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Nonhomogeneous expanding flows in hyperbolic spaces

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Abstract
In the present paper, we consider star-shaped mean convex hypersurfaces of the real, complex and quaternionic hyperbolic space evolving by a class of nonhomogeneous expanding flows. For any choice of the ambient manifold, the initial conditions are preserved and the long-time existence of the flow is proved. The geometry of the ambient space influences the asymptotic behaviour of the flow: after a suitable rescaling, the induced metric converges to a conformal multiple of the standard Riemannian round metric of the sphere if the ambient manifold is the real hyperbolic space; otherwise, it converges to a conformal multiple of the standard sub-Riemannian metric on the odd-dimensional sphere. Finally, in every case, we are able to construct infinitely many examples such that the limit does not have constant scalar curvature.

Keywords Curvature flows · Hyperbolic space · Star-shaped hypersurfaces · Sub-Riemannian geometry

Mathematics Subject Classification 53C17 · 53E10

1 Introduction

In recent years, many results about nonhomogeneous curvature flows in the Euclidean space appear. Different types of speed have been studied and many different problems addressed. Just to mention some of the most recent results, Sinestrari produced convexity estimates with Alessandroni [1] and considered volume and area preserving flows with Bertini [2] and ancient solutions for a very general class of expanding flows with Risa [3]. McCoy considered contracting nonhomogeneous flows [4] and their self-similar solutions [5]. Moreover, Li Chen, Xi Guo and Qiang Tu [6] extended to class of nonhomogeneous expanding speeds the classical result of Gerhardt [7] and Urbas [8].

Despite a growing interest in nonhomogeneous flows, the literature about their evolution in Riemannian manifolds is still at the beginning. To the best of our knowledge, the first paper about nonhomogeneous flows in a curved space is that one of Bertini with the author of the present paper [9], where we consider volume and area preserving flow in the real hyperbolic

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space. The ambient manifold is always a space form in all the recent papers on the subject: [10, 11].

The goal of this paper is to study nonhomogeneous expanding flows in the real, complex and quaternionic hyperbolic spaces exploring the role of the geometry of the ambient space.

Let $\mathbb{K}$ be either the real field $\mathbb{R}$, or the complex field $\mathbb{C}$, or the algebra of quaternions $\mathbb{H}$. Let $\mathbb{KH}^n$ be the $\mathbb{K}$-hyperbolic space endowed with its standard Riemannian metric. Let $F : \mathcal{M} \times [0, T) \to \mathbb{KH}^n$ be a one-parameter family of smooth embeddings such that $F_0(\cdot) = F(\cdot, 0)$ is a given hypersurface and F evolves by

$$\frac{\partial F}{\partial t} = \frac{1}{\psi(H)}\nu, \tag{1.1}$$

where ν is the unit outward normal vector of F, H is its mean curvature and $\psi : [0, \infty) \to \mathbb{R}$ is a continuous function C^2 differentiable in $(0, \infty)$ which satisfies the following structural conditions:

$$\begin{aligned} &\text{(i)} \quad \psi(x) > 0, \ \psi'(x) > 0, \qquad \forall x > 0; \\ &\text{(ii)} \quad \frac{x\psi'(x)}{\psi(x)} \le 1, \qquad \forall x > 0; \\ &\text{(iii)} \quad \psi''(x)\psi(x) - 2(\psi'(x))^2 \le 0, \qquad \forall x > 0. \end{aligned} \tag{1.2}$$

These conditions are part of the properties of the speed considered in [6], but our class is more general. In fact, as a consequence of a richer geometry in the ambient space, we will prove that H cannot converge to zero; hence, we do not need to prescribe the behaviour of ψ and ψ' when x is tends to 0. Clearly the classical inverse mean curvature flow is included in the class of flows that we are considering, but there are many other speeds satisfying (1.2): for example $\ln(1 + x)$, or $\sum_{i=1}^{k} c_i x^{p_i}$ with $c_i > 0$ and $0 < p_i \le 1$.

Before giving the precise statement of our main result, we need to introduce some useful notations in order to consider the three cases at once. For any $\mathbb{K}$, we define:

$$a := \dim_{\mathbb{R}} \mathbb{K} - 1 = \begin{cases} 0 \text{ if } \mathbb{K} = \mathbb{R}, \\ 1 \text{ if } \mathbb{K} = \mathbb{C}, \\ 3 \text{ if } \mathbb{K} = \mathbb{H}; \end{cases} \tag{1.3}$$

$$m := \dim_{\mathbb{R}} \mathbb{KH}^n - 1 = (a + 1)n - 1. \tag{1.4}$$

Clearly m is the dimension of any real hypersurface in $\mathbb{KH}^n$. Let σ be the round metric on the sphere, and if $\mathbb{K} \neq \mathbb{R}$, let $\sigma_{\mathbb{K}}$ be the *standard sub-Riemannian metric* on $\mathbb{S}^m$. See Notation 2.1 for more details. The main result of this paper is the following.

Theorem 1.1 *Let $\mathcal{M}_0$ be a closed star-shaped, mean convex hypersurface of $\mathbb{KH}^n$. If $\mathbb{K} = \mathbb{R}$ suppose that $n \ge 3$, otherwise suppose that $n \ge 2$ and $\mathcal{M}_0$ is $\mathbb{S}^a$-invariant. Let $\mathcal{M}_t$ be the evolution of $\mathcal{M}_0$ along the nonhomogeneous flow* (1.1)*, where ψ satisfies Conditions* (1.2)*. Let g_t be the induced metric on $\mathcal{M}_t$ and consider the rescaled metric*

$$\tilde{g}_t = |\mathcal{M}_t|^{-\frac{2}{m+a}} g_t.$$

Then:

(1) *$\mathcal{M}_t$ is star-shaped, mean convex and $\mathbb{S}^a$-invariant for any time the flow is defined;*
(2) *the flow is defined for any positive time;*
(3a) *if $\mathbb{K} = \mathbb{R}$, there is a smooth function $f : \mathbb{S}^{n-1} \to \mathbb{R}$ such that $\tilde{g}_t$ converges to the Riemannian metric $\tilde{g}_\infty = e^{2f}\sigma$; moreover, there are infinitely many $\mathcal{M}_0$ such that $\tilde{g}_\infty$ does not have constant scalar curvature;*

(3b) *if* $\mathbb{K} = \mathbb{C}$, *there is a smooth* $\mathbb{S}^1$*-invariant function* $f : \mathbb{S}^{2n-1} \to \mathbb{R}$ *such that* $\tilde{g}_t$ *converges to the sub-Riemannian metric* $\tilde{g}_\infty = e^{2f}\sigma_{\mathbb{C}}$*; moreover, there are infinitely many* $\mathcal{M}_0$ *such that* $\tilde{g}_\infty$ *does not have constant Webster scalar curvature;*

(3c) *if* $\mathbb{K} = \mathbb{H}$, *there is a smooth* $\mathbb{S}^3$*-invariant function* $f : \mathbb{S}^{4n-1} \to \mathbb{R}$ *such that* $\tilde{g}_t$ *converges to the sub-Riemannian metric* $\tilde{g}_\infty = e^{2f}\sigma_{\mathbb{H}}$*; moreover, there are infinitely many* $\mathcal{M}_0$ *such that* $\tilde{g}_\infty$ *does not have constant quaternionic contact scalar curvature.*

Our main Theorem 1.1 extends to a bigger class of speeds of a series of results about the inverse mean curvature flow. The case of the real hyperbolic space has been studied by Gerhardt [12], Ding [13], Hung and Wang [14]. In this last paper, the authors showed a fundamental difference with the Euclidean space: in the hyperbolic space, there are infinitely many initial data such that the limit is not round. More recently, the author of the present paper has studied the inverse mean curvature flow in the complex hyperbolic space [15] and in the quaternionic hyperbolic space [16] observing for the first time the presence of a sub-Riemannian limit. A survey about the inverse mean curvature flow in the Euclidean and hyperbolic spaces can be read in [17].

The presence of the nonhomogeneous speed introduces some technical difficulties. Conditions (1.2) are crucial for proving especially part (1) and (2). In fact, (i) guarantees the short-time existence of the flow, (ii) is used for proving that the star-shapeness is preserved and (iii) helps for the mean convexity and the long-time existence of the flow.

It is well known that there are deep interactions between geometric flows and general relativity. In fact, the tools for proving the final parts of Theorem 1.1 come from the general relativity. When $\mathbb{K} = \mathbb{R}$, we adopt the strategy developed in [14] showing that the modified Hawking mass (5.22) is an excellent tool for any speed in the class (1.2): we have that the rescaled induced metric is round if and only if its mass converges to zero; finally, we are able to prove that there are infinitely many examples with a mass that does not converges to zero. If $\mathbb{K} \neq \mathbb{R}$, the modified Hawking mass cannot be used; therefore, in [15, 16] we introduced a weaker notion of mass (5.24) that has the disadvantage that it does not fully classify the initial data with round limit, but it is enough to construct the desired counterexamples. In the final section, we will prove that this mass works very well with the flow (1.1) too, completing the proof of Theorem 1.1 (3b) and (3c).

The paper is organized as follows. In Section 2, we collect some basic notions about the geometry of $\mathbb{K}\mathbb{H}^n$ and its hypersurfaces and we list some general properties of our flows, including the evolution equations of the most significant geometric quantities. In Section 3, we start the proof of Theorem 1.1 showing that the star-shapeness and the mean convexity are preserved. The main results of Section 4 is the long-time existence of the flow. The convergence of the rescaled induced metric and the construction of the examples which develops a limit with not constant scalar curvature is the topic of Section 5.

2 Preliminaries

2.1 Riemannian and sub-Riemannian metric on the sphere

Every hypersurface considered in this paper is closed and star-shaped and so it is an embedding of $\mathbb{S}^m$, the sphere of dimension m into $\mathbb{R}^{m+1} \equiv \mathbb{K}^n$. On that sphere, we will consider different "standard" metrics. In particular, we will use the following notations. When $\mathbb{K} = \mathbb{R}$, they are redundant, but they are useful for giving an unique proof for any value of $\mathbb{K}$.

Notation 2.1 (i) *We denote by σ the round Riemannian metric on the sphere with constant sectional curvature* 1.

