3}x_r^+\right)^{(k)}$$

(see Proposition 5.25);

the expression for $\left(\frac{(1-uT^{-1})^6(1+u^2T^{-2})}{(1-u^2T^{-2})^3}x_r^+\right)^{(k)}$ can be straightened more explicitly: setting $p(t) = (1-t)^6$ we have

$$\begin{aligned}p_+(t) &= 1+15t^2+15t^4+t^6,\\ p_-(t) &= -6-20t^2-6t^4,\\ p_0(t) &= -(1+15t+15t^2+t^3)(3+10t+3t^2),\end{aligned}$$

so that (see Notation 9.1 and Remark 9.2)

$$\exp(x_r^+v)\tilde{h}_+(u) = \tilde{h}_+(u)\exp\left(\frac{(1-uT^{-1})^6(1+u^2T^{-2})}{(1-u^2T^{-2})^3}x_r^+v\right) =$$

$$= \tilde{h}_+(u)\exp\left(\frac{p_-(uT^{-1})(1+u^2T^{-2})}{(1-u^2T^{-2})^3}x_{r+1}^+uv\right)\cdot$$

$$\cdot\exp\left(\frac{(-1)^{r-1}p_0(-u^2T^{-1})(1-u^2T^{-1})^2}{(1+u^2T^{-1})^6}X_{2r+1}^+uv^2\right)\cdot$$

$$\cdot\exp\left(\frac{p_+(uT^{-1})(1+u^2T^{-2})}{(1-u^2T^{-2})^3}x_r^+v\right).$$

Applying the homomorphism λ_{-1} (that is $x_s^+ \mapsto x_{-s}^+$, $X_s^+ \mapsto X_{-s}^+$, $\tilde{h}_+ \mapsto \tilde{h}_-$, $T^{-1} \mapsto T$) one immediately gets the expression for $(X_{2r+1}^+)^{(k)}\tilde{h}_-(u)$ and for $\exp(x_r^+v)\tilde{h}_-(u)$.

Applying the antiautomorphism Ω ($x_s^+ \mapsto x_{-s}^-$, $X_s^+ \mapsto X_{-s}^-$, $\tilde{h}_+ \leftrightarrow \tilde{h}_-$) one gets analogously the expression for $\tilde{h}_\pm(u)(X_{2r+1}^-)^{(k)}$ and for $\tilde{h}_\pm(u)\exp(x_r^- v)$ (see relation (3.2.2)).

VII) Straightening relations for $\tilde{\mathcal{U}}_{\mathbb{Z}}^+\tilde{\mathcal{U}}_{\mathbb{Z}}^-$:

VII,a) $\mathfrak{sl}_2$-like relations (see relation (3.2.2)): $\forall r \in \mathbb{Z}$

$$\exp(x_r^+ u)\exp(x_{-r}^- v) = \exp\Big(\frac{x_{-r}^- v}{1+uv}\Big)(1+uv)^{h_0+rc}\exp\Big(\frac{x_r^+ u}{1+uv}\Big),$$

$$\exp(X_{2r+1}^+ u)\exp(X_{-2r-1}^- v) = \exp\Big(\frac{X_{-2r-1}^- v}{1+4^2uv}\Big)(1+4^2uv)^{\frac{h_0}{2}+\frac{(2r+1)c}{4}}\exp\Big(\frac{X_{2r+1}^+ u}{1+4^2uv}\Big). \quad (9.2.3)$$

VII,b) $\hat{\mathfrak{sl}}_2$-like relations (see Proposition 4.22 and Remark 5.27, eventually applying λ_m and powers of T):

if $r+s\neq 0$ is even

$$\exp(x_r^+ u)\exp(x_s^- v) =$$
$$= \exp\left(\frac{1}{1+uvT^{r+s}}x_s^- v\right)\lambda_{r+s}(\hat{h}_+(uv))\exp\left(\frac{1}{1+uvT^{-r-s}}x_r^+ v\right),$$

while $\forall r+s\neq 0$

$$\exp(X_{2r+1}^+ u)\exp(X_{2s-1}^- v) = \quad (9.2.4)$$
$$\exp\left(\frac{1}{1+4T^{s+r}uv}X_{2s-1}^- v\right)\cdot\lambda_{2(r+s)}(\hat{h}_+(4^2uv)^{\frac{1}{2}})\cdot\exp\left(\frac{1}{1+4uvT^{-s-r}}X_{2r+1}^+ u\right).$$

