UNIVERSITÀ DEGLI STUDI DELL'AQUILA

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DIPARTIMENTO DI INGEGNERIA E SCIENZE
DELL'INFORMAZIONE E MATEMATICA

Dottorato di Ricerca in Matematica e Modelli
38 Ciclo

PI-Superalgebras with Graded Linear Maps

Settore Scientifico Disciplinare:
MATH-02/A - Algebra

Dottoranda
Ginevra Giordani

Coordinatrice del corso
Prof.ssa Debora Amadori

Supervisor
Prof. Antonio Ioppolo

Cosupervisor
Prof. Riccardo Aragona

Anno Accademico 2024/2025

A mia mamma Irene, a mio papà Lorenzo e a mia sorella Giulia,
per il vostro amore, sostegno e appoggio costante:
senza di voi niente di tutto questo sarebbe stato possibile.

A mia nonna Giuliana,
per aver creduto in me ancora prima che io potessi farlo.
A mio nonnno Giuseppe,
perché quella che sono si è ispirata a te.

A mia nonna Roberta,
per il tuo esempio di forza e resilienza.
A mio nonno Roberto,
per il tuo affetto e la tua dedizione.

Matematica: dal greco μάθημα , "Scienza, conoscenza, apprendimento", che a sua volta deriva dal verbo μανθάνω, "imparare". Μαθηματικός - η : desideroso-a di apprendere.

Contents

Abstract

This thesis is about finite-dimensional PI-superalgebras with two additional structures: pseudoautomorphism (p-algebras) and superinvolution ($\diamond$-algebras). After giving the preliminary notions, we will first develop a theory of polynomial identities for p-algebras. Then, we will discuss the behavior of the codimension sequence of central polynomials in the context of $\diamond$-algebras.

Keywords: superalgebras, polynomial identities, graded linear maps, codimension growth, central exponent.

Introduction

This thesis is on polynomial identity (PI) superalgebras with additional structures. The theory of polynomial identities is a branch of algebra arising from group theory. PI-algebras can be seen as class of algebras generalizing the commutative ones and behaving well with respect to some problems in algebra. Indeed, they have been proved to give a positive answer to some famous conjectures in group and ring theory (Kurosh problem and Burnside problem, for example).

Let F be a fixed field of characteristic zero and let $X = \{x_1, x_2, \ldots\}$ be a countable set of non-commuting variables. The free algebra $F\langle X\rangle$ contains all polynomials in the variables from X and coefficients in F.

Given an associative F-algebra A, we say that $f(x_1, \ldots, x_n) \in F\langle X\rangle$ is a polynomial identity of A ($f \equiv 0$ on A) if $f(a_1, \ldots, a_n) = 0$, for any choice of $a_i \in A$. We say that A is a PI-algebra if it satisfies at least one non trivial polynomial identity. The set of all identities satisfied by A is denoted by $\mathrm{Id}(A)$ and it is an ideal invariant under all endomorphisms of $F\langle X\rangle$ (T-ideal). The class of algebras B that satisfy the same identities of an algebra A is called the variety generated by A and denoted by $\mathrm{var}(A)$.

The main goal of the theory of polynomial identities in algebra is, given an algebra A, to study its T-ideal $\mathrm{Id}(A)$. Nevertheless, this is a very difficult problem and it has been completely solved only in very few cases.

In [41], Regev introduced the sequence of codimensions of an algebra A

$$c_n(A) \colon = \dim_F \frac{P_n}{P_n \cap Id(A)},$$

where P_n is the space of multilinear polynomials in n variables. This sequence is an effective way of giving a quantitative measure of the polynomial identities satisfied by an algebra A. The most important feature of the sequence of codimensions, proved in [41], is that, in case A is a PI-algebra, $c_n(A)$ is exponentially bounded. Later, in [30, 31], Kemer showed that, given any PI-algebra A, $c_n(A), n = 1, 2, \ldots$ cannot have intermediate growth, i.e., either it is polynomially bounded or it grows exponentially. We also mention the famous conjecture of Amitsur, proved by Giambruno and Zaicev in 1999, that states for associative PI-algebras the existence of the limit

$$\lim_{n\to\infty} \sqrt[n]{c_n(A)},$$

that is a nonnegative integer, called the exponent of an algebra A.

The growths of codimension sequences of PI-algebras are deeply studied and a celebrated theorem of Kemer ([31]) characterizes them as follows. Given a PI-algebra A, if G is the infinite dimensional Grassmann algebra over F and UT_2 is the algebra of 2×2 upper triangular matrices over F, then $c_n(A), n = 1, 2, \ldots$, is polynomially bounded if and only if G and UT_2 do not satisfy all the identities of A.

We also recall the corresponding characterizations in the context of superalgebras and superalgebras with superinvolution (see [11, 14]). We will prove an equivalent result in the setting of superalgebras with pseudoautomorphism. Indeed, throughout this thesis we will work with finite-dimensional PI-superalgebras with two additional structures: pseudoautomorphism and superinvolution.

Superalgebras play a very important role in PI-theory. Indeed, thanks to a celebrated theorem of Kemer, we are able to recover information about the identities of ordinary algebras by studying the identities of superalgebras.

We recall that a superalgebra is a $\mathbb{Z}_2$-graded algebra. Let A be an associative algebra over a field F of characteristic zero. If $\mathbb{Z}_2$ is the cyclic group of order 2, a $\mathbb{Z}_2$-grading on A is a decomposition of A, as a vector space, into the direct sum of subspaces $A = A_0 \oplus A_1$ such that $A_0A_0 + A_1A_1 \subseteq A_0$ and $A_0A_1 + A_1A_0 \subseteq A_1$. The subspaces A_0 and A_1 are the homogeneous components of A and their elements are called homogeneous of degree zero (even elements) and of degree one (odd elements), respectively.

In the last years one of the main research lines in this theory has developed on PI-superalgebras with additional structures. This happened thanks to a result of Aljadeff, Giambruno and Karasik, who proved that superalgebras with superinvolution can be studied in order to recover information about the identities of algebras with involution (see [2]). Moreover, thanks to the results in [26], we can also link the structures given by linear maps on superalgebras between them.

A pseudoautomorphism p on the superalgebra $A = A_0 \oplus A_1$ is a graded linear map (a map preserving the grading) $p\colon A \to A$ such that $(a^p)^p = (-1)^{|a||b|}a$, for all $a \in A$ and $(ab)^p = (-1)^{|a||b|}a^pb^p$, for any homogeneous elements $a, b \in A_0 \cup A_1$, of degrees $|a|$, $|b|$, respectively. We shorten the notation by referring to a superalgebra with pseudoautomorphism as a p-algebra. Pseudoautomorphisms are useful to link other graded linear maps that can be defined over superalgebras and that are deeply studied (see [26]).

A superinvolution $\diamond$ on the superalgebra $A = A_0 \oplus A_1$ is a graded linear map $\diamond\colon A \to A$ such that $(a^\diamond)^\diamond = a$, for all $a \in A$ and $(ab)^\diamond = (-1)^{|a||b|}b^\diamond a^\diamond$, for any homogeneous elements $a, b \in A_0 \cup A_1$, of degrees $|a|$, $|b|$, respectively. We shall refer to a superalgebra with superinvolution simply as a $\diamond$-algebra. The superinvolutions play a prominent role in the setting of Lie and Jordan superalgebras, beyond $\diamond$-algebras being the graded generalization of algebras with involution.

This thesis has two goals: to develop PI-theory (the theory of polynomial identities) for p-algebras and to study the behavior of the central codimension sequence for $\diamond$-algebras. This latter problem has already been addressed in the field of associative algebras or of algebras with involution or superalgebras with graded involution (see [19, 39, 42]). A central polynomial is a polynomial that has values in the center of an algebra. We can define the central codimension sequence of an algebra A, denoted by $c_n^z(A)$ as the codimension of the T-space of central multilinear polynomials in n variables. The reason of interest in its behavior is that by studying the central codimension sequence we can recover information about the codimension of identities. Indeed, holds

$$c_n(A) = c_n^z(A) + \delta_n(A),$$

where $\delta_n(A) = \dim_F \frac{P_n \cap \mathrm{Id}^z(A)}{P_n \cap \mathrm{Id}(A)}$ is the sequence of proper central polynomials, i.e., central polynomials that are not identities.

The first two chapters are introductory. We start by giving the first notions and stating the main results of the theory of polynomial identities for ordinary associative algebras. Then, in the second chapter we present PI-theory for superalgebras and its main results.

The third chapter is dedicated to p-algebras. After giving the corresponding definitions of PI-theory in this setting (p-identities, p-codimensions, p-varieties and so on), we first prove Wedderburn-Malcev decomposition theorem for p-algebras. We also recall that p-algebras can be seen also as K-graded algebras, where K is the Klein group. Then, we use the Wedderburn-Malcev decomposition theorem we proved to characterize p-algebras with polynomial growth of the p-codimension, getting to prove an equivalent in this context of the celebrated Kemer's theorem mentioned above and stated in the first chapter. After doing this, we discuss the existence of the limit

$$\exp^p(A) \colon = \lim_{n\to\infty} \sqrt[n]{c_n^p(A)},$$

called the p-exponent of a p-algebra A. This limit exists and $\exp^p(A)$ is a nonnegative integer. We close the third chapter by characterizing p-algebras with low p-exponent. We also note that the algebras we studied to give this characterization define minimal varieties with respect to the p-exponent, i.e., varieties $\mathcal{V}$ such that, for every proper subvariety $\mathcal{U}$ generated by a finite dimensional p-algebra, holds $\exp^p(\mathcal{V}) > \exp^p(\mathcal{U})$.

In the last chapter of this thesis we work with superalgebras with superinvolution. After introducing the analogous objects in this setting (the free $\diamond$-algebra, the $\diamond$-polynomial identities $\mathrm{Id}^\diamond(A)$, the sequence of $\diamond$-codimensions $c_n^\diamond(A)$, the concept of $\diamond$-variety $\mathrm{var}^\diamond(A)$ and the $\diamond$-exponent), we introduce central $\diamond$-polynomials $\mathrm{Id}^{\diamond,z}(A)$ and their codimension sequence $c_n^{\diamond,z}(A) = \dim_F \frac{P_n^\diamond}{P_n^\diamond \cap \mathrm{Id}^\diamond(A)}$.

In this chapter we study the existence and the value of the central $\diamond$-exponent of a $\diamond$-algebra A,

$$\exp^{\diamond,z}(A) \colon = \lim_{n\to\infty} \sqrt[n]{c_n^{\diamond,z}(A)}.$$

We will prove that for an associative $\diamond$-algebra A the central $\diamond$-exponent $\exp^{\diamond,z}(A)$ exists, is also a nonnegative integer and its value is less or equal than the $\diamond$-exponent. We close this thesis by giving a couple of examples of cases with different behaviors.

Chapter 1

Preliminary notions from PI-theory

This whole thesis will be about Polynomial Identity algebras (PI-algebras). In this chapter we give the preliminary notions in PI-theory that will be useful in the next chapters to present the main results of this thesis.

Through the whole thesis we will work with associative algebras.

In the first two sections we define polynomial identities, PI-algebras and central polynomials (polynomials with values in the center of an algebra) and we introduce codimension sequences and central codimension sequences of PI-algebras.

Lastly, we state the main results, introducing also the exponent of a PI-algebra. We close this chapter with the definitions of the central exponent and the proper central exponent of an algebra.

1.1 Polynomials, identities and varieties

Let F be a field and let $X = \{x_1, x_2, ...\}$ be a countable set of (non-commuting) variables.

Definition 1.1. *The free associative algebra in X over F is the algebra $F\langle X\rangle$ of all polynomials in the variables of X and with coefficients in F.*

A linear basis of $F\langle X\rangle$ consists of all words in the alphabet X including the empty word 1. Such words are called monomials and the product of two monomials is given by juxtaposition. The elements of $F\langle X\rangle$ are called polynomials, and if $f \in F\langle X\rangle$, we write $f = f(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are the only variables of X occurring in f.

We define $\deg u$, the degree of a monomial u, as the length of the word u and $\deg_{x_i} u$ the degree of u in the indeterminate x_i, i.e. the number of occurrences of x_i in u. In the same way, we define the degree $\deg f$ of a polynomial $f = f(x_1, \dots, x_n)$ as the maximum degree of a monomial in f and the degree $\deg_{x_i} f$ of f in x_i as the maximum degree of $\deg_{x_i} u$ where u is a monomial in f.

The algebra $F\langle X\rangle$ is defined, up to isomorphisms, by the following universal property.

Universal Property 1.2. *Given an associative F-algebra A, any map $X \to A$ can be uniquely extended to a homomorphism of algebras*

$$F\langle X\rangle \to A.$$

Definition 1.3. *The cardinality of X is called the rank of $F\langle X\rangle$.*

Now we introduce the main object of PI-theory.

Definition 1.4. *Let A be an associative F-algebra and $f = f(x_1, ..., x_n) \in F\langle X\rangle$. We say that f is a polynomial identity of A (or just an identity) if $f(a_1, ..., a_n) = 0$, for all $a_1, ..., a_n \in A$. We write $f \equiv 0$ on A.*

As the zero polynomial $f = 0$ is an identity for every algebra, we say that A is a PI-algebra (or a polynomial identity algebra), if it satisfies at least one non-trivial polynomial identity.

Example 1.5. Here are examples of PI algebras:

- every commutative algebra C, since it satisfies the identity $[x_1, x_2] = x_1x_2 - x_2x_1 \equiv 0$;
- every nilpotent algebra N with nilpotency index m, since it satisfies the identity $x_1 \cdots x_m \equiv 0$.

Definition 1.6. *Given an algebra A, we define the two-sided ideal of polynomial identities of A as*

$$\mathrm{Id}(A) := \{f \in F\langle X\rangle : f \equiv 0 \text{ on } A\}.$$

We recall that an ideal I of $F\langle X\rangle$ is a T-ideal if $\varphi(I) \subseteq I$ for every endomorphism φ of $F\langle X\rangle$.
It is easy to check that $\mathrm{Id}(A)$ is a T-ideal. Moreover, given a T-ideal I, it is proved that $\mathrm{Id}(F\langle X\rangle/I) = I$. Then, all T-ideals of $F\langle X\rangle$ are actually ideals of polynomial identities for a suitable algebra A.

It can happen that many algebras correspond to same set of polynomial identities. For this reason, we will define varieties of algebras.

Definition 1.7. *Given a non-empty set $S \subseteq F\langle X\rangle$, the class of all algebras A such that $f \equiv 0$ on A, for all $f \in S$, is called the variety $\mathcal{V} = \mathcal{V}(S)$ determined by S.*

A variety $\mathcal{V}$ is called non-trivial if $S \neq 0$ and $\mathcal{V}$ is proper if it is non-trivial and contains a non-zero algebra.

Example 1.8. The class of all commutative algebras forms a proper variety with $S = \{[x_1, x_2]\}$.

Notice that if $\mathcal{V}$ is the variety determined by the set S and $\langle S\rangle_T$ is the T-ideal of $F\langle X\rangle$ generated by S, then $\mathcal{V}(S) = \mathcal{V}(\langle S\rangle_T)$ and $\langle S\rangle_T = \cap_{A\in\mathcal{V}} \mathrm{Id}(A)$. We write $\langle S\rangle_T = \mathrm{Id}(\mathcal{V})$.
In the following definition we introduce the concept of relatively free algebra.

Definition 1.9. *Let $\mathcal{V}$ be a variety, $A \in \mathcal{V}$ an algebra and $Y \subseteq A$ a subset of A. We say that A is relatively free on Y (with respect to $\mathcal{V}$), if for any algebra $B \in \mathcal{V}$ and for every function $\alpha\colon Y \to B$, there exists a unique homomorphism $\beta\colon A \to B$ extending α.*

When $\mathcal{V}$ is the variety of all algebras, this is just the definition of a free algebra on Y. The cardinality of Y is called the rank of A. Relatively free algebras are easily described in terms of free algebras (see, for instance, [16, Theorem 1.2.4]).

Theorem 1.10. *Let X be a non-empty set, $F\langle X\rangle$ a free algebra on X and $\mathcal{V}$ a variety with corresponding ideal $\mathrm{Id}(\mathcal{V}) \subseteq F\langle X\rangle$. Then $F\langle X\rangle/\mathrm{Id}(\mathcal{V})$ is a relatively free algebra on the set $\overline{X} = \{x + \mathrm{Id}(\mathcal{V}) | x \in X\}$. Moreover, any two relatively free algebras with respect to $\mathcal{V}$ of the same rank are isomorphic.*

The following theorem (see for instance [16, Theorem 1.2.5]) explains the connection between T-ideals and varieties of algebras.

Theorem 1.11. *There is a one-to-one correspondence between T-ideals of $F\langle X\rangle$ and varieties of algebras. More precisely, a variety $\mathcal{V}$ corresponds to the T-ideal of identities $\mathrm{Id}(\mathcal{V})$ and a T-ideal I corresponds to the variety of algebras satisfying all the identities of I.*

If $\mathcal{V}$ is a variety and A is an algebra such that $\mathrm{Id}(A) = \mathrm{Id}(\mathcal{V})$, then we say that $\mathcal{V}$ is the variety generated by A and we write $\mathcal{V} = \mathrm{var}(A)$. Also, we shall refer to $F\langle X\rangle / \mathrm{Id}(\mathcal{V})$ as the relatively free algebra of the variety $\mathcal{V}$ of rank $|X|$.

If the ground field F is infinite, then the study of polynomial identities of an algebra A over F can be reduced to the study of the so-called homogeneous or multilinear polynomials. This reduction is very useful because these kinds of polynomials are easier to deal with.