(ii) *When $\mathbb{K} \neq \mathbb{R}$, we denote by $\sigma_{\mathbb{K}}$ the sub-Riemannian metric on $\mathbb{S}^m$ which coincides with σ on the horizontal distribution $\mathcal{H}$ of the Hopf fibration $\pi : \mathbb{S}^m \to \mathbb{KP}^{n-1}$. They are sub-Riemannian because in both cases $\mathcal{H} + [\mathcal{H}, \mathcal{H}] = T\mathbb{S}^m$.*

(iii) *We define $\sigma_{\mathbb{R}} := \sigma$.*

(iv) *When $\mathbb{K} \neq \mathbb{R}$, fix $\lambda > 0$, we denote with e_λ the Berger metric of parameter λ on $\mathbb{S}^m$ obtained deforming σ with a factor λ in the vertical distribution $\mathcal{V}$ of the Hopf fibration. Note that we used the same symbol for both the value of $\mathbb{K}$, but it is important to keep in mind that we defined two different family of metrics. In fact, for example,* $\dim \mathcal{V} = a$.

(v) *When $\mathbb{K} = \mathbb{R}$, for any $\lambda > 0$ we define $e_\lambda := \sigma$.*

Note that when $\lambda \to \infty$ we have that $e_\lambda \to \sigma_{\mathbb{K}}$. Moreover, we introduce the following notation in order to distinguish between derivatives of a function on the sphere with respect to different metrics.

Notation 2.2 *For any given function $f : \mathbb{S}^m \to \mathbb{R}$, let f_{ij} (resp. $\hat{f}_{ij}$) be the components of the Hessian of f with respect to σ (resp. e_λ). The value of λ and $\mathbb{K}$ will be clear from the context. The indices go up and down with the associated metric: for instance $\hat{f}_i^k = \hat{f}_{ij} e_\lambda^{jk}$, while $f_i^k = f_{ij}\sigma^{jk}$. Analogous notations will be used for higher-order derivatives. Moreover, here and in the following, unless explicitly stated otherwise, we will always use the Einstein convention about the repeated indices.*

For any given function, it will be useful to compare its second derivatives computed with respect to the Berger metric and those determined by σ. This comparison is simpler if we assume the $\mathbb{S}^a$-invariance. In the following result, we summarize Lemma 2.3 of [15] and Lemma 2.3 of [16].

Lemma 2.3 *Fix $\mathbb{K} \neq \mathbb{R}$, $\lambda > 0$ and let $\varphi : \mathbb{S}^m \to \mathbb{R}$ be a smooth function. If φ is $\mathbb{S}^a$-invariant, we have:*

$$\Delta_e \varphi := \hat{\varphi}_i^i = \Delta_\sigma \varphi := \varphi_i^i;$$
$$|\nabla_e^2 \varphi|_e^2 := \hat{\varphi}_i^j \hat{\varphi}_j^i = |\nabla_\sigma^2 \varphi|_\sigma^2 + 2a(\lambda - 1)|\nabla_\sigma \varphi|_\sigma^2 = \varphi_i^j \varphi_j^i + 2a(\lambda - 1)\varphi_i \varphi^i.$$

Each one of the metrics discussed above carries with it a notion of curvature. When the metric is Riemannian it is obvious what we mean by curvature. On the other hand, it can be computed that, if $\mathbb{K} \neq \mathbb{R}$, as $\lambda \to \infty$, the sectional curvature of e_λ diverges. This happens every time we approximate a sub-Riemannian metric with a family of Riemannian metrics. Therefore, when we talk about the curvature of $\sigma_{\mathbb{K}}$ (and their conformal multiples) we need to clarify what we mean.

When $\mathbb{K} = \mathbb{C}$, $\mathbb{S}^m$ has a in a natural way a CR-structure given by the 1-form $\theta(\cdot) = \sigma(J\nu, \cdot)$, where ν is unit normal to $\mathbb{S}^m$ embedded in the standard way in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$, and J is the complex structure of $\mathbb{C}^n$. The sub-Riemannian metric $e^{2f}\sigma_{\mathbb{C}}$ can be thought as the restriction to $\mathcal{H} \times \mathcal{H}$ of the Webster metric of the CR-structure definite by $e^{2f}\theta$. In this context, a fundamental notion is the Tanaka–Webster connection which is the unique connection which satisfies some compatibility conditions with the CR-structure. With this connection, we can define in the usual formal way a curvature, called *Webster curvature*. It is well known that θ has constant Webster curvature (equal to 1), while in general $e^{2f}\theta$ may not. More details and results about CR-geometry can be found in the monograph [18]. In the same spirit, when $\mathbb{K} = \mathbb{H}$, $\mathbb{S}^m$ inherits from $\mathbb{R}^{4n} \equiv \mathbb{H}^n$ a *quaternionic contact*-structure (qc-structure for short). The role of the Tanaka–Webster connection is played by the Biquard connection. It can be used to define a qc-Ricci tensor and a qc-scalar curvature. Once again, the standard qc-structure has constant qc-curvature, but its conformal multiples may not. A good introduction to qc-geometry is [19].

A central and classical subject in Geometric Analysis is the Yamabe problem: find, if they exist, the metrics with constant scalar curvature in a fixed conformal class. It has been solved in great generality for all the three concept of curvature mentioned above. The detailed explanation of its solution goes beyond the purposes of the present work. In the next result, we focus only on the cases of our interest.

Lemma 2.4 *Let $f : \mathbb{S}^m \to \mathbb{R}$ be a smooth function. If $\mathbb{K} \neq \mathbb{R}$, suppose that f is $\mathbb{S}^a$-invariant. The following characterizations of the solution of the Yamabe problem hold.*

(1) *$e^{2f}\sigma$ has constant scalar curvature if and only if e^{-f} is a linear combination of constants and first eigenfunctions on the sphere;*
(2) *$e^{2f}\sigma_{\mathbb{C}}$ has constant Webster scalar curvature if and only if f is constant;*
(3) *$e^{2f}\sigma_{\mathbb{H}}$ has constant qc-scalar curvature if and only if f is constant.*

Part (1) is the content of Lemma 4 of [14], part (2) is Lemma 2.5 of [15] and part (3) is Lemma 2.4 of [16].

2.2 Geometry of hyperbolic spaces

The ambient manifolds that we are considering can be characterized in many ways and they can be described with many different isometric models. Since we wish to work with star-shaped hypersurfaces, the best thing to do is to introduce polar coordinates. The underlying manifold of $\mathbb{KH}^n$ is $\mathbb{R}^{(a+1)n} \equiv \mathbb{K}^n$, where a has been defined in (1.3), equipped with the metric

$$\bar{g} = d\rho^2 + \sinh^2(\rho) e_{\cosh^2(\rho)},$$

where ρ is the radial distance from the centre of the polar coordinates and $e_{\cosh^2(\rho)}$ is the Berger metric of parameter $\cosh^2(\rho)$ as defined in Notation 2.1. Its curvature tensor has the following explicit expression

$$\begin{aligned}\bar{R}(X, Y, Z, W) = & -\bar{g}(X, Z)\bar{g}(Y, W) + \bar{g}(X, W)\bar{g}(Y, Z) \\ & + \sum_{i=1}^{a} [-\bar{g}(X, J_i Z)\bar{g}(Y, J_i W) + \bar{g}(X, J_i W)\bar{g}(Y, J_i Z)] \\ & -2\sum_{i=1}^{a} \bar{g}(X, J_i Y)\bar{g}(Z, J_i W),\end{aligned} \tag{2.5}$$

where $J_1, \ldots, J_a$ are the complex structure of $\mathbb{KH}^n$. Note that if $\mathbb{K} = \mathbb{R}$, and hence, $a = 0$, the sums in the second and third line of (2.5) are empty. From (2.5), it follows that the our ambient manifolds are symmetric, the curvature is constant equal to -1 if $\mathbb{K} = \mathbb{R}$, otherwise is bounded between -4 and -1. Moreover, $\mathbb{KH}^n$ is Einstein with Ricci tensor given by

$$\bar{R}ic = -(m + 3a)\bar{g}. \tag{2.6}$$

2.3 Geometry of hypersurfaces in hyperbolic spaces

Let $\mathcal{M}$ be a real closed star-shaped hypersurface of $\mathbb{KH}^n$, then it is an embedding of the sphere of dimension m in $\mathbb{KH}^n$. Up to an isometry of the ambient manifold, we can always suppose that it is star-shaped with respect to the centre of the polar coordinates. Hence, $\mathcal{M}$ is determined by a positive function $\rho : \mathbb{S}^m \to \mathbb{R}$ such that in polar coordinate $\mathcal{M} = \{(x, \rho(x)) \in \mathbb{KH}^n \mid x \in \mathbb{S}^m\}$. For any given $\mathcal{M}$, we call such ρ the *radial function* associated with $\mathcal{M}$. If $\mathbb{K} \neq \mathbb{R}$, we know that $\mathbb{S}^a$ acts by isometries on $\mathbb{S}^m$. In this case, we say that $\mathcal{M}$ is $\mathbb{S}^a$*-invariant* if its radial function is $\mathbb{S}^a$-invariant. For reasons of synthesis, sometimes will talk about, with an abuse of notation, $\mathbb{S}^a$-invariance even when $\mathbb{K} = \mathbb{R}$: in this case, it has to be considered as an empty condition. With the same proof of Lemma 3.1 of [20], we can prove the following result.

