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Delay-independent conditions of exponential ISS for linear time-varying delay differential systems

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Abstract

This work focuses on linear continuous-time systems with time-varying delays, in the general case of time-varying matrices. Initially, we explore the case of positive systems within this class and introduce two distinct delay-independent conditions of exponential input-to-state stability. Moreover, we provide guaranteed exponential convergence rates for such conditions, explicitly stating the ISS gains. Then, we extend our analysis to systems without sign constraints, adopting a state-bounding approach that takes advantage of the properties of positive systems. Due to the time-varying nature of the systems, all proposed conditions are in terms of an infinite number of inequalities. Hence, implementation issues are discussed and significant special cases in which the conditions can be cast into a linear programming problem of finite dimension are presented.

KEYWORDS

input-to-state stability, positive systems, time-delay systems, time-varying systems

1 | INTRODUCTION

Over the last decades, a large amount of research has been devoted to analysis and control of dynamical systems with time-delays, due to their importance in many fields of applied mathematics and engineering. Although significant challenges arise in analyzing and controlling systems with delays, nowadays one can confidently affirm that a quite general core of standard results has been established for many classes of delay systems, including stability analysis tools, state estimation techniques, and a certain number of stabilization and control results. We refer the reader to the deservedly well-known monographs by Emilia Fridman[1] and Silviu-Iulian Niculescu,[2] among others. On the other hand, establishing results for time-varying delay systems is even more challenging, and this explains why less literature has been devoted to this very general class of systems. This paper focuses on stability analysis for linear continuous-time systems with time-varying matrices and multiple time-varying delays, briefly 'time-varying delay differential systems' henceforth. Analyzing such systems is known to be very challenging, and conservative approaches have been proposed in the literature based on popular extensions of the classical Lyapunov theory established by Krasovskii and Razumikhin. Among these results, we may reference the recent contributions by Zhou and Egorov,[3] Mazenc and Malisoff,[4] and Li et al.[5] Other recent approaches have leveraged trajectory-based methodologies[6] and Halanay-like inequalities[7]; an interesting idea is to look for a continuous-time counterpart[8] of the popular discrete-time approach based on lifted switched representations.[9–11]

This work follows a different approach, which is mainly based on state-bounding and the properties of positive systems. The latter have been popular since the 70's, and held in high esteem due to their "deep and elegant theory – yet pleasantly consistent with intuition" (in the words of David G. Luenberger[12]). Positive systems[13,14] are dynamic systems whose state and output variables can only be nonnegative whenever nonnegative initial conditions and inputs are

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applied. They have a large field of applications that ranges from biology to economics, where positive systems of many different classes can be encountered. Concerning positive delay systems, a prominent literature has shown that they possess remarkable properties that vastly simplify otherwise complex (or even prohibitive) problems. This is the case, for example, for stability analysis, which becomes almost trivial for positive delay systems with constant matrices, as shown in quite a number of works.[15–17] Among such nice properties, two are undeniably appealing: the possibility to use linear Lyapunov functions, yielding stability tests in form of Linear Programming (LP) feasibility problems, and the *robustness* with respect to the magnitude of the delays. In fact, a positive delay system with constant matrices is either stable for all possible delays, or not stable at all,[15,16,18–20] which means that the delay-dependent and delay-independent stability analyses collapse into the latter.

This motivates why researchers have focused on an interesting idea: trying to somehow apply the favorable stability properties of positive systems also to general ones. This idea has been approached with different techniques, whose common denominator is the so-called usage of comparison systems.[19,21,22] Unfortunately, when time-varying matrices are modeled, the above mentioned favorable properties of positive delay systems are greatly diminished: recent work shows, for example, that delay-dependent and delay-independent stability are no longer equivalent.[11,23] This explains the first motivation of this work: to explore the stability properties of time-varying linear positive systems with time-varying delays and to investigate if and how these properties can be applied also to general systems of the same class, possibly with robustness against time-varying delays of any magnitude.

1.1 | Previous work and contributions

This work can be seen as a continuous-time counterpart of a recent contribution[11] which dealt with time-varying discrete-time systems with time-varying delays, and proposed novel delay-dependent and delay-independent conditions of global exponential stability and input-to-state stability for both positive and general systems of that class. We previously explored with similar techniques the class of delay differential systems with time-varying delays, although these previous works[24,25] addressed the simpler case of constant matrices and did not study input-to-state stability.

The contributions introduced in this work are as follows:

- We first introduce two different (although seemingly dual) conditions that define forward invariant sets for linear time-varying positive delay differential systems.
- Then, by using a state-bounding approach that leverages on the properties of positive systems, we show that such forward invariance conditions also prove delay-independent exponential input-to-state stability, and provide explicit gains. The conditions, which remarkably hold for time-varying delays, are first given for positive systems and then extended to general systems. They are computationally appealing: albeit involving infinitely many linear inequalities in the general case of time-varying matrices, they can be checked via LP solvers and are negligibly affected by the number of delays.
- As simple corollaries of the aforementioned conditions, we introduce novel delay-independent conditions of global exponential stability for linear time-varying delay differential systems, with guaranteed bound on the convergence rate as a function of the maximum delay.
- We further delve into computational aspects and implementation issues, discussing how the conditions can be applied in important cases (such as, e.g., the case of systems with periodic matrices), and providing analytical stability conditions for systems with two state components.

The proposed conditions are certainly conservative, still they are among the simplest available conditions of stability for time-varying delay systems in terms of required computational burden and simplicity of presentation. At the end of Section 4, we discuss the position of such conditions in the vast literature on the stability of time-delay systems. In particular, we discuss their relationship with the popular Lyapunov–Krasovskii (LK) and Lyapunov–Razumikhin (LR) approaches (both time-invariant and time-varying), highlighting the fact that one of the conditions introduced in this work for positive systems can also be derived via the Razumikhin approach (although only proving asymptotic stability). When considered for input-free systems, the latter condition bears some resemblance with a similar approach followed in recent contributions[21,26] that are also based on linear Lyapunov functionals and properties of positive systems. Such similarities and differences are discussed in detail in Remarks 6 and 8.

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As briefly mentioned above, we first give stability conditions for positive systems and then extend the analysis to general systems. The tool we exploit to perform such a generalization is that of comparison systems, in the form of Internally Positive Representations, a methodology that can be now considered established to tackle and solve a variety of problems,[25,27–30] and can be compared with similar methodologies that have been used in interval observer design for various classes of delay-free and delayed systems.[31–34]

A limited and preliminary part of the material of this paper was presented in a conference paper.[35] In detail, Corollary 2 and Corollary 5 first appeared in the aforementioned paper, whereas Corollary 4 was stated with no proof. All the other results in this paper are novel and did not appear in the conference version. The latter also described the special case of switching models with delays, which is not reported here to avoid repetition.

1.2 | Paper organization

The material is presented as follows: the Introduction is closed by an overview of the notation used throughout the work. Section 2 introduces the class of systems investigated in this paper and some preliminary definitions. Section 3 develops the stability analysis in the case of positive systems: it starts from some preliminary remarks on the special case of systems with time-invariant matrices and then establishes conditions of forward invariance and delay-independent exponential input-to-state stability for time-varying delay differential systems. Conditions of internal stability are briefly given as corollaries. Section 4 discusses the comparison systems approach, and generalizes all the results of the previous section to the case in which no sign constraint is imposed on system matrices. Finally, Section 5 discusses implementation issues and special cases, with illustrative examples. Conclusions and hints for future work follow.

1.3 | Notation

The following notation is used throughout the work. The symbols $\mathbb{R}_+$ and $\mathbb{R}_{++}$, respectively, denote the set of nonnegative and positive real numbers. $\mathbb{R}^n_+$ ($\mathbb{R}^n_{++}$) is the nonnegative (positive) orthant of $\mathbb{R}^n$. $\mathbb{R}^{m\times n}_+$ is the cone of nonnegative $m \times n$ matrices. $\mathbb{1}_n$ is the column vector consisting of n ones, I_n is the $n \times n$ identity matrix and $0_{m\times n}$ is the $m \times n$ matrix consisting of all zeros. Inequalities among vectors and matrices of the same dimensions are meant to be understood component-wise, that is, $M \leq N$ ($M < N$) if $[M]_{i,j} \leq [N]_{i,j}$ ($[M]_{i,j} < [N]_{i,j}$) for all i, j, where $[M]_{i,j}$ denotes the i, j entry of matrix M. In this sense, a nonnegative matrix $M \in \mathbb{R}^{m\times n}_+$ can also be denoted by $M \geq 0_{m\times n}$, or briefly $M \geq 0$. We will also denote $|M|$ for the component-wise absolute value of matrix M. A square matrix M is *Metzler* if all its off-diagonal components are nonnegative, that is, $[M]_{i,j} \geq 0$ for $i \neq j$. $\overline{\lambda}(M)$ denotes the spectral abscissa of the square matrix M, that is, $\overline{\lambda}(M) = \max_i\{\mathrm{Re}(\lambda_i)\}$, λ_i being the eigenvalues of M. For any $\overline{\delta} > 0$, $C([-\overline{\delta}, 0]; \mathbb{R}^n)$, or the shorthand notation $C^n_{\overline{\delta}}$, denotes the Banach space of continuous functions from the interval $[-\overline{\delta}, 0]$ to $\mathbb{R}^n$, with the usual sup-norm ($\|\phi\|_\infty = \sup_{t\in[-\overline{\delta},0]} \|\phi(t)\|_\infty$), while $C^{n,+}_{\overline{\delta}} = C([-\overline{\delta}, 0]; \mathbb{R}^n_+)$ denotes the cone of nonnegative continuous functions. When considering two functions $\phi_1, \phi_2 \in C^n_{\overline{\delta}}$, the inequality $\phi_1 \leq \phi_2$ means that $\phi_1(\tau) \leq \phi_2(\tau)$, $\forall \tau \in [-\overline{\delta}, 0]$ (component-wise).

2 | PROBLEM FORMULATION AND PRELIMINARIES

This work deals with the stability analysis of linear time-varying delay differential systems with time-varying delays, described by:

$$\begin{aligned} \dot{x}(t) &= A_0(t)x(t) + \sum_{j=1}^{r} A_j(t)x(t - \delta_j(t)) + B(t)u(t), \quad t \geq 0, \\ x(t) &= \phi(t), \qquad\qquad t \in [-\overline{\delta}, 0], \end{aligned} \tag{1}$$

where $u(t) \in \mathbb{R}^p$ is the input, $\delta_j(t)$ are time-varying delays taking values in $[0, \overline{\delta}]$, for $j = 1, \dots, r$, and $\overline{\delta} \in \mathbb{R}_+$ is the maximum delay. All matrices $A_j(t)$, $j = 0, \dots, r$, and $B(t)$, whose sizes are trivially understood, are assumed to be uniformly

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bounded and locally integrable in $[0, \infty)$, so that the solution $x(t)$ exists $\forall t \geq 0$. The function $\phi : [-\overline{\delta}, 0] \mapsto \mathbb{R}^n$ defines the initial condition of system (1), and is assumed to be in $C([-\overline{\delta}, 0]; \mathbb{R}^n)$.

$x_t : [-\overline{\delta}, 0] \mapsto \mathbb{R}^n$ denotes the system state at a given time $t \geq 0$, defined as

$$x_t(\tau) = x(t + \tau), \quad \tau \in [-\overline{\delta}, 0], \tag{2}$$

with $x(t) \in \mathbb{R}^n$. With this notation, the system initialization at time $t = 0$ is compactly written as $x_0 = \phi \in C^n_{\overline{\delta}}$.

Given a subset $S \subset \mathbb{R}^n$, the symbol $S^{[-\overline{\delta},0]}$ denotes the following subset of $C^n_{\overline{\delta}}$

$$S^{[-\overline{\delta},0]} = \left\{ \phi \in C([-\overline{\delta}, 0]; \mathbb{R}^n) : \ \phi(\tau) \in S, \ \forall \tau \in [-\overline{\delta}, 0] \right\}. \tag{3}$$

For compactness of notation, the dummy delay $\delta_0(t) = 0$, $\forall t \geq 0$, is defined, so that the term $A_0(t)x(t)$ in the system equation (1) can be included in the summation:

$$\dot{x}(t) = \sum_{j=0}^{r} A_j(t)x(t - \delta_j(t)) + B(t)u(t), \quad t \geq 0. \tag{4}$$

The compact notation (4) for the state equation of system (1) will be used in the remainder of the paper, although explicit references to the system in statements may point to the equivalent form (1).

Throughout the work, we will employ the usual definitions of stability for the system (1). We refer the reader to Zhou and Egorov,[3] Chaillet et al.,[36] and Mironchenko et al.[37] for further details. In particular, we will briefly write 0-GES to denote the global exponential stability of the origin for $u \equiv 0$, and exp-ISS for exponential input-to-state stability (see Def. 3.3 of Mironchenko et al.[37]). The definition of exp-ISS with linear gains is reported in Def. 2 of Chaillet et al.[38] The terminology "delay-independent" stability will be used to mean that the stability of (1) holds robustly against bounded time-varying delays of any magnitude, that is, for any $\delta_j(t) \in [0, \overline{\delta}]$ and any given $\overline{\delta} \in \mathbb{R}_+$, $j = 1, \ \dots \ , r$.

