Applied Mathematics and Computation 472 (2024) 128610

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Full Length Article

A Magnus-based integrator for Brownian parametric semi-linear oscillators

Raffaele D'Ambrosio [b], Hugo de la Cruz [a], Carmela Scalone [b,*]

[a] School of Applied Mathematics, FGV, Praia de Botafogo 190, Rio de Janeiro, Brazil
[b] Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, University of L'Aquila, Italy

ARTICLE INFO

Keywords:
Stochastic differential equations
Stochastic oscillators
Magnus expansions, exponential integrators

ABSTRACT

We introduce a numerical method for solving second-order stochastic differential equations of the form $\ddot{x} = -\omega^2(t)x + f(t,x) + \sigma(t)\xi(t)$, describing a class of nonlinear oscillators with non-constant frequency, perturbed by white noise $\xi(t)$. The proposed scheme takes advantages of the Magnus approach to construct an integrator for this stochastic oscillator. Its convergence properties are rigorously analyzed and selected numerical experiments on relevant stochastic oscillators are carried out, confirming the effectiveness and the competitive behavior of the proposed method, in comparison with standard integrators in the literature.

1. Introduction

Oscillating phenomena, their modeling and simulation, have a dominant role in applied mathematics. This interest is evident when we look at the vast literature. In recent years, we can also appreciate an increasing attention to numerical issues on stochastic differential equations modeling oscillatory systems. In [27], and reference therein, several models of stochastic oscillators are presented being apparent that many interesting examples of stochastic oscillator are obtained when, in the equation of a deterministic oscillator, a *noisy component* is introduced: this can be, for instance, an additive and/or a multiplicative noise, a random frequency, a random damping, and so on. The aim of the present work is to provide a specific time integrator for Brownian parametric semi-linear oscillators, whose dynamics is described by the following scalar second order equation

$$\ddot{x} = -\omega^2(t)x + f(t,x) + \sigma(t)\xi(t), \tag{1}$$

where $f : [t_0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\xi(t)$ is a white noise process satisfying $\mathbb{E}[\xi(t)\xi(t')] = \delta(t - t')$. In a more general setting, equation (1) may be seen as a stochastic perturbation of several types of deterministic oscillators, see [30,32] and references therein. In the deterministic context, harmonic oscillators with time varying frequencies are deeply studied from several points of view. As pointed out in the work [24], equations of the form (1), with $f = 0$ and $\sigma = 0$ describes the motion of a charged particle in certain types of magnetic fields. Again, in a semiclassical theory of radiation and mechanical problems of oscillating systems, the notion of an oscillator with a variable frequency is used. Moreover, the small oscillations of a pendulum whose length is changing at a uniform rate is one of such problems. For more details, see [24] and reference therein. The need to add white noise to this kind of parametric ODEs, in particular when they vary periodically in time, has been expressed in the works [44,45].

* Corresponding author.
E-mail addresses: raffaele.dambrosio@univaq.it (R. D'Ambrosio), hugo.delacruz@fgv.br (H. de la Cruz), carmela.scalone@univaq.it (C. Scalone).

https://doi.org/10.1016/j.amc.2024.128610
Received 21 August 2023; Received in revised form 1 February 2024; Accepted 14 February 2024

Available online 27 February 2024

The prototype of such kind of oscillators is the stochastic Mathieu equation, which has the general form

$$\ddot{x} = -(1 + b\cos(t))x + f(t,x) + \sigma\xi(t),$$

with $b < 1$, see [27,41] and references therein. In particular, the effect of adding noise to Mathieu equation has been studied in [26], where the authors show that the rich qualitative behavior predicted by Floquet theory is reflected in statistical properties of the noisy system.

From a discrete point of view, in the deterministic scenario, the literature concerning these oscillators is very rich, see [28–34].

On the other hand, in the stochastic setting, several works are devoted to investigate different aspects of stochastic oscillatory problems and the preservation of features of these problems under time discretizations. Specifically, the investigation of long-term properties of the problem, under a numerical point of view is object, for instance, of [5–9,11–21] and references therein.

It is important to note that in the stochastic scenario several numerical methods have been introduced for oscillators with constant frequency. In [43], the authors provide an analysis of long-term features of the linear oscillator

$$\ddot{x} = -x + \sigma\xi(t),$$

i.e., Equation (1) with $\omega = 1$ and $g \equiv 0$. The authors of [11,12] focus on nonlinear equations of the form

$$\ddot{x} = -\omega^2 x + g(x) + \sigma\xi(t),$$

where, the deterministic forcing g is related to a potential function $V(x)$ via $g(x) = -\nabla V(x)$, and present a family of stochastic trigonometric numerical methods based on a variation of constant formula, which is a fundamental tool to design many stochastic exponential integrators, see [22,23,25,35,36,42], and reference therein. The aim of works [18,20], is to join two ingredients, a variation of constant formula and specific quadrature rules, to define numerical schemes for classes of stochastic oscillators.

In spite of the above mentioned literature concerning system (1) with constant frequency, up to our knowledge, there are no specific methods in the literature for the case of time-dependent frequency. Following the principles shared by most of the above-mentioned works, and taking into account that devising numerical methods based on Magnus series expansion is crucial for dealing with deterministic oscillatory systems in order to reproduce meaningful features of the system, see [28,33] and reference therein, we propose here an exponential-type stochastic integrator for (1) by using the Magnus approach in this stochastic context. For previous applications of the Magnus expansions to stochastic problems, see [4].

The paper is organized as follows: in Section 2, we recall the essential aspects of Magnus expansion and the construction of the related approximations to time dependent linear matrix differential equations; in Section 3, we propose our stochastic Magnus-based integrator and provide a convergence analysis. In Section 4, some numerical experiments are carried on in order to show the advantages of the proposed integrator in comparison with other standard methods in the literature.

2. Basics of numerical approximation based on the Magnus series expansion

In this section we summarize some basic results concerning the construction of approximate solution to time-dependent linear matrix differential equations by means of the so-called Magnus expansion, see [37]. Consider the equation

$$\begin{aligned} &\mathrm{d}X(t) = A(t)X(t)\mathrm{d}t, \quad t \in \left[t_n\ t_n + h\right], \\ &X(t_n) = I. \end{aligned} \tag{2}$$

The main idea of the Magnus approach consists in writing the solution of (2) in the form $X(t) = e^{\Omega(t,t_n)}$ with $\Omega(t,t_n)$ a time-dependent matrix and then, after substituting in (2), to derive a differential equation to determine $\Omega(t,t_n)$. In this way, under some appropriate conditions on $A(t)$ (see e.g., [3], [40]), the matrix $\Omega(t,t_n)$ can be expressed as a convergent infinite sum whose terms are given by integrals involving $A(t)$ and its commutators. More precisely,

$$\Omega(t,t_n) = \sum_{k=1}^{\infty} \Omega_k(t,t_n), \tag{3}$$

with the first three terms given by

$$\Omega_1(t,t_n) = \int_{t_n}^{t} A(s)\mathrm{d}s,$$

$$\Omega_2(t,t_n) = \frac{1}{2}\int_{t_n}^{t}\int_{t_n}^{s_1} \left[A(s_1), A(s_2)\right] \mathrm{d}s_2\mathrm{d}s_1,$$

$$\Omega_3(t,t_n) = \frac{1}{6}\int_{t_n}^{t}\int_{t_n}^{s_1}\int_{t_n}^{s_2} \left(\left[A(s_1),\left[A(s_2),A(s_3)\right]\right] + \left[A(s_3),\left[A(s_2),A(s_1)\right]\right]\right) \mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1,$$

and, in general, Ω_k being a k-fold multiple integral of combinations of nested commutators of $A(t)$ at k different times, see [1]. Here, $[A, B] = AB - BA$ is the matrix commutator of A and B. Series (3) is the so-called Magnus expansion for the solution of (2).