Lemma 2.5 *The evolution of an* $\mathbb{S}^a$*-invariant hypersurface of* $\mathbb{KH}^n$ *stays* $\mathbb{S}^a$*-invariant during the whole duration of the flow.*

Now we want to describe the main geometric quantities associated with a star-shaped hypersurface in term of its radial function. For more details and explicit computations we refer to Section 3 of [16] and Section 3 of [15]. We introduce an auxiliary function $\varphi = \varphi(\rho)$ such that $\frac{d\varphi}{d\rho} = \frac{1}{\sinh(\rho)}$. Fix $(Y_1, \dots, Y_m)$ a tangent basis of $\mathbb{S}^m$ and denote with $\rho_i := Y_i(\rho)$ and with $\varphi_i := Y_i(\varphi) = \frac{\rho_i}{\sinh(\rho)}$. When $\mathbb{K} \neq \mathbb{R}$, since the ambient metric is no more isotropic, it is convenient to choose a tangent basis on $\mathbb{S}^m$ adapted to the contact structure: from now on we always suppose that for any $i = 1, \dots, a$ $Y_i = J_i\nu$, where ν is the unit normal of the standard immersion of $\mathbb{S}^m$ in $\mathbb{R}^{m+1} \equiv \mathbb{K}^n$. The use of this base simplifies some computations because, for example, $\rho_i = \varphi_i = 0$ for every $i = 1, \dots, a$. Let $V_i := F_*Y_i = Y_i + \rho_i\partial\rho$, then $(V_1, \dots, V_m)$ is a basis of the tangent space of $\mathcal{M}$. Let $g = F^*\bar{g}$ be the induced metric on $\mathcal{M}$. In coordinates it can be expressed as

$$g_{ij} = \sinh^2(\rho)\left(\varphi_i\varphi_j + e_{ij}\right). \tag{2.7}$$

The outward unit normal vector field of $\mathcal{M}$ is

$$\nu = \frac{1}{v}\left(\partial\rho - \frac{\nabla\varphi}{\sinh(\rho)}\right), \tag{2.8}$$

where

$$v = \bar{g}(\nu, \partial\rho)^{-1} = \sqrt{1 + |\nabla\varphi|^2},$$

and the gradient $\nabla\varphi$ is with respect to σ for any $\mathbb{K}$ because of the $\mathbb{S}^a$-invariance. The inverse of the induced metric is

$$g^{ij} = \frac{1}{\sinh^2(\rho)}\left(e^{ij} - \frac{\varphi^i\varphi^j}{v^2}\right),$$

where we are using Notation 2.1 when $\mathbb{K} = \mathbb{R}$. The second fundamental form of $\mathcal{M}$ is

$$h_i^j = -\frac{\hat{\varphi}_{ik}\tilde{e}^{kj}}{v\sinh(\rho)} + \frac{\cosh(\rho)}{v\sinh(\rho)}\delta_i^j + \frac{\sinh(\rho)}{v\cosh(\rho)}\sum_{k=1}^{a}\delta_i^k\delta_k^j, \tag{2.9}$$

where $\tilde{e}^{ij} = \sinh^2(\rho)g^{ij} = e^{ij} - \frac{\varphi^i\varphi^j}{v^2}$ and we used Notation 2.2 for the second derivative of φ. Taking the trace of (2.9) and using Lemma 2.3, we can compute the mean curvature of $\mathcal{M}$:

$$H = -\frac{\varphi_{ij}\tilde{\sigma}^{ij}}{v\sinh(\rho)} + \frac{\hat{H}}{v}, \tag{2.10}$$

where $\tilde{\sigma}^{ij} = \sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2}$ and

$$\hat{H}(\rho) = m\frac{\cosh(\rho)}{\sinh(\rho)} + a\frac{\sinh(\rho)}{\cosh(\rho)}. \tag{2.11}$$

2.4 Evolution equations

Since by Condition (1.2) we have that $\psi' > 0$, the short-time existence and uniqueness of the solution for the flow (1.1) are guaranteed by standard arguments. Moreover, well-known computations (see for example [21]) can be repeated to compute the evolution equation of the main geometric quantities.

Lemma 2.6 *Since the ambient space is symmetric the following evolution equations hold:*

(1) $\dfrac{\partial g_{ij}}{\partial t} = \dfrac{2}{\psi}h_{ij}, \qquad \dfrac{\partial g^{ij}}{\partial t} = -\dfrac{2}{\psi}h^{ij},$

(2) $\dfrac{\partial H}{\partial t} = \dfrac{\psi'}{\psi^2}\Delta H + \dfrac{\psi''\psi - 2(\psi')^2}{\psi^3}|\nabla H|^2 - \dfrac{1}{\psi}\left(|A|^2 + \bar{R}ic(\nu,\nu)\right),$

(3) $\dfrac{\partial \psi}{\partial t} = \dfrac{\psi'}{\psi^2}\Delta\psi - 2\dfrac{(\psi')^3}{\psi^3}|\nabla H|^2 - \dfrac{\psi'}{\psi}\left(|A|^2 + \bar{R}ic(\nu,\nu)\right),$

(4)
$$\begin{aligned}\frac{\partial h_i^j}{\partial t} &= -\nabla_i\nabla^j\frac{1}{\psi} + \frac{1}{\psi}\left(\bar{R}_{0i0}{}^j - h_i^k h_k^j\right),\\ &= \frac{\psi'}{\psi^2}\Delta h_i^j + \frac{\psi''\psi - 2(\psi')^2}{\psi^3}\nabla_i H\nabla^j H - \left(\frac{1}{\psi} + \frac{H\psi'}{\psi^2}\right)\left(h_i^k h_k^j + \bar{R}_{0i0}{}^j\right),\\ &\quad + \frac{\psi'}{\psi^2}\left(\left(|A|^2 + \bar{R}ic(\nu,\nu)\right)h_i^j + 2\bar{R}^{k\ j}_{\ is}h_k^s - \bar{R}^{k\ s}_{\ si}h_k^j - \bar{R}^{s\ j}_{\ ks}h_i^k\right)\end{aligned}$$

(5) $\dfrac{\partial d\mu}{\partial t} = \dfrac{H}{\psi}d\mu.$

Here and in the following, if there is no risk of confusion, we are using for brevity only ψ for saying $\psi(H)$, and analogously for its derivatives. Note that integrating the evolution equation of the volume form $d\mu$ we get

$$\frac{d\,|\mathcal{M}_t|}{dt} = \int_{\mathcal{M}_t}\frac{H}{\psi}d\mu, \tag{2.12}$$

in particular it follows that (1.1) is an expanding flow, at least as far the evolving hypersurface is mean convex.

Example 2.7 A geodesic sphere is a star-shaped hypersurface with constant radial function. Therefore, by (2.10) its mean curvature is given by $\hat{H}(\rho)$. In particular, it is constant. The evolution of a geodesic sphere is a family of geodesic spheres such that the radius evolves in the following way

$$\frac{d\rho}{dt} = \frac{1}{\psi(\hat{H}(\rho))}.$$

For a general ψ, we cannot find the explicit solution of this ODE, but $\frac{\partial\hat{H}}{\partial\rho} < 0$ by direct computations, $\psi' > 0$ by Conditions (1.2); therefore, ρ is increasing and it blows up in infinite time. Since $\lim_{\rho\to+\infty}\hat{H}(\rho)=m+a$, then

$$\rho \approx \frac{t}{\psi(m+a)}, \text{ as } t\to+\infty.$$

3 First-order estimates

The main goal of this section is to prove that the initial conditions are preserved, i.e. part (1) of Theorem 1.1. The most important technical result is the following.

Proposition 3.1 *There is a positive constant c such that if ψ satisfies Conditions* (1.2) *(i) and (ii), then*

(1) $|\nabla\varphi|^2 \le ce^{-\frac{2}{\psi(m+a)}t}$;

(2) $|H-m-a| \le ce^{-\frac{2}{\psi(m+a)}t}$, $\quad |\psi(H)-\psi(m+a)| \le ce^{-\frac{2}{\psi(m+a)}t}$.

Part (1) of this proposition has a direct important consequences.

Corollary 3.2 *The evolution of any star-shaped $\mathbb{S}^a$-invariant hypersurface stays star-shaped for any time the flow is defined.*

Proof By Proposition 3.1, there exists a positive constant c such that

$$v=\bar{g}\left(\frac{\partial}{\partial\rho},\nu\right)^{-1}=\sqrt{1+|\nabla\varphi|^2}\le c.$$

It follows that $\frac{\partial}{\partial\rho}$ and ν are never orthogonal in $\mathbb{KH}^n$. This means that $\mathcal{M}_t$ is star-shaped for any time t. □

The proof of Proposition 3.1 proceeds by steps: first we prove that $\nabla\varphi$ is just bounded (which is already enough for having Corollary 3.2), then we prove that it decays exponentially fast, and finally, we find the optimal exponent. In the meanwhile, we are able to show that H stays strictly positive and bounded, and converges exponentially fast to $m+a$, i.e. to the mean curvature of a horosphere in $\mathbb{KH}^n$; finally, we can find the optimal exponent for H too. The first crucial step is the following lemma.

Lemma 3.3 *If ψ satisfies Conditions (1.2) i) and ii), then for any (x,t)*

$$|\nabla\varphi(x,t)|^2 \le \max_{y\in\mathbb{S}^m}|\nabla\varphi(y,0)|^2 .$$

Proof Let us define $\omega=\frac{1}{2}|\nabla\varphi|^2=\frac{1}{2}\varphi_k\varphi^k$. We want to compute the evolution equation of ω and apply the maximum principle. We have that the radial function satisfies the scalar evolution equation $\frac{\partial\rho}{\partial t}=\frac{v}{\psi(H)}$; hence, the evolution of φ is given by

$$\frac{\partial\varphi}{\partial t}=G(\varphi_{ij},\varphi_i,\varphi):=\frac{v}{\sinh(\rho)\psi(H)}. \tag{3.13}$$

The original geometric flow (1.1) is defined at least as far the scalar flow (3.13) is defined, and when both are defined, they are equivalent. Therefore, we can work with (3.13).

