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Research Paper

Normality conditions in the Sylow p-subgroup of $\mathrm{Sym}(p^n)$ and its associated Lie algebra ☆

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A B S T R A C T

In this work, we give a description of the structure of the normal subgroups of a Sylow p-subgroup W_n of $\mathrm{Sym}(p^n)$, showing that they contain a term from the lower central series with bounded index. To this end, we explicitly determine the terms of the upper and the lower central series of W_n. We provide a similar description of these series in the Lie algebra associated to W_n, giving a new proof of the equality of their terms in both the group and algebra contexts. Finally, we calculate the growth of the normalizer chain starting from an elementary abelian regular subgroup of W_n.

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1. Introduction

This work studies normality conditions in a Sylow p-subgroup W_n of the symmetric group $\mathrm{Sym}(p^n)$ and relations with the ideals of the Lie algebra associated to the lower central series of this group.

This article aims also to be the conclusion of a series of papers focused on the study of a sequence of normalizers arising from an abelian regular subgroup of the Sylow p-subgroup of $\mathrm{Sym}(p^n)$. More specifically, let T be a vector space of dimension n over $\mathbb{F}_p$. This group acts regularly on itself, so it can be seen as an elementary abelian regular subgroup of $\mathrm{Sym}(p^n)$ via the Cayley embedding. In [3], for the case $p = 2$, the authors prove that any other elementary abelian regular subgroup of $\mathrm{AGL}(2^n)$ intersecting T in a second-maximal subgroup is conjugated to T in $N_{W_n}(T)$. In general for every prime p, the chain of normalizers originating from the subgroup T is defined as follows (see [4–6])

$$N_i = \begin{cases} T & \text{for } i = -1 \\ N_{W_n}(N_{i-1}) & \text{for } i \geq 0 \end{cases} \tag{1.1}$$

We outline the results appearing in literature in the case $p = 2$.

Based on computational experiments, a conjecture in [4] claims that, for large values of n, the index of a normalizer in the consecutive one does not depend on n.

In [5] the authors give the notion of rigid commutators in order to study the normalizer chain (1.1) in the case $p = 2$. They refer to a partition of an integer into at least two distinct parts as a partition having at least two parts and for which all the parts are of different sizes. For $1 \leq i \leq n-2$, they prove that the number $\log_2(|N_i : N_{i-1}|)$ is equal to $q_{2,i+2}$, where $q_{2,i}$ is the i-th partial sum of the sequence of the number of partitions into at least two distinct parts.

In [6], the authors deal with the case $i = n-1$. They call *unrefinable* a partition into at least two distinct parts, if the splitting of any of its parts in two other smaller parts does not produce a new partition into distinct parts. They prove that a transversal of N_{n-2} in N_{n-1} is in bijection with the set of unrefinable partitions into distinct parts with a condition on their minimal excludants, where the minimal excludant of a partition is the least positive integer which does not appear in the partition.

A $(\mathbb{Z}/m\,\mathbb{Z})$-Lie ring counterpart of these results is provided in [2], where the authors introduce the Lie ring $\mathfrak{L}_m(n)$ of partitions with maximal part at most $n-1$ and in which every part has multiplicity at most $m-1$, and construct a chain of idealizer $\{\mathfrak{N}_i\}_{i\geq -1}$, mimicking the aforementioned normalizer chain. When $m = 2$ the idealizer chain growth is shown to be the same as in the group context. This result relies on the fact that also in the Lie ring $\mathfrak{L}_2(n)$, it is possible to introduce the notion of rigid commutators. When $m = p$ is an odd prime, the rank of $\mathfrak{N}_i/\mathfrak{N}_{i-1}$ is shown to be equal to $q_{m,i+1}$, i.e., the $(i+1)$-th partial sum of the sequence of the partitions into at least two distinct parts of i, where each part can be repeated at most $m-1$ times. The authors conjecture that the equality $|N_i : N_{i-1}| = |\mathfrak{N}_i : \mathfrak{N}_{i-1}|$ holds also in the case $p > 2$, without being able to prove it, as they face difficulties in defining a suitable notion of rigid commutators for odd primes.

In the present work, in Subsection 2.2, we introduce the notion of p-degree which provides a way, alternative to the rigid commutator machinery used in characteristic 2, of describing the elements of the Sylow p-subgroup. This tool provides a total ordering of a set of special generators of the Sylow p-subgroup of $\mathrm{Sym}(p^n)$. In our main result, Theorem 6.8, we show that $|N_i : N_{i-1}| = p^{q_{p,i+1}}$, for $1 \le i \le n-1$, proving the mentioned conjecture.

The Sylow p-subgroup $\mathrm{Sym}(p^n)$ can be seen as the iterated wreath product $W_n = \wr_{i=1}^{n} \mathbb{F}_p$ (see [10]). The techniques involved in the present paper make a large use of the central series of the group W_n. This topic is widely studied in literature (see e.g. [8,9, 11–13,16]). In particular, Kaloujnine in [9] proves that the upper and the lower central series of W_n coincide. Sushchanskyy in [16] shows that the same result holds also for wreath products of elementary abelian groups.

We now give a brief outline of the paper. After describing in Section 2 some useful preliminaries, in Section 3 we compute the upper and the lower central series of W_n and we provide a proof, alternative to the Kaloujnine's one, that these two series coincide.

In Section 4, we look at the normal subgroups N of W_n. Specifically, we prove that if N is contained in the last $n-k$ base subgroups of W_n, then it contains a term of the lower central series with bounded index depending only on k.

In Section 5, we introduce the graded Lie algebra $\mathfrak{L}_n$ associated to the lower central series of W_n. In [15], the authors characterize this Lie algebra as the iterated wreath product $\mathfrak{L}_n = \wr^n \mathfrak{L}_1$, where $\mathfrak{L}_1$ is the one-dimensional algebra over $\mathbb{F}_p$. We introduce a map $\varphi\colon W_n \to \mathfrak{L}_n$ establishing a relation between the group and the algebra and which intertwines central series and a special class of normal subgroups with homogeneous ideals. In particular, also the upper and lower central series of $\mathfrak{L}_n$ coincide. Finally, in Section 6, using the map φ, we relate the normalizer chain originating from the canonical regular elementary abelian subgroup of W_n and the idealizer chain introduced in [2], proving that they exhibit the same growth.

2. Preliminaries

In this section, we introduce some notations and definitions that will be used throughout the paper.

