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Nonlinear Young Differential Equations: A Review

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Abstract
Nonlinear Young integrals have been first introduced in Catellier and Gubinelli (Stoch Process Appl 126(8):2323–2366, 2016) and provide a natural generalisation of classical Young ones, but also a versatile tool in the pathwise study of regularisation by noise phenomena. We present here a self-contained account of the theory, focusing on wellposedness results for abstract nonlinear Young differential equations, together with some new extensions; convergence of numerical schemes and nonlinear Young PDEs are also treated. Most results are presented for general (possibly infinite dimensional) Banach spaces and without using compactness assumptions, unless explicitly stated.

Keywords Nonlinear Young integral · Young differential equations · Numerical schemes · Flow property · Transport equations · Parabolic Young equations

Mathematics Subject Classification Primary 60L20; Secondary 60L50 · 34A08

1 Introduction

The main goal of this article is to solve and study differential equations of the form

$$x_t = x_0 + \int_0^t A(\mathrm{d}s, x_s) \tag{1.1}$$

where x is an α-Hölder continuous path taking values in a Banach space V and $A : [0, T] \times V \to V$ is a vector field with suitable space-time Hölder regularity. If A is sufficiently smooth in time, then $A(\mathrm{d}s, x_s)$ can be interpreted as $\partial_t A(s, x_s)\mathrm{d}s$, so that (1.1) can be regarded as an ODE in integral form; here however we are interested in the case $\partial_t A$ does not exist, so that (1.1) does not admit a classical interpretation.

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In the case $A(t, z) = f(z)y_t$, where y is an U-valued α-Hölder continuous path and f maps V into the space of linear maps from U to V, Eq. (1.1) can be rewritten as

$$x_t = x_0 + \int_0^t f(x_s)\mathrm{d}y_s \tag{1.2}$$

which can be regarded as a rough differential equation driven by a signal y.

In the regime $\alpha \in (1/2, 1]$, for sufficiently regular f, Eq. (1.2) can be rigorously interpreted by means of Young integrals, introduced in [44]; wellposedness of Young differential equations (YDEs) was first studied in [34]. After that, several alternative approaches to (1.2) have been developed, either by means of fractional calculus [45] or numerical schemes [14]; see also the review [33] for a self-contained exposition of the main results for YDEs and the paper [13] for some recent developments. YDEs have found several applications in the study of SDEs driven by fractional Brownian motion (fBm) of parameter $H > 1/2$, see for instance [37].

Although Eq. (1.1) may be seen as a natural generalization of (1.2), its development is much more recent. Nonlinear Young integrals of the form

$$\int_0^t A(\mathrm{d}s, x_s)$$

were first defined in [9] in applications to additively perturbed ODEs and subsequently rediscovered in [30], where they were employed to give a pathwise interpretation to Feynman-Kac formulas and SPDEs with random coefficients.

In this paper we will consider exclusively the time regularity regime $\alpha > 1/2$, also known as the Young (or or level-1 rough path) regime. However it is now well known, since the pioneering work of Lyons [35], that it is possible to give meaning to Eq. (1.2) even in the case $\alpha \leq 1/2$ by means of the theory of rough paths, see the monographs [18,19] for a detailed account on the topic. An analogue extesion of (1.1) to the case of *nonlinear rough paths* has been recently achieved in [12,38]; so far however it hasn't found the same variety of applications, discussed below, as the nonlinear Young case. Let us finally mention that all of the above can also be seen as subcases of the theory of rough flows developed in [2,4].

Nonlinear YDEs of the form (1.1) mostly present direct analogue results to their classical counterpart (1.2), but their importance and the main motivation for this work lies in their *versatility*. Indeed, many differential systems which a priori do not present such structure, may be *recast* as nonlinear YDEs; this allows to give them meaning in situations where classical theory breaks down.

This methodology seems seems particularly effective in applications to *regularization by noise* phenomena; to clarify what we mean, let us illustrate the following example, taken from [10,11]. In these works the authors study abstract modulated PDEs of the form

$$\mathrm{d}\varphi_t = A\varphi\dot{w}_t + \mathcal{N}(\varphi_t)\mathrm{d}t \tag{1.3}$$

where $w : [0, T] \to \mathbb{R}$ is a continuous (possibly very rough) path, A is the generator of a group $\{e^{tA}\}_{t\in\mathbb{R}}$ and $\mathcal{N}$ is a nonlinear functional, possibly ill-posed in low regularity spaces. Formally, setting $\psi_t := e^{-w_t A}\varphi_t$, ψ would solve

$$\psi_t = \psi_0 + \int_0^t e^{-w_s A}\mathcal{N}(e^{w_s A}\psi_s)\mathrm{d}s,$$

which can be regarded as an instance of (1.1) for the choice

$$A(t, z) = \int_0^t e^{-w_s A}\mathcal{N}(e^{w_s A} z)\mathrm{d}s. \tag{1.4}$$

Under suitable assumption, even if w is not smooth (actually *exactly* because it is rough, as measured by its ρ-irregularity), it is possibile to rigorously define the field A, even if the integral appearing on the r.h.s. of (1.4) is not meaningful in the Lebesgue sense. As a consequence, the transformation of the state space given by $\varphi \mapsto \psi$ allows to interpret the original PDE (1.3) as a suitable nonlinear YDE; the general abstract theory presented here can then be applied, immediately yielding wellposedness results.

A similar reasoning holds for additively perturbed ODEs of the form

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}s + w_t$$

which were first considered in [9], in which case the transformation amounts to $x \mapsto \theta := x - w$. This case has recently received a lot of attention and developed into a general theory of pathwise regularisation by noise for ODEs and SDEs, see [20–22,26,28] and on a related note [27].

Motivated by the above discussion, we collect here several results for abstract nonlinear YDEs which have appeared in the above references, together with some new extensions; they provide general criteria for existence, uniqueness and stability of solutions to (1.1), as well as convergence of numerical schemes and differentiability of the flow. This work is deeply inspired by the review [33], of which it can be partially regarded as an extension; all the theory is developed in (possibly infinte dimensional) Banach spaces and relies systematically on the use of the sewing lemma, a by now standard feature of the rough path framework. We hope however that the also reader already acquainted with RDEs can find the paper of interest due to later Sects. 5–7, containing less standard results and applications to Young PDEs.

Structure of the paper. In Sect. 2, the nonlinear Young integral is constructed and its main properties are established. Section 3 is devoted to criteria for existence, uniqueness, stability and convergence of numerical schemes for nonlinear YDEs, Sects. 3.4 and 3.5 focusing on several variants of the main case. Section 4 deals continuity of the solutions with respect to the data of the problem, giving conditions for the existence of a flow and differentiability of the Itô map. The results from Sect. 3.3 are revisited in Sect. 5, where more refined criteria for uniqueness of solutions are given; we label them as “conditional uniqueness” results, as they require additional assumptions which are often met in probabilistic applications, but are difficult to check by purely analytic arguments. Sections 6 and 7 deal respectively with Young transport and parabolic type of PDEs. We chose to collect in the “Appendix” some useful tools and further topics.

Notation. Here is a list of the most relevant and frequently used notations and conventions:

- We write $a \lesssim b$ if $a \leq Cb$ for a suitable constant, $a \lesssim_x b$ to stress the dependence $C = C(x)$.
- We will always work on a finite time interval $[0, T]$; the Banach spaces V, W appearing might be infinite dimensional but will be always assumed separable for simplicity.
- Given a Banach space $(E, \|\cdot\|_E)$, we set $C_t^0 E = C([0, T]; E)$ endowed with supremum norm

$$\|f\|_\infty = \sup_{t\in[0,T]} \|f_t\|_E \quad \forall f \in C_t^0 E$$

where $f_t := f(t)$ and we adopt the incremental notation $f_{s,t} := f_t - f_s$. Similarly, for any $\alpha \in (0, 1)$ we set $C_t^\alpha E = C^\alpha([0, T]; E)$ be the space of α-Hölder continuous functions with norm

$$[\![f]\!]_\alpha := \sup_{0 \le s < t \le T} \frac{\|f_{s,t}\|_E}{|t-s|^\alpha}, \qquad \|f\|_\alpha := \|f\|_\infty + [\![f]\!]_\alpha.$$

- The above notation will be applied to several choice of E such as $C^\alpha_t V$, $C^\alpha_t \mathbb{R}^d$ but also $C^\alpha_t C^{\beta,\lambda}_{V,W}$ or $C^\alpha_t C^\beta_{V,W,\mathrm{loc}}$, for which we refer to Definitions 2.3 and 2.5.
- We denote by $\mathcal{L}(V;W)$ the set of all linear bounded operators from V to W, $L(V) = L(V;V)$.
- Whenever we will refer to differentiability this must be understood in the sense of Frechét, unless specified otherwise; given a map $F : V \to W$ we regard its Frechét differential $D^k F$ of order k as a map from V to $\mathcal{L}^k(V;W)$, the set of bounded k-linear forms from V^k to W. We will use indifferently $DF(x,y) = DF(x)(y)$ for the differential at point x evaluated along the direction y.
- Given a linear unbounded operator A, $\mathrm{Dom}(A)$ denotes its domain, $\mathrm{rg}(A)$ its range.
- As a rule of thumb, whenever $J(\Gamma)$ appears, it denotes the sewing of $\Gamma : \Delta_2 \to E$; we refer to Sect. 2.1 for more details on the sewing map. Similarly, in proofs based on a Banach fixed point argument, I will denote the map whose constractivity must be established.
- As a rule of thumb, we will use C_i, $i \in \mathbb{N}$ for the constants appearing in the main statements and κ_i for those only appearing inside the proofs; the numbering restarts at each statement and is only meant to distinguish the dependence of the constants from relevant parameters.

2 The Nonlinear Young Integral

This section is devoted to the construction of nonlinear Young integrals and nonlinear Young calculus more in general, as a preliminary step to the study of nonlinear Young differential equations which will be developed in the next section. We follow the modern rough path approach to abstract integration, based on the sewing lemma as developed in [24] and [17], which is recalled first.

2.1 Preliminaries

This subsections contains an exposition of the sewing lemma and the definition of the joint space-time Hölder continous drifts A we will work with; the reader already acquainted with this concepts may skip it.

Given a finite interval $[0,T]$, consider the n-simplex $\Delta_n := \{(t_1,\dots,t_n) : 0 \le t_1 \le \ldots \le t_n \le T\}$. Let V be a Banach space, for any $\Gamma : \Delta_2 \to V$ we define $\delta\Gamma : \Delta_3 \to V$ by

$$\delta\Gamma_{s,u,t} := \Gamma_{s,t} - \Gamma_{s,u} - \Gamma_{u,t}.$$

We say that $\Gamma \in C^{\alpha,\beta}_2([0,T];V) = C^{\alpha,\beta}_2 V$ if $\Gamma_{t,t} = 0$ for all $t \in [0,T]$ and $\|\Gamma\|_{\alpha,\beta} < \infty$, where

$$\|\Gamma\|_\alpha := \sup_{s<t} \frac{\|\Gamma_{s,t}\|_V}{|t-s|^\alpha}, \quad \|\delta\Gamma\|_\beta := \sup_{s<u<t} \frac{\|\delta\Gamma_{s,u,t}\|_V}{|t-s|^\beta}, \quad \|\Gamma\|_{\alpha,\beta} := \|\Gamma\|_\alpha + \|\delta\Gamma\|_\beta.$$

For a map $f : [0,T] \to V$, we still denote by $f_{s,t}$ the increment $f_t - f_s$.

Lemma 2.1 (Sewing lemma) *Let α, β be such that $0 < \alpha < 1 < \beta$. For any $\Gamma \in C_2^{\alpha,\beta} V$ there exists a unique map $\mathcal{J}(\Gamma) \in C_t^\alpha V$ such that $\mathcal{J}(\Gamma)_0 = 0$ and*

$$\|\mathcal{J}(\Gamma)_{s,t} - \Gamma_{s,t}\|_V \le C_1 \,\|\delta\Gamma\|_\beta |t-s|^\beta \tag{2.1}$$

where the constant C_1 can be taken as $C_1 = (1-2^{\beta-1})^{-1}$. Thus the sewing map $\mathcal{J} : C_2^{\alpha,\beta} V \to C_t^\alpha V$ is linear and bounded and there exists $C_2 = C_2(\alpha, \beta, T)$ such that

$$\|\mathcal{J}\,(\Gamma)\|_\alpha \le C_2 \|\Gamma\|_{\alpha,\beta}. \tag{2.2}$$

For a given Γ, $\mathcal{J}\,(\Gamma)$ is characterized as the unique limit of Riemann-Stjeltes sums: for any $t > 0$

$$\mathcal{J}\,(\Gamma)_t = \lim_{|\Pi|\to 0} \sum_i \Gamma_{t_i,t_{i+1}}.$$

The notation above means that for any sequence of partitions $\Pi_n = \{0 = t_0 < t_1 < \ldots < t_{k_n} = t\}$ with mesh $|\Pi_n| = \sup_{i=1,\ldots,k_n} |t_i - t_{i-1}| \to 0$ as $n \to \infty$, it holds

$$\mathcal{J}\,(\Gamma)_t = \lim_{n\to\infty} \sum_{i=0}^{k_n-1} \Gamma_{t_i,t_{i+1}}.$$

For a proof, see Lemma 4.2 from [18].

Remark 2.2 Let us stress two important aspects of the above result. The first one is that all the estimates do not depend on the Banach space V considered; the second one is that, even when the map $\mathcal{J}(\Gamma)$ is already known to exist, property (2.1) still gives non trivial estimates on its behaviour. In particular, if $f \in C_t^\alpha V$ is a function such that $\|\Gamma_{s,t} - f_{s,t}\|_V \le \kappa |t-s|^\alpha$ for an unknown constant κ, then by the sewing lemma we can deduce that $f = \mathcal{J}(\Gamma)$ and that κ can be taken as $C_1 \,\|\delta\Gamma\|_\beta$.

Next we need to introduce suitable classes of Hölder continuous maps on Banach spaces.

Definition 2.3 Let V, W Banach spaces, $f \in C(V; W)$, $\beta \in (0, 1)$. We say that f is locally β-Hölder continuous and write $f \in C^\beta_{V,W,\mathrm{loc}}$ if for any $R > 0$ the following quantities are finite:

$$[\![f]\!]_{\beta,R} := \sup_{\substack{x \neq y \in V \\ \|x\|_V,\, \|y\|_V \le R}} \frac{\|f(x) - f(y)\|_W}{\|x-y\|_V^\beta}, \quad \|f\|_{\beta,R} := [\![f]\!]_{\beta,R} + \sup_{\substack{x \in V \\ \|x\|_V \le R}} \|f(x)\|_V.$$

For $\lambda \in (0, 1]$, we define the space $C^{\beta,\lambda}_{V,W}$ as the collection of all $f \in C(V; W)$ such that

$$[\![f]\!]_{\beta,\lambda} := \sup_{R\ge 1} R^{-\lambda} [\![f]\!]_{\beta,R}, \quad \|f\|_{\beta,\lambda} := [\![f]\!]_{\beta,\lambda} + \|f(0)\|_V < \infty.$$

Finally, the classical Hölder space $C^\beta_{V,W}$ is defined as the collection of all $f \in C(V; W)$ such that

$$[\![f]\!]_\beta := \sup_{x \neq y \in V} \frac{\|f(x) - f(y)\|_W}{\|x-y\|_V^\beta}, \quad \|f\|_\beta = [\![f]\!]_\beta + \sup_{x\in V} \|f(x)\|_V < \infty.$$

Remark 2.4 We ask the reader to keep in mind that although linked, $[\![f]\!]_{\beta,R}$ and $[\![f]\!]_{\beta,\lambda}$ denote two different quantities. Throughout the paper R will always denote the radius of an open ball in V and consequently all related seminorms are localised on such ball; instead the parameter λ measures the polynomial growth of $[\![\cdot]\!]_{\beta,R}$ as a function of R.

$C^\beta_{V,W,\mathrm{loc}}$ is a Fréchet space with the topology induced by the seminorms $\{\|f\|_{\beta,R}\}_{R\geq 0}$, while $C^{\beta,\lambda}_{V,W}$ and $C^\beta_{V,W}$ are Banach spaces. Observe that if $f \in C^{\beta,\lambda}_{V,W}$, we have an upper bound on its growth at infinity, since for any $x \in V$ with $\|x\|_V \geq 1$ it holds

$$\|f(x)\|_V \leq \|f(x)-f(0)\|_V + \|f(0)\|_V \leq \|x\|_V^\beta [\![f]\!]_{\beta,\|x\|_V} + \|f(0)\|_V \leq \|f\|_{\beta,\lambda}(1+\|x\|_V^{\beta+\lambda}).$$

In particular, if $\beta+\lambda \leq 1$, then f has at most linear growth.

We can now introduce fields $A : [0,T] \times V \to W$ satisfying a joint space-time Hölder continuity. We adopt the incremental notation $A_{s,t}(x) := A(t,x) - A(s,x)$, as well as $A_t(x) = A(t,x)$; from now on, whenever A appears, it is implicitly assumed that $A(0,x) = 0$ for all $x \in V$.

Definition 2.5 Given A as above, $\alpha, \beta \in (0,1)$, we say that $A \in C^\alpha_t C^\beta_{V,W,\mathrm{loc}}$ if for any $R \geq 0$ it holds

$$[\![A]\!]_{\alpha,\beta} := \sup_{0\leq s<t\leq T} \frac{[\![A_{s,t}]\!]_{\beta,R}}{|t-s|^\alpha}, \quad \|A\|_{\alpha,\beta} := \sup_{0\leq s<t\leq T} \frac{\|A_{s,t}\|_{\beta,R}}{|t-s|^\alpha} < \infty.$$

We say that $A \in C^\alpha_t C^{\beta,\lambda}_{V,W}$ if

$$[\![A]\!]_{\alpha,\beta,\lambda} := \sup_{0\leq s<t\leq T} \frac{[\![A_{s,t}]\!]_{\beta,\lambda}}{|t-s|^\alpha}, \quad \|A\|_{\alpha,\beta,\lambda} := \sup_{0\leq s<t\leq T} \frac{\|A_{s,t}\|_{\beta,\lambda}}{|t-s|^\alpha};$$

analogue definitions hold for $C^\alpha_t C^\beta_{V,W}$, $[\![\cdot]\!]_{\alpha,\beta}$, $\|\cdot\|_{\alpha,\beta}$.

The definition can be extended to the cases $\alpha = 0$ or $\beta = 0$ by interpreting the norm in the supremum sense: for instance $A \in C^0_t C^\beta_{V,W}$ if

$$\|A\|_{0,\beta} = \sup_{t\in[0,T]} \|A_t\|_\beta < \infty.$$

Given a smooth $F : V \to W$, we regard its Frechét differential $D^k F$ of order k as a map from V to $\mathcal{L}^k(V;W)$, the set of bounded k-linear forms from V^k to W.

Definition 2.6 We say that $A \in C^\alpha_t C^{n+\beta}_{V,W}$ if $A \in C^\alpha_t C^\beta_{V,W}$ and it is k-times Frechét differentiable in x, with $D^k A \in C^\alpha_t C^\beta_{V,\mathcal{L}^k(V;W)}$ for all $k \leq n$. $C^\alpha_t C^{n+\beta}_{V,W}$ is a Banach space with norm

$$\|A\|_{\alpha,n+\beta} = \sum_{k=0}^n \|D^k A\|_{\alpha,\beta}.$$

Analogue definitions hold for $C^\alpha_t C^{n+\beta}_{V,W,\mathrm{loc}}$ and $C^\alpha_t C^{n+\beta,\lambda}_{V,W}$.

2.2 Construction and First Properties

We are now ready to construct nonlinear Young integrals, following the line of proof from [28, 30]; other constructions are possible, see "Appendix A.2".

Theorem 2.7 *Let $\alpha, \beta, \gamma \in (0,1)$ such that $\alpha + \beta\gamma > 1$, $A \in C^\alpha_t C^\beta_{V,W,\mathrm{loc}}$ and $x \in C^\gamma_t V$. Then for any $[s,t] \subset [0,T]$ and for any sequence of partitions of $[s,t]$ with infinitesimal mesh, the following limit exists and is independent of the chosen sequence of partitions:*

$$\int_s^t A(\mathrm{d}u, x_u) := \lim_{|\Pi| \to 0} \sum_i A_{t_i, t_{t+1}}(x_{t_i}).$$

The limit is usually referred as a nonlinear Young integral. Furthermore:

1. *For all $(s,r,t) \in \Delta_3$ it holds $\int_s^r A(\mathrm{d}u, x_u) + \int_r^t A(\mathrm{d}u, x_u) = \int_s^t A(\mathrm{d}u, x_u)$.*
2. *If $\partial_t A$ exists continuous, then $\int_s^t A(\mathrm{d}u, x_u) = \int_s^t \partial_t A(u, x_u)\mathrm{d}u$.*
3. *There exists a constant $C_1 = C_1(\alpha,\beta,\gamma)$ such that*

$$\left\| \int_s^t A(\mathrm{d}u, x_u) - A_{s,t}(x_s) \right\|_W \le C_1 |t-s|^{\alpha+\beta\gamma} [\![A]\!]_{\alpha,\beta,\|x\|_\infty} [\![x]\!]^\beta_\gamma. \tag{2.3}$$

4. *The map $(A,x) \mapsto \int_0^\cdot A(\mathrm{d}u, x_u)$ is continuous as a function from $C^\alpha_t C^\beta_{V,W,\mathrm{loc}} \times C^\gamma_t V \to C^\alpha_t W$. More precisely, it is a linear map in A and there exists $C_2 = C_2(\alpha,\beta,\gamma,T)$ such that*

$$\left\| \int_0^\cdot A^1(\mathrm{d}u, x_u) - \int_0^\cdot A^2(\mathrm{d}u, x_u) \right\|_\alpha \le C_2 \|A^1 - A^2\|_{\alpha,\beta,\|x\|_\infty} (1 + [\![x]\!]_\gamma); \tag{2.4}$$

it is locally δ-Hölder continuous in x for any $\delta \in (0,1)$ such that $\delta < (\alpha + \beta\gamma - 1)/\gamma$ and there exists $C_3 = C_3(\alpha,\beta,\gamma,\delta,T)$ such that, for any $R \ge \|x\|_\infty \vee \|y\|_\infty$, it holds

$$\left\| \int_0^\cdot A(\mathrm{d}u, x_u) - \int_0^\cdot A(\mathrm{d}u, y_u) \right\|_\alpha \le C_3 \|A\|_{\alpha,\beta,R} (1 + \|x\|_\gamma + \|y\|_\gamma) [\![x-y]\!]^\delta_\gamma. \tag{2.5}$$

Proof In order to show convergence of the Riemann sums, it is enough to apply the sewing lemma to the choice $\Gamma_{s,t} := A_{s,t}(x_s) = A(t, x_s) - A(s, x_s)$. Indeed we have

$$\|\Gamma\|_\alpha = \sup_{s<t} \frac{\|A_{s,t}(x_s)\|_W}{|t-s|^\alpha} \le \sup_{s<t} \frac{\|A_{s,t}\|_{0,\|x\|_\infty}}{|t-s|^\alpha} \le \|A\|_{\alpha,0,\|x\|_\infty}$$

and

$$\begin{aligned} \|\delta\Gamma_{s,u,t}\|_W = \|A_{u,t}(x_s) - A_{u,t}(x_u)\|_W &\le [\![A_{u,t}]\!]_{\beta,\|x\|_\infty} \|x_{u,s}\|^\beta_V \\ &\le |t-u|^\alpha |u-s|^{\beta\gamma} [\![A]\!]_{\alpha,\beta,\|x\|_\infty} [\![x]\!]^\beta_\gamma \end{aligned}$$

which implies $\|\delta\Gamma\|_{\alpha+\beta\gamma} \le [\![A]\!]_{\alpha,\beta,\|x\|_\infty} [\![x]\!]^\beta_\gamma$. In particular $\Gamma \in C^{\alpha,\alpha+\beta\gamma}_2 W$ with $\alpha + \beta\gamma > 1$, therefore by the sewing lemma we can set

$$\int_0^t A(\mathrm{d}s, x_s) := \mathcal{J}(\Gamma)_t = \lim_{|\Pi| \to 0} \sum_i \Gamma_{t_i, t_{t+1}}.$$

Property 1. then follows from $\mathcal{J}(\Gamma)_{s,t} = \mathcal{J}(\Gamma)_{s,r} + \mathcal{J}(\Gamma)_{r,t}$ and Property 3. from the above estimates on $\|\delta\Gamma\|_{\alpha+\beta\gamma}$. Similarly estimate (2.4) is obtained by the previous estimates applied to $A = A^1 - A^2$. Property 2. follows from the fact that if $\partial_t A$ exists continuous, then necessarily

$$\lim_{|\Pi| \to 0} \sum_i A_{t_i, t_{t+1}}(x_{t_i}) = \int_0^t \partial_t A(u, x_u)\mathrm{d}u.$$

It remains to show estimate (2.5). To this end, for fixed $x,\, y \in C^\gamma_t V$ and R as above, we need to provide estimates for $\|\delta\tilde\Gamma\|_{1+\varepsilon}$ for $\tilde\Gamma_{s,t} := A_{s,t}(x_s) - A_{s,t}(y_s)$ and suitable $\varepsilon > 0$. It holds

$$|\delta\tilde\Gamma_{s,u,t}| \le |A_{u,t}(x_u) - A_{u,t}(x_s)| + |A_{u,t}(y_u) - A_{u,t}(y_s)| \le \|A\|_{\alpha,\beta,R}(\llbracket x\rrbracket^\beta_\gamma + \llbracket y\rrbracket^\beta_\gamma)|t-s|^{\alpha+\beta\gamma},$$
$$|\delta\tilde\Gamma_{s,u,t}| \le |A_{u,t}(x_u) - A_{u,t}(y_u)| + |A_{u,t}(x_s) - A_{u,t}(y_s)| \lesssim \|A\|_{\alpha,\beta,R}\|x-y\|^\beta_0|t-s|^\alpha$$

which interpolated together give

$$\|\delta\Gamma\|_{(1-\theta)(\alpha+\beta\gamma)+\theta\alpha} \lesssim \|A\|_{\alpha,\beta,R}(1 + \llbracket x\rrbracket_\gamma + \llbracket y\rrbracket_\gamma)\|x-y\|^{\beta\theta}_0$$

for any $\theta \in (0,1)$ such that $(1-\theta)(\alpha+\beta\gamma) + \theta\alpha = 1 + \varepsilon > 1$, namely such that

$$\beta\theta < \frac{\alpha+\beta\gamma-1}{\gamma}.$$

The sewing lemma then implies that

$$\begin{aligned}\left\|\int_s^t A(\mathrm{d}r, x_r) - \int_s^t A(\mathrm{d}r, y_r)\right\|_W &\lesssim_\theta \left\|\int_s^t A(\mathrm{d}r, x_r) - \int_s^t A(\mathrm{d}r, y_r) - \tilde\Gamma_{s,t}\right\|_W + \|\tilde\Gamma_{s,t}\|_W \\ &\lesssim \|\delta\tilde\Gamma\|_{1+\varepsilon}|t-s|^{1+\varepsilon} + \|A\|_{\alpha,\beta,R}|t-s|^\alpha\|x-y\|^\beta_0 \\ &\lesssim_{\theta,T} |t-s|^\alpha\|A\|_{\alpha,\beta,R}(1+\|x\|_\gamma + \|y\|_\gamma)\|x-y\|^{\beta\theta}_0.\end{aligned}$$

Dividing by $|t-s|^\alpha$ and taking the supremum we obtain (2.5). □

Remark 2.8 Several other variants of the nonlinear Young integral can be constructed. For instance, for A and x as above, we can also define

$$\int_0^\cdot A(s, \mathrm{d}x_s) \in C^{\beta\gamma}_t W$$

as the sewing of $\Gamma_{s,t} := A_s(x_t) - A_s(x_s)$. Another possibility are integrals of the form

$$\int_0^\cdot y_s A(\mathrm{d}s, x_s)$$

for $y \in C^\delta_t\mathbb{R}$ such that $\alpha + \delta > 1$ and A, x as above. This can be either interpreted as a more classical Young integral of the form $\int_0^\cdot y_t \mathrm{d}\left(\int_0^t A(\mathrm{d}s, x_s)\right) = \mathcal{J}(\Gamma)$ for $\Gamma_{s,t} = y_s\int_s^t A(\mathrm{d}r, x_r)$, or as the sewing of $\tilde\Gamma_{s,t} = y_s A_{s,t}(x_s)$;it is immediate to check equivalence of the two definitions. This case can be further extended to consider a bilinear map $G : W \times U \to Z$, where U and Z are other Banach spaces, so that

$$\int_0^\cdot G(y_s, A(\mathrm{d}s, x_s)) \in C^\alpha_t Z$$

is well defined for $y \in C^\delta_t U$, A and x as above, as the sewing of $\Gamma_{s,t} = G(y_s, A_{s,t}(x_s)) \in C^{\alpha,\alpha+\delta}_2 Z$, since

$$\|\Gamma_{s,t}\| \le |t-s|^\alpha\|G\|\|y\|_\infty\|A\|_{\alpha,\beta},$$
$$\|\delta\Gamma_{s,u,t}\| \lesssim |t-s|^{\alpha+\delta}\|G\|\|y\|_\delta\|A\|_{\alpha,\beta}(1 + \llbracket x\rrbracket_\gamma).$$

Nonlinear Young integrals are a generalisation of classical ones, as the next example shows.

Example 2.9 Let $f \in C^\beta(\mathbb{R}^d;\mathbb{R}^{d\times m})$ and $y \in C^\alpha_t\mathbb{R}^m$, then $A(t,x) := f(x)y_t$ is an element of $C^\alpha_t C^\beta_{\mathbb{R}^d}$, since

$$|A_{s,t}(x) - A_{s,t}(y)| = |[f(x) - f(y)]y_{s,t}| \le |f(x) - f(y)||y_{s,t}| \le [\![f]\!]_\beta [\![y]\!]_\alpha |t-s|^\alpha |x-y|^\beta.$$

In particular, for any $x \in C^\gamma_t\mathbb{R}^d$ with $\alpha + \beta\gamma > 1$, we can consider $\int_0^\cdot A(\mathrm{d}s, x_s)$; this corresponds to the classical Young integral $\int_0^\cdot f(x_s)\mathrm{d}y_s$, since both are defined as sewings of

$$A_{s,t}(x_s) = f(x_s)y_t - f(x_s)y_s = f(x_s)y_{s,t}.$$

The previous example generalizes an infinite sum of Young integrals, i.e. considering sequences $f^n \in C^\beta(\mathbb{R}^d;\mathbb{R}^d)$, $y^n \in C^\alpha_t([0,T];\mathbb{R})$ such that (possibly locally)

$$\sum_n \|f^n\|_\beta \|y^n\|_\alpha < \infty.$$

In this case we can define $A(t,x) := \sum_n f^n(x)y^n_t$, which satisfies $\|A\|_{\alpha,\beta} \le \sum_n \|f^n\|_\beta\|y^n\|_\alpha$ and for any $x \in C^\delta_t\mathbb{R}^d$ it holds

$$\int_0^\cdot A(\mathrm{d}s, x_s) = \sum_n \int_0^\cdot f^n(x_s)\mathrm{d}y^n_s.$$

Remark 2.10 In the classical setting (let us take $d = 1$ for simplicity), if $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ satisfies

$$|f(t,z_1) - f(s,z_2)| \le C(|t-s|^{\beta\gamma} + |z_1 - z_2|^\beta), \tag{2.6}$$

$x \in C^\gamma_t$ and $y \in C^\alpha_t$ with $\alpha + \beta\gamma > 1$, then one can define the Young integral $\int_0^\cdot f(s,x_s)\mathrm{d}y_s$. However, $\int_0^\cdot f(s,x_s)\mathrm{d}y_s$ does not coincide with $\int A(\mathrm{d}s,x_s)$ for the choice $A(t,x) := f(t,x)y_t$.

