



Beauville p -groups of wild type and groups of maximal class

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Received: 21 March 2024 / Accepted: 6 April 2024 / Published online: 24 May 2024
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Abstract

Let G be a Beauville p -group. If G exhibits a ‘good behaviour’ with respect to taking powers, then every lift of a Beauville structure of $G/\Phi(G)$ is a Beauville structure of G . We say that G is a Beauville p -group of wild type if this lifting property fails to hold. Our goal in this paper is twofold: firstly, we fully determine the Beauville groups within two large families of p -groups of maximal class, namely metabelian groups and groups with a maximal subgroup of class at most 2; secondly, as a consequence of the previous result, we obtain infinitely many Beauville p -groups of wild type.

Keywords Beauville p -groups · p -Groups of maximal class

Mathematics Subject Classification 20D15 · 14J29

Communicated by John S. Wilson.

The first author is supported by the Spanish Government, Grant PID2020-117281GB-I00, partly with FEDER funds, and by the Basque Government, Grant IT483-22.

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1 Introduction

Beauville groups are finite groups that arise in the construction of an interesting class of complex surfaces, the so-called *Beauville surfaces*, introduced by Catanese in [5] following an example of Beauville [2, page 159]. The question as to which finite groups are Beauville groups has received considerable attention in the past years. The determination of Beauville groups is based on the following purely group-theoretical characterization. Given a group G and $S = \{x, y\} \subseteq G$, we write T for the triple $\{x, y, xy\}$ and we define

$$\Sigma(S) = \bigcup_{\substack{t \in T \\ g \in G}} \langle t \rangle^g. \quad (1)$$

Then G is a Beauville group if and only if it is a 2-generator group and

$$\Sigma(S_1) \cap \Sigma(S_2) = 1 \text{ for some 2-element sets of generators } S_1 \text{ and } S_2 \text{ of } G. \quad (2)$$

We say that S_1 and S_2 form a *Beauville structure* for G .

Research activity around Beauville groups has been very intense since the beginning of this century; see, for example, the survey papers [4, 7, 8, 16]. Catanese [5] showed that a finite abelian group is a Beauville group if and only if it is isomorphic to $C_n \times C_n$, with $n > 1$ and $\gcd(n, 6) = 1$. A remarkable result, proved independently by Guralnick and Malle [13] and by Fairbairn, Magaard and Parker [9], is that every non-abelian finite simple group other than A_5 is a Beauville group.

After abelian groups the most natural class of finite groups to consider are nilpotent groups. The study of nilpotent Beauville groups is reduced to that of Beauville p -groups. The smallest non-abelian Beauville p -groups were determined by Barker, Boston and Fairbairn [1]. Also in [1], it was shown that there are non-abelian Beauville p -groups of order p^n for every $p \geq 5$ and every $n \geq 3$. In [22], Stix and Vdovina constructed infinite series of Beauville p -groups, for every prime p , by considering quotients of ordinary triangle groups. Another infinite family of Beauville p -groups, for p an odd prime, was given by Gül and Uria-Albizuri [14]. Later on, Di Domenico, Gül and Thillaisundaram [6] provided more infinite families of Beauville p -groups, for every prime p . It is also worth mentioning that Fernández-Alcober, Gül and Vannacci [12] showed that the numbers of Beauville and non-Beauville p -groups of fixed order behave asymptotically the same as the number of 2-generator groups of the same order.

On the other hand, in the specific case of p -groups, Fernández-Alcober and Gül [11, Theorem 2.5] extended Catanese's criterion from abelian groups to a much wider family, including all p -groups of exponent p^e having the property that

$$x^{p^{e-1}} = y^{p^{e-1}} \text{ if and only if } (xy^{-1})^{p^{e-1}} = 1$$

for every x, y in the group, and in particular all groups of order at most p^p .

More precisely, if G is such a 2-generator group of exponent p^e , then

$$G \text{ is a Beauville group if and only if } p \geq 5 \text{ and } |G^{p^{e-1}}| \geq p^2. \quad (3)$$

This criterion is not valid for all finite p -groups, and in [11, Corollary 2.12] there are infinitely many examples (among which infinitely many p -groups of maximal class with an abelian maximal subgroup) for which (3) fails to hold.