VII,c) Straightening relations for $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,0}\tilde{\mathcal{U}}_{\mathbb{Z}}^{-,c}$ (and $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,1}\tilde{\mathcal{U}}_{\mathbb{Z}}^{-,c}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{+,c}\tilde{\mathcal{U}}_{\mathbb{Z}}^{-,{0 \atop 1}}$):

$$\exp(x_0^+ u)\exp(X_1^- v) = \quad (9.2.5)$$
$$\exp\left(\frac{4}{1-4^2w^2u^4v^2}x_1^- uv\right)\exp\left(\frac{-4^2w^2}{1-4^2w^2u^4v^2}x_0^- u^3v^2\right)\cdot$$
$$\cdot\exp\left(\frac{1+3\cdot 4^2wu^4v^2}{(1+4^2wu^4v^2)^2}X_1^- v\right)\hat{h}_+(4u^2v)^{\frac{1}{2}}\exp\left(\frac{1-4^2wu^4v^2}{(1+4^2wu^4v^2)^2}X_1^+ u^4v\right)\cdot$$
$$\cdot\exp\left(\frac{-4}{1-4^2w^2u^4v^2}x_1^+ u^3v\right)\exp\left(\frac{1}{1-4^2w^2u^4v^2}x_0^+ u\right)$$

which can be written in a more compact way (thanks to Remark 9.2) observing that

$$\frac{1}{1-4^2t^2} = \left(\frac{1}{1+4t}\right)_+,\ \frac{-4}{1-4^2t^2} = \left(\frac{1}{1+4t}\right)_-,\ \left(\frac{1}{1+4t}\right)_0 = \frac{-2}{(1-4^2t)^2},$$

$$\frac{1-4^2t}{(1+4^2t)^2} - \frac{2}{(1+4^2t)^2} = -\frac{1}{1+4^2t}$$

(these for the component in $\tilde{\mathcal{U}}^+$; for the component in $\tilde{\mathcal{U}}^-$ the computations are similar):

$$\exp(x_0^+ u)\exp(X_1^- v) =$$
$$= \exp\left(\frac{4}{1+4wu^2v}x_1^- uv\right)\exp\left(\frac{1}{1+4^2wu^4v^2}X_1^- v\right)\cdot$$
$$\cdot\hat{h}_+(4u^2v)^{\frac{1}{2}}\exp\left(\frac{1}{1+4wu^2v}x_0^+ u\right)\exp\left(-\frac{1}{1+4^2wu^4v^2}X_1^+ u^4v\right);$$

that is more symmetric but less explicit in terms of the given basis of $\tilde{\mathcal{U}}_{\mathbb{Z}}$.

Applying the homomorphism $T^{-r}\lambda_{2r+2s+1}$ (that is $x_l^{\pm} \mapsto x^{\pm}_{l(2r+2s+1)\pm r}$, $X_1^{\pm} \mapsto (-1)^r X^{\pm}_{2r+2s+1\pm 2r}$, $\hat{h}_k \mapsto \lambda_{2r+2s+1}(\hat{h}_k)$, $w|_{L^{\pm}} \mapsto T^{\mp(2r+2s+1)}$) one deduces the expression for $\exp(x_r^+ u)\exp\big(X^-_{2s+1}v\big)$.

Applying Ω one analogously gets the expression for $\exp\big(X^+_{2r+1}u\big)\exp(x_s^- v)$.