Only in very particular cases it is possible to obtain $\Omega(t, t_n)$ in closed-form. However, precise approximations to the solution of (2) can be constructed from (3) by truncating the Magnus series at some order p:

$$\Omega^{[p]}(t, t_n) = \sum_{k=1}^{p} \Omega_k(t, t_n),$$

and then approximating the integrals that appear in this truncated sum by evaluating $A(t)$ at the nodes of any desired univariate quadrature rule, which is actually sufficient to also approximate all multivariate integrals involved in $\Omega^{[p]}$. In fact, it holds that [2]

$$\Omega^{[2s-2]}(t, t_n) = \Omega(t, t_n) + \mathcal{O}(h^{2s+1}) \quad \text{and} \quad \Omega^{[2s-1]}(t, t_n) = \Omega(t, t_n) + \mathcal{O}(h^{2s+1}),$$

from which it follows that

$$\Omega^{[p]}(t, t_n) = \Omega(t, t_n) + \mathcal{O}(h^{p+2}).$$

If we evaluate $A(t)$ at the corresponding nodes of any quadrature formula of order $p+1$, the resulting approximation $\hat{\Omega}(t, t_n)$ to $\Omega(t, t_n)$ satisfies

$$e^{\Omega(t,t_n)} - e^{\hat{\Omega}(t,t_n)} = \mathcal{O}(h^{p+2}).$$

As a consequence of these results,

$$e^{\Omega(t,t_n)} - e^{\hat{\Omega}^{[p]}(t,t_n)} = \mathcal{O}(h^{p+2}). \tag{4}$$

The advantage of this approach to construct approximations to the solution of (2) is that, although the Magnus series is truncated, the approximation $e^{\hat{\Omega}^{[p]}(t,t_n)}$ still preserves important geometrical properties of the exact solution. For more details see [2] and references therein.

In particular, using the midpoint rule to approximate the integral appearing in $\Omega^{[1]}(t, t_n)$ it is obtained

$$\hat{\Omega}^{[1]}(t, t_n) = \left(t - t_n\right) A\left(t_n + \frac{h}{2}\right).$$

Hence, from (4) we have that

$$e^{\Omega(t,t_n)} - e^{\left(t-t_n\right) A\left(t_n+\frac{h}{2}\right)} = \mathcal{O}(h^3). \tag{5}$$

We will use this approximation $\hat{\Omega}^{[1]}$ to construct the proposed numerical integrator in the next section.

3. Derivation of the method and convergence analysis

In this section we develop a new exponential-type integrator for the numerical approximation of the equation (1). Setting $X^1 := x$ and $X^2 := \dot{x}$, (1) is recast as

$$\mathrm{d}\begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = \left(\begin{pmatrix} 0 & 1 \\ -\omega^2(t) & 0 \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(t, X^1) \end{pmatrix} \right) \mathrm{d}t + \begin{pmatrix} 0 \\ \sigma(t) \end{pmatrix} \mathrm{d}W(t), \quad t_0 \le t \le T, \tag{6}$$

where $f : [t_0, T] \times \mathbb{R} \to \mathbb{R}$, $\sigma(t) \in \mathbb{R}$ and $W(t)$ is a scalar Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$.

Let us consider a partition of the time interval $\left[t_0\ T\right]$, $t_0 < t_1 < \cdots < t_N = T$ with constant stepsize $h = t_{i+1} - t_i$. The solution of equation (6) in $\left[t_n\ t_{n+1}\right]$ satisfies

$$\begin{aligned} \begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} &= e^{\Omega(t,t_n)} \left[\begin{pmatrix} X^1(t_n) \\ X^2(t_n) \end{pmatrix} + \int_{t_n}^{t} e^{-\Omega(s,t_n)} \begin{pmatrix} 0 \\ f(s, X^1(s)) \end{pmatrix} \mathrm{d}s + \int_{t_n}^{t} e^{-\Omega(s,t_n)} \begin{pmatrix} 0 \\ \sigma(s) \end{pmatrix} \mathrm{d}W(s) \right] \\ &= e^{\Omega(t,t_n)} \begin{pmatrix} X^1(t_n) \\ X^2(t_n) \end{pmatrix} + \int_{t_n}^{t} e^{\Omega(t,s)} \begin{pmatrix} 0 \\ f(s, X^1(s)) \end{pmatrix} \mathrm{d}s + \int_{t_n}^{t} e^{\Omega(t,s)} \begin{pmatrix} 0 \\ \sigma(s) \end{pmatrix} \mathrm{d}W(s), \end{aligned} \tag{7}$$

where the matrix $\Omega(t, t_n)$ is such that $e^{\Omega(t,t_n)}$ is the solution in $\left[t_n\ t_{n+1}\right]$ of

$$\begin{aligned} \mathrm{d}Y &= \begin{pmatrix} 0 & 1 \\ -\omega^2(t) & 0 \end{pmatrix} Y \,\mathrm{d}t, \\ Y(t_n) &= I. \end{aligned} \tag{8}$$

Based on the results on Magnus expansions theory and approximation of section 2 we know, see (5), that the solution of (8) can be well approximated by the solution, $e^{A_n(t-t_n)}$, of

$$\mathrm{d}\bar{Y} = A_n \bar{Y} \mathrm{d}t,$$
$$\bar{Y}(t_n) = I,$$

where $A_n = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{pmatrix}$, with $\omega_n := \omega(t_n + \frac{h}{2})$.

Thus, from (7), it is reasonable to approximate the solution of equation (6) in $t = t_{n+1}$ thought the solution of

$$\mathrm{d}\begin{pmatrix} \bar{X}^1 \\ \bar{X}^2 \end{pmatrix} = \left(A_n \begin{pmatrix} \bar{X}^1 \\ \bar{X}^2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(t, \bar{X}^1) \end{pmatrix} \right) \mathrm{d}t + \begin{pmatrix} 0 \\ \sigma(t) \end{pmatrix} \mathrm{d}W(t),$$

that is,

$$\begin{pmatrix} X^1(t_{n+1}) \\ X^2(t_{n+1}) \end{pmatrix} \approx \begin{pmatrix} \bar{X}^1(t_{n+1}) \\ \bar{X}^2(t_{n+1}) \end{pmatrix} = e^{hA_n} \begin{pmatrix} X^1(t_n) \\ X^2(t_n) \end{pmatrix} + \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-s)A_n} \begin{pmatrix} 0 \\ f(s, X^1(s)) \end{pmatrix} \mathrm{d}s$$
$$+ \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-s)A_n} \begin{pmatrix} 0 \\ \sigma(s) \end{pmatrix} \mathrm{d}W(s).$$

Then, by taking a left-rectangular discretization to both integrals and using that

$$e^{hA_n} = \begin{pmatrix} \cos(h\omega_n) & \dfrac{\sin(h\omega_n)}{\omega_n} \\ -\omega_n \sin(h\omega_n) & \cos(h\omega_n) \end{pmatrix},$$

it is obtained the numerical method

$$\begin{pmatrix} X^1_{n+1} \\ X^2_{n+1} \end{pmatrix} = \begin{pmatrix} \cos(h\omega_n) & \dfrac{\sin(h\omega_n)}{\omega_n} \\ -\omega_n \sin(h\omega_n) & \cos(h\omega_n) \end{pmatrix} \begin{pmatrix} X^1_n \\ X^2_n \end{pmatrix} + h \begin{pmatrix} \dfrac{\sin(h\omega_n)}{\omega_n} \\ \cos(h\omega_n) \end{pmatrix} f(t_n, \bar{X}^1_n) + \begin{pmatrix} \dfrac{\sin(h\omega_n)}{\omega_n} \\ \cos(h\omega_n) \end{pmatrix} \sigma(t_n)\Delta W_n, \tag{9}$$

with $\begin{pmatrix} X^1_0 \\ X^2_0 \end{pmatrix} = \begin{pmatrix} X^1(t_0) \\ X^2(t_0) \end{pmatrix}$ and $\Delta W_n = W(t_{n+1}) - W(t_n)$.