Let $a^{ij}=\frac{\partial G}{\partial\varphi_{ij}}=\frac{\psi'}{\psi^2}g^{ij}$: it is a symmetric and positive definite, at least as far $\frac{\psi'}{\psi^2}$ is bounded and strictly positive. Moreover, we denote by $b^i=\frac{\partial G}{\partial\varphi_i}$. From (3.13), we have:

$$\frac{\partial\omega}{\partial t}=\varphi^k\nabla_k\frac{\partial\varphi}{\partial t}=\varphi^k\left(a^{ij}\varphi_{ijk}+b^i\varphi_{ik}+\frac{\partial G}{\partial\varphi}\varphi_k\right).$$

Let R be the Riemannian curvature tensor of σ, then the Ricci identity says

$$\varphi_{ijk} = \varphi_{kij} + R^m_{ijk}\varphi_m = \varphi_{kij} + \varphi_j\sigma_{ik} - \varphi_k\sigma_{ij}. \tag{3.14}$$

Since a^{ij} is symmetric and positive definite, applying (3.14), after some explicit computations we get

$$a^{ij}\varphi_{ijk}\varphi^k = a^{ij}\omega_{ij} - a^{ij}\varphi_{ik}\varphi^k_j + a^{ij}\varphi_i\varphi_j - 2a^i_i\omega \le a^{ij}\omega_{ij}.$$

As a consequence of the $\mathbb{S}^a$-invariance, we have

$$\frac{\partial\tilde{\sigma}}{\partial\varphi} = 0, \quad \frac{\partial v}{\partial\varphi} = 0;$$

hence, using the explicit expression of the mean curvature (2.10) we can compute

$$\frac{\partial G}{\partial\varphi} = \frac{v\cosh(\rho)}{\sinh(\rho)\psi}\left(\frac{H\psi'}{\psi} - 1\right) - \frac{\psi'}{\psi^2}\left(m + a + \frac{a}{\cosh^2(\rho)}\right) \tag{3.15}$$

$$\le -\frac{\psi'}{\psi^2}\left(m + a + \frac{a}{\cosh^2(\rho)}\right) \tag{3.16}$$

where in the last line we used Condition (1.2) (ii). Summarizing we have just found that

$$\frac{\partial\omega}{\partial t} \le a^{ij}\omega_{ij} + b^i\omega_i.$$

The result follows by the maximum principle. □

Now we are able to prove that H is strictly positive and bounded.

Lemma 3.4 *If ψ satisfies Conditions (1.2), then there exist two positive constant c_1, c_2 such that for any time the flow is defined we have*

$$0 < c_1 \le H \le c_2.$$

Proof We can start from the upper bound. Combining the evolution equation of H given in Lemma 2.6, with Condition (1.2) iii), the fact that $H^2 \le m|A|^2$ and (2.6) we have:

$$\frac{\partial H}{\partial t} \le \frac{\psi'}{\psi^2}\Delta H - \frac{1}{\psi}\left(\frac{H^2}{m} - m - 3a\right).$$

By the maximum principle, we have that $H \le c_2$ for some constant c_2 depending only on m, a and $\mathcal{M}_0$. On the other hand, let $\alpha = \frac{\partial\varphi}{\partial t} = \frac{v}{\sinh(\rho)\psi}$, then by (3.16) and (3.13)

$$\frac{\partial\alpha}{\partial t} = a^{ij}\alpha_{ij} + b^i\alpha_i + \frac{\partial G}{\partial\varphi}\alpha \le a^{ij}\alpha_{ij} + b^i\alpha_i.$$

By the maximum principle, α is bounded from above; therefore, there is a positive constant c such that

$$\psi \ge \frac{c}{\sinh(\rho)}.$$

Since ρ does not blow up in finite time, this means that ψ, and hence H, are strictly positive for any finite time. Now we can improve what just said showing that H cannot converge to zero. Let us consider the function $\tilde{\alpha} = \frac{v}{\sinh(\rho)\psi}e^{\frac{t}{\psi(m+a)}}$. Recalling that $\rho \approx \frac{t}{\psi(m+a)}$ as $t \to +\infty$ (if we can take arbitrary big times t), Lemma 3.3, and the fact that $\psi' > 0$, then an upper bound for $\tilde{\alpha}$ implies a strictly positive lower bound for H. We compute the evolution equation of $\tilde{\alpha}$:

$$\begin{aligned}\frac{\partial\tilde{\alpha}}{\partial t} &= a^{ij}\tilde{\alpha}_{ij} + b^i\tilde{\alpha}_i + \frac{\partial G}{\partial\varphi}\tilde{\alpha} + \frac{1}{\psi(m+a)}\tilde{\alpha}\\ &\le a^{ij}\tilde{\alpha}_{ij} + b^i\tilde{\alpha}_i - \tilde{\alpha}(m+a)\frac{\psi'}{\psi^2} + \frac{1}{\psi(m+a)}\tilde{\alpha}\\ &= a^{ij}\tilde{\alpha}_{ij} + b^i\tilde{\alpha}_i - \tilde{\alpha}^2(m+a)\frac{\psi'\sinh(\rho)}{ve^{\frac{t}{\psi(m+a)}}\psi} + \frac{1}{\psi(m+a)}\tilde{\alpha}. \end{aligned} \tag{3.17}$$

We claim that for any $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that $\frac{\psi'(x)}{\psi(x)} > c_\varepsilon$ for any $x \in [0, \varepsilon]$. In fact, by Conditions (1.2) iii) we have

$$\left(\frac{\psi'}{\psi}\right)' \le \left(\frac{\psi'}{\psi}\right)^2.$$

Let y be the solution of

$$y' = y^2, \quad y(\varepsilon) = \frac{\psi'(\varepsilon)}{\psi(\varepsilon)}.$$

Note that $y(\varepsilon) > 0$ by (1.2) i). It follows that for any $x \in [0, \varepsilon]$

$$\frac{\psi'(x)}{\psi(x)} \ge y(x) = \left(\varepsilon + \frac{1}{y(\varepsilon)} - x\right)^{-1} \ge c_\varepsilon := y(0) = \left(\varepsilon + \frac{1}{y(\varepsilon)}\right)^{-1};$$

hence, the claim holds.

Now let $\varepsilon = c_2$ where c_2 is the constant found above, applying the above claim there exists a $\tilde{c} > 0$ such that

$$\frac{\partial\tilde{\alpha}}{\partial t} \le a^{ij}\tilde{\alpha}_{ij} + b^i\tilde{\alpha}_i - \tilde{c}\tilde{\alpha}^2 + \frac{1}{\psi(m+a)}\tilde{\alpha}.$$

By the maximum principle, we can conclude that $\tilde{\alpha}$ is bounded from above from a constant that does not depend on time. □

Now we can improve what said so far showing that $|\nabla\varphi|^2$ decays exponentially fast and that H converges (once we will prove long-time existence for the flow) exponentially fast to $m + a$.

Lemma 3.5 *There exist positive constants β, γ, c such that*

(1) $|\nabla\varphi|^2 \le ce^{-\beta t}$;
(2) $|H - m - a| \le ce^{-\gamma t}$; $\quad |\psi(H) - \psi(m+a)| \le ce^{-\gamma t}$.

Proof (1) The function $\frac{\psi'}{\psi^2}$ is continuous and strictly positive in $[c_1, c_2]$, where c_1 and c_2 are determined in Lemma 3.4. Then we can find a strictly positive constant b such that for any time t

$$\frac{\psi'}{\psi^2} \ge b.$$

Hence, we can define $\beta = (m+a)b$ and repeat the proof of Lemma 3.3 improving the estimates of the reaction term in order to have an exponential decay:

$$\frac{\partial G}{\partial\varphi} \le -(m+a)\frac{\psi'}{\psi^2} \le -\beta.$$

(2) When $\mathbb{K}=\mathbb{R}$, we have $a=0$ and H can be estimate as follows:

$$\frac{\partial H}{\partial t} \leq \frac{\psi'}{\psi^2}\Delta H - \frac{1}{\psi}\left(|A|^2 - m\right) \leq \frac{\psi'}{\psi^2}\Delta H - \frac{1}{m\psi}\left(H^2 - m^2\right).$$

Since in Lemma 3.4 we proved that H is bounded and it cannot be too close to 0, we have that $\frac{1}{\psi}$ is bounded too. Therefore, applying the maximum principle we get that there exists a constant $\gamma > 0$ such that

$$H - m \leq ce^{-\gamma t}$$

When $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$, the proof is more involved. By the $\mathbb{S}^a$-invariance, for any $k = 1, \dots, a$ (2.9) reduced to $h^k_k = \frac{\cosh(\rho)}{v\sinh(\rho)} + \frac{\sinh(\rho)}{v\cosh(\rho)}$. By part (1) of this lemma, we have that for any $k = 1, \dots, a$

$$|h^k_k - 2| \leq ce^{-\beta t}.$$

We can define the tensor $l^j_i := h^j_i + \delta^j_i - \sum_{k=1}^a \delta^k_i \delta^j_k$, and its trace $L := l^i_i = H + m - a$. We get:

$$\begin{aligned}
\frac{\partial H}{\partial t} &\leq \frac{\psi'}{\psi^2}\Delta H - \frac{1}{\psi}\left(|A|^2 + \bar{R}ic(\nu,\nu)\right)\\
&= \frac{\psi'}{\psi^2}\Delta H - \frac{1}{\psi}\left(|l|^2 - 2(H-m-a) + 2\sum_{k=1}^{a}(h^k_k - 2) - 4m\right)\\
&\leq \frac{\psi'}{\psi^2}\Delta H - \frac{1}{\psi}\left(\frac{L^2}{m} - 2(H-m-a) - 4m\right) + ce^{-\beta t}\\
&= \frac{\psi'}{\psi^2}\Delta H - \frac{1}{m\psi}(H-m-a)(H+m-a) + ce^{-\beta t}.
\end{aligned}$$

Since $\frac{H+m-a}{m\psi}$ is strictly positive and, by Lemma 3.4, bounded, by the maximum principle there is a constant $0 < \gamma \leq \beta$ such that

$$H - m - a \leq ce^{-\gamma t}. \tag{3.18}$$

On the other hand, like in the proof of Lemma 3.4, we are able to estimate H from below only considering first ψ with the help of the function $\tilde{\alpha} = \frac{ve^{\frac{t}{\psi(m+a)}}}{\psi\sinh(\rho)}$. We restart from (3.17), but this time we need to use a finer estimate on the reaction term: from (3.15), part (1) of this lemma and (3.18) we get

$$\begin{aligned}
\frac{\partial\tilde{\alpha}}{\partial t} &\leq a^{ij}\tilde{\alpha}_{ij} + b^i\tilde{\alpha}_i + \frac{\tilde{\alpha}}{\psi(m+a)} + \left(\frac{\psi'}{\psi^2}\left(\frac{v\cosh(\rho)}{\sinh(\rho)}H - m - a\right) - \frac{v\cosh(\rho)}{\psi\sinh(\rho)}\right)\tilde{\alpha}\\
&\leq a^{ij}\tilde{\alpha}_{ij} + b^i\tilde{\alpha}_i + \left(\frac{1}{\psi(m+a)} + ce^{-\gamma t} - \frac{\tilde{\alpha}}{2}\right)\tilde{\alpha}.
\end{aligned}$$

Therefore, by the maximum principle $\tilde{\alpha} \leq \frac{2}{\psi(m+a)} + ce^{-\gamma t}$, hence, by definition of $\tilde{\alpha}$ we have

$$\psi(H) - \psi(m+a) \geq -ce^{-\gamma t}. \tag{3.19}$$

We can combine (3.18) and (3.19) to get the result using the mean value theorem.