Note that different Beauville structures in the same group can give rise to non-isomorphic Beauville surfaces. However, determining all Beauville structures in the group is a difficult task in general, due to the effect of conjugacy in (1). In any case, it is interesting to obtain as many Beauville structures as possible in a Beauville group. The finite p -groups considered in [11, Theorem 2.5] have a good behaviour in this respect, since they have the property that every lift of a Beauville structure of $G/\Phi(G) \cong C_p \times C_p$ is a Beauville structure of G . We say that a Beauville p -group G is of tame type if this property holds in G ; otherwise, we say that G is of wild type. Thus, all p -groups with good power structure are of tame type. However, as a careful analysis shows, all counterexamples to (3) given in [11, Corollary 2.12] are also of tame type.

Our goal in this paper is to determine all Beauville groups within two very large families of p -groups of maximal class, namely those which are either metabelian or have a maximal subgroup of class at most 2. These families have been studied in great detail by Miech [20] and by Leedham-Green and McKay [17, 18], respectively; in particular, constructions are given in these papers showing that there is a large number of such groups. Thus we provide a huge extension of Corollary 2.12 of [11], which only covers one p -group of maximal class of every order. An important consequence of our results is that they provide an infinite family of Beauville p -groups of wild type.

Next we state our main theorem in this paper. The Beauville p -groups of maximal class of order at most p^p were already determined in [11, Corollary 2.10]: they are exactly the groups of exponent p , with $p \geq 5$. Thus we restrict to groups of order at least p^{p+1} . In the theory of groups of maximal class, a significant role is played by the maximal subgroup $G_1 = C_G(G'/\gamma_4(G))$, and this is also the case in our result. In the remainder, we use the term maximal branch in a 2-generator finite p -group G to mean a difference of the form $B(M) = M \setminus \Phi(G)$, where M is a maximal subgroup of G .

Main Theorem *Let G be a p -group of maximal class of order p^n , with $n \geq p + 1$, and assume that either G is metabelian or G contains a maximal subgroup of class at most 2. Then G is a Beauville group if and only if $p \geq 5$ and one of the following two cases holds, where X is the set of all elements of $G \setminus \Phi(G)$ of order p :*

- (i) $X = G \setminus G_1$.
- (ii) X is the union of exactly two maximal branches of G , and either $n \not\equiv 2 \pmod{p - 1}$, or $n = p + 1$ and one of those maximal branches is $B(G_1)$.

Also, all groups in (i) are Beauville groups of tame type, all groups in (ii) are of wild type, and there exist infinitely many groups in each of the cases.

Notation. If G is a finitely generated group, $d(G)$ denotes the minimum number of generators of G . On the other hand, if G is a finite p -group, we write $\Omega_i(G)$ for the subgroup generated by all elements of G of order at most p^i . Also, G^{p^i} is the subgroup generated by all powers g^{p^i} , as g runs over G . The exponent of G is the maximum of the orders of all elements of G .

2 Proof of the main theorem

In this section G will always denote a group of order p^n and class $n - 1$, where $n \geq 2$. These groups are called p -groups of maximal class. The Beauville groups of order p^2 and p^3 can be easily determined, so we assume that $n \geq 4$ in the remainder. Then we set $G_i = \gamma_i(G)$ for all $i \geq 2$, and $G_1 = C_G(G_2/G_4)$. We have $|G_i : G_{i+1}| = p$ for all $i = 1, \dots, n - 1$, and the *degree of commutativity* of G is defined as the maximum (non-negative) integer $\ell \leq n - 3$ such that $[G_i, G_j] \leq G_{i+j+\ell}$ for all $i, j \geq 1$. Also, the only normal subgroups of G are G itself, its maximal subgroups, and the subgroups G_i for $i \geq 2$. Observe that $Z(G) = G_{n-1}$, $\Phi(G) = G_2$, and $d(G) = 2$.