This is partially because the domain of definition of the two integrals is different, since condition (2.6) (which is locally equivalent to $f \in C^{\beta\gamma}_t C^0_x \cap C^0_t C^\beta_x$) is not enough to ensure that $A \in C^\alpha_t C^\beta_x$; however, if we additionally assume $f \in C^\alpha_t C^\beta_x$, then so does A, and the relation between the two integrals is given by

$$\int_0^t A(\mathrm{d}s, x_s) = \int_0^t f(s,x_s)\mathrm{d}y_s + \int_0^t y_s f(\mathrm{d}s, x_s). \tag{2.7}$$

To derive (2.7), define $\Gamma^A_{s,t} = A_{s,t}(x_s)$; then

$$\Gamma^A_{s,t} = f(t,x_s)y_t - f(s,x_s)y_s = f(s,x_s)y_{s,t} + y_s f_{s,t}(x_s) + R_{s,t} =: \Gamma^y_{s,t} + \Gamma^f_{s,t} + R_{s,t}$$

where $|R_{s,t}| = |f_{s,t}(x_t) - f_{s,t}(x_s)| \lesssim |t-s|^{\alpha+\beta\gamma}$. This implies $\mathcal{J}(\Gamma^A) = \mathcal{J}(\Gamma^y) + \mathcal{J}(\Gamma^f)$, namely (2.7).

2.3 Nonlinear Young Calculus

Theorem 2.7 establishes continuity of the map $(A,x) \mapsto \int_0^\cdot A(\mathrm{d}s, x_s)$; if A is sufficiently regular, then we can even establish its differentiability.

Proposition 2.11 *Let $\alpha, \beta, \gamma \in (0,1)$ such that $\alpha + \beta\gamma > 1$, $A \in C^\alpha_t C^{1+\beta}_{V,W,\mathrm{loc}}$. Then the nonlinear Young integral, seen as a map $F : C^\gamma_t V \to C^\alpha_t W$, $F(x) = \int_0^\cdot A(\mathrm{d}s, x_s)$, is Frechét differentiable with*

$$DF(x) : y \mapsto \int_0^\cdot DA(\mathrm{d}s, x_s) y_s. \tag{2.8}$$

Proof For notational simplicity we will assume $A \in C^\alpha_t C^{1+\beta}_{V,W}$. It is enough to show that, for any $x, y \in C^\gamma_t V$, the Gateaux derivative of F at x in the direction y is given by the expression above, i.e.

$$\lim_{\varepsilon \to 0} \frac{F(x+\varepsilon y) - F(x)}{\varepsilon} = \int_0^\cdot DA(\mathrm{d}s, x_s) y_s \tag{2.9}$$

where the limit is in the $C^\alpha_t W$-topology. Indeed, once this is shown, it follows easily from reasoning as in Theorem 2.7 that the map $(x, y) \mapsto \int DA(\mathrm{d}s, x_s) y_s$ is jointly uniformly continuous in bounded balls and linear in the second variable; Frechét differentiability then follows from existence and continuity of the Gateaux differential.

In order to show (2.9), setting for any $\varepsilon > 0$

$$\Gamma^\varepsilon_{s,t} := \frac{A_{s,t}(x_s + \varepsilon y_s) - A_{s,t}(x_s)}{\varepsilon} - DA_{s,t}(x_s) y_s,$$

it suffices to show that $\mathcal{J}(\Gamma^\varepsilon) \to 0$ in $C^\alpha_t W$. In particular by Lemma A.2 from the "Appendix", we only need to check that $\|\Gamma^\varepsilon\|_\alpha \to 0$ as $\varepsilon \to 0$ while $\|\delta\Gamma^\varepsilon\|_{\alpha+\beta\gamma}$ stays uniformly bounded. It holds

$$\begin{aligned}
\|\Gamma^\varepsilon_{s,t}\|_W &= \left\| \int_0^1 [DA_{s,t}(x_s + \lambda\varepsilon y_s) - DA_{s,t}(x_s)] y_s \mathrm{d}\lambda \right\|_W \\
&\le \varepsilon^\beta \|DA_{s,t}\|_\beta \|y_s\|_V^{\beta+1} \le \varepsilon^\beta |t-s|^\alpha \|A\|_{\alpha,1+\beta} \|y\|_\delta^{\beta+1}
\end{aligned}$$

which implies that $\|\Gamma^\varepsilon\|_\alpha \lesssim \varepsilon^\beta \to 0$; similar calculations show that

$$\begin{aligned}
\|\Gamma^\varepsilon_{s,u,t}\|_W &= \left\| \int_0^1 [DA_{u,t}(x_s + \lambda\varepsilon y_s) - DA_{u,t}(x_s)] y_s \mathrm{d}\lambda - \int_0^1 [DA_{u,t}(x_u + \lambda\varepsilon y_u) - DA_{u,t}(x_u)] y_u \mathrm{d}\lambda \right\|_W \\
&= \| - \int_0^1 [DA_{u,t}(x_s + \lambda\varepsilon y_s) - DA_{u,t}(x_s)] y_{s,u} \mathrm{d}\lambda \\
&\quad + \int_0^1 [DA_{u,t}(x_s + \lambda\varepsilon y_s) - DA_{u,t}(x_s) - DA_{u,t}(x_u + \lambda\varepsilon y_u) + DA_{u,t}(x_u)] y_u \mathrm{d}\lambda \|_W \\
&\lesssim |t-s|^{\alpha+\gamma} \|DA\|_{\alpha,\beta} \|y\|_\gamma^{1+\beta} + |t-s|^{\alpha+\beta\gamma} \|DA\|_{\alpha,\beta} \|y\|_\gamma ([\![x]\!]_\gamma^\beta + [\![y]\!]_\gamma^\beta)
\end{aligned}$$

which implies that $\|\delta\Gamma\|_{\alpha+\beta\gamma} \lesssim 1$ uniformly in $\varepsilon > 0$. The conclusion the follows. □

Proposition 2.11 allows to give an alternative proof of Lemma 6 from [20].

Corollary 2.12 *Let $\alpha, \beta, \gamma \in (0,1)$ such that $\alpha + \beta\gamma > 1$, $A \in C^\alpha_t C^{1+\beta}_{V,W,\mathrm{loc}}$, $x^1, x^2 \in C^\gamma_t V$. Then*

$$\int_0^\cdot A(\mathrm{d}s, x^1_s) - \int_0^\cdot A(\mathrm{d}s, x^2_s) = \int_0^\cdot v_{\mathrm{d}s}(x^1_s - x^2_s) \tag{2.10}$$

with v given by

$$v_t := \int_0^t \int_0^1 DA(\mathrm{d}s, x^2_s + \lambda(x^1_s - x^2_s)) \mathrm{d}\lambda; \tag{2.11}$$

the above formula meaningfully defines an element of $C_t^\alpha \mathcal{L}(V, W)$ *which satisfies*

$$[\![v]\!]_\alpha \le C \|DA\|_{\alpha,\beta,R}(1 + [\![x^1]\!]_\gamma + [\![x^2]\!]_\gamma) \tag{2.12}$$

where $R \ge \|x\|_\infty \vee \|y\|_\infty$ *and* $C = C(\alpha, \beta, \gamma, T)$.

Proof It follows from the hypothesis on A that the map

$$y \in V \mapsto \int_0^1 \left[\int_0^t DA(\mathrm{d}s, x_s^2 + \lambda(x_s^1 - x_s^2))y\right] \mathrm{d}\lambda \in W \tag{2.13}$$

is well defined, the outer integral being in the Bochner sense, and it is linear in y; moreover estimate (2.3) combined with the trivial inequality $1+[\![x^2+\lambda(x_s^1-x_s^2)]\!]_\gamma^\beta \lesssim 1+[\![x^1]\!]_\gamma+[\![x^2]\!]_\gamma$, valid for any $\lambda, \beta \in [0, 1]$, yields

$$\left\|\int_0^1 \left[\int_0^t DA(\mathrm{d}s, x^2 + \lambda(x_s^1 - x_s^2))y\right] \mathrm{d}\lambda\right\|_W \lesssim \|DA\|_{\alpha,\beta,R}(1 + [\![x^1]\!]_\gamma + [\![x^2]\!]_\gamma)\|y\|_V.$$

In particular, if we define v_t as the linear map appearing (2.13), it is easy to check that similar estimates yield $v \in C_t^\alpha \mathcal{L}(V, W)$. The fact that this definition coincide with the one from (2.11), i.e. that we can exchange integration in dλ and in "ds", follows from the Fubini theorem for the sewing map, see Lemma A.1 in the "Appendix". Inequality (2.12) then follows from estimates analogue to the ones obtained above. Identity (2.10) is an application of the more abstract classical identity

$$F(x^1) - F(x^2) = \left[\int_0^1 DF(x^2 + \lambda(x^1 - x^2))\mathrm{d}\lambda\right](x^1 - x^2)$$

applied to $F(x) = \int_0^\cdot A(\mathrm{d}s, x_s)$, for which the exact expression for DF is given by Proposition 2.11. □

The following Itô-type formula is taken from [30], Theorem 3.4.

Proposition 2.13 *Let* $F \in C_t^\alpha C_{V,W,\mathrm{loc}}^\beta$ *and* $x \in C_t^\gamma V$ *with* $\alpha + \beta\gamma > 1$, *then it holds*

$$F(t, x_t) - F(0, x_0) = \int_0^t F(\mathrm{d}s, x_s) + \int_0^t F(s, \mathrm{d}x_s); \tag{2.14}$$

if in addition $F \in C_t^0 C_{V,W,\mathrm{loc}}^{1+\beta'}$ *with* $\beta' \in (0, 1)$ *s.t.* $\gamma(1 + \beta') > 1$, *then*

$$F(t, x_t) - F(0, x_0) = \int_0^t F(\mathrm{d}s, x_s) + \int_0^t DF(s, x_s)(\mathrm{d}x_s). \tag{2.15}$$

In particular, if $x = \int_0^\cdot A(\mathrm{d}s, y_s)$ *for some* $A \in C_t^\gamma C_V^\delta$, $y \in C_t^\eta V$ *with* $\gamma + \eta\delta > 1$, *then* (2.15) *becomes*

$$F(t, x_t) - F(0, x_0) = \int_0^t F(\mathrm{d}s, x_s) + \int_0^t DF(s, x_s)(A(\mathrm{d}s, y_s)). \tag{2.16}$$

Proof Let $0 = t_0 < t_1 < \cdots < t_n = t$, then it holds

$$\begin{aligned} F(t, x_t) - F(0, x_0) &= \sum_i [F(t_{i+1}, x_{t_{i+1}}) - F(t_i, x_{t_i})] \\ &= \sum_i F_{t_i,t_{i+1}}(x_{t_i}) + \sum_i [F_{t_i}(x_{t_{i+1}}) - F_{t_i}(x_{t_i})] + \sum_i R_{t_i,t_{i+1}} =: I_1^n + I_2^n + I_3^n \end{aligned}$$

where $R_{t_i,t_{i+1}} = F_{t_i,t_{i+1}}(x_{t_i+1}) - F_{t_i,t_{i+1}}(x_{t_i})$ satisfies $\|R_{t_i,t_{i+1}}\| \le \|F\|_{\alpha,\beta,\|x\|_\infty} [\![x]\!]_\gamma^\beta |t_{i+1} - t_i|^{\alpha+\beta\gamma}$, while I_1^n and I_2^n are Riemann-Stjeltes sums associated to $\Gamma^1_{s,t} = F_{s,t}(x_s)$ and $\Gamma^2_{s,t} = F_s(x_t) - F_s(x_s)$. Taking a sequence of partitions Π_n with $|\Pi_n| \to 0$, by the above estimate we have $I_3^n \to 0$ and by the sewing lemma we obtain

$$F(t, x_t) - F(0, x_0) = \mathcal{J}(\Gamma^1)_t + \mathcal{J}(\Gamma^2)_t,$$

which is exactly (2.14). If $F \in C^0_t C^{1+\beta'}_{V,W,\mathrm{loc}}$, then setting $\Gamma^3_{s,t} := DF(s, x_s)(x_{s,t})$, it holds

$$\begin{aligned}
\|\Gamma^2_{s,t} - \Gamma^3_{s,t}\|_V &= \|F(s, x_t) - F(s, x_s) - DF(s, x_s)(x_{s,t})\|_V \\
&= \left\| \int_0^1 [DF(s, x_s + \lambda x_{s,t}) - DF(s, x_s)](x_{s,t})\mathrm{d}\lambda \right\|_V \\
&\lesssim \|DF(s, \cdot)\|_{\beta',\|x\|_\infty} \|x_{s,t}\|^{1+\beta'} \lesssim \|F\|_{0,1+\beta',\|x\|_\infty} [\![x]\!]_\gamma^{\beta'} |t-s|^{\gamma(1+\beta')}
\end{aligned}$$

which under the assumption $\gamma(1 + \beta') > 1$ implies by the sewing lemma that $\mathcal{J}(\Gamma^2) = \mathcal{J}(\Gamma^3)$ and thus (2.15). The proof of (2.16) is analogue, only this time consider $\Gamma^4_{s,t} := DF(s, x_s)(A_{s,t}(y_s))$, then it's easy to check that $\|\Gamma^3_{s,t} - \Gamma^4_{s,t}\|_V \lesssim |t-s|^{\gamma+\eta\delta}$ which implies that $\mathcal{J}(\Gamma^3) = \mathcal{J}(\Gamma^4)$. □

Remark 2.14 The above formulas admit further variants. For instance for any $F \in C^\alpha_t C^\beta_{V,W}$, $x \in C^\gamma_t V$ and $g \in C^\delta_t \mathbb{R}$ with $\alpha + \beta\gamma > 1$, $\alpha + \delta > 1$ and $\beta\gamma + \delta > 1$ it holds

$$\int_0^t g_s \mathrm{d}[F(s, x_s)] = \int_0^t g_s F(\mathrm{d}s, x_s) + \int_0^t g_s F(s, \mathrm{d}x_s)$$

and we have the product rule formula

$$g_t F(t, x_t) - g_0 F(0, x_0) = \int_0^t F(s, x_s)\mathrm{d}g_s + \int_0^t g_s F(\mathrm{d}s, x_s) + \int_0^t g_s F(s, \mathrm{d}x_s).$$

Also observe that, whenever $\partial_t F$ exists continuous, it holds

$$\int_0^t g_s F(\mathrm{d}s, x_s) = \int_0^t g_s \partial_t F(s, x_s)\mathrm{d}s \quad \forall\, g \in C^\delta_t \mathbb{R}.$$

3 Existence, Uniqueness, Numerical Schemes

This section is devoted to the study of nonlinear Young differential equations (YDE for short), defined below; it provides sufficient conditions for existence and uniqueness of solutions, as well as convergence of numerical schemes.

Definition 3.1 Let $A \in C^\alpha_t C^\beta_{V,\mathrm{loc}}$, $x_0 \in V$. We say that x is a solution to the YDE associated to (x_s, A) on an interval $[s, t] \subset [0, T]$ if $x \in C^\gamma([s, t]; V)$ for some γ such that $\alpha + \beta\gamma > 1$ and it satisfies

$$x_r = x_s + \int_s^r A(\mathrm{d}u, x_u) \quad \forall\, r \in [s, t]. \tag{3.1}$$

Before proceeding further, let us point out that by Example 2.9 any Young differential equation

$$x_t = x_0 + \int f(x_s)\mathrm{d}y_s$$

can be reinterpreted as a nonlinear YDE associated to $A := f \otimes y$. Nonlinear YDEs therefore are a natural extension of the standard ones; most results regarding their existence and uniqueness which will be presented are perfect analogues (in terms of regularity requirements) to the well known classical ones (which can be found for instance in [33] or Section 8 of [18]).

Throughout this section, for $x : [0, T] \to V$ and $I \subset [0, T]$, we set

$$[\![x]\!]_{\gamma;I} := \sup_{\substack{s,t \in I \\ s \neq t}} \frac{\|x_{s,t}\|_V}{|t-s|^\gamma}$$

as well as $[\![x]\!]_{\gamma;s,t}$ in the case $I = [s, t]$; similarly for $\|x\|_{\infty;I}$ and $\|x\|_{\gamma;I}$. For any $\Delta > 0$ we also define

$$[\![x]\!]_{\gamma,\Delta,V} = [\![x]\!]_{\gamma,\Delta} := \sup_{\substack{s,t \in [0,T] \\ |s-t| \in (0,\Delta]}} \frac{\|x_{s,t}\|_V}{|t-s|^\gamma}.$$

3.1 Existence

We provide here sufficient conditions for the existence of either local or global solutions to the YDE, under suitable compactness assumptions on A. The proof is based on an Euler scheme for the YDE, in the style of those from [14,33]; its rate of convergence will be studied later on. Other proofs, based on a priori estimates and compactness techniques or an application of Leray–Schauder–Tychonoff fixed point theorem, are possible, see [9,30].

Theorem 3.2 *Let $A \in C^\alpha_t C^\beta_{V,W}$ where W is compactly embedded in V and $\alpha(1+\beta) > 1$. Then for any $s > 0$ and $x_s \in V$ there exists a solution to the YDE*

$$x_t = x_s + \int_s^t A(\mathrm{d}s, x_s) \quad \forall t \in [s, T]. \tag{3.2}$$

Proof The proof is based on the application of an Euler scheme. Up to rescaling and shifting, we can assume for simplicity $T = 1$ and $s = 0$.

Fix $N \in \mathbb{N}$, set $t^n_k = k/n$ for $k \in \{0, \dots, n\}$ and define recursively $(x^n_k)_{k=1}^n$ by $x^n_0 = x_0$ and

$$x^n_{k+1} = x^n_k + A_{t^n_k, t^n_{k+1}}(x^n_k).$$

We can embed $(x^n_k)_{k=1}^n$ into $C^0_t V$ by setting

$$x^n_t := x_0 + \sum_{0 \le k \le \lfloor nt \rfloor} A_{t^n_k, t \wedge t^{n+1}_k}(x^n_k);$$

note that by construction $x^n - x_0$ is a path in $C^\alpha_t W$. Using the identity

$$A_{s,t}(x^n_s) = \int_s^t A(\mathrm{d}r, x^n_r) + \int_s^t [A(\mathrm{d}r, x^n_s) - A(\mathrm{d}r, x^n_r)]$$

we deduce that x^n satisfies a YDE of the form

$$x^n_t = x_0 + \int_0^t A(\mathrm{d}s, x^n_s) + \psi^n_t \tag{3.3}$$

where

$$\psi^n_t = \sum_{0\le k\le n} \psi^{n,k}_t = \sum_{0\le k\le n} \int_{t^n_k}^{(t\wedge t^n_{k+1})\vee t^n_k} [A(\mathrm{d}r, x^n_{t^n_k}) - A(\mathrm{d}r, x^n_r)].$$

By the properties of Young integrals, ψ^n satisfies

$$\|\psi^n_{t^n_k,t^n_{k+1}}\|_W = \left\| \int_{t^n_k}^{t^n_{k+1}} [A(\mathrm{d}r, x^n_{t^n_k}) - A(\mathrm{d}r, x^n_r)] \right\|_W \lesssim n^{-\alpha(1+\beta)} \|A\|_{\alpha,\beta} [\![x^n]\!]^\beta_{\alpha,1/n,V}. \tag{3.4}$$

We first want to obtain a bound for $[\![\psi^n]\!]_{\gamma,\Delta,W}$; we can assume wlog $\Delta > 1/n$, since we want to take $n \to \infty$. Estimates depend on whether s and t lie on the same interval $[t^n_k, t^n_{k+1}]$ or not; assume first this is the case, then

$$\begin{aligned}
\|\psi^n_{s,t}\|_W &= \left\| \int_s^t [A(\mathrm{d}r, x^n_{t^n_k}) - A(\mathrm{d}r, x^n_r)] \right\|_W \\
&\lesssim \|A_{s,t}(x^n_{t^n_k}) - A_{s,t}(x^n_s)\|_W + |t-s|^{\alpha(1+\beta)} \|A\|_{\alpha,\beta} [\![x^n]\!]^\beta_{\alpha,\Delta,V} \\
&\lesssim n^{-\alpha\beta} |t-s|^\alpha \|A\|_{\alpha,\beta} [\![x^n]\!]^\beta_{\alpha,\Delta,V}.
\end{aligned}$$

Next, given $s < t$ such that $|t-s| < \Delta$ which are not in the same interval, there are around $n|t-s|$ intervals separating them, i.e. there exist $l < m$ such that $m - l \sim n|t-s|$ and $s \le t^n_l < \cdots < t^n_m \le t$. Therefore in this case we have

$$\begin{aligned}
\|\psi^n_{s,t}\|_W &\le \|\psi^n_{s,t^n_l}\|_W + \sum_{k=l}^{m-1} \|\psi^n_{t^n_k,t^n_{k+1}}\|_W + \|\psi^n_{t^n_m,t}\|_W \\
&\lesssim \|A\|_{\alpha,\beta} [\![x^n]\!]^\beta_{\alpha,\Delta,V} [|t-s|^\alpha n^{-\alpha\beta} + (m-l) n^{-\alpha(1+\beta)}] \\
&\lesssim \|A\|_{\alpha,\beta} [\![x^n]\!]^\beta_{\alpha,\Delta,V} [|t-s|^\alpha n^{-\alpha\beta} + |t-s| n^{1-\alpha(1+\beta)}] \\
&\lesssim \|A\|_{\alpha,\beta} [\![x^n]\!]^\beta_{\alpha,\Delta,V} |t-s|^\alpha n^{1-\alpha(1+\beta)}
\end{aligned}$$

where in the second line we used both (3.4) and the previous bound for ψ^n_{s,t^n_l} and $\psi^n_{t^n_m,t}$, while in the last one the fact that $-\alpha\beta \le 1 - \alpha(1+\beta)$. Overall we conclude that

$$[\![\psi^n]\!]_{\alpha,\Delta,W} \le \kappa_1 n^{1-\alpha(1+\beta)} \|A\|_{\alpha,\beta} [\![x^n]\!]^\beta_{\alpha,\Delta,V} \tag{3.5}$$

for a suitable constant $\kappa_1 = \kappa_1(\alpha, \beta)$ independent of Δ and n.

Our next goal is a uniform bound for $[\![x^n]\!]_{\alpha,\Delta,W}$. Since x^n solves (3.3), it holds

$$\begin{aligned}
\|x^n_{s,t}\|_W &\lesssim \|A_{s,t}(x^n_s)\|_W + |t-s|^{\alpha(1+\beta)} \|A\|_{\alpha,\beta} [\![x^n]\!]^\beta_{\alpha,\Delta,W} + \|\psi^n_{s,t}\|_W \\
&\lesssim |t-s|^\alpha \|A\|_{\alpha,\beta} + |t-s|^\alpha \Delta^{\alpha\beta} \|A\|_{\alpha,\beta} [\![x^n]\!]^\beta_{\alpha,\Delta,W} + |t-s|^\alpha [\![\psi^n]\!]_{\alpha,\Delta,W} \\
&\lesssim |t-s|^\alpha \|A\|_{\alpha,\beta} + |t-s|^\alpha \|A\|_{\alpha,\beta} [\![x^n]\!]^\beta_{\alpha,\Delta,W} (\Delta^{\alpha\beta} + n^{1-\alpha(1+\beta)})
\end{aligned}$$

and so dividing by $|t-s|$ and taking the supremum over all $|t-s| < \Delta$, choosing Δ such that $\Delta^{\alpha\beta} \|A\|_{\alpha,\beta} \le 1/4$, then for all n big enough such that $n^{1-\alpha(1+\beta)} \|A\|_{\alpha,\beta} \le 1/4$ it holds

$$[\![x^n]\!]_{\alpha,\Delta,W} \lesssim \|A\|_{\alpha,\beta} + \frac{1}{2} [\![x^n]\!]^\beta_{\alpha,\Delta,W} \lesssim \|A\|_{\alpha,\beta} + \frac{1}{2} + \frac{1}{2} [\![x^n]\!]_{\alpha,\Delta,W}$$

by the trivial bound $a^\beta \le 1 + a$, which holds for all $\beta \in [0, 1]$ and $a \ge 0$. This implies the uniform bound $[\![x^n]\!]_{\alpha,\Delta,W} \lesssim 1 + \|A\|_{\alpha,\beta}$ for all n big enough.

The subspace $\{y \in C^\alpha([0,1];W) : y_0 = 0\}$ is a Banach space endowed with the seminorm $[\![y]\!]_{\alpha,\Delta,W}$, which in this case is equivalent to the norm $\|y\|_{\alpha,W}$; $\{x_n - x_0\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence in this space. By Ascoli–Arzelà, since W compactly embeds in V, we can extract a subsequence (not relabelled for simplicity) such that $x_n - x_0 \to x - x_0$ in $C_t^{\alpha-\varepsilon}V$ for any $\varepsilon > 0$, for some $x \in C_t^\alpha V$ such that $x(0) = x_0$. Observe that ψ^n satisfy (3.5) and $[\![x^n]\!]^\beta_{\alpha,\Delta,V}$ are uniformly bounded, therefore $\psi^n \to 0$ in $C_t^\alpha W$ as $n \to \infty$; choosing ε small enough s.t. $\alpha + \beta(\alpha - \varepsilon) > 1$, by continuity of the non-linear Young integral it holds

$$\int_0^\cdot A(\mathrm{d}s, x_s^n) \to \int_0^\cdot A(\mathrm{d}s, x_s) \quad \text{in } C_t^\alpha W$$

and therefore passing to the limit in (3.3) we obtain the conclusion. □

Remark 3.3 If V is finite dimensional, the compactness condition is trivially satisfied by taking $V = W$. The proof also works for non uniform partitions Π_n of $[0,T]$, under the condition that their mesh $|\Pi_n| \to 0$ and that there exists $c > 0$ such that $|t^n_{i+1} - t^n_i| \geq c|\Pi_n|$ for all $n \in \mathbb{N}$, $i \in \{0,\dots,N_n\}$.

Remark 3.4 The proof provides several estimates, some of which are true even without the compactness assumption. For instance, by $[\![x^n]\!]_{\alpha,\Delta} \lesssim 1 + \|A\|_{\alpha,\beta}$ and Exercise 4.24 from [18], choosing Δ s.t. $\Delta^{\alpha\beta}\|A\|_{\alpha,\beta} \sim 1$, we deduce that there exists $C_1 = C_1(\alpha,\beta,T)$ such that

$$[\![x^n]\!]_\alpha \leq C_1 \left(1 + \|A\|_{\alpha,\beta}^{1+\frac{1-\alpha}{\alpha\beta}}\right) \quad \forall n \in \mathbb{N}.$$

Estimate (3.5) is true for any choice of $\Delta > 0$, in particular for $\Delta = T$, which gives a global bound; combining it with the above one, we deduce that

$$[\![\psi^n]\!]_\alpha \leq C_2 n^{1-\alpha(1+\beta)} \left(1 + \|A\|_{\alpha,\beta}^{\frac{1+\alpha\beta}{\alpha}}\right) \quad \forall n \in \mathbb{N}$$

for some $C_2 = C_2(\alpha,\beta,T)$. Also observe that from the assumptions on α and β it always holds

$$1 + \frac{1-\alpha}{\alpha\beta} \leq 2, \qquad \frac{1+\alpha\beta}{\alpha} \leq 3.$$

Under the compactness assumption, since $x^n \to x$ in $C_t^0 V$, the solution x obtained also satisfies

$$[\![x]\!]_\alpha \leq \liminf_{n\to\infty} [\![x^n]\!]_\alpha \leq C_1 \left(1 + \|A\|_{\alpha,\beta}^{1+\frac{1-\alpha}{\alpha\beta}}\right) \leq 2C_1(1 + \|A\|^2_{\alpha,\beta}). \tag{3.6}$$

Finally observe that by going through the same proof of (3.5), for any $T > 0$ and α, β, γ such that $\alpha + \beta\gamma > 1$, there exists $C_3 = C_3(\alpha,\beta,\gamma,T)$ such that

$$[\![\psi^n]\!]_{\alpha,\Delta,V} \leq C_3 n^{1-\alpha-\beta\gamma} \|A\|_{\alpha,\beta} [\![x^n]\!]^\beta_{\gamma,\Delta,V} \quad \forall n \in \mathbb{N}. \tag{3.7}$$

This estimate is rather useful when A enjoys different space-time regularity at different scales, see the discussion at Sect. 3.4.

Corollary 3.5 *Let $A \in C_t^\alpha C^\beta_{V,W,\mathrm{loc}}$ where W is compactly embedded in V and $\alpha(1+\beta) > 1$. Then for any $s \in [0,T)$ and any $x_s \in V$, there exists $\tau^* \in (s,T]$ and a solution to the YDE (3.2) defined on $[s,T^*)$, with the property that either $T^* = T$ or*

$$\lim_{t\uparrow T^*} \|x_t\|_V = +\infty.$$

Proof As before it is enough to treat the case $s = 0, T = 1$. Fix $R > 0$ and consider $A^R \in C^\alpha_t C^\beta_{V,W}$ such that $A^R(t,x) = A(t,x)$ for any (t,x) with $\|x\|_V \le 2R$ and $A^R(t,x) \equiv 0$ for $\|x\|_V \ge 3R$; let $C_R := C(1 + \|A\|^2_{\alpha,\beta,3R})$, where C is the constant appearing in (3.6).

For any $x_0 \in V$ with $\|x_0\| \le R$, by Theorem 3.2 there exists a solution $x_\cdot$ to the YDE associated to (x_0, A^R) on the interval $[0,1]$; setting $\tau_1 := \inf\{t \in [0,1] : \|x_t\|_V \ge 2R\}$, by (3.6) it holds $[\![x]\!]_{\alpha;[0,\tau_1]} \le C_R$, and so

$$2R = \|x_{\tau_1}\|_V \le \|x_0\|_V + \tau_1^\alpha [\![x]\!]_{\alpha;[0,\tau_1]} \le R + \tau_1^\alpha C_R$$

which implies

$$\tau_1 \ge \left(\frac{C_R}{R}\right)^{-\alpha}. \tag{3.8}$$

In particular, since $A = A^R$ on $[0,T] \times B_{2R}$, we conclude that $x_\cdot$ is also a solution to the YDE associated to (x_0, A) on the interval $[0, \tau_1]$.

We can now iterate this procedure, i.e. set $x^1 := x_{\tau_1}$ and construct another solution to (3.2), defined on an interval $[\tau_1, \tau_2]$, and so on; by "gluing" these solutions together, we obtain an increasing sequence $\{\tau_n\} \subset [0,1]$ and a solution $x_\cdot$ defined on $[0, T^*)$, where $T^* = \lim_n \tau_n$.

Now suppose that $T^* < T$ and $\liminf_{t \to T^*} \|x_t\|_V < \infty$, then we can find a sequence $t_n \to T^*$ such that $\|x_{t_n}\|_V \le M$ for some $M > 0$; but then starting from any of this x_{t_n} we can construct another solution y^n defined on $[t_n, t_n + \varepsilon]$, where ε is uniform in n since $\|x_{t_n}\| \le M$ and ε can be estimated by (3.8) with R replaced by M. By replacing the solution $x_\cdot$ on $[t_n, T^*)$ with y^n, choosing n big enough, we can construct a solution defined on $[0, T^* + \varepsilon/2)$. Reiterating this procedure we obtain the conclusion. □

3.2 A Priori Estimates

A classical way to pass from local to global solutions is to establish suitable a priori estimates, which are also of fundamental importance for compactness arguments. Throughout this section, we assume that a solution x to the YDE is already given and focus exclusively on obtainig bounds on it; for simplicity we work on $[0,T]$, but all the statements immediately generalise to $[s,T]$.