We will gradually unveil a number of results on system (1). First, in Section 3 we will start from the positive case, illustrating conditions of forward invariance and exponential input-to-state stability; conditions of global exponential stability for the case of zero input will be given as corollaries. Then, in Section 4 we will generalize such results to systems with no sign constraints exploiting comparison systems and the state-bounding approach.

3 | STABILITY ANALYSIS OF POSITIVE TIME-VARYING DELAY DIFFERENTIAL SYSTEMS

Let us start considering positive systems as in (1). The usual definition of positivity applies to system (1). Namely, such system is positive if $\phi \in C^{n,+}_{\overline{\delta}}$ and $u(t) \geq 0, \forall t \geq 0$, imply that $x(t) \geq 0, \forall t \geq 0$. To be precise, one should distinguish among internal positivity (in which the state must be nonnegative) and external positivity (where the output is required to be nonnegative but the state might not be). Due to the fact that in this paper we focus on stability and do not model outputs, the specification is removed and we briefly write "positive system" to mean "internally positive system." Simple necessary and sufficient conditions do exist on system matrices $A_j(t), B(t)$ to ensure that the aforementioned definition is fulfilled. The next Proposition is a special case (no distributed delays) of Lemma 1 in De Iuliis et al.,[25] which itself is a generalization of Thm. II.2 by Ngoc[16] to the case of time-varying delays.

Proposition 1. *System (1) is positive if and only if $A_0(t)$ is Metzler for all $t \geq 0$, and $A_j(t), B(t)$ are nonnegative for all $t \geq 0$, and $j = 1, \ \dots \ , r$.*

This section introduces stability results for positive delay systems. As a follow up to our preliminary contributions of De Iuliis and Manes,[35] which only dealt with internal stability, we here provide delay-independent conditions of exp-ISS. It has to be remarked that the fact that internal stability implies input-to-state stability is well-known for the context of linear delay systems (1) in the case of time-invariant matrices (see, e.g., Thm. 22 in Chaillet et al.[36]). Although less known, the same implication for time-varying delay systems can be deduced by Thm. 3.4 in Yeganefar et al.,[39] exploiting a converse Lyapunov theorem proved in Lemma 33.1 of Krasovskii.[40] On the other hand, what we show here is that the conditions of internal stability first introduced in De Iuliis and Manes,[35] which do not explicitly rely on the existence of a

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Lyapunov function, imply exp-ISS with specified linear gains. This contribution cannot be seen as a trivial consequence of the aforementioned results.

We will now illustrate some preliminary facts about the stability of (1) and then proceed to the main results of this section.

For the system (1) satisfying the positiveness conditions of Proposition 1, consider the two following conditions on the system matrices $A_j(t), j = 0, \dots, r$:

Right-positive vector condition

$$\begin{matrix} \exists q \in \mathbb{R}^n_{++} \\ \exists \beta \in \mathbb{R}_{++} \end{matrix} \quad : \quad \left(\sum_{j=0}^{r} A_j(t)\right) q \leq -\beta\, q, \quad \forall t \geq 0; \tag{5}$$

Left-positive vector condition

$$\begin{matrix} \exists h \in \mathbb{R}^n_{++} \\ \exists \beta \in \mathbb{R}_{++} \end{matrix} \quad : \quad h^\top \left(\sum_{j=0}^{r} A_j(t)\right) \leq -\beta\, h^\top, \quad \forall t \geq 0. \tag{6}$$

It is important to note that in the case of *zero-delay* systems (i.e., $\delta_j(t) \equiv 0$ in (1)), both conditions (5) and (6) ensure the 0-GES of the system (and hence ISS). For instance, if the left-condition (6) holds true, then $V(x) = h^\top x$ is a Linear Co-positive Lyapunov Function (LCLF) for the zero-delay positive system $\dot{x}(t) = \sum_{j=0}^{r} A_j(t)x(t)$. Indeed,

$$\dot{V}(x(t)) = h^\top \left(\sum_{j=0}^{r} A_j(t)\right) x(t) \leq -\beta\, h^\top x(t) = -\beta V(x(t)), \quad t \geq 0, \tag{7}$$

so that $h^\top x(t) \leq e^{-\beta t} h^\top x(0)$, which implies $\|x(t)\|_1 \leq e^{-\beta t}\|h\|_\infty \|x(0)\|_1$. Analogously, the right-condition (5) implies the existence of a LCLF on the transposed zero-delay system. Notice that the terminology *transposed system* here only stands for the simple transposition of system matrices, with no reference to the adjoint system.

The stability properties of the zero-delay system associated to (1) are remarkably significant in the special case of constant matrices A_j; in this case, under positivity assumptions, it is known[17,25] that the 0-GES of the zero-delay system is necessary and sufficient for the robust (i.e., delay-independent) 0-GES of the delay system. In this special case, the right-condition (5) and the left-condition (6) are equivalent to $\overline{\lambda}(\sum_{j=0}^{r} A_j) < 0$.

Unfortunately, the stability of the zero-delay system is not sufficient anymore to prove the stability of a positive delay system, in general, when time-varying matrices are considered (for a simple counterexample in discrete time, see, e.g., Ex. 3 in De Iuliis and Manes[23]). We may then wonder whether any of conditions (5) and (6) may still be sufficient to prove the robust stability of a time-varying positive delay system. The remainder of this section provides an answer to this question.

Interestingly, we will see below that in the case of delay systems with time-varying matrices the left- and right-conditions are not equivalent: the right-condition (5) is still sufficient to prove delay-independent stability, while a stronger assumption than the left-condition (6) is needed to prove delay-independent stability.

Firstly, we provide a result concerning the determination of a forward invariant set (or positively invariant set) for a positive delay differential system as in (1) under bounded input $u(t)$.

Theorem 1. *Consider a system described by (1), where $A_0(t)$ is Metzler and $A_j(t)$ are nonnegative, for $j = 1, \dots, r$, with bounded initial state $\phi : [-\overline{\delta}, 0] \to \mathbb{R}^n_+$. Assume that $B(t)$ and the input $u(t)$ are nonnegative and uniformly bounded, with $\overline{B} \in \mathbb{R}^{n\times p}_+$ and $\bar{u} \in \mathbb{R}^p_+$ denoting the corresponding bounds, that is, $0 \leq B(t) \leq \overline{B}$ and $u(t) \leq \bar{u}$, $\forall t \geq 0$. Moreover, assume that the right-condition (5) holds true, and define $\overline{q}$ as*:

$$\overline{q} = \frac{1}{q_{min}} q, \quad with \quad q_{min} = \min_{i\in[1,n]} (q_i). \tag{8}$$

and let

$$\rho = \frac{1}{\beta} \|\overline{B}\bar{u}\|_\infty. \tag{9}$$

Then, the set (hyperbox) $H_\rho = \{x \in \mathbb{R}^n_+ : \; x \leq \overline{q}\rho\} \subset \mathbb{R}^n_+$ is a forward invariant set for the system (1), that is, if, for some $\bar{t} \geq 0$, $x(t) \leq \overline{q}\rho$, $\forall t \in [\bar{t} - \overline{\delta}, \bar{t}]$, then $x(t) \leq \overline{q}\rho$, $\forall t \in [\bar{t}, \infty)$.

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Remark 1. A more rigorous formulation of Theorem 1 should employ the system state x_t as defined in (2) and subsets of $C_{\bar{\delta}}^{n,+}$ of the type defined in (3). Using these notations, the thesis of the Theorem states that the set $H_\rho^{[-\bar{\delta},0]} \subset C_{\bar{\delta}}^{n,+}$ is a forward invariant set for the system (1), that is, if, for some $\bar{t} \geq 0$, $x_{\bar{t}} \in H_\rho^{[-\bar{\delta},0]}$, then $x_t \in H_\rho^{[-\bar{\delta},0]}$, $\forall t \geq \bar{t}$.

Proof. Note first that, thanks to the inequality $\mathbb{1}_n \leq \overline{q}$ and by the definition of ρ, we have

$$\overline{B}\bar{u} \leq \mathbb{1}_n \|\overline{B}\bar{u}\|_\infty \leq \overline{q}\|\overline{B}\bar{u}\|_\infty = \overline{q}\rho\beta, \tag{10}$$

from which

$$-\overline{q}\rho\beta + \overline{B}\bar{u} \leq 0. \tag{11}$$

On the other hand, from the right-condition (5)

$$\left(\sum_{j=0}^{r} A_j(t)\right)\overline{q}\rho + \overline{B}\bar{u} \leq -\beta\ \overline{q}\rho + B\bar{u} \leq 0. \tag{12}$$

Now consider the positive system with an exogenous input $v(t) \geq 0$

$$\dot{\bar{x}}(t) = A_0(t)\bar{x}(t) + \sum_{j=1}^{r} A_j(t)\bar{x}(t - \delta_j(t)) + \overline{B}\bar{u} + v(t), \quad t \geq 0. \tag{13}$$

It is easily understood that, for any nonnegative $v(t)$, the state $\bar{x}_t$ of system (13) is an upper bound of the state x_t of (1). Indeed, comparing the solutions $\bar{x}(t)$ of (13) and $x(t)$ of (1) we get that if at a given $\bar{t} \geq 0$ it happens that $x(\tau) \leq \bar{x}(\tau)$, for $\tau \in [\bar{t} - \bar{\delta}, \bar{t}]$ (i.e., $x_{\bar{t}} \leq \bar{x}_{\bar{t}}$), then $x(t) \leq \bar{x}(t)$ for all $t \in [\bar{t}, \infty)$ (i.e., $x_t \leq \bar{x}_t$, $\forall t \geq \bar{t}$).

Thus, the proof of the Theorem is obtained by showing that with the choice

$$v(t) = -\left(\left(\sum_{j=0}^{r} A_j(t)\right)\overline{q}\rho + \overline{B}\bar{u}\right) \geq 0, \tag{14}$$

which is nonnegative thanks to (12), the solution $\bar{x}(t)$ of (13) for $\bar{x}(\tau) = \overline{q}\rho$, $\tau \in [-\bar{\delta}, 0]$ is the constant $\bar{x}(t) = \overline{q}\rho$. Indeed, replacing $\bar{x}(t) = \overline{q}\rho$ in (13), with $v(t)$ as in (14), we easily obtain $\dot{\bar{x}}(t) = 0$. This implies that if $x(\tau) \leq \overline{q}\rho$, $\tau \in [\bar{t} - \bar{\delta}, \bar{t}]$ (i.e., $x_{\bar{t}} \in H_\rho^{[-\bar{\delta},0]}$), then $x(t) \leq \overline{q}\rho$, $t \in [\bar{t}, +\infty)$ (i.e., $x_t \in H_\rho^{[-\bar{\delta},0]}$, $\forall t \geq \bar{t}$), which concludes the proof. ∎

Corollary 1. *Under the same assumptions of Theorem 1, the system (1) is such that when $x(\tau) = 0$, $\tau \in [-\bar{\delta}, 0]$, it follows*

$$x(t) \leq \frac{\|\overline{B}\bar{u}\|_\infty}{\beta}\overline{q}, \quad \forall t \geq 0, \tag{15}$$

where $\overline{q}$ is defined as in (8).

Now we can show that the right-condition (5) implies exp-ISS for the system (1) under positivity conditions.

Theorem 2. *Consider system (1) under the same positivity condition of Theorem 1 and uniformly bounded $B(t)$ and $u(t)$. If the right-condition (5) holds true, then the system (1) is delay-independent exp-ISS.*

More specifically, denoting with $\overline{B}$ and $\bar{u}$ the upper bounds of $B(t)$ and $u(t)$, the state evolves according to the following bound:

$$x(t) \leq \overline{q}\|\phi\|_\infty e^{-\alpha t} + \frac{\|\overline{B}\bar{u}\|_\infty}{\beta}\overline{q}, \quad \forall t \geq 0, \tag{16}$$

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where $\overline{q}$ as in (8) and $\alpha > 0$ is any positive real that satisfies

$$(\alpha I_n + A_0(t))q \leq -e^{\alpha\overline{\delta}}\left(\sum_{j=1}^{r} A_j(t)\right)q. \tag{17}$$

Proof. The proof is divided in two parts. The first part proves stability with guaranteed exponential bound for zero input and nonzero initial state (0-GES). This part of the proof first appeared in Thm. 1 of De Iuliis and Manes,[35] and is here reported for reader's convenience. The second part of the proof concerns the case of zero initial state and nonzero input. The complete statement of the theorem directly follows by the linearity of system (1).

Part I. Note first that if the right-condition (5) holds true, then for any $\overline{\delta}$ there exists $\overline{\alpha}_{\overline{\delta}} \in (0, \beta)$, such that (17) is verified for all $\alpha \in [0, \overline{\alpha}_{\overline{\delta}}]$. This can be easily proved by rewriting (5) as

$$(\beta I_n + A_0(t))q \leq -\left(\sum_{j=1}^{r} A_j(t)\right)q. \tag{18}$$

The inequality (5), written in the form (18), clearly implies that (17) is *strictly* satisfied for $\alpha = 0$. Since both terms of inequality (17) are continuous functions of α, it follows that there exists a positive $\overline{\alpha}_{\overline{\delta}} < \beta$ such that the strict inequality holds true for any $\alpha \in [0, \overline{\alpha}_{\overline{\delta}})$, and the nonstrict version (17) holds in $[0, \overline{\alpha}_{\overline{\delta}}]$.