Let $F_n = F\langle x_1, \dots, x_n\rangle$ be the free algebra of rank $n \geq 1$ over F. This algebra can be naturally decomposed as

$$F_n = F_n^{(1)} \oplus F_n^{(2)} \oplus \cdots ,$$

where, for every $k \geq 1$, $F_n^{(k)}$ is the subspace spanned by all monomials of total degree k. The $F_n^{(i)}$s are called the homogeneous components of F_n. This decomposition can be further refined as follows: for every $k \geq 1$, write

$$F_n^{(k)} = \bigoplus_{i_1+\cdots+i_n=k} F_n^{(i_1,\dots,i_n)},$$

where $F_n^{(i_1,\dots,i_n)}$ is the subspace spanned by all monomials of degree i_1 in x_1,..., i_n in x_n.

Definition 1.12. *A polynomial $f \in F_n^{(k)}$, for some $k \geq 1$, is called homogeneous of degree k. If $f \in F_n^{(i_1,\dots,i_n)}$ it will be called multihomogeneous of multidegree $(i_1, \dots, i_n)$. We also say that a polynomial f is homogeneous in the variable x_i if x_i appears with the same degree in every monomial of f.*

Among multihomogeneous polynomials a special role is played by the multilinear ones.

Definition 1.13. *A polynomial $f \in F\langle X\rangle$ is called linear in the variable x_i if x_i occurs with degree 1 in every monomial of f. Moreover, f is called multilinear if f is linear in each of its variables (multihomogeneous of multidegree $(1, \dots, 1)$).*

It is always possible to reduce an arbitrary polynomial to a multilinear one. This so-called process of multilinearization can be found, for instance, in [16, Chapter 1]. The following theorem explains the importance of multilinear polynomials. For a proof of it see [16, Corollary 1.3.9] and previous results.

Theorem 1.14. *If $\mathrm{char}F = 0$, every T-ideal is generated, as a T-ideal, by the multilinear polynomials it contains.*

Now we will define central polynomials. They will be one of the main objects of study of the last part of this thesis. Let us denote by $Z(A)$ the center of an algebra A.

Definition 1.15. *We say that $f \in F\langle X\rangle$ is a central polynomial of an algebra A if*

1. *$f(a_1, \dots, a_n) \in Z(A)$, for any $a_1, \dots, a_n$ elements of A;*

2. *its constant term is zero.*

It is easy to see that all the identities of A are central polynomials. A central polynomial f such that $f \notin \mathrm{Id}(A)$ is called a proper central polynomial of A. Here, it is important to highlight that even if an algebra A has a nonzero center, it can happen that A has no proper central polynomials.

We will denote by $\mathrm{Id}^z(A)$ the set of all central polynomials of a given algebra A and it will be called the T-space of A. In fact, $\mathrm{Id}^z(A)$ is a vector subspace of $F\langle X\rangle$ which is invariant under the endomorphisms of $F\langle X\rangle$.

1.2 Codimensions and central codimensions

The main goal of the theory of PI algebras is to study the ideal of identities of algebras. To explicitly compute $\mathrm{Id}(A)$ given an associative PI-algebra A is in general a very difficult problem that is completely solved only in very few cases. So, a combinatorial approach to this problem consists of studying how multilinear polynomials and identities lying in a given T-ideal grow as we increase the number of variables we consider. So, we will define a numerical invariant of a T-ideal: its codimension sequence. This object has been introduced by Regev in [41]. In order to give its definition, we have to set some preliminary notations.

Let A be a PI-algebra over a field F of characteristic zero and let $\mathrm{Id}(A)$ be the T-ideal of identities of A. We define

$$P_n = \mathrm{span}_F\{x_{\sigma(1)}, \dots, x_{\sigma(n)} \mid \sigma \in S_n\}$$

the vector space of multilinear polynomials in the variables $x_1, \dots, x_n$ in the free algebra $F\langle X\rangle$. From Theorem 1.14, $\mathrm{Id}(A)$ is determined by its multilinear polynomials, since $\mathrm{char} F = 0$. So, in the free associative algebra $F\langle X\rangle$, $\mathrm{Id}(A)$ is generated by

$$(P_1 \cap \mathrm{Id}(A)) \oplus (P_2 \cap \mathrm{Id}(A)) \oplus \cdots \oplus (P_n \cap \mathrm{Id}(A)) \oplus \cdots .$$

If the algebra A satisfies all the identities of some other algebra B then $(P_n \cap \mathrm{Id}(A)) \supseteq (P_n \cap \mathrm{Id}(B))$ and so $\dim((P_n \cap \mathrm{Id}(A)) \geq \dim(P_n \cap \mathrm{Id}(B))$ for every $n \geq 1$. In this sense, the dimensions of $P_n \cap \mathrm{Id}(A)$ give us the growth of the identities of an algebra A.

Finally, we can define the following object.

Definition 1.16. *Let $n \geq 1$. The nonnegative integer*

$$c_n(A) = \dim_F \frac{P_n}{P_n \cap \mathrm{Id}(A)}$$

is called the n-th codimension of the algebra A. The sequence $\{c_n(A)\}_{n\geq 1}$ is the codimension sequence of A.

In the same way, we can study the growth of the spaces of the central polynomials of an algebra and then compare it with the growth of the spaces of its identities.

Definition 1.17. *Let $n \geq 1$. The nonnegative integer*

$$c_n^z(A) = \dim_F \frac{P_n}{P_n \cap \mathrm{Id}^z(A)}$$

is called the n-th central codimension of the algebra A. The sequence $\{c_n(A)\}_{n\geq 1}$ is the central codimension sequence of A.

Note that since $\mathrm{Id}(A) \subseteq \mathrm{Id}^z(A)$, then $c_n^z(A) \leq c_n(A)$.

For $n \geq 1$, we can also define the following object.

Definition 1.18. *The sequence of proper central codimensions of an algebra A is*

$$c_n^\delta(A) = \dim_F \frac{P_n \cap \mathrm{Id}^z(A)}{P_n \cap \mathrm{Id}(A)}.$$

Lastly, from the definitions of these objects, we can see that

$$c_n(A) = c_n^z(A) + c_n^\delta(A). \tag{1.1}$$

1.3 Classical results

In this section we want to state some results in PI-theory that are fundamental in the study of codimensions and central codimensions. We start by presenting the most important results about codimensions and then we will pass to central codimensions.

By definition, the growth of the codimension sequence is bounded by n factorial. In [41], Regev showed that such a growth is much slower, provided that the algebra is PI.

Theorem 1.19 (Regev, 1972). *If A is a PI-algebra then $c_n(A)$, $n \geq 1$, is exponentially bounded.*

Definition 1.20. *Let A be a PI-algebra. A has polynomial growth if its sequence of codimensions $c_n(A)$, $n = 1, 2 \ldots$, is polynomially bounded, i.e., $c_n(A) \leq an^b$, for some constants a and b.*

If $\mathcal{V}$ is a variety of algebras, then the growth of $\mathcal{V}$ is defined as the growth of the sequence of codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}(A)$. So we define

Definition 1.21. *A variety $\mathcal{V}$ has polynomial growth if its sequence of codimensions $c_n(\mathcal{V})$, $n = 1, 2, \ldots$ is polynomially bounded. We say that $\mathcal{V}$ has almost polynomial growth if $c_n(\mathcal{V})$, $n = 1, 2, \ldots$, is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.*

Now we will see that $c_n(\mathcal{V})$ not polynomially bounded means that the growth of $c_n(\mathcal{V})$ is exponential. The reason is the following result of Kemer (see [30, 31]).

Theorem 1.22 (Kemer, 1978-79). *Let A be a PI-algebra. Then $c_n(A)$, $n \geq 1$, either is polynomially bounded or it grows exponentially.*

Now we want to present another very important theorem of Kemer. In order to do it, we first define two algebras generating varieties of almost polynomial growth. First consider the infinite-dimensional Grassmann algebra over F, that is the algebra generated by a countable set of elements

$\{e_1, e_2, \dots\}$ with the relation $e_ie_j = -e_je_i$. Then, consider UT_2, the algebra of upper triangular 2×2 matrices over the field F. These algebras have been studied in [34] and [35], so we have an explicit description of the T-ideals of identities of these algebras.

Let $\langle f_1, \dots, f_n\rangle_T$ denote the T-ideal generated by the polynomials $f_1, \dots, f_n \in F\langle X\rangle$.

Theorem 1.23. *We have*

- $\mathrm{Id}(UT_2) = \langle [x_1, x_2][x_3, x_4]\rangle_T$;
- $\mathrm{Id}(G) = \langle [[x_1, x_2], x_3]\rangle_T$.

Now we can state the theorem of Kemer.

Theorem 1.24 (Kemer, 1979). *A variety of algebras $\mathcal{V}$ has polynomial growth if and only if G, $UT_2 \notin \mathcal{V}$.*

Corollary 1.25. *The varieties* $\mathrm{var}(G)$ *and* $\mathrm{var}(UT_2)$ *are the only varieties of almost polynomial growth.*

A natural question that could come out about codimension sequence is what happens to $c_n(A)$ when n goes to infinity. More precisely, if the limit of this sequence exits. This question is known as the Amitsur's conjecture. For associative algebras, the answer is positive and we have a very strong result, proved by Giambruno and Zaicev.

Theorem 1.26 (Giambruno, Zaicev, 1999). *Let A be an associative algebra and let $c_n(A)$ be its codimension sequence. Then the limit*

$$\exp(A) \colon = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

exists and is a nonnegative integer. This number is called the exponent *of the algebra A.*

We conclude this section by recalling some results concerning the central codimension sequence. From (1.1) and by Theorem 1.19, it follows that also $c_n^z(A)$ is exponentially bounded. So, again, a natural question that comes out is asking if the corresponding limit exists. Also here, the answer is positive for associative algebras (see [18, 19]).

Theorem 1.27. *Let A be an associative algebra and let $c_n^z(A)$ and $c_n^\delta(A)$ be its central codimension sequence and proper central codimension sequence, respectively. Then the limits*

$$\exp^z(A) \colon = \lim_{n\to\infty} \sqrt[n]{c_n^z(A)} \quad \textit{and} \quad \exp^\delta(A) \colon = \lim_{n\to\infty} \sqrt[n]{c_n^\delta(A)}$$

exist and are nonnegative integers. These numbers are called the central exponent *and the* proper central exponent *of the algebra A, respectively.*

We close this chapter by noting that, from (1.1), we have that

$$\exp^z(A) \leq \exp(A) \quad \text{and} \quad \exp^\delta(A) \leq \exp(A).$$

Chapter 2

Superalgebras

In this chapter we define graded algebras in order to introduce $\mathbb{Z}_2$-graded algebras, also called superalgebras. Indeed, superalgebras will be the base object of research in this thesis. We give some examples and present the main results in PI-theory concerning them, also highlighting the importance of their role.

2.1 Graded algebras and superalgebras

Definition 2.1. *Let A be an algebra over a field F and let G be a group. The algebra A is G-graded if it can be written as the direct sum of subspaces*

$$A = \bigoplus_{g \in G} A_g$$

such that for all $g, h \in G$, $A_g A_h \subseteq A_{gh}$. The subspaces A_g are called the homogeneous components of A and, in the same way, an element $a \in A$ is homogeneous if $a \in A_g$ for some $g \in G$.

From the definition follows that any element $a \in A$ can be written as a sum

$$a = \bigoplus_{g \in G} a_g$$

with $a_g \in A_g$. In other words, a group decomposes an algebra into its homogeneous components.

In this thesis we are mainly interested in algebras graded by $\mathbb{Z}_2 \cong \{0, 1\}$.

Definition 2.2. *A superalgebra A is a $\mathbb{Z}_2$-graded algebra, i.e., an algebra that can be written as the direct sum of subspaces $A = A_0 \oplus A_1$ such that*

- $A_0 A_0 + A_1 A_1 \subseteq A_0$;
- $A_0 A_1 + A_1 A_0 \subseteq A_1$.

The subspaces A_0 and A_1 are the homogeneous components of A and their elements are called homogeneous of degree zero (even elements) and of degree one (odd elements), respectively. If a is a homogeneous element, we shall write $\deg a$ or $|a|$ to indicate its homogeneous degree. A subspace $B \subseteq A$ is graded if $B = (B \cap A_0) \bigoplus (B \cap A_1)$. In a similar way one can define graded subalgebra, graded ideals, and so on.

Note that any algebra A can be viewed as a superalgebra with trivial grading by setting $A_0 = A$ and $A_1 = 0$. Now we present some examples of superalgebras.

Example 2.3. Let $A = M_n(F)$ be the algebra of the $n \times n$ matrices on F. Given any n-tuple $(g_1, \dots, g_n) \in \mathbb{Z}_2^n$, one can define a $\mathbb{Z}_2$-grading on A, called elementary, by setting

$$(M_n(F))_0 = \operatorname{span}_F\{e_{ij} \mid g_i + g_j \equiv 0 \pmod 2\}$$
$$(M_n(F))_1 = \operatorname{span}_F\{e_{ij} \mid g_i + g_j \equiv 1 \pmod 2\}$$

where e_{ij}'s are the usual unitary matrices.

Elementary gradings play a main role for matrix algebras. In fact, from [5] we have the following result.

Theorem 2.4 (Bahturin, Zaicev, 2003). *Let F be an algebraically closed field. Then any $\mathbb{Z}_2$-grading on the matrix algebra $M_n(F)$ is an elementary grading.*

Let now $(g_1, \dots, g_n) \in \mathbb{Z}_2^n$ be an n-tuple defining an elementary grading on $M_n(F)$. It is obvious that the n-tuple $(g + g_1, \dots, g + g_n)$, $g \in \mathbb{Z}_2$, defines the same grading. In particular, one may always assume that $g_1 = 0$. Furthermore, if we permute $g_1, \dots, g_n$, according to some permutation $\sigma \in S_n$, then we get the isomorphic $\mathbb{Z}_2$-grading defined by the n-tuple $(g_{\sigma(1)}, \dots, g_{\sigma(n)})$. In fact, if A and B are the matrix algebra $M_n(F)$ with the grading defined by the n-tuple $(g_1, \dots, g_n)$ and $(g_{\sigma(1)}, \dots, g_{\sigma(n)})$, respectively, then $\varphi_\sigma \colon A \to B$ defined by $\varphi(e_{ij}) = e_{\sigma(i)} e_{\sigma(j)}$ is an isomorphism of superalgebras. In conclusion, up to isomorphism, any elementary $\mathbb{Z}_2$-grading on $M_n(F)$ is defined by the n-tuple

$$(0^k, 1^h) \colon = (\underbrace{0, \dots, 0}_{k}, \underbrace{1, \dots, 1}_{h})$$

with $k + h = n$. In this case $M_n(F)$ is the superalgebra denoted by $M_{k,h}(F)$, whose decomposition is given by

$$(M_{k,h}(F))_0 = \left\{ \begin{pmatrix} K & 0 \\ 0 & H \end{pmatrix} \mid K \in M_k(F),\ H \in M_h(F) \right\},$$
$$(M_{k,h}(F))_1 = \left\{ \begin{pmatrix} 0 & R \\ S & 0 \end{pmatrix} \mid R \in M_{k \times h}(F),\ S \in M_{h \times k}(F) \right\}.$$

Note that if $h = 0$ we have trivial grading.

Elementary gradings can also be defined on subalgebras of $M_{k,h}(F)$, so we define them in the same way on the algebra of upper triangular matrices $UT_n(F)$. We give a concrete example with $n = 2$.

Example 2.5. Let $A = UT_2(F)$. We set $h = k = 1$ and define the non trivial grading given by $(0, 1)$. The decomposition is

$$(UT_2(F))_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \mid a, c \in F \right\}, \qquad (UT_2(F))_1 = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in F \right\}.$$

We denote with UT_2 and UT_2^{sup} the algebra $UT_2(F)$ with trivial and non-trivial grading, respectively.

In the following two examples we present some last superalgebras that will be useful later on.

Example 2.6. Let F be algebraically closed. We define the superalgebra with $Q(n) = M_n(F) \oplus cM_n(F)$, where $M_n(F)$ is the algebra of $n \times n$ matrices over F and $c^2 = 1$. We can consider this algebra as a superalgebra with trivial grading or with natural grading defined by

$$(Q(n))_0 = M_n(F), \qquad (Q(n))_1 = cM_n(F).$$

Note that if $n = 1$, we get the superalgebra $F \oplus F$, that can have trivial grading or natural grading given by

$$(F \oplus F)_0 = F, \quad (F \oplus F)_1 = F.$$

This grading can be written as

$$(F \oplus F)_0 = \{(a, a) \mid a \in F\} = F(1, 1),$$
$$(F \oplus F)_1 = \{(a, -a) \mid a \in F\} = F(1, -1).$$

Example 2.7. Consider the infinite dimensional Grassmann (exterior) algebra G that we introduced in the previous chapter, i.e., the algebra generated by a countable set of elements $\{e_1, e_2, \dots\}$ satisfying the relations $e_ie_j = -e_je_i$, for all i, j. It is well known that G has a natural $\mathbb{Z}_2$-grading $G = G_0 \oplus G_1$, where G_0 is the span of all monomials in the e_is of even length and G_1 is the span of all monomials in the e_is of odd length. Since we can define, as we did before, on G also the trivial grading, in order to distinguish between the two, we denote with G^{sup} the Grassmann algebra with its natural $\mathbb{Z}_2$-grading and with G the Grassmann algebra with trivial grading.

2.2 PI-theory for superalgebras

Superalgebras are very important in PI-theory. Indeed, by studying the identities of superalgebras, we can recover information about the identities of ordinary algebras. This link is explained in the following theorem of Kemer (see [32]).