3.1. Convergence results

We now analyze the convergence of the method (9). We will show, assuming standard smoothness conditions on f and σ that the method has order 1 of strong convergence. We note that, for the sake of simplicity in the forthcoming exposition, the same letter K will be given for all the constants that will appear in the sequel. Throughout the paper, $\|\cdot\|$ is the Euclidean norm or its induced matrix norm.

Theorem 3.1. *Suppose that the function f and σ are Lipschitz and f also fulfill a Lipschitz continuity condition in the first variable:*

$$|f(t,x) - f(s,x)| \le K \left(1 + |x|^2\right)^{\frac{1}{2}} |t-s|. \tag{10}$$

Then,

$$\max_n \left(\mathbb{E}\left(\|X(t_n) - X_n\|^2 \,\middle|\, \mathcal{F}_{t_0} \right) \right)^{\frac{1}{2}} \le K \left(1 + \|X_0\|^2\right)^{\frac{1}{2}} h$$

Proof. Let $X_n = \begin{pmatrix} X^1_n \\ X^2_n \end{pmatrix}$ and let $X_{t_n,X_n}(t)$ denote the solution of Equation (6) at time t which starts in X_n at time t_n. Then, we obtain from (7)

$$X_{t_n,X_n}(t_{n+1}) = e^{\Omega(t_{n+1},t_n)} X_n + \int_{t_n}^{t_{n+1}} \mathbf{E}^{\langle 2\rangle}(t_{n+1}, s) f(s, X^1_{t_n,X_n}(s)) \mathrm{d}s + \int_{t_n}^{t_{n+1}} \mathbf{E}^{\langle 2\rangle}(t_{n+1}, s)\sigma(s)\mathrm{d}W(s),$$

where $\mathbf{E}^{\langle 2\rangle}(t_{n+1}, s)$ is the second column of $e^{\Omega(t_{n+1},s)}$. Let $\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)$ be the second column of

$$\mathbf{M}((t_{n+1}-s)\omega_n) := \begin{pmatrix} \cos((t_{n+1}-s)\omega_n) & \dfrac{\sin((t_{n+1}-s)\omega_n)}{\omega_n} \\ -\omega_n \sin((t_{n+1}-s)\omega_n) & \cos((t_{n+1}-s)\omega_n) \end{pmatrix}.$$

Termwise subtracting (9) from the above equation, and taking norm, we obtain

$$\begin{aligned} \|X_{t_n,X_n}(t_{n+1}) - X_{n+1}\| = &\left\| \left(e^{\Omega(t_{n+1},t_n)} - \mathbf{M}(h\omega_n)\right) X_n + \int_{t_n}^{t_{n+1}} \left(\mathbf{E}^{\langle 2\rangle}(t_{n+1},s) f(s, X^1_{t_n,X_n}(s)) - \mathbf{M}^{\langle 2\rangle}(h\omega) f(t_n, X^1_n)\right) \mathrm{d}s \right. \\ &\left. + \int_{t_n}^{t_{n+1}} \left(\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)\sigma(s) - \mathbf{M}^{\langle 2\rangle}(h\omega)\sigma(t_n)\right) \mathrm{d}W(s) \right\| \le \\ &\left\| \left(e^{\Omega(t_{n+1},t_n)} - \mathbf{M}(h\omega_n)\right) X_n \right\| + \left\| \int_{t_n}^{t_{n+1}} \left(\mathbf{E}^{\langle 2\rangle}(t_{n+1},s) f(s, X^1_{t_n,X_n}(s)) - \mathbf{M}^{\langle 2\rangle}(h\omega) f(t_n, X^1_n)\right) \mathrm{d}s \right\| \\ &+ \left\| \int_{t_n}^{t_{n+1}} \left(\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)\sigma(s) - \mathbf{M}^{\langle 2\rangle}(h\omega)\sigma(t_n)\right) \mathrm{d}W(s) \right\|. \end{aligned} \tag{11}$$

Squaring both sides of the last inequality and applying the Cauchy-Schwartz inequality yields

$$\begin{aligned} \left\|X_{t_n,X_n}(t_{n+1}) - X_{n+1}\right\|^2 \le\ & 3\left\| \left(e^{\Omega(t_{n+1},t_n)} - \mathbf{M}(h\omega_n)\right) X_n \right\|^2 \\ &+ 3\left(\int_{t_n}^{t_{n+1}} \left\|\mathbf{E}^{\langle 2\rangle}(t_{n+1},s) f(s, X^1_{t_n,X_n}(s)) - \mathbf{M}^{\langle 2\rangle}(h\omega_n) f(t_n, X^1_n)\right\| \mathrm{d}s \right)^2 \\ &+ 3\left\| \int_{t_n}^{t_{n+1}} \left(\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)\sigma(s) - \mathbf{M}^{\langle 2\rangle}(h\omega_n)\sigma(t_n)\right) \mathrm{d}W(s) \right\|^2 \\ \le\ & 3\left\| e^{\Omega(t_{n+1},t_n)} - \mathbf{M}(h\omega_n) \right\|^2 \|X_n\|^2 \\ &+ 3h \int_{t_n}^{t_{n+1}} \left\|\mathbf{E}^{\langle 2\rangle}(t_{n+1},s) f(s, X^1_{t_n,X_n}(s)) - \mathbf{M}^{\langle 2\rangle}(h\omega_n) f(t_n, X^1_n)\right\|^2 \mathrm{d}s \\ &+ 3\left\| \int_{t_n}^{t_{n+1}} \left(\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)\sigma(s) - \mathbf{M}^{\langle 2\rangle}(h\omega_n)\sigma(t_n)\right) \mathrm{d}W(s) \right\|^2. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}\left(\left\|X_{t_n,X_n}(t_{n+1}) - X_{n+1}\right\|^2 \middle| \mathcal{F}_{t_n} \right) \le\ & 3\left\| e^{\Omega(t_{n+1},t_n)} - \mathbf{M}(h\omega_n) \right\|^2 \mathbb{E}\left(\|X_n\|^2 \middle| \mathcal{F}_{t_n} \right) \\ &+ 3h \int_{t_n}^{t_{n+1}} \mathbb{E}\left(\left\|\mathbf{E}^{\langle 2\rangle}(t_{n+1},s) f(s, X^1_{t_n,X_n}(s)) - \mathbf{M}^{\langle 2\rangle}(h\omega_n) f(t_n, X^1_n)\right\|^2 \middle| \mathcal{F}_{t_n} \right) \mathrm{d}s \\ &+ 3 \int_{t_n}^{t_{n+1}} \left\|\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)\sigma(s) - \mathbf{M}^{\langle 2\rangle}(h\omega_n)\sigma(t_n)\right\|^2 \mathrm{d}s. \end{aligned} \tag{12}$$

In view of (5), we obtain

$$\left\| e^{\Omega(t_{n+1},t_n)} - \mathbf{M}(h\omega_n) \right\|^2 \mathbb{E}\left(\|X_n\|^2 \middle| \mathcal{F}_{t_n} \right) \le K h^6 \left(1 + \|X_n\|^2\right). \tag{13}$$