Given the assumptions on the matrices $A_j(t)$, for any nonnegative initial condition ϕ, the solution $x(t)$ of system (1) is nonnegative for all $t \geq -\overline{\delta}$. Let us define the variable $\eta(t) \in \mathbb{R}^n$ as follows

$$\eta(t) = \overline{q}\|\phi\|_\infty e^{-\alpha t} - x(t), \quad t \geq -\overline{\delta}. \tag{19}$$

Note that $\overline{q}$ defined in (8) is such that $\overline{q} \geq \mathbb{1}_n$, and therefore $\phi(t) \leq \overline{q}\|\phi\|_\infty$, $\forall t \in [-\overline{\delta}, 0]$. Thus, the inequality

$$x(t) \leq \overline{q}\|\phi\|_\infty e^{-\alpha t}, \tag{20}$$

is satisfied in $[-\overline{\delta}, 0]$ and the function $\eta(t)$ is nonnegative in the same interval $[-\overline{\delta}, 0]$. Computing the time derivative of $\eta(t)$, for $t \geq 0$, we get

$$\dot{\eta}(t) = -\alpha\overline{q}\|\phi\|_\infty e^{-\alpha t} - \dot{x}(t) = -\alpha\overline{q}\|\phi\|_\infty e^{-\alpha t} - \sum_{j=0}^{r} A_j(t)x(t - \delta_j(t)). \tag{21}$$

By the definition (19) of $\eta(t)$ we have

$$x(t - \delta_j(t)) = \overline{q}\|\phi\|_\infty e^{-\alpha(t-\delta_j(t))} - \eta(t - \delta_j(t)). \tag{22}$$

Substitution into (21) gives

$$\dot{\eta}(t) = -\alpha\overline{q}\|\phi\|_\infty e^{-\alpha t} - \|\phi\|_\infty \sum_{j=0}^{r} A_j(t)\overline{q}\, e^{-\alpha(t-\delta_j(t))} + \sum_{j=0}^{r} A_j(t)\eta(t - \delta_j(t)). \tag{23}$$

Now let us define function $v(t)$, for $t \geq 0$, as

$$v(t) = -\alpha\, \overline{q}\|\phi\|_\infty\, e^{-\alpha t} - \|\phi\|_\infty \sum_{j=0}^{r} A_j(t)\overline{q}\, e^{-\alpha(t-\delta_j(t))} = \left(-\alpha\, \overline{q} - \sum_{j=0}^{r} A_j(t)\overline{q}\, e^{\alpha\delta_j(t)}\right)\|\phi\|_\infty e^{-\alpha t}. \tag{24}$$

Recalling that we are using the compact notation (4), with dummy delay $\delta_0(t) = 0$, and noting that

$$e^{\alpha\overline{\delta}} \geq e^{\alpha\delta_j(t)}, \quad \forall\delta_j(t) \in [0, \overline{\delta}], \tag{25}$$

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from inequality (17), which holds true by assumption, it easily follows that

$$0 \leq \left(-\alpha\ I_n - A_0(t) - e^{\alpha\overline{\delta}} \left(\sum_{j=1}^{r} A_j(t) \right) \right) \overline{q} \leq \left(-\alpha\ \overline{q} - \sum_{j=0}^{r} A_j(t) \overline{q}\ e^{\alpha\delta_j(t)} \right), \tag{26}$$

so that the function $v(t)$ defined in (24) is nonnegative. Thus, function $\eta(t)$ obeys the following differential equation:

$$\begin{aligned} \dot{\eta}(t) &= \sum_{j=0}^{r} A_j(t)\eta(t-\delta_j(t)) + v(t), \quad t \geq 0 \\ \eta(t) &= \overline{q}\|\phi\|_\infty\ e^{-\alpha t} - \phi(t) \geq 0, \quad t \in [-\overline{\delta}, 0], \end{aligned} \tag{27}$$

which defines a positive system, with nonnegative initial condition and nonnegative input $v(t)$. Thus, $\eta(t) \geq 0$, $\forall t \geq 0$, which implies that for zero input and nonzero initial state (20) is satisfied for all $t \geq -\overline{\delta}$, thanks to the definition (19) of $\eta(t)$.

Part II. In the case of zero initial state and nonzero input, Corollary 1 provides the bound (15), which coincides with the second term in the right-hand side of inequality (16).

In the case of nonzero initial state and nonzero input the bound (16) directly follows by the linearity of system (1). ∎

Remark 2. Note that any α satisfying (17) is a guaranteed exponential convergence rate, and there is a clear trade-off between the maximum delay $\overline{\delta}$ and the rate α: the smaller is $\overline{\delta}$, the larger α that satisfies (17) can be. It is easy to see that $\alpha \to \beta$ when $\overline{\delta} \to 0$. Also, note that exp-ISS with linear gains can be straightforwardly obtained on the lines of Remark 9.

It is clear that Theorem 2 implies the following corollary of 0-GES for a positive system as in (1) for $u \equiv 0$.

Corollary 2. *Consider system (1), where $A_0(t)$ is Metzler and $A_j(t)$ are nonnegative, for $j = 1, \dots, r$, with bounded initial state $\phi : [-\overline{\delta}, 0] \mapsto \mathbb{R}^n_+$. If the right-condition (5) holds true, then the system is delay-independent 0-GES, and for any given maximum delay $\overline{\delta} \in \mathbb{R}_+$, the following exponential bound holds true $\forall t \geq -\overline{\delta}$*

$$x(t) \leq \overline{q}\|\phi\|_\infty e^{-\alpha t}, \tag{28}$$

where $\overline{q}$ is as (8), and $\alpha > 0$ is any positive real satisfying (17).

As discussed above, if there was a perfect analogy with the case of constant matrices, also (6) could allow to conclude stability for (1) with time-varying matrices. However, this is not the case.

A stronger left-condition, which implies (6), is needed to prove the delay-independent stability in the case of time-varying matrices, as will be shown below. The condition is as follow:

Strong left-positive vector condition

$$\exists \beta \in \mathbb{R}_{++}, \qquad h^\top A_0(t) \leq -\tilde{\alpha}_0(t)\ h^\top, \tag{29}$$

$$\exists h \in \mathbb{R}^n_{++}, \qquad : \qquad h^\top A_j(t) \leq\ \tilde{\alpha}_j(t)\ h^\top, \quad j = 1, \dots, r, , \quad \forall t \geq -\overline{\delta}, \tag{30}$$

$$\exists \tilde{\alpha}_j(t) \geq 0,\ j = 0, \dots, r \qquad \tilde{\alpha}_0(t) \geq \beta + \sum_{j=1}^{r} \tilde{\alpha}_j(t), \tag{31}$$

Remark 3. It is easy to check that the set of strong left-conditions (29)–(31) imply the left-condition (6) (simply sum-up the conditions (29) and (30) and exploit (31)), but, in general, the left-condition (6) does not imply the conditions (29)–(31). This explains why we use the adjective *strong* to refer to (29)–(31). Note that a stronger condition is the one that requires that all $\tilde{\alpha}_j$ in the inequalities (29)–(31) are constant.

We first proceed establishing a forward invariance property for sets related to such strong left-conditions.

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Theorem 3. *Consider a system described by (1), where $A_0(t)$ is Metzler and $A_j(t)$ are nonnegative, for $j = 1, \dots, r$, with bounded initial state $\phi : [-\overline{\delta}, 0] \to \mathbb{R}^n_+$. Assume that $B(t)$ and the input $u(t)$ are nonnegative and uniformly bounded, with $\overline{B} \in \mathbb{R}^{n\times p}_+$ and $\bar{u} \in \mathbb{R}^p_+$ denoting the corresponding bounds, that is, $0 \le B(t) \le \overline{B}$ and $u(t) \le \bar{u}$, $\forall t \ge 0$. Under the strong left-condition (29)–(31), the set*

$$S_{\bar{u}} = \left\{ x \in \mathbb{R}^n_+ : \ h^\top x \le \frac{h^\top \overline{B}\bar{u}}{\beta} \right\}, \tag{32}$$

is a forward invariant set for the system (1), that is, if, for some $\bar{t} \ge 0$, $x(\tau) \in S_{\bar{u}}$, $\forall \tau \in [\bar{t} - \overline{\delta}, \bar{t}]$, then $x(t) \in S_{\bar{u}}$, $\forall t \in [\bar{t}, \infty)$.

Remark 4. As done in Remark 1, by using the system state x_t as defined in (2) and subsets of $C^{n,+}_{\overline{\delta}}$ of the type defined in (3), the statement of (3) can be rephrased as follows: the set $S^{[-\overline{\delta},0]}_{\bar{u}} \subset C^{n,+}_{\overline{\delta}}$ is a forward invariant set for the system (1), i.e. if, for some $\bar{t} \ge 0$, $x_{\bar{t}} \in S^{[-\overline{\delta},0]}_{\bar{u}}$, then $x_t \in S^{[-\overline{\delta},0]}_{\bar{u}}$, $\forall t \ge \bar{t}$.

Proof. Consider the positive system

$$\dot{\overline{x}}(t) = A_0(t)\overline{x}(t) + \sum_{j=1}^{r} A_j(t)\overline{x}(t - \delta_j(t)) + \overline{B}\bar{u}, \quad t \ge 0, \tag{33}$$

which is a majorant of system (1), that is, if $x_{\bar{t}} \le \overline{x}_{\bar{t}}$ at a given $\bar{t} \ge 0$, then $x(t) \le \overline{x}(t)$, $\forall t \ge \bar{t}$. Now define the nonnegative variable

$$\zeta(t) = h^\top \overline{x}(t), \tag{34}$$

which obeys the differential equation

$$\dot{\zeta}(t) = h^\top A_0(t)\overline{x}(t) + \sum_{j=1}^{r} h^\top A_j(t)\overline{x}(t - \delta_j(t)) + h^\top \overline{B}\bar{u}, \quad t \ge 0. \tag{35}$$

Thanks to inequalities (29)–(30) we get

$$\dot{\zeta}(t) \le -\tilde{\alpha}_0(t)h^\top \overline{x}(t) + \sum_{j=1}^{r} \tilde{\alpha}_j(t)h^\top \overline{x}(t - \delta_j(t)) + h^\top \overline{B}\bar{u} = -\tilde{\alpha}_0(t)\zeta(t) + \sum_{j=1}^{r} \tilde{\alpha}_j(t)\zeta(t - \delta_j(t)) + h^\top \overline{B}\bar{u}. \tag{36}$$

Thus, there exists a nonnegative function $\gamma(t)$ such that the nonnegative variable $\zeta(t)$ satisfies the differential equation

$$\dot{\zeta}(t) = -\tilde{\alpha}_0(t)\zeta(t) + \sum_{j=1}^{r} \tilde{\alpha}_j(t)\zeta(t - \delta_j(t)) + h^\top \overline{B}\bar{u} - \gamma(t). \tag{37}$$

Now, consider the variable $\overline{\zeta}(t)$ satisfying, for $t \ge 0$ and $\nu(t) \ge 0$,

$$\dot{\overline{\zeta}}(t) = -\tilde{\alpha}_0(t)\overline{\zeta}(t) + \sum_{j=1}^{r} \tilde{\alpha}_j(t)\overline{\zeta}(t - \delta_j(t)) + h^\top \overline{B}\bar{u} + \nu(t). \tag{38}$$

It can be promptly shown that, for any nonnegative $\nu(t)$, if $\zeta_{\bar{t}} \le \overline{\zeta}_{\bar{t}}$ for some $\bar{t} \ge 0$, then $\zeta(t) \le \overline{\zeta}(t)$ for all $t \ge 0$. This result is simply proved by defining the variable $\eta(t) = \overline{\zeta}(t) - \zeta(t)$, which solves the following differential equation for $t \ge \bar{t}$

$$\dot{\eta}(t) = -\tilde{\alpha}_0(t)\eta(t) + \sum_{j=1}^{r} \tilde{\alpha}_j(t)\eta(t - \delta_j(t)) + \nu(t) + \gamma(t), \tag{39}$$

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starting from a nonnegative initialization $\eta_{\bar{t}} = \overline{\zeta}_{\bar{t}} - \zeta_{\bar{t}}$. As a consequence the solution $\eta(t)$ remains nonnegative for all $t \geq \bar{t}$.