□

Finally we can look for the optimal exponent.

Proof of Proposition 3.1 (1) As in Lemma 3.3, we consider again the evolution of w. Using Lemma 3.5 and estimating the reaction terms of the evolution equation of ω with (3.15), we get

$$\frac{\partial w}{\partial t} \leq a^{ij} w_{ij} + b^i w_i + \left(ce^{-\gamma t} - \frac{2}{\psi(m+a)}\right) w.$$

The result follows from the maximum principle.

(2) The proofs of Lemma 3.5, part (2) can be repeated using $\beta = \frac{2}{\psi(m+a)}$ and noting that $\frac{H+m-a}{m\psi}$ converges exponentially fast to $\frac{2m-a}{m\psi(m+a)}$ which is smaller than $\frac{2}{\psi(m+a)}$; therefore, we can take $\lambda = \frac{2}{\psi(m+a)}$.

□

4 Higher-order estimate and long-time existence

The main goal of this section is to prove part (2) of Theorem 1.1, i.e. the long-time existence of the flow. Moreover, we will show some other important auxiliary results, such as the convergence of the second fundamental form to that of a horosphere of $\mathbb{KH}^n$.

Proposition 4.1 *The principal curvatures of the evolving hypersurface are uniformly bounded for any time.*

Proof Since by the results of the previous section H is bounded from below, it is sufficient to prove that the principal curvatures are bounded from above. We define the tensor $M_i^j = \psi(H)h_i^j$. By Lemma 2.6 and after some standard computations, we have that the evolution equation of M_i^j is

$$\begin{aligned}\frac{\partial M_i^j}{\partial t} &= \frac{\psi'}{\psi^2}\Delta M_i^j - 2\frac{\psi'}{\psi^3}\left\langle \nabla M_i^j, \nabla\psi\right\rangle + \frac{\psi''\psi - 2(\psi')^2}{\psi^2}\nabla_i H \nabla^j H \\ &\quad - \frac{1}{\psi^2}\left(1 + \frac{H\psi'}{\psi}\right) M_i^k M_k^j - \left(1 + \frac{H\psi'}{\psi}\right)\bar{R}_{0i0}{}^{j} \\ &\quad + \frac{\psi'}{\psi^2}\left(2\bar{R}_{is}^{k\ j} M_k^s - \bar{R}_{si}^{k\ s} M_k^j - \bar{R}_{ks}^{s\ j} M_i^k\right).\end{aligned}$$

Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$ be the eigenvalues of M_i^j. Since $\sum_{i=1}^m \mu_i = H\psi(H) > 0$, then $\mu_m > 0$ everywhere. The goal is to prove that μ_m is bounded from above. Fix any time T^* strictly smaller than the maximal time T. We can find a point (x_0, t_0) where μ_m reaches its maximum in $\mathbb{S}^m \times [0, T^*]$. At this point, we can fix an orthonormal basis which diagonalizes M_i^j; then, in this system of coordinates we have $\mu_m = M_M^M$ at this point; hence, μ_m satisfies the same evolution equation of M_m^m. By Condition (1.2), the term $\frac{\psi''\psi-2(\psi')^2}{\psi^2}\nabla_m H \nabla^m H$ is negative and it can be ignored. Moreover, the curvature of $\mathbb{KH}^n$ is bounded; hence, all the coefficients involving $\bar{R}$ are bounded. Furthermore, H, ψ and ψ' are uniformly bounded too; therefore, there are positive constants C_0, C_1, C_2 independent on the choice of T^* such that in (x_0, t_0) the following holds

$$\frac{\partial \mu_m}{\partial t} \leq \frac{\psi'}{\psi^2}\Delta\mu_m - 2\frac{\psi'}{\psi^3}\langle \nabla\mu_m, \nabla\psi\rangle - C_2\mu_m^2 + C_1\mu_m + C_0.$$

Since μ_m is positive, it follows by Hamilton's maximum principle [22] that μ_m is bounded by a constant depending on $\mathcal{M}_0$, but not on the choice of T^*. □

Corollary 4.2 *The flow exists for any positive time. Moreover, the flow is expanding, and as t diverges, the volume goes like*

$$|\mathcal{M}_t| \approx e^{\frac{m+a}{\psi(m+a)}t}.$$

Proof From Proposition 4.1, it follows the uniform parabolicity of Equation (3.13) and a uniform C^2-estimate for the function φ. Arguing as in Chapter 2.6 of [23], we can apply the $C^{2,\alpha}$ estimates of [24] and higher-order estimates as Theorem 2.5.9 of [23]; hence, the solution is smooth up to the maximal time. Moreover, this maximal time cannot be finite in order to not have contradiction with the short-time existence of the flow.

For the growth of the volume, consider the quantity $V = |\mathcal{M}_t|e^{-\frac{m+a}{\psi(m+a)}t}$. By Lemma 2.6 and Proposition 3.1, we have

$$\left|\frac{dV}{dt}\right| = e^{-\frac{m+a}{\psi(m+a)}t}\left|\int_{\mathcal{M}_t}\left(\frac{H}{\psi} - \frac{m+a}{\psi(m+a)}\right)d\mu_t\right| \le ce^{-\frac{2}{\psi(m+a)}t}V.$$

Hence, $0 < |\mathcal{M}_0|e^{-\frac{c\psi(m+a)}{2}} \le V \le |\mathcal{M}_0|e^{\frac{c\psi(m+a)}{2}}$. □

Lemma 4.3 *For every $N \in \mathbb{N}$, there is a positive constant c such that the Nth derivative of φ satisfies*

$$|\nabla^N\varphi|^2 \le ce^{-\frac{2}{\psi(m+a)}t}.$$

Proof Fix $N \in \mathbb{N}$ and consider the quantity $\omega = \frac{1}{2}|\nabla^N\varphi|^2 = \frac{1}{2}\varphi^{k_1\ldots k_N}\varphi_{k_1\ldots k_N}$. With the notations of the proof of Lemma 3.3, its evolution equation is

$$\begin{aligned}\frac{\partial\omega}{\partial t} &= \varphi^{k_1\ldots k_N}\nabla_{k_1}\ldots\nabla_{k_N}\frac{\partial\varphi}{\partial t}\\ &= \varphi^{k_1\ldots k_N}\left(a^{ij}\varphi_{ijk_1\ldots k_N} + b^i\varphi_{ik_1\ldots k_N}\right) + 2\frac{\partial G}{\partial\varphi}\omega\end{aligned}$$

By (3.15) and Proposition 3.1, we have that $\frac{\partial G}{\partial\varphi} \le C_1e^{-\frac{2}{\psi(m+a)}t} - \frac{1}{\psi(m+a)}$ for some constant $C_1 > 0$. Applying a finite number of times the Ricci identity (3.14), we have

$$\begin{aligned}\frac{\partial\omega}{\partial t} &\le a^{ij}\omega_{ij} + b^i\omega_i + \left(2C_1e^{-\frac{2}{\psi(m+a)}t} - \frac{2}{\psi(m+a)}\right)\omega\\ &+a*\nabla^N\varphi*\nabla^N\varphi + b*\nabla^{N-1}\varphi*\nabla^N\varphi,\end{aligned}$$

where given two tensors S and T, $S*T$ denotes any linear combination obtained contracting S and T by σ. We have that

$$a^{ij} = \frac{\psi'}{\psi^2}g^{ij} = \frac{\psi'}{\sinh^2(\rho)\psi^2}\tilde{\sigma}^{ij} \le C_2e^{-\frac{2}{\psi(m+a)}t}\sigma^{ij}. \tag{4.20}$$

Moreover, by direct computations we have that

$$b^i = \frac{1}{\sinh(\rho)}\nabla\varphi*\nabla^2\varphi. \tag{4.21}$$

The case $N = 1$ has been already proved in Proposition 3.1. Suppose by induction that the result holds for $N-1$, then by (4.20) and (4.21) we have that there is a positive constant C_3 such that

$$a * \nabla^N \varphi * \nabla^N \varphi + b * \nabla^{N-1}\varphi * \nabla^N \varphi \leq C_3 e^{-\frac{2}{\psi(m+a)}t}\omega.$$

Therefore, there is a $C_4 > 0$ such that

$$\frac{\partial \omega}{\partial t} \leq a^{ij}\omega_{ij} + b^i \omega_i + \left(C_4 e^{-\frac{2}{\psi(m+a)}t} - \frac{2}{\psi(m+a)}\right)\omega.$$

The desired estimates follow by the maximum principle. □

A consequence of this lemma is the convergence of the second fundamental form to that of a horosphere.