VII,d) The remaining relations (see Notation 5.38):

$$\exp\big(x_0^+ u\big)\exp\big(x_1^- v\big) =$$

$$= \exp\left(\frac{1+w^2u^2v^2}{1-6w^2u^2v^2+w^4u^4v^4}x_1^- v\right)\exp\left(\frac{-3+w^2u^2v^2}{1-6w^2u^2v^2+w^4u^4v^4}x_2^- uv^2\right)\cdot$$

$$\cdot\exp\left(-\frac{1-4wu^2v^2-w^2u^4v^4}{(1+6wu^2v^2+w^2u^4v^4)^2}X_3^- uv^3\right)\hat{h}_+^{\{d\}}(uv)\cdot$$

$$\cdot\exp\left(\frac{1-4wu^2v^2-w^2u^4v^4}{(1+6wu^2v^2+w^2u^4v^4)^2}X_1^+ u^3 v\right)\cdot$$

$$\cdot\exp\left(\frac{-3+w^2u^2v^2}{1-6w^2u^2v^2+w^4u^4v^4}x_1^+ u^2 v\right)\exp\left(\frac{1+w^2u^2v^2}{1-6w^2u^2v^2+w^4u^4v^4}x_0^+ u\right)$$

or, as well (using Remark 9.2),

$$\exp\big(x_0^+ u\big)\exp\big(x_1^- v\big) =$$

$$= \exp\left(\frac{1-wuv}{1+2wuv-w^2u^2v^2}x_1^- v\right)\exp\left(\frac{1}{2(1+6wu^2v^2+w^2u^4v^4)}X_3^- uv^3\right)\cdot$$

$$\cdot\hat{h}_+^{\{d\}}(uv)\cdot$$

$$\cdot\exp\left(\frac{1-wuv}{1+2wuv-w^2u^2v^2}x_0^+ u\right)\left(\frac{-1}{2(1+6wu^2v^2+w^2u^4v^4)}X_1^+ u^3 v\right).$$

It can be helpful in the computations observing that if $p(t) = \frac{1-t}{1+2t-t^2}$ then :

$$p_+(t) = \frac{1+t^2}{1-6t^2+t^4},\ p_-(t) = \frac{-3+t^2}{1-6t^2+t^4},\ p_0(t) = \frac{(1+t)(-3+t)}{(1-6t+t^2)^2},$$

$$\frac{(1-4t-t^2)}{(1+6t+t^2)^2} + \frac{(1-t)(-3-t)}{2(1+6t+t^2)^2} = -\frac{1}{2(1+6t+t^2)}.$$

The general straightening formula for $\exp(x_r^+ u)\exp(x_s^- v)$ when $r+s$ is odd is obtained from the case $r=0$, $s=1$ applying $T^{-r}\lambda_{r+s}$, remarking that $w|_{L^{\pm}} \mapsto T^{\mp(r+s)}$.

9.B Garland description of $\mathcal{U}^{im,+}$

In this appendix we discuss Garland's description of the imaginary positive part $\mathcal{U}^{im,+}_{\mathbb{Z}}$ of $\mathcal{U}_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z}}(\mathfrak{g})$ (see the Introduction) when $\mathfrak{g}$ is an affine Kac-Moody algebra it is enough to understand the rank 1 case, that is $\mathfrak{g} = \hat{\mathfrak{sl}}_2$ or $\mathfrak{g} = \hat{\mathfrak{sl}}_3^{\,\chi}$ and some interpretation appearing in successive works.

Then, with the notation introduced in 1.21, Garland's description of $\mathcal{U}^{im,+}_{\mathbb{Z}}$ can be stated as follows:

Theorem 9.3 (Garland). *$\mathcal{U}^{im,+}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module with basis B_λ (see Definition ii.).*
Equivalently:
i) $\mathcal{U}^{im,+}_{\mathbb{Z}} = \mathbb{Z}_\lambda[h_r \mid r > 0]$;
ii) B_λ is linearly independent.

Remark 9.4. *Once proved that $\mathcal{U}_{\mathbb{Z}}^{im,+}$ is the $\mathbb{Z}$-subalgebra of $\mathcal{U}$ generated by $\{\lambda_m(\hat{h}_k) \mid m > 0, k \geq 0\}$ (hence by B_λ or equivalently by $\mathbb{Z}_\lambda[h_r \mid r > 0]$), proceeding in two different directions leads to the two descriptions of $\mathcal{U}_{\mathbb{Z}}^{im,+}$ that turned out to be the same:*