2.1. Wreath product

Let $W_n = \wr^n \mathbb{F}_p$, where p is an odd prime integer. Notice that $W_n = \mathrm{Fun}(\mathbb{F}_p^{n-1}, \mathbb{F}_p) \rtimes W_{n-1}$ and that the group of functions $\mathrm{Fun}(\mathbb{F}_p^{n-1}, \mathbb{F}_p)$ can be identified with the additive group of the polynomials in $n-1$ variables in which every variable appears with degree at most $p-1$. More precisely, the k-th base subgroup of W_n is defined as

$$B_k := \mathrm{Fun}(\mathbb{F}_p^{k-1}, \mathbb{F}_p) \cong \mathbb{F}_p[x_1, \dots, x_{k-1}]/(x_1^p - x_1, \dots, x_{k-1}^p - x_{k-1})$$

and $W_n = \rtimes_{k=0}^{n-1} B_{n-k}$. We will call *homogeneous* an element f of W_n contained in a base subgroup B_k and we will denote it as $f\Delta_k$ to avoid confusion, since the same polynomial may belong to different base subgroups. In particular, every element of $w \in W_n$ can be uniquely written as a product of the form $w = \prod_{k=0}^{n-1} f_{n-k}\Delta_{n-k}$, where $f_i \in \mathrm{Fun}(\mathbb{F}_p^{i-1}, \mathbb{F}_p)$.

Let i and k be integers such that $i < k$. Let $x = (x_1, \dots, x_{k-1})$ and e_i be the i-th vector of the canonical basis of $\mathbb{F}_p^{k-1}$. For each $h\Delta_i \in B_i$, we define

$$\Delta_i(h)(f(x)\Delta_k) = (f(x + he_i) - f(x))\Delta_k.$$

The operator Δ can be used to express the conjugation action of an element $f_i\Delta_i \in B_i$ on an element $f_k\Delta_k \in B_k$ by way of the commutator

$$[f_k\Delta_k, f_i\Delta_i] = \Delta_i(f_i)\big(f_k\Delta_k\big).$$

Since the functions in the base subgroups are polynomials in which every variable appears with degree at most $p-1$, we can use Taylor formula to write the above commutator as follows

$$[f_k\Delta_k, f_i\Delta_i] = \sum_{j=1}^{p-1} \frac{1}{j!} \frac{\partial^j f_k}{\partial x_i^j} f_i^j \Delta_k. \tag{2.1}$$

Kaloujnine in [9] proved that W_n is the Sylow p-subgroup of $\mathrm{Sym}(p^n)$. The element $f_i\Delta_i \in B_i$ acts on $(x_1, \dots, x_n) \in \mathbb{F}_p^n$ via the translation

$$(x_1, \dots, x_n) \to (x_1, \dots, x_n) - e_i f_i(x_1, \dots, x_{i-1})$$

In this action, the subgroup $T = \langle \Delta_1, \dots, \Delta_n \rangle \leq W_n$ acts regularly and is called *the canonical elementary abelian regular subgroup of* W_n.

2.2. Power monomials

Let $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ be a sequence of non-negative integers with finite support and weight

$$\mathrm{wt}(\Lambda) := \sum_{i=1}^{\infty} i\lambda_i < \infty. \tag{2.2}$$

We shall say that Λ is a partition of N if $\mathrm{wt}(\Lambda) = N$. The maximal part of Λ is the integer $\max\{i \mid \lambda_i \neq 0\}$. The set of the partitions whose maximal part is less than or equal to k is denoted by $\mathcal{P}(k)$ and we define for each $m \geq 1$

$$\mathcal{P}_m(k) = \{\Lambda \in \mathcal{P}(k) \mid \lambda_i \leq m-1 \text{ for all } i\}.$$

Let $\Lambda \in \mathcal{P}(k)$. We define the *power monomial* x^Λ by

$$x^\Lambda = \prod_{i=1}^{\infty} x_i^{\lambda_i}.$$

Throughout this paper, unless otherwise stated, we will consider partitions in $\mathcal{P}_p(k)$ for $k = 1, \ldots, n$. The set $\mathcal{B}$ consists of all power monomials in W_n, i.e.

$$\mathcal{B} = \left\{x^\Lambda \Delta_k \mid \Lambda \in \mathcal{P}_p(k) \text{ and } 1 \leq k \leq n\right\}. \tag{2.3}$$

Definition 2.1. We define the *p-degree* of the power monomial $x^\Lambda = x_1^{\lambda_1} \cdots x_{n-1}^{\lambda_{n-1}}$, written $\operatorname{pdeg}(x^\Lambda)$, by

$$\operatorname{pdeg}(x^\Lambda) = \lambda_{n-1}p^{n-2} + \cdots + \lambda_2 p + \lambda_1.$$

Moreover, let $1 \leq j \leq n$, we set $\mu_j = p^{n-1} - p^{j-1}$ and we define

$$\operatorname{pdeg}(x^\Lambda \Delta_j) = \operatorname{pdeg}(x^\Lambda) + \mu_j. \tag{2.4}$$

Let $f\Delta_k = (c_1 x^{\Lambda_1} + \cdots + c_t x^{\Lambda_t})\Delta_k \in B_k$ be an homogeneous element with $c_i \in \mathbb{F}_p$. We define $\operatorname{pdeg}(f\Delta_k)$ as the max $\left\{\operatorname{pdeg}(x_i^\Lambda \Delta_k) \mid i = 1, \ldots, t\right\}$ and refer to the *leading term* $\operatorname{lt}(f\Delta_k)$ of $f\Delta_k$ as the element of W_n which realizes that maximum.

Notice that if $\operatorname{pdeg}(x^\Lambda \Delta_k) \leq \mu_k$, then $x^\Lambda \Delta_k = 1$.

The following definition is already given in [7].

Definition 2.2. A subgroup $S \leq W_n$ is said to be saturated if

(1) $S = S_1 \cdots S_n$, where $S_i \leq B_i$,
(2) if $f\Delta_k \in S$, then for each monomial cx^Λ of f, with $c \in \mathbb{F}_p^*$, the element $x^\Lambda \Delta_k$ is in S.

Notice that a saturated subgroup S of W_n is spanned by the set $S \cap \mathcal{B}$.

3. The lower and the upper central series of W_n

In this section we compute the lower and the upper central series of W_n and we give a proof of the equality between the terms of the two series.