Applying the Cauchy-Schwartz inequality and taking expectation yields

$$\begin{aligned} &\mathbb{E}\left(\left\|\mathbf{E}^{\langle 2\rangle}(t_{n+1},s) f(s, X^1_{t_n,X_n}(s)) - \mathbf{M}^{\langle 2\rangle}(h\omega_n) f(t_n, X^1_n)\right\|^2 \middle| \mathcal{F}_{t_n} \right) \\ &\le 3\mathbb{E}\left(\left\|\mathbf{E}^{\langle 2\rangle}(t_{n+1},s) f(s, X^1_{t_n,X_n}(s)) - \mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n) f(s, X^1_{t_n,X_n}(s))\right\|^2 \middle| \mathcal{F}_{t_n} \right) \end{aligned}$$

$$
\begin{aligned}
&+3\mathbb{E}\left(\left\|\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)f(s,X^1_{t_n,X_n}(s))-\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)f(s,X^1_n)\right\|^2\Big|\mathcal{F}_{t_n}\right)\\
&+3\mathbb{E}\left(\left\|\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)f(s,X^1_n)-\mathbf{M}^{\langle 2\rangle}(h\omega_n)f(t_n,X^1_n)\right\|^2\Big|\mathcal{F}_{t_n}\right)\\
\le{}& 3\left\|\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)-\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)\right\|^2\mathbb{E}\left(\left|f(s,X^1_{t_n,X_n}(s))\right|^2\Big|\mathcal{F}_{t_n}\right)\\
&+3\left\|\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)\right\|^2\mathbb{E}\left(\left|f(s,X^1_{t_n,X_n}(s))-f(s,X^1_n)\right|^2\Big|\mathcal{F}_{t_n}\right)\\
&+6\mathbb{E}\left(\left\|\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)f(s,X^1_n)-\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)f(t_n,X^1_n)\right\|^2\Big|\mathcal{F}_{t_n}\right)\\
&+6\mathbb{E}\left(\left\|\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)f(t_n,X^1_n)-\mathbf{M}^{\langle 2\rangle}(h\omega_n)f(t_n,X^1_n)\right\|^2\Big|\mathcal{F}_{t_n}\right).
\end{aligned}
$$

Now, we estimate the four terms in the right-side of the inequality above as follows. Since f is Lipschitz, it fulfills a linear growth condition that, together with (5) and the L^2-estimates of $X^1_{t_n,X_n}(s)$ (see Theorem 4.1 in [38]), implies

$$
\left\|\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)-\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)\right\|^2\mathbb{E}\left(\left|f(s,X^1_{t_n,X_n}(s))\right|^2\Big|\mathcal{F}_{t_n}\right)\le Kh^6\left(1+\|X_n\|^2\right).
$$

Moreover, from Theorem 4.3 in [38] and since f is Lipschitz, we obtain

$$
\begin{aligned}
\left\|\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)\right\|^2\mathbb{E}\left(\left|f(s,X^1_{t_n,X_n}(s))-f(s,X^1_n)\right|^2\Big|\mathcal{F}_{t_n}\right)&\le K\mathbb{E}\left(\left\|X_{t_n,X_n}(s)-X_n\right\|^2\Big|\mathcal{F}_{t_n}\right)\\
&\le K\left(1+\|X_n\|^2\right)(s-t_n).
\end{aligned}
$$

Applying (10) yields

$$
\begin{aligned}
&\mathbb{E}\left(\left\|\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)f(s,X^1_n)-\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)f(t_n,X^1_n)\right\|^2\Big|\mathcal{F}_{t_n}\right)\\
&\qquad\le\left\|\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)\right\|^2\mathbb{E}\left(\left|f(s,X^1_{t_n,X_n}(s))-f(t_n,X^1_n)\right|^2\Big|\mathcal{F}_{t_n}\right)\\
&\qquad\le K\left(1+\|X_n\|^2\right)(s-t_n)^2.
\end{aligned}
$$

In addition, mean-value theorem and Theorem 4.3 in [38] imply that

$$
\begin{aligned}
&\mathbb{E}\left(\left\|\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)f(t_n,X^1_n)-\mathbf{M}^{\langle 2\rangle}(h\omega_n)f(t_n,X^1_n)\right\|^2\Big|\mathcal{F}_{t_n}\right)\\
&\qquad\le\left\|\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)-\mathbf{M}^{\langle 2\rangle}(h\omega_n)\right\|^2\mathbb{E}\left(\left|f(t_n,X^1_n)\right|^2\Big|\mathcal{F}_{t_n}\right)\\
&\qquad\le Kh^2\left(1+\|X_n\|^2\right).
\end{aligned}
$$

Then, from the estimates above we conclude that

$$
\int_{t_n}^{t_{n+1}}\mathbb{E}\left(\left\|\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)f(s,X^1_{t_n,X_n}(s))-\mathbf{M}^{\langle 2\rangle}(h\omega_n)f(t_n,X^1_n)\right\|^2\Big|\mathcal{F}_{t_n}\right)\mathrm{d}s\le K\left(1+\|X_n\|^2\right)h^2. \tag{14}
$$

Finally, to estimate the second integral in (12) we use Cauchy-Schwarz inequality and the Lipschitz continuity of σ to obtain

$$
\begin{aligned}
\left\|\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)\sigma(s)-\mathbf{M}^{\langle 2\rangle}(h\omega_n)\sigma(t_n)\right\|^2\le{}&3\left\|\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)\sigma(s)-\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)\sigma(s)\right\|^2\\
&+3\left\|\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)\sigma(s)-\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)\sigma(t_n)\right\|^2\\
&+3\left\|\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)\sigma(t_n)-\mathbf{M}^{\langle 2\rangle}(h\omega_n)\sigma(t_n)\right\|^2
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\left\|\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)\sigma(s)-\mathbf{M}^{\langle 2\rangle}(h\omega_n)\sigma(t_n)\right\|^2\le{}&3\left\|\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)-\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)\right\|^2\|\sigma(s)\|^2\\
&+3\left\|\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)\right\|^2\|\sigma(s)-\sigma(t_n)\|^2\\
&+3\left\|\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)-\mathbf{M}^{\langle 2\rangle}(h\omega_n)\right\|^2\|\sigma(t_n)\|^2
\end{aligned}
$$

$$\leq K h^4 + K(s-t_n)^2 + K h^2 \leq K\left(1+\|X_n\|^2\right)h^2.$$

Consequently,

$$\int_{t_n}^{t_{n+1}} \left\|\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)\sigma(s) - \mathbf{M}^{\langle 2\rangle}(h\omega_n)\sigma(t_n)\right\|^2 \mathrm{d}s \leq K\left(1+\|X_n\|^2\right)h^3. \tag{15}$$

Substituting (13), (14) and (15) in (12) it follows that the mean-square local error satisfies

$$\mathbb{E}\left(\left\|X_{t_n,X_n}(t_{n+1}) - X_{n+1}\right\|^2 \middle| \mathcal{F}_{t_n}\right)^{\frac{1}{2}} \leq K\left(1+\|X_n\|^2\right)^{\frac{1}{2}} h^{\frac{3}{2}}. \tag{16}$$

Now, let us analyze the mean deviation of one-step of (9): taking expectation in (11) leads to

$$\begin{aligned} \left\|\mathbb{E}\left(X_{t_n,X_n}(t_{n+1}) - X_{n+1}\middle|\mathcal{F}_{t_n}\right)\right\| &\leq \left\|e^{\Omega(t_{n+1},t_n)} - \mathbf{M}(h\omega_n)\right\| \|X_n\| \\ &+ \int_{t_n}^{t_{n+1}} \left\|\mathbb{E}\left(\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)f(s,X^1_{t_n,X_n}(s)) - \mathbf{M}^{\langle 2\rangle}(h\omega_n)f(t_n,X^1_n)\middle|\mathcal{F}_{t_n}\right)\right\| \mathrm{d}s. \end{aligned} \tag{17}$$

Following arguments similar to those used in the derivation of (13) we have

$$\left\|e^{\Omega(t_{n+1},t_n)} - \mathbf{M}(h\omega_n)\right\| \|X_n\| \leq K\left(1+\|X_n\|^2\right)^{\frac{1}{2}} h^3. \tag{18}$$