In his seminal paper [3], Blackburn established many of the cornerstones of the theory of groups of maximal class. More precisely, he showed that $\ell > 0$ if and only if $G_1 = C_G(G_{n-2})$, and that this is the case whenever $n \geq p + 2$. An element $s \in G$ is called *uniform* if $s \notin G_1 \cup C_G(G_{n-2})$. Then $s^p \in Z(G)$ and the order of s is at most p^2 . Also, $|C_G(s)| = p^2$ and the conjugates of s are exactly the elements in the coset $s\Phi(G)$. If we choose an element $s_1 \in G_1 \setminus G_2$ and we define $s_i = [s_1, s, \overset{i-1}{\cdot}, s]$, then $s_i \in G_i \setminus G_{i+1}$ for all $i = 1, \dots, n - 1$. Thus $Z(G) = \langle s_{n-1} \rangle$. Finally, we recall a couple of facts about the power structure of G . If $|G| \leq p^{p+1}$ then $\exp G/Z(G) = \exp \Phi(G) = p$, while for $|G| \geq p^{p+2}$ we have

$$x^p \in G_{i+p-1} \setminus G_{i+p}, \quad \text{for all } x \in G_i \setminus G_{i+1} \text{ and all } i = 1, \dots, n - p. \quad (4)$$

These properties of p -groups of maximal class can be found in [10], [15, Chapter 4, Section 14], or [19, Chapter 3].

Lemma 2.1 *Let G be a p -group of maximal class, and let M be a maximal subgroup of G . Then all elements in $B(M)$ have the same order, and if $M \neq G_1$, this order is either p or p^2 .*

Proof Assume first that $M \neq G_1, C_G(G_{n-2})$. If $s \in B(M)$ then we have $B(M) = \cup_{i=1}^{p-1} s^i \Phi(G)$. All powers s^i are uniform elements of G for $i = 1, \dots, p - 1$, and they have the same order p or p^2 . Since all elements in the coset $s^i \Phi(G)$ are conjugate to s^i , the result follows in this case.

Now suppose that $M = G_1$ or $C_G(G_{n-2})$. If $n \geq p + 2$ then $M = G_1$ and by (4), every element in $B(M)$ has order p^k , where $k = \lceil \frac{n-1}{p-1} \rceil$. If $n \leq p + 1$ then M is a regular p -group in which $\Phi(G)$ is a maximal subgroup of exponent p . Thus $\Omega_1(M) = \Phi(G)$ or M , and all elements of $B(M)$ have order p or p^2 , respectively. \square

If G is a p -group of maximal class, we denote by $\mu(G)$ the number of maximal subgroups $M \neq G_1$ for which all elements in $B(M)$ have order p . The next result shows that the value of $\mu(G)$ is quite restricted if either G is metabelian or G_1 is of class at most 2. Its proof relies on two technical results of Miech regarding the calculation of p th powers in groups of maximal class that can be found in [20] and [21].

Theorem 2.2 *Let G be a p -group of maximal class. If either G is metabelian or G_1 is of class at most 2, then $\mu(G) = 0, 1, 2$ or p . Furthermore, if G has an abelian maximal subgroup then $\mu(G) = 0, 1$ or p .*

Proof If $p = 2$ then G has only 2 maximal subgroups different from G_1 and the result is obviously true. Thus we consider p to be an odd prime in the remainder of the proof.

Let us assume that $\mu(G) \geq 2$ and prove that $\mu(G) = 2$ or p . Since $\Phi(G) = G_2$, we can choose a uniform element s and an element $s_1 \in G_1 \setminus G_2$ so that s and ss_1 belong to two maximal branches with all elements of order p .

If G is metabelian, then by [20, Lemma 8] and since $s^p = 1$, we can write $(ss_1^i)^p = a^i b^{i^2}$ for all $i = 1, \dots, p - 1$, where

$$a = s_1^p s_2^{\binom{p}{2}} \dots s_p^{\binom{p}{p}}$$

and b is a power of s_{n-1} independent of i . Since $(ss_1)^p = 1$, we have $a = b^{-1}$ and consequently

$$(ss_1^i)^p = b^{i^2-i} = b^{i(i-1)}.$$

Since the order of b is a power of p and $\gcd(i(i - 1), p) = 1$ for $i = 2, \dots, p - 1$ (observe that p is odd), $b^{i(i-1)}$ is 1 if and only if $b = 1$. It then follows that $\mu(G)$ is either 2 or p .