Let us consider $x^\Lambda \Delta_k \in B_k$ and $x^\Theta \Delta_\ell \in B_\ell$ with $k > \ell$. By [7, Lemma 2.8], it is easy to see that

$$\operatorname{pdeg}([x^\Lambda\Delta_k, x^\Theta\Delta_\ell]) = \operatorname{pdeg}\left(\frac{\partial x^\Lambda}{\partial x_\ell} x^\Theta \Delta_k\right).$$

Thus, if $[x^\Lambda\Delta_k, x^\Theta\Delta_\ell] \neq 0$, then

$$\operatorname{pdeg}([x^\Lambda\Delta_k, x^\Theta\Delta_\ell]) < \operatorname{pdeg}(x^\Lambda\Delta_k). \tag{3.1}$$

Lemma 3.1. *Let $x^\Lambda\Delta_k \in B_k$. There exists a monic monomial element $w \in W_n$ such that $[x^\Lambda\Delta_k, w]$ lies in B_k and*

$$\operatorname{pdeg}([x^\Lambda\Delta_k, w]) = \operatorname{pdeg}(x^\Lambda\Delta_k) - 1.$$

Proof. Let $\Lambda = (\lambda_1, \ldots, \lambda_{k-1})$ and $j = \min\{j \mid \lambda_j \neq 0\}$. If $x_j = x_1$, then $w = \Delta_1$. Indeed, we have that

$$\operatorname{pdeg}([x^\Lambda\Delta_k, \Delta_1]) = \operatorname{pdeg}\big(\frac{\partial x^\Lambda}{\partial x_1}\Delta_k\big) = \operatorname{pdeg}(x^\Lambda\Delta_k) - 1.$$

If $x_j \neq x_1$, then $w = x_1^{p-1}\cdots x_{j-1}^{p-1}\Delta_j$. Indeed

$$\begin{aligned}\operatorname{pdeg}([x^\Lambda\Delta_k, x_1^{p-1}\cdots x_{j-1}^{p-1}\Delta_j]) &= \operatorname{pdeg}\big(\frac{\partial x^\Lambda}{\partial x_j} x_1^{p-1}\cdots x_{j-1}^{p-1}\Delta_k\big)\\ &= (\operatorname{pdeg}(x^\Lambda) - p^{j-1}) + (p-1)\sum_{i=0}^{j-2} p^i + \mu_k\\ &= \operatorname{pdeg}(x^\Lambda\Delta_k) - 1. \quad \square\end{aligned}$$

Corollary 3.2. *If $x^\Lambda\Delta_k \in B_k$, then there exist monic monomial elements $w_1, \ldots, w_\ell \in W_n$ such that the commutator $[x^\Lambda\Delta_k, w_1, \ldots, w_\ell]$ lies in B_k and*

$$\operatorname{pdeg}([x^\Lambda\Delta_k, w_1, \ldots, w_\ell]) = \operatorname{pdeg}(x^\Lambda\Delta_k) - \ell \tag{3.2}$$

where $1 \leq \ell \leq \operatorname{pdeg}(x^\Lambda)$.

We use the standard notation $\gamma_i(W_n)$ to indicate the i-th term of the lower central series of W_n.

Lemma 3.3. *Let $i \geq 1$, then $\gamma_i(W_n) \cap B_k = \langle x^\Lambda\Delta_k \mid \operatorname{pdeg}(x^\Lambda\Delta_k) \leq p^{n-1} - i\rangle$.*

Proof. By Equation (3.1) and Corollary 3.2, it is enough to notice that $\max\{\operatorname{pdeg}(x^\Lambda\Delta_k) \mid x^\Lambda\Delta_k \in \gamma_i(W_n) \cap B_k\} = p^{n-1} - i$. That maximum is reached by applying Corollary 3.2 to the maximum element $x_1^{p-1}\cdots x_{k-1}^{p-1}$ of B_k. Moreover, by applying Corollary 3.2 to the monic monomial element of B_k with pdeg equal to $s + i$, we get the monic monomial element with pdeg equal to s in $\gamma_i(W_n) \cap B_k$, for each $s < p^{n-1} - i$. $\square$

Lemma 3.4. *Let A an abelian subgroup of G and $B \trianglelefteq G$ such that*

(1) *$AB = A \ltimes B$ is normal in G,*
(2) *there exists $H \leq G$ such that $H \leq N_G(A)$ and $G = H(AB)$.*

Then $[AB, G] = ([A, G] \cap A)[B, G]$.

Proof. Let $g = h\bar{a}\bar{b} \in G$ with $h \in H$, $\bar{a} \in A$, $\bar{b} \in B$ and $ab \in AB$. Note that the commutator

$$[ab, g] = [a, g][a, g, b][b, g] \in [a, g][B, G].$$

Moreover, since A is abelian we have

$$[g, a] = [h, a][h, a, \bar{b}][\bar{b}, a] \in ([G, A] \cap A)[B, G]$$

Hence $[ab, g] \in ([G, A] \cap A)[B, G]$, so $[AB, G] \leq ([G, A] \cap A)[B, G]$. Since the opposite inclusion is trivial we have the claim. □

Corollary 3.5. *In the same hypotheses of the previous lemma the following equality holds*

$$[AB, \underbrace{G, \ldots, G}_{k \textit{ times}}] = ([A, \underbrace{G, \ldots, G}_{k \textit{ times}}] \cap A)[B, \underbrace{G, \ldots, G}_{k \textit{ times}}]$$

The following is another straight consequence of Lemma 3.4.

Corollary 3.6. $\gamma_i(W_n) = (\gamma_i(W_n) \cap B_n) \rtimes \cdots \rtimes (\gamma_i(W_n) \cap B_1)$.

Notice that the terms of the lower central series of W_n are saturated subgroups and

$$\gamma_i(W_n) = \gamma_{i+1}(W_n) \rtimes \langle cx^\Lambda \Delta_k \mid \operatorname{pdeg}(x^\Lambda) = p^{n-1} - i,\ 1 \leq k \leq n \text{ and } c \in \mathbb{F}_p \rangle.$$

The remaining part of this section is devoted to prove the following theorem.

Theorem 3.7. *The upper and the lower central series of W_n coincide.*

To prove this theorem, we need to prove some preliminary results.

Lemma 3.8. $Z_i(W_n) \cap B_n = \gamma_{p^{n-1}-i}(W_n) \cap B_n = [(Z_{i+1}(W_n) \cap B_n), W_n]$.

Proof. We know that $Z_i(W_n) \cap B_n \geq \gamma_{p^{n-1}-i}(W_n) \cap B_n$. Since B_n is a uniserial W_n-module, for all $i, j \geq 1$ we have the followings

(1) $\gamma_j(W_n) \cap B_n = [B_n, \underbrace{W_n, \ldots, W_n}_{j-1 \text{ times}}]$;

(2) $|(\gamma_{p^{n-1}-i}(W_n) \cap B_n) : (\gamma_{p^{n-1}-i-1}(W_n) \cap B_n)| = p$;
(3) $|Z_i(W_n) : Z_{i-1}(W_n)| \geq p$;
(4) $Z_{p^{n-1}-1}(W_n) \cap B_n \neq B_n$;
(5) $Z_{p^{n-1}}(W_n) \cap B_n = B_n$.