Corollary 4.4 *There is a positive constant c such that*

(1) *if* $\mathbb{K} = \mathbb{R}$, *we have*

$$\left|h_i^j - \delta_i^j\right|^2 \leq ce^{-\frac{4}{\psi(m)}t}, \quad |\mathring{A}|^2 \leq ce^{-\frac{4}{\psi(m)}t};$$

(2) *if* $\mathbb{K} \neq \mathbb{R}$, *we have*

$$\left|h_i^j - \delta_i^j - \sum_{k=1}^{a} \delta_i^k \delta_k^j\right|^2 \leq ce^{-\frac{2}{\psi(m+a)}t},$$

while on the horizontal distribution we have a faster convergence

$$\sum_{i,j=a+1}^{m} (h_i^j - \delta_i^j)(h_j^i - \delta_j^i) \leq ce^{-\frac{4}{\psi(m+a)}t}.$$

Proof (1) From (2.9), Proposition 3.1 and Lemma 4.3, we get

$$\begin{aligned}\left|h_i^j - \delta_i^j\right|^2 &= \frac{1}{v^2\sinh^2(\rho)}\left|\varphi_{ik}\tilde{\sigma}^{kj}\right|^2 + m\left(\frac{\cosh(\rho)}{v\sinh(\rho)} - 1\right)^2 \\ &\quad + 2\left(\frac{\cosh(\rho)}{v\sinh(\rho)} - 1\right)\left(H - \frac{\hat{H}}{v}\right) \leq ce^{-\frac{4}{\psi(m)}t}.\end{aligned}$$

Therefore,

$$|\mathring{A}|^2 = \left|h_i^j - \frac{H}{m}\delta_i^j\right|^2 \leq \left|h_i^j - \delta_i^j\right|^2 + m\left(\frac{H}{m} - 1\right)^2 \leq ce^{-\frac{4}{\psi(m)}t}.$$

(2) By (2.9), Lemma 2.3, Proposition 3.1, Proposition 3.1 and Lemma 4.3, we have

$$\begin{aligned}\left|h_i^j - \delta_i^j - \sum_{k=1}^{a}\delta_i^k\delta_k^j\right|^2 &= \frac{1}{v^2\sinh^2(\rho)}\left|\varphi_{ik}\tilde{\sigma}^{kj}\right|^2 \\ &\quad + \frac{2a}{v^2}|\nabla\varphi|^2 + 2\left(\frac{\cosh(\rho)}{v\sinh(\rho)} - 1\right)\left(H - \frac{\hat{H}}{v}\right) \\ &\quad + m\left(\frac{\cosh(\rho)}{v\sinh(\rho)} - 1\right)^2 + a\left(\frac{\sinh(\rho)}{v\cosh(\rho)} - 1\right)^2\end{aligned}$$

$$+2a\left(\frac{\cosh(\rho)}{v\sinh(\rho)}-1\right)\left(\frac{\sinh(\rho)}{v\cosh(\rho)}-1\right)$$
$$\leq ce^{-\frac{2}{\psi(m+a)}t}.$$

On the horizontal distribution, the computations are similar to those of part (1) of this corollary; hence, we have a faster convergence.

□

5 Convergence and curvature of the induced metric

We finish this paper with the proof of statements (3a), (3b) and (3c) of Theorem 1.1. This is the part where the geometries of the ambient manifolds influence mostly the result. Each geometry produces its own typical behaviour, and all of them are very different from what found in the Euclidean case in [6]. We recall that in the Euclidean case the limit is always σ. We will show in a while that in the hyperbolic spaces the limit is not necessarily round and even not necessarily Riemannian. We start from the convergence of the rescaled induced metric.

Theorem 5.1 *For any given $\mathbb{K}$, there is a smooth $\mathbb{S}^a$-invariant function $f : \mathbb{S}^m \to \mathbb{R}$ such that the rescaled induced metric $\tilde{g}_t = |\mathcal{M}_t|^{-\frac{2}{m+a}} g_t$ converges, as t goes to infinity, to the metric $e^{2f}\sigma_{\mathbb{K}}$.*

Proof For any time t, let $\tilde{\rho}(t)$ be the radius of a geodesic sphere $\mathcal{B}_t$ such that $|\mathcal{M}_t| = |\mathcal{B}_t|$. The mean curvature of $\mathcal{B}_t$ is $\tilde{H} = \hat{H}(\tilde{\rho})$; hence,

$$\frac{d\tilde{\rho}}{dt} = \frac{1}{\psi(\tilde{H})},$$

then $\tilde{\rho} = \frac{t}{\psi(m+a)} + o(1)$ as $t \to \infty$. Consider the function $\tilde{f}(x,t) = \rho(x,t) - \tilde{\rho}(t)$. We claim that $\tilde{f}$ converges to a smooth function $\tilde{f}_\infty$. In fact, as a consequence of Lemma 4.3 we know that for any $N \in \mathbb{N}$ $\nabla^N \tilde{f} = \nabla^N \rho$ is uniformly bounded. Moreover, by Proposition 3.1 we have

$$\left|\frac{\partial \tilde{f}}{\partial t}\right| = \left|\frac{v}{\psi(H)} - \frac{1}{\psi(\tilde{H})}\right|$$
$$\leq \frac{1}{\psi(H)}|v-1| + \frac{1}{\psi(H)\psi(\tilde{H})}\left(|\psi(H)-\psi(m+a)| + \left|\psi(\tilde{H})-\psi(m+a)\right|\right)$$
$$\leq ce^{-\frac{2}{\psi(m+a)}t}.$$

When $\mathbb{K} = \mathbb{R}$ $e = \sigma_{\mathbb{R}}$ by Notation 2.1. When $\mathbb{K} \neq \mathbb{R}$ e converges to $\sigma_{\mathbb{K}}$. Moreover, by definition of $\tilde{\rho}$ we have that $|\mathcal{M}_t| = \omega_m \sinh^m(\tilde{\rho})\cosh^a(\tilde{\rho})$ for some constant ω_m. Therefore, there is a constant $c > 0$ such that when $t \to \infty$ we have

$$\sinh(\rho)|\mathcal{M}_t|^{-\frac{1}{m+a}} \approx ce^{\rho-\tilde{\rho}} = ce^{\tilde{f}}.$$

Recalling that in Proposition 3.1 we proved that $|\nabla\varphi|$ decays to 0, it follows by (2.7) that there exists a positive constant c such that

$$\lim_{t\to\infty} \tilde{g}_t = ce^{2\tilde{f}_\infty}\sigma_{\mathbb{K}}.$$

The function f that we are looking for is the solution of $e^{2f} = ce^{2f_\infty}$. □

We want to describe the construction of examples such that the associated limit $\tilde{g}_\infty$ has not constant scalar curvature. The proof is an adaptation to a general ψ of the techniques developed for the inverse mean curvature flow. As previous literature suggests, we need different tools for different values of $\mathbb{K}$. When $\mathbb{K} = \mathbb{R}$, we use the modified Hawking mass taken from [14], but it is not useful in the other two ambient manifolds. In the other cases, we use the Brown–York-like masses introduced in [15, 16].

5.1 The case of the real hyperbolic space

In this subsection, we will focus only on the real hyperbolic space. Following Hung and Wang [14], for any hypersurface $\mathcal{M}$ in $\mathbb{RH}^n$ we consider the modified Hawking mass

$$Q(\mathcal{M}) := |\mathcal{M}|^{-1+\frac{4}{m}} \int_{\mathcal{M}} |\mathring{A}|^2 d\mu, \tag{5.22}$$

where $|\mathring{A}|^2$ is the norm of the trace free part of the second fundamental form.

We can compute the evolution of the modified Hawking mass under the flow (1.1).

Lemma 5.2 *Let $\mathcal{M}_t$ be a closed hypersurface of $\mathbb{RH}^n$ evolving according to (1.1), then*

$$|\mathcal{M}_t|^{1-\frac{4}{m}} \frac{dQ(\mathcal{M}_t)}{dt} = \left(\frac{4}{m} - 1\right)\left[|\mathcal{M}_t|^{-1} \int_{\mathcal{M}_t} |\mathring{A}|^2 d\mu_t \int_{\mathcal{M}_t} \left(\frac{H}{\psi}\right) d\mu_t - \int_{\mathcal{M}_t} |\mathring{A}|^2 \frac{H}{\psi} d\mu_t\right]$$
$$- \int_{\mathcal{M}_t} \frac{2}{\psi} \mathring{h}_i^j \mathring{h}_j^k \mathring{h}_k^i d\mu_t - 2 \int_{\mathcal{M}_t} \frac{\psi'}{\psi^2} \nabla_i H \nabla^j \mathring{h}_j^i d\mu_t,$$

where $\mathring{h}$ denotes trace free part of the shape operator.

Proof By Lemma 2.6 and the explicit expression of the curvature tensor of the ambient space, we have

$$\frac{\partial \mathring{h}_i^j}{\partial t} = -\nabla_i \nabla^j \frac{1}{\psi} - \frac{1}{\psi}\left(\mathring{h}_i^k \mathring{h}_k^j + \frac{2H}{m} \mathring{h}_i^j\right) + \left(\frac{1}{\psi} + \frac{\partial H}{\partial t}\right) \delta_i^j.$$

Therefore,

$$\frac{\partial |\mathring{A}|^2}{\partial t} = 2\nabla^j \left(\frac{\psi' \nabla_i H}{\psi^2}\right) \mathring{h}_j^i - \frac{4H}{m\psi} |\mathring{A}|^2 - \frac{2}{\psi} \mathring{h}_i^j \mathring{h}_j^k \mathring{h}_k^i.$$

The result follows easily considering the evolution of the volume form in Lemma 2.6. □

The goal is to show that if Q is decreasing, then it does so very slowly. Looking at the evolution of Q, we need to add an estimate to those of the previous sections.

Lemma 5.3 *If $\mathbb{K} = \mathbb{R}$, there is a positive constant c such that*

$$|\nabla \mathring{A}|^2 \le ce^{-\frac{6}{\psi(m)} t}.$$

Proof In this proof, the C_i will be positive constants. By Lemma 2.6, Proposition 3.1 and Corollary 4.4, we can compute

$$\frac{\partial |\mathring{A}|^2}{\partial t} = \frac{\psi'}{\psi^2} \Delta |\mathring{A}|^2 - 2\frac{\psi'}{\psi^2} |\nabla \mathring{A}|^2 + 2\frac{\psi'' \psi - 2(\psi')^2}{\psi^3} \nabla_i H \nabla_j H \mathring{h}_j^i$$

$$-2\left(\frac{1}{\psi}+\frac{H\psi'}{\psi^2}\right)\mathring{h}_i^j\mathring{h}_j^k\mathring{h}_k^i+2\left(\frac{\psi'}{\psi^2}\left(|A|^2+m-\frac{2}{m}H\right)-\frac{2H}{m\psi}\right)|\mathring{A}|^2$$
$$\leq\frac{\psi'}{\psi^2}\Delta|\mathring{A}|^2-C_1|\nabla\mathring{A}|^2+\left(C_2e^{-\frac{2}{\psi(m)}t}-\frac{4}{\psi(m)}\right)|\mathring{A}|^2+C_3e^{-\frac{6}{\psi(m)}t}.$$