Proposition 3.6 *Let $\alpha > 1/2$, $\beta \in (0,1)$ such that $\alpha(1+\beta) > 1$, $A \in C^\alpha_t C^\beta_V$, $x_0 \in V$ and $x \in C^\alpha_t V$ be a solution to the associated YDE. Then there exists $C = C(\alpha, \beta, T)$ such that*

$$[\![x]\!]_\alpha \le C(1 + \|A\|^2_{\alpha,\beta}), \qquad \|x\|_\alpha \le C(1 + \|x_0\|_V + \|A\|^2_{\alpha,\beta}). \tag{3.9}$$

Proof Let $\Delta \in (0,T]$ be a parameter to be chosen later. For any $s < t$ such that $|s-t| \le \Delta$, using the fact that x is a solution, it holds

$$\begin{aligned}
\|x_{s,t}\|_V &= \left\| \int_s^t A(\mathrm{d}u, x_u) \right\|_V \\
&\le \|A_{s,t}(x_s)\|_V + \kappa_1 |t-s|^{\alpha(1+\beta)} [\![A]\!]_{\alpha,\beta} [\![x]\!]^\beta_{\alpha,\Delta} \\
&\le |t-s|^\alpha \|A\|_{\alpha,\beta} (1 + \kappa_1 \Delta^{\alpha\beta} [\![x]\!]^\beta_{\alpha,\Delta}) \\
&\le |t-s|^\alpha \|A\|_{\alpha,\beta} (1 + \kappa_1 \Delta^{\alpha\beta} + \kappa_1 \Delta^{\alpha\beta} [\![x]\!]_{\alpha,\Delta})
\end{aligned}$$

were we used the trivial inequality $a^\beta \le 1 + a$. Dividing both sides by $|t-s|^\alpha$ and taking the supremum over $|s-t| \le \Delta$, we get

$$[\![x]\!]_{\alpha,\Delta} \le \|A\|_{\alpha,\beta}(1+\kappa_1\Delta^{\alpha\beta}) + \kappa_1\Delta^{\alpha\beta}\|A\|_{\alpha,\beta}[\![x]\!]_{\alpha,\Delta}.$$

Choosing Δ small enough such that $\kappa_1\Delta^{\alpha\beta}\|A\|_{\alpha,\beta} \le 1/2$, we obtain

$$[\![x]\!]_{\alpha,\Delta} \le 2\|A\|_{\alpha,\beta}(1+\kappa_1\Delta^{\alpha\beta}) \lesssim 1 + \|A\|_{\alpha,\beta}.$$

If we can take $\Delta = T$, we get an estimate for $[\![x]\!]_\alpha$, which gives the conclusion. If this is not the case, we can choose Δ such that in addition $\kappa_1\Delta^{\alpha\beta}\|A\|_{\alpha,\beta} \ge 1/4$ and then as before, by Exercise 4.24 from [18] it holds $[\![x]\!]_\alpha \lesssim_T \Delta^{\alpha-1}[\![x]\!]_{\alpha,\Delta}$, so that

$$\begin{aligned}
[\![x]\!]_\alpha &\lesssim (1+\|A\|_{\alpha,\beta})\Delta^{\alpha-1}\\
&\lesssim (1+\|A\|_{\alpha,\beta})\|A\|_{\alpha,\beta}^{(1-\alpha)/(\alpha\beta)}\\
&\lesssim 1+\|A\|_{\alpha,\beta}^2
\end{aligned}$$

where we used the fact that $\alpha(1+\beta) > 1$ implies $(1-\alpha)/(\alpha\beta) < 1$. The conclusion follows by the standard inequality $\|x\|_\alpha \lesssim_T \|x_0\|_V + [\![x]\!]_\alpha$. □

The assumption of a global bound on A of the form $A \in C_t^\alpha C_V^\beta$ is sometimes too strong for practical applications. It can be relaxed to suitable growth conditions, as the next result shows; it is taken from [30], Theorem 3.1 (see also Theorem 2.9 from [9]).

Proposition 3.7 *Let $A \in C_t^\alpha C_V^{\beta,\lambda}$ with $\alpha(1+\beta) > 1$, $\beta+\lambda \le 1$. Then there exists a constant $C = C(\alpha,\beta,T)$ such that any solution x on $[0,T]$ to the YDE associated to (x_0, A) satisfies*

$$\|x\|_\alpha \le C\exp\left(\|A\|_{\alpha,\beta,\lambda}^{1+\frac{1-\alpha}{\alpha\beta}}\right)(1+\|x_0\|_V). \tag{3.10}$$

Proof Fix an interval $[s,t] \subset [0,T]$, set $R = \|x\|_{\infty;s,t}$. Since x is a solution, for any $[u,r] \subset [s,t]$ it holds

$$\begin{aligned}
\|x_{u,r}\|_V &\lesssim \|A_{u,r}(x_u)\|_V + |r-u|^{\alpha(1+\beta)}[\![A]\!]_{\alpha,\beta,R}[\![x]\!]_{\alpha;s,t}^\beta\\
&\lesssim \|A_{u,r}(x_u) - A_{u,r}(x_s)\|_V + |r-u|^\alpha\|A\|_{\alpha,\beta,\lambda}(1+\|x_s\|_V)\\
&\quad + |r-u|^\alpha|t-s|^{\alpha\beta}\|A\|_{\alpha,\beta,\lambda}(1+\|x\|_{\infty;s,t}^\lambda)[\![x]\!]_{\alpha;s,t}^\beta\\
&\lesssim |r-u|^\alpha\|A\|_{\alpha,\beta,\lambda}[1+\|x_s\|_V + |t-s|^{\alpha\beta}(1+\|x\|_{\infty;s,t}^\lambda)[\![x]\!]_{\alpha;s,t}^\beta]
\end{aligned}$$

which implies, dividing by $|r-u|^\alpha$ and taking the supremum, that

$$[\![x]\!]_{\alpha;s,t} \lesssim \|A\|_{\alpha,\beta,\lambda}(1+\|x_s\|_V) + |t-s|^{\alpha\beta}\|A\|_{\alpha,\beta,\lambda}(1+\|x\|_{\infty;s,t}^\lambda)[\![x]\!]_{\alpha;s,t}^\beta.$$

By an application of Young's inequality, for any $a, b \ge 0$ it holds $a^\lambda b^\beta \le a^{\beta+\lambda} + b^{\beta+\lambda}$; moreover $\beta+\lambda \le 1$ so that $a^{\beta+\lambda} \le 1+a$ for any $\theta \in [0,1]$, therefore we obtain

$$\begin{aligned}
[\![x]\!]_{\alpha;s,t} &\lesssim \|A\|_{\alpha,\beta,\lambda}(1+\|x_s\|_V) + |t-s|^{\alpha\beta}\|A\|_{\alpha,\beta,\lambda}(1+\|x\|_{\infty;s,t} + [\![x]\!]_{\alpha;s,t})\\
&\lesssim \|A\|_{\alpha,\beta,\lambda}(1+\|x_s\|_V) + \|A\|_{\alpha,\beta,\lambda}|t-s|^{\alpha\beta}[\![x]\!]_{\alpha;s,t}
\end{aligned}$$

where in the second passage we used the estimate $\|x\|_{\infty;s,t} \lesssim_T \|x_s\|_V + [\![x]\!]_{\alpha;s,t}$. Overall we deduce the existence of a constant $\kappa_1 = \kappa_1(\alpha,\beta,T)$ such that

$$[\![x]\!]_{\alpha;s,t} \le \frac{\kappa_1}{2}\|A\|_{\alpha,\beta,\lambda}(1+\|x_s\|_V) + \frac{\kappa_1}{2}\|A\|_{\alpha,\beta,\lambda}|t-s|^{\alpha\beta}[\![x]\!]_{\alpha;s,t}.$$

Choosing $[s,t]$ such that $|t-s|=\Delta$ satisfies $\kappa_1\|A\|_{\alpha,\beta,\lambda}\Delta^{\alpha\beta}\le 1$, we obtain

$$[\![x]\!]_{\alpha;s,t}\le \kappa_1\|A\|_{\alpha,\beta,\lambda}(1+\|x_s\|_V). \tag{3.11}$$

If T satisfies $\kappa_1\|A\|_{\alpha,\beta,\lambda}T^{\alpha\beta}\le 1$, then we can take $\Delta=T$, which gives a global estimate and thus the conclusion. If this is not the case, then we can choose $\Delta<T$ s.t. $\kappa_1\|A\|_{\alpha,\beta,\lambda}\Delta^{\alpha\beta}=1$ and (3.11) implies that

$$[\![x]\!]_{\alpha,\Delta}\le \kappa_1\|A\|_{\alpha,\beta,\lambda}(1+\|x\|_\infty) \tag{3.12}$$

and thus

$$[\![x]\!]_{\alpha}\lesssim \Delta^{\alpha-1}[\![x]\!]_{\alpha,\Delta}\lesssim \|A\|_{\alpha,\beta,\lambda}^{\frac{1-\alpha}{\alpha\beta}}\|A\|_{\alpha,\beta,\lambda}(1+\|x\|_\infty).$$

Therefore

$$[\![x]\!]_{\alpha}\le \kappa_2\|A\|_{\alpha,\beta,\lambda}^{1+\frac{1-\alpha}{\alpha\beta}}(1+\|x\|_\infty)$$

where again $\kappa_2=\kappa_2(\alpha,\beta,T)$. In particular, in order to obtain the final estimate, we only need to focus on $\|x\|_\infty$. Let us consider, for Δ as above, the intervals $I_n:=[(n-1)\Delta,n\Delta]$ and set $J_n:=1+\|x\|_{\infty;I_n}$, with the convention $J_0=1+\|x_0\|_V$. Then estimates analogue to (3.11) yield

$$\begin{aligned} J_n &\le 1+\|x_{(n-1)\Delta}\|_V+\Delta^\alpha[\![x]\!]_{\alpha;I_n}\\ &\le (1+\kappa_1\Delta^\alpha\|A\|_{\alpha,\beta,\lambda})(1+\|x_{(n-1)\Delta}\|_V)\\ &\le (1+\kappa_1\Delta^\alpha\|A\|_{\alpha,\beta,\lambda})J_{n-1}\end{aligned}$$

which iteratively implies

$$J_n\le[1+\kappa_1\Delta^\alpha\|A\|_{\alpha,\beta,\lambda}]^nJ_0\le\exp(\kappa_1 n\Delta^\alpha\|A\|_{\alpha,\beta,\lambda})(1+\|x_0\|_V),$$

where we used the basic inequality $1+x\le e^x$. Since $[0,T]$ is covered by $N\sim T\Delta^{-1}$ intervals and we chose $\Delta^{-1}\sim\|A\|^{1/\alpha\beta}$, up to relabelling κ_1 into a new constant κ_3 we obtain

$$1+\|x\|_\infty=\sup_{n\le N}J_n\le\exp\left(\kappa_3\|A\|_{\alpha,\beta,\lambda}^{1+\frac{1-\alpha}{\alpha\beta}}\right)(1+\|x_0\|_V).$$

Finally, combining this with the estimate for $[\![x]\!]_\alpha$ above we obtain

$$\begin{aligned}[\![x]\!]_\alpha &\le \kappa_2\|A\|_{\alpha,\beta,\lambda}^{1+\frac{1-\alpha}{\alpha\beta}}\exp\left(\kappa_3\|A\|_{\alpha,\beta,\lambda}^{1+\frac{1-\alpha}{\alpha\beta}}\right)(1+\|x_0\|_V)\\ &\le \kappa_4\exp\left(\kappa_4\|A\|_{\alpha,\beta,\lambda}^{1+\frac{1-\alpha}{\alpha\beta}}\right)(1+\|x_0\|_V)\end{aligned}$$

where we used the inequality $xe^{\lambda x}\le\lambda^{-1}e^{2\lambda x}$. The conclusion follows. □

Remark 3.8 Since $\alpha(1+\beta)>1$, it holds $1+\|A\|_{\alpha,\beta,\lambda}^{1+(1-\alpha)/(\alpha\beta)}\lesssim 1+\|A\|_{\alpha,\beta,\lambda}^2$ and so

$$\|x\|_\alpha\le C\exp(C\|A\|_{\alpha,\beta,\lambda}^2)(1+\|x_0\|_V) \tag{3.13}$$

up to possibly changing constant $C=C(\alpha,\beta,T)$.

The dependence of C on T can be established by a rescaling argument: if x is a solution on $[0,T]$ to the YDE associated to (x_0,A), then $x_t=\tilde{x}_{t/T}$ where $\tilde{x}$ is a solution on $[0,1]$ to the YDE associated to $(x_0,\tilde{A})$, $\tilde{A}(t,z)=A(Tt,z)$. Therefore one can apply the estimates to $\tilde{x}$, $\tilde{A}$ and $T = 1$ and then write explicitly how $\|x\|_\alpha$, $\|A\|_{\alpha,\beta,\lambda}$ depend on $\|\tilde{x}\|_\alpha$, $\|\tilde{A}\|_{\alpha,\beta,\lambda}$. The same reasoning applies to several other estimates appearing later on, for which the dependence of C on T is not made explicit.

In classical ODEs, a key role in establishing a priori estimates (as well as uniqueness) is played by Gronwall's lemma; the following result can be regarded as a suitable replacement in the Young setting. One of the main cases of applicability is for $A \in C^\alpha_t L(V; V)$.

Theorem 3.9 *Let $\alpha > 1/2$, $A \in C^\alpha_t \mathrm{Lip}_V$ such that $A(t, 0) = 0$ for all $t \in [0, T]$ and $h \in C^\alpha_t V$. Then there exists a constant $C = C(\alpha)$ such that any solution x to the YDE*

$$x_t = x_0 + \int_0^t A(\mathrm{d}s, x_s) + h_t \tag{3.14}$$

satisfies the a priori bounds

$$\llbracket x \rrbracket_\alpha \le C(\llbracket A \rrbracket_{\alpha,1} \|x\|_\infty + \llbracket h \rrbracket_\alpha); \tag{3.15}$$

$$\|x\|_\infty \le C \exp(CT \llbracket A \rrbracket_{\alpha,1}^{1/\alpha})(\|x_0 + h_0\|_V + T^\alpha \llbracket h \rrbracket_\alpha); \tag{3.16}$$

$$\|x\|_\alpha \le C \exp(CT(1 + \llbracket A \rrbracket_{\alpha,1}^2))[\|x_0 + h_0\|_V + (1 + T^\alpha)\llbracket h \rrbracket_\alpha]. \tag{3.17}$$

Proof We can assume without loss of generality that $T = 1$, as the general case follows by rescaling. It is also clear that, up to changing constant C, inequality (3.17) follows from combining together (3.15) and (3.16) and using the fact that $\llbracket A \rrbracket_{\alpha,1}^{1/\alpha} \lesssim 1 + \llbracket A \rrbracket_{\alpha,1}^2$ since $\alpha > 1/2$. Up to renaming x_0, we can also assume $h_0 = 0$. The proof is similar to that of Proposition 3.7, but we provide it for the sake of completeness.

Let $\Delta > 0$ to be chosen later, $s < t$ such that $|t - s| \le \Delta$, then by (3.14) it holds

$$\begin{aligned}
\|x_{s,t}\|_V &\le \left\| \int_s^t A(\mathrm{d}u, x_u) \right\|_V + \|h_{s,t}\|_V \\
&\le \|A_{s,t}(x_s)\|_V + \kappa_1 |t - s|^{2\alpha} \llbracket A \rrbracket_{\alpha,1} \llbracket x \rrbracket_{\alpha,\Delta} + |t - s|^\alpha \llbracket h \rrbracket_\alpha \\
&\le |t - s|^\alpha (\llbracket A \rrbracket_{\alpha,1} \|x\|_\infty + \llbracket h \rrbracket_\alpha + \kappa_1 \Delta^\alpha \llbracket A \rrbracket_{\alpha,1} \llbracket x \rrbracket_{\alpha,\Delta})
\end{aligned}$$

and so dividing both sides by $|t - s|^\alpha$, taking the supremum over s, t and choosing Δ such that $\kappa_1 \Delta^\alpha \llbracket A \rrbracket_{\alpha,1} \le 1/2$ we obtain

$$\llbracket x \rrbracket_{\alpha,\Delta} \le 2(\llbracket A \rrbracket_{\alpha,1} \|x\|_\infty + \llbracket h \rrbracket_\alpha). \tag{3.18}$$

As usual, if $\kappa_1 \llbracket A \rrbracket_{\alpha,1} \le 1/2$, then the conclusion follows from (3.18) with the choice $\Delta = 1$ and the trivial estimate $\|x\|_\infty \le \|x_0\|_V + \llbracket x \rrbracket_\alpha$. Suppose instead the opposite, choose $\Delta < 1$ such that $\kappa_1 \Delta^\alpha \llbracket A \rrbracket_{\alpha,1} = 1/2$; define $I_n = [(n - 1)\Delta, n\Delta]$, $J_n = \|x\|_{\infty;I_n}$, then estimates similar to the ones done above show that

$$\begin{aligned}
J_{n+1} &\le \|x_{n\Delta}\|_V + \Delta^\alpha \llbracket x \rrbracket_{\alpha;I_n} \\
&\le \|x_{n\Delta}\|_V (1 + 2\Delta^\alpha \llbracket A \rrbracket_{\alpha,1}) + 2\llbracket h \rrbracket_\alpha \\
&\lesssim J_n + \llbracket h \rrbracket_\alpha
\end{aligned}$$

which implies recursively that for a suitable constant κ_2 it holds $J_n \lesssim e^{\kappa_2 n}(\|x_0\|_V + \llbracket h \rrbracket_\alpha)$. Since $n \sim \Delta^{-1} \sim \llbracket A \rrbracket_{\alpha,1}^{1/\alpha}$ we deduce that

$$\|x\|_\infty = \sup_n J_n \lesssim \exp(\kappa_3 \llbracket A \rrbracket_{\alpha,1}^{1/\alpha})(\|x_0\|_V + \llbracket h \rrbracket_\alpha)$$

which gives (3.16); combined with $\Delta^{-\alpha} \sim [\![A]\!]_{\alpha,1}$, estimate (3.18) and the basic inequality

$$[\![x]\!]_\alpha \lesssim \Delta^{-\alpha}\|x\|_\infty + [\![x]\!]_{\alpha,\Delta}$$

it also yields estimate (3.15). □

Another way to establish that solutions don't blow-up in finite time is to the show that the YDE admits (coercive) invariants. The next lemma gives simple conditions to establish their existence.

Lemma 3.10 *Let $A \in C^\alpha_t C^\beta_V$ with $\alpha(1+\beta) > 1$, $x \in C^\alpha_t V$ be a solution to the YDE associated to (x_0, A) and assume $F \in C^2(V;\mathbb{R})$ is such that*

$$DF(z)(A_{s,t}(z)) = 0 \quad \forall z \in V, 0 \le s \le t \le T.$$

Then F is constant along x, i.e. $F(x_t) = F(x_0)$ for all $t \in [0,T]$.

Proof It follows immediately from the Itô-type formula (2.16), since it holds

$$F(x_t) - F(x_0) = \int_0^t DF(x_s)(A(\mathrm{d}s, x_s)) = \mathcal{J}(\Gamma)$$

for the choice $\Gamma_{s,t} = DF(x_s)(A_{s,t}(x_s)) \equiv 0$ by hypothesis. □

Remark 3.11 If V is an Hilbert space with $\|z\|_V^2 = \langle z, z\rangle_V$, then $\|\cdot\|_V$ is constant along solutions of the YDE under the condition $\langle z, A_{s,t}(z)\rangle_V = 0$ for all $z \in V$ and $s \le t$. In this case, blow up cannot occurr, thus under the hypothesis of Corollary 3.5, global existence of solutions holds. Similarly, if in addition $A \in C^\alpha_t C^{1+\beta}_{V,\mathrm{loc}}$, then by Corollary 3.13 below, global existence and uniqueness holds.

3.3 Uniqueness

We now turn to sufficient conditions for uniqueness of solutions; some of the results below also establish existence under different sets of assumptions than those from Sect. 3.1.

Theorem 3.12 *Let $A \in C^\alpha_t C^{1+\beta}_V$, $\alpha(1+\beta) > 1$. Then for any $x_0 \in V$ there exists a unique global solution to the YDE associated to (x_0, A).*

Proof The proof is based on an application of Banach fixed point theorem. Let M, τ be positive parameters to be fixed later and set

$$E := \left\{x \in C^\alpha([0,\tau]; V) \,:\, x(0) = x_0,\ [\![x]\!]_\alpha \le M\right\},$$

which is complete metric space with the metric $d(x,y) = [\![x-y]\!]_\alpha$; define the map $\mathcal{I}$ by

$$x \mapsto \mathcal{I}(x)_\cdot = x_0 + \int_0^\cdot A(\mathrm{d}s, x_s).$$

We want to show that $\mathcal{I}$ is a contraction from E to itself, for suitable choice of M and τ. It holds

$$\begin{aligned}\|\mathcal{I}(x)_{s,t}\|_V &\le \|A_{s,t}(x_s)\|_V + \kappa_1 [\![A]\!]_{\alpha,1}[\![x]\!]_\alpha |t-s|^{2\alpha}\\ &\le \|A_{s,t}(x_s) - A_{s,t}(x_0)\|_V + \|A_{s,t}(x_0)\|_V + \kappa_1[\![A]\!]_{\alpha,1}[\![x]\!]_\alpha|t-s|^{2\alpha}\\ &\le \|A\|_{\alpha,1}[\![x]\!]_\alpha s^\alpha|t-s|^\alpha + \|A\|_{\alpha,1}|t-s|^\alpha + \kappa_1[\![A]\!]_{\alpha,1}[\![x]\!]_\alpha|t-s|^{2\alpha}\\ &\le \tau^\alpha(1+\kappa_1)\|A\|_{\alpha,1}[\![x]\!]_\alpha|t-s|^\alpha + \|A\|_{\alpha,1}|t-s|^\alpha.\end{aligned}$$

Choosing τ and M such that

$$\tau^\alpha(1+\kappa_1)\|A\|_{\alpha,1} \le \frac{1}{2}, \quad M \ge 2\|A\|_{\alpha,1},$$

for any $x \in V$ it holds

$$\|\mathcal{I}(x)\|_\alpha \le \tau^\alpha \|A\|_{\alpha,1}(1+\kappa_1)[\![x]\!]_\alpha + \|A\|_{\alpha,1} \le M/2 + M/2 \le M$$

which shows that $\mathcal{I}$ maps E into itself.

By the hypothesis and Corollary 2.12, for any $x, y \in V$ it holds

$$\begin{aligned}
\|\mathcal{I}(x)_{s,t} - \mathcal{I}(y)_{s,t}\|_V &= \left\| \int_s^t v_{\mathrm{d}u}(x_u - y_u) \right\|_V \\
&\le \|v_{s,t}(x_s - y_s)\|_V + \kappa_1 [\![v]\!]_\alpha [\![x-y]\!]_\alpha |t-s|^{2\alpha} \\
&\le [\![v]\!]_\alpha [\![x-y]\!]_\alpha (s^\alpha + \kappa_1|t-s|^\alpha)|t-s|^\alpha \\
&\le \kappa_2\|A\|_{\alpha,1+\beta}(1+[\![x]\!]_\alpha + [\![y]\!]_\alpha)[\![x-y]\!]_\alpha \tau^\alpha |t-s|^\alpha,
\end{aligned}$$

which implies

$$[\![\mathcal{I}(x) - \mathcal{I}(y)]\!]_\alpha \le \kappa_2\|A\|_{\alpha,1+\beta}(1+2M)\tau^\alpha [\![x-y]\!]_\alpha < [\![x-y]\!]_\alpha$$

as soon as we choose τ such that $\kappa_2\|A\|_{\alpha,1+\beta}(1+2M)\tau^\alpha < 1$. Therefore in this case $\mathcal{I}$ is a contraction from E to itself; for any $x_0 \in V$ there exists a unique solution $x \in C^\alpha([0,\tau]; V)$ starting from x_0. The same procedure allows to show existence and uniqueness of solutions $x \in C^\alpha([s, s+\tau] \cap [0,T]; V)$ for any $s \in [0,T]$ and any $x_s \in V$, where τ does not depend on (s, x_s); by iteration, global existence and uniqueness follows. □

Corollary 3.13 *Let $A \in C^\alpha_t C^{1+\beta}_{V,\mathrm{loc}}$, $\alpha(1+\beta) > 1$. Then for any $x_0 \in V$ there exists a unique maximal solution x to the YDE associated to (x_0, A), defined on $[0, T^*) \subset [0,T]$, such that either $T^* = T$ or*

$$\lim_{t\to T^*} \|x_t\|_V = +\infty.$$

In particular if $A \in C^\alpha_t C^{\beta,\lambda}_V \cap C^\alpha_t C^{1+\beta}_{V,\mathrm{loc}}$ with $\alpha(1+\beta) > 1$, $\beta+\lambda \le 1$, then global existence and uniqueness holds.

Proof We only sketch the proof, as it follows from classical ODE arguments and is similar to that of Corollary 3.5.

By localization, given any $s \in [0,T)$ and any $x_s \in V$, there exists $\tau = \tau(s, x_s)$ such that there exists a unique solution to the YDE associated to (x_s, A) on the interval $[s, s+\tau]$. Therefore given two solutions x^i defined on intervals $[s, T_i]$ with $x^1_s = x^2_s$, they must coincide on $[s, T_1 \wedge T_2]$; in particular, any extension procedure of a given solution to a larger interval is consistent, which allows to define the maximal solution as the maximal extension of any solution starting from x_0 at $t = 0$.

The blow-up alternative can be established reasoning by contradiction as in Corollary 3.5. If $A \in C^\alpha_t C^{\beta,\lambda}_V$, then by the a priori estimate (3.10) blow-up cannot occur and so global well-posedness follows. □

Once existence of solutions is established, their uniqueness can be alternatively shows by means of a Comparison Principle, which is the analogue of a Gronwall type estimate for classical ODEs. Such results are of independent interest as they also allow to compare solutions to different YDEs; they were first introduced in [9] and later revisited in [20].

Theorem 3.14 *Let $R, M > 0$ fixed. For $i = 1, 2$, let $x_0^i \in V$ such that $\|x_0^i\|_V \le R$, $A^i \in C_t^\alpha C_V^{\beta,\lambda}$ with $\alpha(1+\beta) > 1$, $\beta+\lambda \le 1$ and $\|A^i\|_{\alpha,\beta,\lambda} \le M$, as well as $A^1 \in C_t^\alpha C_V^{1+\beta,\lambda}$ with $\|A^1\|_{\alpha,1+\beta,\lambda} \le M$; let x^i be two given solutions associated respectively to (x_0^i, A^i). Then it holds*

$$[\![x^1 - x^2]\!]_\alpha \le C(\|x_0^1 - x_0^2\|_V + \|A^1 - A^2\|_{\alpha,\beta,\lambda})$$

for a constant $C = C(\alpha, \beta, T, R, M)$ increasing in the last two variables.

Proof Let x^i be the two given solutions and set $e_t := x_t^1 - x_t^2$, then e satisfies

$$\begin{aligned} e_t &= e_0 + \int_0^t A^1(\mathrm{d}s, x_s^1) - \int_0^t A^2(\mathrm{d}s, x_s^2) \\ &= e_0 + \int_0^t A^1(\mathrm{d}s, x_s^1) - \int_0^t A^1(\mathrm{d}s, x_s^2) + \int_0^t (A^1 - A^2)(\mathrm{d}s, x_s^2) \\ &= e_0 + \int_0^t v_{\mathrm{d}s}(e_s) + \psi_t \end{aligned}$$

for the choice

$$v_t := \int_0^t \int_0^1 DA^1(\mathrm{d}s, x_s^2 + \lambda(x_s^1 - x_s^2))\mathrm{d}\lambda, \quad \psi_t := \int_0^t (A^1 - A^2)(\mathrm{d}s, x_s^2)$$

where we applied Corollary 2.12. By the same result, combined with estimate (3.13), it holds

$$\begin{aligned} [\![v]\!]_{\alpha,1} &\le \kappa_1 \|DA^1\|_{\alpha,\beta,\lambda}(1 + \|x^1\|_\alpha + \|x^2\|_\alpha) \\ &\le \kappa_2 \exp(\kappa_2(\|A^1\|^2_{\alpha,1+\beta,\lambda} + \|A^2\|^2_{\alpha,\beta,\lambda}))(1+R) \\ &\le \kappa_2 \exp(2\kappa_2 M^2)(1+R); \end{aligned}$$

similarly, by Point 4. of Theorem 2.7,

$$\begin{aligned} [\![\psi]\!]_\alpha &\le \kappa_3 \|A^1 - A^2\|_{\alpha,\beta,\lambda}(1 + \|x^2\|_\infty^\lambda)(1 + [\![x^2]\!]_\alpha) \\ &\le \kappa_4 \|A^1 - A^2\|_{\alpha,\beta,\lambda} \exp(\kappa_4(1 + M^2))(1+R). \end{aligned}$$

Applying Theorem 3.9 to e, we have

$$[\![x^1 - x^2]\!]_\alpha \le \kappa_5 e^{\kappa_5 [\![v]\!]^2_{\alpha,1}}(\|x_0^1 - x_0^2\|_V + [\![\psi]\!]_\alpha)$$

which combined with the previous estimates implies the conclusion. □

Remark 3.15 If $A \in C_t^\alpha C_V^{1+\beta}$ and we consider solutions x^i associated to (x_0^i, A), going through the same proof but applying instead estimate (3.9), we obtain

$$[\![v]\!]_{\alpha,1} \lesssim \|DA\|_{\alpha,\beta}(1 + \|x^1\|_\alpha + \|x^2\|_\alpha) \lesssim 1 + \|A\|^3_{\alpha,1+\beta}$$

which combined with (3.17) implies the existence of a constant $C = C(\alpha, \beta, T)$ such that

$$[\![x^1 - x^2]\!]_\alpha \le C \exp(C\|A\|^6_{\alpha,1+\beta})\, \|x_0^1 - x_0^2\|_V. \tag{3.19}$$

As a consequence, the solution map $F[A] : x_0 \mapsto x$ associated to A, seen as a map from V to $C_t^\alpha V$, is globally Lipschitz. Similar estimates show that, if $\{A_n\}_n$ is a sequence such that $A_n \to A$ in $C_t^\alpha C_V^{1+\beta}$, then $F[A_n] \to F[A]$ uniformly on bounded sets.

As a corollary, we obtain convergence of the Euler scheme introduced in Sect. 3.1, with rate $2\alpha - 1$. For simplicity we state the result in the case $A \in C_t^\alpha C_V^{1+\beta}$, but the same results follow for $A \in C_t^\alpha C_V^{1+\beta,\lambda}$ by the usual localization procedure.

Corollary 3.16 *Given $A \in C_t^\alpha C_V^{1+\beta}$ with $\alpha(1+\beta) > 1$ and $x_0 \in V$, denote by x^n the element of $C_t^\alpha V$ constructed by the n-step Euler approximation from* Theorem 3.2, *and by x the unique solution associated to (x_0, A). Then there exists a constant $C = C(\alpha, \beta, T)$ such that*

$$\|x - x^n\|_\alpha \le C \exp(C\|A\|_{\alpha,1+\beta}^6) n^{1-2\alpha} \quad \text{as } n \to \infty.$$

Proof Recall that by Theorem 3.2, x^n satisfies the YDE

$$x_t^n = x_0 + \int_0^t A(\mathrm{d}s, x_s^n) + \psi_t^n,$$

where by Remark 3.4, for the choice $\beta = 1$, it holds

$$[\![\psi^n]\!] \lesssim (1 + \|A\|_{\alpha,1}^{1+1/\alpha}) n^{1-2\alpha}.$$

Define $e^n := x - x^n$, then by Corollary 2.12 it satisfies

$$e_t^n = \int_0^t A(\mathrm{d}s, x_s^n) - A(\mathrm{d}s, x_s) + \psi_t^n = \int_0^t v_{\mathrm{d}s}^n(e_s^n) + \psi_t^n$$

where again by Remark 3.4 it holds

$$[\![v^n]\!]_{\alpha,1} \lesssim \|A\|_{\alpha,1+\beta}(1 + [\![x]\!]_\alpha + [\![x^n]\!]_\alpha) \lesssim 1 + \|A\|_{\alpha,1+\beta}^3.$$

Applying Theorem 3.9, we deduce the existence of $\kappa_1 = \kappa_1(\alpha, \beta, T)$ such that

$$\|e^n\|_\alpha \le \kappa_1 \exp(\kappa_1 \|A\|_{\alpha,1+\beta}^6)[\![\psi^n]\!]_\alpha,$$

which combined with the estimate for $[\![\psi^n]\!]_\alpha$ yields the conclusion. □

3.4 The Case of Continuous $\partial_t A$

In this section we study how the well-posedness theory changes when, in addition to the regularity condition $A \in C_t^\alpha C_t^\beta$, we impose $\partial_t A : [0, T] \times V \to V$ to exist continuous and uniformly bounded (we assume boundedness for simplicity, but it could be replaced by a growth condition).

The key point is that, by Point 2. from Theorem 2.7, any solution to the YDE is also a solution to the classical ODE associated to $\partial_t A$; as such, it is Lipschitz continuous with constant $\|\partial_t A\|_\infty$. We can exploit this additional time regularity, combined with nonlinear Young theory, to obtain well-posedness under weaker conditions than those from Theorem 3.12.

While the existence of $\partial_t A$ is not a very meaningful requirement for classical YDEs, i.e. for $A(t, x) = f(x) y_t$, as it would imply that $y \in C_t^1$, there are other situations in which it becomes a natural assumption. One example is for perturbed ODEs $\dot{x} = b(x) + \dot{w}$, in which the associated A is the averaged field

$$A(t, x) = \int_0^t b(s, x + w_s)\mathrm{d}s$$

for which $\partial_t A$ exists continuous as soon as b is continuous field; still classical wellposedness is not is not guaranteed under the sole continuity of b.