Thus, $p^{p^{n-1}} = \prod_{j=1}^{p^{n-1}} |(Z_j(W_n) \cap B_n) : (Z_{j-1}(W_n) \cap B_n)| \geq \prod_{j=1}^{p^{n-1}} p = p^{p^{n-1}}$ and so $|(Z_j(W_n) \cap B_n) : (Z_{j-1}(W_n) \cap B_n)| = p$ for all j. The statement follows inductively noting that $Z_1(W_n) \cap B_n = \gamma_{p^{n-1}}(W_n) \cap B_n$ □

Lemma 3.9. *If $g = g_k \dots g_n \in Z_i(W_n)$ with $g_s \in B_s$ then $g_k \in Z_i(W_n)$.*

Proof. Let $\psi \colon W_n \to W_k$ be the canonical map whose kernel is $B_{k+1} \cdots B_n$. Note that $\psi(Z_i(W_n)) \leq Z_{i-p^{n-1}+p^{k-1}}(W_k)$. By Lemma 3.8 it follows that

$$\psi(g) = g_k \in Z_{i-p^{n-1}+p^{k-1}}(W_k) \cap B_k = \gamma_{p^{n-1}-i}(W_k) \cap B_k = \gamma_{p^{n-1}-i}(W_n) \cap B_k.$$

Thus, $g_k \in \gamma_{p^{n-1}-i}(W_n) \cap B_k \leq Z_i(W_n) \cap B_k$. □

Corollary 3.10. *If $g = g_1 \dots g_n \in Z_i(W_n)$ with $g_s \in B_s$ then $g_s \in Z_i(W_n)$ for all s. In particular*

$$Z_i(W_n) = (Z_i(W_n) \cap B_1) \cdots (Z_i(W_n) \cap B_n).$$

Proof of Theorem 3.7. Notice that

$$(Z_i(W_n) \cap B_1) \cdots (Z_i(W_n) \cap B_n) = (\gamma_{p^{n-1}-i}(W_n) \cap B_1) \cdots (\gamma_{p^{n-1}-i}(W_n) \cap B_n).$$

Indeed, by Lemma 3.8, we have that $(Z_i(W_n) \cap B_n) = \gamma_{p^{n-1}-i}(W_n) \cap B_n$ and the claim follows easily by arguing induction considering the quotient $W_{n-1} = W_n/B_n$.

Finally, by Corollaries 3.6 and 3.10, we have

$$\begin{aligned} Z_i(W_n) &= (Z_i(W_n) \cap B_1) \cdots (Z_i(W_n) \cap B_n) \\ &= (\gamma_{p^{n-1}-i}(W_n) \cap B_1) \cdots (\gamma_{p^{n-1}-i}(W_n) \cap B_n) = \gamma_{p^{n-1}-i}(W_n). \quad \square \end{aligned}$$

4. Normal subgroups of W_n

In this section we characterize the normal subgroups of W_n showing that they contain a term of the lower central series with bounded index.

Lemma 4.1. *Let N be a normal subgroup of W_n and $f\Delta_k \in N$. Every monomial element $x^\Lambda \Delta_k$ of p-degree at most $\operatorname{pdeg}(f\Delta_k)$ belongs to N.*

Proof. We argue by induction on the p-degree of $f\Delta_k$. The base of the induction is when f is constant of p-degree minimum possible μ_k. In this case, the claim is trivial. Otherwise $t = \mathrm{pdeg}(f\Delta_k) \geq \mu_k$. By Lemma 3.1 applied to the leading term $x^\Lambda \Delta_k$ of $f\Delta_k$, there exists an element in N of p-degree equal to $t-1$. By induction, $N \cap B_k$ contains every monomial element of p-degree at most $t-1$. In particular, $f\Delta_k - x^\Lambda\Delta_k$ lies in N and so also $x^\Lambda\Delta_k \in N$, proving the statement. □

Lemma 4.2. *If $N \trianglelefteq W_n$, then $(N \cap B_k)(N \cap B_{k+1}) \cdots (N \cap B_n) \trianglelefteq W_n$ for all $1 \leq k \leq n$.*

Proof. Since the elements in N of the form $f\Delta_h$, where $h \geq k$, generate $(N \cap B_k)(N \cap B_{k+1}) \cdots (N \cap B_n)$, it suffices to note that the commutator $[f\Delta_h, x^\Lambda\Delta_s]$ belongs to $N \cap B_\ell$ for each generator $x^\Lambda\Delta_s$ of W_n, where $\ell = \max(s, h)$. □

Lemma 4.3. *Let N be the normal closure of $\langle f\Delta_k \rangle$. Then*

$$N = (N \cap B_k)(N \cap B_{k+1}) \cdots (N \cap B_n).$$

Proof. On the one hand, note that $N \geq (N \cap B_k)(N \cap B_{k+1}) \cdots (N \cap B_n)$. On the other hand, by Lemma 4.2, $(N \cap B_k)(N \cap B_{k+1}) \cdots (N \cap B_n)$ is a normal subgroup of W_n containing $f\Delta_k$, hence it contains its normal closure N. Thus we have the equality. □

Lemma 4.4. *If $1 \neq x^\Lambda\Delta_k \in \gamma_t(W_n) \setminus \gamma_{t+1}(W_n)$, then*

$$[\langle x^\Lambda\Delta_k \rangle, W_n] = \big(\gamma_{t+1}(W_n) \cap B_k\big)\big(\gamma_{p^{k-1}+1}(W_n) \cap (B_{k+1} \cdots B_n)\big).$$

Proof. The inclusion $[\langle x^\Lambda\Delta_k \rangle, W_n] \leq \big(\gamma_{t+1}(W_n) \cap B_k\big)\big(\gamma_{p^{k-1}+1}(W_n) \cap (B_{k-1} \cdots B_n)\big)$ is trivial. In order to prove the opposite inclusion, consider the monomial element $x^\Theta\Delta_h \in \big(\gamma_{t+1}(W_n) \cap B_k\big)\big(\gamma_{p^{k-1}+1}(W_n) \cap (B_{k-1} \cdots B_n)\big)$. Let us first analyze the case $h = k$. We know that $\mathrm{pdeg}(x^\Lambda\Delta_k) = \mathrm{pdeg}(x^\Lambda) + \mu_k = p^{n-1} - t$ and $\mathrm{pdeg}(x^\Theta\Delta_k) = \mathrm{pdeg}(x^\Theta) + \mu_k \leq p^{n-1} - t - 1 = \mathrm{pdeg}(x^\Lambda\Delta_k) - 1$. By Lemma 4.1, $\gamma_{t+1}(W_n) \cap B_k \leq [\langle x^\Lambda\Delta_k \rangle, W_n]$.