Now, to estimate the integral in (17) we note that

$$\begin{aligned} &\left\|\mathbb{E}\left(\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)f(s,X^1_{t_n,X_n}(s)) - \mathbf{M}^{\langle 2\rangle}(h\omega_n)f(t_n,X^1_n)\middle|\mathcal{F}_{t_n}\right)\right\| \\ &\leq \left\|\mathbb{E}\left(\mathbf{M}^{\langle 2\rangle}(h\omega_n)f(t_n,X^1_{t_n,X_n}(s)) - \mathbf{M}^{\langle 2\rangle}(h\omega_n)f(t_n,X^1_n)\middle|\mathcal{F}_{t_n}\right)\right\| \\ &\qquad + \left\|\mathbb{E}\left(\mathbf{E}^{\langle 2\rangle}(t_{n+1},s)f(s,X^1_{t_n,X_n}(s)) - \mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)f(s,X^1_{t_n,X_n}(s))\middle|\mathcal{F}_{t_n}\right)\right\| \\ &\qquad + \left\|\mathbb{E}\left(\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)f(s,X^1_{t_n,X_n}(s)) - \mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)f(t_n,X^1_{t_n,X_n}(s))\middle|\mathcal{F}_{t_n}\right)\right\| \\ &\qquad + \left\|\mathbb{E}\left(\mathbf{M}^{\langle 2\rangle}((t_{n+1}-s)\omega_n)f(t_n,X^1_{t_n,X_n}(s)) - \mathbf{M}^{\langle 2\rangle}(h\omega_n)f(t_n,X^1_{t_n,X_n}(s))\middle|\mathcal{F}_{t_n}\right)\right\| \\ &= I + II + III + IV. \end{aligned} \tag{19}$$

Now we estimate I. Assuming that the partial derivative of f with respect to the second variable is bounded and Lipschitz, we obtain

$$f(t_n,X^1_{t_n,X_n}(s)) - f(t_n,X^1_n) \leq f'_x(t_n,X^1_n)\left(X^1_{t_n,X_n}(s) - X^1_n\right) + \rho, \text{ with } |\rho| \leq K\left|X^1_{t_n,X_n}(s) - X_n\right|^2.$$

As a consequence,

$$\left|\mathbb{E}\left(f(t_n,X^1_{t_n,X_n}(s)) - f(t_n,X^1_n)\middle|\mathcal{F}_{t_n}\right)\right| \leq K\left|\mathbb{E}\left(X^1_{t_n,X_n}(s) - X^1_n\middle|\mathcal{F}_{t_n}\right)\right| + K\mathbb{E}\left(\left|X^1_{t_n,X_n}(s) - X^1_n\right|^2\middle|\mathcal{F}_{t_n}\right).$$

Since from (6)

$$\left|\mathbb{E}\left(X^1_{t_n,X_n}(s) - X^1_n\middle|\mathcal{F}_{t_n}\right)\right| = \int_{t_n}^{s} \mathbb{E}\left(\left|X^2_{t_n,X_n}(s)\right|\middle|\mathcal{F}_{t_n}\right)ds \leq \int_{t_n}^{s} \mathbb{E}\left(\left|X_{t_n,X_n}(s)\right|\middle|\mathcal{F}_{t_n}\right)ds \leq K\left(1+\|X_n\|^2\right)^{\frac{1}{2}}\left(s-t_n\right),$$

and, in view of the L^2 estimate in [38],

$$\mathbb{E}\left(\left|X^1_{t_n,X_n}(s) - X^1_n\right|^2\middle|\mathcal{F}_{t_n}\right) \leq K\left(1+\|X_n\|^2\right)\left(s-t_n\right)$$

we conclude that

$$I \leq \left\|\mathbf{M}^{\langle 2\rangle}(h\omega_n)\right\| \left|\mathbb{E}\left(f(t_n,X^1_{t_n,X_n}(s)) - f(t_n,X^1_n)\middle|\mathcal{F}_{t_n}\right)\right| \leq K\left(1+\|X_n\|^2\right)^{\frac{1}{2}}\left(s-t_n\right).$$

On the other hand, following arguments similar to those used in the derivation of (14) and (15), we obtain

$$II \leq K\left(1+\|X_n\|^2\right)^{\frac{1}{2}} h^3,$$

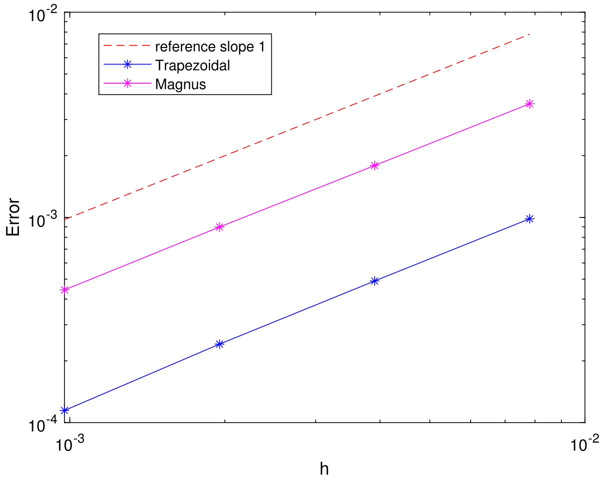


Fig. 1. Comparison between method (9) and Trapezoidal method on Mathieu with $\sigma = 0.5$ and $b = 0.1$. The reference solution is computed by the Trapezoidal method with stepsize $h = 2^{-12}$.

$$III \le K\left(1 + \|X_n\|^2\right)^{\frac{1}{2}}\left(s - t_n\right),$$

$$IV \le K\left(1 + \|X_n\|^2\right)^{\frac{1}{2}} h.$$

Substituting $I - IV$ in (19) and using the resulting estimate together with (18) in (17), we obtain that the mean deviation satisfies

$$\left\|\mathbb{E}\left(X_{t_n,X_n}(t_{n+1}) - X_{n+1}\left|\mathcal{F}_{t_n}\right.\right)\right\| \le K\left(1 + \|X_n\|^2\right)^{\frac{1}{2}} h^2. \tag{20}$$

To conclude, we use the estimates (16) and (20) to apply the fundamental convergence theorem which states the strong order of convergence of a numerical method from the mean and mean-square deviation of one-step approximation (see [39]). In this way, we finally prove the strong order 1 of the method. □

Remark 3.1. We note that taking more terms in the Magnus expansion of $\Omega(t, t_n)$ does not improve the order of convergence of the proposed method. In fact, in general, it is not possible to construct methods of higher order using only the information provided by increments of the driving Wiener process. See [10] for a discussion on this.

4. Numerical experiments

This section is dedicated to a selection of numerical experiments to show the performance of the introduced integrator (9). We select two models of oscillator widely considered in the literature.

4.1. A Mathieu's oscillator

A particular example of stochastic Mathieu's oscillator with additive noise is described by the equation

$$\ddot{x} = -(1 + b\cos(t))x + \sin(x) + \sigma\xi_t, \tag{21}$$

where ξ_t is a Gaussian white noise. This kind of model, with all its variants from deterministic to stochastic with different kinds of noises and non-linearities, is widespread in the scientific literature. We aim to provide a numerical study of convergence properties for the introduced method (9). We start setting the parameter $b = 0.1$. Reference solutions of (21) are computed by the θ-Maruyama method with $\theta = 0.5$ (the Trapezoidal method) with stepsize $h = 2^{-12}$ over the interval [0 1]. The experiments are conducted over $M = 10^3$ paths. Setting $\sigma = 0.5$, we show in Fig. 1 a comparison between the mean-square order of convergence for (9) and Trapezoidal method.

Since it may be interesting to observe the behavior for different values of the noise amplitude σ, we show in Fig. 2 the estimates of the order of convergence for the proposed method, for different values of σ. Also, in Table 2, the estimated order of convergence for (9) (p) and for the Trapezoidal rule (q) are displayed. Table 1 and Fig. 3 show the numerical estimation of the order of convergence of the proposed integrator (9), for several values of the parameter b and $\sigma = 0.2$.