Theorem 3.17 *Let A be such that $A \in C_t^\alpha C_V^{1+\beta}$ and $\partial_t A \in C_b([0,T]\times V; V)$ with $\alpha+\beta > 1$. Then for any $x_0 \in V$ there exists a unique global solution to the YDE associated to (x_0, A).*

Proof Similarly to Theorem 3.12, the proof is by Banach fixed point theorem. For suitable values of $M, \tau > 0$ to be fixed later, consider the space $E := \{x \in \mathrm{Lip}([0,\tau]; V) : x(0) = x_0, [\![x]\!]_{\mathrm{Lip}} \le M\}$; it is a complete metric space with the metric $d(x,y) = [\![x-y]\!]_\gamma$ (the condition $[\![x]\!]_{\mathrm{Lip}} \le M$ is essential for this to be true). Define the map $\mathcal{I}$ by

$$\mathcal{I}(x)_t = x_0 + \int_0^t \partial_t A(s, x_s)\mathrm{d}s = x_0 + \int_0^t A(\mathrm{d}s, x_s)$$

and observe that under the condition $\|\partial_t A\|_\infty \le M$ it maps E into itself. By the hypothesis and Corollary 2.12, for any $x, y \in E$ it holds

$$\begin{aligned}
\|\mathcal{I}(x)_{s,t} - \mathcal{I}(y)_{s,t}\|_V &= \left\| \int_s^t v_{\mathrm{d}u}(x_u - y_u) \right\|_V \\
&\le \|v_{s,t}(x_s - y_s)\|_V + \kappa_1 [\![v]\!]_\alpha [\![x-y]\!]_{\mathrm{Lip}} |t-s|^{2\alpha} \\
&\le [\![v]\!]_\alpha [\![x-y]\!]_\alpha (s^\alpha + \kappa_1 |t-s|^\alpha)|t-s|^\alpha \\
&\le \kappa_2 \tau^\alpha \|A\|_{\alpha,1+\beta}(1 + [\![x]\!]_{\mathrm{Lip}} + [\![y]\!]_{\mathrm{Lip}})[\![x-y]\!]_\alpha |t-s|^\alpha
\end{aligned}$$

which implies

$$[\![\mathcal{I}(x) - \mathcal{I}(y)]\!]_\alpha \le \kappa_2 \tau^\alpha \|A\|_{\alpha,1+\beta}(1+2M)[\![x-y]\!]_\alpha < [\![x-y]\!]_\alpha$$

as soon as we choose τ small enough such that $\kappa_2 \tau^\alpha \|A\|_{\alpha,1+\beta}(1+2M) < 1$. Therefore $\mathcal{I}$ is a contraction on E and for any $x_0 \in V$ there exists a unique associated solution $x \in C^\gamma([0,\tau]; V)$. Global existence and uniqueness then follows from the usual iterative argument. □

We can also establish an analogue of Theorem 3.14 in this setting.

Theorem 3.18 *Let $M > 0$ fixed. For $i = 1, 2$, let $A^i \in C_t^\alpha C_V^\beta$ such that $\partial_t A^i \in C^0([0,T] \times V; V)$, $\alpha + \beta > 1$ and $\|A^i\|_{\alpha,\beta} + \|\partial_t A\|_\infty \le M$, as well as $A^1 \in C_t^\alpha C_V^{1+\beta}$ with $\|A^1\|_{\alpha,1+\beta} \le M$, and $x_0^i \in V$; let x^i be two given solutions associated respectively to (x_0^i, A^i). Then it holds*

$$[\![x^1 - x^2]\!]_\alpha \le C(\|x_0^1 - x_0^2\|_V + \|A^1 - A^2\|_{\alpha,\beta})$$

for a constant $C = C(\alpha, \beta, T, M)$ increasing in the last variable. A more explicit formula for C is given by (3.20).

Proof The proof is analogous to that of Theorem 3.14, so we will mostly sketch it; it is based on an application of Corollary 2.12 and Theorem 3.9.

Given two solutions as above, their difference $e = x^1 - x^2$ satisfies the affine YDE

$$e_t = e_0 + \int_0^t v_{\mathrm{d}s} e_s + \psi_t$$

with

$$v_t = \int_0^t \int_0^1 DA^1(\mathrm{d}s, x_s^2 + \lambda e_s)\mathrm{d}\lambda, \quad \psi_t = \int_0^t (A^1 - A^2)(\mathrm{d}s, x_s^2).$$

We have the estimates

$$\|v\|_{\alpha,1} \lesssim_{\alpha,\beta,T} \|A^1\|_{\alpha,1+\beta}(1+[\![x^1]\!]_{\mathrm{Lip}}+[\![x^2]\!]_{\mathrm{Lip}}) \lesssim \|A^1\|_{\alpha,1+\beta}(1+\|\partial_t A^1\|_\infty+\|\partial_t A^2\|_\infty)$$
$$\|\psi_t\|_\alpha \lesssim_{\alpha,\beta,T} \|A^1-A^2\|_{\alpha,\beta}(1+[\![x^2]\!]_{\mathrm{Lip}}) \lesssim \|A^1-A^2\|_{\alpha,\beta}(1+\|\partial_t A^2\|_\infty)$$

which, combined with Theorem 3.9, yield

$$\begin{aligned}\|e\|_\alpha &\le \kappa_1 e^{\kappa_1(1+\|A^1\|^2_{\alpha,1+\beta})(1+\|\partial_t A^1\|^2_\infty+\|\partial_t A^2\|^2_\infty)}(\|e_0\|_V+\|A^1-A^2\|_{\alpha,\beta}(1+\|\partial_t A^2\|_\infty))\\ &\le \kappa_2 e^{\kappa_2(1+\|A^1\|^2_{\alpha,1+\beta})(1+\|\partial_t A^1\|^2_\infty+\|\partial_t A^2\|^2_\infty)}(\|e_0\|_V+\|A^1-A^2\|_{\alpha,\beta})\end{aligned}$$

for some $\kappa_2=\kappa_2(\alpha,\beta,T)$. In particular, C can be taken of the form

$$C(\alpha,\beta,T,M)=\kappa_3(\alpha,\beta,T)\exp(\kappa_3(\alpha,\beta,T)(1+M^4)). \tag{3.20}$$

□

Corollary 3.19 *Given A as in Theorem 3.17, denote by x^n the element of $C^\alpha_t V$ constructed by the n-step Euler approximation from Theorem 3.2 and by x the solution associated to (x_0,A). Then there exists a constant $C=C(\alpha,\beta,T,\|A\|_{\alpha,1+\beta},\|\partial_t A\|_\infty)$ such that*

$$\|x-x^n\|_\alpha \le Cn^{-\alpha} \quad \text{as } n\to\infty.$$

A more explicit formula for C is given by (3.21).

Proof By Theorem 3.2, x^n satisfies the YDE

$$x^n = x_0+\int_0^t A(\mathrm{d}s,x^n_s)+\psi^n_t = x_0+\int_0^t A^n(\mathrm{d}s,x^n_s)$$

where $A^n(t,z):=A(t,z)+\psi^n_t$ and that by estimate (3.7), for the choice $\Delta=T$, $\beta=\gamma=1$, we have

$$[\![\psi^n]\!]_\alpha \lesssim_{\alpha,T} \|A\|_{\alpha,1}[\![x^n]\!]_{\mathrm{Lip}} n^{-\alpha} \lesssim \|A\|_{\alpha,1}\|\partial_t A\|_\infty n^{-\alpha}.$$

Defining $e^n:=x-x^n$, by the basic estimates $\|A-A^n\|_{\alpha,\beta}\lesssim_T [\![\psi^n]\!]_\alpha$ and $\|\partial_t A^n\|_\infty \lesssim \|\partial_t A\|_\infty$, going through the same proof as in Theorem 3.18 we deduce that

$$\|e^n\|_\alpha \le \kappa_1 e^{\kappa_1(1+\|A\|^2_{\alpha,1})(1+\|\partial_t A\|^2_\infty)}\|A-A^n\|_{\alpha,\beta}$$

and so finally that, for a suitable constant $\kappa_2=\kappa_2(\alpha,T)$, it holds

$$\|e^n\|_\alpha \le \kappa_2\exp(\kappa_2(1+\|A\|^2_{\alpha,1})(1+\|\partial_t A\|^2_\infty))n^{-\alpha}. \tag{3.21}$$

□

3.5 Further Variants

Several other kinds of differential equations involving a nonlinear Young integral term can be studied. In this section we focus on two cases: nonlinear YDEs involving a classical drift term and fractional YDEs.

3.5.1 Mixed Equations

Let us consider now an equation of the form

$$x_t = x_0 + \int_0^t F(s, x_s)\mathrm{d}s + \int_0^t A(\mathrm{d}s, x_s). \tag{3.22}$$

where $F : [0, T] \times V \to V$ is continuous function; the first integral is meaningful as a classical one.

Proposition 3.20 *Let $A \in C^\alpha_t C^{1+\beta}_V$ with $\alpha(1+\beta) > 1$, F be bounded and globally Lipschitz, namely*

$$\|F(t, y)\|_V \le C_F, \qquad \|F(t, y) - F(t, z)\|_V \le C_F\|y - z\|_V \qquad \textit{for all } t \in [0, T],\ y, z \in V$$

for some constant $C_F > 0$. Then global well-posedness holds for (3.22).

Proof For simplicity we will use the notation $\|A\| = \|A\|_{\alpha,1+\beta}$; the proof is analogue to that of Theorem 3.12. Let M, τ be positive parameters to be fixed later and define as usual

$$E = \left\{x \in C^\alpha([0, \tau]; V) : x(0) = x_0,\ [\![x]\!]_\alpha \le M\right\}.$$

A path x solves (3.22) if and only if it belongs to E and is a fixed point for the map

$$x \mapsto \mathcal{I}(x)_\cdot = x_0 + \int_0^\cdot F(s, x_s) + \int_0^\cdot A(\mathrm{d}s, x_s).$$

We have the estimates

$$\begin{aligned}\|\mathcal{I}(x)_{s,t}\|_V &\le \int_s^t \|F(r, x_r)\|_V \mathrm{d}r + \|A_{s,t}(x_s)\|_V + \kappa_1|t-s|^{2\alpha}\|A\|[\![x]\!]_\alpha\\ &\le |t-s|C_F + \|A_{s,t}(x_s) - A_{s,t}(x_0)\|_V + \|A_{s,t}(x_0)\|_V + \kappa_1|t-s|^{2\alpha}\|A\|[\![x]\!]_\alpha\\ &\le |t-s|^\alpha[C_F\tau^{1-\alpha} + \|A\|\tau^\alpha M + \|A\| + \kappa_1\|A\|\tau^\alpha M],\end{aligned}$$

which imply

$$[\![\mathcal{I}(x)]\!]_\alpha \le C_F\tau^{1-\alpha} + \|A\| + [\tau + \|A\|(1+\kappa_1)\tau^\alpha]M.$$

In order for $\mathcal{I}$ to map E into itself, it suffices to choose τ and M such that

$$\tau \le 1, \quad \tau + \|A\|(1+\kappa_1)\tau^\alpha \le 1/2, \quad M \ge 2(C_F + \|A\|).$$

Next we check contractivity of $\mathcal{I}$; given $x, y \in E$, it holds

$$\begin{aligned}\|\mathcal{I}(x)_{s,t} - \mathcal{I}(y)_{s,t}\|_V &\le \int_s^t \|F(r, x_r) - F(r, y_r)\|_V\, \mathrm{d}r + \left\|\int_s^t v_{\mathrm{d}r}(x_r - y_r)\right\|_V\\ &\le C_F|t-s|\tau^\alpha[\![x-y]\!]_\alpha + \|v_{s,t}(x_s - y_s)\|_V + \kappa_2|t-s|^{2\alpha}[\![v]\!]_\alpha[\![x-y]\!]_\alpha\\ &\le \kappa_3\tau^\alpha[C_F + \|A\|(1 + [\![x]\!]_\alpha + [\![y]\!]_\alpha)][\![x-y]\!]_\alpha|t-s|^\alpha\end{aligned}$$

which implies

$$[\![\mathcal{I}(x) - \mathcal{I}(y)]\!]_\alpha \le \kappa_3\tau^\alpha[C_F + \|A\|(1+2M)]$$

thus choosing τ small enough we deduce contractivity. Therefore existence and uniqueness of solutions holds on the interval $[0, \tau]$; as the choice of τ does not depend on x_0, we can iterate the reasoning to cover the whole interval $[0, T]$. ◻

Theorem 3.21 *Let $A \in C^\alpha_t C^{1+\beta}_{V,\mathrm{loc}}$ with $\alpha(1+\beta) > 1$ and F be a continuous locally Lipschitz function, in the sense that for any $R > 0$ there exist a constant C_R such that*

$$\|F(t,y) - F(t,z)\|_V \le C_R \|y-z\|_V \quad \text{for all } t \in [0,T] \text{ and } y,z \in V \text{ such that } \|y\|_V, \|z\|_V \le R.$$

Then for any $x_0 \in V$ there exists a unique maximal solution x to (3.22), defined on $[0,T^*) \subset [0,T]$ such that either $T = T^*$ or

$$\lim_{t\to T^*} \|x_t\|_V = +\infty.$$

If in addition $A \in C^\alpha_t C^{\beta,\lambda}_V$ with $\beta + \lambda \le 1$ and F has at most linear growth, i.e. there exists $C_F > 0$ s.t.

$$\|F(t,z)\|_V \le C_F(1+\|z\|_V) \quad \forall\, (t,z) \in [0,T] \times V,$$

then global wellposedness holds. Moreover in this case there exists $C = C(\alpha,\beta,T)$ such that, setting $\theta = 1 + \frac{1-\alpha}{\alpha\beta}$, any solution to (3.22) satisfies the a priori estimate

$$\|x\|_\alpha \le C \exp(C(C_F^\theta + \|A\|^\theta_{\alpha,\beta,\lambda}))(1+\|x_0\|_V). \tag{3.23}$$

Proof The first part of the statement, regarding local wellposedness and the blow-up alternative, follows from the usual localisation arguments, so we omit its proof.

The proof of a priori estimate (3.23) is analogue to that of Proposition 3.7, so we will mostly sketch it; as before $\|A\| = \|A\|_{\alpha,\beta,\lambda}$ for simplicity. Let x be a solution to (3.22) defined on $[0,T^*)$, then for any $[r,u] \subset [s,t] \subset [0,T^*)$ it holds

$$\begin{aligned}
\left\| \int_u^r F(a,x_a)\mathrm{d}a \right\|_V &\le C_F|r-u| + C_F \int_u^r \|x_a\|\mathrm{d}a \\
&\le |r-u| C_F(1+\|x_s\|_V) + |r-u||t-s|^\alpha C_F [\![x]\!]_{\alpha;s,t} \\
&\lesssim |r-u|^\alpha C_F[1 + \|x_s\|_V + |t-s|[\![x]\!]_{\alpha;s,t}].
\end{aligned}$$

Together with the estimates from the proof of Proposition 3.7 and the fact that $|t-s| \lesssim |t-s|^{\alpha\beta}$, this implies the existence of $\kappa_1 = \kappa_1(\alpha,\beta,T)$ such that any solution x to (3.22) satisfies

$$[\![x]\!]_{\alpha;s,t} \le \frac{\kappa_1}{2}(C_F + \|A\|)(1+\|x_s\|_V) + \frac{\kappa_1}{2}(C_F + \|A\|)|t-s|^{\alpha\beta}[\![x]\!]_{\alpha;s,t}.$$

The rest of the proof is identical, up to replacing $\|A\|$ with $C_F + \|A\|$ in all the passages. Specifically, if T is such that $\kappa_1(C_F + \|A\|)T^{\alpha\beta} < 2$, then we obtain a global estimate by choosing $s = 0$, $t = T$, which shows that $T^* = T$ and gives the conclusion in this case. Otherwise, taking $\Delta < T$ such that $\kappa_1(C_F + \|A\|)\Delta^{\alpha\beta} = 1$ and defining J_n as before, we obtain the recurrent estimate

$$J_n \le [1 + \kappa_1 \Delta^\alpha (C_F + \|A\|)] J_{n-1}$$

and going through the same reasoning the conclusion follows. □

3.5.2 Fractional Young Equations

We restrict in this subsection to the finite dimensional case $V = \mathbb{R}^d$ for some $d \in \mathbb{N}$; as usual we work on a finite time interval $[0,T]$. We are interested in studying a fractional type of equation of the form

$$D^\delta_{0+} x_t = A(\mathrm{d}t, x_t) \quad \forall\, t \in [0,T] \tag{3.24}$$

for a suitable parameter $\delta \in (0,1)$. Here D^δ_{0+} denotes a Riemann–Liouville type of fractional derivative on $[0,T]$; for more details on fractional derivatives and fractional calculus we refer the reader to [40]. In the case $\delta = 1$, formally $D^\delta x_s = \mathrm{d}x_s$ and we recover the class of YDEs studied so far.

In order to study (3.24), it is more convenient to write it in integral form, using the fact that D^δ_{0+} is the inverse operator of the fractional integral I^δ_{0+} given by

$$(I^\delta_{0+} f)_t = \frac{1}{\Gamma(\delta)} \int_0^\delta (t-s)^{\delta-1} f_s \mathrm{d}s$$

(being interpreted componentwise if $f : [0,T] \to \mathbb{R}^d$). From now on we will for simplicity drop the constant $1/\Gamma(\delta)$, which can be incorporated in the drift A. We need the following lemma.

Lemma 3.22 *For $\delta \in (0,1)$, consider the functional Ξ defined for smooth f by*

$$\Xi[f]_t := (I^\delta_{0+} \dot f)_t = \int_0^t (t-s)^{\delta-1} \dot f_s \mathrm{d}s.$$

For any $\alpha \in (0,1)$ such that $\alpha + \delta > 1$ and any $\varepsilon > 0$, Ξ extends uniquely to a continuous linear map from $C^\alpha([0,T];\mathbb{R}^d)$ to $C^{\alpha+\delta-1-\varepsilon}([0,T];\mathbb{R}^d)$; in particular, there exists $C = C(\alpha,\delta,\varepsilon,T)$, which will be denoted by $\|\Xi\|$, such that

$$\|\Xi[f]\|_{\alpha+\delta-1-\varepsilon} \le \|\Xi\| [\![f]\!]_\alpha \quad \textit{for all } f \in C^\alpha([0,T];\mathbb{R}^d). \tag{3.25}$$

Proof Up to multiplicative constant, $\Xi = I^\alpha_{0+} D$. Recall that fractional integrals and fractional derivatives, on their domain of definition, satisfy the following properties, for $\alpha, \beta, \alpha+\beta \in [0,1]$:

i. $I^\alpha_{0+} \circ I^\beta_{0+} = I^{\alpha+\beta}_{0+}$, $I^0_{0+} = \mathrm{Id}$, similarly for D^α_{0+};
ii. $I^\alpha_{0+} \circ D^\alpha_{0+} = D^\alpha_{0+} \circ I^\alpha_{0+} = \mathrm{Id}$, $D^1_{0+} = D$.

Let f be a smooth function, then $\Xi[f] = I^\delta_{0+} D f = D^{1-\delta}_{0+} f$; moreover for any $\gamma < \alpha$, we can write f as $f = I^\gamma_{0+} \tilde f$ with $\|\tilde f\|_\infty \lesssim \|f\|_\alpha$; choosing $\gamma > 1-\delta$, we obtain $\Xi[f] = I^{\gamma+\delta-1}_{0+} \tilde f$ and so overall $\Xi[f] \in I^{\gamma+\delta-1}_{0+}(L^\infty_t) \hookrightarrow C^{\gamma+\delta-1-\varepsilon}_t$ with

$$\|\Xi[f]\|_{\gamma+\delta-1-\varepsilon} \lesssim \|I^{\gamma+\delta-1}_{0+} \tilde f\|_{I^{\gamma+\delta-1}_{0+}(L^\infty_t)} \lesssim \|\tilde f\|_\infty \lesssim \|f\|_\alpha.$$

The conclusion for general f follows from an approximation procedure. Indeed, since all inequalities are strict, we can replace α with $\alpha - \varepsilon$ and use the fact that functions in C^α_t can be approximated by smooth functions in the $C^{\alpha-\varepsilon}_t$-norm.

The fact that in (3.25) only the seminorm $[\![f]\!]$ appears is a consequence of the fact that by definition $\Xi[1] = 0$ and so we can always shift f in such a way that $f_0 = 0$. □

Remark 3.23 Let us point out two properties of the operator Ξ. The first one is that, if $f \equiv g$ on $[0,\tau]$ with $\tau \le T$, the same holds for $\Xi[f] \equiv \Xi[g]$; in particular, since we can always extend $f \in C^\alpha([0,\tau];\mathbb{R}^d)$ to $C^\alpha([0,T];\mathbb{R}^d)$ by setting $f_t = f_\tau$ for all $t \ge \tau$, we can consider Ξ as an operator from $C^\alpha([0,\tau];\mathbb{R}^d)$ to $C^{\alpha+\delta-1-\varepsilon}([0,\tau];\mathbb{R}^d)$. As long as $\tau \le T$, the operator norm of this restricted functional is still controlled by $\|\Xi\|$.

The second one is that if $h \equiv 0$ on $[0, \tau]$, then $\Xi[h]_{\cdot+\tau} = \Xi[h_{\cdot+\tau}]$. Indeed for h smooth it holds

$$\begin{aligned}\Xi[h]_{t+\tau} &= \int_0^{t+\tau} (t+\tau-s)^{\delta-1}\dot{h}_s \mathrm{d}s = \int_\tau^{t+\tau} (t+\tau-s)^{\delta-1}\dot{h}_s \mathrm{d}s \\ &= \int_0^t (t-s)^{\delta-1}\dot{h}_{s+\tau}\mathrm{d}s = \Xi[h_{\cdot+\tau}]_t.\end{aligned}$$

The general case follows from an approximation procedure.

Thanks to Lemma 3.22 we can give a proper meaning to the fractional YDE.

Definition 3.24 We say that x is a solution to (3.24) if $\int_0^\cdot A(\mathrm{d}s, x_s)$ is well defined as a nonlinear Young integral in C_t^α for some $\alpha > 1-\delta$ and x satisfies the identity

$$x_\cdot = x_0 + \Xi\left[\int_0^\cdot A(\mathrm{d}s, x_s)\right].$$

Proposition 3.25 *Let $A \in C_t^\alpha C_x^\beta$ with $\alpha, \beta \in (0,1)$ satisfying*

$$\alpha + \delta - 1 > \frac{1-\alpha}{\beta}. \tag{3.26}$$

Then for any $x_0 \in \mathbb{R}^d$ and any $\gamma < \alpha + \delta - 1$ there exists a solution $x \in C_t^\gamma$ to (3.24), in the sense of Definition 3.24.

Proof Due to condition (3.26), we can find $\gamma \in (0,1)$, $\varepsilon > 0$ sufficiently small satisfying

$$\alpha + \delta - 1 > \gamma > \gamma - \varepsilon > \frac{1-\alpha}{\beta}.$$

The existence of a solution is then equivalent to the existence of a fixed point in C_t^γ for the map

$$I(x) := x_0 + \Xi\left[\int_0^\cdot A(\mathrm{d}s, x_s)\right].$$

The above conditions imply $\alpha + \beta(\gamma - \varepsilon) > 1$, so by Theorem 2.7 the map $x \mapsto A(\mathrm{d}s, x_s)$, from $C_t^{\gamma-\varepsilon}$ to C_t^α is continuous and satisfies

$$\left[\!\!\left[\int_0^\cdot A(\mathrm{d}s, x_s)\right]\!\!\right]_\alpha \lesssim \|A\|_{\alpha,\beta}(1 + [\![x]\!]_{\gamma-\varepsilon}^\beta)$$

which together with estimate (3.25) implies that I is continuous from $C_t^{\gamma-\varepsilon}$ to C_t^γ with

$$\|I(x)\|_\gamma \le \|x_0\| + \kappa_1 \|\Xi\| \|A\|_{\alpha,\beta}(1 + [\![x]\!]_{\gamma-\varepsilon}^\beta)$$

for suitable $\kappa_1 = \kappa_1(T, \alpha + \beta(\gamma - \varepsilon))$. It follows by Ascoli-Arzelà that I is compact from $C_t^{\gamma-\varepsilon}$ to itself; for any $\lambda \in (0,1)$, if x solves $x = \lambda I(x)$, then

$$\|x\|_{\gamma-\varepsilon} \le \|x\|_\gamma = \lambda \|T(x)\|_\gamma \le \|x_0\| + \kappa_1 \|\Xi\| \|A\|_{\alpha,\beta}(1 + \|x\|_{\gamma-\varepsilon}^\beta).$$

Since $\beta < 1$, any such solution x must satisfy (for instance)

$$\|x\|_{\gamma-\varepsilon} \le \max\left\{2(\|x_0\| + \kappa_1\|\Xi\|\|A\|_{\alpha,\beta}),\ (2\kappa_1\|\Xi\|\|A\|_{\alpha,\beta})^{\frac{1}{1-\beta}}\right\}$$

where the estimate is uniform in $\lambda \in [0,1]$. We can thus apply Schaefer's theorem to deduce the existence of a fixed point for I in $C_t^{\gamma-\varepsilon}$, which also belongs to C_t^γ since $I(x)$ does so. □

Theorem 3.26 *Let $A \in C_t^\alpha C_x^{1+\beta}$ with α, β, δ satisfying (3.26). Then for any $x_0 \in \mathbb{R}^d$ there exists a unique solution $x \in C_t^\gamma$ to (3.24), for any γ satisfying*

$$\alpha + \delta - 1 > \gamma > \frac{1-\alpha}{\beta}.$$

Proof Existence is granted by Proposition 3.25, so we only need to check uniqueness. Let x and y be two solutions, say with $\|x\|_\alpha, \|y\|_\alpha \le M$ for suitable $M > 0$; we are first going to show that they must coincide on an interval $[0, \tau]$ with τ sufficiently small. It holds

$$\begin{aligned} [\![x - y]\!]_{\gamma;0,\tau} &= \left[\!\!\left[\Xi \left[\int_0^\cdot A(\mathrm{d}s, x_s) - \int_0^\cdot A(\mathrm{d}s, y_s) \right] \right]\!\!\right]_{\gamma;0,\tau} \\ &\le \|\Xi\| \left[\!\!\left[\int_0^\cdot A(\mathrm{d}s, x_s) - \int_0^\cdot A(\mathrm{d}s, y_s) \right]\!\!\right]_{\alpha;0,\tau} \\ &= \|\Xi\| \left[\!\!\left[\int_0^\cdot v_{\mathrm{d}s}(x_s - y_s) \right]\!\!\right]_{\alpha;0,\tau} \end{aligned}$$

where v is given by

$$v_t = \int_0^1 \int_0^t \nabla A(\mathrm{d}s, y_s + \lambda(x_s - y_s))\mathrm{d}\lambda$$

and satisfies $\|v\|_{\alpha;0,T} \le \kappa_1 \|A\|_{\alpha,1+\beta}(1 + M)$. Since $x_0 = y_0$, for any $[s, t] \subset [0, \tau]$ it holds

$$\begin{aligned} \left\| \int_s^t v_{\mathrm{d}r}(x_r - y_r) \right\| &\le \|v_{s,t}(x_s - y_s)\| + \kappa_2 |t - s|^{\alpha+\gamma} \|v\|_\alpha [\![x - y]\!]_{\gamma;0,\tau} \\ &\le |t - s|^\alpha \tau^\gamma (1 + \kappa_2) \|v\|_\alpha [\![x - y]\!]_{\gamma;0,\tau}; \end{aligned}$$

combined with the previous estimates we obtain

$$\begin{aligned} [\![x - y]\!]_{\gamma;0,\tau} &\le \|\Xi\| \tau^\gamma (1 + \kappa_2) \|v\|_\alpha [\![x - y]\!]_{\gamma;0,\tau} \\ &\le \kappa_3 \|\Xi\| \|A\|_{\alpha,1+\beta} (1 + M) \tau^\gamma [\![x - y]\!]_{\gamma;0,\tau}. \end{aligned}$$

Choosing τ small enough such that $\kappa_3 \|\Xi\| \|A\|_{\alpha,1+\beta}(1 + M)\tau^\gamma < 1$, we conclude that $x \equiv y$ on $[0, \tau]$.

As a consequence, $\int_0^\cdot A(\mathrm{d}s, x_s) = \int_0^\cdot A(\mathrm{d}s, y_s)$ on $[0, \tau]$ as well; define $v_t = x_{t+\tau} - y_{t+\tau}$, then applying Remark 3.23 to v we obtain

$$\begin{aligned} v_t &= \Xi \left[\int_0^\cdot A(\mathrm{d}s, x_s) - A(\mathrm{d}s, y_s) \right]_{t+\tau} \\ &= \Xi \left[\int_\tau^{\cdot+\tau} A(\mathrm{d}s, x_s) - \int_\tau^{\cdot+\tau} A(\mathrm{d}s, y_s) \right]_t \\ &= \Xi \left[\int_0^\cdot \tilde{A}(\mathrm{d}s, x_{s+\tau}) - \int_0^\cdot \tilde{A}(\mathrm{d}s, y_{s+\tau}) \right]_t \end{aligned}$$

where $\tilde{A}(t, x) = A(t + \tau, x)$ has the same regularity properties of A. We can therefore iterate the previous argument, applied this time to $\tilde{A}$, $x_{\cdot+\tau}$ and $y_{\cdot+\tau}$, to deduce that x and y also coincide on $[\tau, 2\tau]$; repeating this procedure we can cover the whole interval $[0, T]$. □

4 Flow

Having established sufficient conditions for the existence and uniqueness of solutions to the YDE associated to (x_0, A), it is natural to study their dependence on the data of the problem. This section is devoted to the study of the flow, seen as the ensemble of all possible solutions, and its Frechét differentiability w.r.t. both (x_0, A).

In order to avoid technicalities we will only consider the case of $A \in C^\alpha_t C^{1+\beta}_V$ with global bounds, but everything extends easily by localisation arguments to $A \in C^\alpha_t C^{\beta,\lambda}_V \cap C^\alpha_t C^{1+\beta}_{V,\mathrm{loc}}$; similar results can also be established for the type of equations considered respectively in Sects. 3.4 and 3.5.

4.1 Flow of Diffeomorphisms

We start by giving a proper definition of a flow for the YDE associated to A; recall here that Δ_n denotes the n-simplex on $[0, T]$.

Definition 4.1 Given $A \in C^\alpha_t C^\beta_V$ with $\alpha(1+\beta) > 1$, we say that $\Phi : \Delta_2 \times V \to V$ is a flow of homeomorphisms for the YDE associated to A if the following hold:

i. $\Phi(t, t, x) = x$ for all $t \in [0, T]$ and $x \in V$;
ii. $\Phi(s, \cdot, x) \in C^\alpha([s, T]; V)$ for all $s \in [0, T]$ and $x \in V$;
iii. for all $(s, t, x) \in \Delta_2 \times \mathbb{R}^d$ it holds

$$\Phi(s, t, x) = x + \int_s^t A(\mathrm{d}r, \Phi(s, r, x));$$

iv. Φ satisfies the group property, namely

$$\Phi(u, t, \Phi(s, u, x)) = \Phi(s, t, x) \quad \text{for all } (s, u, t) \in \Delta_3 \text{ and } x \in V;$$

v. for any $(s, t) \in \Delta_2$, the map $\Phi(s, t, \cdot)$ is an homeomorphism of V, i.e. it is continuous with continuous inverse.

From now on, whenever talking about a flow Φ, we will use the notation $\Phi_{s\to t}(x) = \Phi(s, t, x)$; we will denote by $\Phi_{s\leftarrow t}(\cdot)$ the inverse of $\Phi_{s\to t}(\cdot)$ as a map from V to itself.

Definition 4.2 Given A as above, $\gamma \in (0, 1)$, we say that it admits a locally γ-Hölder continuous flow Φ, Φ is C^γ_{loc} for short, if for any $(s, t) \in \Delta_2$ it holds $\Phi_{s\to t}, \Phi_{s\leftarrow t} \in C^\gamma_{\mathrm{loc}}(V; V)$; we say that Φ is a flow of diffeomorphisms if $\Phi_{s\to t}, \Phi_{s\leftarrow t} \in C^1_{\mathrm{loc}}(V; V)$ for any $(s, t) \in \Delta_2$. Similar definitions hold for a locally Lipschitz flow, or a $C^{n+\gamma}_{\mathrm{loc}}$-flow with $\gamma \in [0, 1)$ and $n \in \mathbb{N}$.