If $h > k$ it suffices to consider the commutator

$$[x^\Lambda\Delta_k, x_1^{p-1-\lambda_1} \cdots x_{h-1}^{p-1-\lambda_{h-1}}\Delta_h]$$

which has pdeg equal to $\mu_k - 1$ and apply Lemma 4.1 as above. □

Proposition 4.5. *The normal closure of the subgroup of W_n generated by $x^\Lambda\Delta_k$ is*

$$\langle x^\Lambda\Delta_k \rangle [\langle x^\Lambda\Delta_k \rangle, W_n] = \langle x^\Lambda\Delta_k \rangle \big(\gamma_{t+1}(W_n) \cap B_k\big)\big(\gamma_{p^{k-1}+1}(W_n) \cap (B_{k+1} \cdots B_n)\big),$$

where $t = \mathrm{pdeg}(x^\Lambda\Delta_k)$.

Proof. Since by Lemmas 4.2 and 4.4 the subgroup $[\langle x^\Lambda\Delta_k \rangle, W_n]$ is normal in W_n, the claim follows. □

The proposition above together with Lemma 4.1 gives the following results.

Corollary 4.6. *The normal closure of the subgroup of W_n generated by $f\Delta_k$ is the saturated subgroup*

$$\langle f\Delta_k\rangle[\langle f\Delta_k\rangle, W_n] = \langle f\Delta_k\rangle\big(\gamma_{t+1}(W_n)\cap B_k\big)\big(\gamma_{p^{k-1}+1}(W_n)\cap(B_{k+1}\cdots B_n)\big),$$

where $t = \mathrm{pdeg}(f\Delta_k)$.

Proposition 4.7. *Let $g = f\Delta_k \cdot h$, with $h \in B_{k+1}\cdots B_n$. The normal closure $\langle g\rangle^{W_n}$ contains $\gamma_{p^{k-1}+1}(W_n)$.*

Proof. If $h = 1$ the statement follows by Corollary 4.6. If $h \neq 1$, let $x^\Lambda\Delta_k$ be the leading term of $f\Delta_k$. Observe that $[x_1^{p-1-\lambda_1}\dots x_{n-1}^{p-1-\lambda_{n-1}}\Delta_n, g]$ has p-degree $p^{n-1}-p^{k-1}-1$ and so we can apply Lemma 4.1 to get $\gamma_{p^{k-1}+1}(W_n)\cap B_n \leq \langle g\rangle^{W_n}$. Next, the commutator $[x_1^{p-1-\lambda_1}\cdots x_{n-2}^{p-1-\lambda_{n-2}}\Delta_{n-1}, g] = s\Delta_n q\Delta_{n-1}$ is such that

$$\mathrm{pdeg}(s\Delta_n) \leq \mathrm{pdeg}([x_1^{p-1-\lambda_1}\cdots x_{n-2}^{p-1-\lambda_{n-2}}\Delta_{n-1}, x_1^{p-1}\dots x_{n-1}^{p-1}\Delta_n]) \leq p^{n-1}-p^{k-1}-1.$$

It follows that $s\Delta_n \in \langle g\rangle^{W_n}$ by the argument above. In particular, $q\Delta_{n-1} \in \langle g\rangle^{W_n}$ and has p-degree equal to $p^{n-2}-p^{k-1}-1$. Thus, by Lemma 4.1, $\gamma_{p^{k-1}+1}(W_n)\cap B_{n-1} \leq \langle g\rangle^{W_n}$. The rest of the proof is obtained by iterating inductively this argument. □

A straightforward consequence is the following estimate.

Corollary 4.8. *If $N \trianglelefteq W_n$ is a normal subgroup such that $N \subseteq (B_k\cdots B_n)\backslash(B_{k+1}\cdots B_n)$, then, N contains $\gamma_{p^{k-1}+1}(W_n)$ and*

$$|N : \gamma_{p^{k-1}+1}(W_n)| \leq \big(p^{p^{k-1}}\big)^{n-k+1}.$$

In particular this index is bounded above by a function depending only on p and k.

Remark 4.9. Notice that if $k = n$, then N coincides with a term of the lower central series depending only on the maximal monomial term appearing in N.

5. The associated Lie algebra of W_n

In this section, we introduce the Lie algebra $\mathfrak{L}_n$ associated to the group W_n and we define a map between these structures. We also compute the upper and the lower central series of $\mathfrak{L}_n$.

We start noting that each base subgroup B_i is a uniserial module for W_n so that by Corollary 3.6 the quotient of two consecutive terms of the lower central series is an elementary abelian group. This implies that the graded Lie ring $\mathfrak{L}_n$ associated to the lower central series of W_n is a Lie algebra over the field $\mathbb{F}_p$. In [15] this Lie algebra is characterized as an iterated wreath product $\mathfrak{L}_n = \wr^n \mathfrak{L}_1$, where $\mathfrak{L}_1$ is the one dimensional algebra over $\mathbb{F}_p$. In particular, let ∂_k be the derivation given by the standard partial derivative with respect to the variable x_k, with $1 \leq k \leq n$. We identify $\mathfrak{L}_n$ as the subalgebra of the Witt algebra over $\mathbb{F}_p$ in n variables (see [14, Chapter 2]) spanned by the basis

$$\mathfrak{B} = \bigcup_{k=1}^{n} \mathfrak{B}_k \text{ where } \mathfrak{B}_k = \left\{ x^\Lambda \partial_k \mid \Lambda \in \mathcal{P}_p(k-1) \right\}.$$

The product of $\mathfrak{L}_n$ is defined on the basis $\mathfrak{B}$ as follows

$$\begin{aligned}\left[x^\Lambda \partial_k, x^\Theta \partial_j\right] :=& \ \partial_j(x^\Lambda) x^\Theta \partial_k - x^\Lambda \partial_k(x^\Theta)\partial_j \\ =& \begin{cases} \partial_j(x^\Lambda) x^\Theta \partial_k & \text{if } j < k, \\ -x^\Lambda \partial_k(x^\Theta)\partial_j & \text{if } j > k, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

This operation is then extended by bilinearity on the whole $\mathfrak{L}_n$.