Now assume that G_1 is of class at most 2. By applying [21, Theorem 4], we get

$$(ss_1^i)^p = s_1^{ip} \sigma_{i,1}^{\binom{p}{2}} \dots \sigma_{i,p-1}^{\binom{p}{p}} x_{i,p}, \tag{5}$$

where $\sigma_{i,k} = [s_1^i, s, \dots, s]$ and

$$x_{i,p} = \prod_{k=1}^{p-1} \prod_{\ell=0}^{k-1} [\sigma_{i,k}, \sigma_{i,\ell}]^{B(p,k,\ell)}, \tag{6}$$

for some integer $B(p, k, \ell)$ depending on p, k and ℓ , but not on i .

Let us show, by induction on k , that

$$\sigma_{i,k} = s_{k+1}^i t_{k+1}^{\binom{i}{2}}, \quad \text{for some } t_{k+1} \in [G_1, G_1] \leq Z(G_1). \tag{7}$$

First of all, observe that the basis of the induction is given by

$$\sigma_{i,1} = [s_1^i, s] = s_1^{-i} (s_1^s)^i = s_1^{-i} (s_1 s_2)^i = s_2^i [s_2, s_1]^{\binom{i}{2}}, \tag{8}$$

where the last equality follows from the condition that the class of G_1 is at most 2. Now if $k \geq 2$, as in (8), we have

$$[s_k^i, s] = s_{k+1}^i [s_{k+1}, s_k]^{\binom{i}{2}},$$

and then by the induction hypothesis we have

$$\begin{aligned} \sigma_{i,k} &= [s_k^i, s][t_k^{(i)}, s] = [s_k^i, s][t_k, s]^{(i)} = s_{k+1}^i [s_{k+1}, s_k]^{(i)} [t_k, s]^{(i)} \\ &= s_{k+1}^i ([s_{k+1}, s_k][t_k, s])^{(i)}, \end{aligned}$$

and the induction is complete.

Now from (7) we get

$$[\sigma_{i,k}, \sigma_{i,\ell}] = [s_{k+1}^i, s_{\ell+1}^i] = [s_{k+1}, s_{\ell+1}]^{i^2},$$

and this implies that $x_{i,p} = x_{1,p}^{i^2}$ for all $i = 1, \dots, p - 1$. Hence (5) yields

$$(ss_1^i)^p = (s_1^p s_2^{(p)} \dots s_p)^i \left(t_2^{(2)} \dots t_p^{(p)} \prod_{j=1}^{p-1} \prod_{k=j+1}^p [s_j, s_k]^{(j)(k)} \right)^{(i)} x_{1,p}^{i^2}.$$

Since p is odd, we can write again $(ss_1^i)^p = a^i b^{i^2}$ for all $i = 1, \dots, p - 1$, with a and b not depending on i . As above, we conclude that $\mu(G) = 2$ or p also in this case.

Finally, if G has an abelian maximal subgroup then obviously that subgroup must be G_1 . Observe that in the above discussion we have $b \in [G_1, G_1]$. Hence $b = 1$ in the present case and $(ss_1^i)^p = 1$ for all $i = 1, \dots, p - 1$. In other words, if $\mu(G) \geq 2$ then necessarily $\mu(G) = p$. This completes the proof. \square

We also need the following lemma about maximal subgroups of class ≤ 2 in a group of maximal class.

Lemma 2.3 *Let G be a p -group of maximal class of order at least p^5 . If G has a maximal subgroup M of class at most 2 then $M = G_1$.*

Proof Suppose for a contradiction that $M \neq G_1$. By definition, we have $G_1 = C_G(G_2/G_4)$, and consequently the commutator subgroup $[M, G_2]$ is not contained in G_4 . Since the only normal subgroups of G contained in G_2 are of the form G_i , it follows that $[M, G_2] = G_3$. On the other hand, by [10, Theorem 4.6], the degree of commutativity of G/G_5 is positive. As a consequence, we have $[G_1, G_3] \leq G_5$ i.e. $G_1 = C_G(G_3/G_5)$. Arguing as above, this implies that $[M, G_3] = G_4$. Now since $|M : G_2| = p$, we have $M' = [M, G_2]$ and then $\gamma_3(M) = [M, G_2, M] = G_4 \neq 1$, taking into account that $|G| \geq p^5$. This contradicts the assumption that the class of M is at most 2. \square

Now we can proceed to the proof of our main theorem.