If $V = \mathbb{R}^d$, we say that Φ is a Lagrangian flow if there exists a constant C such that

$$C^{-1}\lambda_d(E) \le \lambda_d(\Phi_{s\leftarrow t}(E)) \le C\lambda_d(E) \qquad \forall\, E \in \mathcal{B}(\mathbb{R}^d),\ \forall\,(s, t) \in \Delta_2,$$

where λ_d denotes the Lebesgue measure on $\mathbb{R}^d$ and $\mathcal{B}(\mathbb{R}^d)$ the collection of Borel sets.

It follows from Remark 3.15 that, if $A \in C^\alpha_t C^{1+\beta}_V$ with $\alpha(1+\beta) > 1$, then the solution map $(x_0, t) \mapsto x_t$ is Lipschitz in space, uniformly in time. However we cannot yet talk about a flow, as we haven't shown the invertibility of the solution map, nor the flow property; this is accomplished by the following lemma.

Lemma 4.3 *Let $A \in C^\alpha_t C^\beta_V$ and $x \in C^\alpha_t V$ such that $\alpha(1+\beta) > 1$, x be a solution of the YDE associated to (x_0, A). Then setting $\tilde{A}(t,z) := A(T-t,z)$ and $\tilde{x}_t := x_{T-t}$, $\tilde{x}$ is a solution to the time-reversed YDE*

$$\tilde{x}_t = \tilde{x}_0 + \int_0^t \tilde{A}(\mathrm{d}s, \tilde{x}_s).$$

Similarly, setting $\tilde{x}_t = x_{t-s}$, $\tilde{A}(t,x) = A(t-s,x)$ for $t \in [s,T]$, then $\tilde{x}$ is a solution to the time-shifted YDE

$$\tilde{x}_t = \tilde{x}_0 + \int_0^t \tilde{A}(\mathrm{d}r, \tilde{x}_r) \quad \forall t \in [s,T].$$

The proof is elementary but a bit tedious, so we omit it; we refer the interested reader to Lemma 2, Section 6.1 from [33] or Lemmas 11 and 12, Section 4.3.1 from [20].

As a consequence, we immediately deduce conditions for the existence of a Lipschitz flow.

Corollary 4.4 *Let $A \in C^\alpha_t C^{1+\beta}_V$ with $\alpha(1+\beta) > 1$, then the associated YDE admits a locally Lipschitz flow Φ. Moreover there exists $C = C(\alpha, \beta, T, \|A\|_{\alpha,1+\beta})$ such that*

$$\|\Phi_{s\to\cdot}(x) - \Phi_{s\to\cdot}(y)\|_{\alpha;s,T} \le C\|x-y\|_V, \quad [\![\Phi_{s\to\cdot}(x)]\!]_{\alpha;s,T} \le C \quad \forall s \in [0,T],\ x,y \in V \tag{4.1}$$

together with a similar estimate for $\Phi_{\cdot\leftarrow t}(\cdot)$.

Proof The proof is a straightforward application of Remark 3.15 and Lemma 4.3. In both cases of time reversal and translation we have $\|\tilde{A}\|_{\alpha,1+\beta} \le \|A\|_{\alpha,1+\beta}$ so that uniqueness holds also for the reversed/translated YDE, with the same continuity estimates; this provides respectively invertibility of the solution map and flow property. □

Actually, under the same hypothesis it is possible to prove that the YDE admits a flow of diffeomorphisms, which satisfies a variational equation.

Theorem 4.5 *Let $A \in C^\alpha_t C^{1+\beta}_V$ with $\alpha(1+\beta) > 1$, then the YDE associated to A admits a flow of diffeomorphisms. For any $x \in V$, $D_x\Phi_{s\to t}(x) = J^x_{s\to t}$, where $J^x_{s\to\cdot} \in C^\alpha_t \mathcal{L}(V;V)$ is the unique solution to the variational equation*

$$J^x_{s\to t} = I + \int_s^t DA(\mathrm{d}r, \Phi_{s\to r}(x)) \circ J^x_{s\to r} \qquad \forall t \in [s,T] \tag{4.2}$$

where $\circ$ denotes the composition of linear operators.

We postpone the proof of this result to Sect. 4.2, as the variation equation will follow from a more general result on the differentiability of the Itô map. Following [30], we give an alternative proof in the case of finite dimensional V, in which more precise information on Φ is known.

Theorem 4.6 *Let A satisfy the hypothesis of* Theorem 4.5*, $V = \mathbb{R}^d$ for some $d \in \mathbb{N}$; then the associated YDE admits a flow of diffeomorphisms and the following hold:*

i. For any $x \in \mathbb{R}^d$ and $s \in [0,T]$, $D_x\Phi_{s\to\cdot}(x)$ corresponds to $J^x_{s\to\cdot} \in C^\alpha([s,T];\mathbb{R}^{d\times d})$ satisfying

$$J^x_{s\to t} = I + \int_s^t DA(\mathrm{d}r, \Phi_{s\to r}(x)) J^x_{s\to r}. \tag{4.3}$$

ii. The Jacobian $J_{s\to t}(x) := \det(D_x \Phi_{s\to t}(x))$ *satisfies the identity*

$$J_{s\to t}(x) = \exp\left(\int_s^t \operatorname{div} A(\mathrm{d}r, \Phi_{s\to r}(x))\right) \tag{4.4}$$

and there exists a constant $C = C(\alpha, \beta, T, \|A\|_{\alpha,1+\beta}) > 0$ *such that*

$$C^{-1} \le J_{s\to t}(x) \le C \quad \forall (s,t,x) \in \Delta_2 \times \mathbb{R}^d.$$

In particular, Φ *is a Lagrangian flow of diffeomorphisms.*

Proof For simplicity we will prove all the statements for $s = 0$, the general case being similar. By Corollary 4.4, the existence of a locally Lipschitz flow Φ is known; to show differentiability, it is enough to establish existence and continuity of the Gateaux derivatives.

Fix $x, v \in \mathbb{R}^d$ and consider for any $\varepsilon > 0$ the map $\eta^\varepsilon_t := \varepsilon^{-1}(\Phi_{0\to\cdot}(x+\varepsilon_n v) - \Phi_{0\to\cdot}(x))$; by estimate (4.1), the family $\{\eta^\varepsilon\}_{\varepsilon>0}$ is bounded in $C^\alpha_t \mathbb{R}^d$. Thus by Ascoli-Arzelà we can extract a subsequence $\varepsilon_n \to 0$ such that $\eta^\varepsilon \to \eta$ in $C^{\alpha-\delta}_t$ for some $\eta \in C^\alpha_t$ and any $\delta > 0$. Choose $\delta > 0$ small enough such that $(\alpha - \delta)(1+\beta) > 1$; using the fact that the map $F(y) = \int_0^\cdot A(\mathrm{d}s, y_s)$ is differentiable from $C^{\alpha-\delta}_t$ to itself by Proposition 2.11, with DF given by (2.8), by chain rule we deduce that

$$\begin{aligned}
\eta_\cdot &= \lim_{\varepsilon_n\to 0} \frac{\Phi_{0\to\cdot}(x+\varepsilon_n v) - \Phi_{0\to\cdot}(x)}{\varepsilon_n}\\
&= v + \lim_{\varepsilon_n\to 0} \frac{F(\Phi_{0\to\cdot}(x+\varepsilon_n v)) - F(\Phi_{0\to\cdot}(x))}{\varepsilon_n}\\
&= v + DF(\Phi_{0\to\cdot}(x))(\eta_\cdot);
\end{aligned}$$

namely, η satisfies the YDE

$$\eta_t = v + \int_0^t D_x A(\mathrm{d}r, \Phi_{0\to r}(x))\eta_r \tag{4.5}$$

whose meaning was defined in Remark 2.8. Equation (4.5) is an affine YDE, which admits a unique solution by Corollary 3.13; moreover it's easy to check that the unique solution must have the form $\eta_t = J^x_{0\to t} v$, where $J^x_{0\to\cdot} \in C^\alpha_t \mathbb{R}^{d\times d}$ is the unique solution to the affine $\mathbb{R}^{d\times d}$-valued YDE

$$J^x_{0\to t} = I + \int_0^t D_x A(\mathrm{d}r, \Phi_{0\to r}(x)) J^x_{0\to r},$$

whose global existence and uniqueness follows from Corollary 3.13 and Theorem 3.9. As the reasoning holds for any subsequence ε_n we can extract and any $v \in \mathbb{R}^d$, we conclude that $\Phi_{0\to t}(\cdot)$ is Gateaux differentiable with $D\Phi_{0\to t}(x) = J^x_{0\to t}$ which satisfies (4.3). A similar argument shows that $J^x_{0\to t}$ depends continuously on x, from which Frechét differentiability follows.

Part *ii.* can be established for instance by means of an approximation procedure; indeed by Lemma A.4, given $A \in C^\alpha_t C^{1+\beta}_x$, we can find $A^n \in C^1_t C^{1+\beta}_x$ such that $A^n \to A$ in $C^{\alpha-}_t C^{1+\beta-}_x$ and by Theorem 3.14, the solutions $y^n_\cdot = \Phi^n_{0\to\cdot}(x)$ associated to (x, A^n) converge to $\Phi_{0\to\cdot}(x)$ associated to (x, A). Moreover for A^n the YDE is meaningful as the more classical ODE associated to $\partial_t A^n$, so we can apply to it all the classical results from ODE theory; the Jacobian associated to A^n is given by

$$\det(D_x \Phi^n_{0\to t}(x)) = \exp\left(\int_0^t \operatorname{div} \partial_t A^n(r, \Phi^n_{0\to r}(x))\mathrm{d}r\right) = \exp\left(\int_0^t \operatorname{div} A^n(\mathrm{d}r, \Phi^n_{0\to r}(x))\right).$$

Passing to the limit as $n \to \infty$, by the continuity of nonlinear Young integrals, we obtain (4.4). Moreover by Eq. (4.1) we have the estimate

$$\sup_{t\in[0,T]} \left| \int_0^t \operatorname{div} A(\mathrm{d}r, \Phi_{0\to r}(x)) \right| \lesssim \| \operatorname{div} A\|_{\alpha,\beta}(1 + [\![\Phi_{0\to\cdot}(x)]\!]_{\alpha}) \lesssim \|A\|_{\alpha,1+\beta},$$

which gives Lagrangianity. □

It's possible to show that the flow inherits regularity from the drift, namely that to a spatially more regular A corresponds a more regular Φ.

Theorem 4.7 *Let $n \in \mathbb{N}$, $\alpha, \beta \in (0, 1)$ be such that $\alpha(1 + \beta) > 1$ and assume $A \in C^\alpha_t C^{n+\beta}_V$. Then the flow Φ associated to A is locally C^n-regular.*

We omit the proof, which follows similar lines to those of Theorems 4.5 and 4.6 and is mostly technical; we refer the interested reader to [20,28] and the discussion at the end of Section 3 from [33].

Remark 4.8 In line with Sect. 3.4, one can obtain sufficient conditions for the existence of a regular flow under the additional assumption $\partial_t A \in C([0, T] \times V; V)$; in this case if $A \in C^\alpha_t C^{n+\beta}_V$, then it has a locally C^n-regular flow, see the discussion in Section 4.3 from [20]. Similar reasonings allow to establish existence of a flow also for the equations treated in Sect. 3.5.

4.2 Differentiability of the Itô map

Denote by $\Phi^A_{s\to\cdot}(x)$ the solution to the YDE associated to (x, A); the aim of this section is to study the dependence of the flow Φ^A as a function of $A \in C^\alpha_t C^{1+\beta}_V$, namely to identify $D_A\Phi^A_{s\to\cdot}(x)$.

For simplicity we will restrict to the case $s = 0$; we will actually fix $A \in C^\alpha_t C^{1+\beta}_V$, consider $\Phi^{A+\varepsilon B}$ with B varying and set $X^x_t := \Phi^A_{0\to t}(x)$.

Theorem 4.9 *Let $\alpha(1 + \beta) > 1$, $x_0 \in V$ and consider the Itô map $\Phi^{\cdot}_{0\to\cdot}(x) : C^\alpha_t C^{1+\beta}_V \to C^\alpha_t V$, $A \mapsto \Phi^A_{0\to\cdot}(x)$. Then $\Phi^{\cdot}_{0\to\cdot}(x)$ is Frechét differentiable and for any $B \in C^\alpha_t C^{1+\beta}_V$ the Gateaux derivative*

$$D_A\Phi^A_{0\to\cdot}(x)(B) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(\Phi^{A+\varepsilon B}_{0\to\cdot}(x) - \Phi^A_{0\to\cdot}(x)) \in C^\alpha_t V$$

satisfies the affine YDE

$$Y^x_t = \int_0^t DA(\mathrm{d}s, X^x_s)(Y^x_s) + \int_0^t B(\mathrm{d}s, X^x_s) \quad \forall t \in [0, T] \tag{4.6}$$

and is given explicitly by

$$D_A\Phi^A_{0\to t}(x)(B) = J^x_{0\to t} \int_0^t (J^x_{0\to s})^{-1} B(\mathrm{d}s, X^x_s) \quad \forall t \in [0, T] \tag{4.7}$$

where $J^x_{0\to\cdot}$ is the unique solution to (4.2) and $(J^x_{0\to s})^{-1}$ denotes its inverse as an element of $L(V)$.

The proof requires the following preliminary lemma.

Lemma 4.10 *For any $L \in C^\alpha_t L(V)$, there exists a unique solution $M \in C^\alpha_t L(V)$ to the YDE*

$$M_t = \mathrm{Id}_V + \int_0^t L_{\mathrm{d}s} \circ M_s \qquad \forall t \in [0, T]; \tag{4.8}$$

moreover M_t is invertible for any $t \in [0, T]$ and $N_\cdot := (M_\cdot)^{-1} \in C^\alpha_t L(V)$ is the unique solution to

$$N_t = \mathrm{Id}_V - \int_0^t N_s \circ L_{\mathrm{d}s} \qquad \forall t \in [0, T]. \tag{4.9}$$

Finally, for any $y_0 \in V$ and any $\psi \in C^\alpha_t V$, the unique solution to the affine YDE

$$y_t = y_0 + \int_0^t L_{\mathrm{d}s} y_s + \psi_t \tag{4.10}$$

is given by

$$y_t = M_t y_0 + M_t \int_0^t N_s \mathrm{d}\psi_s. \tag{4.11}$$

Proof Setting $A(t, M) := L_t \circ M$, $A \in C^\alpha_t C^2_{L(V),\mathrm{loc}}$ and so existence and uniqueness of a global solution to (4.8) follows from Corollary 3.13 and Theorem 3.9; similarly for (4.9) with $\tilde{A}(t, N) = N \circ L_t$. Let $M_\cdot, N_\cdot \in C^\alpha_t L(V)$ be solution respectively to (4.8), (4.9), we claim that they are inverse of each other. Indeed by the product rule for Young integrals it holds

$$\mathrm{d}(N_t \circ M_t) = (\mathrm{d}N_t) \circ M_t + N_t \circ (\mathrm{d}M_t) = -N_t \circ L_{\mathrm{d}t} \circ M_t + N_t \circ L_{\mathrm{d}t} \circ M_t = 0$$

which implies $N_t \circ M_t = N_0 \circ M_0 = \mathrm{Id}_V$ and thus $N_t = (M_t)^{-1}$. Let $y_\cdot \in C^\alpha_t V$ be the unique solution to (4.10), whose global existence and uniqueness follows as above, and set $z_t = N_t y_t$; then again by Young product rule it holds $\mathrm{d}z_t = N_t \mathrm{d}\psi_t$ and thus

$$N_t y_t = z_t = z_0 + \int_0^t \mathrm{d}z_s = y_0 + \int_0^t N_s \mathrm{d}\psi_s$$

which gives (4.11). □

Proof of Theorem 4.9 Given $A, B \in C^\alpha_t C^{1+\beta}_V$, it is enough to show that

$$\lim_{\varepsilon \to 0} \frac{\Phi^{A+\varepsilon B}_{0\to\cdot}(x) - \Phi^A_{0\to\cdot}(x)}{\varepsilon} \text{ exists in } C^\alpha_t V$$

and that it is a solution to (4.6). Once this is shown, we can apply Lemma 4.10 for the choice $L_t = \int_0^t D_x A(\mathrm{d}s, X^x_s)$, $y_0 = 0$ and $\psi_t = \int_0^t B(\mathrm{d}s, X^x_s)$ to deduce that the limit is given by formula (4.7), which is meaningful since $J^x_{0\to\cdot}$ is defined as the solution to (4.8) for such choice of L and is therefore invertible. The explicit formula (4.7) for the Gateaux derivatives readily implies existence and continuity of the Gateaux differential $D_A \Phi^A_{0\to\cdot}(x)$ and thus also Frechét differentiability.

In order to prove the claim, let $Y^x \in C^\alpha_t V$ be the solution to (4.6), which exists and is unique by Lemma 4.10; then we need to show that

$$\lim_{\varepsilon \to 0} \left\| \frac{\Phi^{A+\varepsilon B}_{0\to\cdot}(x) - X^x_\cdot}{\varepsilon} - Y^x_\cdot \right\|_\alpha = 0.$$

Set $X_{\cdot}^{\varepsilon,x} := \Phi_{0\to\cdot}^{A+\varepsilon B}(x)$; recall that by the Comparison Principle (Theorem 3.14), we have

$$\|X^{\varepsilon,x} - X^x\|_\alpha \lesssim \varepsilon \|B\|_{\alpha,\beta}. \tag{4.12}$$

Setting $e^\varepsilon := \varepsilon^{-1}[X^{\varepsilon,x} - X^x] - Y^x$, it holds

$$\begin{aligned}
e_t^\varepsilon &= \frac{1}{\varepsilon}\left[\int_0^t (A+\varepsilon B)(\mathrm{d}s, X_s^{\varepsilon,x}) - A(\mathrm{d}s, X_s^x)\right] - \int_0^t DA(\mathrm{d}s, X_s^x)(Y_s^x) - \int_0^t B(\mathrm{d}s, X_s^x)\\
&= \int_0^t \left[\frac{A(\mathrm{d}s, X_s^{\varepsilon,x}) - A(\mathrm{d}s, X_s^x)}{\varepsilon} - DA(\mathrm{d}s, X_s^x)(Y_s)\right] + \int_0^t [B(\mathrm{d}s, X_s^{\varepsilon,x}) - B(\mathrm{d}s, X_s^x)]\\
&= \int_0^t DA(\mathrm{d}s, X_s^x)(e_s^\varepsilon) + \psi_t^\varepsilon
\end{aligned}$$

where ψ^ε is given by

$$\begin{aligned}
\psi_t^\varepsilon &= \int_0^t \frac{A(\mathrm{d}s, X_s^{\varepsilon,x}) - A(\mathrm{d}s, X_s^x) - DA(\mathrm{d}s, X_s^x)(X_s^{\varepsilon,x} - X_s^x)}{\varepsilon} + \int_0^t B(\mathrm{d}s, X_s^{\varepsilon,x}) - B(\mathrm{d}s, X_s^x)\\
&=: \psi_t^{\varepsilon,1} + \psi_t^{\varepsilon,2}.
\end{aligned}$$

In order to conclude, it is enough to show that $\|\psi^\varepsilon\|_\alpha \to 0$ as $\varepsilon \to 0$, since then we can apply the usual a priori estimates from Theorem 3.9 to e^ε, which solves an affine YDE starting at 0. We already know that $X^{\varepsilon,x} \to X^x$ as $\varepsilon \to 0$, which combined with the continuity of nonlinear Young integrals implies that $\psi_t^{\varepsilon,2} \to 0$ as $\varepsilon \to 0$. Observe that $\psi^{\varepsilon,1} = \mathcal{J}(\Gamma^\varepsilon)$ for

$$\Gamma_{s,t}^\varepsilon = \varepsilon^{-1}[A_{s,t}(X_s^{\varepsilon,x}) - A_{s,t}(X_s^x) - DA_{s,t}(X_s^x)(X_s^{\varepsilon,x} - X_s^x)]$$

which by virtue of (4.12) satisfies

$$\|\Gamma_{s,t}^\varepsilon\|_V \lesssim \varepsilon^{-1}\|A_{s,t}\|_{C_V^{1+\beta}}\|X_s^{\varepsilon,x} - X_s^x\|_V^{1+\beta} \lesssim \varepsilon^\beta |t-s|^\alpha \|A\|_{\alpha,1+\beta}$$

which implies that $\|\Gamma^\varepsilon\|_\alpha \to 0$ as $\varepsilon \to 0$. On the other hand we have

$$\begin{aligned}
\|\delta\Gamma_{s,u,t}^\varepsilon\|_V &= \varepsilon^{-1}\Big\| \int_0^1 [DA_{u,t}(X_s^x + \lambda(X_s^{\varepsilon,x} - X_s^x)) - DA_{u,t}(X_s^x)](X_s^{\varepsilon,x} - X_s^x)\mathrm{d}\lambda\\
&\qquad - \int_0^1 [DA_{u,t}(X_u^x + \lambda(X_u^{\varepsilon,x} - X_u^x)) - DA_{u,t}(X_u^x)](X_u^{\varepsilon,x} - X_u^x)\mathrm{d}\lambda\Big\|_V\\
&\le \varepsilon^{-1}\left\|\int_0^1 [DA_{u,t}(X_s^x + \lambda(X_s^{\varepsilon,x} - X_s^x)) - DA_{u,t}(X_s^x)](X_{s,u}^{\varepsilon,x} - X_{s,u}^x)\mathrm{d}\lambda\right\|_V\\
&\quad + \varepsilon^{-1}\left\|\int_0^1 [DA_{u,t}(X_u^x + \lambda(X_u^{\varepsilon,x} - X_u^x)) - DA_{u,t}(X_s^x + \lambda(X_s^{\varepsilon,x} - X_s^\varepsilon))](X_u^{\varepsilon,x} - X_u^x)\mathrm{d}\lambda\right\|_V\\
&\quad + \varepsilon^{-1}\left\|\int_0^1 [DA_{u,t}(X_u^x) - DA_{u,t}(X_s^x)](X_u^{\varepsilon,x} - X_u^x)\mathrm{d}\lambda\right\|_V\\
&\lesssim \varepsilon^{-1}|t-s|^{\alpha(1+\beta)}\|A\|_{\alpha,1+\beta}[\![X^{\varepsilon,x} - X^x]\!]_\alpha(1 + [\![X^{\varepsilon,x} - X^x]\!]_\alpha + [\![X^x]\!]_\alpha)\\
&\lesssim |t-s|^{\alpha(1+\beta)}\|A\|_{\alpha,1+\beta}(1 + [\![X^x]\!]_\alpha)
\end{aligned}$$

which implies that $\|\delta\Gamma^\varepsilon\|_{\alpha(1+\beta)}$ are uniformly bounded in ε. We can therefore apply Lemma A.2 from the "Appendix" to conclude. □

Remark 4.11 Although $A \mapsto \Phi^A$ is defined only on $C_t^\alpha C_V^{1+\beta}$, observe that $(A, B) \mapsto D_A\Phi_{0\to\cdot}^A(x)(B)$ as given by formula (4.7) is well defined and continuous for any $(A, B) \in C_t^\alpha C_V^{1+\beta} \times C_t^\alpha C_V^\beta$.

We can use Theorem 4.9 to complete the proof of Theorem 4.5.

Proof of Theorem 4.5 The existence of a Lipschitz flow Φ is granted by Corollary 4.4, so it suffices to show its differentiability and the variational equation; for simplicity we take $s = 0$. Existence of a unique solution $J^x_{0\to\cdot} \in C^\alpha_t L(V)$ to (4.2) follows from Lemma 4.10 applied to

$$L_t = \int_0^t DA(\mathrm{d}r, \Phi_{0\to r}(x))$$

and by linearity it's easy to check that for any $h \in V$, $Y^h_t := J^x_{0\to t}(h)$ is the unique solution to

$$Y^h_t = h + \int_0^t DA(\mathrm{d}r, \Phi_{0\to r}(x))(Y^h_r). \tag{4.13}$$

Therefore in order to conclude it suffices to show that the directional derivatives

$$D_x\Phi^A_{0\to\cdot}(x)(h) = \lim_{\varepsilon\to 0}\frac{\Phi^A_{0\to\cdot}(x+\varepsilon h) - \Phi^A_{0\to\cdot}(x)}{\varepsilon}$$

exist in $C^\alpha_t V$ and are solutions to (4.13), as this implies that $D_x\Phi^A_{0\to\cdot}(x) = J^x_{0\to\cdot}$. Now fix $x, h \in V$ and let $y^\varepsilon = \Phi^A_{0\to\cdot}(x+\varepsilon h)$, then $z^\varepsilon := y^\varepsilon - \varepsilon h$ solves

$$z^\varepsilon_t = x + \int_0^t A^\varepsilon(\mathrm{d}s, z^\varepsilon_s)$$

with $A^\varepsilon(t, v) = A(t, v+\varepsilon h)$, i.e. $z^\varepsilon_\cdot = \Phi^{A^\varepsilon}_{0\to\cdot}(x)$. It's easy to see that, if the first limit below exists, then

$$\lim_{\varepsilon\to 0}\frac{z^\varepsilon - z^0}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{y^\varepsilon - y^0}{\varepsilon} - h, \quad \lim_{\varepsilon\to 0}\frac{A^\varepsilon - A}{\varepsilon} = B, \quad B(t,x) = DA(t,x)(h).$$

By the Frechét differentiability of $A \mapsto \Phi^A_{0\to\cdot}(x)$ and the chain rule, it holds

$$\lim_{\varepsilon\to 0}\frac{z^\varepsilon - z^0}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{\Phi^{A^\varepsilon}_{0\to\cdot}(x) - \Phi^A_{0\to\cdot}(x)}{\varepsilon} = D_A\Phi^A_{0\to\cdot}(x)(B)$$

which is characterized as the unique solution Z^h to

$$Z^h_t = \int_0^t DA(\mathrm{d}r, \Phi^A_{0\to r}(x))(Z^h_r) + \int_0^t DA(\mathrm{d}r, \Phi^A_{0\to r}(x))(h).$$

This implies by linearity that $Y^h = Z^h_t + h = \lim_\varepsilon \varepsilon^{-1}(y^\varepsilon - y) = D_x\Phi^A_{0\to\cdot}(x)(h)$ solves exactly (4.13). The conclusion follows. □

Example 4.12 Here are some examples of applications of Theorem 4.9.

i. Consider the simple case of an additive perturbation, i.e. for fixed (x_0, A) we want to understand how the solution x of

$$x_t = x_0 + \int_0^t A(\mathrm{d}s, x_s) + \psi_t$$

depends on ψ, where $\psi \in C^\alpha_t V$ with $\psi_0 = 0$. Identifying ψ with $B^\psi(t, z) = \psi_t$ for all $z \in V$, it holds $x_\cdot = \Phi^{A+B^\psi}_{0\to\cdot}(x_0) =: F(\psi)$, which implies that F is Frechét differentiable in 0 with

$$DF(0)(\psi)_\cdot = J^x_{0\to\cdot}\int_0^\cdot (J^x_{0\to s})^{-1}\mathrm{d}\psi_s.$$

ii. Consider the classical Young case, namely $V = \mathbb{R}^d$, with

$$A(t,z) = A[\omega](t,z) = \sigma(z)\omega_t = \sum_{i=1}^m \sigma_i(z)\omega_t^i, \quad (t,z) \in [0,T] \times \mathbb{R}^d$$

for regular vector fields $\sigma_i : \mathbb{R}^d \to \mathbb{R}^d$ and $\omega \in C_t^\alpha \mathbb{R}^m$, $\alpha > 1/2$; assume σ_i are fixed and we are interested in the dependence on the drivers ω, namely the map $\Phi^\omega_{0\to\cdot}(x) := \Phi^{A[\omega]}_{0\to\cdot}(x)$. For fixed $\omega \in C_t^\alpha \mathbb{R}^m$ and $x \in \mathbb{R}^d$, setting $X_t^x := \Phi^{A[\omega]}_{0\to t}(x)$, $J^x_{0\to t} := D_x \Phi^{A[\omega]}_{0\to t}(x)$, $\Phi^{A[\cdot]}_{0\to\cdot}(x)$ is Frechét differentiable at ω with directional derivatives

$$D_\omega \Phi^{A[\cdot]}_{0\to t}(x)(\psi) = J^x_{0\to t} \int_0^t \sum_{i=1}^m (J^x_{0\to r})^{-1} \sigma_i(X_r^x) \mathrm{d}\psi_r^i. \tag{4.14}$$

The above formula uniquely extends by continuity to the case $\psi \in W_t^{1,1}$, in which case we can write it in compact form as

$$D_\omega \Phi^{A[\cdot]}_{0\to t}(x)(\psi) = \int_0^T K(t,r)\dot{\psi}_r \mathrm{d}r, \quad K(t,r) = 1_{r\le t} J^x_{0\to t} (J^x_{0\to r})^{-1} \sigma(X_r^x). \tag{4.15}$$

Formulas (4.14) and (4.15) are well known by Malliavin calculus, mostly in the case ω is sampled as an fBm of parameter $H > 1/2$, see Section 11.3 from [18]; formula (4.7) can be regarded as a generalisation of them.

5 Conditional Uniqueness

This section provides several criteria for uniqueness of the YDE, under additional assumptions on the properties of the associated solutions. Typically such properties can't be established directly, at least not under mild regularity assumptions on A; yet the criteria are rather useful in application to SDEs, where the analytic theory can be combined with more probabilistic techniques.

5.1 A Van Kampen Type Result for YDEs

The following result is inspired by the analogue results for ODEs in the style of van Kampen and Shaposhnikov, see [41,42].

Theorem 5.1 *Suppose $A \in C_t^\alpha C_V^{\beta,\lambda}$ with $\alpha(1+\beta) > 1$, $\beta + \lambda \le 1$ and that the associated YDE admits a spatially locally γ-Hölder continuous flow. If*

$$\alpha\gamma(1+\beta) > 1,$$

then for any $x_0 \in V$ there exists a unique solution to the YDE in the class $x \in C_t^\alpha V$.

Proof Let $x_0 \in V$ and x be a given solution to the YDE starting at x_0. By the a priori estimate (3.10), we can always find $R = R(x_0)$ big enough such that

$$\sup_{s\in[0,T]} \{\|x\|_\alpha + \|\Phi(s,\cdot,x_s)\|_{\alpha;s,T}\} \le R;$$

therefore in the following computations, up to a localisation argument, we can assume without loss of generality that $A \in C_t^\alpha C_V^\beta$ and that Φ is globally γ-Hölder.

It suffices to show that $f_t := \Phi(t, T, x_t) - \Phi(0, T, x_0)$ satisfies $\| f_{s,t}\|_V \lesssim |t-s|^{1+\varepsilon}$ for some $\varepsilon > 0$; if that's the case, then $f \equiv 0$, $\Phi(t, T, x_t) = \Phi(0, T, x_0)$ for all $t \in [0, T]$ and so inverting the flow $x_t = \Phi(0, t, x_0)$, which implies that $\Phi(0, \cdot, x_0)$ is the unique solution starting from x_0.

By the flow property

$$\begin{aligned} \| f_{s,t}\|_V &= \|\Phi(t, T, x_t) - \Phi(s, T, x_s)\|_V \\ &= \|\Phi(t, T, x_t) - \Phi(t, T, \Phi(s, t, x_s))\|_V \\ &\lesssim \|x_t - \Phi(s, t, x_s)\|_V^{\gamma}. \end{aligned}$$

Since both x and $\Phi(s, \cdot, x_s)$ are solutions to the YDE starting from x_s, it holds

$$\begin{aligned} \|x_t - \Phi(s, t, x_s)\|_V &= \left\| \int_s^t A(\mathrm{d}r, x_r) - \int_s^t A(\mathrm{d}r, \Phi(s, r, x_s)) \right\|_V \\ &\lesssim \|A_{s,t}(x_s) - A_{s,t}(\Phi(s, s, x_s))\|_V \\ &\quad + |t-s|^{\alpha(1+\beta)} \|A\|_{\alpha,\beta} (1 + [\![x]\!]_\alpha + [\![\Phi(s, \cdot, x_s)]\!]_\alpha) \\ &\lesssim |t-s|^{\alpha(1+\beta)} \end{aligned}$$

and so overall we obtain $\| f_{s,t}\|_V \lesssim |t-s|^{\gamma\alpha(1+\beta)}$, which implies the conclusion. □

Remark 5.2 The assumption can be weakened in several ways. For instance, the existence of a γ-Hölder regular semiflow is enough to establish that $\Phi(t, T, x_t) = \Phi(0, T, x_0)$, even when Φ is not invertible. Uniqueness only requires $\Phi(t, T, \cdot)$ to be invertible for $t \in D$, D dense subset of $[0, T]$; indeed this implies $x_t = \Phi(0, t, x_0)$ on D and then by continuity the equality can be extended to the whole $[0, T]$. Similarly, it is enough to require

$$\sup_{t\in D} \|\Phi(t, T, \cdot)\|_{\gamma,R} < \infty \quad \text{for all } R \geq 0$$

for D dense subset of $[0, T]$ as before.