For an element $x^\Lambda \partial_k$ of $\mathfrak{B}_k$, we define $\operatorname{pdeg}(x^\Lambda \partial_k)$ in a way completely analogous to what was done in Definition 2.1 for the element $x^\Lambda \Delta_k$ of B_k.

Let $cx^\Lambda \Delta_k \in W_n$, where $c \in \mathbb{F}_p$, we define

$$\varphi_i(cx^\Lambda \Delta_k) = \begin{cases} cx^\Lambda \partial_k & \text{if } x^\Lambda \Delta_k \in \gamma_i(W_n) \setminus \gamma_{i+1}(W_n) \\ 0 & \text{otherwise} \end{cases}$$

Notice that $\varphi_i(x^\Lambda \Delta_k) \neq 0$ if and only if $\operatorname{pdeg}(x^\Lambda \Delta_k) = p^{n-1} - i$. Let $f\Delta_k$ be an homogeneous element of $\gamma_i(W_n)$, we define $\varphi_i(f\Delta_k) := \varphi_i(\operatorname{lt}(f\Delta_k))$. Let $g = g_1 \dots g_n \in W_n$, we set

$$\varphi_i(g) = \begin{cases} \sum_{j=1}^{n} \varphi_i(\operatorname{lt}(g_j)) & \text{if } g \in \gamma_i(W_n) \setminus \gamma_{i+1}(W_n), \\ 0 & \text{otherwise.} \end{cases}$$

More generally we define $\varphi\colon W_n \to \mathfrak{L}_n$ by setting $\varphi(g) = \varphi_i(g)$ if $g \in \gamma_i(W_n) \setminus \gamma_{i+1}(W_n)$.

Definition 5.1. A Lie subring $\mathfrak{h}$ of $\mathfrak{L}_n$ is said to be homogeneous if it is the span over $\mathbb{F}_p$ of some subset $\mathfrak{H}$ of $\mathfrak{B}$.

Let S be a saturated subgroup of W_n. We shall denote by S^φ the homogeneous Lie subring of $\mathfrak{L}_n$ spanned by the set $\varphi(S \cap \mathcal{B})$.

Lemma 5.2. *Let $x^\Lambda \Delta_k \in \gamma_i(W_n) \setminus \gamma_{i+1}(W_n)$ and $x^\Theta \Delta_h \in \gamma_j(W_n) \setminus \gamma_{j+1}(W_n)$. If the commutator $[x^\Lambda \Delta_k, x^\Theta \Delta_h]$ is not trivial, then it lies in $\gamma_{i+j}(W_n) \setminus \gamma_{i+j+1}(W_n)$ and the following equality holds*

$$\varphi_{i+j}([x^\Lambda \Delta_k, x^\Theta \Delta_h]) = [\varphi_i(x^\Lambda \Delta_k), \varphi_j(x^\Theta \Delta_h)]. \tag{5.1}$$

Proof. Without lost of generality we can assume $k > h$, that $\varphi_i(x^\Lambda \Delta_k) \neq 0 \neq \varphi_j(x^\Theta \Delta_h)$ and that $\dfrac{\partial x^\Lambda}{\partial x_h} \neq 0$. Notice that $\operatorname{pdeg}([x^\Lambda \Delta_k, x^\Theta \Delta_h]) = \operatorname{pdeg}\big(\dfrac{\partial x^\Lambda}{\partial x_h} x^\Theta \Delta_k\big) = p^{n-1} - i - j$, thus

$$\begin{aligned}
\varphi_{i+j}([x^\Lambda \Delta_k, x^\Theta \Delta_h]) &= \varphi_{i+j}\left(\frac{\partial x^\Lambda}{\partial x_h} x^\Theta \Delta_k\right) \\
&= \frac{\partial x^\Lambda}{\partial x_h} x^\Theta \partial_k \\
&= [x^\Lambda \partial_k, x^\Theta \partial_h] \\
&= [\varphi_i(x^\Lambda \Delta_k), \varphi_j(x^\Theta \Delta_h)]. \quad \square
\end{aligned}$$

Remark 5.3. A straightforward consequence of the previous lemma is that if S is a saturated subgroup of W_n, then $|S| = |S^\varphi|$.

Corollary 5.4. *For each $i \geq 1$ we have $\gamma_i(W_n)^\varphi = \mathfrak{L}_n^i$ is the i-th Lie power of $\mathfrak{L}_n$.*

We now compute the upper central series of $\mathfrak{L}_n$.

Definition 5.5. Let $1 \leq j \leq n$. We define ξ_m as the $\mathbb{F}_p$-span of the set

$$\left\{x^{\Lambda_n} \partial_n, \dots, x^{\Lambda_1} \partial_1 \mid \operatorname{pdeg}(x^{\Lambda_i} \partial_i) < m \text{ for all } i = 1, \dots, n\right\}.$$

Notice that, by Lemma 3.3, $\xi_m = \gamma_{p^{n-1}-m}(W_n)^\varphi$, and so $\xi_m \subseteq \xi_{m+1}$.

Lemma 5.6. *For each $x^\Theta \partial_\ell \in \mathfrak{L}_n$ and $x^\Lambda \partial_i \in \xi_m$ the commutator $[x^\Lambda \partial_i, x^\Theta \partial_\ell]$ belongs to ξ_{m-1}.*

Proof. Without loss of generality, we may assume $x^\Lambda \partial_i \neq 0 \neq x^\Theta \partial_\ell$. For $\ell < i$, the assertion is easily verified noting that

$$\operatorname{pdeg}\big(\frac{\partial x^\Lambda}{\partial x_\ell} x^\Theta \partial_i\big) \leq \operatorname{pdeg}(x^\Lambda \partial_i) - 1 < m - 1.$$

If $\ell > i$, we have that $[x^\Theta \partial_\ell, x^\Lambda \partial_i] = \dfrac{\partial x^\Theta}{\partial x_i} x^\Lambda \partial_\ell$. We observe that, since $x^\Lambda \partial_i \neq 0$, we must have $m > \mu_i$. Hence

$$\operatorname{pdeg}\left(\frac{\partial x^\Theta}{\partial x_i} x^\Lambda \partial_\ell\right) \leq p^{n-1} - 1 - p^{i-1} = \mu_i - 1 < m - 1.$$

Thus, $x^\Lambda \partial_i \in \xi_{m-1}$. □

Corollary 5.7. *For all* $i = 1, \dots, n$ *we have* $\xi_m \cap \mathfrak{B}_i \subseteq Z_m(\mathfrak{L}_n) \cap \mathfrak{B}_i$.