Now, consider system (38) such that at a given time $\bar{t}$ the state $\overline{\zeta}_{\bar{t}}$ is as follows

$$\overline{\zeta}_{\bar{t}}(\tau) = \overline{\zeta}(\bar{t} + \tau) = \frac{h^\top \overline{B}\bar{u}}{\beta}, \quad \tau \in [-\overline{\delta}, 0], \tag{40}$$

and with the following input function:

$$\nu(t) = -\left(-\tilde{\alpha}_0(t) + \sum_{j=1}^{r} \tilde{\alpha}_j(t) + \beta \right) \frac{h^\top \overline{B}\bar{u}}{\beta} \geq 0, \tag{41}$$

which is nonnegative thanks to the property (31). Then, after replacing $\nu(t)$ into (38), we get that the solution $\overline{\zeta}(t)$ obeys the following differential equation

$$\dot{\overline{\zeta}}(t) = -\tilde{\alpha}_0(t)\left(\overline{\zeta}(t) - \frac{h^\top \overline{B}\bar{u}}{\beta} \right) + \sum_{j=1}^{r} \tilde{\alpha}_j(t)\left(\overline{\zeta}(t - \delta_j(t)) - \frac{h^\top \overline{B}\bar{u}}{\beta} \right), \tag{42}$$

whose unique solution for $t \geq \bar{t}$, compatible with the initial condition (40), is

$$\overline{\zeta}(t) = \frac{h^\top \overline{B}\bar{u}}{\beta}, \quad t \geq \bar{t}. \tag{43}$$

Thus, we have proved that $\zeta(\tau) \leq h^\top \overline{B}\bar{u}/\beta$ for $\tau \in [\bar{t} - \overline{\delta}, \bar{t}]$, for a given $\bar{t} \geq 0$, implies $\zeta(t) \leq h^\top \overline{B}\bar{u}/\beta$, $\forall t \geq \bar{t}$. Recalling the definition of $\zeta(t) = h^\top x(t)$, we have that if $h^\top x(\tau) \leq h^\top \overline{B}\bar{u}/\beta$, for $\tau \in [\bar{t} - \overline{\delta}, \bar{t}]$, for a given $\bar{t} \geq 0$, then $h^\top x(t) \leq h^\top \overline{B}\bar{u}/\beta$, $\forall t \geq \bar{t}$. This is exactly the forward invariance of the set $\mathcal{S}_{\bar{u}}$ (more exactly, of $\mathcal{S}_{\bar{u}}^{[-\overline{\delta},0]}$) and the Theorem is proved. ∎

Corollary 3. *Under the same assumptions of Theorem 3, the system (1) is such that when* $x(\tau) = 0$, $\tau \in [-\overline{\delta}, 0]$, *it follows*

$$h^\top x(t) \leq \frac{h^\top \overline{B}\bar{u}}{\beta}, \quad \forall t \geq 0. \tag{44}$$

Now we are ready to show that the strong left-conditions (29)–(31) also imply the exp-ISS for the system (1) under positivity conditions.

Theorem 4. *Consider system (1) under the same positivity condition of Theorem 3 and uniformly bounded* $B(t)$ *and* $u(t)$. *If the strong left-conditions (29)–(31) hold true, then the system (1) is delay-independent exp-ISS.*

More specifically, denoting with $\overline{B}$ *and* $\bar{u}$ *the upper bounds of* $B(t)$ *and* $u(t)$, *the state evolves according to the following bound*:

$$\|x(t)\|_1 \leq \frac{h_{max}}{h_{min}} \|\phi\|_1 \, e^{-\alpha t} + \frac{h^\top \overline{B}\bar{u}}{h_{min}\, \beta}, \quad \forall t \geq 0, \tag{45}$$

where $\alpha > 0$ *is any positive real that satisfies*

$$\tilde{\alpha}_0(t) \geq \alpha + \left(\sum_{j=1}^{r} \tilde{\alpha}_j(t) \right) e^{\alpha \overline{\delta}}, \tag{46}$$

and $h_{\max}, h_{\min}, \|\phi\|_1$ *are defined as*

$$h_{max} = \max_{i \in [1,n]}(h_i), \quad h_{min} = \min_{i \in [1,n]}(h_i), \quad \|\phi\|_1 = \sup_{t \in [-\overline{\delta}, 0]} \|\phi(t)\|_1. \tag{47}$$

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Proof. As for Theorem 2, the proof will be given in two parts. The first part proves stability with guaranteed exponential bound for zero input and nonzero initial state (0-GES), the second part concerns the case of zero initial state and nonzero input. The complete statement follows from the linearity of (1).

Part I. Firstly, note that inequality (31) ensures that for any $\overline{\delta} > 0$ there exists $\overline{\alpha}_{\overline{\delta}} \in (0, \beta)$ such that (46) is satisfied for any $\alpha \in [0, \overline{\alpha}_{\overline{\delta}}]$. Indeed, thanks to (31), the inequality (46) is trivially *strictly verified* for $\alpha = 0$. Since its right term is continuous and strictly increasing, there exists an open interval $[0, \overline{\alpha}_{\overline{\delta}})$ where the strict inequality is still verified.

Note also that, from definitions (47), it follows that $h_{\min}\mathbb{1}_n \leq h \leq h_{\max}\mathbb{1}_n$ and therefore, for a given $x \in \mathbb{R}^n_+$, we have $h_{\min}\mathbb{1}_n^\top x \leq h^\top x \leq h_{\max}\mathbb{1}_n^\top x$. Recalling that $\mathbb{1}_n^\top x = \|x\|_1$ we have

$$h_{\min}\|x\|_1 \leq h^\top x \leq h_{\max}\|x\|_1. \tag{48}$$

Thus, we can prove that for zero-input the following bound holds:

$$\|x(t)\|_1 \leq \frac{h_{\max}}{h_{\min}}\|\phi\|_1 \, e^{-\alpha t}, \quad t \geq 0. \tag{49}$$

By recalling that $x(t) \geq 0$, it is sufficient to show that

$$h^\top x(t) \leq h_{\max}\|\phi\|_1 \, e^{-\alpha t}, \quad t \geq 0. \tag{50}$$

For this purpose, let us define the following functions

$$\zeta(t) = h_{\max}\|\phi\|_1 \, e^{-\alpha t} - h^\top x(t), \quad t \in [-\overline{\delta}, \infty), \tag{51}$$

$$\begin{aligned} \varepsilon_0^\top(t) &= -\tilde{\alpha}_0(t) \, h^\top - h^\top A_0(t), \\ \varepsilon_j^\top(t) &= \tilde{\alpha}_j(t) \, h^\top - h^\top A_j(t), \quad j = 1, \dots, r. \end{aligned} \tag{52}$$

By assumptions (29) and (30), $\varepsilon_j(t) \geq 0, \forall t \geq 0$. The theorem is proved simply by showing that the function $\zeta(t)$ is nonnegative $\forall t \geq 0$, which implies (50). Differentiating $\zeta(t)$ we get

$$\dot{\zeta}(t) = -\alpha h_{\max}\|\phi\|_1 \, e^{-\alpha t} - h^\top A_0(t)x(t) - \sum_{j=1}^{r} h^\top A_j(t)x(t - \delta_j(t)). \tag{53}$$

From (52) we have

$$\begin{aligned} h^\top A_0(t) &= -\tilde{\alpha}_0(t) \, h^\top - \varepsilon_0^\top(t), \\ h^\top A_j(t) &= \tilde{\alpha}_j(t) \, h^\top - \varepsilon_j^\top(t), \quad j = 1, \dots, r. \end{aligned} \tag{54}$$

Replacing these into (53) we get

$$\dot{\zeta}(t) = -\alpha h_{\max}\|\phi\|_1 \, e^{-\alpha t} + \tilde{\alpha}_0(t)h^\top x(t) + \varepsilon_0^\top(t)x(t) - \sum_{j=1}^{r} \tilde{\alpha}_j(t)h^\top x(t - \delta_j(t)) + \varepsilon_j^\top(t)x(t - \delta_j(t)). \tag{55}$$

Replacing into (55) the following terms, for $j = 0, \dots, r$,

$$h^\top x(t - \delta_j(t)) = h_{\max}\|\phi\|_1 \, e^{-\alpha(t - \delta_j(t))} - \zeta(t - \delta_j(t)), \tag{56}$$

we get

$$\begin{aligned} \dot{\zeta}(t) = &-\alpha h_{\max}\|\phi\|_1 \, e^{-\alpha t} + \tilde{\alpha}_0(t)h_{\max}\|\phi\|_1 \, e^{-\alpha t} - \tilde{\alpha}_0(t)\zeta(t) \\ &- \sum_{j=1}^{r} \tilde{\alpha}_j(t)h_{\max}\|\phi\|_1 \, e^{-\alpha(t - \delta_j(t))} + \sum_{j=1}^{r} \tilde{\alpha}_j(t)\zeta(t - \delta_j(t)) + w_1(t), \end{aligned} \tag{57}$$

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where

$$w_1(t) = \sum_{j=0}^{r} \varepsilon_j^\top(t) x(t-\delta_j(t)) \geq 0, \quad \forall t \in [0,\infty). \tag{58}$$

Collecting the terms, Equation (57) can be rewritten as

$$\begin{aligned} \dot{\zeta}(t) &= -\tilde{\alpha}_0(t)\zeta(t) + \sum_{j=1}^{r} \tilde{\alpha}_j(t)\zeta(t-\delta_j(t)) + w_1(t) + w_2(t), \\ \text{where} \qquad w_2(t) &= h_{\max}\|\phi\|_1 \left(-\alpha + \tilde{\alpha}_0(t) - \sum_{j=1}^{r} \tilde{\alpha}_j(t) e^{\alpha\delta_j(t)} \right) e^{-\alpha t} \geq 0. \end{aligned} \tag{59}$$

The nonnegativity of $w_2(t)$ follows from assumption (46): since $e^{\alpha\bar{\delta}} \geq e^{\alpha\delta_j(t)}$ for all $\delta_j(t) \in [0,\bar{\delta}]$, we have

$$-\alpha + \tilde{\alpha}_0(t) - \sum_{j=1}^{r} \tilde{\alpha}_j(t) e^{\alpha\delta_j(t)} \geq -\alpha + \tilde{\alpha}_0(t) - \left(\sum_{j=1}^{r} \tilde{\alpha}_j(t) \right) e^{\alpha\bar{\delta}} \geq 0. \tag{60}$$

Since $w_1(t)$ and $w_2(t)$ are nonnegative, the dynamics (57) ensures nonnegativity of the variable $\zeta(t)$. By definition (51) of $\zeta(t)$, its nonnegativity implies the inequality (50), which in turn implies (49).

Part II. For the case of zero initial condition and nonzero input, it suffices to note that, thanks to the inequalities

$$h_{\min}\|x\|_1 \leq h^\top x \leq h_{\max}\|x\|_1, \tag{61}$$

the bound (44) of Corollary 3 implies that

$$\|x(t)\|_1 \leq \frac{h^\top \bar{B}\bar{u}}{h_{\min}\ \beta}, \quad t \geq 0. \tag{62}$$

Finally, the inequality (45) directly follows from the linearity of system (1). ∎

As for Theorem 2, it is clear that also Theorem 4 implies a corollary of 0-GES for a positive system as in (1) for $u \equiv 0$.

Corollary 4. *Consider system (1), where $A_0(t)$ is Metzler and $A_j(t)$ are nonnegative, for $j = 1, \dots, r$, with bounded initial state $\phi : [-\bar{\delta}, 0] \mapsto \mathbb{R}^n_+$. If the strong left-condition (29)–(31) hold true, then the system is delay-independent 0-GES, and for any given maximum delay $\bar{\delta} \in \mathbb{R}_+$, the following exponential bound holds true*:

$$\|x(t)\|_1 \leq \frac{h_{max}}{h_{min}} \|\phi\|_1\ e^{-\alpha t}, \quad t \geq 0, \tag{63}$$

where α is any positive real that satisfies (46), and h_{max}, h_{min}, $\|\phi\|_1$ are as in (47).

Remark 5. It is interesting to note that, under the *strong left-positive vector condition* (29)–(31), it can be shown that the function $V(x) = h^\top x$ is a LR function that allows to prove asymptotic stability of the trivial solution of (1) for $u \equiv 0$. However, in order to prove exponential stability and to provide an explicit exponential bound in (45) for zero input, a bounding approach has been followed in the proof of Part I of Theorem 4.

Remark 6. Note that, in the case of only one delay ($r = 1$), the strong left-condition (29)–(31) collapse into the left-condition (6) that can be simply written as

$$\begin{matrix} \exists h \in \mathbb{R}^n_{++} \\ \exists \beta \in \mathbb{R}_{++} \end{matrix} \quad : \quad h^\top (A_0(t) + A_1(t)) \leq -\beta\ h^\top, \quad \forall t \geq 0, \tag{64}$$

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and is sufficient to prove exp-ISS (and 0-GES) of system (1), under positivity assumption. The 0-GES part is indeed the result provided by Thm. 2 of Mazenc and Malisoff,[21] though not exactly for the same class of systems: the latter considers possibly state-dependent matrices in (1), but in the special case of a single constant delay.

Remark 7. The discussed asymmetry between the right- and left-conditions (5) and (6) also appears in the case of discrete-time positive delay systems, investigated in De Iuliis and Manes,[11] where the left- and right-positive vector conditions are as follows:

Right-positive vector condition

$$\begin{matrix}\exists q \in \mathbb{R}^n_{++} \\ \exists \beta \in (0,1)\end{matrix} \quad : \quad \left(\sum_{j=1}^{r} A_j(k)\right) q \leq \beta q, \quad \forall k \geq 0, \tag{65}$$

Left-positive vector condition

$$\begin{matrix}\exists h \in \mathbb{R}^n_{++} \\ \exists \beta \in (0,1)\end{matrix} \quad : \quad h^\top \left(\sum_{j=1}^{r} A_j(k)\right) \leq \beta h^\top, \quad \forall k \geq 0. \tag{66}$$

Also in the discrete-time framework, the right-condition (65) is sufficient for exp-ISS, while an additional (and, remarkably, delay-dependent) condition is needed on the left-condition (66) to ensure exp-ISS.