5.2 Averaged Translations and Conditional Comparison Principle

The concept of averaged translation has been introduced in [9], Definition 2.13. We provide here a different construction based on the sewing lemma (although with the same underlying idea).

Definition 5.3 Let $A \in C^\alpha_t C^\beta_V$, $y \in C^\gamma_t V$ with $\alpha + \beta\gamma > 1$. The averaged translation $\tau_x A$ is defined as

$$\tau_y A(t, x) = \int_0^t A(\mathrm{d}s, z + y_s) \quad \forall t \in [0, T],\ z \in V.$$

Lemma 5.4 *Let $A \in C^\alpha_t C^{n+\beta}_V$, $y \in C^\gamma_t V$ with $\alpha + \beta\gamma > 1$, $\eta \in (0, 1)$ satisfying $\eta < n + \beta$, $\alpha + \eta\gamma > 1$. The operator τ_y is continuous from $C^\alpha_t C^{n+\beta}_V$ to $C^\alpha_t C^{n+\beta-\eta}_V$ and there exists $C = C(\alpha, \beta, \gamma, \eta, T)$ s.t.*

$$\|\tau_y A\|_{\alpha,n+\beta-\eta} \leq C \|A\|_{\alpha,n+\beta} (1 + [\![y]\!]_\gamma). \tag{5.1}$$

Proof Observe that $\tau_y A$ corresponds to the sewing of $\Gamma : \Delta_2 \to C^{n+\beta}_V$ given by

$$\Gamma_{s,t} := A_{s,t}\,(\cdot + y_s)\,.$$

It holds $\|\Gamma_{s,t}\|_{n+\beta} \le |t-s|^{\alpha}\|A\|_{\alpha,n+\beta}$; moreover by Lemma A.3 in "Appendix A.1" it holds

$$\begin{aligned}\|\delta\Gamma_{s,u,t}\|_{n+\beta-\eta} &= \left\|A_{u,t}(\cdot+y_s)-A_{u,t}(\cdot+y_u)\right\|_{n+\beta-\eta}\\ &\lesssim \|y_s-y_u\|_V^{\eta}\|A_{u,t}\|_{n+\beta}\\ &\lesssim |t-s|^{\alpha+\gamma\eta}[\![y]\!]_{\gamma}\|A\|_{\alpha,n+\beta}.\end{aligned}$$

Since $\alpha+\gamma\eta>1$, by the sewing lemma we deduce that $\mathcal{J}(\Gamma)=\tau_y A\in C_t^{\alpha}C_V^{n+\beta-\eta}$, together with estimate (5.1). □

Young integrals themselves can indeed be regarded as averaged translations evaluated at $z=0$. Moreoveor iterating translations is a consistent procedure, as the following lemma shows.

Lemma 5.5 *Assume that $\alpha+\beta\gamma>1$ and $A\in C_t^{\alpha}C_V^{\beta}$, $x\in C_t^{\gamma}V$ and $\tau_x A\in C_t^{\alpha}C_V^{\beta}$. Then for any $y\in C_t^{\gamma}V$ it holds*

$$\int_0^t(\tau_x A)(\mathrm{d}s,y_s)=\int_0^t A(\mathrm{d}s,x_s+y_s)\quad\forall t\in[0,T].$$

Proof The statement follows immediately from the observation that for any $s\le t$ it holds

$$\begin{aligned}\left\|\int_s^t(\tau_x A)(\mathrm{d}r,y_r)-\int_s^t A(\mathrm{d}r,x_r+y_r)\right\| &\lesssim \|(\tau_x A)_{s,t}(y_s)-A_{s,t}(x_s+y_s)\|+|t-s|^{\alpha+\beta\gamma}\\ &\lesssim \left\|\left(A_{s,t}(\cdot+x_s)\right)(y_s)-A_{s,t}(x_s+y_s)\right\|+|t-s|^{\alpha+\beta\gamma}\\ &\lesssim |t-s|^{\alpha+\beta\gamma}\end{aligned}$$

so that the two integrals must coincide. □

The main reason for introducing averaged translations is the following key result.

Theorem 5.6 (Conditional Comparison Principle) *Let $A^1,A^2\in C_t^{\alpha}C_V^{\beta}$ with $\alpha(1+\beta)>1$ for some $\alpha,\beta\in(0,1)$ and let $x^i\in C_t^{\alpha}V$ be given solutions respectively to the YDE associated to (x_0^i,A^i). Suppose in addition that x^1 is such that $\tau_{x^1}A^1\in C_t^{\alpha}\mathrm{Lip}_V$. Then there exists $C=C(\alpha,\beta,T)$ s.t.*

$$\|x^1-x^2\|_{\alpha}\le C\exp(C\|\tau_{x^1}A^1\|_{\alpha,1}^{1/\alpha})(1+\|A^2\|_{\alpha,\beta}^2)(\|x_0^1-x_0^2\|+\|A^1-A^2\|_{\alpha,\beta}).\quad(5.2)$$

In particular, uniqueness holds in the class $C_t^{\alpha}V$ to the YDE associated to (x_0^1,A^1).

Proof The final uniqueness claim immediately follows from inequality (5.2), since in that case we can consider $A^1=A^2$, $x_0^1=x_0^2$. Now let x^i be two solutions as above, then their difference $v=x^1-x^2$ satisfies

$$\begin{aligned}v_t &= v_0+\int_0^t A^1(\mathrm{d}s,x_s^1)-\int_0^t A^2(\mathrm{d}s,x_s^2)\\ &= v_0+\int_0^t A^1(\mathrm{d}s,x_s^1)-\int_0^t A^1(\mathrm{d}s,v_s+x_s^1)+\int_0^t(A^2-A^1)(\mathrm{d}s,x_s^2)\\ &= v_0-\int_0^t\tau_{x^1}A^1(\mathrm{d}s,v_s)+\int_0^t\tau_{x^1}A^1(\mathrm{d}s,0)+\int_0^t(A^2-A^1)(\mathrm{d}s,x_s^2)\\ &= v_0+\int_0^t B(\mathrm{d}s,v_s)+\psi_t\end{aligned}$$

where in the third line we applied Lemma 5.5 and we take

$$B(t, z) = -\tau_{x_1} A^1(t, z) + \tau_{x_1} A^1(t, 0), \qquad \psi_\cdot = \int_0^\cdot (A^2 - A^1)(\mathrm{d}s, x_s^2).$$

By the hypothesis, $B \in C_t^\gamma \operatorname{Lip}_V$ with $B(t, 0) = 0$ for all $t \in [0, T]$, while $\psi \in C_t^\alpha V$. Therefore from Theorem 3.9 applied to v we deduce the existence of a constant $\kappa_1 = \kappa_1(\alpha, T)$ such that

$$\|x^1 - x^2\|_\alpha \le \kappa_1 \exp(\kappa_1 [\![\tau_{x^1} A^1]\!]_{1,\alpha}^{1/\alpha})(\|x_0^1 - x_0^2\|_V + [\![\psi]\!]_\alpha).$$

On the other hand, estimates (2.4) and (3.6) imply that

$$[\![\psi]\!]_\alpha \le \kappa_2 \|A^1 - A^2\|_{\alpha,\beta}(1 + \|A^2\|_{\alpha,\beta}^2)$$

for some $\kappa_2 = \kappa_2(\alpha, \beta, T)$. Combining the above estimates the conclusion follows. □

Remarkably, the hypothesis $\tau_x A \in C_t^\alpha \operatorname{Lip}_V$ allows not only to show that this is the unique solution starting at x_0, but also that any other solution will not get too close to it. In the next lemma, in order to differentiate $\|\cdot\|_V$, we assume for simplicity V to be a Hilbert space, but a uniformly smooth Banach space would suffice.

Lemma 5.7 *Let V be a Hilbert space, $A \in C_t^\alpha C_V^\beta$ with $\alpha(1+\beta) > 1$, $x, y \in C_t^\alpha V$ solutions respectively to the YDEs associated to (x_0, A), (y_0, A) and assume that $\tau_x A \in C_t^\alpha \operatorname{Lip}_V$. Then there exists $C = C(\alpha, T)$ s.t.*

$$\sup_{t\in[0,T]} \frac{\|x_t - y_t\|_V}{\|x_0 - y_0\|_V} \le C \exp(C \|\tau_x A\|_{\alpha,1}^{1/\alpha}), \quad \sup_{t\in[0,T]} \frac{\|x_0 - y_0\|_V}{\|x_t - y_t\|_V} \le C \exp(C \|\tau_x A\|_{\alpha,1}^{1/\alpha}).$$

Proof The first inequality is an immediate consequence of Theorem 5.6, so we only need to prove the second one. By the same computation as in Theorem 5.6, the map $v = y - x$ satisfies

$$\mathrm{d}v_t = A(\mathrm{d}t, y_t) - A(\mathrm{d}t, x_t) = \tau_x A(\mathrm{d}t, v_t) - \tau_x A(\mathrm{d}t, 0) = B(\mathrm{d}t, v_t)$$

where $B(t, z) := \tau_x A(t, z) - \tau_x A(t, 0)$, which by hypothesis belongs to $C_t^\alpha \operatorname{Lip}_V$ with $[\![B]\!]_{\alpha,1} = [\![\tau_x A]\!]_{\alpha,1}$; moreover $B(t, 0) = 0$ for all $t \in [0, T]$.

Now for $0 < \varepsilon < \|x_0 - y_0\|_V$, define $T^\varepsilon = \inf\{t \in [0, T] : \|x_t - y_t\|_V \le \varepsilon\}$, with the convention that $\inf \emptyset = T$; then on $[0, \tau_\varepsilon]$ the map $z_t := \|y_t - x_t\|_V^{-1} = \|v_t\|_V^{-1}$ is in $C_t^\alpha \mathbb{R}$ and by Young chain rule

$$\mathrm{d}z_t = -\|v_t\|_V^{-3} \langle v_t, \tilde{A}(\mathrm{d}t, v_t)\rangle_V.$$

We are going to show that z satisfies a bound from above which does not depend on the interval $[0, T^\varepsilon]$; as a consequence, for all $\varepsilon > 0$ small enough it must hold $T^\varepsilon = T$, which yields the conclusion.

For any $[u, r] \subset [s, t] \subset [0, T^\varepsilon]$ it holds

$$\begin{aligned}
|z_{u,r}| &\le \|v_u\|_V^{-3} |\langle v_u, B_{u,r}(v_u)\rangle_V| + \kappa_1 [\![z]\!]_{\alpha;s,t} [\![B]\!]_{\alpha,1} |u - r|^{2\alpha} \\
&\le \|v_u\|_V^{-1} [\![B]\!]_{\alpha,1} |u - r|^\alpha + \kappa_1 [\![z]\!]_{\alpha;s,t} [\![B]\!]_{\alpha,1} |t - s|^\alpha |u - r|^\alpha \\
&\le |z_u| [\![\tau_x A]\!]_{\alpha,1} |u - r|^\alpha + \kappa_1 [\![z]\!]_{\alpha;s,t} [\![\tau_x A]\!]_{\alpha,1} |t - s|^\alpha |u - r|^\alpha \\
&\le |u - r|^\alpha [\![\tau_x A]\!]_{\alpha,1} [|z_s| + (1 + \kappa_1) [\![z]\!]_{\alpha;s,t} |t - s|^\alpha];
\end{aligned}$$

dividing by $|u-r|^\alpha$ and taking the supremum we obtain

$$\llbracket z \rrbracket_{\alpha;s,s+\Delta} \le \llbracket \tau_x A \rrbracket_{\alpha,1} |z_s| + \kappa_2 \Delta^\alpha \llbracket \tau_x A \rrbracket_{\alpha,1} \llbracket z \rrbracket_\alpha.$$

The rest of the proof follows exactly the same calculations as in the proof of Theorem 3.9: taking Δ such that $\kappa_2 \Delta^\alpha \llbracket \tau_x A \rrbracket_{\alpha,1} \le 1/2$, $\kappa_2 \Delta^\alpha \llbracket \tau_x A \rrbracket_{\alpha,1} \sim 1$, we deduce that

$$\llbracket z \rrbracket_{\alpha;s,s+\Delta} \le 2 \llbracket \tau_x A \rrbracket_{\alpha,1} |z_s|;$$

setting $J_n = \|z\|_{\infty;I_n}$ with $I_n = [(n-1)\Delta, n\Delta] \cap [0, T^\varepsilon]$, $J_0 = |z_0|$, it holds

$$J_n \le J_{n-1} + \Delta^\alpha \llbracket z \rrbracket_{\alpha;I_n} \le (1 + 2\Delta^\alpha \llbracket \tau_x A \rrbracket_{\alpha,1}) J_{n-1},$$

which implies recursively

$$\|z\|_{\infty;0,T^\varepsilon} = \sup_n J_n \le (1 + 2\Delta^\alpha \llbracket \tau_x A \rrbracket_{\alpha,1})^N |z_0| \le \exp(2N\Delta^\alpha \llbracket \tau_x A \rrbracket_{\alpha,1}) |z_0|.$$

Since $T^\varepsilon \le T$, it takes at most $N \sim T/\Delta$ intervals of size Δ to cover $[0, T^\varepsilon]$, and $\Delta \sim \llbracket \tau_x A \rrbracket_{\alpha,1}^{1/\alpha}$, therefore overall we have found a constant $C = C(\alpha, T)$ such that

$$\sup_{t\in[0,T^\varepsilon]} \frac{1}{\|x_s - y_s\|_V} = \sup_{t\in[0,T^\varepsilon]} |z_t| \le C \exp(C \llbracket \tau_x A \rrbracket_{\alpha,1}^{1/\alpha}) |z_0| = C \exp(C \llbracket \tau_x A \rrbracket_{\alpha,1}^{1/\alpha}) \frac{1}{\|x_0 - y_0\|_V}.$$

As the estimate does not depend on ε, the conclusion follows. □

5.3 Conditional Rate of Convergence for the Euler Scheme

Remarkably, under the assumption of regularity of $\tau_x A$, convergence of the Euler scheme to the unique solution can be established, with the same rate $2\alpha - 1$ as in the more regular case of $A \in C_t^\alpha C_V^{1+\beta}$. The following results are direct analogues of Corollaries 3.16 and 3.19.

Corollary 5.8 *Let $A \in C_t^\alpha \mathrm{Lip}_V$ with $\alpha > 1/2$, $x_0 \in V$ and suppose there exists a solution x associated to (x_0, A) such that $\tau_x A \in C_t^\alpha \mathrm{Lip}_V$ (which is therefore the unique solution); denote by x^n the element of $C_t^\alpha V$ constructed by the n-step Euler approximation from* Theorem 3.2. *Then there exists $C = C(\alpha, T)$ such that*

$$\|x - x^n\|_\alpha \le C \exp(C \|\tau_x A\|_{\alpha,1}^{1/\alpha})(1 + \|A\|_{\alpha,1}^3) n^{1-2\alpha} \quad \text{as } n \to \infty.$$

Proof As in the proof of Corollary 3.16, recall that x^n satisfies the YDE

$$x^n = x_0 + \int_0^t A(\mathrm{d}s, x_s^n) + \psi_t^n, \qquad \llbracket \psi^n \rrbracket_\alpha \lesssim_{\alpha,T} (1 + \|A\|_{\alpha,1}^3) n^{1-2\alpha}.$$

Therefore $v^n = x^n - x$ satisfies

$$v_t^n = \int_0^t B(\mathrm{d}s, v_s^n) + \psi_t^n, \quad B(t,z) = \tau_x A(t,z) - \tau_x A(t,0), \quad \llbracket B \rrbracket_{\alpha,1} = \llbracket \tau_x A \rrbracket_{\alpha,1}.$$

Applying Theorem 3.9 we obtain that, for suitable $\kappa = \kappa(\alpha, T)$ it holds

$$\|x - x^n\|_\alpha \le \kappa \exp(\kappa \|\tau_x A\|_{\alpha,1}^{1/\alpha}) \llbracket \psi^n \rrbracket_\alpha$$

which combined with the above inequality for $\llbracket \psi^n \rrbracket_\alpha$ gives the conclusion. □

Corollary 5.9 *Let A be such that $A \in C^\alpha_t C^\beta_V$ and $\partial_t A \in C^0([0,T]\times V; V)$ with $\alpha(1+\beta) > 1$, $x_0 \in V$ and suppose there exists a solution x associated to (x_0, A) such that $\tau_x A \in C^\alpha_t \operatorname{Lip}_V$ (which is therefore the unique solution); denote by x^n the element of $C^\alpha_t V$ constructed by the n-step Euler approximation from* Theorem 3.2. *Then there exists $C = C(\alpha, T)$ such that*

$$\|x - x^n\|_\alpha \le C \exp(C\|\tau_x A\|_{\alpha,1}^{1/\alpha})\|A\|_{\alpha,1}\|\partial_t A\|_\infty n^{-\alpha} \quad \text{as } n \to \infty.$$

Proof Recall that x^n satisfies the YDE

$$x^n = x_0 + \int_0^t A(\mathrm{d}s, x^n_s) + \psi^n_t, \qquad [\![\psi^n]\!]_\alpha \lesssim_{\alpha,T} \|A\|_{\alpha,1}\|\partial_t A\|_\infty n^{-\alpha}.$$

The rest of the proof is mostly identical to that of Corollary 5.8. □

6 Young Transport Equations

This section is devoted to the study of Young transport equations of the form

$$u_{\mathrm{d}t} + A_{\mathrm{d}t} \cdot \nabla u_t + c_{\mathrm{d}t} u_t = 0. \tag{6.1}$$

which we will refer to as the YTE associated to (A, c).

We restrict here to the case $V = \mathbb{R}^d$; as in Sect. 4 for simplicity we will assume on A global bounds like $A \in C^\alpha_t C^{1+\beta}_x$, but slightly more tedious localisation arguments allow to relax them to growth conditions and local regularity requirements.

Classical results on weak solutions to (6.1) in the case $A_{\mathrm{d}t} = b_t \mathrm{d}t$, $c_{\mathrm{d}t} = \tilde{c}_t \mathrm{d}t$ can be found in [1,16]. Our approach here mostly follows the one given in [20], although slightly less based on the method of characteristics and more on a duality approach; other works concerning transport equations in the Young (or "level-1") regime are given by [8,30] and Chapter 9 from [36]. Let us also mention on a different note the works [3,5,15] which treat with different techniques and in various regularity regimes rough trasnport equations of "level-2" or higher (namely corresponding to a time regularity $\alpha \le 1/2$).

Before explaining the meaning of (6.1), we need some preparations. Given any compact $K \subset \mathbb{R}^d$, we denote by $C^\beta_K = C^\beta_K(\mathbb{R}^d)$ the Banach space of $f \in C^\beta(\mathbb{R}^d)$ with supp $f \subset K$; $C^\beta_c = C^\beta_c(\mathbb{R}^d)$ is the set of all compactly supported β-Hölder continuous functions. C^β_c is a direct limit of Banach spaces and thus it is locally convex; we denote its topological dual by $(C^\beta_c)^*$. Given $\gamma, \beta \in (0,1)$, we say that $f \in C^\alpha_t C^\beta_c$ if there exists a compact K such that $f \in C^\alpha_t C^\beta_K$; similarly, a distribution $u \in C^\gamma_t (C^\beta_c)^*$ if $u \in C^\gamma_t (C^\beta_K)^*$ for all compact $K \subset \mathbb{R}^d$. We will use the bracket $\langle \cdot, \cdot \rangle$ to denote both the classical L^2-pairing and the one between C^β_c and its dual. Finally, M_{loc} denotes the space of Radon measures on $\mathbb{R}^d$, M_K the space of finite signed measure supported on K; observe that the above notation is consistent with $M_{\mathrm{loc}} = (C^0_c)^*$.

We are now ready to give a notion of solution to the YTE.

Definition 6.1 Let $\alpha, \beta \in (0,1)$ such that $\alpha(1+\beta) > 1$.We say that $u \in L^\infty_t M_{\mathrm{loc}} \cap C^{\alpha\beta}_t (C^\beta_c)^*$ is a weak solution to the YTE associated to $A \in C^\alpha_t C^\beta_x$, $c \in C^\alpha_t C^\beta_x$ with div $A \in C^\alpha_t C^\beta_x$ if

$$\langle u_t, \varphi\rangle - \langle u_0, \varphi\rangle = \int_0^t \langle A_{\mathrm{d}s} \cdot \nabla\varphi + (\operatorname{div} A_{\mathrm{d}s} - c_{\mathrm{d}s})\varphi, u_s\rangle \quad \forall \varphi \in C^\infty_c. \tag{6.2}$$

Observe that under the above assumptions, for any $\varphi \in C_c^\infty$, $A \cdot \nabla\varphi$ and $(\operatorname{div} A - c)\varphi$ belong to $C_t^\alpha C_c^\beta$; since $u \in C_t^{\alpha\beta}(C_c^\beta)^*$ with $\alpha(1+\beta) > 1$, the integral appearing in (6.2) is meaningful as a functional Young integral.

Remark 6.2 For practical purposes, it is useful to consider the following equivalent characterization of solutions: under the above regularity assumptions on u, A, c, u is a solution if and only if for any compact $K \subset \mathbb{R}^d$ and $\varphi \in C_K^\infty$ it holds

$$\begin{aligned}|\langle u_{s,t}, \varphi\rangle - \langle A_{s,t} \cdot \nabla\varphi + (\operatorname{div} A_{s,t} - c_{s,t})\varphi, u_s\rangle| \lesssim_K \|\varphi\|_{C_K^{1+\beta}} |t-s|^{\alpha(1+\beta)} [\![u]\!]_{C_t^{\alpha\beta}(C_K^\beta)^*} \\ \times (\|A\|_{\alpha,\beta} + \|\operatorname{div} A - c\|_{\alpha,\beta}).\end{aligned} \tag{6.3}$$

Clearly in the l.h.s. above one can replace u_s with u_t to get a similar estimate.

Remark 6.3 The presence of c in (6.1) allows to also consider nonlinear Young continuity equations (YCE for short) of the form

$$v_{\mathrm{d}t} + \nabla \cdot (A_{\mathrm{d}t} v_t) + c_{\mathrm{d}t} v_t = 0;$$

weak solutions to the above equation must be understood as weak solutions to the YTE associated to $(A, \tilde{c})$ with $\tilde{c} = c + \nabla \cdot A$.

Let us quickly recall some results from Sect. 4: given $A \in C_t^\alpha C_x^{1+\beta}$, the YDE admits a flow of diffeomorphisms $\Phi_{s\to t}(x)$ and there exists $C = C(\alpha, \beta, T, \|A\|_{\alpha,1+\beta})$ such that

$$\begin{aligned}\|\Phi_{s\to\cdot}(x) - \Phi_{s\to\cdot}(y)\|_{\alpha;s,T} &\le C|x-y| \\ |\Phi_{s\to t}(x) - x| &\le C|t-s|^\alpha \\ [\![\Phi_{s\to\cdot}(x)]\!]_{\alpha;s,T} + |D_x\Phi_{s\to t}(x)| &\le C\end{aligned}$$

for all $x, y \in \mathbb{R}^d$, $(s,t) \in \Delta_2$, together with similar estimates for $\Phi_{\cdot\leftarrow t}$. Moreover

$$\det D\Phi_{s\to t}(x) = \exp\left(\int_s^t \operatorname{div} A(\mathrm{d}r, \Phi_{s\to r}(x))\right)$$

and similarly

$$\det D\Phi_{s\leftarrow t}(x) = (\det D\Phi_{s\to t}(\Phi_{s\leftarrow t}(x)))^{-1} = \exp\left(-\int_s^t \operatorname{div} A(\mathrm{d}r, \Phi_{r\leftarrow t}(x))\right).$$

Proposition 6.4 *Let $A \in C_t^\alpha C_x^{1+\beta}$, $c \in C_t^\alpha C_x^\beta$. Then for any $\mu_0 \in M_{\mathrm{loc}}$, a solution to the YTE is given by the formula*

$$\langle u_t, \varphi\rangle = \int \varphi(\Phi_{0\to t}(x)) \exp\left(\int_0^t (\operatorname{div} A - c)(\mathrm{d}s, \Phi_{0\to s}(x))\right) \mu_0(\mathrm{d}x) \quad \forall\, \varphi \in C_c^\infty. \tag{6.4}$$

If $\mu_0(\mathrm{d}x) = u_0(x)\mathrm{d}x$ for $u_0 \in L_{\mathrm{loc}}^p$, then u_t corresponds to the measurable function

$$u(t,x) = u_0(\Phi_{0\leftarrow t}(x)) \exp\left(-\int_0^t c(\mathrm{d}s, \Phi_{s\leftarrow t}(x))\right) \tag{6.5}$$

which belongs to $L_t^\infty L_{\mathrm{loc}}^p$ and satisfies

$$\int_K |u(t,x)|^p \mathrm{d}x = \int_{\Phi_{0\leftarrow t}(K)} |u_0(x)|^p \exp\left(\int_0^t (\operatorname{div} A - c)(\mathrm{d}s, \Phi_{0\to s}(x))\right).$$

If in addition $c \in C_t^\alpha C_x^{1+\beta}$, then for any $u_0 \in C_{\mathrm{loc}}^1$ it holds $u \in C_t^\alpha C_{\mathrm{loc}}^0 \cap C_t^0 C_{\mathrm{loc}}^1$.

Proof Since $|\Phi_{0\to t}(x)-x|\lesssim T^\alpha$, it is always possible to find $R\ge 0$ big enough such that $\operatorname{supp}\varphi(\Phi_{0\to t}(\cdot))\subset \operatorname{supp}\varphi + B_R$ for all $t\in[0,T]$; by estimates (2.4) and (3.9), it holds

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\left|\int_0^t (\operatorname{div} A-c)(\mathrm{d}s,\Phi_{0\to s}(x))\right| \lesssim \|\operatorname{div} A-c\|_{\alpha,\beta}\sup_{x\in\mathbb{R}^d}(1+[\![\Phi_{0\to\cdot}(x)]\!]_\alpha)<\infty.$$

It is therefore clear that u_t defined as in (6.4) belongs to $L^\infty_t(C^0_c)^*$. Similarly, combining the estimates

$$\begin{aligned}
&|\varphi(\Phi_{0\to t}(x))-\varphi(\Phi_{0\to s}(x))|\le |t-s|^{\alpha\beta}[\![\varphi]\!]_\beta[\![\Phi_{0\to\cdot}(x)]\!]_\alpha^\beta\lesssim |t-s|^{\alpha\beta}[\![\varphi]\!]_\beta\\
&\left|\int_s^t(\operatorname{div} A-c)(\mathrm{d}s,\Phi_{0\to s}(x))\right|\lesssim |t-s|^\alpha\|\operatorname{div} A-c\|_{\alpha,\beta}(1+[\![\Phi_{0\to\cdot}(x)]\!]_\alpha)\lesssim|t-s|^\alpha,
\end{aligned}$$

it is easy to check that $u\in C^{\alpha\beta}_t(C^\beta_c)^*$.

Let us show that it is a solution to the YTE in the sense of Definition 6.1. Given $\varphi\in C^\infty_K$ and $x\in\mathbb{R}^d$, define

$$z_t(x):=\varphi(\Phi_{0\to t}(x))\exp\left(\int_0^t(\operatorname{div} A-c)(\mathrm{d}s,\Phi_{0\to s}(x))\right).$$

By Itô formula, z satisfies

$$\begin{aligned}
z_{s,t}(x)&=\int_s^t\varphi(\Phi_{0\to r}(x))\exp\left(\int_0^r(\operatorname{div} A-c)(\mathrm{d}s,\Phi_{0\to s}(x))\right)(\operatorname{div} A-c)(\mathrm{d}r,\Phi_{0\to r}(x))\\
&\quad+\int_s^t\exp\left(\int_0^r(\operatorname{div} A-c)(\mathrm{d}s,\Phi_{0\to s}(x))\right)\nabla\varphi(\Phi_{0\to r}(x))\cdot A(\mathrm{d}r,\Phi_{0\to r}(x)).
\end{aligned}$$

By the properties of Young integrals and the above estimates, which are uniform in x, it holds

$$\begin{aligned}
z_{s,t}(x)&\sim\exp\left(\int_0^s(\operatorname{div} A-c)(\mathrm{d}r,\Phi_{0\to r}(x))\right)\times\\
&\quad\times[\varphi(\Phi_{0\to s}(x))(\operatorname{div} A-c)_{s,t}(\Phi_{0\to s}(x))+\nabla\varphi(\Phi_{0\to s}(x))\cdot A_{s,t}(\Phi_{0\to s}(x))]
\end{aligned}$$

in the sense that the two quantities differ by $O(|t-s|^{\alpha(1+\beta)})$, uniformly in $x\in\mathbb{R}^d$. Therefore

$$\begin{aligned}
\langle u_{s,t},\varphi\rangle&=\int_{K+B_R}z_{s,t}(x)\mu_0(\mathrm{d}x)\\
&\sim\int_{K+B_R}[A_{s,t}\cdot\nabla\varphi+(\operatorname{div} A-c)_{s,t}\varphi](\Phi_{0\to t}(x))\exp\left(\int_0^s(\operatorname{div} A-c)(\mathrm{d}r,\Phi_{0\to r}(x))\right)\mu_0(\mathrm{d}x)\\
&\sim\langle u_s,A_{s,t}\cdot\nabla\varphi+(\operatorname{div} A-c)_{s,t}\varphi\rangle
\end{aligned}$$

where the two quantities differ by $O(\|\varphi\|_{C^{1+\beta}_K}|t-s|^{\alpha(1+\beta)})$. By Remark 6.2 we deduce that u is indeed a solution.

The statements for $u_0\in L^p_{\mathrm{loc}}$ are an easy application of formula (4.4); it remains to prove the claims for $u_0\in C^1_{\mathrm{loc}}$, under the additional assumption $c\in C^\alpha_t C^{1+\beta}_x$. First of all observe that, for any $(s,t)\in\Delta_2$, it holds

$$\|\Phi_{\cdot\leftarrow t}(x)-\Phi_{\cdot\leftarrow s}(x)\|_\alpha=\|\Phi_{\cdot\leftarrow s}(\Phi_{s\leftarrow t}(x))-\Phi_{\cdot\leftarrow s}(x)\|_\alpha\lesssim|\Phi_{s\leftarrow t}(x)-x|\lesssim|t-s|^\alpha; \tag{6.6}$$

as a consequence, the map $(t,x)\mapsto u_0(\Phi_{0\leftarrow t}(x))$ belongs to $C^\alpha_t C^0_{\text{loc}}$. Consider now the map

$$g(t,x) := \int_0^t c(\mathrm{d}r, \Phi_{r\leftarrow t}(x)).$$

It holds

$$\int_0^t c(\mathrm{d}r, \Phi_{r\leftarrow t}(x)) - \int_0^s c(\mathrm{d}r, \Phi_{r\leftarrow s}(x)) = \int_s^t c(\mathrm{d}r, \Phi_{r\leftarrow t}(x)) + \int_0^s [c(\mathrm{d}r, \Phi_{r\leftarrow t}(x)) - c(\mathrm{d}r, \Phi_{r\leftarrow s}(x))];$$

by Corollary 2.12 and estimate (6.6) we have

$$\begin{aligned}\left\| \int_0^\cdot [c(\mathrm{d}r, \Phi_{r\leftarrow t}(x)) - c(\mathrm{d}r, \Phi_{r\leftarrow s}(x))] \right\|_\alpha &\lesssim \|c\|_{\alpha,1+\beta}(1 + [\![\Phi_{\cdot\leftarrow t}(x)]\!]_\alpha + [\![\Phi_{\cdot\leftarrow s}(x)]\!]_\alpha) \times \\ &\quad \times \|\Phi_{\cdot\leftarrow t}(x) - \Phi_{\cdot\leftarrow s}(x)\|_\alpha \\ &\lesssim |t-s|^\alpha.\end{aligned}$$

As a consequence, $g \in C^\alpha_t C^0_{\text{loc}}$ and so does u. The verification that $u \in C^0_t C^1_{\text{loc}}$ is similar and thus omitted. □

Remark 6.5 Analogous computations show that a solution to the YTE with terminal condition $u(T,\cdot) = \mu_T(\cdot)$ is given by

$$\langle u_t, \varphi\rangle = \int \varphi(\Phi_{t\leftarrow T}(x)) \exp\left(\int_t^T (c - \operatorname{div} A)(\mathrm{d}s, \Phi_{s\leftarrow T}(x))\right) \mu_T(\mathrm{d}x) \quad \forall \varphi \in C^\infty_c;$$

in the case $\mu_T(\mathrm{d}x) = u_T(x)\mathrm{d}x$ with $u_T \in L^p_{\text{loc}}$ it corresponds to

$$u_t(x) = u_T(\Phi_{t\to T}(x)) \exp\left(\int_t^T c(\mathrm{d}s, \Phi_{t\to s}(x))\right).$$

This solution satisfies the same space-time regularity as in Proposition 6.4. Moreover by the properties of the flow, if μ_0 (resp. μ_T) has compact support, then it's possible to find $K \subset \mathbb{R}^d$ compact such that $\operatorname{supp} u_t \subset K$ uniformly in $t \in [0,T]$. In particular if $c \in C^\alpha_t C^{1+\beta}_x$ and $u_0 \in C^1_c$ (resp. $u_T \in C^1_c$), then the associated solution belongs to $C^\alpha_t C^0_c \cap C^0_t C^1_c$.