Lemma 5.8. *Let* $x^\Lambda \partial_i$ *be such that* $\operatorname{pdeg}(x^\Lambda \partial_i) = r + \mu_i$. *Then for each* $k + 1 < r$ *there exists* $x^\Theta \partial_\ell \in \mathfrak{L}_n$ *such that* $\operatorname{pdeg}([x^\Theta \partial_\ell, x^\Lambda \partial_i]) > k + \mu_i$.

Proof. Let $r = \lambda_1 + \lambda_2 p + \cdots + \lambda_{i-1} p^{i-2}$ and $k + 1 = \gamma_1 + \gamma_2 p + \cdots + \gamma_{i-1} p^{i-2}$. Let j be the maximum index such that $\lambda_j > \gamma_j$. If $\lambda_j - \gamma_j > 1$, then

$$\operatorname{pdeg}([x^\Lambda \partial_i, \partial_j]) > k + \mu_i.$$

If $\lambda_j - \gamma_j = 1$ and there exists $s < j$ such that $\lambda_s \neq 0$, then $\operatorname{pdeg}([x^\Lambda \partial_i, \partial_s]) > k + \mu_i$. If such s does not exist and $j \neq 1$, then $\operatorname{pdeg}([x^\Lambda \partial_i, x_1^{p-1} \dots x_{j-1}^{p-1} \partial_j]) = r - 1 + \mu_i > k + \mu_i$. If $j = 1$, then $\operatorname{pdeg}([x^\Lambda \partial_i, \partial_1]) = r - 1 + \mu_i > k + \mu_i$. □

Corollary 5.9. *For all* $i = 1, \dots, n$ *the following equalities hold*

$$\xi_m \cap \mathfrak{B}_i = Z_m(\mathfrak{L}_n) \cap \mathfrak{B}_i = \mathfrak{L}_n^{p^{n-1}-m} \cap \mathfrak{B}_i.$$

In particular, $\xi_m = Z_m(\mathfrak{L}_n) = \mathfrak{L}_n^{p^{n-1}-m}$.

6. A chain of normalizers

In this section, we compute the growth of the normalizer chain $\{N_i\}_{i \geq -1}$ starting from the canonical elementary abelian regular subgroup $T = \langle \Delta_1, \dots, \Delta_n \rangle \leq W_n$, and defined as follows

$$N_i = \begin{cases} T & \text{if } i = -1 \\ N_{W_n}(N_{i-1}) & \text{if } i \geq 0. \end{cases} \tag{6.1}$$

Proposition 6.1. *Let* $S \leq W_n$ *be a saturated subgroup. The normalizer* $N_{W_n}(S)$ *of* S *in* W_n *is a saturated subgroup.*

Proof. Let $g = hg_k \in N_{W_n}(S)$ with $g_k \in B_k \setminus \{1\}$ and $h \in B_{k+1} \cdots B_n$. Since S is saturated, we have that the condition $g \in N_{W_n}(S)$ is equivalent to require that $[g, s] \in S$ for every $s \in B_i \cap S$ and all $i \in \{1, \dots, n\}$. Notice that in order to prove that $N_{W_n}(S)$ is saturated, it is enough to prove that $g_k \in N_{W_n}(S)$ since then $h = g g_k^{-1} \in N_{W_n}(S)$ and we can argue by induction on k. Let $s \in B_i \cap S$, we have

$$S \ni [g, s] = [h, s]^{g_k} [g_k, s].$$

If $k = i$, then $[g_k, s] = 1$ and we are done.

Without loss of generality we can suppose $[g_k, s] \neq 1$. If $k > i$, then $[h, s]^{g_k} \in B_{k+1} \cdots B_n$ and $[g_k, s] \in B_k$. In particular, since S is saturated, we get $[g_k, s] \in S$.

If $k < i$, let $\bar{h} \in B_{k+1} \cdots B_{i-1}$ and $\widehat{h} \in B_i \cdots B_n$ such that $h = \widehat{h}\bar{h}$. We have that

$$S \ni [g, s] = [\widehat{h}\bar{h}g_k, s] = [\widehat{h}, s]^{\bar{h}g_k}[\bar{h}g_k, s]$$

and that $[\bar{h}g_k, s] \in B_i$, $[\widehat{h}, s]^{\bar{h}g_k} \in B_{i+1} \cdots B_n$. It follows that $[\bar{h}g_k, s] \in B_i \cap S$. Let now consider the decomposition $\bar{h} = h_{i-1} \cdots h_{k+1}$ with $h_j \in B_j$. We get that

$$S \ni [\bar{h}g_k, s] = [h_{i-1}, s]^{h_{i-2}\cdots h_{k+1}g_k} \cdots [h_{k+1}, s]^{g_k}[g_k, s].$$

Notice that, on the one hand the variable x_j appears in each monomial element of $[g_k, s]$ with degree equal to the degree of the variable x_j in s. On the other hand, the variable x_j appears in each monomial element of $[h_j, s]^{h_{j-1}\cdots h_{k+1}g_k}$ with degree strictly less than the degree of the variable x_j in s. Consequently, each monomial element in the decomposition of the commutator $[g_k, s]$ differs from each monomial element in the decomposition of the commutators $[h_j, s]^{h_{j-1}\cdots h_{k+1}g_k}$, for $j = k+1, \dots i-1$. Thus, since S is saturated, each monomial element of the decomposition $[g_k, s]$ belongs to S, proving the statement. □

Bearing in mind the study of the idealizer chain introduced in [2], we will compute the growth normalizer chain of Equation (6.1) using the function φ previously introduced.

Definition 6.2. If $\mathfrak{U}$ is a subset of $\mathfrak{B}$, then its idealizer is defined as

$$\mathfrak{N}_{\mathfrak{B}}(\mathfrak{U}) = \{b \in \mathfrak{B} \mid [b, u] \in \mathbb{F}_p\mathfrak{U} \text{ for all } u \in \mathfrak{U}\}.$$

We refer to [2, Theorem 2.5] for a proof of the following result.

Theorem 6.3. *Let $\mathfrak{H}$ be a homogeneous subring of $\mathfrak{L}_n$ having basis $\mathfrak{U} \subseteq \mathfrak{B}$. The idealizer $\mathfrak{N}_{\mathfrak{L}_n}(\mathfrak{H})$ of $\mathfrak{H}$ in $\mathfrak{L}_n$ is the homogeneous subring of $\mathfrak{L}_n$ spanned over $\mathbb{F}_p$ by $\mathfrak{N}_{\mathfrak{B}}(\mathfrak{U})$.*

The following result is a technical lemma aiming to intertwine idealizers and normalizers.