4 | STABILITY ANALYSIS OF GENERAL TIME-VARYING DELAY DIFFERENTIAL SYSTEMS

The results introduced in the previous section only pertain positive systems. Thus, they may seem to be of limited interest and limited applicability in general cases. Nevertheless, they have a remarkably simple formulation. For these reasons, one may wonder whether there is a possibility to extend all such results to general systems as in (1), removing any positivity assumption. As already illustrated in the Introduction, the idea of exploiting positivity-based stability analysis is not new, and a quite large literature has appeared in the recent decades on the topic (see the Introduction for representative references). In this paper, we further explore the so-called 'comparison systems' approach, by means of the Internally Positive Representation (IPR) technique.

For systems as in (1), the IPR methodology was introduced in De Iuliis et al.[25] Still, the stability analysis reported in the referenced work only considers the special case of constant matrices and focuses only on internal stability. In this section, expanding over the preliminary results of De Iuliis and Manes,[35] we generalize the stability results to the way more general case of time-varying matrices, and also provide exponential input-to-state stability criteria. To help the reader, we here briefly recall the IPR construction of De Iuliis et al.[25]

Let us start recalling operators and notation. The first operator is denoted $(\cdot)^{\mathcal{M}}$, and transforms a square matrix A into a Metzler matrix: for any given $A \in \mathbb{R}^{n\times n}$, $A^{\mathcal{M}} \in \mathbb{R}^{n\times n}$ is the Metzler matrix obtained from A by replacing all its off-diagonal entries with their absolute values, i.e.

$$A^{\mathcal{M}} = \mathrm{d}(A) + |A - \mathrm{d}(A)|, \tag{67}$$

where $\mathrm{d}(A)$ is a diagonal matrix with the same diagonal of matrix A. Moreover, let us denote with A^+ the component-wise nonnegative part of A, and with A^- the component-wise nonpositive part of A with changed sign, respectively defined as:

$$[A^+]_{i,j} = \max\{[A]_{i,j}, 0\}, \quad [A^-]_{i,j} = \max\{-[A]_{i,j}, 0\}. \tag{68}$$

Then we can define the following operators:

$$\tilde{\mu}(A) = \mathrm{d}(A) + (A - \mathrm{d}(A))^+, \quad \tilde{\nu}(A) = (A - \mathrm{d}(A))^-, \tag{69}$$

where we note that $\tilde{\mu}(A)$ is Metzler, $\tilde{\nu}(A)$ is nonnegative, and they allow to write

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$$A^{\mathcal{M}} = \tilde{\mu}(A) + \tilde{\nu}(A), \quad A = \tilde{\mu}(A) - \tilde{\nu}(A). \tag{70}$$

Also, we define:

$$\Gamma(A) = \begin{bmatrix} \tilde{\mu}(A) & \tilde{\nu}(A) \\ \tilde{\nu}(A) & \tilde{\mu}(A) \end{bmatrix}, \quad \Pi(A) = \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix}. \tag{71}$$

Note that $\Gamma(A)$ is a $2n \times 2n$ Metzler matrix, while $\Pi(A)$ is $2n \times 2n$ nonnegative.

The following operator $\pi : \mathbb{R}^n \mapsto \mathbb{R}^{2n}$, and matrix Δ_n are also needed

$$\pi(x) = \begin{bmatrix} x^+ \\ x^- \end{bmatrix}, \quad \Delta_n = \begin{bmatrix} I_n & -I_n \end{bmatrix}. \tag{72}$$

It is easy to see that they are such that $x = \Delta_n \pi(x)$. Note also that $|x| = x^+ + x^- = [I_n \;\; I_n]\pi(x)$.

We refer the reader to Def. 3 in De Iuliis et al.[25] for a precise definition of IPR for the class of systems under investigation. Namely, an IPR consists of:

- A novel system, of larger dimension, which is positive by construction;
- a set of transformations which allow to shift from the original system to its positive representation, and vice versa.

The following theorem introduces a simple IPR for system (1).

Theorem 5. *Consider system (1), with initial state function $\phi(t)$, $t \in [-\overline{\delta}, 0]$. The following positive system*

$$\begin{aligned} &\dot{\mathcal{X}}(t) = \Gamma(A_0(t))\mathcal{X}(t) + \sum_{j=1}^{r} \Pi\big(A_j(t)\big)\mathcal{X}(t - \delta_j(t)) + \Pi(B(t))\pi(u(t)), \;\; t \geq 0, \\ &\mathcal{X}(t) = \pi(\phi(t)), \qquad\qquad t \in [-\overline{\delta}, 0], \end{aligned} \tag{73}$$

is such that

$$x(t) = \Delta_n \mathcal{X}(t), \quad t \geq -\overline{\delta}. \tag{74}$$

Theorem 5 states that system (73), together with the state transformations $T_X^f(x) = \pi(x)$ and $T_X^b(\mathcal{X}) = \Delta_n \mathcal{X}$, and the input transformation $T_U(u) = \pi(u)$ is an IPR for system (1), and is a straightforward consequence of Thm. 2 of De Iuliis et al.[25] Indeed, Theorem 5 is a simplified version of the referenced one, since no output is considered in (73), due to the focus on internal and input-to-state stability.

In this section we will state the generalization of both delay-independent conditions of exp-ISS introduced in Theorems 2 and 4 to systems with no positivity constraints, yielding the most general result of this work. Results of 0-GES will then be stated as corollaries.

Theorem 6. *Consider system (1), with $B(t)$ and $u(t)$ satisfying $|B(t)| \leq \overline{B} \in \mathbb{R}_+^{n \times p}$ and $|u(t)| \leq \bar{u} \in \mathbb{R}_+^p$, $\forall t \geq 0$. If the following right-condition holds true*

$$\begin{matrix} \exists q \in \mathbb{R}_{++}^n \\ \exists \beta \in \mathbb{R}_{++} \end{matrix} \quad : \quad \left(A_0^{\mathcal{M}}(t) + \sum_{j=1}^{r} |A_j(t)| \right) q \leq -\beta q, \quad \forall t \geq 0, \tag{75}$$

then the system (1) is delay-independent exp-ISS.

More specifically, the state evolves according to the following bound:

$$|x(t)| \leq 2\overline{q}\|\phi\|_\infty e^{-\alpha t} + \frac{2\|\overline{B}\bar{u}\|_\infty}{\beta}\overline{q}, \quad \forall t \geq 0, \tag{76}$$

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where $\overline{q}$ is defined as in (8), and $\alpha > 0$ is a guaranteed exponential convergence rate that satisfies

$$\alpha\ q + A_0^{\mathcal{M}}(t)q + e^{\alpha\overline{\delta}}\left(\sum_{j=1}^{r}|A_j(t)|\right)q \leq 0, \quad \forall t \geq 0. \tag{77}$$

Proof. We start by showing that condition (75) is equivalent to the following condition:

$$\exists p \in \mathbb{R}^{2n}_{++}\ ,\ \exists \beta \in \mathbb{R}_{++}\ :\ \left(\Gamma(A_0(t)) + \sum_{j=1}^{r}\Pi\big(A_j(t)\big)\right)p \leq -\beta\ p, \quad \forall t \geq 0. \tag{78}$$

Before proving the equivalence, notice that condition (78) is the delay-independent exp-ISS condition under positivity assumption (Theorem 2) applied to the IPR of (1) given in Theorem 5 (thus, it involves order $2n$ matrices); on the other hand, condition (75) involves order n matrices (so, it is a favorable test in terms of computational burden).

(78) $\Rightarrow$ (75). If (78) holds, it is easy to check that (75) is verified with $q = [I_n\ I_n]p$, by simply left-multiplying both terms in (78) by $[I_n\ I_n]$ and noting that

$$\Big[I_n \quad I_n\Big]\Gamma(A_0(t))p = A_0^{\mathcal{M}}(t)\Big[I_n \quad I_n\Big]p, \quad \text{and} \quad \Big[I_n \quad I_n\Big]\Pi\big(A_j(t)\big)p = |A_j(t)|\Big[I_n \quad I_n\Big]p, \quad j = 1, \dots, r. \tag{79}$$

(75) $\Rightarrow$ (78). If (75) holds true, then (78) is verified with $p = [q^\top\ q^\top]^\top$, as it is easy to check, since

$$\Gamma(A_0(t))p = \Gamma(A_0(t))\begin{bmatrix}q\\ q\end{bmatrix} = \begin{bmatrix}A_0^{\mathcal{M}}(t)\ q\\ A_0^{\mathcal{M}}(t)\ q\end{bmatrix} \quad \text{and} \quad \Pi\big(A_j(t)\big)p = \Pi\big(A_j(t)\big)\begin{bmatrix}q\\ q\end{bmatrix} = \begin{bmatrix}|A_j(t)|q\\ |A_j(t)|q\end{bmatrix} \quad j = 1, \dots, r. \tag{80}$$

Now, consider that the IPR of (1) is such that its state bounds the state of (1) at any time, and in particular:

$$x(t) = \Delta_n\mathcal{X}(t) \Rightarrow \|x(t)\| \leq \|\mathcal{X}(t)\|, \quad \forall t \geq 0. \tag{81}$$

It is then clear that condition (75) implies the delay-independent exp-ISS of the IPR and, in turn, the exp-ISS of the original system (1).

Due to Theorem 2 the system variable of the IPR (73) satisfies:

$$[I_n\ \ I_n]\ \mathcal{X}(t) \leq [I_n\ \ I_n]\begin{bmatrix}\overline{q}\\ \overline{q}\end{bmatrix}\|\pi(\phi)\|_\infty e^{-\alpha t} + [I_n\ \ I_n]\begin{bmatrix}\overline{q}\\ \overline{q}\end{bmatrix}\frac{\|\overline{B}\overline{u}\|_\infty}{\beta}, \quad \forall t \geq -\overline{\delta}, \tag{82}$$

that is the thesis (76), simply noting that $\|\pi(\phi)\|_\infty = \|\phi\|_\infty$, and due to the bound $|x(t)| \leq [I_n\ \ I_n]\mathcal{X}(t)$ for all $t \geq \overline{\delta}$.

For what concerns the exponential bound α, consider that if the inequality (75) is true, then for any $\overline{\delta}$ there exists $\overline{\alpha}_{\overline{\delta}} \in (0, \beta)$, such that (77) is verified for all $\alpha \in [0, \overline{\alpha}_{\overline{\delta}}]$. This can be easily proved by rewriting (75) as

$$\big(\beta I_n + A_0^{\mathcal{M}}(t)\big)q \leq -\left(\sum_{j=1}^{r}|A_j(t)|\right)q. \tag{83}$$

The inequality (75), written as (83), clearly implies that (77) strictly holds for $\alpha = 0$. Since both terms of inequality (77) are continuous functions of α, there exists a positive $\overline{\alpha}_{\overline{\delta}} < \beta$ such that the strict inequality holds for any $\alpha \in [0, \overline{\alpha}_{\overline{\delta}})$, and the nonstrict inequality (77) holds in $[0, \overline{\alpha}_{\overline{\delta}}]$. This concludes the proof. ∎

The following corollary is a straightforward consequence of Theorem 6 and extends Corollary 2 to general systems.

Corollary 5. *Consider system (1). If the right-condition (75) holds true, then the system is delay-independent 0-GES. Moreover, for any given maximum delay $\overline{\delta} \in \mathbb{R}_+$ the following exponential bound holds true*

$$|x(t)| \leq \overline{q}\|\phi\|_\infty e^{-\alpha t}, \quad \forall t \geq -\overline{\delta}, \tag{84}$$

where α is any positive real satisfying (77) and and $\overline{q}$ is defined in (8).

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The following result extends the strong left-condition of Theorem 4 to general systems, and thus provides a different condition of exp-ISS for system (1).

Theorem 7. *Consider system (1), with $B(t)$ and $u(t)$ satisfying $|B(t)| \leq \overline{B} \in \mathbb{R}_+^{n\times p}$ and $|u(t)| \leq \bar{u} \in \mathbb{R}_+^p$, $\forall t \geq 0$. Let there exist $h \in \mathbb{R}_{++}^n$, $\beta > 0$, and nonnegative functions $\tilde{\alpha}_j(t), j = 0, \dots, r$ (possibly constant), such that $\forall t \geq 0$ the strong left-positive vector condition*

$$h^\top A_0^{\mathcal{M}}(t) \leq -\tilde{\alpha}_0(t)\, h^\top, \tag{85}$$

$$h^\top |A_j(t)| \leq \tilde{\alpha}_j(t)\, h^\top, \quad j = 1, \dots, r, \tag{86}$$

$$\tilde{\alpha}_0(t) \geq \beta + \sum_{j=1}^{r} \tilde{\alpha}_j(t), \tag{87}$$

is satisfied.

Then, the system (1) is delay-independent exp-ISS. More specifically, the state evolves according to the following bound:

$$\|x(t)\|_1 \leq \frac{h_{max}}{h_{min}} \|\phi\|_1\, e^{-\alpha t} + \frac{2h^\top \overline{B}\bar{u}}{h_{min}\, \beta}, \quad \forall t \geq 0, \tag{88}$$

where $h_{\max}, h_{\min}$ and $\|\phi\|_1$ are defined as in (47), and and $\alpha > 0$ is a guaranteed exponential convergence rate that satisfies (46).