$\star$) $\mathbb{Z}_\lambda[h_r \mid r > 0]$ is a $\mathbb{Z}$-subalgebra of $\mathbb{Q}[h_r \mid r > 0]$ (that is $\mathbb{Z}_\lambda[h_r \mid r > 0]$ is closed under multiplication): this implies that

$$\mathcal{U}_{\mathbb{Z}}^{im,+} = \mathbb{Z}_\lambda[h_r \mid r > 0];$$

it also implies that $\mathbb{Z}[\hat{h}_k \mid k > 0] \subseteq \mathbb{Z}_\lambda[h_r \mid r > 0]$;

$\star\star$) $\mathbb{Z}[\hat{h}_k \mid k > 0]$ is λ_m-stable for all $m > 0$ this implies that

$$\mathcal{U}_{\mathbb{Z}}^{im,+} = \mathbb{Z}[\hat{h}_k \mid k > 0];$$

it also implies that $\mathbb{Z}_\lambda[h_r \mid r > 0] \subseteq \mathbb{Z}[\hat{h}_k \mid k > 0]$.

Hence $\star$) and $\star\star$) imply that $\mathcal{U}_{\mathbb{Z}}^{im,+} = \mathbb{Z}_\lambda[h_r \mid r > 0] = \mathbb{Z}[\hat{h}_k \mid k > 0]$: that is what we proved in Proposition 1.13 and Theorem $\star$). In [6] Garland proved only by induction on a suitably defined degree. The first step of the induction is the second assertion of [6]-Lemma 5.11(b), proved in [6]-Section 9: for all $k, l \in \mathbb{N}$ $\hat{h}_k\hat{h}_l - \binom{k+l}{k}\hat{h}_{k+l}$ is a linear combination with integral coefficients of elements of B_λ of degree lower than the degree of $\hat{h}_{k+l}$.
In the proof the author uses that B_λ is a $\mathbb{Q}$-basis of $\mathbb{Q}[h_r \mid r > 0]$ and concentrates on the integrality of the coefficients: he studies the action of $\mathfrak{h}$ on $\hat{\mathfrak{sl}}_3^{\otimes N}$ where $\mathfrak{h}$ is the commutative Lie-algebra with basis $\{h_r \mid r > 0\}$ and $N \in \mathbb{N}$ is large enough (N is the maximum among the degrees of the elements of B_λ appearing in $\hat{h}_k\hat{h}_l$ with non-integral coefficient, assuming that such an element exists): $\mathfrak{h}$ is a subalgebra of $\hat{\mathfrak{sl}}_2$ and there is an embedding of $\hat{\mathfrak{sl}}_2$ in $\hat{\mathfrak{sl}}_3$ for every vertex of the Dynkin diagram of $\mathfrak{sl}_3$, so that fixing a vertex of the Dynkin diagram of $\mathfrak{sl}_3$ induces an embedding $\mathfrak{h} \subseteq \hat{\mathfrak{sl}}_2 \hookrightarrow \hat{\mathfrak{sl}}_3$, hence an action of $\mathfrak{h}$ on $\hat{\mathfrak{sl}}_3$. But the integral form of $\hat{\mathfrak{sl}}_3$ defined as the $\mathbb{Z}$-span of a Chevalley basis is $\mathcal{U}_{\mathbb{Z}}(\hat{\mathfrak{sl}}_3)$-stable; since the stability under $\mathcal{U}_{\mathbb{Z}}(\hat{\mathfrak{sl}}_3)$ is preserved by tensor products ([6]-Section 6), the author can finally deduce the desired integrality property of $\hat{h}_k\hat{h}_l$ from the study of the $\mathfrak{h}$-action on $\hat{\mathfrak{sl}}_3^{\otimes N}$.

Garland's argument has been sometimes misunderstood: it is the case for instance of [3] where the authors affirm (in Lemma 1.6) that [6]-Lemma 5.11(b) implies that $\mathcal{U}_{\mathbb{Z}}^{im,+} = \mathbb{Z}[\hat{h}_k \mid k > 0]$, while, as discussed above, it just implies the inclusion $\mathbb{Z}[\hat{h}_k \mid k > 0] \subseteq \mathcal{U}_{\mathbb{Z}}^{im,+} = \mathbb{Z}_\lambda[h_r \mid r > 0]$.