Moreover,

$$\begin{aligned}\frac{\partial|\nabla\mathring{A}|^2}{\partial t}&=2\nabla^s\mathring{h}_j^i\nabla_s\frac{\partial\mathring{h}_i^j}{\partial t}+\frac{\partial g^{rs}}{\partial t}\nabla_s\mathring{h}_i^j\nabla_r\mathring{h}_j^i\\&=2\nabla^s\mathring{h}_j^i\nabla_s\left[\frac{\psi'}{\psi^2}\Delta\mathring{h}_i^j+\frac{\psi''\psi-2(\psi')^2}{\psi^3}\nabla_iH\nabla^jH-\left(\frac{1}{\psi}+\frac{H\psi'}{\psi^2}\right)\mathring{h}_i^r\mathring{h}_r^j\right.\\&\qquad\left.+\left(\frac{\psi'}{\psi^2}\left(|A|^2+m-\frac{2H^2}{m\psi}\right)-\frac{2H}{m\psi}\right)\mathring{h}_i^j\right]-\frac{2}{\psi}h^{rs}\nabla_s\mathring{h}_i^j\nabla_r\mathring{h}_j^i\\&\leq\frac{\psi'}{\psi^2}\Delta|\mathring{A}|^2+\left(C_4e^{-\frac{2}{\psi(m)}t}-\frac{6}{\psi(m)}\right)|\nabla\mathring{A}|^2+C_5|\nabla\mathring{A}|^4.\end{aligned}$$

Consider the auxiliary function $\beta=\log|\nabla\mathring{A}|^2+K|\mathring{A}|^2$ for some positive constant K to be determined later. From the above two evolution equations, we get

$$\frac{\partial\beta}{\partial t}\leq\frac{\psi'}{\psi^2}\Delta\beta+\frac{\psi'}{\psi^2}\left|\nabla\log|\nabla\mathring{A}|^2\right|^2-\frac{6}{\psi(m)}+C_5|\nabla\mathring{A}|^2-KC_1|\nabla\mathring{A}|^2+C_6e^{-\frac{2}{\psi(m)}t}.$$

By Corollary 4.4, there is a constant C_7 such that for any time t we have $|\mathring{A}|^2\leq C_7e^{-\frac{4}{\psi(m)}t}$. Fix t^* big enough such that

$$16C_5C_7\frac{\psi'}{\psi^2}e^{-\frac{4}{\psi(m)}t^*}\leq C_1^2.$$

Consider β only for times $t\geq t^*$. In a point (x_0,t_0) where β attains its maximum, we have

$$\left|\nabla\log|\nabla\mathring{A}|^2\right|^2=K^2\left|\nabla|\mathring{A}|^2\right|^2\leq4K^2|\mathring{A}|^2|\nabla\mathring{A}|^2\leq4K^2C_7e^{-\frac{4}{\psi(m)}t^*}|\nabla\mathring{A}|^2.$$

By the choice of t^*, we can find a $K>0$ such that

$$4K^2C_7\frac{\psi'}{\psi^2}e^{-\frac{4}{\psi(m)}t^*}-KC_1+C_5\leq0.$$

By the maximum principle, we have $\beta\leq-\frac{6}{\psi(m)}t+C_8$, and by the definition of β, the desired result follows. □

Proposition 5.4 *Let $\mathcal{M}_t$ be the evolution of a star-shaped, mean convex hypersurface of $\mathbb{RH}^n$, then there is a positive constant c such that*

$$\frac{dQ(\mathcal{M}_t)}{dt}\geq-ce^{-\frac{2}{\psi(m)}t}.$$

Proof By Corollary 4.4, we know that in the real hyperbolic space $|\mathring{A}|^2\leq ce^{-\frac{4}{\psi(m)}t}$. Moreover, by Corollary 3.1 we have

$$\left||\mathcal{M}|^{-1}\int\frac{H}{\psi}d\mu-\frac{H}{\psi}\right|\leq\left||\mathcal{M}|^{-1}\int\frac{H}{\psi}d\mu-\frac{m}{\psi(m)}\right|+\left|\frac{m}{\psi(m)}-\frac{H}{\psi}\right|\leq ce^{-\frac{2}{\psi(m)}t}.$$

Classical inequalities say that there is a positive constant c such that

$$|\nabla H|^2 \leq m|\nabla A|^2 \leq c|\nabla \mathring{A}|^2.$$

The first inequality is trivial, and the second one follows by Lemma 2.2 of [25]. Finally, we can use the Cauchy–Schwarz inequality and Lemma 5.3 to estimate the last term in the evolution equation of Q:

$$\left|\nabla_i H \nabla^j \mathring{h}^i_j\right| \leq |\nabla H| \cdot |\nabla \mathring{A}| \leq c|\nabla \mathring{A}|^2 \leq ce^{-\frac{6}{\psi(m)}t}.$$

□

Now we have all the ingredients to repeat the construction of the not round examples described in [14]. We recall it briefly for completeness. Let $f : \mathbb{S}^m \to \mathbb{R}$ be a smooth function and $\tau > 0$, let $\widetilde{M}^\tau$ be the star-shaped hypersurface of $\mathbb{RH}^n$ defined by the radial function $\tilde{\rho}(z) = \tau + f(z) + o(1)$. Proposition 5 of [14] says that

$$\lim_{\tau\to\infty} Q(\widetilde{M}^\tau) = \left(\int_{\mathbb{S}^m} e^{mf} d\sigma\right)^{-1+\frac{4}{m}} \int_{\mathbb{S}^m} e^{(m-2)f} \left|\mathring{\nabla}^2 e^{-f}\right|^2 d\sigma, \tag{5.23}$$

where $\mathring{\nabla}^2$ is the trace free part of the Hessian.

Pick a constant $c_0 > 0$ and a function $\bar{f}$ such that

$$\left(\int_{\mathbb{S}^m} e^{m\bar{f}} d\sigma\right)^{-1+\frac{4}{m}} \int_{\mathbb{S}^m} e^{(m-2)\bar{f}} \left|\mathring{\nabla}^2 e^{-\bar{f}}\right|^2 d\sigma > 4c_0.$$

Let $\widetilde{M}^\tau$ be the star-shaped hypersurface of $\mathbb{RH}^n$ defined by the radial function $\tilde{\rho}(z) = \tau + \bar{f}(z)$. By (5.23) we can choose τ big enough such that $Q(\widetilde{M}^\tau) > 2c_0$. Moreover, for such hypersurface, the first addendum in the right-hand side of (2.10) is negligible for τ big; therefore, we can choose τ big enough such that $\widetilde{M}^\tau$ is mean convex too.

Let $\mathcal{M}^\tau_t$ be the evolution according to the flow (1.1) with initial data $\widetilde{M}^\tau$. In the previous sections, we proved that this evolution is defined for any positive time, and as t diverges, the rescaled induced metric converges to $e^{2f}\sigma$ for some function $f : \mathbb{S}^m \to \mathbb{R}$. By Proposition 5.4, we have that if c_0 is big enough then $\lim_{t\to\infty} Q(\mathcal{M}^\tau_t) > c_0 > 0$, and by Proposition 5 of [14], we can conclude that $e^{2f}\sigma$ is not round.

5.2 The cases of the complex and quaternionic hyperbolic space

The richer geometry of $\mathbb{CH}^n$ and $\mathbb{HH}^n$ makes the research of a not round limit harder. In fact, it is well known that in these spaces there are no totally umbilical hypersurfaces (see Theorem 5.1 of [26] for a proof). Therefore, $|\mathring{A}|^2$ is always bounded away from zero and the modified Hawking mass is no more useful in this context. On the other hand, the complexity of the problem can be reduced using the $\mathbb{S}^a$-invariance. In fact, under this further hypothesis, Lemma 2.4 suggests that the limits with constant scalar curvature should be very rare. In order to overcome all these difficulties in [15, 16], we defined the following weaker notion of mass.

$$Q(\mathcal{M}) = |\mathcal{M}|^{-1+\frac{2}{m+a}} \int_{\mathcal{M}} \left(H - \hat{H}\right) d\mu, \tag{5.24}$$

where $\hat{H}$ is the function defined in (2.11). Note that Q in general does not have a sign, but it is bounded. Moreover, it is important to keep in mind that, even if we are using an unique symbol, Q depends on the choice of $\mathbb{K}$. Its evolution is given by the following result.

Lemma 5.5 *Let $\mathcal{M}_t$ be the evolution of a star-shaped, mean convex, $\mathbb{S}^a$-invariant hypersurface of $\mathbb{KH}^n$, then*

$$|\mathcal{M}_t|^{1-\frac{2}{m+a}} \frac{dQ(\mathcal{M}_t)}{dt} = \left(-1 + \frac{2}{m+a}\right) |\mathcal{M}_t|^{-1} \int_{\mathcal{M}_t} \frac{H}{\psi} d\mu \int_{\mathcal{M}_t} (H - \hat{H}) d\mu + \int_{\mathcal{M}_t} \left[\frac{H}{\psi}(H - \hat{H}) - \frac{1}{\psi}\left(|A|^2 + \bar{R}ic(\nu,\nu)\right)\right] d\mu + \int_{\mathcal{M}_t} \frac{v}{\psi}\left(\frac{m}{\sinh^2(\rho)} - \frac{a}{\cosh^2(\rho)}\right) d\mu.$$

Proof The result follows combining Lemma 2.6 with

$$\frac{\partial \hat{H}}{\partial t} = \frac{v}{\psi}\left(\frac{a}{\cosh^2(\rho)} - \frac{m}{\sinh^2(\rho)}\right),$$

and the fact that for any t

$$\int_{\mathcal{M}_t} \left(\frac{\psi'}{\psi^2}\Delta H + \frac{\psi''\psi - 2(\psi')^2}{\psi^3}|\nabla H|^2\right) d\mu = -\int_{\mathcal{M}_t} \Delta\left(\frac{1}{\psi}\right) d\mu = 0.$$

□

With the help of the results of the previous sections, we can estimate the evolution equation of Q proving that if Q decays, then it does so very slowly.