Theorem 2.8 (Kemer, 1984). *For every PI-algebra B, there exists a finite dimensional superalgebra $A = A_0 \oplus A_1$ such that*

$$\mathrm{Id}(B) = \mathrm{Id}(G(A)),$$

where $G(A) := (A_0 \otimes G_0) \oplus (A_1 \otimes G_1)$ and G is the Grassmann superalgebra $G = G_0 \oplus G_1$.

Now, let us introduce the main concepts of the theory of polynomial identities in the setting of superalgebras.

The free associative algebra $F\langle X\rangle$ on a countable set $X = \{x_1, x_2, \dots\}$ can be seen as a superalgebra as follows. We write $X = Y \cup Z$, the disjoint union of two sets $Y = \{y_1, y_2, \dots\}$ and $Z = \{z_1, z_2, \dots\}$, and we denote by $\mathcal{F}_0$ the subspace of $F\langle Y \cup Z\rangle$ spanned by all monomials in the variables of X having even degree in the variables of Z and by $\mathcal{F}_1$ the subspace spanned by all monomials of odd degree in Z. Then $F\langle Y \cup Z\rangle = \mathcal{F}_0 \oplus \mathcal{F}_1$ is a superalgebra called the free superalgebra on $X = Y \cup Z$ over F.

Definition 2.9. *A superpolynomial $f(y_1, \dots, y_n, z_1, \dots, z_m) \in F\langle Y \cup Z\rangle$ is a superidentity of the superalgebra $A = A_0 \oplus A_1$, and we write $f \equiv 0$, if, for all $v_1, \dots, v_n \in A_0$, $u_1, \dots, u_n \in A_1$, we have*

$$f(v_1, \dots, v_n, u_1, \dots, u_m) = 0.$$

We denote by $\mathrm{Id}^{sup}(A) = \{f \in F\langle Y \cup Z\rangle \mid f \equiv 0 \text{ on } A\}$ the T_2-ideal of superpolynomial identities of A (i.e., an ideal invariant under all graded endomorphisms of $F\langle Y \cup Z\rangle$).

It is well known that in characteristic zero $\mathrm{Id}^{sup}(A)$ is completely determined by its multilinear polynomials and we denote by P_n^{sup} the vector space of all multilinear polynomials of degree n in the variables $y_1, z_1, \dots, y_n, z_n$.

Definition 2.10. *The non-negative integer*

$$c_n^{sup}(A) = \dim_F \frac{P_n^{sup}}{P_n^{sup} \cap \mathrm{Id}^{sup}(A)}, \quad n \geq 1$$

is called the n-th supercodimension of A.

As showed in [15], we have an exponential bound also for supercodimensions.

Theorem 2.11 (Giambruno, Regev, 1985). *The sequence $c_n^{sup}(A)$, $n \geq 1$, is exponentially bounded.*

Given a non-empty set $S \subseteq F\langle Y \cup Z\rangle$, the class of all superalgebras $A = A_0 \oplus A_1$ such that $f \equiv 0$ on A, for all $f \in S$, is called the supervariety $\mathcal{V} = \mathcal{V}(S)$ determined by S. Moreover, if $A = A_0 \oplus A_1$ is a superalgebra, we write $\mathrm{var}^{sup}(A)$ to denote the supervariety generated by A, i.e., the class of all superalgebras $B = B_0 \oplus B_1$ such that $f \equiv 0$ on B, for all $f \in \mathrm{Id}^{sup}(A)$.

Definition 2.12. *Let $A = A_0 \oplus A_1$ be a superalgebra. A has polynomial growth if its sequence of supercodimensions $c_n^{sup}(A)$, $n \geq 1$, is polynomially bounded.*

In the same way we define the growth of a supervariety $\mathcal{V}$ as the growth of the sequence of supercodimensions of any superalgebra $A = A_0 \oplus A_1$ generating $\mathcal{V}$. Then a supervariety $\mathcal{V}$ has polynomial growth if $c_n^{sup}(\mathcal{V})$, $n \geq 1$, is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n^{sup}(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

Now we will present some important results concerning superidentities and supercodimensions of some superalgebras that we have introduced in the previous section. Recall that $\langle f_1, \dots, f_n\rangle_{T_2}$ denotes the T_2-ideal generated by the polynomials $f_1, \dots, f_n \in F\langle Y \cup Z\rangle$.

Proposition 2.13. *The superalgebra $F \oplus F$ generates a supervariety of almost polynomial growth. Moreover*

$$\mathrm{Id}^{sup}(F \oplus F) = \langle [y_1, y_2], [y, z], [z_1, z_2]\rangle_{T_2}.$$

Let us now consider the superalgebras G and UT_2 with trivial grading. Since $z \equiv 0$ on G and UT_2, then we are dealing with ordinary identities and, by Theorem 1.23, we get the following result.

Proposition 2.14. *The superalgebras G and UT_2 generate corresponding supervarieties of almost polynomial growth. Moreover*

- $\mathrm{Id}^{sup}(UT_2) = \langle [y_1, y_2][y_3, y_4], z\rangle_{T_2}$;

- $\mathrm{Id}^{sup}(G) = \langle [[y_1, y_2], y_3], z \rangle_{T_2}$.

We now consider G^{sup}. In [14] the authors proved the following result.

Proposition 2.15. *The superalgebra G^{sup} generates a supervariety of almost polynomial growth. Moreover*

$$\mathrm{Id}^{sup}(G^{sup}) = \langle [y_1, y_2], [y, z], z_1z_2 + z_2z_1 \rangle_{T_2}.$$

The following proposition collects the results concerning the superalgebra UT_2^{sup}, proved by Valenti in [43].

Proposition 2.16. *The superalgebra UT_2^{sup} generates a supervariety of almost polynomial growth. Moreover*

$$\mathrm{Id}^{sup}(UT_2^{sup}) = \langle [y_1, y_2], z_1z_2 \rangle_{T_2}.$$

Finally, we present the characterization of the supervarieties of polynomial growth (see [14]).

Theorem 2.17 (Giambruno, Mishchenko, Zaicev, 2001). *Let $\mathcal{V}$ be a supervariety of superalgebras. Then $\mathcal{V}$ has polynomial growth if and only if $F \oplus F$, G, G^{sup}, UT_2, $UT_2^{sup} \notin \mathcal{V}$.*

Corollary 2.18. *The supervarieties* $\mathrm{var}(F \oplus F)$, $\mathrm{var}(G)$, $\mathrm{var}(G^{sup})$, $\mathrm{var}(UT_2)$, $\mathrm{var}(UT_2^{sup})$ *are the only supervarieties of almost polynomial growth. Moreover, $c_n^{sup}(A)$, $n \geq 1$, either is polynomially bounded or it grows exponentially.*

The Amitsur's conjecture has a positive answer also in the setting of superalgebras (see [3, 6, 12, 23]).

Theorem 2.19 (Super exponent). *Let $A = A_0 \oplus A_1$ be an associative superalgebra and let $c_n^{sup}(A)$ be its codimension sequence. Then the limit*

$$\exp^{sup}(A) \colon = \lim_{n \to \infty} \sqrt[n]{c_n^{sup}(A)}$$

exists and is a nonnegative integer. This number is called the super exponent *of the algebra A.*

Now we want to develop the same theory for central superpolynomials.

Definition 2.20. *We say that $f \in F\langle Y \cup Z \rangle$ is a* central superpolynomial *of a superalgebra $A = A_0 \oplus A_1$ if*

1. *$f(a_1, \ldots, a_n) \in Z(A)$, for any $a_1, \ldots, a_n$ homogeneous elements in $A_0 \cup A_1$;*
2. *its constant term is zero.*

It is easy to see that all superidentities of A are central superpolynomials. A central superpolynomial f such that $f \notin \mathrm{Id}^{sup}(A)$ is called a *proper central superpolynomial* of A.

Now it is important to highlight that, even if a superalgebra $A = A_0 \oplus A_1$ has a non-zero center, it can happen that A has no proper central superpolynomials.

We will denote by $\mathrm{Id}^{sup,z}(A)$ the set of all central superpolynomials of a given superalgebra $A = A_0 \oplus A_1$ and it will be called the T_2-space of A. In fact, $\mathrm{Id}^{sup,z}(A)$ is a vector subspace of $F\langle Y \cup Z \rangle$ which is invariant by endomorphisms of $F\langle Y \cup Z \rangle$.

We want to state some results about the growth of the spaces of the central superpolynomials of a superalgebra (see [37]) and we want to compare it with the growth of the spaces of its superidentities. To this end, for $n \geq 1$ we define the *n-th central supercodimension* of A as

$$c_n^{sup,z}(A) = \dim_F \frac{P_n^{sup}}{P_n^{sup} \cap \mathrm{Id}^{sup,z}(A)}.$$

Remark 2.21. Since $\mathrm{Id}^{sup}(A) \subseteq \mathrm{Id}^{sup,z}(A)$, then $c_n^{sup,z}(A) \leq c_n^{sup}(A)$.

For $n \geq 1$, we can also define the sequence of *proper central supercodimensions of A* as

$$c_n^{sup,\delta}(A) = \dim_F \frac{P_n^{sup} \cap \mathrm{Id}^{sup,z}(A)}{P_n^{sup} \cap \mathrm{Id}^{sup}(A)}.$$

It is clear from the definitions of this objects that

$$c_n^{sup}(A) = c_n^{sup,z}(A) + c_n^{sup,\delta}(A). \tag{2.1}$$

By Remark 2.21 and considering that $c_n^{sup}(A)$ is exponentially bounded, it follows that also $c_n^{sup,z}(A)$ is exponentially bounded. So, a natural question that comes out is asking if the corresponding limit exists. The following result is contained in [37].

Theorem 2.22 (La Mattina, Martino, Rizzo, 2022). *Let $A = A_0 \oplus A_1$ be an associative PI superalgebra over a field F with characteristic zero. Then, the limits*

$$\exp^{sup,z}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{sup,z}(A)} \qquad \textit{and} \qquad \exp^{sup,\delta}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{sup,\delta}(A)}$$

exist and are a non-negative integers. These numbers are called the central exponent *and the* proper central exponent *of A, respectively.*

From Remark 2.21 and (2.1), we have that

$$\exp^{sup,z}(A) \leq \exp^{sup}(A) \qquad \text{and} \qquad \exp^{sup,\delta}(A) \leq \exp^{sup}(A).$$

Chapter 3

Superalgebras with pseudoautomorphism

In this chapter we focus on superalgebras with pseudoautomorphism.

We start by defining superalgebras with pseudoautomorphism and giving some examples. We recall that p-algebras can be also seen as K-graded algebras, where K is the Klein group. Then, after proving Wedderburn-Malcev decomposition theorem in this setting, we study the sequence of p-codimensions and develop PI-theory for p-algebras, presenting our results through all this chapter.

Assume that our fixed field F of characteristic zero contains an element i such that $i^2 = -1$.

Definition 3.1. *Let $A = A_0 \oplus A_1$ be a $\mathbb{Z}_2$-graded algebra. A pseudoautomorphism on A is a linear map*

$$p : A \to A$$

that is graded, i.e., $p(A_i) \subseteq A_i$ for $i = 0, 1$, and such that

$$p(ab) = (-1)^{|a||b|} p(a)p(b) \quad \text{and} \quad p(p(a)) = (-1)^{|a|} a,$$

where $|a|$ is the homogeneous degree of a. If A is endowed with a pseudoautomorphism, we say that A is a superalgebra with pseudoautomorphism.

We will often use the notation a^p to indicate $p(a)$. From this definition we can see that a pseudoautomorphism on the even part of a superalgebra is just an automorphism. We say that an algebra A has trivial pseudoautomorphism if A has trivial grading and $p = id$ is the identity map.

Remark 3.2. According to the hypotheses on the field F, the pseudoautomorphism p allows us to decompose the superalgebra $A = A_0 \oplus A_1$ as follows:

$$A = A_0^+ \oplus A_0^- \oplus A_1^i \oplus A_1^{-i}$$

where

$$\begin{aligned} A_0^+ &:= \{a \in A_0 : p(a) = a\}, \quad & A_0^- &:= \{a \in A_0 : p(a) = -a\}, \\ A_1^i &:= \{a \in A_1 : p(a) = ia\}, \quad & A_1^{-i} &:= \{a \in A_1 : p(a) = -ia\}, \end{aligned}$$

where $i^2 = -1$.

Definition 3.3. *An element $a \in A$ is called symmetric if $a \in A_0^+$, skew-symmetric if $a \in A_0^-$, i-symmetric if $a \in A_1^i$, and i-skew-symmetric if $a \in A_1^{-i}$.*

With the term p-algebra we will indicate a finite-dimensional PI-superalgebra with pseudoautomorphism.

Here are some examples of p-algebras.

Example 3.4. Consider the algebra $UT_2(F)$ of 2×2 upper triangular matrices over F. Recall that with e_{ij} we indicate the unitary matrices.

- The algebra UT_2 is the algebra $UT_2(F)$ with trivial pseudoautomorphism.
- The algebra UT_2^- is the algebra $UT_2(F)$ with trivial grading and (pseudo)automorphism given by
$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^p = \begin{pmatrix} a & -b \\ 0 & d \end{pmatrix}. \tag{3.1}$$
We can see that
$$UT_2^- = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in F \right\} \oplus \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in F \right\} \oplus \{0\} \oplus \{0\}.$$
- The algebra UT_2^i is the algebra $UT_2(F)$ with natural grading
$$\left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in F \right\} \oplus \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in F \right\}$$
and pseudoautomorphism given by
$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^p = \begin{pmatrix} a & ib \\ 0 & d \end{pmatrix}. \tag{3.2}$$
We have
$$UT_2^i = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in F \right\} \oplus \{0\} \oplus \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in F \right\} \oplus \{0\}.$$
- The algebra UT_2^{-i} is the algebra $UT_2(F)$ with natural grading and pseudoautomorphism given by
$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^p = \begin{pmatrix} a & -ib \\ 0 & d \end{pmatrix}. \tag{3.3}$$
So, we have
$$UT_2^{-i} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in F \right\} \oplus \{0\} \oplus \{0\} \oplus \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in F \right\}.$$

Example 3.5. Consider the algebra $F \oplus F$.

- D is the algebra $F \oplus F$ with trivial grading and (pseudo)automorphism
$$(a, b)^{ex} = (b, a).$$

Clearly
$$D = \{(a,a) \mid a \in F\} \oplus \{(a,-a) \mid a \in F\} \oplus \{0\} \oplus \{0\}.$$

- D^i is the algebra $F \oplus F$ with natural grading and pseudoautomorphism given by
$$(a,b)^{ex_i} = i^{|(a,b)|}(b,a).$$
So, we have
$$D^i = \{(a,a) \mid a \in F\} \oplus \{0\} \oplus \{0\} \oplus \{(a,-a) \mid a \in F\}.$$

- D^{-i} is the algebra $F \oplus F$ with natural grading and pseudoautomorphism given by
$$(a,b)^{ex_{-i}} = (-i)^{|(a,b)|}(b,a).$$
We can see that
$$D^{-i} = \{(a,a) \mid a \in F\} \oplus \{0\} \oplus \{(a,-a) \mid a \in F\} \oplus \{0\}.$$

Let $F\langle X\rangle$ be the free algebra on X over F. We already said that $F\langle X\rangle$ has a natural structure of superalgebra. Write $X = Y \cup Z$, the disjoint union of two sets. Then $F\langle X\rangle = F\langle Y \cup Z\rangle = \mathcal{F}_0 \oplus \mathcal{F}_1$ is a superalgebra, as we explained in the previous section. Now we define a pseudoautomorphism on $F\langle Y \cup Z\rangle$. We write $F\langle Y \cup Z, p\rangle$ for the free superalgebra with pseudoautomorphism on the countable set $Y \cup Z$ over F and we regard it as generated by even symmetric and even skew elements and by odd i-symmetric and odd i-skew elements: if for $j = 1, 2, \dots$, we let y_j^+ be the symmetric elements, y_j^- be the skew-symmetric elements, z_j^+ be the i-symmetric elements and z_j^- be the i-skew elements, then
$$F\langle Y \cup Z\rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \dots\rangle$$

Definition 3.6. *A polynomial $f(y_1^+,\dots,y_m^+,y_1^-,\dots,y_n^-,z_1^+,\dots,z_r^+,z_1^-,\dots,z_s^-) \in F\langle Y \cup Z, p\rangle$ is a p-polynomial identity of $A = A_0^+ \oplus A_0^- \oplus A_1^i \oplus A_1^{-i}$ (or simply a p-identity), and we write $f \equiv 0$, if, for all $u_1^+,\dots,u_m^+ \in A_0^+$, $u_1^-,\dots,u_n^- \in A_0^-$, $v_1^+,\dots,v_r^+ \in A_1^i$, $v_1^-,\dots,v_s^- \in A_1^{-i}$, we have that*
$$f(u_1^+,\dots,u_m^+,u_1^-,\dots,u_n^-,v_1^+,\dots,v_r^+,v_1^-,\dots,v_s^-) = 0.$$

We denote by $\mathrm{Id}^p(A) = \{f \in F\langle Y \cup Z, p\rangle | f \equiv 0 \text{ on } A\}$ the T_2^p-ideal of p-identities of A, i.e., $\mathrm{Id}^p(A)$ is an ideal of $F\langle Y \cup Z, p\rangle$ invariant under all graded endomorphisms of $F\langle Y \cup Z\rangle$ commuting with the pseudoautomorphism p. We denote by $\langle f_1,\dots,f_n\rangle_{T_2^p}$ the T_2^p-ideal generated by $f_1,\dots,f_n \in F\langle Y \cup Z, p\rangle$.