The following result is at the heart of the duality approach and our main tool to establish uniqueness.

Proposition 6.6 *Let $u \in C^\alpha_t C^0_c \cap C^0_t C^1_c$ be a solution of the YTE*

$$u_{\mathrm{d}t} + A_{\mathrm{d}t}\cdot\nabla u_t + c_{\mathrm{d}t} u_t = 0 \tag{6.7}$$

and let $v \in L^\infty_t (C^0_c)^* \cap C^{\alpha\beta}_t (C^\beta_c)^*$ be a solution to the YCE

$$v_{\mathrm{d}t} + \nabla\cdot(A_{\mathrm{d}t} v_t) - c_{\mathrm{d}t} v_t = 0. \tag{6.8}$$

Then it holds $\langle v_t, u_t\rangle = \langle v_s, u_s\rangle$ for all $(s,t)\in\Delta_2$. A similar statement holds for $u \in C^\alpha_t C^0_{\text{loc}} \cap C^0_t C^1_{\text{loc}}$ and v as above and compactly supported uniformly in time.

The proof requires some preparations. Let $\{\rho_\varepsilon\}_{\varepsilon>0}$ be a family of standard spatial mollifiers (say ρ_1 supported on B_1 for simplicity) and define the R^ε, for sufficiently regular g and h, as the following bilinear operator:

$$R^\varepsilon(g,h) = (g\cdot\nabla h)^\varepsilon - g\cdot\nabla h^\varepsilon = \rho^\varepsilon * (g\cdot\nabla h) - g\cdot\nabla(\rho^\varepsilon * h); \tag{6.9}$$

the following commutator lemma is a slight variation on Lemma 16, Section 5.2 from [20], which in turn is inspired by the general technique first introduced in [16].

Lemma 6.7 *The operator $R^\varepsilon : C^{1+\beta}_{\rm loc} \times C^1_{\rm loc} \to C^\beta_{\rm loc}$ defined by (6.9) satisfies the following.*

i. *There exists a constant C independent of ε and R such that*

$$\|R^\varepsilon(g,h)\|_{\beta,R} \le C\|g\|_{1+\beta,R+1}\|h\|_{\beta,R+1}.$$

ii. *For any fixed $g \in C^{1+\beta}_{\rm loc}$, $h \in C^\beta_{\rm loc}$ it holds $R^\varepsilon(g,h) \to 0$ in $C^{\beta'}_{\rm loc}$ as $\varepsilon \to 0$, for any $\beta' < \beta$.*

Proof It holds

$$\begin{aligned} R^\varepsilon(g,h)(x) &= \int_{B_1} h(x-\varepsilon z)\frac{g(x-\varepsilon z)-g(x)}{\varepsilon}\cdot\nabla\rho(z)\mathrm{d}z - (h\,\mathrm{div}\, g)^\varepsilon(x)\\ &=: \tilde R^\varepsilon(g,h)(x) - (h\,\mathrm{div}\, g)^\varepsilon(x). \end{aligned}$$

Thus claim *i.* follows from $\|(h\,\mathrm{div}\, g)^\varepsilon\|_{\beta,R} \le \|h\|_{1,R+1}\|g\|_{1+\beta,R+1}$ and

$$\begin{aligned} |\tilde R^\varepsilon(g,h)(x)-\tilde R^\varepsilon(g,h)(y)| &\le \left|\int_{B_1}[h(x-\varepsilon z)-h(y-\varepsilon z)]\frac{g(x-\varepsilon z)-g(x)}{\varepsilon}\cdot\nabla\rho(z)\mathrm{d}z\right|\\ &\quad+\left|\int_{B_1}h(x-\varepsilon z)\left[\frac{g(x-\varepsilon z)-g(x)}{\varepsilon}-\frac{g(y-\varepsilon z)-g(y)}{\varepsilon}\right]\cdot\nabla\rho(z)\mathrm{d}z\right|\\ &\le |x-y|^\beta\|h\|_{\beta,R+1}\|g\|_{1,R+1}\|\nabla\rho\|_{L^1}\\ &\quad+\|h\|_{0,R+1}\int_{B_1}\left|\int_0^1[\nabla g(x-\varepsilon\theta z)-\nabla g(x)-\nabla g(y-\varepsilon\theta z)+\nabla g(y)]\right|\times\\ &\quad\times|z||\nabla\rho(z)|\mathrm{d}z\\ &\lesssim |x-y|^\beta\|h\|_{\beta,R+1}\|g\|_{1+\beta,R+1} \end{aligned}$$

where the estimate is uniform in $x, y \in B_R$ and in $\varepsilon > 0$. Claim *ii.* follows from the above uniform estimate, the fact that $R^\varepsilon(g,h) \to 0$ $C^0_{\rm loc}$ by Lemma 16 from [20] and an interpolation argument. □

Proof of Proposition 6.6 We only treat the case $u \in C^\alpha_t C^0_c \cap C^0_t C^1_c$, $v \in L^\infty_t (C^0_c)^* \cap C^{\alpha\beta}_t (C^\beta_c)^*$, the other one being similar. Applying a mollifier ρ^ε on both sides of (6.7), it holds

$$u^\varepsilon_{\mathrm{d}t} + A_{\mathrm{d}t}\cdot\nabla u^\varepsilon_t + (c_{\mathrm{d}t}u_t)^\varepsilon + R^\varepsilon(A_{\mathrm{d}t}, u_t) = 0$$

where we used the definition of R^ε; equivalently by Remark 6.2, the above expression can be interpreted as

$$\|u^\varepsilon_{s,t} + A_{s,t}\cdot\nabla u^\varepsilon_s + (c_{s,t}u_s)^\varepsilon + R^\varepsilon(A_{s,t},u_s)\|_{C^0} \lesssim_\varepsilon |t-s|^{\alpha(1+\beta)} \quad \text{uniformly in } (s,t)\in\Delta_2$$

Since v is a weak solution to (6.8), it holds

$$\begin{aligned} \langle u^\varepsilon_t, v_t\rangle - \langle u^\varepsilon_s, v_s\rangle &= \langle u^\varepsilon_{s,t}, v_s\rangle + \langle u^\varepsilon_t, v_{s,t}\rangle\\ &\sim_\varepsilon -\langle A_{s,t}\cdot\nabla u^\varepsilon_t + (c_{s,t}u_t)^\varepsilon + R^\varepsilon(A_{s,t},u_t), v_s\rangle + \langle A_{s,t}\cdot\nabla u^\varepsilon_t + c_{s,t}u^\varepsilon_t, v_s\rangle\\ &\sim \langle c_{s,t}u^\varepsilon_t - (c_{s,t}u_t)^\varepsilon - R^\varepsilon(A_{s,t},u_t), v_s\rangle \end{aligned}$$

where by $a \sim_\varepsilon b$ we mean that $|a-b| \lesssim_\varepsilon |t-s|^{\alpha(1+\beta)}$. As a consequence, defining $f^\varepsilon_t := \langle u^\varepsilon_t, v_t\rangle$, we deduce that $f^\varepsilon_t - f^\varepsilon_0 = J(\Gamma^\varepsilon_{s,t})$ for the choice

$$\Gamma^\varepsilon_{s,t} := \langle c_{s,t}u^\varepsilon_t - (c_{s,t}u_t)^\varepsilon - R^\varepsilon(A_{s,t},u_t), v_s\rangle.$$

Our aim is to show that $J(\Gamma^{\varepsilon}_{s,t}) \to 0$ as $\varepsilon \to 0$; to this end, we start estimating $\|\Gamma^{\varepsilon}\|_{\alpha,\alpha(1+\beta)}$.
It holds

$$\begin{aligned}\delta\Gamma^{\varepsilon}_{s,r,t} &= \langle c_{s,r}u^{\varepsilon}_{r,t}, v_s\rangle - \langle c_{r,t}u^{\varepsilon}_t, v_{s,r}\rangle\\ &\quad+\langle c_{r,t}u_{s,r}, v^{\varepsilon}_t\rangle - \langle c_{s,r}u_s, v^{\varepsilon}_{r,t}\rangle\\ &\quad+\langle R^{\varepsilon}(A_{r,t}, u_t), v_{s,r}\rangle - \langle R^{\varepsilon}(A_{s,r}, u_{r,t}), v_s\rangle.\end{aligned}$$

Therefore, up to choosing a suitable compact $K \subset \mathbb{R}^d$, we have the estimates

$$\begin{aligned}|\Gamma^{\varepsilon}_{s,t}| &\le (\|c_{s,t}u^{\varepsilon}_t\|_{C^0_K} + \|(c_{s,t}u^{\varepsilon}_t)\|_{C^0_K} + \|R^{\varepsilon}(A_{s,t}, u_t)\|_{C^0_K})\|v_s\|_{(C^0_K)^*}\\ &\lesssim |t-s|^{\alpha}(\|c\|_{\alpha,\beta} + \|A\|_{\alpha,1})\|u\|_{C^0_tC^0_c}\|v_s\|_{(C^0_K)^*}\end{aligned}$$

as well as

$$\begin{aligned}|\delta\Gamma^{\varepsilon}_{s,r,t}| &\le \|c_{s,r}u^{\varepsilon}_{r,t}\|_{C^0_K}\|v_s\|_{(C^0_K)^*} + \|c_{r,t}u^{\varepsilon}_t\|_{C^{\beta}_K}\|v_{s,r}\|_{(C^{\beta}_K)^*}\\ &\quad+\|c_{r,t}u_{s,r}\|_{C^0_K}\|v^{\varepsilon}_t\|_{(C^0_K)^*} + \|c_{s,r}u_s\|_{C^{\beta}_K}\|v^{\varepsilon}_{r,t}\|_{(C^{\beta}_K)^*}\\ &\quad+\|R^{\varepsilon}\|\|A_{r,t}\|_{1+\beta}\|u_t\|_{C^1_K}\|v_{s,r}\|_{(C^{\beta}_K)^*} + \|R^{\varepsilon}\|\|A_{s,r}\|_{1+\beta}\|u_{r,t}\|_{C^0_K}\|v_s\|_{(C^0_K)^*}\\ &\lesssim |t-s|^{\alpha(1+\beta)}(\|c\|_{\alpha,\beta} + \|R^{\varepsilon}\|\|A\|_{\alpha,1+\beta}) \times\\ &\quad\times(\|u\|_{C^0_tC^1_K}\|v\|_{L^{\infty}_t(C^0_K)^*} + \|u\|_{C^{\alpha}_tC^0_K}\|v\|_{C^{\alpha\beta}_t(C^{\beta}_K)^*}).\end{aligned}$$

Overall we deduce that $\|\Gamma^{\varepsilon}\|_{\alpha}$ and $\|\delta\Gamma^{\varepsilon}\|_{\alpha(1+\beta)}$ are bounded uniformly in $\varepsilon > 0$; moreover by properties of convolutions and Lemma 6.7, it holds $\Gamma^{\varepsilon}_{s,t} \to 0$ as $\varepsilon \to 0$ for any $(s,t) \in \Delta_2$ fixed. By Lemma 2.1 it holds

$$|f^{\varepsilon}_{s,t} - \Gamma^{\varepsilon}_{s,t}| \lesssim |t-s|^{\alpha(1+\beta)}$$

uniformly in $\varepsilon > 0$ and so passing to the limit as $\varepsilon \to 0$ we deduce that

$$|\langle u_t, v_t\rangle - \langle u_s, v_s\rangle| \lesssim |t-s|^{\alpha(1+\beta)} \quad \forall\,(s,t) \in \Delta_2$$

which implies the conclusion. □

We are now ready to establish uniqueness of solutions to the YTE and YCE under suitable regularity conditions on (A, c).

Theorem 6.8 *Let $A \in C^{\alpha}_t C^{1+\beta}_x$, $c \in C^{\alpha}_t C^{1+\beta}_x$ with $\alpha(1+\beta) > 1$. Then for any $u_0 \in C^1_{\mathrm{loc}}$ there exists a unique solution to the YTE (6.7) with initial condition u_0 in the class $C^{\alpha}_t C^0_{\mathrm{loc}} \cap C^0_t C^1_{\mathrm{loc}}$, which is given by formula (6.5); similarly, for any $\mu_0 \in M_{\mathrm{loc}}$ there exists a unique solution to the YCE (6.8) with initial condition μ_0 in the class $L^{\infty}_t(C^0_c)^* \cap C^{\alpha\beta}_t(C^{\beta}_c)^*$, which is given by formula (6.4).*

Proof Existence follows from Proposition 6.4, so we only need to establish uniqueness. By linearity of YTE, it suffices to show that the only solution u to (6.7) in the class $C^{\alpha}_t C^0_{\mathrm{loc}} \cap C^0_t C^1_{\mathrm{loc}}$ with $u_0 \equiv 0$ is given by $u \equiv 0$. Let u be such a solution and fix $\tau \in [0, T]$; since $(\operatorname{div} A - c) \in C^{\alpha}_t C^{\beta}_x$, by Proposition 6.4 and Remark 6.5, for any compactly supported $\mu \in M$ there exists a solution $v \in L^{\infty}_t M_K \cap C^{\alpha\beta}_t(C^{\beta}_c)^*$ to (6.8) with terminal condition $v_{\tau} = \mu$, up to taking a suitable compact set K. By Proposition 6.6 it follows that

$$\langle u_{\tau}, \mu\rangle = \langle u_{\tau}, v_{\tau}\rangle = \langle u_0, v_0\rangle = 0;$$

as the reasoning holds for any compactly supported $\mu \in M$, $u_{\tau} \equiv 0$ and thus $u \equiv 0$.

Uniqueness of solutions to YCE (6.8) in the class $L^{\infty}_t(C^0_c)^* \cap C^{\alpha\beta}_t(C^{\beta}_c)^*$ follows similatly. □

7 Parabolic Nonlinear Young PDEs

We present in this section a generalization to the nonlinear Young setting of some of the results contained in [25]. Specifically, we are interested in studying a parabolic nonlinear evolutionary problem of the form

$$\mathrm{d}x_t = -Ax_t\mathrm{d}t + B(\mathrm{d}t, x_t) \tag{7.1}$$

where $-A$ is the generator of an analytical semigroup.

In order not to create confusion, in this section the nonlinear Young term will be always denoted by B. As we will use a one-parameter family of spaces $\{V_\alpha\}_{\alpha\in\mathbb{R}}$, the regularity of B will be denoted by $B \in C^\gamma_t C^\beta_{W,U}$, with W and U being taken from that family; whenever it doesn't create confusion, we will still denote the associated norm by $\|B\|_{\gamma,\beta}$.

Let us first recall the functional setting from [25], Section 2.1. It is based on the theory of analytical semigroups and infinitesimal generators, see [39] for a general reference, but the reader not acquainted with the topic may consider for simplicity $A = I - \Delta$, $V = L^2(\mathbb{R}^d)$ and $V_\alpha = H^{2\alpha}(\mathbb{R}^d)$ fractional Sobolev spaces.

Let $(V, \|\cdot\|_V)$ be a separable Banach space, $(A, \mathrm{Dom}(A))$ be an unbounded linear operator on V, $\mathrm{rg}(A)$ be its range; suppose its resolvent set is contained in $\Sigma = \{z \in \mathbb{C} : |\arg(z)| > \pi/2 - \delta\} \cup U$ for some $\delta > 0$ and some neighbourhood U of 0 and that there exist positive constants C, η such that its resolvent R_α satisfies

$$\|R_\alpha\|_{\mathcal{L}(V;V)} \le C(\eta + |\alpha|)^{-1} \quad \forall \alpha \in \Sigma.$$

Under these assumptions, $-A$ is the infinitesimal generator of an analytical semigroup $(S(t))_{t\ge 0}$ and there exist positive constants M, λ such that

$$\|S(t)\|_{\mathcal{L}(V;V)} \le Me^{-\lambda t} \quad \forall t \ge 0.$$

Moreover, $-A$ is one-to-one from $\mathrm{Dom}(A)$ to V and the fractional powers $(A^\alpha, \mathrm{Dom}(A^\alpha))$ of A can be defined for any $\alpha \in \mathbb{R}$; if $\alpha < 0$, then $\mathrm{Dom}(A^\alpha) = V$ and A^α is a bounded operator, while for $\alpha \ge 0$ $(A^\alpha, \mathrm{Dom}(A^\alpha))$ is a closed operator with $\mathrm{Dom}(A^\alpha) = \mathrm{rg}(A^{-\alpha})$ and $A^\alpha = (A^{-\alpha})^{-1}$.

For $\alpha \ge 0$, let V_α be the space $\mathrm{Dom}(A^\alpha)$ with norm $\|x\|_{V_\alpha} = \|A^\alpha x\|_V$; for $\alpha = 0$ it holds $A^0 = \mathrm{Id}$ and $V_0 = V$. For $\alpha < 0$, let V_α be the completion of V w.r.t. the norm $\|x\|_{V_\alpha} = \|A^\alpha x\|_V$, which is thus a bigger space than V. The one-parameter family of spaces $\{V_\alpha\}_{\alpha\in\mathbb{R}}$ is such that V_δ embeds continuously in V_α whenever $\delta \ge \alpha$ and $A^\alpha A^\delta = A^{\alpha+\delta}$ on the common domain of definition; moreover $A^{-\delta}$ maps V_α onto $V_{\alpha+\delta}$ for all $\alpha \in \mathbb{R}$ and $\delta \ge 0$.

The operator $S(t)$ can be extended to V_α for all $\alpha < 0$ and $t > 0$ and maps V_α to V_δ for all $\alpha \in \mathbb{R}$, $\delta \ge 0$, $t > 0$; finally, it satisfies the following properties:

$$\|A^\alpha S(t)\|_{\mathcal{L}(V;V)} \le M_\alpha t^{-\alpha} e^{-\lambda t} \text{ for all } \alpha \ge 0, t > 0; \tag{7.2}$$

$$\|S(t)x - x\|_V \le C_\alpha t^\alpha \|A^\alpha x\|_V \text{ for all } x \in V_\alpha, \alpha \in (0, 1]. \tag{7.3}$$

Remark 7.1 It follows from the statements above and the semigroup property of $S(t)$ that for any $\alpha \in \mathbb{R}$, $\delta > 0$, $x \in V_\alpha$ and any $s \le t$ it holds

$$\|S(t)x - S(s)x\|_{V_\alpha} = \|S(s)[S(t-s)x - x]\|_{V_\alpha} \lesssim_{\alpha,\delta} |t-s|^\delta \|x\|_{V_{\alpha+\delta}}$$

which implies that $\|S(t)-S(s)\|_{\mathcal{L}(V_{\alpha+\delta};V_\alpha)} \lesssim |t-s|^\delta$, equivalently $S(\cdot) \in C_t^\delta \mathcal{L}(V_{\alpha+\delta}; V_\alpha)$. It also follows that for any given $x_0 \in V_{\alpha+\delta}$, the map $t \mapsto S(t)x_0$ belongs to $C_t^\delta V_\alpha$ with

$$[\![S(\cdot)x_0]\!]_{\delta,V_\alpha} \lesssim_{\alpha,\delta} \|x_0\|_{V_{\alpha+\delta}}. \tag{7.4}$$

The following result shows that the mild solution formula for the linear equation

$$\mathrm{d}x_t = -Ax_t\mathrm{d}t + \mathrm{d}y_t,$$

which is formally given by

$$x_t = S(t)x_0 + \int_0^t S(t-s)\mathrm{d}y_s,$$

can be extended by continuity to suitable non differentiable functions $y \in C([0,T]; V)$.

Theorem 7.2 *Let $\alpha \in \mathbb{R}$ and consider the map Ξ defined for any $y \in C_t^1 V_{-\alpha}$ by*

$$\Xi(y)_t = \int_0^t S(t-s)\dot{y}_s \mathrm{d}s.$$

Then for any $\gamma > \alpha$, Ξ extends uniquely to a map $\Xi \in \mathcal{L}(C_t^\gamma V_{-\alpha}; C_t^\kappa V_\delta)$ for all $\delta \in (0, \gamma-\alpha)$ and all $\kappa \in (0, (\gamma-\alpha-\delta)\wedge 1)$. Moreover there exists a constant $C = C(\alpha,\kappa,\delta,\gamma)$ such that

$$[\![\Xi(y)]\!]_{\kappa,V_\delta} \le C[\![y]\!]_{\gamma,V_{-\alpha}}, \quad \sup_{t\in[0,T]} \|\Xi(y)_t\|_{V_\delta} \le CT^{\gamma-\delta-\alpha}[\![y]\!]_{\gamma,V_{-\alpha}}. \tag{7.5}$$

We omit the proof, for which we refer to Theorem 1 from [25]. Let us only provide an heuristic derivation of the relation between the parameters $\alpha,\kappa,\delta,\gamma$ based on a regularity counting argument. It follows from Remark 7.1 that $\|S(t-s)\|_{\mathcal{L}(V_{-\alpha};V_\delta)} \lesssim |t-s|^{-\delta-\alpha}$; if it's possible to define the map $\Xi(y)$ taking values in V_δ, then we would expect its time regularity to be analogue to that of

$$g_t := \int_0^t |t-s|^{-\delta-\alpha}\mathrm{d}f_s, \tag{7.6}$$

where now f, g are real valued functions, $f \in C_t^\gamma$; indeed, considering a fixed $y_0 \in V_{-\alpha}$, the result should also apply to $y_t := f_t y_0$. The integral in (7.6) is a type of fractional integral of order $1-\delta-\alpha$ and by hypothesis $\mathrm{d}f \in C_t^{\gamma-1}$, therefore g should have regularity $\gamma-\delta-\alpha$, which is exactly the threshold parameter for κ (this is because Hölder spaces do not behave well under fractional integration and one must always give up an ε of regularity by embedding them in nicer spaces).

Definition 7.3 Given A as above and $B \in C_t^\gamma C^\beta_{V_\delta,V_\rho}$, $\rho \le \delta$, we say that $x \in C_t^\kappa V_\delta$ is a mild solution to Eq. (7.1) with initial data $x_0 \in V_\delta$ if $\gamma + \beta\kappa > 1$, so that $\int_0^\cdot B(\mathrm{d}s, x_s)$ is well defined as a nonlinear Young integral, and if x satisfies

$$x_t = S(t)x_0 + \int_0^t S(t-s)B(\mathrm{d}s, x_s) = S(t)x_0 + \Xi\left(\int_0^\cdot B(\mathrm{d}s,x_s)\right)_t \quad \forall t \in [0,T] \tag{7.7}$$

where Ξ is the map defined by Theorem 7.2 and the equality holds in V_α for suitable α.

We are now ready to prove the main result of this section.

Theorem 7.4 *Assume A as above, $B \in C_t^\gamma C^{1+\beta}_{V_\delta,V_\rho}$ with $\rho > \delta - 1$ and suppose there exists $\kappa \in (0,1)$ such that*

$$\begin{cases} \gamma + \beta\kappa > 1 \\ \kappa < \gamma + \rho - \delta \end{cases}. \tag{7.8}$$

Then for any $x_0 \in V_{\delta+\kappa}$ there exists a unique solution with initial data x_0 to (7.1), in the sense of Definition 7.3, *in the class $C_t^\kappa V_\delta \cap C_t^0 V_{\delta+\kappa}$.*

Moreover, the solution depends in a Lipschitz way on (x_0, B), in the following sense: for any $R > 0$ exists a constant $C = C(\beta,\gamma,\delta,\rho,\kappa,T,R)$ such that for any (x_0^i, B^i), $i = 1,2$, satisfying $\|x_0^i\|_{V_{\delta+\kappa}} \vee \|B^i\|_{\gamma,1+\beta} \le R$, denoting by x^i the associated solutions, it holds

$$[\![x^1 - x^2]\!]_{\kappa,V_\rho} \le C(\|x_0^1 - x_0^2\|_{V_{\delta+\kappa}} + \|B^1 - B^2\|_{\gamma,1+\beta}).$$

Remark 7.5 If $B \in C_t^\gamma C^2_{V_\delta,V_\rho}$, then it is possible to find κ satisfying (7.8) if and only if

$$2\gamma + \rho - \delta > 1.$$

Proof The basic idea is to apply a Banach fixed point argument to the map

$$x \mapsto \mathcal{I}(x)_t := S(t)x_0 + \Xi\left(\int_0^\cdot B(\mathrm{d}s, x_s)\right)_t \tag{7.9}$$

defined on a suitable domain.

By Remark 7.1, if $x_0 \in V_{\delta+\kappa}$, then $S(\cdot)x_0 \in C_t^\kappa V_\delta$; moreover $B \in C_t^\gamma C^1_{V_\delta,V_\rho}$, so under the condition $\gamma + \kappa > 1$ the nonlinear Young integral in (7.9) is well defined for $x \in C_t^\kappa V_\delta$, $y_t = \int_0^t B(\mathrm{d}s, x_s) \in C_t^\gamma V_\rho$ and then $\Xi(y) \in C_t^\kappa V_\delta$ under the condition $\kappa < \gamma + \rho - \delta$. So under our assumptions $\mathcal{I}$ maps $C_t^\kappa V_\delta$ into itself; our first aim is to find a closed bounded subset which is invariant under I.

For suitable τ, M to be fixed later, consider the set

$$E := \{x \in C^\kappa([0,\tau]; V_\delta) : x(0) = x_0, [\![x]\!]_{\kappa,V_\delta} \le M, \sup_{t\in[0,\tau]} \|x_t\|_{V_{\delta+\kappa}} \le M\};$$

E is a complete metric space endowed with the distance $d_E(x_1,x_2) = [\![x_1 - x_2]\!]_{\kappa,V_\delta}$. It holds

$$[\![\mathcal{I}(x)]\!]_{\kappa,V_\delta} \le [\![S(\cdot)x_0]\!]_{\kappa,V_\delta} + \left[\!\!\left[\Xi\left(\int_0^\cdot B(\mathrm{d}s, x_s)\right)\right]\!\!\right]_{\kappa,V_\delta} \lesssim \|x_0\|_{V_{\delta+\rho}} + \left[\!\!\left[\int_0^\cdot B(\mathrm{d}s, x_s)\right]\!\!\right]_{\gamma,V_\rho};$$

for the nonlinear Young integral we have the estimate

$$\begin{aligned}\left\|\int_s^t B(\mathrm{d}r, x_r)\right\|_{V_\rho} &\lesssim \|B_{s,t}(x_s)\|_{V_\rho} + |t-s|^{\gamma+\kappa}[\![B]\!]_{\gamma,1}[\![x]\!]_{\kappa,V_\delta} \\ &\lesssim \|B_{s,t}(x_s) - B_{s,t}(x_0)\|_{V_\rho} + |t-s|^\gamma \|B\|_{\gamma,0} + |t-s|^\gamma \tau^\kappa [\![B]\!]_{\gamma,1}[\![x]\!]_\kappa \\ &\lesssim |t-s|^\gamma \|B\|_{\gamma,1}(1 + \tau^\kappa [\![x]\!]_{\kappa,V_\delta})\end{aligned}$$

and so

$$\left[\!\!\left[\int_0^\cdot B(\mathrm{d}r, x_r)\right]\!\!\right]_{\gamma,V_\rho} \lesssim \|B\|_{\gamma,1}(1 + \tau^\kappa [\![x]\!]_{\kappa,V_\delta}).$$

Overall, we can find a constant κ_1 such that

$$[\![\mathcal{I}(x)]\!]_{\kappa,V_\delta} \le \kappa_1 \|x_0\|_{V_{\delta+\kappa}} + \kappa_1 \|B\|_{\gamma,1}(1 + \tau^\kappa [\![x]\!]_{\kappa,V_\delta}).$$

Similar computations, together with estimate (7.5), show the existence of κ_2 such that

$$\sup_{t\in[0,\tau]} \|I(x)_t\|_{V_{\delta+\kappa}} \le \kappa_2\|x_0\|_{V_{\delta+\kappa}} + \kappa_2\|B\|_{\gamma,1}\tau^{\gamma-\delta+\rho}(1+\tau^{\kappa}[\![x]\!]_{\kappa,V_\delta}).$$

Therefore takng $\tau \le 1$, $\kappa_3 = \kappa_1 \vee \kappa_2$, in order for $\mathcal{I}$ to map E into itself it suffices

$$\kappa_3\|x_0\|_{V_{\delta+\kappa}} + \kappa_3\|B\|_{\gamma,1}(1+\tau^{\kappa}M) \le M,$$

which is always possible, for instance by requiring

$$2\kappa_3\|B\|_{\gamma,1}\tau^{\kappa} \le 1, \quad 2\kappa_3\|x_0\|_{V_{\delta+\kappa}} + 2\kappa_3\|B\|_{\gamma,1} \le M.$$

Observe that τ can be chosen independently of $\|x_0\|_{V_{\delta+\kappa}}$; moreover for the same choice of τ, analogous computations show that any solution x to (7.1) defined on $[0,\tilde{\tau}]$ with $\tilde{\tau} \le \tau$ satisfies the a priori estimate

$$[\![x]\!]_{\kappa,V_\delta;0,\tilde{\tau}} + \sup_{t\in[0,\tilde{\tau}]} \|x_t\|_{V_{\delta+\kappa}} \le \kappa_4(\|x_0\|_{V_{\delta+\kappa}} + \|B\|_{\gamma,1}) \tag{7.10}$$

for another constant κ_4, independent of x_0.

We now want to find $\tilde{\tau} \in [0,\tau]$ such that I is a contraction on $\tilde{E}$, $\tilde{E}$ being defined as E in terms of $\tilde{\tau}$, M. Given $x^1, x^2 \in \tilde{E}$, it holds

$$\begin{aligned} d_E(\mathcal{I}(x^1),\mathcal{I}(x^2)) &= \left[\!\!\left[\Xi\left(\int_0^{\cdot} B(\mathrm{d}s,x^1_s) - \int_0^{\cdot} B(\mathrm{d}s,x^2_s)\right)\right]\!\!\right]_{\kappa,V_\delta} \\ &\lesssim \left[\!\!\left[\left(\int_0^{\cdot} B(\mathrm{d}s,x^1_s) - \int_0^{\cdot} B(\mathrm{d}s,x^2_s)\right)\right]\!\!\right]_{\kappa,V_\rho} \end{aligned}$$

and under the assumptions we can apply Corollary 2.12, so we have

$$\begin{aligned} \left\| \int_s^t B(\mathrm{d}r,x^1_r) - \int_s^t B(\mathrm{d}r,x^2_r)\right\|_{V_\rho} &= \left\|\int_s^t v_{\mathrm{d}r}(x^1_r - x^2_r)\right\|_{V_\rho} \\ &\lesssim |t-s|^{\gamma}[\![v]\!]_{\gamma,\mathcal{L}}\|x^1_s - x^2_s\|_{V_\rho} + |t-s|^{\gamma+\kappa}[\![v]\!]_{\gamma,\mathcal{L}}[\![x^1-x^2]\!]_{\kappa,V_\rho} \\ &\lesssim |t-s|^{\gamma}\|B\|_{\gamma,1+\beta}(1+M)\tilde{\tau}^{\kappa}[\![x^1-x^2]\!]_{\kappa,V_\rho}. \end{aligned}$$

This implies

$$\left[\!\!\left[\int_0^{\cdot} B(\mathrm{d}r,x^1_r) - B(\mathrm{d}r,x^2_r)\right]\!\!\right]_{\gamma,V_\rho} \lesssim \|B\|_{\gamma,1+\beta}(1+M)\tilde{\tau}^{\kappa}[\![x^1-x^2]\!]_{\kappa,V_\rho}$$

and so overall, for a suitable constant κ_5,

$$d_E(\mathcal{I}(x^1),\mathcal{I}(x^2)) \le \kappa_5\|B\|_{\gamma,1+\beta}(1+M)\tilde{\tau}^{\kappa}d_E(x^1,x^2).$$

Choosing $\tilde{\tau}$ small enough such that $\kappa_5\|B\|_{\gamma,1+\beta}(1+M)\tilde{\tau}^{\kappa} < 1$, we deduce that there exists a unique solution to (7.1) defined on $[0,\tilde{\tau}]$. Since we have the uniform estimate (7.10), we can iterate the contraction argument to construct a unique solution on $[0,\tau]$; but since the choice of τ does not depend on x_0 and $x_\tau \in V_{\delta+\kappa}$, we can iterate further to cover the whole interval $[0,T]$ with subintervals of size τ.