Proof. We start by showing that conditions (85) and (86) are respectively equivalent, with unchanged $\tilde{\alpha}_j(t)$ and $\beta > 0$, to the following conditions: $\exists l \in \mathbb{R}_{++}^{2n}$ s.t. $\forall t \geq 0$

$$l^\top \Gamma(A_0(t)) \leq -\tilde{\alpha}_0(t)\, h^\top, \tag{89}$$

$$l^\top \Pi\big(A_j(t)\big) \leq \tilde{\alpha}_j(t)\, h^\top, \quad j = 1, \dots, r. \tag{90}$$

Indeed, if (85) holds then (89) holds as well, with $h^\top = l^\top [I_n \;\; I_n]^\top$. This is seen by right-multiplying both terms in (85) by $[I_n \;\; I_n]^\top$ and noting that

$$l^\top \Gamma(A_0(t)) \begin{bmatrix} I_n \\ I_n \end{bmatrix} = l^\top \begin{bmatrix} A_0^{\mathcal{M}}(t) \\ A_0^{\mathcal{M}}(t) \end{bmatrix} = l^\top \begin{bmatrix} I_n \\ I_n \end{bmatrix} A_0^{\mathcal{M}}(t) = h^\top A_0^{\mathcal{M}}(t). \tag{91}$$

Analogously, if (86) is satisfied, then (90) is also verified, with $h^\top = l^\top [I_n \;\; I_n]^\top$. This is again seen by right-multiplying both terms in (86) by $[I_n \;\; I_n]^\top$ and noting that

$$l^\top \Pi\big(A_j(t)\big) \begin{bmatrix} I_n \\ I_n \end{bmatrix} = l^\top \begin{bmatrix} |A_j(t)| \\ |A_j(t)| \end{bmatrix} = l^\top \begin{bmatrix} I_n \\ I_n \end{bmatrix} |A_j(t)| = h^\top |A_j(t)|, \;\; j = 1, \dots, r. \tag{92}$$

Conversely, if (89) and (90) hold, then (85)-(86) are also verified with $l^\top = [h^\top \; h^\top]$, as can be straightforwardly seen:

$$\begin{bmatrix} h^\top & h^\top \end{bmatrix} \begin{bmatrix} A_0^{\mathcal{M}}(t) & 0 \\ 0 & A_0^{\mathcal{M}}(t) \end{bmatrix} = \begin{bmatrix} h^\top & h^\top \end{bmatrix} \Gamma(A_0(t)) = l^\top\, \Gamma(A_0(t)), \tag{93}$$

$$\begin{bmatrix} h^\top & h^\top \end{bmatrix} \begin{bmatrix} |A_j(t)| & 0 \\ 0 & |A_j(t)| \end{bmatrix} = \begin{bmatrix} h^\top & h^\top \end{bmatrix} \Pi\big(A_j(t)\big) = l^\top\, \Pi(A_j(t)), \qquad j = 1, \dots, r. \tag{94}$$

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At this point, it suffices to note that conditions (89)-(90)-(87) are the delay-independent exp-ISS condition of Theorem 4 applied to the IPR of (1), and thus imply the exp-ISS stability of (1) due to (81). Also, Theorem 4 implies that the state of the IPR evolves according to the exponential bound:

$$\|\mathcal{X}(t)\|_1 \leq \frac{h_{\max}}{h_{\min}}\|\phi\|_1\, e^{-\alpha t} + \frac{l^\top \overline{v}}{h_{\min}\,\beta}, \quad \forall t \geq 0, \tag{95}$$

where α is any positive real that satisfies (46), $h_{\max}$ and $h_{\min}$ are defined in (47), and $\overline{v}$ is defined so to bound the input term of the IPR:

$$\overline{v} = \begin{bmatrix} \overline{B}\overline{u} \\ \overline{B}\overline{u} \end{bmatrix}, \quad \text{so that} \quad \Pi(B(t))\pi(u(t)) \leq \overline{v}, \quad t \geq 0. \tag{96}$$

Also, the following fact has been used:

$$\|\phi\|_1 = \sup_{t\in[-\overline{\delta},0]} \|\phi(t)\|_1 = \sup_{t\in[-\overline{\delta},0]} \|\pi(\phi(t))\|_1. \tag{97}$$

The thesis (88) follows from (95) by noticing that $l^\top \overline{v} = [h^\top \;\; h^\top]\overline{v} = 2h^\top \overline{B}\overline{u}$ and by $\|\Delta_n\|_1 = \|[I_n \;\; -I_n]\|_1 = 1$, that implies $\|x(t)\|_1 = \|\mathcal{X}(t)\|_1$ for all $t \geq 0$ (see (81). ∎

Finally, the following corollary is a ready consequence of Theorem 7 and extends Corollary 4 to general systems.

Corollary 6. *Consider system (1). If the strong left-conditions (85)–(87) hold true, then the system is delay-independent 0-GES. Moreover, for any given maximum delay $\overline{\delta} \in \mathbb{R}_+$ the following exponential bound holds true*

$$\|x(t)\|_1 \leq \frac{h_{max}}{h_{min}}\|\phi\|_1\, e^{-\alpha t}, \quad \forall t \geq 0, \tag{98}$$

where h_{max}, h_{min} and $\|\phi\|_1$ are given by (47), and and $\alpha > 0$ is a guaranteed exponential convergence rate that satisfies (46).

Remark 8. Even though some similarity could be seen (mainly, the fact that the stability condition involves linear inequalities due to the positivity-based analysis), this Corollary is not fairly comparable with Thm. 1 of Mazenc and Malisoff,[21] which is interesting and worthy of remark. The latter considers systems with a single constant delay, although with state-dependent matrices. Moreover, it provides a delay-dependent stability condition. On the other hand, Corollary 6 applies to systems with multiple time-varying delays, provides an explicit bound on the exponential convergence rate, and is delay-independent. To sum up, neither the system class nor the kind of analysis are strictly comparable. Finally, Corollary 6 is simply a consequence to the more general result of exponential ISS given in Theorem 7.

Remark 9. Although not explicitly remarked for all results of Sections 3 and 4, exp-ISS with linear gains (see Def. 2 of Chaillet et al.[38]) can be specifically stated for Theorems 2,4,6,7.

To avoid repetition, just consider the general case of Theorems 6 and 7 given in this section. Considering Theorem 6, setting $\bar{u} = \mathbb{1}_p\|u\|_\infty$ one can simply set the vectors $k_1 = 2\overline{q}$ and $\mu_1 = \frac{2\|\overline{B}\|_\infty}{\beta}\overline{q}$ as linear gains in (76), yielding:

$$|x(t)| \leq k_1\|\phi\|_\infty e^{-\alpha t} + \mu_1\|u\|_\infty. \tag{99}$$

In Theorem 7, considering that $\|\phi\|_1 \leq n\|\phi\|_\infty$, setting the scalars $k_2 = \frac{h_{\max}}{h_{\min}}n$ and $\mu_2 = \frac{2h^\top\overline{B}}{h_{\min}\beta}$ in (88) yields

$$\|x(t)\|_1 \leq k_2\|\phi\|_\infty e^{-\alpha t} + \mu_2\|u\|_\infty. \tag{100}$$

As a final comment for this section, we would like to spend a few words on the position of the proposed conditions in the rich literature about stability analysis of delay systems. As first mentioned in the introduction, there is a now

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established *corpus* of stability results for time-invariant delay differential systems. The most popular approach is certainly based on extensions of the second method of Lyapunov, in the form of LK and LR theorems. Extensions of such methods to time-varying delay differential systems have been proposed in the recent literature. For a comprehensive reference see the work by Zhou and Egorov,[3] where the Section 3 is specifically devoted to linear systems. Hence, the reader might wonder what is the purpose of conservative conditions as the ones proposed in Theorems 6 and 7, when time-varying Lyapunov functionals could be used to analyze stability, for example, via the LK delay-dependent method of Thm. 3 in Zhou and Egorov,[3] or the LR delay-independent method of Theorem 4 in the same work (actually, the only one among the two that allows for time-varying delays). The key point here is that there is no general method to systematically find a function to be used as a LK or LR in a stability proof. In particular, for linear systems there is no method to solve differential Lyapunov equations, like the one in Zhou and Egorov,[3] and find a positive definite time-varying matrix $P(t)$, uniformly lower and upper bounded, that allows to conclude stability. There is no such result even for the "simpler" case of delay-free time-varying systems, as admirably discussed in Zhou.[41] In this scenario, the contribution of this work lies in the simplicity of the proposed, albeit conservative, conditions, that can be used as a first analysis tool that can be tested, in principle, on any time-varying delay system as in (1). If the proposed tests fail, one may investigate stability with the far more complex tool of time-varying Lyapunov methods, with no existing systematic and satisfactory approach available to date to test the existence of suitable solutions. In this regard, we think it is worth emphasizing that even for the far simpler case of time-invariant delay differential systems, there is no formal guarantee that LK or LR methods would generally be less conservative than Theorems 6 and 7. Simple counterexamples are reported in the Ex. 2 and 3 in De Iuliis et al.[25]

Finally, although this might be seen as a minor point of comparison, the criteria introduced in Theorems 6 and 7 prove exponential input-to-state stability with explicit gains.

Of course, as computationally simpler as they are, also the conditions proposed in this work generally require the solution of infinitely many inequalities. This feature is discussed in the next section, where implementation issues and further relaxations are introduced to increase the applicability of such conditions, and illustrative examples are proposed.

5 | IMPLEMENTATION ISSUES AND EXAMPLES

In this section some issues arising when attempting to solve the inequalities that prove the delay-independent 0-GES and exp-ISS of system (1) are discussed. For simplicity, only the right-condition (5), for positive systems, or (75), for general systems, are considered here.

First, it is important to point out that although the conditions (5) and (75) are not linear in the pair (q, β), these can still be checked/solved by using LP solvers. To see this, consider a constant Metzler $n \times n$ matrix $\widetilde{A}$, and the problem of finding a feasible solution $(q, \beta) \in \mathbb{R}^n \times \mathbb{R}$ to

$$\widetilde{A}q \leq -\beta q, \quad q > 0, \quad \beta > 0. \tag{101}$$

Of course, this is not a LP problem, due to the presence of the product βq in the right-hand side of the first inequality. Nevertheless, it is easy to show that the feasibility of (101) is equivalent to the feasibility of the following LP problem

$$\widetilde{A}q \leq -\mathbb{1}_n, \qquad q \geq \mathbb{1}_n, \tag{102}$$

where the only unknown is $q \in \mathbb{R}^n$ ($\mathbb{1}_n$ denotes the vector of ones in $\mathbb{R}^n$). The feasibility of (102) can be checked by any LP solver, which provides a solution, if it exists, or certifies its infeasibility. If a solution q to (102) is found, then a feasible solution to (101) is given by the pair (q, β), where β is computed as

$$\beta = \min_{i=1,\dots,n} \frac{-(\widetilde{A}q)_i}{q_i}. \tag{103}$$

Now that we have established that inequalities (5) and (75) can be solved by means of LP solvers, we have to deal with the fact that the coefficient matrices are time-varying (as it should be, since the conditions are for time-varying systems).

Clearly, it is not obvious how to check the feasibility of (5) or (75) for all $t \geq 0$, since it amounts to check the feasibility of infinitely many LP problems. In the following we will refer only to the feasibility problem (75), which of course includes (5), and will define the following compact notation for the time-varying Metzler coefficient matrix in (75):

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$$\widetilde{A}(t) = A_0^{\mathcal{M}}(t) + \sum_{j=1}^{r} |A_j(t)|. \tag{104}$$

Since it is impossible to test the feasibility of (75) for all $t \in [0, \infty)$, a viable strategy is to test it on a finite number of upper bounds of the matrix $\widetilde{A}(t)$. More in detail, by properly choosing a finite set of times $\{t_i\}_0^N$, with $t_0 = 0$, $t_N = \infty$ and $t_i < t_{i+1}$, and a finite set of Metzler matrices $\{\overline{A}_i\}_{i=0}^{N-1}$ majorizing $\widetilde{A}(t)$ in the intervals $[t_i, t_{i+1}]$, that is,

$$A_0^{\mathcal{M}}(t) + \sum_{j=1}^{r} |A_j(t)| \leq \overline{A}_i, \quad \begin{matrix} \forall t \in [t_i, t_{i+1}], \\ \forall i \in [0, N-1], \end{matrix} \tag{105}$$

it is easy to show that the feasibility of (75), for a given β, is implied by the the feasibility of the N linear inequalities

$$\overline{A}_i q \leq -\beta q, \quad \forall i \in [0, N-1]. \tag{106}$$

The aforementioned strategy includes the favorable case in which a Metzler Hurwitz majorant $\overline{A}$ exists on $\widetilde{A}(t)$, for all $t \geq 0$, as illustrated by the following example. Note that, when a system admits a Metzler Hurwitz bounding matrix, then the *right-positive vector condition* (75) is satisfied, while the *strong left-positive vector condition* (85)–(87) may not be satisfied.