It is not difficult to prove that every p-identity is equivalent to a system of multilinear ones. Hence if we denote by
$$P_n^p = \mathrm{span}_F\{w_{\sigma(1)}\cdots w_{\sigma(n)} | \sigma \in S_n,\ w_i \in \{y_i^+, y_i^-, z_i^+, z_i^-\}, i = 1,\dots,n\}$$
the space of multilinear polynomials of degree n in the variables $y_1^+, y_1^-, z_1^+, z_1^-, \dots, y_n^+, y_n^-, z_n^+, z_n^-$ (i.e., y_i^+ or y_i^- or z_i^+ or z_i^- appears in each monomial at degree 1), the study of $\mathrm{Id}^p(A)$ is equivalent to the study of $P_n^p \cap \mathrm{Id}^p(A)$, for all $n \geq 1$.

Definition 3.7. *The non-negative integer*

$$c_n^p(A) = \dim_F \frac{P_n^p}{P_n^p \cap \mathrm{Id}^p(A)}, \quad n \geq 1,$$

is called the n-th p-codimension of A.

Let $n \geq 1$ and write $n = n_1 + \cdots + n_4$ as a sum of four nonnegative integers. We denote by $P_{n_1,\dots,n_4} \subseteq P_n^p$ the vector space of multilinear polynomials in which the first n_1 variables are symmetric, the next n_2 variables are skew, the next n_3 variables are i-symmetric and the last n_4 variables are i-skew.

Now, if we set $c_{n_1,\dots,n_4}(A) = \dim_F \frac{P_{n_1,\dots,n_4}}{P_{n_1,\dots,n_4} \cap \mathrm{Id}^p(A)}$, it is immediately seen that

$$c_n^p(A) = \sum_{n_1+\cdots+n_4=n} \binom{n}{n_1,\dots,n_4} c_{n_1,\dots,n_4}(A), \tag{3.4}$$

where $\binom{n}{n_1,\dots,n_4} = \frac{n!}{n_1!\cdots n_4!}$ stands for the multinomial coefficient. Hence, the growth of $c_n^p(A)$ is related to the growth of multinomial coefficients and of $c_{n_1,\dots,n_4}(A)$, for any $n = n_1 + \cdots + n_4$. Since for any $n = n_1 + \cdots + n_4$,

$$c_{n_1,\dots,n_4}(A) \leq c_n(A),$$

where $c_n(A)$ is the n-th ordinary codimension of A (see [10], Remark 2.1), we get that $c_n(A) \leq c_n^p(A) \leq 4^n c_n(A)$, $n = 1, 2, \dots$. So the following corollary holds.

Corollary 3.8. *Let A be a p-algebra satisfying a non-trivial ordinary identity. Then $c_n^p(A)$, $n = 1, 2, \dots,$ is exponentially bounded.*

Now we introduce the notion of variety of superalgebras with pseudoautomorphism. Roughly speaking, a p-variety $\mathcal{V}$ is a class of superalgebras with pseudoautomorphism sharing the same set of p-identities. The growth of $\mathcal{V}$ is defined as the growth of the sequence of p-codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}^p(A)$.

Definition 3.9. *A p-variety $\mathcal{V}$ has polynomial growth if $c_n^p(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n^p(\mathcal{V})$ is not polynomially bounded but every proper p-subvariety of $\mathcal{V}$ has polynomial growth.*

We close this section with a remark linking p-algebras and Klein-graded algebras.

Remark 3.10. We can see that a p-algebra A can be also viewed as a K-graded algebra, where K is the Klein group. Let $A = A_0^+ \oplus A_0^- \oplus A_1^i \oplus A_1^{-i}$ and $K = \{e, r, s, rs\}$. Then,

$$A = A_e \oplus A_r \oplus A_s \oplus A_{rs}$$

where

$$A_e = A_0^+, \quad A_r = A_0^-, \quad A_s = A_1^i, \quad A_{rs} = A_1^{-i}.$$

Vice versa, consider a K-graded algebra $B_e \oplus B_r \oplus B_s \oplus B_{rs}$, where $B_0 = B_e \oplus B_r$ and $B_1 = B_s \oplus B_{rs}$.

Define the following pseudoautomorphism p

$$\begin{aligned} p\colon B &\to B \\ b = b_e + b_r + b_s + b_{rs} &\mapsto b_e - b_r + ib_s - ib_{rs}. \end{aligned}$$

We can easily see that

$$B_0^+ = B_e, \quad B_0^- = B_r, \quad B_1^i = B_s, \quad B_1^{-i} = B_{rs}.$$

So, any K-grading on an algebra A is equivalent to a suitable pseudoautomorphism defined on the superalgebra $A = A_0 \oplus A_1$ with a certain $\mathbb{Z}_2$-grading. For more details, see [29].

3.1 Wedderburn-Malcev decomposition for p-algebras

In this section we prove a Wedderburn-Malcev theorem for p-algebras (see also [20]).

Recall that a subset (subalgebra, ideal) S of a superalgebra $A = A_0 \oplus A_1$ is said to be graded if $S = (S \cap A_0) \oplus (S \cap A_1)$.

We denote by $J(A)$ the Jacobson radical of A, which is the intersection of all maximal left ideals of A. It is well-known that $J(A)$ is a graded nilpotent ideal of A, containing any other nilpotent ideal. If $J(A) = 0$ we say that the algebra A is semisimple.

Now assume that the superalgebra A is endowed with a pseudoautomorphism p. In what follows we will often use of the exponential notation: so A^p stands for $p(A)$. We are ready to prove the following lemma.

Lemma 3.11. *Let A be a superalgebra endowed with a pseudoautomorphism p. Assume that $B = B_0 \oplus B_1$ is a graded subalgebra of A and let $I = I_0 \oplus I_1$ be a graded ideal of B. The following hold:*

1. *$B^p = B_0^p \oplus B_1^p$ is a graded subalgebra of A;*
2. *$I^p = I_0^p \oplus I_1^p$ is a graded ideal of B^p;*
3. *If I is minimal, then I^p is minimal;*
4. *If B is semisimple, then B^p is semisimple.*

Proof. The first three items are trivial. For the last one we just need to show that $J(B^p) = 0$, where $J(B^p)$ is the Jacobson radical of B^p. Since $J(B^p)$ is nilpotent, there exist the smallest positive integer m such that $J(B^p)^m = 0$. Now take $a_1, \ldots, a_m \in J(B^p)^p$. Since $J(B^p)$ is a graded ideal of B^p, for all i, we have that $a_i = (b_i + c_i)^p$, where $b_i \in J(B^p)_0$ and $c_i \in J(B^p)_1$. Then

$$a_1 \cdots a_m = \sum \alpha_j (d_1^j \cdots d_m^j)^p,$$

where $d_i^j = b_i$ or $d_i^j = c_i$, $\alpha_j = \pm 1$. But $J(B^p)^m = 0$ and so we get that $a_1 \cdots a_m = 0$. We have proved that $J(B^p)^p$ is nilpotent. Hence $J(B^p)^p \subseteq J(B)$. Since B is semisimple, it follows that $J(B^p)^p = J(B) = 0$ and so $J(B^p) = 0$. ■

An ideal I of A is called a p-ideal if it is graded and $I^p = I$.

Definition 3.12. *Let A be a p-algebra such that $A^2 \neq 0$. We say that A is*

- *simple if it has no non-trivial ideals;*
- *super simple if it has no non-trivial graded ideals;*
- *p-simple if it has no non-trivial p-ideals.*

Now we can prove the Wedderburn-Malcev theorem for p-algebras.

Theorem 3.13. *Let A be a finite dimensional p-algebra over F. Then there exists a semisimple superalgebra B with pseudoautomorphism such that*

$$A = B + J(A).$$

Moreover $J(A)$ is a p-ideal of A and B can be further decomposed in the direct sum of simple p-algebras:

$$B = B_1 \oplus \cdots \oplus B_m.$$

Proof. By the Wedderburn-Malcev theorem for superalgebras ([17]) we can write

$$A = B + J = B_1 \oplus \cdots \oplus B_m + J,$$

where B is a semisimple graded subalgebra of A, the B_i's are simple superalgebras and the Jacobson radical $J = J(A) = J_0 \oplus J_1$ is a graded ideal of A. Since J is nilpotent, following the proof of Lemma 3.11, we get that also J^p is a nilpotent ideal of A. Being J the maximal nilpotent ideal of A, then $J^p \subseteq J$ and so $J^p = J$. In conclusion J is a p-ideal of A.

If $J = 0$ or $B = B^p$ there is nothing to prove. So assume that $J \neq 0$ and $B \neq B^p$.

First consider the case in which $J^2 = 0$. By item (4) of Lemma 3.11, B^p is a semisimple graded subalgebra of A. Moreover, according to [10, Lemma 4.2], there exists $x_0 \in J_0$ such that

$$B^p = (1 + x_0)B(1 - x_0).$$

For any homogeneous element $b \in B$ we have that

$$b^p = (1 + x_0)\bar{b}(1 - x_0),$$

for some element $\bar{b} \in B$ with the same homogeneous degree as b^p and b. Hence

$$\begin{aligned}(-1)^{|b|}b = (b^p)^p = ((1 + x_0)\bar{b}(1 - x_0))^p &= (1 + x_0^p)\bar{b}^p(1 - x_0^p)\\ &= (1 + x_0^p)(1 + x_0)\tilde{b}(1 - x_0)(1 - x_0)^p\\ &= (1 + x_0^p + x_0)\tilde{b}(1 - x_0^p - x_0)\\ &= \tilde{b} + x_0^p\tilde{b} + x_0\tilde{b} - \tilde{b}x_0^p - \tilde{b}x_0\end{aligned}$$

for some $\tilde{b} \in B_0 \cup B_1$ with the same homogeneous degree as b. Since $J \cap B = 0$, we obtain that

$$(-1)^{|b|}b = \tilde{b} \quad \text{and} \quad (x_0^p + x_0)\tilde{b} = \tilde{b}(x_0 + x_0^p).$$

It follows that

$$\begin{aligned} b^p &= (1+x_0)\bar{b}(1-x_0) \\ &= \left(1+\frac{x_0+x_0^p}{2}+\frac{x_0-x_0^p}{2}\right)\bar{b}\left(1-\frac{x_0+x_0^p}{2}-\frac{x_0-x_0^p}{2}\right) \\ &= \left(1+\frac{x_0-x_0^p}{2}\right)\ \bar{b}\ \left(1-\frac{x_0-x_0^p}{2}\right) \\ &= \left(1+x_0^-\right)\bar{b}\left(1-x_0^-\right) \end{aligned}$$

where $x_0^- = \dfrac{x_0 - x_0^p}{2} \in J_0^-$.

To complete the proof of this case consider the following subalgebra of A:

$$C = \left(1+\frac{x_0^-}{2}\right) B \left(1-\frac{x_0^-}{2}\right).$$

Clearly C is a graded subalgebra of A and by the above it is also a p-subalgebra. Moreover, since $C \cong B$, it is a semisimple p-subalgebra of A and we are done.

Now let us consider the general case in which $J^2 \neq 0$. Choose $m \geq 2$ such that $J^m \neq 0$ and $J^{m+1} = 0$. It is easy to see that J^m is a p-ideal of A and, so, A/J^m is a superalgebra with induced pseudoautomorphism. Its Jacobson radical $J(A/J^m) = J(A)/J^m$ is such that $J(A/J^m)^m = 0$. Hence, by induction on m, we have that there exists a semisimple p-subalgebra B'/J^m such that

$$A/J^m = B'/J^m + J/J^m.$$

From $J(B'/J^m) = 0$ it follows that $J(B') = J^m$ and, so, we can write

$$B' = C + J^m,$$

where C is a semisimple graded subalgebra of B'. Since $(J^m)^2 \subseteq J^{2m} = 0$, by the first part of the proof we can assume $C^p = C$, i.e., C is a semisimple p-subalgebra of A and we are done since

$$A = B' + J = C + J^m + J = C + J.$$

In order to complete the proof of the theorem we are left to show that the semisimple part can be further decomposed in the direct sum of p-simple algebras. By the Wedderburn-Malcev theorem for superalgebras, we can write

$$C = D_1 \oplus \cdots \oplus D_h,$$

where $D_1, \ldots, D_h$ are all the minimal graded ideals of C. Hence, by Lemma 3.11, for every i, D_i^p is also a minimal graded ideal of C and, so, $D_i^p = D_j$, for some $j \in \{1, \ldots, h\}$.

We now rename $D_1, \ldots, D_h$ and we write

$$C = C_1 \oplus \cdots \oplus C_m,$$

where either $C_i = D_j$ with $D_j = D_j^p$ or $C_i = D_j \oplus D_j^p$, with $D_j \neq D_j^p$. Thus $C_1, \ldots, C_m$ are minimal p-ideals of C, i.e., p-simple algebras. ■

We conclude this section by refining the above decomposition in light of the classification of p-simple algebras obtained by Ioppolo and La Mattina in [27]. In order to do it, we need some preliminary notions and to keep in mind the superalgebras we presented in the previous section.

Given a superalgebra B, let $\bar{B}$ denote the superalgebra with the same graded vector space structure as B but with product $\circ$ given on homogeneous elements a, b by the formula

$$a \circ b := (-1)^{|a||b|} ab.$$

Two p-algebras (A, p) and (A', p') are isomorphic if there exists an isomorphism of superalgebras $\tau \colon A \to A'$ such that $\tau(a^p) = \tau(a)^{p'}$, for any $a \in A$.

Assume that our field F is also algebraically closed.

Theorem 3.14. *Let A be a finite dimensional p-simple superalgebra with pseudoautomorphism over F. Then A is isomorphic to one of the following:*

1. $M_{k,h}(F) \oplus \overline{M_{k,h}(F)}$ *with the pseudoautomorphism pex given on homogeneous elements by*

$$(a, b)^{pex} = \left((-1)^{|(a,b)|} b, a\right).$$

2. $Q(n) \oplus \overline{Q(n)}$*, with the pseudoautomorphism pex defined above.*

3. $M_{k,h}(F)$ *endowed with the following pseudoautomorphism*

$$\begin{pmatrix} K & R \\ S & H \end{pmatrix}^p = \begin{pmatrix} PKP & \pm iPRQ \\ \pm iQSP & QHQ \end{pmatrix},$$

where $P = \begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{pmatrix}, Q = \begin{pmatrix} I_{h_1} & 0 \\ 0 & -I_{h_2} \end{pmatrix}, I_{k_1}, I_{h_1}, I_{k_2}, I_{h_2}$ *are the identity matrices of orders* k_1, k_2, h_1, h_2*, resp.,* $k = k_1 + k_2, h = h_1 + h_2$*.*

4. $M_{k,k}(F)$ *endowed with the following pseudoautomorphism*

$$\begin{pmatrix} K & R \\ S & H \end{pmatrix}^p = \begin{pmatrix} H & iS \\ iR & K \end{pmatrix}.$$

5. $Q(n)$ *endowed with the following pseudoautomorphism*

$$p(a + cb) = f(a) \pm icf(b),$$

where f is an automorphism of order ≤ 2 on $M_n(F)$.

3.2 A Kemer-like theorem

This section is dedicated to the characterization of the varieties of superalgebras with pseudoautomorphism of polynomial codimension growth generated by finite dimensional algebras. We will prove a theorem whose role is analogous as the one of the Kemer's theorem for ordinary PI-algebras (Theorem 1.24). We follow [20].

In the following proposition we collect some results concerning some suitable p-algebras.

Proposition 3.15. *We have that*

- $\mathrm{Id}^p(D) = \langle [x_1, x_2], z^+, z^- \rangle_{T_2^p}$.
- $\mathrm{Id}^p(D^i) = \langle [x_1, x_2], y^-, z^+ \rangle_{T_2^p}$.
- $\mathrm{Id}^p(D^{-i}) = \langle [x_1, x_2], y^-, z^- \rangle_{T_2^p}$.
- $\mathrm{Id}^p(UT_2) = \langle [y_1^+, y_2^+][y_3^+, y_4^+], y^-, z^+, z^- \rangle_{T_2^p}$.
- $\mathrm{Id}^p(UT_2^-) = \langle [y_1^+, y_2^+], y_1^- y_2^-, z^+, z^- \rangle_{T_2^p}$.
- $\mathrm{Id}^p(UT_2^i) = \langle [y_1^+, y_2^+], y^-, z_1^+ z_2^+, z^- \rangle_{T_2^p}$.
- $\mathrm{Id}^p(UT_2^{-i}) = \langle [y_1^+, y_2^+], y^-, z^+, z_1^- z_2^- \rangle_{T_2^p}$.
- $c_n^p(D) = c_n^p(D^i) = c_n^p(D^{-i}) = 2^n$.
- $c_n^p(UT_2) = 2^{n-1}(n-2) + 2$.
- $c_n^p(UT_2^-) = c_n^p(UT_2^i) = c_n^p(UT_2^{-i}) = n2^{n-1} + 1$.
- *All these algebras generate varieties of almost polynomial growth.*

Proof. The algebra D has trivial grading and it is commutative. Hence its pseudoautomorphism ex can be seen as an involution and this case was completely treated in [13]. The same approach can be used to deal with the algebras D^i and D^{-i}. Since UT_2 has trivial grading and trivial pseudoautomorphism, one can see it just as an ordinary algebra (see [30, 38]). The case of the algebra UT^- is analogous to that one treated by Valenti in [43]. The same approach can be used to complete the cases of UT^i and UT^{-i}. ■

In order to prove the last theorem of this section we need two lemmas.

Lemma 3.16. *Assume that F is algebraically closed. Let A be a finite dimensional p-algebra over F. Suppose that D, D^i, D^{-i}, UT_2, UT_2^-, UT_2^i, $UT_2^{-i} \notin \mathrm{var}^p(A)$. Then $A = B + J(A)$, where $B \cong F \oplus \cdots \oplus F$ is endowed with trivial (induced) pseudoautomorphism.*

Proof. By Theorem 3.13,

$$A = B_1 \oplus \cdots \oplus B_m + J,$$

where $B_1, \ldots, B_m$ are finite dimensional p-simple superalgebras and J is the Jacobson radical of A. According to Theorem 3.14, we have to consider the following cases.