To check the Lipschitz dependence on (x_0, B), one can reason using the Comparison Principle as usual, but let us give an alternative proof; we only check Lipschitz dependence on B, as the proof for x_0 is similar.

Given B^i, $i = 1, 2$ as above, denote by $\mathcal{I}_{B^i}$ the map associated to B^i defined as in (7.9); we can choose $\tilde{\tau}$ and M such that they are both strict contractions of constant $\kappa_6 < 1$ on E defined as before. Observe that for any $z \in E$ it holds

$$\begin{aligned} d_E(\mathcal{I}_{B^1}(z), \mathcal{I}_{B^2}(z)) &= \left[\!\left[\Xi\left(\int_0^\cdot B^1(\mathrm{d}s, z_s) - \int_0^\cdot B^2(\mathrm{d}s, z_s)\right)\right]\!\right]_{\kappa, V_\delta} \\ &\lesssim \left[\!\left[\int_0^\cdot B^1(\mathrm{d}s, z_s) - \int_0^\cdot B^2(\mathrm{d}s, z_s)\right]\!\right]_{\gamma, V_\rho} \\ &\lesssim (1+M)\|B^1 - B^2\|_{\gamma,\beta}. \end{aligned}$$

Denote by x^i the unique solutions on E associated to B^i, then by the above computation we get

$$\begin{aligned} [\![x^1 - x^2]\!]_{\kappa, V_\delta} &= d_E(\mathcal{I}_{B^1}(x^1), \mathcal{I}_{B^2}(x^2)) \\ &\le d_E(\mathcal{I}_{B^1}(x^1), \mathcal{I}_{B^1}(x^2)) + d_E(\mathcal{I}_{B^1}(x^2), \mathcal{I}_{B^2}(x^2)) \\ &\le \kappa_6 [\![x^1 - x^2]\!]_{\kappa, V_\delta} + \kappa_7 (1+M)\|B^1 - B^2\|_{\gamma,\beta} \end{aligned}$$

which implies that

$$[\![x^1 - x^2]\!]_{\kappa, V_\delta} \le \frac{\kappa_7}{1-\kappa_6}(1+M)\|B^1 - B^2\|_{\gamma,\beta}$$

which shows Lipschitz dependence on B^i on the interval $[0, \tilde{\tau}]$. As before, a combination of a priori estimates and iterative arguments allows to extend the estimate to a global one. □

By the usual localization and blow-up alternative arguments, we obtain the following result.

Corollary 7.6 *Assume A as above, $B \in C_t^\gamma C^{1+\beta}_{V_\delta, V_\rho, \mathrm{loc}}$ with $\rho > \delta - 1$ and suppose there exists $\kappa \in (0, 1)$ satisfying (7.8). Then for any $x_0 \in V_{\delta+\kappa}$ there exists a unique maximal solution x starting from x_0, defined on an interval $[0, T^*) \subset [0, T]$, such that either $T^* = T$ or*

$$\lim_{t \uparrow T^*} \|x_t\|_{V_{\delta+\kappa}} = +\infty.$$

Remark 7.7 For simplicity we have only treated here uniqueness results, but if the embedding $V_\delta \hookrightarrow V_\alpha$ for $\delta > \alpha$ is compact, as is often the case, one can use compactness arguments to deduce existence of solutions under weaker regularity conditions on B, in analogy with Theorem 3.2. Once can also consider equations of the form

$$\mathrm{d}x_t = -Ax_t \mathrm{d}t + F(x_t)\mathrm{d}t + B(\mathrm{d}t, x_t),$$

in which case uniqueness can be achieved under the same conditions on B as above and a Lipschitz condition on F, see also Remark 1 from [25].

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A Appendix

A.1 Some Useful Lemmas

We collect in this appendix some basic tools; we start with a Fubini-type result for the sewing map. In the following, the separable Banach space V is endowed with its Borel σ-algebra, the space $C_2^{\alpha,\beta}V$ with the σ-algebra induced by the norm $\|\cdot\|_{\alpha,\beta}$; recall that by the sewing lemma, $\mathcal{J}: C_2^{\alpha,\beta}V \to C_t^\alpha V$ is linear and continuous.

Lemma A.1 (Fubini for sewing map) *Let V as above, $(S, \mathcal{A}, \mu)$ a measure space and consider a measurable map $\Gamma: S \to C_2^{\alpha,\beta}V$, $\theta \mapsto \Gamma(\theta)$, such that*

$$\int_S \|\Gamma(\theta)\|_{\alpha,\beta}\mu(\mathrm{d}\theta) < \infty.$$

Then the map $\mathcal{J}\circ\Gamma : S \to C_t^\alpha V$ is measurable and it holds

$$\mathcal{J}\left(\int_S \Gamma(\theta)\mu(\mathrm{d}\theta)\right) = \int_S \mathcal{J}(\Gamma(\theta))\mu(\mathrm{d}\theta). \tag{A.1}$$

Proof Since $\mathcal{J}$ is continuous, in particular it is measurable, and so is $\mathcal{J}\circ\Gamma$ being a composition of measurable functions; it also follows that for any fixed $(s,t)\in\Delta_2$, the map $\theta\mapsto\mathcal{J}(\Gamma(\theta))_{s,t}$ is measurable from S to V. We can therefore define both integrals appearing in (A.1) as Bochner integrals, by considering them for any fixed pair $(s,t)\in\Delta_2$. For instance it holds

$$\left\|\int_S \Gamma(\theta)_{s,t}\mu(\mathrm{d}\theta)\right\|_V \le \int_S \|\Gamma(\theta)_{s,t}\|_V\mu(\mathrm{d}\theta) \le |t-s|^\alpha \int_S \|\Gamma(\theta)\|_{\alpha,\beta}\mu(\mathrm{d}\theta) < \infty$$

which also shows that the map $(s,t)\mapsto \int_S \Gamma(\theta)_{s,t}\mu(\mathrm{d}\theta)$ belongs to $C_2^{\alpha,\beta}V$ with

$$\left\|\int_S \Gamma(\theta)\mu(\mathrm{d}\theta)\right\|_{\alpha,\beta} \le \int_S \|\Gamma(\theta)\|_{\alpha,\beta}\mu(\mathrm{d}\theta).$$

In order to show that (A.1) holds, by the sewing lemma it suffices to prove that

$$\left\|\left(\int_S \Gamma(\theta)\mu(\mathrm{d}\theta)\right)_{s,t} - \int_S \mathcal{J}(\Gamma(\theta))_{s,t}\mu(\mathrm{d}\theta)\right\|_V \lesssim |t-s|^\beta \quad \forall\,(s,t)\in\Delta_2;$$

from the properties of $\mathcal{J}(\Gamma(\theta))$, we have the estimate

$$\begin{aligned}\left\|\left(\int_S \Gamma(\theta)\mu(\mathrm{d}\theta)\right)_{s,t} - \int_S \mathcal{J}(\Gamma(\theta))_{s,t}\mu(\mathrm{d}\theta)\right\|_V &\le \int_S \|\Gamma(\theta)_{s,t} - \mathcal{J}(\Gamma(\theta))_{s,t}\|_V\mu(\mathrm{d}\theta)\\ &\lesssim |t-s|^\beta \int_S \|\Gamma(\theta)\|_{\alpha,\beta}\mu(\mathrm{d}\theta)\end{aligned}$$

and the conclusion follows. □

Lemma A.2 *Let $\{\Gamma^n\}_n \subset C_2^{\alpha,\beta}V$ be a sequence such that $\sup_n \|\delta\Gamma^n\|_\beta \le R$ and $\lim_n \|\Gamma^n\|_\alpha \to 0$. Then $\mathcal{J}\Gamma^n \to 0$ in $C_t^\alpha V$ and for all n big enough it holds*

$$[\![\mathcal{J}\Gamma^n]\!]_\alpha \lesssim_{T,\beta} (1+R)\|\Gamma^n\|_\alpha^{(\beta-1)/(\beta-\alpha)}. \tag{A.2}$$

Proof Fix an interval $[s,t] \subset [0,T]$. By hypothesis, it holds

$$\|(\mathcal{J}\Gamma^n)_{s,t}\|_V \le \|\Gamma^n\|_\alpha |t-s|^\alpha + \kappa_\beta \|\delta\Gamma^n\|_\beta |t-s|^\beta;$$

splitting the interval in m subintervals of size $|t-s|/m$, applying the estimate to each of them and summing over we also have

$$\|(\mathcal{J}\Gamma^n)_{s,t}\|_V \le \|\Gamma^n\|_\alpha m^{1-\alpha}|t-s|^\alpha + \kappa_\beta \|\delta\Gamma^n\|_\beta m^{1-\beta}|t-s|^\beta. \tag{A.3}$$

By hypothesis it holds

$$\lim_{n\to\infty} \frac{\|\delta\Gamma^n\|_\beta}{\|\Gamma^n\|_\alpha} = +\infty,$$

therefore for all n big enough we can choose $m \in \mathbb{N}$ such that $m^{1-\alpha} \sim (\|\delta\Gamma^n\|_\beta/\|\Gamma^n\|_\alpha)^\theta$ for some $\theta \in (0,1)$. Then in estimate (A.3), diving by $|t-s|^\alpha$ and taking the supremum, we obtain

$$\begin{aligned} [\![\mathcal{J}\Gamma^n]\!]_\alpha &\lesssim_{T,\beta} \|\Gamma^n\|_\alpha^{1-\theta}\|\delta\Gamma^n\|_\beta^\theta + \|\Gamma^n\|_\alpha^{\theta(\beta-1)/(1-\alpha)}\|\delta\Gamma^n\|_\beta^{1-\theta(\beta-1)/(1-\alpha)} \\ &\lesssim_{T,\beta} (1+R)[\|\Gamma^n\|_\alpha^{1-\theta} + \|\Gamma^n\|_\alpha^{\theta(\beta-1)/(1-\alpha)}]. \end{aligned}$$

The conclusion follows choosing $\theta = (1-\alpha)/(\beta-\alpha)$. □

The following basic result was used in Sect. 5.2.

Lemma A.3 *Let $f \in C_V^{n+\beta}$, $z_1, z_2 \in V$. Then for any $\eta \in (0,1)$ with $\eta < n+\beta$ it holds*

$$\|f(\cdot + z_1) - f(\cdot + z_2)\|_{n+\beta-\eta} \lesssim \|z_1 - z_2\|_V^\eta \, \|f\|_{n+\beta}.$$

Proof It is enough to prove the claim in the cases $n=0$ and $n=1$, the others being similar. Assume first $n=0$, then we have the elementary estimates

$$\begin{aligned} &\|f(x+z_1) - f(y+z_1) - f(x+z_2) + f(y+z_2)\|_V \le 2\|f\|_\beta \|x-y\|_V^\beta, \\ &\|f(x+z_1) - f(y+z_1) - f(x+z_2) + f(y+z_2)\|_V \le 2\|f\|_\beta \|z_1-z_2\|_{C_V^\beta}^\beta \end{aligned}$$

which interpolated together give the conclusion.
Now consider $n=1$ and $\eta \in (\beta, 1+\beta)$, then

$$\begin{aligned} &\|f(x+z_1) - f(y+z_1) - f(x+z_2) + f(y+z_2)\|_V \\ &\qquad = \left\| \int_0^1 [Df(z_1 + y + \theta(x-y), x-y) - Df(z_2 + y + \theta(x-y), x-y)] \mathrm{d}\theta \right\|_V \\ &\qquad \lesssim \|x-y\|_V \|z_1 - z_2\|^\beta \|f\|_{1+\beta}; \end{aligned}$$

inverting the roles of z_1 and x (respectively z_2 and y) we also obtain

$$\|f(x+z_1) - f(y+z_1) - f(x+z_2) + f(y+z_2)\|_V \lesssim \|z_1 - z_2\|_V \|x-y\|^\beta \|f\|_{1+\beta}.$$

Interpolating the two inequalities again yields the conclusion. □

A.2 Alternative Constructions of Young Integrals

We collect in this appendix several other constructions of the nonlinear Young integral, although mostly equivalent to the one from Sect. 2.

In Sect. 2 we constructed the nonlinear Young integral following the modern approach based on an application of the sewing lemma, but this is not how it was first introduced in [9]. The approach therein was instead based on combining property *4.* of Theorem 2.7 with estimate (2.3); namely, the classical integral $\int_0^{\cdot} \partial_t A(s, x_s)\mathrm{d}s$ can be controlled by $\|A\|_{\alpha,\beta}$ and $\|x\|_\gamma$, and thus its definition can be extended by an approximation procedure, as the following lemma shows.

Lemma A.4 *Any $A \in C^\alpha_t C^\beta_{V,W}$ can be approximated in $C^{\alpha-}_t C^{\beta-}_{V,W}$ by a sequence A^n such that $\partial_t A^n$ exists and is continuous.*

Proof Extend A to $t \in (-\infty, \infty)$ by

$$A(t, x) = A(0, x)\mathbb{1}_{t<0} + A(t, x)\mathbb{1}_{t\in[0,T]} + A(T, x)\mathbb{1}_{t>T}$$

and consider $\rho \in C^\infty_c(\mathbb{R})$ s.t. $\rho \geq 0$, $\rho(0) = 1$ and $\int \rho(t)\mathrm{d}t = 1$. Setting $\rho^\varepsilon(t) = \varepsilon^{-1}\rho(t/\varepsilon)$ and

$$A^\varepsilon(t, x) = \int_{\mathbb{R}} \rho^\varepsilon(t - s)A(s, x)\mathrm{d}s,$$

it's immediate to check that $\sup_{(t,x)} \|A - A^\varepsilon\| \to 0$ as $\varepsilon \to 0$ by the uniform continuity of A (which is granted from the fact that $A \in C^\alpha_t C^\beta_{V,W}$). We also have the uniform bound $[\![A^\varepsilon]\!]_{\alpha,\beta} \leq [\![A]\!]_{\alpha,\beta}$, since

$$\begin{aligned}\|A^\varepsilon_{s,t}(x) - A^\varepsilon_{s,t}(y)\|_W &= \left\| \int_{\mathbb{R}} \rho^\varepsilon(u)[A(t-u, x) - A(s-u, x) - A(t-u, y) + A(s-u, y)]\mathrm{d}u \right\|_W \\ &\leq [\![A]\!]_{\alpha,\beta}|t - s|^\alpha \|x - y\|^\beta_V,\end{aligned}$$

as well as similar uniform bounds for $\|A_{s,t}\|_\beta$, etc. Interpolating these estimates together, convergence of A^ε to A in $C^{\alpha-\delta}_t C^{\beta-\delta}_{V,W}$ as $\varepsilon \to 0$, for any $\delta > 0$, immediately follows. $\square$

Observe that in the above giving up a δ of regularity is not an issue in terms of defining $\int_0^{\cdot} A(\mathrm{d}s, x_s)$, since we can always find $\delta > 0$ small enough such that it still holds $\alpha - \delta + (\beta - \delta)\gamma > 1$.

Another more functional way to define nonlinear Young integrals is the following one: for any $\beta > 0$, consider the map $J : V \to \mathcal{L}(C^\beta_{V,W}; W)$ given by $x \mapsto \delta_x$; such a map is trivially β-Hölder regular, since

$$\|Jx - Jy\|_{\mathcal{L}(C^\beta_{V,W};W)} = \sup_{g\in C^\beta_{V,W}} \frac{\|\langle Jx - Jy, g\rangle\|_W}{\|g\|_\beta} = \sup_{g\in C^\beta_{V,W}} \frac{\|g(x) - g(y)\|_W}{\|g\|_\beta} \leq \|x - y\|^\beta_V.$$

where we denoted by $\langle\cdot,\cdot\rangle$ the pairing between $\mathcal{L}(C^\beta_{V,W}; W)$ and $C^\beta_{V,W}$. Therefore for any $x \in C^\gamma_t V$, the map $t \mapsto Jx_t = \delta_{x_t}$ is now an element of $C^{\gamma\beta}_t \mathcal{L}(C^\beta_{V,W}; W)$. If on the other hand $A \in C_t^\alpha C_{V,W}^\beta$ and $\alpha + \gamma\beta > 1$, then we can define the (linear) Young integral

$$\int_0^t \langle \delta_{x_s}, A_{\mathrm{d}s} \rangle = \lim_{|\Pi|\to 0} \sum_i \langle \delta_{x_{t_i}}, A_{t_i,t_{i+1}} \rangle = \lim_{|\Pi|\to 0} \sum_i A_{t_i,t_{i+1}}(x_{t_i})$$

which immediately shows that it coincides with the definition from Sect. 2.

While this construction might seem unnecessarily abstract, it shows that nonlinear Young integrals can be regarded as linear ones, after the nonlinear transformation $x \mapsto \delta_x$ has been applied. It also allows to give intuitive derivations of several integral relations: for instance by Young product rule it must hold

$$\langle \delta_{x_t}, A_t \rangle - \langle \delta_{x_0}, A_0 \rangle = \int_0^t \langle \delta_{x_s}, A_{\mathrm{d}s} \rangle + \int_0^t \langle \mathrm{d}\delta_{x_s}, A_s \rangle$$

which is the abstract analogue of the Itô-like formula from Proposition 2.13.

We finally mention a third construction of nonlinear Young integrals, given in [29] by means of fractional calculus, in the spirit of the results by Zähle [45] for the classical Young integral. Fractional calculus is a powerful tool in the study of detailed properties of solutions to classical YDEs, see [31,32] and the references therein.

The statement therein is restricted to the case $V = \mathbb{R}^d$, although we believe the same proof extends to more general Banach spaces.

Theorem A.5 *Let $A \in C_t^\alpha C_{\mathrm{loc}}^\beta$, $x \in C_t^\gamma$ with $\alpha + \beta\gamma > 1$ and $\delta \in (1-\alpha, \beta\gamma)$. Then the following identity holds:*

$$\begin{aligned}\int_0^T A(\mathrm{d}s, x_s) = &- \frac{1}{\Gamma(\delta)\Gamma(1-\delta)} \left\{ \int_0^T \frac{A_{T-}(t, x_t)}{(T-t)^{1-\delta}} \mathrm{d}t \right. \\ &+ \delta \int_0^T \int_0^t \frac{A_{T-}(t, x_t) - A_{T-}(t, x_r)}{(T-t)^{1-\delta}(t-r)^{1+\delta}} \mathrm{d}r\mathrm{d}t \\ &+ (1-\delta) \int_0^T \int_s^T \frac{A(t, x_t) - A(s, x_t)}{(t-s)^{2-\delta} s^\delta} \mathrm{d}t\mathrm{d}s \\ &- \delta(1-\delta) \int_0^T \int_0^s \int_s^T \frac{A_{s,t}(x_t) - A_{s,t}(x_r)}{(t-s)^{2-\delta}(t-r)^{1+\delta}} \mathrm{d}t\mathrm{d}r\mathrm{d}s\end{aligned}$$

where $A_{T-}(t, z) := A(t, z) - A(T, z)$.

See Theorem 1 from [29] for a proof.

A.3 The Set of Solutions to Nonlinear YDEs

We restrict here to the case $V = \mathbb{R}^d$. Inspired by a series of results by Stampacchia, Vidossich, Browder, Gupta and others (see [43] and the references therein), we want to study the topological structure of the set

$$C(x_0, A) = \left\{ x \in C_t^\alpha \text{ such that } x_t = x_0 + \int_0^t A(\mathrm{d}s, x_s) \text{ for all } t \in [0, T] \right\}$$

where $A \in C_t^\alpha C_x^{\beta,\lambda}$ with $\alpha(1+\beta) > 1$ and $\beta + \lambda \le 1$; namely, $C(x_0, A)$ is the set of solutions to the Cauchy problem associated to (x_0, A). Recall that by Corollary 3.5 and Proposition 3.7, existence of global solutions is granted, but uniqueness is not unless $A \in C_t^\alpha C_{\mathrm{loc}}^{1+\beta}$; therefore

$C(x_0, A)$ may not consist of a singleton. The following result is an extension of Proposition 43 from [22], where the structure of the set $C(x_0; A)$ was already addressed.

Theorem A.6 *Let $A \in C_t^\alpha C_x^{\beta,\lambda}$ with α, β, λ as above, $x_0 \in \mathbb{R}^n$; then the set $C(x_0, A)$ is nonempty, compact and simply connected. Moreover, for any fixed $y \in \mathbb{R}^d$, the map*

$$\mathbb{R}^d \times C_t^\alpha C_x^{\beta,\lambda} \ni (x_0, A) \mapsto d(y, C(x_0, A)) \in \mathbb{R}$$

is lower semincontinuous.

Here we recall that for $y \in C_t^\alpha$, $K \subset C_t^\alpha$, the distance of an element from a set is defined by

$$d(y, K) = \inf_{z \in K} \|y - z\|_\alpha.$$

A main tool in the proof of Theorem A.6 is the use of the Browder–Gupta theorem from [6]; we recite here a slight modification due to Gorniewicz.

Theorem A.7 (Theorem 69.1, Chapter VI from *[23]*) *Let X be a metric space, $(E, \|\cdot\|)$ a Banach space and $f : X \to E$ a proper map, i.e. f is continuous and for every compact $K \subset E$ the set $f^{-1}(K)$ is compact. Assume further that for each $\varepsilon > 0$ a proper map $f_\varepsilon : X \to E$ is given and the following two conditions are satisfied:*

i. $\|f_\varepsilon(x) - f(x)\| \le \varepsilon$ for all $x \in X$;
ii. for any $\varepsilon > 0$ and $u \in E$ such that $\|u\| \le \varepsilon$, the equation $f_\varepsilon(x) = u$ has exactly one solution.

Then the set $S = f^{-1}(0)$ is R^δ in the sense of Aronszajn.

Recall that an R^δ set is the intersection of a decreasing sequence of compact absolute retracts, thus always simply connected.

In order to prove Theorem A.6 we need the a preliminary lemma.

Lemma A.8 *For A as above and for any $y \in C_t^\alpha$, there exists at least one solution $x \in C_t^\alpha$ to*

$$x_t = y_t + \int_0^t A(\mathrm{d}s, x_s) \qquad \forall\, t \in [0, T]; \tag{A.4}$$

moreover, there exists $C = C(\alpha, \beta, T)$ such that any solution satisfies the a priori estimate

$$\|x\|_\alpha \le C \exp(C\|A\|^2_{\alpha,\beta,\lambda} + \|y\|^2_\alpha)(1 + |y_0|). \tag{A.5}$$

If in addition $A \in C_t^\alpha C_{\mathrm{loc}}^{1+\beta}$, then the solution is unique.

Proof Set $\tilde{A}(t, x) = A(t, x) + y_t$, then x is a solution to (A.4) if and only if it solves

$$x_t = y_0 + \int_0^t \tilde{A}(\mathrm{d}s, x_s)$$

where $\tilde{A} \in C_t^\alpha C_x^{\beta,\lambda}$ with $\|\tilde{A}\|_{\alpha,\beta,\lambda} \le \|A\|_{\alpha,\beta,\lambda} + \|y\|_\alpha$. Existence and the estimate (A.5) then follow from Corollary 3.5 and Proposition 3.7; $A \in C_t^\alpha C_{\mathrm{loc}}^{1+\beta}$ implies $\tilde{A} \in C_t^\alpha C_{\mathrm{loc}}^{1+\beta}$ and so uniqueness follows from Corollary 3.13. ◻

Proof of Theorem A.6 We divide the proof in several steps.

Step 1: $C(x_0, A)$ nonempty, compact. Nonemptiness follows immediately from Lemma A.8 applied to $y \equiv x_0$; let x^n be a sequence of elements of $C(x_0, A)$, then by (A.5) they are uniformly bounded in C^α_t and so by Ascoli–Arzelà we can extract a (not relabelled) subsequence $x^n \to x$ in $C^{\alpha-\varepsilon}_t$ for all $\varepsilon > 0$, for some $x \in C^\alpha_t$. Choosing $\varepsilon > 0$ sufficiently small such that $\alpha + \beta(\alpha - \varepsilon) > 1$, by Theorem 2.7 the map $z_\cdot \mapsto \int_0^\cdot A(\mathrm{d}s, z_s)$ is continuous from $C^{\alpha-\varepsilon}_t$ to C^α_t, therefore

$$x^n_\cdot = x_0 + \int_0^\cdot A(\mathrm{d}s, x^n_s) \to x_0 + \int_0^\cdot A(\mathrm{d}s, x_s) = x_\cdot \text{ in } C^\alpha_t,$$

which shows compactness.

Step 2: $C(x_0, A)$ connected. Given $A \in C^\alpha_t C^{\beta,\lambda}_x$, consider a sequence $A^\varepsilon \in C^\alpha_t C^{1+\beta,\lambda}_x$ such that

$$\|A^\varepsilon\|_{\alpha,\beta,\lambda} \le 2\|A\|_{\alpha,\beta,\lambda}, \quad A^\varepsilon \to A \text{ in } C^\alpha_t C^\beta_{\mathrm{loc}} \quad \text{as } \varepsilon \to 0;$$

this is always possible, for instance by taking $A^\varepsilon = \rho^\varepsilon * A$, $\{\rho^\varepsilon\}_{\varepsilon>0}$ being a family of standard spatial mollifiers. For $x_0 \in \mathbb{R}^d$ fixed, take $R > 0$ big enough such that

$$C\exp(C\|A^\varepsilon\|^2_{\alpha,\beta,\lambda} + \|x_0 + y\|^2_\alpha)(1 + |y_0 + x_0|) \le R \quad \forall\, \varepsilon \in (0,1),\ y \in C^\alpha_t \text{ s.t. } \|y\|_\alpha \le 1,$$

where C is the constant appearing in (A.5); this is always possible due to the uniform bound on $\|A^\varepsilon\|_{\alpha,\beta,\lambda}$. Define the metric space E to be

$$E = \left\{z \in C^\alpha_t : \|z\|_\alpha \le R\right\}, \quad d_E(z^1, z^2) = \|z^1 - z^2\|_\alpha;$$

and define maps $f, f_\varepsilon : E \to C^\alpha_t$ by

$$f(x) = x_\cdot - x_0 - \int_0^\cdot A(\mathrm{d}s, x_s), \quad f_\varepsilon(x) = x_\cdot - x_0 - \int_0^\cdot A^\varepsilon(\mathrm{d}s, x_s).$$

By Theorem 2.7, they are continuous maps from E to C^α_t; by reasoning exactly as in Step 1 it is easy to check thar they are proper. Observe that an element $x \in E$ satisfies $f(x) = y$ if and only if it satisfies

$$x \in C^\alpha_t, \quad x_t = x_0 + y_t + \int_0^t A(\mathrm{d}s, x_s) \quad \forall\, t \in [0, T], \quad \|x\|_\alpha \le R,$$

similarly for f_ε; moreover the bound $\|x\|_\alpha \le R$ is trivially satisfied for all y such that $\|y\|_\alpha \le 1$, by our choice of R and Lemma A.8. It follows that, for any such y, $f_\varepsilon(x) = y$ has exactly one solution $x \in E$. In order to apply Theorem A.7 and get the conclusion, it remains to show that $f_\varepsilon \to f$ uniformly in E; but by Theorem 2.7 it holds

$$\begin{aligned}\|f(z) - f_\varepsilon(z)\|_\alpha &= \left\|\int_0^\cdot A(\mathrm{d}s, x_s) - \int_0^\cdot A^\varepsilon(\mathrm{d}s, x_s)\right\|_\alpha \\ &\lesssim \|A - A^\varepsilon\|_{\alpha,\beta,R}(1 + \|z\|_\alpha) \\ &\lesssim \|A - A^\varepsilon\|_{\alpha,\beta,R}(1 + R) \to 0 \text{ as } \varepsilon \to 0\end{aligned}$$

and the can conclude that $f^{-1}(0) = C(x_0, A)$ is simply connected in E, thus also in C^α_t.

Step 3: lower semicontinuity. Consider now a sequence $(x^n_0, A^n) \to (x_0, A)$ in $\mathbb{R}^d \times C^\alpha_t C^{\beta,\lambda}_x$, we need to show that for any fixed $y \in C^\alpha_t$ it holds

$$d(y, C(x_0, A)) \le \liminf_{n\to\infty} d(y, C(x^n_0, A^n)).$$

Since by Step 1 the set $C(x_0^n, A^n)$ is compact, it is always possible to find $x^n \in C(x_0^n, A^n)$ such that

$$\|y - x_0^n\| = (y, C(x_0^n, A^n));$$

we can assume wlog that $\lim d(y, C(x_0^n, A^n))$ exists, since otherwise we can extract a subsequence realizing the liminf. Since (x_0^n, A^n) is convergent, it is also bounded in $\mathbb{R}^d \times C_t^\alpha C_x^{\beta,\lambda}$, which implies by estimate (A.5) that the sequence $\{x^n\}_n$ is bounded in C_t^α. It is not difficult to see, invoking Ascoli–Arzelà and going through the same reasoning as in Step 1, that we can extract a (not relabelled) subsequence such that $x^n \to x$ in C_t^α where $x \in C(x_0, A)$. As a consequence

$$d(y, C(x_0, A)) \le \|y - x\|_\alpha = \lim_{n\to\infty} \|y - x^n\|_\alpha = \liminf_{n\to\infty} d(y, C(x_0^n, A^n))$$

which gives the conclusion. □

Theorem A.6 has relevant consequence when considering $C(x_0, A)$ as a multivalued map; we refer the reader to [7] for an overview on the topic.

Recall that, given a complete metric space (E, d), the space

$$K(E) = \{K \subset E \,:\, K \text{ is compact}\}$$

is itself a complete metric space with the Hausdorff metric

$$d_H(K_1, K_2) = \max\{\sup_{a\in K_1} d(a, K_2), \sup_{b\in K_2} d(b, K_1)\}$$

and that moreover

$$d_H(K_1, K_2) = \sup_{a\in E} |d(a, K_1) - d(a, K_2)| = \max_{a\in K_1\cup K_2} |d(a, K_1) - d(a, K_2)|.$$

If we endow the space $(K(E), d_H)$ with its Borel σ-algebra, then it's possible to show that a map $F : (\Omega, A) \to (K(E), d_H)$ is measurable if and only if, for all $a \in E$, the map

$$\Omega \ni \omega \mapsto d(a, F(\omega)) \in \mathbb{R}$$

is measurable.

Corollary A.9 *The map from $\mathbb{R}^d \times C_t^\alpha C_x^{\beta,\lambda}$ to $K(C_t^\alpha)$ given by $(x_0, A) \mapsto C(x_0, A)$ is a measurable multifunction.*

Proof It follows immediately from Theorem A.6 and the fact that lower semicontinuous maps are measurable. □

Remark A.10 For simplicity we have only treated the case $V = \mathbb{R}^d$, but it's clear that Theorem A.6 admits several extensions; for instance it can be readapted to the case of equations of the form (3.22) with $A \in C_t^\alpha C_x^{\beta,\lambda}$ and F continuous of linear growth. In alternative, one can consider a general Banach space V and $A \in C_t^\alpha C_{V,W}^{\beta,\lambda}$ with W compactly embedded in V; this is enough to grant global existence by Corollary 3.5 and the usual a priori estimates.

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