Lemma 6.4. *Let H be a saturated subgroup of W_n and let $n \in B_\ell$. If $\mathrm{lt}([n, h]) \in H$ for all $h \in H$, then $[n, h] \in H$.*

Proof. Since H is a saturated subgroup, without loss of generality we may assume that $h = g\Delta_i \in B_i$ for some i, and $n = f\Delta_\ell$. If $\ell > i$, then

$$[n, h] = \sum_{s=1}^{p-1} \frac{1}{s!} \frac{\partial^s f}{\partial x_i^s} g^s \Delta_\ell$$

and $\mathrm{lt}([n,h]) = \dfrac{\partial f}{\partial x_i} g \Delta_\ell \in H$. The statement follows noting that $[\mathrm{lt}([n,h]), h] \in H$ so that $\dfrac{\partial^s f}{\partial x_i^s} g^s \Delta_\ell \in H$ for $s = 2, \dots, p-1$. If $\ell < i$, then

$$[n,h] = \sum_{s=1}^{\infty} \frac{1}{s!} \frac{\partial^s g}{\partial x_\ell^s} f^s \Delta_i$$

and $\mathrm{lt}([n,h]) = \dfrac{\partial g}{\partial x_\ell} f \Delta_i \in H$. By hypothesis, $\mathrm{lt}\big([n, \mathrm{lt}([n,h])]\big) = \dfrac{\partial^2 g}{\partial x_\ell^2} f^2 \Delta_i \in H$. Iterating the process we obtain the desired result. □

Proposition 6.5. *Let H be a saturated subgroup of W_n. The following equality holds*

$$(N_{W_n}(H))^\varphi = \mathfrak{N}_{\mathfrak{L}_n}(H^\varphi).$$

Proof. By Proposition 6.1 we know that $N_{W_n}(H)$ is a saturated subgroup of W_n. Let $n \in N_{W_n}(H) \cap \mathcal{B}$ and i an integer such that $n \in \gamma_i(W_n) \setminus \gamma_{i+1}(W_n)$. For every $h \in H \cap \mathcal{B}$, there exists an integer j such that $h \in \gamma_j(W_n) \setminus \gamma_{j+1}(W_n)$ and we have the following equality by Lemma 5.2

$$\varphi([n,h]) = \varphi_{i+j}([n,h]) = [\varphi_i(n), \varphi_j(h)].$$

Since $\varphi([n,h]) \in H^\varphi$ for all $h \in H \cap \mathcal{B}$, it follows that $\varphi_i(n) = \varphi(n) \in \mathfrak{N}_{\mathfrak{L}_n}(H^\varphi)$.

We prove now the opposite inclusion. Let $t \in \mathfrak{N}_{\mathfrak{L}_n}(H^\varphi) \cap \mathfrak{B}$. For some positive integer i, there exists $n \in \mathcal{B} \cap \big(\gamma_i(W_n) \setminus \gamma_{i+1}(W_n)\big)$ such that $\varphi(n) = t$. For all $\varphi(h) \in H^\varphi \cap \mathfrak{B}$ there exists an integer j such that $\varphi(h) = \varphi_j(h)$ and

$$H^\varphi \ni [\varphi_i(n), \varphi_j(h)] = \varphi_{i+j}([n,h] = \varphi(\mathrm{lt}([n,h]))).$$

Thus, $\mathrm{lt}([n,h]) \in H$ for all $h \in H$ and, by Lemma 6.4, we have $[n,h] \in H$. □

Remark 6.6. By Proposition 6.5, we obtain that the correspondence sending H to H^φ maps normal saturated subgroups of W_n into homogeneous ideals of $\mathfrak{L}_n$. Moreover, we define a new map $\varepsilon \colon \mathfrak{B} \to \mathcal{B}$ by $x^\Lambda \partial_k \mapsto x^\Lambda \Delta_k$. If $\mathfrak{I}$ is an homogeneous ideal of $\mathfrak{L}_n$, we denote by $\mathfrak{I}^\varepsilon$ the saturated subgroup of W_n generated by $\varepsilon(\mathfrak{I} \cap \mathfrak{B})$. Since

$$\varphi[\varepsilon(x^\Lambda \partial_k), \varepsilon(x^\Theta \partial_h)] = [\varphi\varepsilon(x^\Lambda \partial_k), \varphi\varepsilon(x^\Theta \partial_h)] = [x^\Lambda \partial_k, x^\Theta \partial_h],$$

it follows that $\mathfrak{I}^\varepsilon$ is a normal saturated subgroup of W_n such that $\big(\mathfrak{I}^\varepsilon\big)^\varphi = \mathfrak{I}$. Similarly, if N is a saturated normal subgroup of W_n, then $\big(N^\varphi\big)^\varepsilon = N$. This shows that the maps $(\cdot)^\varphi$ and $(\cdot)^\varepsilon$ realize a bijection between the poset of normal saturated subgroups of W_n and the poset of homogeneous ideals of $\mathfrak{L}_n$.

By Remark 5.3 and Proposition 6.5 we have that

$$|\mathfrak{N}_{\mathfrak{L}_n}(H^\varphi)| = |N_{W_n}(H)|. \tag{6.2}$$

Thus, the growth of the normalizer chain defined in Equation (6.1) is equal to the growth of the following idealizer chain

$$\mathfrak{N}_j = \begin{cases} \mathfrak{T} & \text{if } j = -1 \\ \mathfrak{N}_{\mathfrak{L}_n}(\mathfrak{N}_{j-1}) & \text{if } j \geq 0 \end{cases}$$

where $\mathfrak{T}$ is the homogeneous subring of $\mathfrak{L}_n$ spanned by the set $\{\partial_1, \dots, \partial_n\}$, which has been already described in [2]. More in details, let $t_{p,i}$ be the number of partitions of i into at least two parts, where each part can be repeated at most $p-1$ times, and $q_{p,i} = \sum_{j=1}^{i} t_{p,j}$. The growth of the idealizer chain is then given in the following result.

Theorem 6.7. *[2, Theorem 2.15] Let $1 \leq i \leq n-1$. The $\mathbb{F}_p$-vector space $\mathfrak{N}_i/\mathfrak{N}_{i-1}$ has dimension $q_{p,i+1}$.*

By way of Equation (6.2), this theorem can be immediately restated in the group case

Theorem 6.8. *Let $1 \leq i \leq n-1$, then $|N_i/N_{i-1}| = p^{q_{p,i+1}}$.*

The first numbers of the sequences $t_{p,i}$ and $q_{p,i}$ for $p = 3, 5$ are available in OEIS [1], respectively under the labels A000726 and A317910.

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Data availability

No data was used for the research described in the article.

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