Case 1. $B_i \cong (B \oplus \overline{B}, pex)$, where $B = M_{k,h}(F)$ or $B = Q(n)$.

Consider the subalgebra $C = \langle a, b\rangle$ of B_i generated by the even elements $a = (e_{1,1}, e_{1,1})$ and $b = (e_{1,1}, -e_{1,1})$ with induced grading and pseudoautomorphism. The linear map

$$\begin{aligned} g\colon C &\to D \\ a &\mapsto (1,1) \\ b &\mapsto (1,-1) \end{aligned}$$

is an isomorphism of superalgebras (trivial grading) with pseudoautomorphism. Hence, $D \in \mathrm{var}^p(B_i) \subseteq \mathrm{var}^p(A)$ and we have a contradiction.

Case 2. $B_i \cong M_{k,h}(F)$ with pseudoautomorphism

$$\begin{pmatrix} K & R \\ S & H \end{pmatrix}^p = \begin{pmatrix} PKP & \pm iPRQ \\ \pm iQSP & QHQ \end{pmatrix}$$

where $P = \begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{pmatrix}$, $Q = \begin{pmatrix} I_{h_1} & 0 \\ 0 & -I_{h_2} \end{pmatrix}$, $k_1 + k_2 = k$ and $h_1 + h_2 = h$.

Assume first $h = 0$ and consider the subalgebra $C = \langle a, b, c\rangle$ of B_i generated by the elements $a = e_{1,1}$, $b = e_{1,k_1+k_2}$ and $c = e_{k_1+k_2,k_1+k_2}$. It is clearly a graded subalgebra with (induced) pseudoautomorphism isomorphic to UT_2 in case $k_2 = 0$ and to UT_2^- in case $k_2 > 0$. Hence UT_2 or UT_2^- belongs to $\mathrm{var}^p(B_i) \subseteq \mathrm{var}^p(A)$, a contradiction.

Now suppose that $h > 0$. Consider the subalgebra $C = \langle a, b, c\rangle$, where $a = e_{1,1}$, $b = e_{1,k+h}$ and $c = e_{k+h,k+h}$. Clearly C is a graded subalgebra with pseudoautomorphism.

Since $p(e_{1,k+h}) = \pm ie_{1,k+h}$, we get that C is isomorphic (as a p-algebra) to UT_2^i or UT_2^{-i}, via the map g given by

$$g(a) = e_{1,1}, \quad g(b) = e_{1,2}, \quad g(c) = e_{2,2}.$$

This implies that UT_2^i or UT_2^{-i} belongs to $\mathrm{var}^p(B_i) \subseteq \mathrm{var}^p(A)$, a contradiction.

Case 3. $B_i \cong M_{k,k}(F)$, with pseudoautomorphism

$$\begin{pmatrix} K & R \\ S & H \end{pmatrix}^p = \begin{pmatrix} H & iS \\ iR & K \end{pmatrix}.$$

We define the subalgebra $C := \langle e_{1,1}, e_{k+1,k+1}\rangle$ of B_i with induced (trivial) grading and pseudoautomorphism. The linear map

$$\begin{aligned} f\colon C &\to D \\ e_{1,1} &\mapsto (1,0) \\ e_{k+1,k+1} &\mapsto (0,1) \end{aligned}$$

is an isomorphism of p-algebras. So we have a contradiction, as $D \in \mathrm{var}^p(B_i) \subseteq \mathrm{var}^p(A)$.

Case 4. $B_i = Q(n)$ with pseudoautomorphism

$$p(a + cb) = f(a) \pm icf(b),$$

where f is an automorphism of order ≤ 2 on $M_n(F)$.

Assume first that we have a $+$ in the pseudoautomorphism p. If $n = 1$, we have that $Q(1) = \langle 1, c \rangle \cong D^{-i}$ via the map

$$\begin{aligned} f\colon Q(1) &\to D^{-i} \\ 1 &\mapsto (1,1) \\ c &\mapsto (1,-1) \end{aligned}$$

and so we get immediately a contradiction. Suppose then that $n > 1$ and write $I_n = e_{1,1} + \cdots + e_{n,n}$. Consider the subalgebra $C := \langle I_n, cI_n \rangle$ of B_i with induced grading and pseudoautomorphism. The linear map

$$\begin{aligned} g\colon C &\to D^{-i} \\ I_n &\mapsto (1,1) \\ cI_n &\mapsto (1,-1) \end{aligned}$$

is an isomorphism of p-algebras. In conclusion we reach a contradiction, as $D^{-i} \in \mathrm{var}^p(B_i) \subseteq \mathrm{var}^p(A)$.

Finally, in case we have $-$ in the pseudoautomorphism p, we obtain the corresponding contradiction with the algebra D^i.

We have showed that, for all $1 \leq i \leq m$, $B_i \cong F$ with trivial pseudoautomorphism. The proof is complete. ■

Lemma 3.17. *Assume that F is algebraically closed. Let $A = B_1 \oplus \cdots \oplus B_m + J$ be a finite dimensional p-algebra over F, where for every $i = 1, \ldots, m$, $B_i \cong F$ is endowed with the trivial pseudoautomorphism. If $UT_2, UT_2^-, UT_2^i, UT_2^{-i} \notin \mathrm{var}^p(A)$ then $B_i J B_k = 0$, for all $1 \leq i, k \leq m$, $i \neq k$.*

Proof. Suppose that there exist $i, k \in \{1, \ldots, m\}$, $i \neq k$, such that $B_i J B_k \neq 0$. Then there exist $a \in B_i$, $b \in B_k$, $j \in J$ such that $ajb \neq 0$, with $a^2 = a = a^p$, $b^2 = b = b^p$ and $|a| = |b| = 0$. We may assume without loss of generality that j is an homogeneous element.

Let C be the graded subalgebra of A with induced pseudoautomorphism generated by a, b, ajb. Consider the linear map $f\colon UT_2(F) \to C$ defined by

$$\begin{aligned} e_{1,1} &\mapsto a \\ e_{2,2} &\mapsto b \\ e_{1,2} &\mapsto ajb. \end{aligned}$$

Clearly f is an isomorphism of ordinary algebras. Moreover, if $|j| = 0$, we have isomorphisms of p-algebras between C and UT_2 or UT_2^-, according as j is symmetric or skew. Analogously if $|j| = 1$, we get that C is isomorphic as a p-algebra to UT_2^i or UT_2^{-i}, according as j is i-symmetric or i-skew.

In all the four cases, we reach a contradiction since by hypothesis UT_2, UT_2^-, UT_2^i, $UT_2^{-i} \notin \mathrm{var}^p(A)$. ■

Finally we are in a position to prove the main result of this section.

Theorem 3.18. *Let A be a finite dimensional p-algebra over F. Then the sequence $c_n^p(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if D, D^i, D^{-i}, UT_2, UT_2^-, UT_2^i, $UT_2^{-i} \notin \mathrm{var}^p(A)$.*

Proof. By Proposition 3.15, all the above algebras generate varieties of exponential growth. Hence, if $c_n^p(A)$ is polynomially bounded, they cannot belong to the variety generated by A.

In order to prove the other direction, notice that, since we are dealing with codimensions that do not change by extending the base field, we may assume that the field F is algebraically closed. Hence, by Theorem 3.13 and Lemmas 3.16 and 3.17,

$$A = B_1 \oplus \cdots \oplus B_m + J,$$

where for every $i = 1, \ldots, m$, $B_i \cong F$ is endowed with the trivial pseudoautomorphism and $B_i J B_k = 0$, for all $1 \leq i, k \leq m$, $i \neq k$. Hence $A_0^- \oplus A_1^i \oplus A_1^{-i} \subseteq J$ and, if q is the least positive integer such that $J^q = 0$, then $A_0^- \oplus A_1^i \oplus A_1^{-i}$ generates a nilpotent ideal of A of nilpotency index $\leq q$. This says that $c_{n_1,n_2,n_3,n_4}(A) = 0$ as soon as $n_2 + n_3 + n_4 \geq q$ (see [33, Theorem 2.2]). Hence, by (3.4), we get:

$$c_n^p(A) = \sum_{\substack{n_1+\cdots+n_4 \\ n_2+n_3+n_4<q}} \binom{n}{n_1, \ldots, n_4} c_{n_1,\ldots,n_4}(A). \tag{3.5}$$

Notice that the number of non-zero summands in (3.5) is bounded by q^3 and that $\binom{n}{n_1,\ldots,n_4} < n^q$ ([33, Proposition 2.2]). Hence, since $c_{n_1,n_2,n_3,n_4}(A) \leq c_n(A)$ and $c_n(A) \leq an^t$ (see [17, Theorem 7.2.14]), we get the desired conclusion. ∎

As a consequence we have the following corollaries.

Corollary 3.19. *The algebras D, D^i, D^{-i}, UT_2, UT_2^-, UT_2^i, UT_2^{-i} are the only finite dimensional superalgebras with pseudoautomorphism generating varieties of almost polynomial growth.*

Corollary 3.20. *If A is a p-algebra, the sequence $c_n^p(A)$, $n = 1, 2, \ldots$, either is polynomially bounded or grows exponentially.*

3.3 The p-exponent of a p-algebra

In this section we discuss the existence of the limit

$$\lim_{n\to\infty} \sqrt[n]{c_n^p(A)}.$$

Also in the case of p-algebras, we have a positive answer. Indeed, the result can be proved in the same way as it has been done in other settings (for example, see [25] for superalgebras with superinvolution).

We shortly recall the main points of how to prove this result. We start with the following definition.

Definition 3.21. *Let $A = B_1 \oplus \cdots \oplus B_k + J$ be a finite dimensional p-algebra and let $C_1, \ldots, C_h$ be distinct simple p-subalgebras of A from $\{B_1, \ldots, B_k\}$. The p-algebra $C = C_1 \oplus \cdots \oplus C_h$ is called p-admissible if*

$$C_1 J \cdots J C_{h-1} J C_h \neq 0.$$

Now we can define the following integer.

Definition 3.22. *Let A be a finite dimensional p-algebra. We set*

$$d = d(A) := \max\{\dim_F C : \ C \textit{ is an admissible } p\textit{-subalgebra of } A\}.$$

The role of such an integer in the description of the asymptotic behavior of the p-codimensions is explained in the following result (see [7, Theorem 5]).

Theorem 3.23. *Let A be a finite dimensional p-algebra over F and consider the integer d of Definition 3.22. Then there exist constants $a_1 > 0$ and a_2, b_1, b_2 such that*

$$a_1 n^{b_1} d^n \leq c_n^p(A) \leq a_2 n^{b_2} d^n.$$

Hence the p-exponent of A, $\exp^p(A) = \lim_{n\to\infty} \sqrt[n]{c_n^p(A)}$ exists and it is a non-negative integer.

Remark 3.24. Note that, as explained in Remark 3.10, p-algebras are equivalent to K-graded algebras, where K is the Klein group. So, the above theorem can also be obtained by following this direction (see [1, 3, 12, 24]).

3.4 Characterizing algebras according to the p-exponent

In this section we aim to characterize p-algebras according to the p-exponent.

As an immediate consequence of Theorem 3.23 we get the following corollary.

Corollary 3.25. *Under the hypotheses of Theorem 3.23, the sequence $c_n^p(A)$, $n = 1, 2, \ldots$, either is polynomially bounded (i.e., $\exp^p(A) \leq 1$) or it grows exponentially (i.e., $\exp^p(A) \geq 2$).*

In the previous section, we described the varieties of p-algebras of polynomial growth by giving a finite list of p-algebras to be excluded from the variety. Hence we can state the following theorem.

Theorem 3.26. *Let A be a finite dimensional p-algebra over F. The following are equivalent:*

1. $\exp^p(A) \leq 1$;
2. $c_n^p(A)$ *is polynomially bounded;*
3. $D, D^i, D^{-i}, UT_2, UT_2^-, UT_2^i, UT_2^{-i} \notin var^p(A)$.

Definition 3.27. *A variety $\mathcal{V}$ of p-algebras is minimal with respect to the p-exponent if for any proper subvariety $\mathcal{U}$, we have that $\exp^p(\mathcal{V}) > \exp^p(\mathcal{U})$. Here the p-exponent of a variety is the p-exponent of a generating algebra.*

Since the above algebras have p-exponent equal to 2 (see previous section and [20]), by using this definition we get the following result.

Corollary 3.28. *The algebras UT_2, UT_2^-, UT_2^i, UT_2^{-i}, D, D^i, D^{-i} are the only finite dimensional p-algebras, up to equivalence, generating minimal varieties of p-exponent 2.*

Now, in order to classify p-algebras with p-exponent bounded by 2, we construct a suitable finite list of p-algebras generating varieties of p-exponent ≥ 2. This list is presented in [7].

Let us consider the following simple p-algebras:

- C_1, the superalgebra $M_{2,0}(F)$ with trivial pseudoautomorphism *id*;
- C_2, the superalgebra $M_{2,0}(F)$ with pseudoautomorphism p given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^p = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$;
- C_3, the superalgebra $M_{1,1}(F)$ with pseudoautomorphism p given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^p = \begin{pmatrix} a & ib \\ ic & d \end{pmatrix}$;
- C_4, the superalgebra $M_{1,1}(F)$ with pseudoautomorphism p given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^p = \begin{pmatrix} a & -ib \\ -ic & d \end{pmatrix}$;
- C_5, the superalgebra $M_{1,1}(F)$ with pseudoautomorphism p given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^p = \begin{pmatrix} d & ic \\ ib & a \end{pmatrix}$;
- C_6, the superalgebra $Q(1) \oplus \overline{Q(1)}$ with the pseudoautomorphism $pex(a,b) = \big((-1)^{|(a,b)|}b, a\big)$.

By Theorem 3.23 we get the following result.

Remark 3.29. For any $i = 1, \dots, 6$, $\exp^p(C_i) = 4$.

The above p-algebras allow us to prove the following lemma.

Lemma 3.30. *Let B be a simple p-algebra with $\dim_F B \geq 4$. Then $C_j \in \mathrm{var}^p(B)$, for some $j \in \{1, \dots, 6\}$.*

Proof. We shall prove the lemma by constructing a p-subalgebra of B isomorphic to C_j for some $j \in \{1, \dots, 6\}$.

Case 1. $B = (M_{k,h}(F), p)$ with $\begin{pmatrix} K & R \\ S & H \end{pmatrix}^p = \begin{pmatrix} PKP & \pm iPRQ \\ \pm iQSP & QHQ \end{pmatrix}$, where P and Q are $P = \begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{pmatrix}$ and $Q = \begin{pmatrix} I_{h_1} & 0 \\ 0 & -I_{h_2} \end{pmatrix}$.

Suppose first that $h = 0$. This means that the superalgebra B has trivial grading and the pseudoautomorphism p is just a graded automorphism. If $k_2 = 0$, p is the identity map and it is immediate to see that the p-subalgebra of B generated by the elements $a_1 = e_{1,1}$, $a_2 = e_{1,k_1}$, $a_3 = e_{k_1,1}$, $a_4 = e_{k_1,k_1}$ is isomorphic to C_1. Now assume $k_2 > 0$. We easily get that the p-subalgebra C' of B generated by the elements $a_1 = e_{1,1}$, $a_2 = e_{k_1+1,k_1+1}$, $a_3 = e_{1,k_1+1}$, $a_4 = e_{k_1+1,1}$ is isomorphic to C_2 through the isomorphism $f \colon C' \to C_2$, given by

$$f(a_1) = e_{1,1}, \quad f(a_2) = e_{2,2}, \quad f(a_3) = e_{1,2}, \quad f(a_4) = e_{2,1}.$$

We are left to deal with the case $h > 0$. Let C' be the p-subalgebra of B generated by the elements $a_1 = e_{1,1}$, $a_2 = e_{k+1,k+1}$, $a_3 = e_{1,k+1}$, $a_4 = e_{k+1,1}$. The linear map f given by

$$f(a_1) = e_{1,1}, \quad f(a_2) = e_{2,2}, \quad f(a_3) = e_{1,2}, \quad f(a_4) = e_{2,1},$$

is an isomorphism of p-algebras between C' and C_3 or C_4, according to the sign ($\pm i$) of the pseudoautomorphism p.

Case 2. $B = (M_{k,k}(F), p)$ with $p\left(\begin{pmatrix} K & R \\ S & H \end{pmatrix}\right) = \begin{pmatrix} H & iS \\ iR & K \end{pmatrix}$.

Let C' be the p-subalgebra of B generated by $a_1 = e_{1,1}$, $a_2 = e_{k+1,k+1}$, $a_3 = e_{1,k+1}$, $a_4 = e_{k+1,1}$. Hence we obtain an isomorphism of p-algebras between C' and C_5 via the linear map $f\colon C' \to C_5$ given by

$$f(a_1) = e_{1,1}, \quad f(a_2) = e_{2,2}, \quad f(a_3) = e_{1,2}, \quad f(a_4) = e_{2,1}.$$

Case 3. $B = \left(M_{k,h}(F) \oplus \overline{M_{k,h}(F)}, pex\right)$, $k \geq h \geq 0$.