Example 1. Consider a system as in (1) with two time-varying delays, described by:

$$A_0(t) = \begin{bmatrix} -\frac{1}{t^4+1} - 1 & \frac{1}{10} \\ \frac{1}{10} & -\frac{2}{t^2+4} - 2 \end{bmatrix}, \quad A_1(t) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{5(t^2+1)} \\ -\frac{1}{10(t^2+2)} & \frac{1}{4} \end{bmatrix}, \quad A_2(t) = \begin{bmatrix} 0 & \frac{7}{10} - e^{-2t} \\ \frac{1}{10}e^{-t} & 0 \end{bmatrix}. \tag{107}$$

We can readily prove the delay-independent exp-ISS (and 0-GES) of this system by simply noticing that

$$A_0^{\mathcal{M}}(t) + |A_1(t)| + |A_2(t)| \leq \bar{A} = \begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{4} & -\frac{7}{4} \end{bmatrix}, \quad t \geq 0, \tag{108}$$

and verifying that $\overline{A}$ is Hurwitz with spectral abscissa $\overline{\lambda}(\overline{A}) = -0.3246$, implying the feasibility of $\overline{A}q \leq -\beta q$ with, for example, $q = [1 \;\; 0.2]^\top$, and $\beta = 0.1$. Indeed, for this example the strong left-positive vector condition is not satisfied, so that Theorem 7 cannot be invoked to prove stability.

The trajectories of the system with zero input are shown in Figure 1, for a choice of the delays given by $\delta_1(t) = \frac{4}{5}\sin(t) + 1$, $\delta_2(t) = \sin(t) + \frac{3}{2}$, and initial conditions $x(t) = [2 \;\; 1]^\top$ in $t \in \left[-\frac{5}{2}, 0\right]$. ■

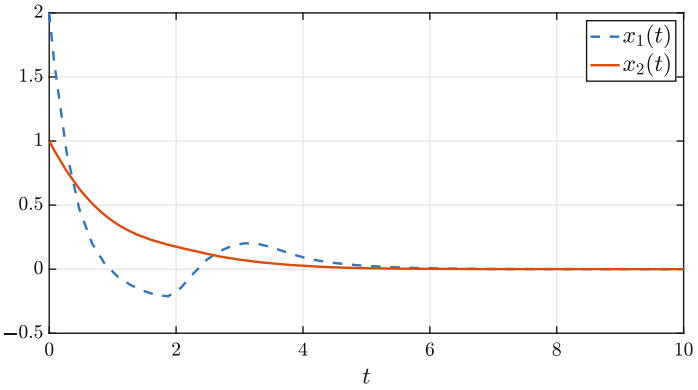

FIGURE 1 Time evolution of $x(t)$ for the system of Example 1, for $t \geq 0$.

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5.1 | The case of periodic matrices

Consider the case in which the system matrices $A_j(t)$ are periodic. This means that there exists a minimal time interval $T > 0$, the period, such that $A_j(t) = A_j(t+T)$, for any $t \geq 0$. In this case, a strategy to check the delay-independent stability condition (75) by means of a finite number of inequalities is to try different partitions of the period T. Using the previously introduced notation, the strategy consists in choosing sets of times $\{t_i\}_0^N$, with $t_0 = 0$, $t_N = T$ and $t_i < t_{i+1}$, and finding majorizations $\overline{A}_i$ of the matrix $\widetilde{A}(t)$ in the intervals $[t_i, t_{i+1}]$, then testing the feasibility of the N linear inequalities (106). If inequalities are feasible for a given partition, the delay-independent stability is proven, and the calculated β can be used to derive guaranteed convergence rates. On the other hand, if for a given partition the inequalities are unfeasible, one can decide to test a finer partition or to give up.

To illustrate the case of periodic matrices, let us consider the following example.

Example 2. Consider a system as in (1) with one delay, described by:

$$A_0(t) = \begin{bmatrix} -4 + \cos(\omega t) & 0 \\ 1 & -3 + \sin(\omega t) \end{bmatrix}, \quad A_1(t) = \begin{bmatrix} 0.8 & \sin(\omega t) \\ \sin(\omega t) & 1 + 0.5\cos(\omega t) \end{bmatrix}, \tag{109}$$

for a given $\omega = 2\pi/T$. The matrix $\widetilde{A}(t) = A_0^{\mathcal{M}}(t) + |A_1(t)|$ is:

$$\widetilde{A}(t) = \begin{bmatrix} -3.2 + \cos(\omega t) & |\sin(\omega t)| \\ 1 + |\sin(\omega t)| & -3 + \sin(\omega t) + |1 + 0.5\cos(\omega t)| \end{bmatrix}, \quad t \geq 0.$$

A trivial majorization in the interval $[0, T]$ is obtained simply by majorizing all sines and cosines with 1, obtaining:

$$\widetilde{A}(t) \leq \overline{A} = \begin{bmatrix} -2.2 & 1 \\ 2 & -0.5 \end{bmatrix}, \quad t \geq 0. \tag{110}$$

This bound is not useful, since the matrix $\overline{A}$ is unstable (spectral abscissa $\overline{\lambda}(\overline{A}) = 0.3508$), and therefore the inequality $\overline{A}q \leq -\beta q$ is unfeasible for any $\beta > 0$.

In order to find useful bounds on $\widetilde{A}(t)$, three different partitions of the interval $[0, T]$ have been tried, by choosing sub-intervals of length $T/2$, $T/4$, and $T/8$, respectively, and computing the majorizing matrices $\overline{A}_i$ within these intervals. We obtained the following result: considering majorizing matrices in the two intervals $[0, T/2]$, $[T/2, T]$, the corresponding two inequalities are unfeasible for any $\beta > 0$. Also considering majorizing matrices dividing $[0, T]$ into four intervals of length $T/4$, we get unfeasible inequalities. However, when considering the partition of $[0, T]$ into eight intervals of size $T/8$, we find that the eight inequalities are feasible with $\beta = 0.04432$ and $q = [1 \;\; 2.3874]^\top$, implying that (75) is satisfied for all $t \in [0, T]$, thus ensuring the delay-independent exp-ISS (and 0-GES) of the time-varying system.

Figures 2 and 3 show the upper-bounds of the components of the matrix $\widetilde{A}(t)$ in the interval $[0, T]$ (red lines) for the uniform partitions of sizes $T/4$ and $T/8$. For a clear graphic presentation, a period $T = 8\,s$ has been chosen. ■

We would like to conclude this first part of Section 5 remarking that the choice of $n = 2$ in Examples 1 and 2 is only adopted for ease of visualization, especially to limit the number of plots in Figures 2 and 3. It is clear that, in principle, the same strategies described in these examples for systems admitting a single Metzler Hurwitz majorizing matrix $\overline{A}$ (Example 1), or to find periodic majorizations for the case of periodic matrices (Example 2), can be adopted for any value of n.

On the other hand, something more can be said about the conditions of Theorem 6 when $n = 2$, and this will be the scope of the next section.

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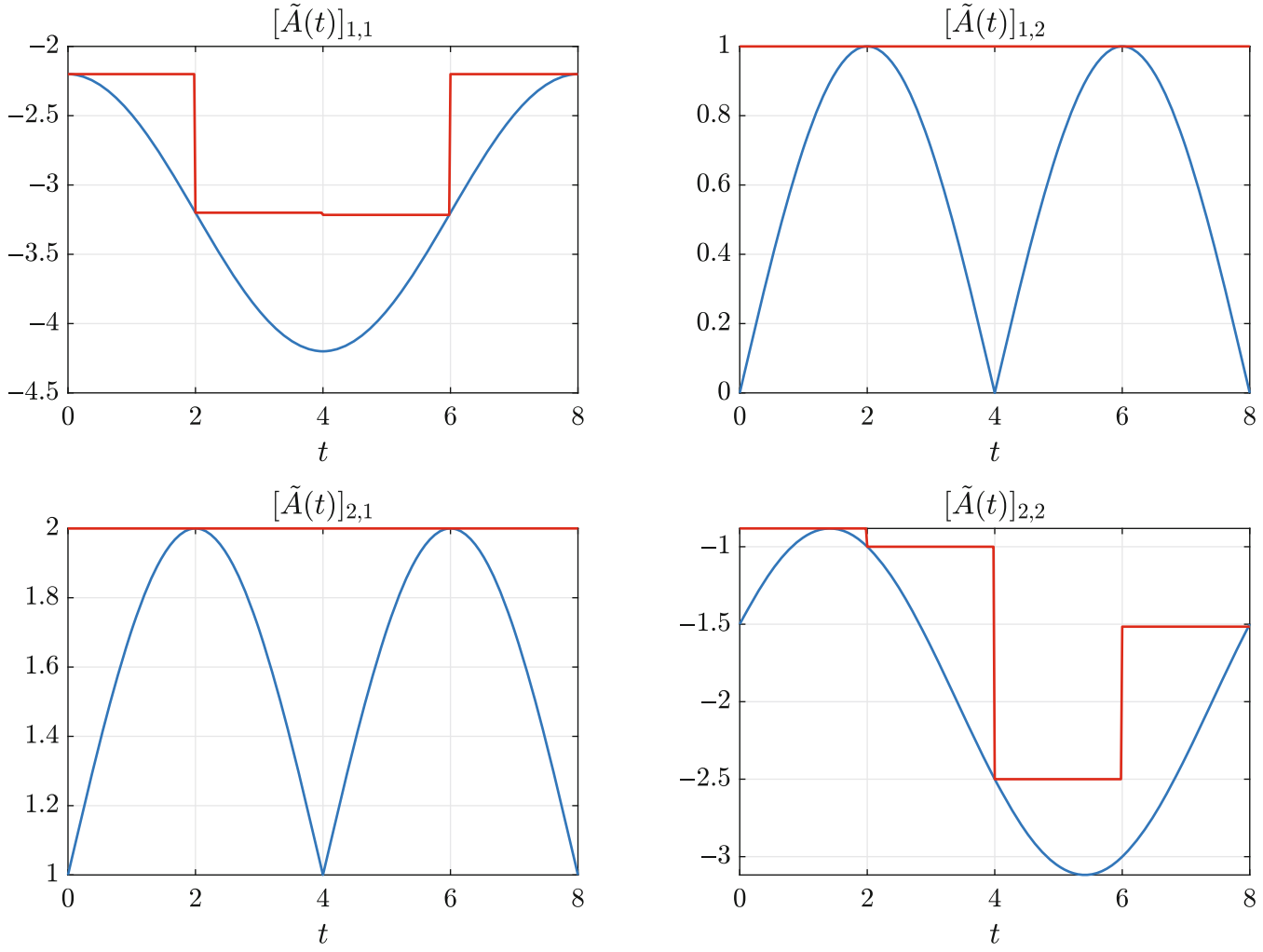

FIGURE 2 Upper bounds (red lines) on the components of $\widetilde{A}(t)$ (blue lines) over intervals of size $T/4$.

5.2 | Analytical conditions for the case $n = 2$

In this section we show that for $n = 2$, that is, for systems whose state consists of two components, the stability conditions (75) of Theorem 6 can be studied and solved analytically. Let

$$\begin{bmatrix} \widetilde{a}_{11}(t) & \widetilde{a}_{12}(t) \\ \widetilde{a}_{21}(t) & \widetilde{a}_{22}(t) \end{bmatrix} = \widetilde{A}(t) = A_0^{\mathcal{M}}(t) + \sum_{j=1}^{r} |A_j(t)| \in \mathbb{R}^{2\times 2}.$$

The problem of finding $q \in \mathbb{R}^2_{++}$ and $\beta > 0$ such that $\widetilde{A}(t)q \leq -\beta q$, $\forall t \geq 0$, can be simplified by setting, without loss of generality, $q = [1 \;\; q_2]^\top$, so that the positive pair (q_2, β) is the unknown in the inequality

$$\begin{bmatrix} \widetilde{a}_{11}(t) & \widetilde{a}_{12}(t) \\ \widetilde{a}_{21}(t) & \widetilde{a}_{22}(t) \end{bmatrix} \begin{bmatrix} 1 \\ q_2 \end{bmatrix} \leq -\beta \begin{bmatrix} 1 \\ q_2 \end{bmatrix}, \quad \forall t \geq 0, \tag{111}$$

where $\widetilde{a}_{12}(t) \geq 0$ and $\widetilde{a}_{21}(t) \geq 0$ due to the fact that $\widetilde{A}(t)$ is a Metzler matrix. Necessary conditions for the existence of the solution of (111) are $\widetilde{a}_{11}(t) + \beta \leq 0$ and $\widetilde{a}_{22}(t) + \beta \leq 0$. A sufficient and rather conservative condition for the existence of a positive solution pair (q_2, β) is the Hurwitz stability of the matrix $\overline{A}$ defined as