On the other hand Garland's argument strongly involves many results of the (integral) representation theory of the Kac-Moody algebras, while $\star$) is a property of the algebra $\mathbb{Q}[h_r \mid r > 0]$ and of its integral forms that can be stated in a way completely independent of the Kac-Moody algebra setting:

$$\mathbb{Z}^{(sym)}[h_r \mid r > 0] \subseteq \mathbb{Z}_\lambda[h_r \mid r > 0].$$

The above considerations motivate the present care that this thesis dedicates to provide a complete proof of the description of $\mathcal{U}_{\mathbb{Z}}^{im,+}$ and also to propose a self-contained proof of $\star$), independent of the Kac-Moody algebra context: on one hand we think that a direct proof can help highlight the essential structure of the integral form of $\mathbb{Q}[h_r \mid r > 0]$ arising from our study; on the other hand the idea of isolating the single pieces and gluing them together after studying them separately is much in the spirit of this work, so that it is natural for us to explain also Garland's basis of $\mathcal{U}_{\mathbb{Z}}^{im,+}$ through this approach; and finally we hope that presenting a different proof can also help to clarify the steps which appear more difficult in Garland's proof.

9.C List of Symbols

Lie Algebras and Commutative Algebras:

$S^{(div)}$ Definition 1.6

$S^{(bin)}$ Example 1.9

$S^{(sym)}$ Example 1.2

$\mathfrak{sl}_2$ Definition 3.1

$\hat{\mathfrak{sl}}_2$ Definition 4.1

${\hat{\mathfrak{sl}}_3}^{\chi}$ Definition 5.1

${\hat{\mathfrak{sl}}_3}^{\chi}$ Definition 8.1

Enveloping Algebras:

$\mathcal{U}_{\mathbb{Z}}^{re,\pm}$, $\mathcal{U}_{\mathbb{Z}}^{im,\pm}$, $\mathcal{U}_{\mathbb{Z}}^{\mathfrak{h}}$, ${}^*\mathcal{U}_{\mathbb{Z}}$, ${}^*\mathcal{U}_{\mathbb{Z}}^{im,\pm}$ Chapter Introduction

$\mathcal{U}(\mathfrak{sl}_2)$, $\mathcal{U}_{\mathbb{Z}}(\mathfrak{sl}_2)$ Definiton 3.1

$\mathcal{U}^+$, $\mathcal{U}^-$, $\mathcal{U}^0$ Theorem 3.2

$\hat{\mathcal{U}}$, $\hat{\mathcal{U}}^+$, $\hat{\mathcal{U}}^-$, $\hat{\mathcal{U}}^0$, $\hat{\mathcal{U}}^{0,\pm}$, $\hat{\mathcal{U}}^{\mathfrak{h}}$ Definition 4.1

$\hat{\mathcal{U}}_{\mathbb{Z}}$, $\hat{\mathcal{U}}_{\mathbb{Z}}^{\pm}$, $\hat{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$, $\hat{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ Definition 4.7

$\tilde{\mathcal{U}}$, $\tilde{\mathcal{U}}^{\pm}$, $\tilde{\mathcal{U}}^0$, $\tilde{\mathcal{U}}^{\pm,0}$, $\tilde{\mathcal{U}}^{\pm,1}$, $\tilde{\mathcal{U}}^{\pm,c}$, $\tilde{\mathcal{U}}^{0,\pm}$, $\tilde{\mathcal{U}}^{\mathfrak{h}}$ Definition 5.1

$\tilde{\mathcal{U}}_{\mathbb{Z}}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,0}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,1}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\pm,c}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$, $\tilde{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ Definition 5.12

$\bar{\mathcal{U}}_{\mathbb{Z}}$, $\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm}$, $\bar{\mathcal{U}}_{\mathbb{Z}}^{0}$, $\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm,0}$, $\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm,1}$, $\bar{\mathcal{U}}_{\mathbb{Z}}^{\pm,c}$, $\bar{\mathcal{U}}_{\mathbb{Z}}^{0,\pm}$, $\bar{\mathcal{U}}_{\mathbb{Z}}^{\mathfrak{h}}$ Definition 8.10