Proposition 5.6 *Let $\mathcal{M}_t$ be the evolution of a star-shaped, mean convex, $\mathbb{S}^a$-invariant hypersurface of $\mathbb{KH}^n$, then there is a positive constant c such that*

$$\frac{dQ(\mathcal{M}_t)}{dt} \geq -ce^{-\frac{2}{\psi(m+a)}t}.$$

Proof By the explicit expressions of the second fundamental form (2.9), of the mean curvature (2.10) and by Lemma 2.3, we have

$$\begin{aligned} |A|^2 + \bar{R}ic(\nu,\nu) &= \frac{\hat{\varphi}_{ik}\hat{\varphi}_{js}\tilde{e}^{kj}\tilde{e}^{si}}{v^2\sinh^2(\rho)} + \frac{2\cosh(\rho)}{v\sinh(\rho)}\left(H - \frac{\hat{H}}{v}\right) \\ &\quad + m\left(\frac{\cosh^2(\rho)}{v^2\sinh^2(\rho)} - 1\right) + a\left(\frac{\sinh^2(\rho)}{v^2\cosh^2(\rho)} + \frac{2}{v^2} - 3\right) \\ &= \frac{\varphi_{ik}\varphi_{js}\tilde{\sigma}^{kj}\tilde{\sigma}^{si}}{v^2\sinh^2(\rho)} + \frac{2\cosh(\rho)\hat{H}}{v^2(v+1)\sinh(\rho)}|\nabla\varphi|^2 - \frac{m+a}{v^2}|\nabla\varphi|^2 \\ &\quad + \frac{2\cosh(\rho)}{v\sinh(\rho)}\left(H - \hat{H}\right) + \frac{m}{v^2\sinh^2(\rho)} - \frac{a}{v^2\cosh^2(\rho)}. \end{aligned}$$

By substituting this formula in the result of Lemma 5.5, we can rearrange the terms of the evolution equation of Q as follows:

$$\begin{aligned} |\mathcal{M}_t|^{1-\frac{2}{m+a}} \frac{dQ(\mathcal{M}_t)}{dt} &= 2\int \left(H - \hat{H}\right)\left(\frac{1}{(m+a)|\mathcal{M}_t|}\int \frac{H}{\psi}d\mu - \frac{\cosh(\rho)}{v\psi\sinh(\rho)}\right) d\mu \\ &\quad + \int \left(H - \hat{H}\right)\left(\frac{H}{\psi} - \frac{1}{|\mathcal{M}_t|}\int \frac{H}{\psi}d\mu\right) d\mu \\ &\quad + \int \frac{1}{\psi}\left(v - \frac{1}{v^2}\right)\left(\frac{m}{\sinh^2(\rho)} - \frac{a}{\cosh^2(\rho)}\right) d\mu \\ &\quad + \int \frac{|\nabla\varphi|^2}{v^2\psi}\left(m + a - \frac{2\cosh(\rho)\hat{H}}{(v+1)\sinh(\rho)}\right) d\mu - \int \frac{\varphi_{ik}\varphi_{js}\tilde{\sigma}^{ij}\tilde{\sigma}^{ks}}{v^2\psi\sinh^2(\rho)} d\mu. \end{aligned} \tag{5.25}$$

We claim that every term in the right-hand side of (5.25) is smaller than $ce^{-\frac{4}{\psi(m+a)}t}|\mathcal{M}_t|$, for some constant $c > 0$. In fact, by Proposition 3.1, Lemma 4.3, the facts that ρ grows like $\frac{t}{\psi(m+a)}$, and $\frac{1}{\psi}$ is bounded, the various terms of (5.25) can be estimate as follow:

$$|H - \hat{H}| \leq |H - m - a| + |m + a - \hat{H}| \leq ce^{-\frac{2}{\psi(m+a)}t};$$

$$\begin{aligned}
\left|\frac{1}{(m+a)|\mathcal{M}_t|}\int \frac{H}{\psi}d\mu - \frac{\cosh(\rho)}{v\psi\sinh(\rho)}\right| &\leq \frac{1}{(m+a)|\mathcal{M}_t|}\int\left|\frac{H}{\psi} - \frac{m+a}{\psi(m+a)}\right|d\mu \\
&\quad + \frac{\cosh(\rho)}{v\sinh(\rho)}\left|\frac{1}{\psi} - \frac{1}{\psi(m+a)}\right| \\
&\quad + \frac{1}{v\psi(m+a)}\left|\frac{\cosh(\rho)}{\sinh(\rho)} - 1\right| \\
&\leq ce^{-\frac{2}{\psi(m+a)}t}; \\
\left|\frac{H}{\psi} - \frac{1}{|\mathcal{M}_t|}\int\frac{H}{\psi}d\mu\right| &\leq \left|\frac{H}{\psi} - \frac{m+a}{\psi(m+a)}\right| + \frac{1}{|\mathcal{M}_t|}\int\left|\frac{H}{\psi} - \frac{m+a}{\psi(m+a)}\right|d\mu \\
&\leq ce^{-\frac{2}{\psi(m+a)}t}; \\
v - \frac{1}{v^2} &= \frac{|\nabla\varphi|^2}{v^2(v+1)} \leq ce^{-\frac{2}{\psi(m+a)}t}; \\
\left|m + a - \frac{2\cosh(\rho)\hat{H}}{(v+1)\sinh(\rho)}\right| &\leq \left|m + a - \hat{H}\right| + \hat{H}\left|1 - \frac{2\cosh(\rho)}{(1+v)\sinh(\rho)}\right| \\
&\leq ce^{-\frac{2}{\psi(m+a)}t}; \\
\left|\varphi_{ik}\varphi_{js}\tilde{\sigma}^{ij}\tilde{\sigma}^{ks}\right| &\leq ce^{-\frac{2}{\psi(m+a)}t}.
\end{aligned}$$

□

We can construct many $\mathbb{S}^a$-invariant examples $\mathcal{M}_0$ such that $\lim_{t\to\infty} Q(\mathcal{M}_t) > 0$. By Proposition 9.4 of [15] and Proposition 7.1 of [16], we get that the sub-Riemannian limit metric $\tilde{g}_\infty$ cannot have constant (Webster or qc) scalar curvature. The strategy is analogous to that of Hung and Wang [14] described in the previous subsection and it use the mass (5.24) instead of the modified Hawking mass (5.22).

Data availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Alessandroni, R., Sinestrari, C.: Convexity estimates for a nonhomogeneous mean curvature flow. Math. Z. **266**, 65–82 (2010)
2. Bertini, M.C., Sinestrari, C.: Volume-preserving nonhomogeneous mean curvature flow of convex hypersurfaces. Ann. Mat. Pura Appl. **197**(4), 1295–1309 (2018)
3. Risa, S., Sinestrari, C.: Strong spherical rigidity of ancient solutions of expansive curvature flows. Bull. Lond. Math. Soc. **52**(1), 94–99 (2020)
4. McCoy, J.A.: Contraction of convex hypersurfaces by nonhomogeneous functions of curvature. J. Evol. Equ. **21**(2), 2265–2291 (2021)
5. McCoy, J.A.: Contracting self-similar solutions of nonhomogeneous curvature flows. J. Geom. Anal. **31**(6), 6410–6426 (2021)
6. Li, C.H.E.N., Xi, G.U.O., Quiang, T.U.: Nonhomogeneous inverse mean curvature flow in Euclidean space. Proc. Amer. Math. Soc. **148**(10), 4557–4571 (2020)
7. Gerhardt, C.: Flow of nonconvex hypersurfaces into spheres. J. Differ. Geom. **32**, 299–314 (1990)
8. Urbas, J.I.E.: On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures. Math. Z. **205**, 355–372 (1990)
9. Bertini, M.C., Pipoli, G.: Volume preserving non homogeneous mean curvature flow in hyperbolic space. Differ. Geom. Appl. **54**, 448–463 (2017)
10. Li, G., Lv, Y.: Contracting convex hypersurfaces in space form by non-homogeneous curvature function. J. Geom. Anal. **30**(1), 417–447 (2020)
11. Li, H., Zhou, T.: Nonhomogeneous inverse Gauss curvature flow in $\mathbb{H}^3$. Proc. Amer. Math. Soc. **147**, 3995–4005 (2019)
12. Gerhardt, C.: Inverse mean curvature flow in hyperbolic space. J. Differ. Geom. **89**, 487–527 (2011)
13. Ding, Q.I.: The inverse mean curvature flow in rotationally symmetric spaces. Chin. Ann. Math. **32B**(1), 27–44 (2011)
14. Hung, P.K., Wang, M.T.: Inverse mean curvature flow in the hyperbolic 3-space revisited. Calc. Var Part. Differ. Equ. **54**(1), 119–126 (2015)
15. Pipoli, G.: Inverse mean curvature flow in complex hyperbolic space. Ann. Scient. Èc. Norm. Sup. **52**(5), 1107–1135 (2019)
16. Pipoli, G.: Inverse mean curvature flow in quaternionic hyperbolic space. Rend. Lincei Mat. Appl. **29**, 153–171 (2018)
17. Pipoli, G.: A survey on Inverse mean curvature flow in ROSSes. Complex Manifolds **4**, 245–262 (2017)
18. Dragomir, S., Tomassini, G.: Differential geometry and analysis on CR manifolds, Progress in math. vol 246, Birkhäuser (2006)
19. Ivanov, S., Vassilev, D.: Extremal for the Sobolev inequality and the quaternionic contanct Yamabe problem, Imperial College Press Lecture Notes, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2011)
20. Pipoli, G.: Mean curvature flow and Riemannian submersions. Geom. Dedicata **184**(1), 67–81 (2016)
21. Huisken, G., Polden, A.: Geometric evolution equations for hypersurfaces, (Cetraro,: Lecture Notes in Math. 1713. Springer, Berlin **1999**, 45–84 (1996)
22. Hamilton, R.S.: Four-manifolds with positive curvature operator. J. Differ. Geom. **24**, 153–179 (1986)
23. Gerhardt, C.: Curvature problems series in geometry and topology, vol. 39. International Press, Somerville, MA (2006)
24. Krylov, N.V.: Nonlinear elliptic and parabolic equations of the secondorder, Mathematics and its Applications (Soviet Series), 7. D. Reidel Publishing Co., Dordrecht (1987)
25. Huisken, G.: Flow by mean curvature of convex surfaces into spheres. J. Differ. Geom. **20**, 237–266 (1984)
26. Niebergall, R., Ryan, P.J.: Tight and taut Submanifolds, MSRI Pubblications **32**, (1997)