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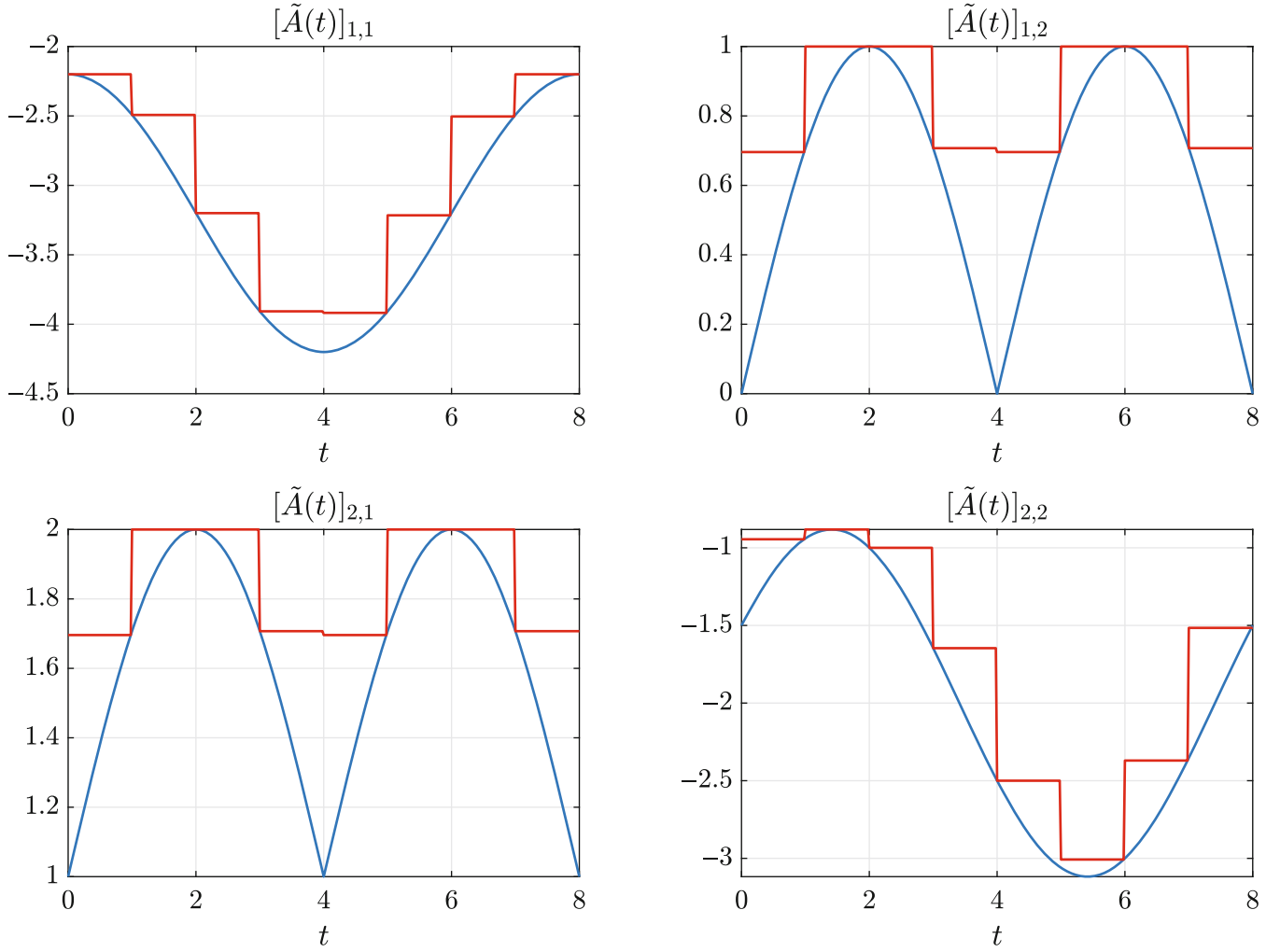

FIGURE 3 Upper bounds (red lines) on the components of $\tilde{A}(t)$ (blue lines) over intervals of size $T/8$.

$$\overline{A} = \begin{bmatrix} \sup_{t\geq 0}(\tilde{a}_{11}(t)) & \sup_{t\geq 0}(\tilde{a}_{12}(t)) \\ \sup_{t\geq 0}(\tilde{a}_{21}(t)) & \sup_{t\geq 0}(\tilde{a}_{22}(t)) \end{bmatrix}, \tag{112}$$

which is a tight upper bound of $\tilde{A}(t)$. Indeed, the Hurwitz stability of the Metzler matrix $\overline{A}$ is equivalent to the existence of a positive solution pair (q, β) to the inequality $\overline{A}q \leq -\beta q$, and hence to the fulfillment of the condition of exp-ISS (75) of Theorem 6, as shown in Example 1 above.

On the other hand, if the Hurwitz property of matrix $\overline{A}$ is not satisfied, the less conservative stability condition (111) can be tested. The condition (111) boils down to the following inequalities

$$\begin{aligned} &\tilde{a}_{11}(t) + \beta + \tilde{a}_{12}(t)q_2 \leq 0, \\ &\tilde{a}_{21}(t) + (\tilde{a}_{22}(t) + \beta)q_2 \leq 0. \end{aligned} \tag{113}$$

From these it follows that the solution pair (q_2, β) must satisfy the inequalities

$$\frac{\tilde{a}_{21}(t)}{-(\tilde{a}_{22}(t) + \beta)} \leq q_2 \leq \frac{-(\tilde{a}_{11}(t) + \beta)}{\tilde{a}_{12}(t)}. \tag{114}$$

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It is easy to see that for the existence of a solution pair (q_2, β) the following condition is necessary and sufficient

$$\sup_{t\geq 0}\left(\frac{\widetilde{a}_{21}(t)}{-\widetilde{a}_{22}(t)}\right) < \inf_{t\geq 0}\left(\frac{-\widetilde{a}_{11}(t)}{\widetilde{a}_{12}(t)}\right), \tag{115}$$

as illustrated by the following example.

Example 3. Consider a system as in (1) with a single time-varying delay, where:

$$A_0(t) = \begin{bmatrix} -6+\cos(2\pi\ t)-2e^{-t} & 1-e^{-t}\sin(2\pi\ t) \\ 1.5\sin(2\pi\ t) & -6+\sin(2\pi\ t) \end{bmatrix}, \quad A_1(t) = \begin{bmatrix} 1+e^{-t} & \cos(2\pi\ t) \\ 2\big(1+e^{-t}\cos(2\pi\ t)\big) & 2+e^{-t} \end{bmatrix}. \tag{116}$$

The computation of $\widetilde{A}(t) = A_0^{\mathcal{M}}(t) + |A_1(t)|$ provides

$$\widetilde{A}(t) = \begin{bmatrix} -5+\cos(2\pi\ t)-e^{-t} & 1-e^{-t}\sin(2\pi\ t)+|\cos(2\pi\ t)| \\ 1.5|\sin(2\pi\ t)|+2\big(1+e^{-t}\cos(2\pi\ t)\big) & -4+\sin(2\pi\ t)+e^{-t} \end{bmatrix}, \quad t\geq 0. \tag{117}$$

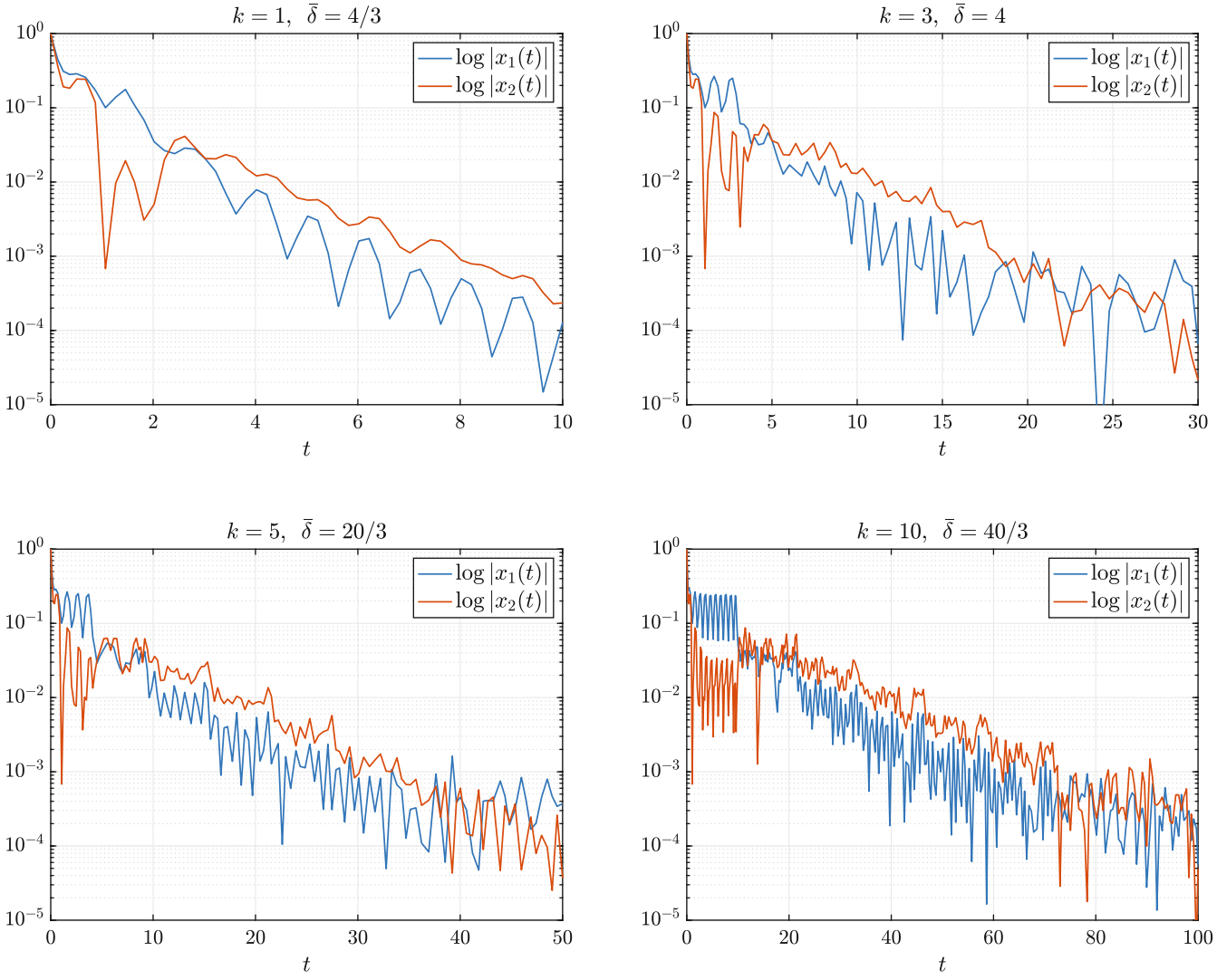

FIGURE 4 Time evolution of $\log(|x(t)|)$ for the system of Example 3, for different values of the time-varying delay $\delta_1(t) = \frac{k}{3}\sin(t) + k$, illustrating delay-independent 0-GES and slower convergence for larger delay bound $\bar{\delta}$.

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Computing the suprema of the components, we get the following upper bound of $\widetilde{A}(t)$:

$$\widetilde{A}(t) \leq \overline{A} = \begin{bmatrix} -4 & 2.146 \\ 4.347 & -2.213 \end{bmatrix}, \quad t \geq 0, \tag{118}$$

which is unstable (positive spectral abscissa $\overline{\lambda}(\overline{A}) = 0.0758$). As a consequence, a positive solution pair (q, β) of $\overline{A}q \leq -\beta q$ does not exist. However, the inequality (111) admits a positive solution pair (q_2, β) since the necessary and sufficient condition (115) is satisfied:

$$\sup_{t\geq 0}\left(\frac{\widetilde{a}_{21}(t)}{-\widetilde{a}_{22}(t)}\right) = 1.815 < 2 = \inf_{t\geq 0}\left(\frac{-\widetilde{a}_{11}(t)}{\widetilde{a}_{12}(t)}\right). \tag{119}$$

Indeed, the inequalities (114) are satisfied choosing $q_2 = 1.9$ and $\beta = 0.1$:

$$\sup_{t\geq 0}\left(\frac{\widetilde{a}_{21}(t)}{-\left(\widetilde{a}_{22}(t)+0.1\right)}\right) = 1.897 < 1.9 < 1.95 = \inf_{t\geq 0}\left(\frac{-\left(\widetilde{a}_{11}(t)+0.1\right)}{\widetilde{a}_{12}(t)}\right). \tag{120}$$

Thus, the exp-ISS for the system is proved for any bounded delay.

The trajectories of the system for zero input are shown in Figure 4, for different choices of the time-varying delay given by $\delta_1(t) = \frac{k}{3}\sin(t) + k$, for $k = 1, 3, 5, 10$, and initial conditions $x(t) = [-1 \;\; 1]^\top$ in $t \in [-\overline{\delta}, 0] = \left[-\frac{4k}{3}, 0\right]$.

Remark 10. It should be remarked that the stability of the systems in Examples 1 and 3 could not have been analyzed by means of Thm. 1 in Mazenc and Malisoff,[6] due to the presence of time-varying delays (multiple, in the case of Example 1).

6 | CONCLUSION

We studied the delay-independent stability of linear delay differential systems, in the general case of time-varying matrices and delays. Starting from the positive case, we introduced two distinct delay-independent conditions of exponential input-to-state stability. These conditions, formulated as linear inequalities, provide explicit bounds on the exponential convergence rate based on the maximum allowable delay, and allow to establish exp-ISS with linear gains. To broaden our analysis, we then extended all such conditions to systems without sign constraints, leveraging the state-bounding approach and comparison systems. As the systems considered exhibit time-varying characteristics, all proposed conditions are in terms of an infinite number of inequalities. Therefore, we devoted our attention to discussing implementation issues and some significant special cases that permit finite testing of the conditions. Moreover, it has been shown that for systems whose state has only two components the feasibility of the stability conditions can be analytically tested. Future work may concern the investigation of similar approaches for delay systems with nonlinear perturbations.

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CONFLICT OF INTEREST STATEMENT

The authors have no conflicts of interest to declare.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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