Theorem 2.4 *Let G be a p -group of maximal class of order p^n , with $n \geq p + 1$, and assume that either G is metabelian or G contains a maximal subgroup of class at most 2. Then G is a Beauville group if and only if $p \geq 5$ and one of the following two cases holds:*

- (i) $\mu(G) = 2$, and either $n \not\equiv 2 \pmod{p-1}$, or $n = p + 1$ and one of the maximal branches consisting of elements of order p is $B(G_1)$.
- (ii) $\mu(G) = p$.

In the first case, all Beauville groups are of wild type, and in the second case, all Beauville groups are of tame type.

Proof We first prove the ‘only if’ part of the statement. Let us suppose that S_1 and S_2 form a Beauville structure for G , with associated triples T_1 and T_2 . We start by showing that $p \geq 5$. By way of contradiction, assume that $p = 2$ or 3 . Then G has positive degree of commutativity [10, Theorem 4.6], and consequently all elements in $G \setminus G_1$ are uniform. Now the elements in T_1 , and similarly those in T_2 , lie in different maximal branches of G . Since G has at most 4 maximal subgroups, some $x_1 \in T_1$ and $x_2 \in T_2$ lie in the same maximal branch $B(M)$, where $M \neq G_1$. Now observe that for every $x \in B(M)$, all elements in the coset $x\Phi(G)$ are conjugate to x . Hence some power of x_1 is conjugate to some power of x_2 . This is contrary to the condition $\Sigma(S_1) \cap \Sigma(S_2) = 1$, which holds according to (2). Thus we have $p \geq 5$ and, in particular, $n \geq 6$. By Lemma 2.3, in the case where G has a maximal subgroup of class ≤ 2 , that subgroup must be G_1 .

Let us prove that $\mu(G) \geq 2$. Otherwise there exist $x_1 \in T_1$ and $x_2 \in T_2$ which are uniform elements of order p^2 . It follows that $\langle x_1^p \rangle = Z(G) = \langle x_2^p \rangle$, again a contradiction. By applying Theorem 2.2, we conclude that either $\mu(G) = 2$ or $\mu(G) = p$.

Assume now that $\mu(G) = 2$, and either that $n = k(p - 1) + 2$ for some $k \geq 2$ or that $n = p + 1$ and the maximal branch $B(G_1)$ consists of elements of order p^2 . Again, we seek a contradiction. In this case, each of the triples T_1 and T_2 contains an element which lies either in $B(G_1)$ or in a maximal branch $B(M)$ consisting of uniform elements of order p^2 , with $M \neq G_1$. Let x be any element of $B(G_1)$. If $n = k(p - 1) + 2$ for some $k \geq 2$ then we have $x^{p^k} \in Z(G) \setminus 1$, by (4). On the other hand, if $n = p + 1$ and the elements of $B(G_1)$ are of order p^2 , then since $\exp G/Z(G) = p$, we have $x^p \in Z(G) \setminus 1$. In any case, we get $Z(G) \subseteq \Sigma(S_1) \cap \Sigma(S_2)$, which is contrary to our assumption that S_1 and S_2 form a Beauville structure. This completes the proof of the first implication.

Let us prove the converse. Thus we assume that $p \geq 5$ in the remainder. Suppose first that $\mu(G) = p$. In this case, we have to show that G is a Beauville group of tame type, i.e. that every lift to G of a Beauville structure of $G/\Phi(G)$ yields a Beauville structure of G . Let us consider then two minimal sets of generators S_1 and S_2 of G that map onto a Beauville structure of $G/\Phi(G)$. Then at most one of the elements in $T_1 \cup T_2$ lies in $B(G_1)$. Hence for every choice of $x_1 \in T_1$ and $x_2 \in T_2$, at least one of the elements, say x_1 , is of order p . Thus if $\langle x_1^g \rangle$ and $\langle x_2^h \rangle$ have non-trivial intersection for some $g, h \in G$, then $\langle x_1^g \rangle \subseteq \langle x_2^h \rangle$ and $\langle x_1\Phi(G) \rangle = \langle x_2\Phi(G) \rangle$. This is impossible, since $x_1\Phi(G)$ and $x_2\Phi(G)$ participate in different triples of a Beauville structure of $G/\Phi(G)$. Hence S_1 and S_2 form a Beauville structure of G .