To give more complete results with respect to the accuracy of the integrator (9), we provide the Table 3.
To elaborate the results shown in this table, we simulate for each of the values of h ($h = 2^{-5}, 2^{-6}, 2^{-7}$) 2000 trajectories computed by

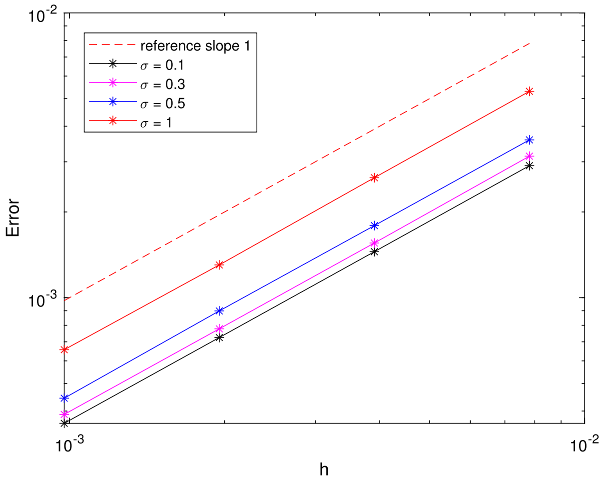


Fig. 2. Order of method (9) on Mathieu's oscillator, with reference solution computed by the trapezoidal method, for different values of σ.

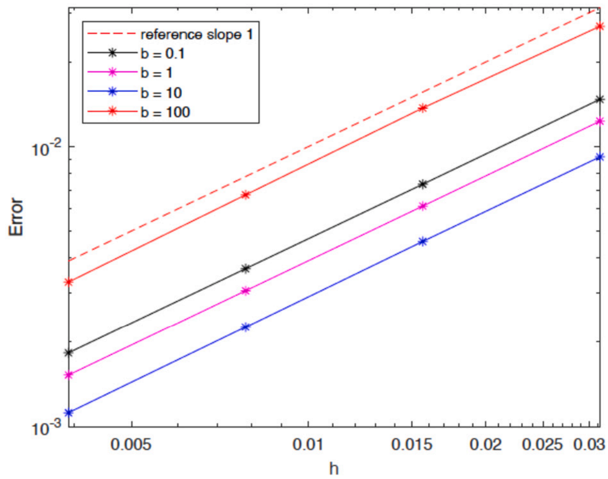


Fig. 3. Order of method (9) on Mathieu's oscillator, with reference solution computed by the trapezoidal method, for different values of b and $\sigma = 0.2$.

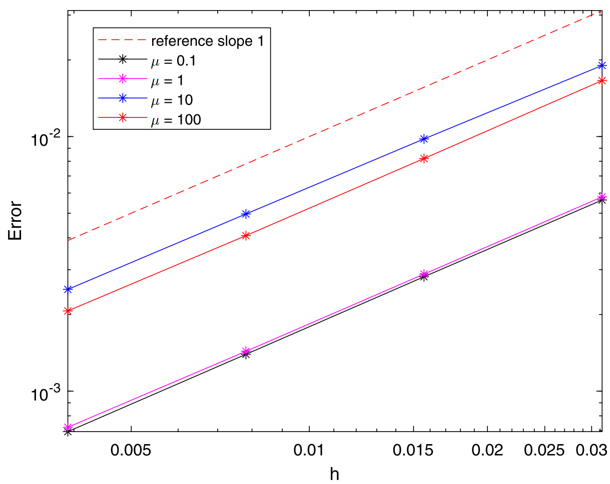


Fig. 4. Order of method (9) on the oscillator (22), with reference solution computed by the trapezoidal method, for different values of μ and $\sigma = 0.2$.

Table 1
Numerical estimation of the order of convergence of method (9) (p) and of Trapezoidal rule (q) for the Mathieu's oscillator (21), with several values of b and $\sigma = 0.2$.

b	p	q
0.1	1.0014	1.0572
1	1.0031	1.0570
10	1.0198	1.0550
100	1.0188	1.2879

Table 2
Numerical estimation of the order of convergence of method (9) (p) and of Trapezoidal rule (q) for the Mathieu's oscillator (21), with $b = 0.1$ and different values of σ.

σ	p	q
0.1	1.0451	1.0451
0.3	1.0051	1.0416
0.5	1.0058	1.0611
1	1.0038	1.0270

Table 3
Estimated values $\hat{e}(h)$ and their respective 95% confidence intervals computed for method (9), considering the Mathieu's oscillator (21).

σ \ stepsize	$h = 2^{-5}$ $\hat{e} \pm \Delta$	$h = 2^{-6}$ $\hat{e} \pm \Delta$	$h = 2^{-7}$ $\hat{e} \pm \Delta$
0.1	0.0583 ± 0.0003	0.0291 ± 0.0002	0.0145 ± 0.0001
0.3	0.0621 ± 0.0010	0.0311 ± 0.0006	0.0155 ± 0.0002
0.7	0.0833 ± 0.0016	0.0415 ± 0.0009	0.0209 ± 0.0005
1	0.1088 ± 0.0030	0.0541 ± 0.0013	0.0268 ± 0.0006

the method (9), and grouped them into $M = 20$ batches with $L = 100$ realization in each. Then, we consider the error $e_{i,j}(h)$ between the j-th trajectory and the reference solution in the i-th batch, and estimate the mean error of the i-th batch by

$$\hat{e}_i(h) = \frac{1}{L}\sum_{j=1}^{L} e_{i,j}(h),$$

their average by

$$\hat{e}(h) = \frac{1}{M}\sum_{j=1}^{M} \hat{e}_j(h),$$

and their variance by

$$\hat{\sigma}_e^2(h) = \frac{1}{M-1}\sum_{i=1}^{M} |\hat{e}_i(h) - \hat{e}(h)|^2.$$

The $100(1-\alpha)\%$ confidence interval for the mean error is then given by

$$[\hat{e}(h) - \Delta(h), \hat{e}(h) + \Delta(h)],$$

where

$$\Delta(h) = t_{1-\alpha,M-1}\sqrt{\frac{\hat{\sigma}_e^2(h)}{M}}$$

and $t_{1-\alpha,M-1}$ denotes the Student's t distribution with $M-1$ degrees of freedom. In accordance with the theoretical analysis, Table 3, show that the reduction of the stepsize provokes an improvement of the behavior of the error.

Table 4
Table of mean runtimes at various values of the stepsize h, for the Mathieu's oscillator (21) with $\sigma = 0.3$.

h	0.0005	0.0010	0.0020	0.0039	0.0078
Trapezoidal time	313, 7	308.4	277.5	299.7	304.7
Magnus time (ref)	1	1	1	1	1

Table 5
Table of mean runtimes at various values of the stepsize h, for the Mathieu's oscillator with $\sigma = 1$.

h	0.0005	0.0010	0.0020	0.0039	0.0078
Trapezoidal time	335.8	341.3	345.4	352.8	280
Magnus time (ref)	1	1	1	1	1

Table 6
Estimated values $\hat{e}(h)$ and their respective 95% confidence intervals computed for method (9), considering the oscillator (22).

σ \ stepsize	$h = 2^{-5}$ $\hat{e} \pm \Delta$	$h = 2^{-6}$ $\hat{e} \pm \Delta$	$h = 2^{-7}$ $\hat{e} \pm \Delta$
0.1	0.0253 ± 0.0002	0.0126 ± 0.0001	0.0062 ± 0.0001
0.3	0.0338 ± 0.0007	0.0168 ± 0.0004	0.0084 ± 0.0002
0.5	0.0462 ± 0.0013	0.0229 ± 0.0006	0.0115 ± 0.0003
1	0.0784 ± 0.0025	0.0400 ± 0.0011	0.0196 ± 0.0005

Table 7
Table of mean runtimes for various values of the stepsize h, for the oscillator (22) with $\sigma = 0.3$ and $\mu = 1$.

h	0.0005	0.0010	0.0020	0.0039	0.0078
Runtime Trapezoidal	159.5	132.7	109.2	110.1	107.1
Runtime for (9)	1	1	1	1	1

Table 8
Table of mean runtimes for various values of the stepsize h, for the oscillator (22) with $\sigma = 0.5$ and $\mu = 10$.

h	0.0005	0.0010	0.0020	0.0039	0.0078
Runtime for Trapezoidal	316	324	325.6	122.8	277.3
Runtime for (9)	1	1	1	1	1

Tables 4 and 5 show the running mean-time spent to compute, for different values of parameters σ, one solution path for some values of the stepsize for the Mathieu's oscillator (21). The results are shown assuming the running time of the solution computed by method (9) as reference.