Suppose first $k = h = 1$. The p-subalgebra C' generated by the elements $a_1 = (e_{1,1} + e_{2,2}, 0)$, $a_2 = (e_{1,2}+e_{2,1}, 0)$, $a_3 = (0, e_{1,1}+e_{2,2})$, $a_4 = (0, e_{1,2}+e_{2,1})$ is isomorphic to $C_6 = (F \oplus cF) \oplus \overline{(F \oplus cF)}$ through the isomorphism of p-algebras: $f\colon C' \to C_6$, given by

$$f(a_1) = (1, 0), \quad f(a_2) = (c1, 0), \quad f(a_3) = (0, 1), \quad f(a_4) = (0, c1).$$

Now assume $k > 1$. In this case, we get that the p-subalgebra of B generated by $a_1 = (e_{1,1}, e_{1,1})$, $a_2 = (e_{2,2}, e_{2,2})$, $a_3 = (e_{1,2}, e_{1,2})$, $a_4 = (e_{2,1}, e_{2,1})$ is isomorphic to C_1.

Case 4. $B = \left(Q(n) \oplus \overline{Q(n)}, pex\right)$.

If $n = 1$, then $B = C_6$ and there is nothing to prove. Now, let $n > 1$. Then C_1 is isomorphic to the p-subalgebra of B generated by the elements $a_1 = (e_{1,1}, e_{1,1})$, $a_2 = (e_{2,2}, e_{2,2})$, $a_3 = (e_{1,2}, e_{1,2})$ and $a_4 = (e_{2,1}, e_{2,1})$.

Case 5. $B = Q(n)$ with pseudoautomorphism $p(a + cb) = f(a) \pm icf(b)$, f automorphism of order ≤ 2 on $M_n(F)$.

Since $\dim_F B \geq 4$, we have that $n \geq 2$. Hence we can consider the p-subalgebra C' of B generated by $a_1 = e_{1,1}$, $a_2 = e_{2,2}$, $a_3 = e_{1,2}$ and $a_4 = e_{2,1}$. C' has trivial grading and induced pseudoautomorphism p. Since f is an order 2 automorphism of $M_n(F)$, it is well-known that it acts trivially on $M_2(F)$ or

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

According to the action of f, we get a p-algebras isomorphism between C' and C_1 or C_2 and we are done. ■

Next we need to consider some suitable $\mathbb{Z}_2$-gradings and pseudoautomorphisms on the algebra UT_3 of 3×3 upper triangular matrices. Recall that an arbitrary triple (g_1, g_2, g_3) of elements of $\mathbb{Z}_2$ defines an elementary $\mathbb{Z}_2$-grading on UT_3 by setting:

$$(UT_3)_0 = \text{span}\{e_{i,j} \mid g_i + g_j = 0 \ (\text{mod } 2)\} \text{ and } (UT_3)_1 = \text{span}\{e_{i,j} \mid g_i + g_j = 1 \ (\text{mod } 2)\}.$$

On UT_3 we can define the following automorphisms (of order ≤ 2):

$$id\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\right) = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}, \qquad \varphi_1\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\right) = \begin{pmatrix} a & -b & -c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix},$$

$$\varphi_2\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\right) = \begin{pmatrix} a & b & -c \\ 0 & d & -e \\ 0 & 0 & f \end{pmatrix}, \qquad \varphi_3\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\right) = \begin{pmatrix} a & -b & c \\ 0 & d & -e \\ 0 & 0 & f \end{pmatrix}.$$

If UT_3 is endowed with trivial grading, the above automorphisms can be seen as pseudoautomorphisms.

Given any superalgebra $A = A_0 \oplus A_1$, one can consider the following pseudoautomorphism (recall that $i^2 = -1$)

$$\begin{aligned} p\colon A_0 \oplus A_1 &\to A_0 \oplus A_1 \\ a_0 + a_1 &\mapsto a_0 + ia_1. \end{aligned}$$

Notice that in case the superalgebra has trivial grading, the pseudoautomorphism p is actually the identity map.

According to the result of [29], it is not difficult to see that the composition between p and a graded automorphism of order ≤ 2 on UT_3 is a pseudoautomorphism of UT_3. Hence we have the following p-algebras:

- C_7, with trivial grading and trivial pseudoautomorphism id;
- C_8, with trivial grading and pseudoautomorphism φ_1;
- C_9, with trivial grading and pseudoautomorphism φ_2;
- C_{10}, with trivial grading and pseudoautomorphism φ_3;
- C_{11}, with grading induced by $(0,0,1)$ and pseudoautomorphism p;
- C_{12}, with grading induced by $(0,0,1)$ and pseudoautomorphism $p \circ \varphi_1$;
- C_{13}, with grading induced by $(0,0,1)$ and pseudoautomorphism $p \circ \varphi_2$;
- C_{14}, with grading induced by $(0,0,1)$ and pseudoautomorphism $p \circ \varphi_3$;
- C_{15}, with grading induced by $(0,1,1)$ and pseudoautomorphism p;
- C_{16}, with grading induced by $(0,1,1)$ and pseudoautomorphism $p \circ \varphi_1$;

- C_{17}, with grading induced by $(0,1,1)$ and pseudoautomorphism $p \circ \varphi_2$;
- C_{18}, with grading induced by $(0,1,1)$ and pseudoautomorphism $p \circ \varphi_3$;
- C_{19}, with grading induced by $(0,1,0)$ and pseudoautomorphism p;
- C_{20}, with grading induced by $(0,1,0)$ and pseudoautomorphism $p \circ \varphi_1$;
- C_{21}, with grading induced by $(0,1,0)$ and pseudoautomorphism $p \circ \varphi_2$;
- C_{22}, with grading induced by $(0,1,0)$ and pseudoautomorphism $p \circ \varphi_3$.

Remark 3.31. For $j = 7, \ldots, 22$, we have that $\exp^p(C_j) = 3$.

Proof. All the p-algebras C_j have the same Wedderburn-Malcev decomposition:

$$C_j = A_1 \oplus A_2 \oplus A_3 + J,$$

where $A_1 = Fe_{1,1}$, $A_2 = Fe_{2,2}$, $A_3 = Fe_{3,3}$ and $J = Fe_{1,2} \oplus Fe_{1,3} \oplus Fe_{2,3}$. Since $A_1JA_2JA_3 \neq 0$, $A_1 \oplus A_2 \oplus A_3$ is a maximal dimensional p-admissible subalgebra and the result follows by Theorem 3.23. ■

The above p-algebras allow us to prove the following lemma.

Lemma 3.32. *Assume the field F to be algebraically closed. Let $A = B_1 \oplus \cdots \oplus B_k + J$ be a finite dimensional p-algebra over F. If there exist three distinct simple components $B_{i_1} \cong B_{i_2} \cong B_{i_3} \cong F$ such that $B_{i_1}JB_{i_2}JB_{i_3} \neq 0$, then $C_j \in \mathrm{var}^p(A)$, for some $j \in \{7, \ldots, 22\}$.*

Proof. Let e_1, e_2, e_3 be the unit elements of $B_{i_1}, B_{i_2}, B_{i_3}$, respectively. Then $e_l^2 = e_l^p = e_l \in (B_{i_l})_0$ and $e_re_s = \delta_{rs}e_r$, for $r, s, l = 1, 2, 3$. Since $B_{i_1}JB_{i_2}JB_{i_3} \neq 0$ then $e_1Je_2Je_3 \neq 0$. So we may assume that there exist homogeneous (symmetric, skew, i-symmetric or i-skew) elements $j, j' \in J$ such that

$$e_1je_2j'e_3 \neq 0.$$

Consider the p-subalgebra U of A linearly generated by

$$e_1, \quad e_2, \quad e_3, \quad e_1je_2, \quad e_2j'e_3, \quad e_1je_2j'e_3.$$

The linear map $f \colon U \to UT_3$, defined by

$$f(e_1) = e_{1,1}, \quad f(e_2) = e_{2,2}, \quad f(e_3) = e_{3,3}, \quad f(e_1je_2) = e_{1,2}, \quad f(e_2j'e_3) = e_{2,3}, \quad f(e_1je_2j'e_3) = e_{1,3},$$

is an isomorphism of algebras. Now, by taking into account the homogeneous degrees of j and j' and their symmetry with respect to the pseudoautomorphism p, we get an isomorphism of p-algebras between U and C_j, for some $j = 7, \ldots, 22$. ■

Now, let us consider the algebra

$$M = \left\{ \begin{pmatrix} a+\alpha a' & e+\alpha e' & 0 & 0 \\ 0 & b+\alpha b' & 0 & 0 \\ 0 & 0 & c+\alpha c' & 0 \\ 0 & 0 & f+\alpha f' & d+\alpha d' \end{pmatrix} \mid a,a',b,b',c,c',d,d',e,e',f,f' \in F,\ \alpha^2 = 1 \right\}$$

and the automorphism † on it given by

$$\dagger\left(\begin{pmatrix} a+\alpha a' & e+\alpha e' & 0 & 0 \\ 0 & b+\alpha b' & 0 & 0 \\ 0 & 0 & c+\alpha c' & 0 \\ 0 & 0 & f+\alpha f' & d+\alpha d' \end{pmatrix}\right) = \begin{pmatrix} d+\alpha d' & f+\alpha f' & 0 & 0 \\ 0 & c+\alpha c' & 0 & 0 \\ 0 & 0 & b+\alpha b' & 0 \\ 0 & 0 & e+\alpha e' & a+\alpha a' \end{pmatrix}.$$

We denote by M_1 the algebra M such that $a' = b' = c' = d' = e' = f' = 0$, endowed with trivial grading and pseudoautomorphism †. Instead we use the symbol M_2 in case the grading is the elementary one induced by $(0,1,1,0)$ and the pseudoautomorphism is $p \circ \dagger$. We need the following p-algebras:

- C_{23}, the subalgebra of M_1 with $b = c$;
- C_{24}, the subalgebra of M_1 with $a = d$;
- C_{25}, the subalgebra of M_2 with $b = c$;
- C_{26}, the subalgebra of M_2 with $a = d$.

Now notice that the algebra M can be seen as a superalgebra also with the following grading:

$$\left\{ \begin{pmatrix} a & e & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & f & d \end{pmatrix} \mid a,b,c,d,e,f \in F \right\} \oplus \left\{ \begin{pmatrix} \alpha a' & \alpha e' & 0 & 0 \\ 0 & \alpha b' & 0 & 0 \\ 0 & 0 & \alpha c' & 0 \\ 0 & 0 & \alpha f' & \alpha d' \end{pmatrix} \mid a',b',c',d',e',f' \in F \right\}.$$

We denote by M_i the superalgebra M with the pseudoautomorphism ρ_i given by

$$\rho_i\left(\begin{pmatrix} a+\alpha a' & e+\alpha e' & 0 & 0 \\ 0 & b+\alpha b' & 0 & 0 \\ 0 & 0 & c+\alpha c' & 0 \\ 0 & 0 & f+\alpha f' & d+\alpha d' \end{pmatrix}\right) = \begin{pmatrix} d+i\alpha d' & f+i\alpha f' & 0 & 0 \\ 0 & c+i\alpha c' & 0 & 0 \\ 0 & 0 & b+i\alpha b' & 0 \\ 0 & 0 & e+i\alpha e' & a+i\alpha a' \end{pmatrix}.$$

Analogously, M_{-i} denote the superalgebra M with pseudoautomorphism ρ_{-i} defined as ρ_i but with $-i$ instead of i.

The last p-algebras we have to consider are the following:

- C_{27} is the subalgebra of M_i with $a = d$, $b = c$, $a' = d'$ and $c' = b' = 0$;
- C_{28} is the subalgebra of M_{-i} with $a = d$, $b = c$, $a' = d'$ and $c' = b' = 0$;

- C_{29} is the subalgebra of M_i with $a = d$, $b = c$, $b' = c'$ and $a' = d' = 0$;

- C_{30} is the subalgebra of M_{-i} with $a = d$, $b = c$, $b' = c'$ and $a' = d' = 0$.

Using the same approach as in Remark 3.31, we get the following result.

Remark 3.33. For $j = 23, \dots, 30$, we have that $\exp^p(C_j) = 3$.

Now we can prove the following lemma.

Lemma 3.34. *Assume that the field F is also algebraically closed. Let $A = B_1 \oplus \cdots \oplus B_k + J$ be a finite dimensional p-algebra over F such that $B_l J B_m \neq 0$ with (B_l, B_m) in*

$$\{(F, D), (D, F), (F, D^i), (D^i, F), (F, D^{-i}), (D^{-i}, F)\}$$

with $l \neq m$. Then $C_j \in \mathrm{var}^p(A)$, for some $j \in \{23, \dots, 30\}$.

Proof. Suppose first that $(B_l, B_m) = (D, F)$. Let $e_l = e_1 + e_2$ and $e_m = e_3$ be the unit elements of B_l and B_m, respectively. Clearly $p(e_1) = e_2$ and $p(e_3) = e_3$. Since $B_l J B_m \neq 0$, there exists a homogeneous element $j \in J$ such that

$$e_l j e_m = (e_1 + e_2) j e_3 \neq 0.$$

Without loss of generality, we may assume that $e_1 j e_3 \neq 0$. Let U be the p-algebra linearly generated by the elements

$$e_1, \quad e_2, \quad e_3, \quad e_1 j e_3, \quad e_2 p(j) e_3.$$

When the homogeneous degree of j is 0, the map f defined by

$$f(e_1) = e_{1,1}, \quad f(e_2) = e_{4,4}, \quad f(e_3) = e_{2,2} + e_{3,3}, \quad f(e_1 j e_3) = e_{1,2}, \quad f(e_2 p(j) e_3) = e_{4,3},$$

is an isomorphism of p-algebras between U and C_{23}. When the homogeneous degree of j is 1, the map f defined by

$$f(e_1) = e_{1,1}, \quad f(e_2) = e_{4,4}, \quad f(e_3) = e_{2,2} + e_{3,3}, \quad f(e_1 j e_3) = e_{1,2}, \quad f(e_2 p(j) e_3) = i e_{4,3},$$

is an isomorphism of p-algebras between U and C_{25}.

In a similar way, when $(B_l, B_m) = (F, D)$ we obtain that $C_{24}, C_{26} \in \mathrm{var}^p(A)$.

Suppose now that $(B_l, B_m) = (D^i, F)$. Let $e_l = e_1$ and $e_m = e_2$ be the unit elements of B_l and B_m, respectively. Clearly $p(e_1) = e_1$ and $p(e_2) = e_2$. Since $B_l J B_m \neq 0$, there exists a homogeneous element $j \in J$ such that

$$e_l j e_m = e_1 j e_2 \neq 0.$$

Let U be the p-algebra linearly generated by the elements

$$e_1, \quad e_2, \quad c e_1, \quad e_1 j e_2, \quad e_1 p(j) e_2, \quad c e_1 j e_2, \quad c e_1 p(j) e_2.$$

Then U is isomorphic to C_{27} as p-algebras, via f defined by

$$f(e_1) = e_{1,1} + e_{4,4}, \quad f(e_2) = e_{2,2} + e_{3,3}, \quad f(ce_1) = c(e_{1,1} + e_{4,4}), \quad f(e_1 j e_2) = e_{1,2},$$
$$f(e_1 p(j) e_2) = e_{4,3}, \quad f(ce_1 j e_2) = ce_{1,2}, \quad f(ce_1 p(j) e_2) = ce_{4,3},$$

when the homogeneous degree of j is 0, and f defined by

$$f(e_1) = e_{1,1} + e_{4,4}, \quad f(e_2) = e_{2,2} + e_{3,3}, \quad f(ce_1) = c(e_{1,1} + e_{4,4}), \quad f(e_1 j e_2) = ce_{1,2},$$
$$f(e_1 p(j) e_2) = ice_{4,3}, \quad f(ce_1 j e_2) = e_{1,2}, \quad f(ce_1 p(j) e_2) = ie_{4,3},$$

when the homogeneous degree of j is 1.

In a similar way, when $(B_l, B_m) = (D^{-i}, F)$, (F, D^i) or (F, D^{-i}), we obtain that C_{28}, C_{29} or $C_{30} \in \mathrm{var}^p(A)$, respectively. ■

The following proposition proves that the list of p-algebras $C_1, \ldots, C_{30}$ cannot be reduced.

Proposition 3.35. *For all $l, j \in \{1, \ldots, 30\}$, $l \neq j$, $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$.*

Proof. By the classification of pseudoautomorphisms on UT_n given in [29], we have that the p-algebras C_i, $i = 7, \ldots, 22$ are not pairwise equivalent. Now, let y denote an even variable, z an odd one and x any variable. The proof is completed by putting together the following facts.

- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 1, 3, 4, 7, 11, 13, 15, 16, 19, 22$, $j \neq l$: in fact $y^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 1, 2, 7, 8, 9, 10, 23, 24$, $j \neq l$: in fact $z \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 3$, $j = 4$: in fact $z^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 4$, $j = 3$: in fact $z^+ \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 2, \ldots, 6, 12, 14, 17, 18, 20, 21, 25, 26$, $j \neq l$: in fact $[y_1^+, y_2^+] \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 1, \ldots, 5$, $j = 7, \ldots, 22$,: in fact $\big[[x_1, x_2]^2, x_3\big] \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 5$, $j = 2, 3, 4, 25, 26$: in fact $[y^+, x] \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 5$, $j = 6$: in fact $y^- z^- + z^- y^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 6$, $j = 1, \ldots, 5, 7 \ldots, 30$: in fact $[x_1, x_2] \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 7, \ldots, 30$, $j = 1, \ldots, 6$: in fact $\exp^p(C_l) < \exp^p(C_j)$.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 6, 23, \ldots, 30$, $j = 1, \ldots, 5, 7, \ldots, 22$: in fact $[x_1, x_2][x_3, x_4] \equiv 0$ on C_l but not on C_j.

- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 7, \ldots, 22$, $j = 23, \ldots, 26$: in fact $y_1^- y_2^- y_3^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 23$, $j = 24$: in fact $[y_1^-, y_2^-]y_3^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 24$, $j = 23$: in fact $y_3^-[y_1^-, y_2^-] \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 25$, $j = 26$: in fact $z^+ y^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 26$, $j = 25$: in fact $y^- z^+ \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 27, \ldots, 30$, $j = 23, \ldots, 26$: in fact $y_1^- y_2^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 28, 30$, $j = 27, 29$: in fact $z_1^+ z_2^+ \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 27, 29$, $j = 28, 30$: in fact $z_1^- z_2^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 27, 30$, $j = 28, 29$: in fact $z^- z^+ \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 28, 29$, $j = 27, 30$: in fact $z^+ z^- \equiv 0$ on C_l but not on C_j.

■

Now we are in a position to characterize the p-algebras A with $\exp^p(A) \leq 2$.

Theorem 3.36 ([7]). *Let A be a finite dimensional p-algebra over F. Then $\exp^p(A) \leq 2$ if and only if $C_j \notin \mathrm{var}^p(A)$, for any $j \in \{1, \ldots, 30\}$.*

Proof. Since we are dealing with p-codimensions that do not change by extending the base field, in what follows we may assume that the field F is algebraically closed.

First let $\exp^p(A) \leq 2$. Since $\exp^p(C_i) > 2$, by Remarks 3.29, 3.31 and 3.33, we get $C_j \notin \mathrm{var}^p(A)$, $j \in \{1, \ldots, 30\}$.

Conversely, let $C_j \notin \mathrm{var}^p(A)$, for any $j \in \{1, \ldots, 30\}$. Hence by Theorem 3.13 we can write $A = B_1 \oplus \cdots \oplus B_m + J$, where the B_j's are simple p-algebras isomorphic to those ones given in Theorem 3.14. Since $C_1, \ldots, C_6 \notin \mathrm{var}^p(A)$, according to Lemma 3.30, we have that $\dim_F B_l < 4$, for any l.

Suppose by contradiction that $\exp^p(A) > 2$. Then by Theorem 3.23, one of the following possibilities occurs:

1. there exist distinct $B_{i_1}, B_{i_2}, B_{i_3}$ such that $B_{i_1} J B_{i_2} J B_{i_3} \neq 0$ and $B_{i_1} \cong B_{i_2} \cong B_{i_3} \cong F$,
2. for some $i_1 \neq i_2$, $B_{i_1} J B_{i_2} \neq 0$ and $B_{i_1} \cong F$ and $B_{i_2} \cong D$ or D^i or D^{-i},
3. for some $i_1 \neq i_2$, $B_{i_1} J B_{i_2} \neq 0$ and $B_{i_1} \cong D$ or D^i or D^{-i} and $B_{i_2} \cong F$.

Notice that when we are in the situation $B_{i_1} J B_{i_2} \neq 0$ with $B_{i_1} \cong D$ or D^i or D^{-i} and $B_{i_2} \cong D$ or D^i or D^{-i}, it easily follows that one of the last two cases occurs.

If 1. holds, then, by Lemma 3.32, $C_j \in \mathrm{var}^p(A)$, for some $j \in \{7, \ldots, 22\}$, a contradiction. We reach a contradiction also in all the other cases, since by Lemma 3.34, we should have that $C_j \in \mathrm{var}^p(A)$, for some $j \in \{23, \ldots, 30\}$. ■

In light of Theorems 3.18 and 3.36, we get the characterization of p-algebras with p-exponent equal to two.

Corollary 3.37. *Let A be a finite dimensional p-algebra over F. Then* $\exp^p(A) = 2$ *if and only if*

- $C_j \notin \mathrm{var}^p(A)$, *for all* $j \in \{1, \dots, 30\}$ *and*
- *either* UT_2 *or* UT_2^- *or* UT_2^i *or* UT_2^{-i} *or* D *or* D^i *or* $D^{-i} \in \mathrm{var}^p(A)$.

Now let us slightly change the definition of minimal varieties given in the previous sections: a variety $\mathcal{V}$ of p-algebras is minimal with respect to the p-exponent if, for any proper subvariety $\mathcal{U}$, generated by a finite dimensional p-algebra, we have that $\exp^p(\mathcal{V}) > \exp^p(\mathcal{U})$. By using this definition we get the following.

Corollary 3.38.

1. *The p-algebras C_j, $j = 1, \dots, 6$, generate minimal varieties of p-exponent* 4.
2. *The p-algebras C_j, $j = 7, \dots, 30$, are the only finite dimensional algebras, up to equivalence, generating minimal varieties of p-exponent* 3.

Proof. We prove just item 2. (the proof of 1. is similar).

Let $\mathcal{V}$ be a proper subvariety of $\mathrm{var}^p(C_j)$, $j = 7, \dots, 30$. Clearly $C_j \notin \mathcal{V}$. Also, by Proposition 3.35, we get that $C_l \notin \mathcal{V}$, for any $l = 1, \dots, 30$. Then, from Theorem 3.36, $\exp^p(\mathcal{V}) \leq 2$ and we are done.

Now suppose that there exists a minimal variety $\mathcal{U}$ of p-exponent 3 which is not generated by any of the algebras in 2. Since $\mathcal{U}$ is minimal and its p-exponent is 3, $C_j \notin \mathcal{U}$, for any j. Then by Theorem 3.36 we should have $\exp^p(\mathcal{U}) \leq 2$, a contradiction. ∎

So, we close this section by stating the complete result

Theorem 3.39. *Let A be a finite dimensional p-algebra over F. Then* $\exp^p(A)$

- < 2 *if* UT_2, UT_2^-, UT_2^i, UT_2^{-i}, D, D^i, $D^{-i} \notin \mathrm{var}^p(A)$.
- $= 2$ *if*
  - *either* UT_2 *or* UT_2^- *or* UT_2^i *or* UT_2^{-i} *or* D *or* D^i *or* $D^{-i} \in \mathrm{var}^p(A)$ *and*
  - $C_j \notin \mathrm{var}^p(A)$, *for all* $j \in \{1, \dots, 30\}$.
- > 2 *if* $C_j \in \mathrm{var}^p(A)$, *for some* $j \in \{1, \dots, 30\}$.

Remark 3.40. Note that another way to prove this result can be using the correspondence (Remarks 3.10 and 3.24) between p-algebras and K-graded algebras and looking at the characterization of G-graded algebras of exponent two given in [28].

Chapter 4

Superalgebras with superinvolution

This last chapter is dedicated to superalgebras with superinvolution.

We start by giving the definition of superalgebras with superinvolution and some examples, recalling that these algebras are the natural graded generalization of algebras with involution. Then, we state the main results in the theory of polynomial identities in this context. Lastly, we discuss the behavior of the central $\diamond$- codimension sequence and the existence of the central $\diamond$-exponent. We present our results in the last section of this chapter.

4.1 Definitions and examples

From now on, unless otherwise stated, the field F is of characteristic zero.

Definition 4.1. *Let $A = A_0 \oplus A_1$ be a $\mathbb{Z}_2$-graded algebra. We say that A is a superalgebra with superinvolution $\diamond$ if it is endowed with a graded linear map $\diamond\colon A \to A$ with the following properties:*

- $(ab)^\diamond = (-1)^{|a||b|} b^\diamond a^\diamond$ *for every $a \in A$,*
- $(a^\diamond)^\diamond = a$ *for all homogeneous elements $a, b \in A_0 \cup A_1$.*

where $|a|$ is the homogeneous degree of a.

Note that with the notation $a^\diamond$ we indicate $\diamond(a)$. We can immediately see from the definition that a superinvolution acting on the even part of a superalgebra is just an involution. We say that an algebra A has trivial superinvolution if A has trivial grading and $\diamond = id$ is the identity map.

Remark 4.2. Let $A = A_0 \oplus A_1$ be a superalgebra with superinvolution $\diamond$. Since F has characteristic zero, we can write

$$A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-,$$

where, for $i = 0, 1$,

$$A_i^+ := \{a \in A_i : a^\diamond = a\} \quad \text{and} \quad A_i^- := \{a \in A_i : a^\diamond = -a\}$$

denote the sets of symmetric and skew-symmetric elements of A_i, respectively.

From now on we will refer to a finite dimensional PI-superalgebra with superinvolution simply as a $\diamond$-algebra.

Recall that an ideal (subalgebra) I of a $\diamond$-algebra A is a $\diamond$-ideal (subalgebra) of A if it is a graded ideal (subalgebra) and $I^\diamond = I$. The $\diamond$-algebra A is a simple $\diamond$-algebra if $A^2 \neq 0$ and A has no non-trivial $\diamond$-ideals.

Now we will give some examples of superinvolutions.

Remark 4.3. On the superalgebra $M_{k,l}(F)$ we can define two types of superinvolutions. They have been described by Racine in [40].

- The *orthosymplectic* superinvolution osp can be defined on $M_{k,l}(F)$ if and only if $l = 2s$:

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{osp} = \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix}^{-1} \begin{pmatrix} X & -Y \\ Z & T \end{pmatrix}^{t} \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix},$$

  where t denotes the usual transpose, $Q = \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix}$, I_k, I_s are identity matrices of orders k, s. We have

$$(M_{k,2s}(F), osp)_0^+ = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X = X^t, \quad T = -QT^tQ, \quad X \in M_k(F), \quad T \in M_{2s}(F) \right\},$$

$$(M_{k,2s}(F), osp)_0^- = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X = -X^t, \quad T = QT^tQ, \quad X \in M_k(F), \quad T \in M_{2s}(F) \right\},$$

$$(M_{k,2s}(F), osp)_1^+ = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Y = Z^tQ \quad (\text{or } Z = QY^t), \quad Y \in M_{k,2s}(F), \quad Z \in M_{2s,k}(F) \right\},$$

$$(M_{k,2s}(F), osp)_1^- = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Y = -Z^tQ \quad (\text{or } Z = -QY^t), \quad Y \in M_{k,2s}(F), \quad Z \in M_{2s,k}(F) \right\}.$$

- The *supertranspose* superinvolution trp can be defined on $M_{k,l}(F)$ if and only if $k = l$ by the formula

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{trp} = \begin{pmatrix} T^t & -Y^t \\ Z^t & X^t \end{pmatrix}.$$

  We see that

$$(M_{k,k}(F), trp)_0^+ = \left\{ \begin{pmatrix} X & 0 \\ 0 & X^t \end{pmatrix} \mid X \in M_k(F) \right\},$$

$$(M_{k,k}(F), trp)_0^- = \left\{ \begin{pmatrix} X & 0 \\ 0 & -X^t \end{pmatrix} \mid X \in M_k(F) \right\},$$

$$(M_{k,k}(F), trp)_1^+ = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Y = -Y^t, \quad Z = Z^t, \quad Y, Z \in M_k(F) \right\},$$

$$(M_{k,k}(F), trp)_1^- = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Y = Y^t, \quad Z = -Z^t, \quad Y, Z \in M_k(F) \right\}.$$

Other examples of superalgebras with superinvolutions are the following, presented in [11].

Example 4.4.

- Let $F \oplus F$ be the two dimensional group algebra of $\mathbb{Z}_2$. We denote by D the algebra $F \oplus F$ with trivial grading and exchange superinvolution $\diamond$ given by

$$(a,b)^\diamond = (b,a),$$

for all $(a,b) \in D$. We can decompose D in the following way

$$D_0^+ = \{(a,a) : a \in F\}, \quad D_0^- = \{(a,-a) : a \in F\}, \quad D_1^+ = D_1^- = \{0\}.$$

- Let $M = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33}) \oplus Fe_{12} \oplus Fe_{34}$ be a subalgebra of UT_4, the algebra of 4×4 upper-triangular matrices, with trivial grading and endowed with the *reflection superinvolution*, i.e., the superinvolution obtained by reflecting a matrix along its secondary diagonal.

  Hence, if $a = \alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) + \gamma e_{12} + \delta e_{34}$, then

$$a^\diamond = \alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) + \delta e_{12} + \gamma e_{34}.$$

  So we have

$$M_0^+ = \{\alpha(e_{11}+e_{44})+\beta(e_{22}+e_{33})+\gamma(e_{12}+e_{34})\}, \quad M_0^- = \{\gamma(e_{12}-e_{34})\}, \quad M_1^+ = M_1^- = \{0\}.$$

- Now we consider a non-trivial grading on M: we denote by M^{sup} the algebra M with grading

$$M_0^{sup} = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33}), \qquad M_1^{sup} = Fe_{12} \oplus Fe_{34},$$

  endowed with reflection superinvolution. Hence

$$\begin{aligned}
(M^{sup})_0^+ &= \{\alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) \mid \alpha, \beta \in F\},\\
(M^{sup})_0^- &= \{0\},\\
(M^{sup})_1^+ &= \{\gamma(e_{12} + e_{34}) \mid \gamma \in F\},\\
(M^{sup})_1^- &= \{\delta(e_{12} - e_{34}) \mid \delta \in F\}.
\end{aligned}$$

- Let $G = \langle 1, e_1, e_2, \dots \mid e_ie_j = -e_je_i\rangle$ be the infinite dimensional Grassmann algebra over F with its natural grading $G = G_0 \oplus G_1$, defined as we explained in Section 2.

  - We let $G^\#$ be the algebra G with natural grading and superinvolution $\diamond$ induced by setting $e_i^\diamond = e_i$. Hence

$$(G^\#)_0^+ = (G^\#)_0, \quad (G^\#)_0^- = \{0\}, \quad (G^\#)_1^+ = (G^\#)_1, \quad (G^\#)_1^- = \{0\}.$$

  - We denote by G^* the algebra G with natural grading and superinvolution $\diamond$ induced by setting $e_i^\diamond = -e_i$. In this case

$$(G^*)_0^+ = G_0^*, \quad (G^*)_0^- = \{0\}, \quad (G^*)_1^+ = \{0\}, \quad (G^*)_1^- = G_1^*.$$

The structure of a finite-dimensional $\diamond$-algebra is known due to a generalization of the Wedderburn - Malcev decomposition (see [10, Theorem 4.1]).

Theorem 4.5 (Giambruno, Ioppolo, La Mattina, 2016). *Let A be a finite-dimensional $\diamond$-algebra over a field F of characteristic 0. Then there exists a semisimple $\diamond$-subalgebra B such that*

$$A = B \oplus J$$

as vector spaces and the Jacobson radical J is a $\diamond$-ideal of A. Moreover

$$B \cong B_1 \oplus \cdots \oplus B_m,$$

where $B_1, \ldots, B_m$ are simple $\diamond$-algebras.

We can further refine this decomposition since the classification of the $\diamond$-simple algebras is known (see [4, 22, 40]). Before doing it, we introduce some preliminary notions.

Given a superalgebra $A = A_0 \oplus A_1$, we define the corresponding superalgebra A^{sop} having the same graded vector space structure as A and product given on homogeneous elements $a, b \in A^{sop}$ by

$$a \circ b = (-1)^{|a||b|} ba.$$

Moreover $\mathcal{B} = A \oplus A^{sop}$ is a superalgebra with $\mathcal{B}_0 = A_0 \oplus A_0^{sop}$ and $\mathcal{B}_1 = A_1 \oplus A_1^{sop}$ and it becomes a $\diamond$-algebra since it is possible to define the so-called *exchange superinvolution ex* as below

$$(a, b)^{ex} = (b, a)\,.$$

Consider again the superalgebra from Example 2.6, $Q(n) = M_n(F \oplus cF)$, where $c^2 = 1$, with grading $Q(n)_0 = M_n(F)$ and $Q(n)_1 = cM_n(F)$. Then, the algebra $Q(n) \oplus Q(n)^{sop}$ is a $\diamond$-algebra with exchange superinvolution.

Now, we can state the following theorem that gives us the classification of the finite-dimensional simple $\diamond$-algebras (see [4]).

Recall that, if A and B are two superalgebras endowed with superinvolutions $\diamond$ and $\star$, respectively, then $(A, \diamond)$ and $(B, \star)$ are isomorphic, as $\diamond$-algebras, if there exists an isomorphism of superalgebras $\psi\colon A \to B$ such that $\psi(x^\diamond) = \psi(x)^\star$ for all $x \in A$.

Theorem 4.6 (Bahturin, Tvalvadze, Tvalvadze, 2009). *Let A be a finite-dimensional simple $\diamond$-algebra over an algebraically closed field F of characteristic different from 2. Then A is isomorphic to one of the following*

1. *$M_{k,k}(F)$ with supertranspose superinvolution,*
2. *$M_{k,2s}(F)$ with orthosymplectic superinvolution,*
3. *$M_{k,l}(F) \oplus M_{k,l}(F)^{sop}$ with exchange superinvolution,*
4. *$Q(k) \oplus Q(k)^{sop}$ with exchange superinvolution.*

4.2 $\diamond$-Identities

Our next step is to define a superinvolution on the free associative superalgebra $F\langle Y \cup Z\rangle$. We write the sets Y and Z as the disjoint union of two other infinite sets of symmetric and skew-symmetric elements, respectively. Hence the free superalgebra with superinvolution (free $\diamond$-algebra), denoted by $F\langle Y \cup Z, \diamond\rangle$, is generated by symmetric and skew variables of even and odd degree. We write

$$F\langle Y \cup Z, \diamond\rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \dots\rangle,$$

where y_i^+ stands for a symmetric variable of even degree, y_i^- for a skew variable of even degree, z_i^+ for a symmetric variable of odd degree and z_i^- for a skew variable of odd degree. In order to simplify the notation, sometimes we denote by y any even variable, by z any odd variable and by x an arbitrary variable. The elements of $F\langle Y \cup Z, \diamond\rangle$ are called $\diamond$-polynomials.