Now suppose that we are in case (i), and choose a uniform element s and an element $s_1 \in G_1 \setminus G_2$ such that both s and ss_1 are of order p . We claim that $S_1 = \{s, s_1\}$ and $S_2 = \{ss_1^2, ss_1^4\}$ is a Beauville structure of G . Let $T_1 = \{s, s_1, ss_1\}$ and $T_2 = \{ss_1^2, ss_1^4, ss_1^2ss_1^4\}$. We need to see that $\langle x_1^g \rangle \cap \langle x_2^h \rangle = 1$ for all $x_1 \in T_1, x_2 \in T_2$ and

$g, h \in G$. Observe that, since

$$ss_1^2ss_1^4 \equiv s^2s_1^6 \equiv (ss_1^3)^2 \pmod{\Phi(G)}$$

and $p \geq 5$, x_1^s and x_2^h lie in different maximal subgroups of G . Thus if $x_1 = s$ or ss_1 , or if $x_1 = s_1$ and $B(G_1)$ consists of elements of order p , we can argue as in the previous paragraph. Hence we assume that $x_1 = s_1$ and that $|G| = p^n$ with $n \geq p + 2$ and $n \neq k(p - 1) + 2$ for all $k \geq 2$. By applying repeatedly (4), we have $\langle s_1 \rangle \cap Z(G) = 1$. Since $\Omega_1(\langle x_2 \rangle) = Z(G)$ for all $x_2 \in T_2$, the claim follows. Thus G is a Beauville group.

Let us finally see that G is of wild type in case (i). There are at least two maximal branches $B(M_1)$ and $B(M_2)$ which consist of uniform elements of order p^2 . It is always possible to construct a Beauville structure in $G/\Phi(G) \cong C_p \times C_p$ in which the first set of generators contains an element from $M_1/\Phi(G)$ and the second set an element from $M_2/\Phi(G)$. However, no lift of this structure can be a Beauville structure of G , since $\langle x_1^p \rangle = \langle x_2^p \rangle = Z(G)$ for all $x_1 \in B(M_1)$ and $x_2 \in B(M_2)$. □

Remark 2.5 Actually, the ‘if part’ of the previous theorem is valid for all p -groups of maximal class of order at least p^{p+1} , without requiring that G is metabelian or that G has a maximal subgroup of class at most 2.

The case of groups of maximal class with an abelian maximal subgroup is especially easy to describe.

Corollary 2.6 *Let G be a p -group of maximal class of order at least p^{p+1} . If G has an abelian maximal subgroup, then G is a Beauville group if and only if every element of $G \setminus G_1$ is of order p . All these Beauville groups are of tame type.*

Proof This follows immediately by combining Theorems 2.2 and 2.4. □

We remark that the only infinite pro- p group of maximal class P has an abelian subgroup A of index p with the property that all elements in $P \setminus A$ are of order p . Thus by taking finite quotients of P we get infinitely many examples of Beauville groups of tame type in our main theorem. The existence of infinitely many groups of wild type follows from the construction of metabelian p -groups of maximal class given by Miech in [20]. More precisely, if we consider metabelian groups satisfying the condition $[G_1, G_2] = G_{n-p+2}$, then Lemma 8 of [20] together with our proof of Theorem 2.2 shows the existence of infinitely many groups with $\mu(G) = 2$. If $n \not\equiv 2 \pmod{p - 1}$ then these groups are Beauville of wild type, according to Theorem 2.4.

Every quotient of order $\geq p^2$ of a p -group of maximal class is obviously again of maximal class. We conclude with the following surprising consequence of our main theorem.

Corollary 2.7 *Let G be a p -group of maximal class, where $p \geq 5$. Then every proper quotient of G is a Beauville group, and it is of tame type.*

Proof It suffices to show that $G/Z(G)$ is a Beauville group. If $|G/Z(G)| \leq p^p$ then, since $\exp G/Z(G) = p$, we only need to apply Corollary 2.10 of [11]. Assume now

that $|G/Z(G)| \geq p^{p+1}$. Since the p th powers of all uniform elements of G lie in $Z(G)$, it follows that $\mu(G/Z(G)) = p$. Now the result follows from Theorem 2.4 and Remark 2.5. \square

Funding Open access funding provided by the Scientific and Technological Research Council of Türkiye (TÜBİTAK).

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