4.2. Airy

This section is dedicated to numerical experiments referred to the model

$$\ddot{x} = -\mu t x + \cos^2(x) + \sigma \xi_t. \tag{22}$$

We start presenting Table 6, showing estimated values $\hat{e}(h)$ and their respective 95% confidence intervals computed for method (9), considering the oscillator (22) (computed in a similar way to Table 3). Also in this case, we consider as stepsizes $h = 2^{-5}, 2^{-6}, 2^{-7}$ and some values of σ.

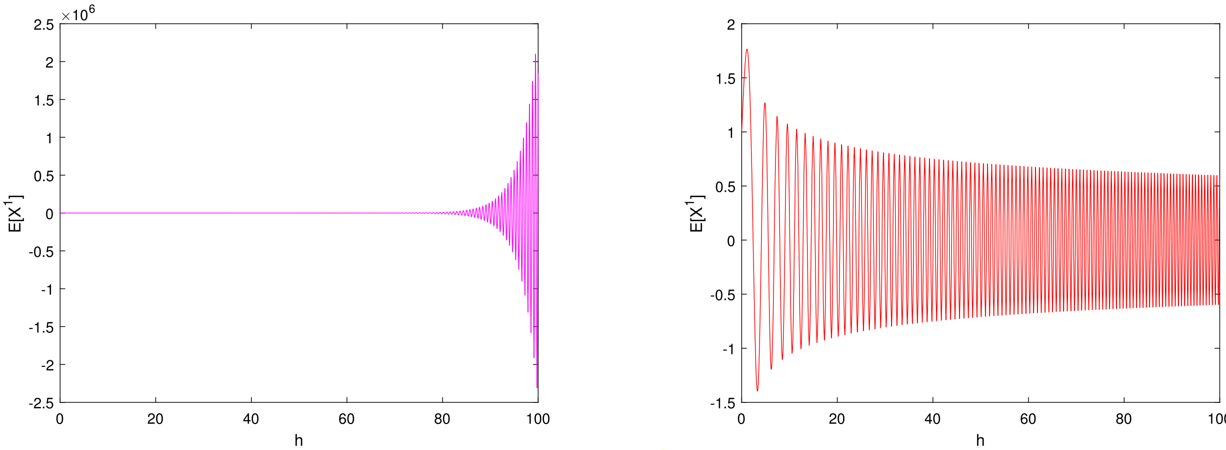
Tables 7 and 8 show the running mean-time spent to compute, for different values of parameters σ and μ, one solution path for some values of the stepsize. The results are shown assuming the running time of the solution computed by method (9) as reference.

As expected, the performances of method (9) in terms of computational time, are much better than Trapezoidal method.

Table 9 and Fig. 4 show the numerical estimation of the order of convergence of the proposed integrator (9), for several values of the parameter μ and $\sigma = 0.2$ in Equation (22).

Table 9
Numerical estimation of the order of convergence of method (9) (p) and of Trapezoidal rule (q) for the oscillator (22), with several values of μ and $\sigma = 0.2$.

μ	p	q
0.1	1.0072	1.0352
1	1.0028	1.0311
10	0.9801	1.1390
100	1.0076	2.0560



(a) Euler Maruyama method.   (b) Method (9).

Fig. 5. Mean reproduction over the interval [0 100], choosing the stepsize $h = 2^{-7}$.

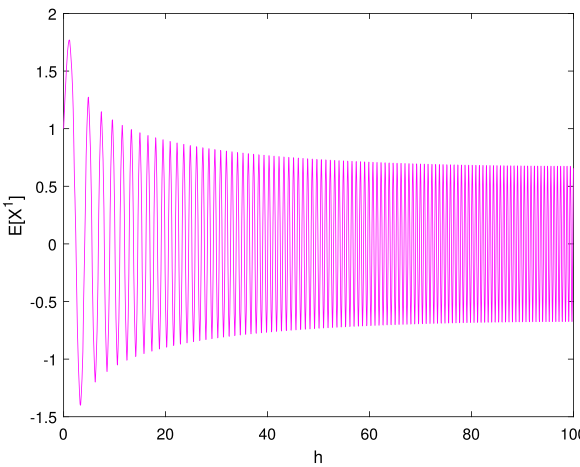


Fig. 6. Plot of the mean of X^1 as function of the time for the explicit Euler Maruyama method with $h = 2^{-13}$.

4.3. Statistical behavior

The next numerical experiments aim to illustrate the long-term probabilistic behavior of the proposed method. For this, we consider the equation

$$\ddot{x} = -\mu t x + \sigma \xi_t, \quad (\mu > 0) \tag{23}$$

which is equivalent to

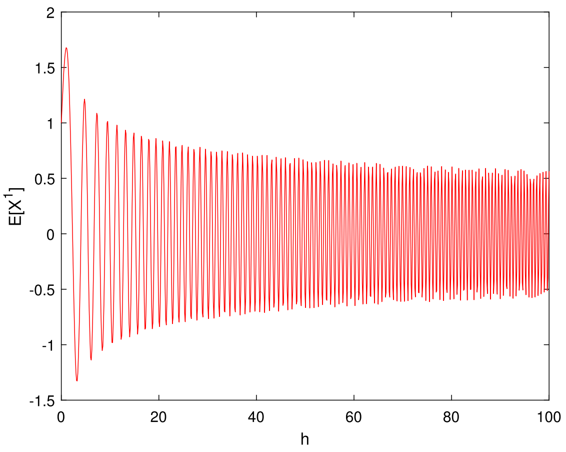


Fig. 7. Plot of the mean of X^1 as function of the time for the Magnus method with $h = 2^{-3}$.

$$\mathrm{d}X = \begin{pmatrix} 0 & 1 \\ -\mu t & 0 \end{pmatrix} X\mathrm{d}t + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} \mathrm{d}W(t). \tag{24}$$

Let $Y = \mathbb{E}[X]$, then Y satisfies the following ODE

$$Y' = \begin{pmatrix} 0 & 1 \\ -\mu t & 0 \end{pmatrix} Y. \tag{25}$$

Thus, Y displays oscillations of progressively increasing frequency [32]. In Fig. 5, we plot the reproduction of the mean of the solution for Euler-Maruyama and for method (9), for $t \in [0\,100]$, $\mu = 1$ and for $h = 2^{-7}$. The instability of Euler-Maruyama appears clear for this choice of the stepsize. In fact, we need to reduce up to $h = 2^{-13}$, to have a correct reproduction of the mean, as shown in Fig. 6. It is worth to note that even for bigger stepsizes, like $h = 2^{-3}$, the proposed integrator shows a stable performance, still reproducing the behavior of the exact solution, as displayed in Fig. 7.

5. Conclusions

In this paper, we introduce a numerical integrator based on Magnus' expansions specifically designed to deal with scalar stochastic oscillators of the form (1). A complete analysis of the convergence of the method is provided and several numerical experiments were carried out to confirm the good properties of the proposed method in terms of accuracy and computational cost, in comparison with standard integrators in the literature. Moreover, the possibility to conserve probabilistic features, like the mean, is shown. Motivated by the excellent properties of the Magnus' integrators in the deterministic setting, the search for similar properties and possible extensions to more complicated models can provide key insights for future research.

Data availability

No data was used for the research described in the article.

Acknowledgements

R. D'Ambrosio and C. Scalone are supported by GNCS-INDAM project and by PRIN2017-MIUR project (2017JY-CLSF "Structure preserving approximation of evolutionary problems"). R. D'Ambrosio and C. Scalone are members of the INDAM Research group GNCS.