$\bar{\mathcal{U}}_{\mathbb{Z},1}$, $\bar{\mathcal{U}}_{\mathbb{Z},1}^{\pm}$, $\bar{\mathcal{U}}_{\mathbb{Z},1}^{0}$, $\bar{\mathcal{U}}_{\mathbb{Z},1}^{\pm,0}$, $\bar{\mathcal{U}}_{\mathbb{Z},1}^{\pm,1}$, $\bar{\mathcal{U}}_{\mathbb{Z},1}^{\pm,c}$, $\bar{\mathcal{U}}_{\mathbb{Z},1}^{0,\pm}$, $\bar{\mathcal{U}}_{\mathbb{Z},1}^{\mathfrak{h}}$ Definition 8.10

$\check{\mathcal{U}}_{\mathbb{Z},1}^{0}$, $\check{\mathcal{U}}_{\mathbb{Z},1}^{0,\pm}$, $\check{\mathcal{U}}_{\mathbb{Z},1}^{\mathfrak{h}}$ Definition 8.10

$\mathcal{U}_{\mathbb{Z},M}$ Definition 6.2

$\mathcal{U}_{\mathbb{Z},M}^{-}$, $\mathcal{U}_{\mathbb{Z},M}^{0}$, $\mathcal{U}_{\mathbb{Z},M}^{+}$ Theorem 6.7

Bases:

$B^{re,\pm}$, $B^{im,\pm}$, $B^{\mathfrak{h}}$ Chapter Introduction

$\hat{B}^{\pm}$, $\hat{B}^{0,\pm}$, $\hat{B}^{\mathfrak{h}}$ Theorem 4.25

$B^{\pm,0}$, $B^{\pm,1}$, $B^{\pm,c}$, $B^{0,\pm}$, $B^{\mathfrak{h}}$ Theorem 5.44

B_{λ}, $B_{\lambda}^{[n]}$, B_x, $B_x^{[n]}$ Definitions 1.21 and 1.26

Elements and their generating series:

$\Lambda_r(\xi(k))$	Chapter Introduction
$a^{(k)}$, $\exp(au)$	Notation 1.7
$\binom{a}{k}$, $(1+u)^a$	Notation 1.10
$\hat{p}(u)$, $\hat{p}_r$	Example 1.2
$\hat{h}_r^{\{a\}}$, $\hat{h}^{\{a\}}(u)$	Notation 1.15
$\bar{h}^{\pm}(u), \check{h}^{\pm}(u), \bar{h}_r(u), \check{h}_r$	Definition 1.31
$\mathfrak{g}(A)$	Definition 2.2
Γ, χ	Definition 2.3
$\Phi^s, \Phi^l, \Phi^{im}, \Phi^{re}, \Phi^{im,\pm}, \Phi^{re,\pm}, D, d_i$	Section 2.4
τ_i	Section 2.3
$\hat{\mathfrak{g}}^{\chi}, \tilde{d}_i$	Definition 2.2
$x_r^{\pm}$, h_r, c	Definition 4.1 and Definition 5.1
$X_{2r+1}^{\pm}$	Definition 5.1
$x^{\pm}(u)$, $h_{\pm}(u)$, $\hat{h}_{\pm}(u)$, $\hat{h}_r$	Notation 4.9
$\tilde{h}_{\pm}(u), \tilde{h}_{\pm r}$	Definition 5.12
e_i, f_i, h_i	Remark 6.1
$y_{2r+1}^{\pm}$, $\mathtt{k}_r$, $\tilde{c}$	Notation 6.4
$\mathtt{k}_{\pm}(u)$	Remark 6.10

Anti/auto/homomorphisms:

λ_m, $\lambda_m^{[n]}$	Proposition 1.13
ev	Notation (4.0.1)
σ, Ω, T, λ_m	Definition 4.3 and Definition 5.4
$\tilde{\lambda}_m$	Lemma 5.16

Other symbols:

$\mathbb{1}$, $\mathbb{1}^{(m)}$, $\mathbb{1}_r$, $\mathbb{1}_r^{(m)}$ Notation 1.15

L_a, R_a Notation 1.54

ε_r Definition 5.12

L, $L^{\pm}$, L^0, $L^{\pm,0}$, $L^{\pm,1}$, $L^{\pm,c}$ Definition 5.8

$w.$ Definition 5.10

d, $\tilde{d}$, d_n, $\tilde{d}_n$ Notation 5.38

δ_n Remark 5.39

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