H. de la Cruz thanks the financial support of a PPA project-EMAp FGV. Brazil.

References

[1] A. Arnal, F. Casas, C. Chiralt, A general formula for the Magnus expansion in terms of iterated integrals and right-nested commutators, J. Phys. Commun. 2 (2018) 035024.
[2] S. Blanes, F. Casas, J.A. Oteo, J. Ros, The Magnus expansion and some of its applications, Phys. Rep. 470 (2009) 151–238.
[3] F. Casas, Sufficient conditions for the convergence of the Magnus expansion, J. Phys. A: Math. Theor. 40 (2007) 15001–15017.
[4] K. Burrage, P.M. Burrage, High strong order methods for non-commutative-stochastic ordinary differential equation systems and the Magnus formula, Physica D 133 (1999) 34–48.
[5] P.M. Burrage, K. Burrage, Structure-preserving Runge-Kutta methods for stochastic Hamiltonian equations with additive noise, Numer. Algorithms 65 (2012) 519–532.
[6] P.M. Burrage, K. Burrage, Low rank Runge-Kutta methods, symplecticity and stochastic Hamiltonian problems with additive noise, J. Comput. Appl. Math. 236 (2014) 3920–3930.

[7] K. Burrage, I. Lenane, G. Lythe, Numerical methods for second-order stochastic differential equations, SIAM J. Sci. Comput. 29 (2007) 245–264.
[8] K. Burrage, G. Lythe, Accurate stationary densities with partitioned numerical methods for stochastic differential equations, SIAM J. Numer. Anal. 47 (2009) 1601–1618.
[9] V. Citro, R. D'Ambrosio, Long-term analysis of stochastic theta-methods for damped stochastic oscillators, Appl. Numer Math. 150 (2020) 18–26.
[10] J.M. Clark, R.J. Cameron, The Maximum Rate of Convergence of Discrete Approximations for Stochastic Differential Equations, LNCIS, 2005, https://doi.org/10.1007/BFb000400.
[11] D. Cohen, On the numerical discretisation of stochastic oscillators, Math. Comput. Simul. 82 (2009) 1478–1495.
[12] D. Cohen, M. Sigg, Convergence analysis of trigonometric methods for stiff second-order stochastic differential equations, Numer. Math. 121 (2012) 1–29.
[13] R. D'Ambrosio, Numerical Approximation of Ordinary Differential Problems - From Deterministic to Stochastic Numerical Methods, Springer, 2023.
[14] R. D'Ambrosio, S. Di Giovacchino, Mean-square contractivity of stochastic theta-methods, Commun. Nonlinear Sci. Numer. Simul. 96 (2021) 105671.
[15] R. D'Ambrosio, S. Di Giovacchino, Nonlinear stability issues for stochastic Runge-Kutta methods, Commun. Nonlinear Sci. Numer. Simul. 94 (2021) 105549.
[16] R. D'Ambrosio, M. Moccaldi, B. Paternoster, Numerical preservation of long-term dynamics by stochastic two-step methods, Discrete Contin. Dyn. B 23 (2018) 2763–2773.
[17] R. D'Ambrosio, C. Scalone, On the numerical structure preservation of nonlinear damped stochastic oscillators, Numer. Algorithms 86 (2021) 933–952.
[18] R. D'Ambrosio, C. Scalone, Filon quadrature for stochastic oscillators driven by time-varying forces, Appl. Numer. Math. 169 (2021) 21–31.
[19] R. D'Ambrosio, C. Scalone, Two-step Runge-Kutta methods for stochastic differential equations, Appl. Math. Comput. 403 (2021) 125930.
[20] H. de la Cruz, J.C. Jimenez, J.P. Zubelli, Locally linearized methods for the simulation of stochastic oscillators driven by random forces, BIT Numer. Math. 57 (2017) 123–151.
[21] H. de la Cruz, J.C. Jimenez, R.J. Biscay, On the oscillatory behavior of coupled stochastic harmonic oscillators driven by random forces, Stat. Probab. Lett. 146 (2019) 85–89.
[22] H. de la Cruz Cancino, R.J. Biscay, J.C. Jimenez, F. Carbonell, T. Ozaki, High order local linearization methods: an approach for constructing A-stable explicit schemes for stochastic differential equations with additive noise, BIT Numer. Math. 50 (2010) 509–539.
[23] H. de la Cruz, Stabilized explicit methods for the approximation of stochastic systems driven by small additive noises, Chaos Solit. Fractals 140 (2020) 110195.
[24] C.J. Eleizer, A. Gray, A note on the time-dependent harmonic oscillator, SIAM J. Appl. Math. 30 (3) (1976) 463–468.
[25] U. Erdogan, G. Lord, A new class of exponential integrators for SDEs with multiplicative noise, IMA J. Numer. Anal. 39 (2019) 820–846.
[26] M. Gitterman, R.I. Shrager, G.H. Weiss, Influence of noise on systems described by Mathieu equation, Phys. Lett. A 142 (1989) 84–88.
[27] M. Gitterman, The Noisy Oscillator, The First Hundred Years, from Einstein Until Now, World Scientific, 2005.
[28] M. Hochbruck, C. Lubich, On Magnus integrators for time-dependent Schroedinger equations, SIAM J. Numer. Anal. 3 (2004) 945–963.
[29] M. Hochbruck, A. Ostermann, Exponential integrators, Acta Numer. 19 (2010) 209–286.
[30] A. Iserles, On the global error of discretization methods for highly-oscillatory ordinary differential equations, BIT Numer. Math. 42 (2002) 561–599.
[31] A. Iserles, H.Z. Munthe-Kaas, S.P. Nørsett, A. Zanna, Lie group methods, Acta Numer. 9 (2005) 1–148.
[32] A. Iserles, S.P. Nørsett, On the solution of linear differential equations in Lie groups, Philos. Trans. R. Soc. Lond. Ser. A 357 (1999) 983–1019.
[33] A. Iserles, A. Marthinsen, S.P. Nørsett, On the implementation of the method of Magnus series for linear differential equations, BIT Numer. Math. 357 (1999) 983–1019.
[34] C. González, A. Ostermann, M. Thalhammer, A second-order Magnus-type integrator for nonautonomous parabolic problems, J. Comput. Appl. Math. 189 (2006) 142–156.
[35] J.C. Jimenez, H. de la Cruz Cancino, Convergence rate of strong Local Linearization schemes for stochastic differential equations with additive noise, BIT Numer. Math. 52 (2012) 357–382.
[36] Y. Komori, K. Burrage, A stochastic exponential Euler scheme for simulation of stiff biochemical reaction systems, BIT Numer. Math. 54 (2014) 1067–1085.
[37] W. Magnus, On the exponential solution of differential equations for a linear operator, Commun. Pure Appl. Math. 7 (1954) 649–673.
[38] X. Mao, Stochastic Differential Equations and Applications, Woodhead Publishing, 2008.
[39] G. Milstein, M. Tretyakov, Stochastic Numerics for Mathematical Physics, Springer, 2004.
[40] P.C. Moan, J. Niesen, Convergence of the Magnus series, Found. Comput. Math. 8 (2008) 291–301.
[41] F.J. Poulin, G.R. Flierl, The stochastic Mathieu's equation, Proc. R. Soc. A 464 (2008) 1885–1904.
[42] C. Shi, Y. Xiao, C. Zhang, The convergence and MS stability of exponential Euler method for semilinear stochastic differential equations, Abstr. Appl. Anal. (2012) 1–12.
[43] A.H. Strömmen Melbö, D.J. Higham, Numerical simulation of a linear stochastic oscillator with additive noise, Appl. Numer. Math. 51 (2004) 89–99.
[44] C. Zerbe, P. Jung, P. Hanggi, Brownian parametric oscillators, Phys. Rev. E 49 (1994) 3626–3636.
[45] C. Zerbe, P. Hanggi, Brownian parametric oscillators with dissipation, Phys. Rev. E 52 (1995) 1533–1543.