Definition 4.7. *A $\diamond$-polynomial $f(y_1^+, \dots, y_n^+, y_1^-, \dots, y_m^-, z_1^+, \dots, z_t^+, z_1^-, \dots, z_s^-) \in F\langle Y \cup Z, \diamond\rangle$ is a $\diamond$-identity of the $\diamond$-algebra $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, and we write $f \equiv 0$, if, for all $u_1^+, \dots, u_n^+ \in A_0^+$, $u_1^-, \dots, u_m^- \in A_0^-$, $v_1^+, \dots, v_t^+ \in A_1^+$ and $v_1^-, \dots, v_s^- \in A_1^-$, we have*

$$f(u_1^+, \dots, u_n^+, u_1^-, \dots, u_m^-, v_1^+, \dots, v_t^+, v_1^-, \dots, v_s^-) = 0.$$

We denote by $\mathrm{Id}^\diamond(A) = \{f \in F\langle Y \cup Z, \diamond\rangle \mid f \equiv 0 \text{ on } A\}$ the $T_2^\diamond$-ideal of $\diamond$-identities of A, i.e., $\mathrm{Id}^\diamond(A)$ is an ideal of $F\langle Y \cup Z, \diamond\rangle$ invariant under all $\mathbb{Z}_2$-graded endomorphisms of the free superalgebra $F\langle Y \cup Z\rangle$ commuting with the superinvolution $\diamond$.

We denote by $\langle f_1, \dots, f_n\rangle_{T_2^\diamond}$ the $T_2^\diamond$-ideal generated by $f_1, \dots, f_n \in F\langle Y \cup Z, \diamond\rangle$.

As in the ordinary case, it is easily seen that, in characteristic zero, every $\diamond$-identity is equivalent to a system of multilinear $\diamond$-identities. Hence if we denote by

$$P_n^\diamond = \mathrm{span}_F\Big\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n,\ w_i \in \{y_i^+, y_i^-, z_i^+, z_i^-\},\ i = 1, \dots, n\Big\}$$

the space of multilinear polynomials of degree n in $y_1^+, y_1^-, z_1^+, z_1^-, \dots, y_n^+, y_n^-, z_n^+, z_n^-$ (i.e., y_i^+ or y_i^- or z_i^+ or z_i^- appears in each monomial with degree 1), the study of $\mathrm{Id}^\diamond(A)$ is equivalent to the study of $P_n^\diamond \cap \mathrm{Id}^\diamond(A)$, for all $n \geq 1$.

Definition 4.8. *The non-negative integer*

$$c_n^\diamond(A) = \dim_F \frac{P_n^\diamond}{P_n^\diamond \cap \mathrm{Id}^\diamond(A)},\ n \geq 1,$$

is called the n-th $\diamond$-codimension of A.

In [10], the authors proved the following result.

Theorem 4.9 (Giambruno, Ioppolo, La Mattina, 2016). *The sequence $c_n^\diamond$ of $\diamond$-codimensions of a $\diamond$-algebra is exponentially bounded, provided that the $\diamond$-algebra satisfies a non-trivial ordinary identity.*

In case A is a finitely generated superalgebra with superinvolution and satisfies a non-trivial ordinary polynomial identity, in [25] the author proved the following result.

Theorem 4.10 (Ioppolo, 2018). *Let A be a finitely generated superalgebra with superinvolution and satisfies a non-trivial ordinary polynomial identity. Then, there exist constants $C_1 > 0, C_2, t_1, t_2$ and a non-negative integer d such that*

$$C_1 n^{t_1} d^n \leq c_n^{\diamond}(A) \leq C_2 n^{t_2} d^n.$$

Thus we can conclude that

$$\lim_{n\to\infty} \sqrt[n]{c_n^{\diamond}(A)} = d.$$

Definition 4.11. *The non-negative integer*

$$\exp^{\diamond}(A) \colon = d$$

is called the $\diamond$-exponent of A.

The class of $\diamond$-algebras satisfying the identities in the $T_2^{\diamond}$-ideal $\mathrm{Id}^{\diamond}(A)$ is called the $\diamond$-variety generated by A and denoted by $\mathrm{var}^{\diamond}(A)$. If $\mathcal{V} = \mathrm{var}^{\diamond}(A)$, then we consider that $c_n^{\diamond}(\mathcal{V}) = c_n^{\diamond}(A)$ and $\exp^{\diamond}(\mathcal{V}) = \exp^{\diamond}(A)$.

Definition 4.12. *A $\diamond$-variety $\mathcal{V}$ has polynomial growth if $c_n^{\diamond}(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n^{\diamond}(\mathcal{V})$ is not polynomially bounded but every proper $\diamond$-subvariety of $\mathcal{V}$ has polynomial growth.*

In [11], the authors characterized $\diamond$-varieties with polynomial and almost polynomial growth.

Proposition 4.13 (Giambruno, Ioppolo, La Mattina, 2019).

- $\mathrm{Id}^{\diamond}(M) = \langle y_1^- y_2^-, z^+, z^- \rangle_{T_2^{\diamond}}$.
- $\mathrm{Id}^{\diamond}(M^{sup}) = \langle y^-, z_1 z_2 \rangle_{T_2^{\diamond}}$.
- $\mathrm{Id}^{\diamond}(D) = \langle [x_1, x_2], z^+, z^- \rangle_{T_2^{\diamond}}$.
- $\mathrm{Id}^{\diamond}(G^{\#}) = \langle [y, x], z_1 z_2 + z_2 z_1, y^-, z^- \rangle_{T_2^{\diamond}}$.
- $\mathrm{Id}^{\diamond}(G^{*}) = \langle [y, x], z_1 z_2 + z_2 z_1, y^-, z^+ \rangle_{T_2^{\diamond}}$.

Theorem 4.14 (Giambruno, Ioppolo, La Mattina, 2019). *Let $\mathcal{V}$ be a variety of algebras with superinvolution. Then $\mathcal{V}$ has polynomial growth if and only if M, M^{sup}, D, $G^{\#}$, $G^* \notin \mathcal{V}$.*

From this theorem follows that there is no $\diamond$-variety of intermediate growth between polynomial and exponential. Moreover, this theorem implies the following corollary.

Corollary 4.15 (Giambruno, Ioppolo, La Mattina, 2019). *The varieties of algebras with superinvolution $\mathrm{var}^{\diamond}(D)$, $\mathrm{var}^{\diamond}(M)$, $\mathrm{var}^{\diamond}(M^{sup})$, $\mathrm{var}^{\diamond}(G^{\#})$ and $\mathrm{var}^{\diamond}(G^*)$ are the only ones of almost polynomial growth.*

In [25], the author showed how to explicitly compute the $\diamond$-exponent of a $\diamond$-algebra. Let $A = \bar{A} + J = A_1 \oplus \cdots \oplus A_n + J$ be a finite-dimensional $\diamond$-algebra, with the A_i's $\diamond$-simple. A $\diamond$-subalgebra

B is *admissible* if either $B = B_i$ or if $B = B_1 \oplus \cdots \oplus B_r$ where $B_1, \ldots, B_r$ are distinct $\diamond$-subalgebras taken from the set $\{A_1, \ldots, A_n\}$ and

$$B_1 J B_2 J \cdots J B_r \neq 0. \tag{4.1}$$

The $\diamond$-exponent $\exp^\diamond(A)$ is the maximal dimension of an admissible $\diamond$-subalgebra of A:

$$\exp^\diamond(A) = \max \{\dim_F B \mid B \text{ admissible}\}.$$

It easily follows that, for a finite-dimensional $\diamond$-algebra A,

$$\exp^\diamond(A) \leq \dim_F \bar{A}.$$

In case $\exp^\diamond(A) = \dim_F \bar{A}$ the $\diamond$-algebra is called *reduced*.

4.3 Central $\diamond$-polynomials

Now we introduce central $\diamond$-polynomials, the main object of this chapter.

Let us denote by $Z(A)$ the center of an algebra A.

Definition 4.16. *We say that $f \in F\langle Y \cup Z, \diamond\rangle$ is a* central $\diamond$-polynomial *of a $\diamond$-algebra A if*

1. *$f(a_1, \ldots, a_n) \in Z(A)$, for any $a_1, \ldots, a_n$ homogeneous elements of A;*

2. *its constant term is zero.*

It is easy to see that all $\diamond$-identities of A are central $\diamond$-polynomials. A central $\diamond$-polynomial f such that $f \notin \mathrm{Id}^\diamond(A)$ is called a *proper central $\diamond$-polynomial* of A.

Here it is important to highlight that, even if a $\diamond$-superalgebra A has a non-zero center, it can happen that A has no proper central $\diamond$-polynomials. We will see some examples later on.

We will denote by $\mathrm{Id}^{\diamond,z}(A)$ the set of all central $\diamond$-polynomials of a given $\diamond$-algebra A and it will be called the $T_2^\diamond$-space of A. In fact, $\mathrm{Id}^{\diamond,z}(A)$ is a vector subspace of $F\langle Y \cup Z, \diamond\rangle$ which is invariant by endomorphisms of $F\langle Y \cup Z, \diamond\rangle$.

The main goal of this chapter is to study the growth of the spaces of the central $\diamond$-polynomials of a $\diamond$-algebra and to compare it with the growth of the spaces of its $\diamond$-identities. To this end, we define, for $n \geq 1$, the *n-th central $\diamond$-codimension* of A as

$$c_n^{\diamond,z}(A) = \dim_F \frac{P_n^\diamond}{P_n^\diamond \cap \mathrm{Id}^{\diamond,z}(A)}.$$

Remark 4.17. Since $\mathrm{Id}^\diamond(A) \subseteq \mathrm{Id}^{\diamond,z}(A)$, then $c_n^{\diamond,z}(A) \leq c_n^\diamond(A)$.

By the previous remark and according to the fact that $c_n^\diamond(A)$ is exponentially bounded, it follows that $c_n^{\diamond,z}(A)$ is exponentially bounded. Hence the following holds.

Remark 4.18. If A is a finite-dimensional $\diamond$-algebra with $\exp^\diamond(A) = d$, then there exist constants $C_2 > 0, t_2$, such that, for n large enough,

$$c_n^{\diamond,z}(A) \leq C_2 n^{t_2} d^n.$$

A natural question that comes out is asking if the corresponding limit

$$\exp^{\diamond,z}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{\diamond,z}(A)}$$

exists.

In the next section we will find a lower bound for $c_n^{\diamond,z}(A)$ and conclude that for any finite-dimensional $\diamond$-algebra A such that $\exp^\diamond(A) \geq 5$ the central $\diamond$-exponent $\exp^{\diamond,z}(A)$ is equal to the $\diamond$-exponent $\exp^\diamond(A)$. Also, we will observe that the same is true in case $\exp^\diamond(A) = 3$. Then, we will study the remaining possibilities for $\exp^\diamond(A)$ and prove the existence of the central $\diamond$-exponent, presenting the possible values for $\exp^{\diamond,z}(A)$.

To this end, we define proper central $\diamond$-codimensions.

Definition 4.19. *For $n \geq 1$, the sequence of proper central $\diamond$-codimensions of A is defined as*

$$c_n^{\diamond,\delta}(A) = \dim_F \frac{P_n^\diamond \cap \mathrm{Id}^{\diamond,z}(A)}{P_n^\diamond \cap \mathrm{Id}^\diamond(A)}.$$

It is clear from the definitions of these objects that $c_n^\diamond(A) = c_n^{\diamond,z}(A) + c_n^{\diamond,\delta}(A)$.

In [36], La Mattina, dos Santos and Vieira provided information about the existence of the limit

$$\lim_{n\to\infty} \sqrt[n]{c_n^{\diamond,\delta}(A)}.$$

4.4 Central exponent for superalgebras with superinvolutions

As explained in the previous section and in Chapter 1, understanding the behavior of the codimension of central polynomials in PI algebras can help us to recover information about the codimension sequence. Indeed, the the growth of the central codimension has been studied in different settings. Firstly, in [18] the authors studied the behavior of central and proper central codimensions of ordinary PI algebras. Then, [11] the authors studied central polynomials of associative algebras and their growth, they proved the existence of the central exponent and they compared it with the PI-exponent of the algebra. Moreover, in [37] and [39] there are results about the growth of central polynomials of algebras with involution and of graded algebras, respectively. Finally, the authors of [42] proved the existence of the central exponent in the context of algebras with graded involution.

Here we present our results on the behavior of the central codimension sequence for superalgebras with superinvolution (see [21]).

4.4.1 The $\diamond$-algebras $UT^\diamond(B_1, \ldots, B_m)$

Let A be a finite-dimensional $\diamond$-algebra. The goal of this subsection is to introduce a suitable $\diamond$-algebra $UT_g^\diamond(B_1, \ldots, B_m)$ belonging to $\mathrm{var}^\diamond(A)$ and having the same $\diamond$-exponent as A. Such a construction was obtained by Di Vincenzo and Nardozza (see [8, 9]).

Let $s_1, \ldots, s_q$ be positive integers and denote by $UT(s_1, \ldots, s_q)$ the upper block triangular matrix algebra of size $s_1, \dots, s_q$ over F, namely

$$\left\{ \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1q} \\ 0 & B_{22} & \cdots & B_{2q} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_{qq} \end{pmatrix} \mid B_{ij} \in M_{s_i \times s_j}(F),\ 1 \le i \le j \le q \right\}.$$

As anticipated, the $\diamond$-algebra whose construction we are recalling is built from $\diamond$-simple algebras. As observed in [9], the idea of constructing $UT_g^{\diamond}(B_{k_1}, \dots, B_{k_m})$ is to find isomorphic matrix algebras with elementary grading and superinvolution given, up to sign, by reflection along the secondary diagonal for each simple $\diamond$-algebra $B_1, \dots, B_m$. In order to do it, we recall some preliminary notions and results from [8].

Let $g = (g_1, \dots, g_p)$ be an elementary grading on $M_p(F)$. Define $\operatorname{Rev} g := (g_p, \dots, g_1)$ and $\mathscr{C} g := (g_1 + 1, \dots, g_p + 1)$ the complementary word of g. We denote by ϑ_p the reflection involution along the secondary diagonal. Finally, define $\tilde{g} := (g|\mathscr{C}(\operatorname{Rev} g))$, where $|$ denotes the juxtaposition of words.

From now on, assume the field F to be algebraically closed. We are ready to recall the $\diamond$-embedding φ according to the specific type of simple $\diamond$-algebra between the ones in Theorem 4.6.

Type 1-2: If A is isomorphic to either $M_{k,k}(F)$ with supertranspose superinvolution, or $M_{k,2s}(F)$ with orthosymplectic superinvolution, then it has an elementary grading given by $g = (g_1, \dots, g_p)$, where $p = k + l$, with $l = k$ in the former case and $l = 2s$ in the latter one.

Consider now $M_{2p}(F)$ with grading word $(g|\operatorname{Rev} g) = (0^k, 1^{2l}, 0^k)$ and denote $L = \{i \in (g|\operatorname{Rev} g) \colon g_i = 0\}$ and $R = \{i \in (g|\operatorname{Rev} g) \colon g_i = 1\}$. Then, $|L| = 2k$ and $|R| = 2l$. The fiber L is divided into $L_1 = \{1, \dots, k\}$ and $L_2 = \{k + 2l + 1, \dots, 2k + 2l\}$.

To construct a signed version of ϑ_{2p}, we assign a signature η according to the following table (see [8, 9])

η	L_1	R	L_2
L_1	$+1$	$+1$	-1
R	-1	$+1$	$+1$
L_2	-1	-1	$+1.$

Then we define $\overline{\vartheta}_{2p}$ by

$$e_{ij}^{\overline{\vartheta}_{2p}} \colon = \eta(i,j) e_{ij}^{\vartheta_{2p}}.$$

We call $\overline{\vartheta}_{2p}$ the *orthosymplectic superinvolution* for $(M_{2p}(F), g)$. So, the embedding is described by the monomorphism φ from $A = M_{k,l}(F)$ to $M_{2p}(F)$ with grading given by $(g|\operatorname{Rev} g)$ and superinvolution $\overline{\vartheta}_{2p}$:

$$a \longmapsto \begin{pmatrix} a & 0 \\ 0 & (a^{\diamond})^{\vartheta_p} \end{pmatrix}$$

where $\diamond$ denotes the superinvolution of $M_{k,l}(F)$.

Type 3: If $A = M_{k,l}(F) \oplus M_{k,l}(F)^{sop}$, then it has an elementary grading given by $g = (0^k, 1^l)$, where $p = k + l$. In this case we can consider the monomorphism φ from A to $M_{2p}(F)$ with elementary grading given by $(g|\operatorname{Rev} g) = (0^k, 1^{2l}, 0^k)$ and superinvolution $\overline{\vartheta}_{2p}$:

$$(a,b) \longmapsto \begin{pmatrix} a & 0 \\ 0 & b^{\overline{\vartheta}_{2p}} \end{pmatrix}$$

where $\overline{\vartheta}_{2p}$ is defined as in the previous case.

Type 4: If $A = Q(k) \oplus Q(k)^{sop}$, then it has an elementary grading given by $g = (0^k, 1^k)$, where $p = 2k$. In this case we can consider $M_{2p}(F)$ with grading given by $(g|\operatorname{Rev} g) = (0^k, 1^{2k}, 0^k)$. The fibers L and R have the same cardinality (see [8]). So, $2k = |L| = |R|$. We define the sign map (see [9]) $\varepsilon\colon [2k] \times [2k] \to \{\pm 1\}$ by

$$\varepsilon(i,j) = \begin{cases} -1 & \text{if } (i,j) \in L \times R \\ +1 & \textit{otherwise.} \end{cases}$$

Then the map $\overline{\vartheta} = \overline{\vartheta}_{2p}$ defined by

$$e_{ij}^{\overline{\vartheta}_{2p}} : \ = \varepsilon(i,j) e_{ij}^{\vartheta_{2p}}$$

is a superinvolution and it is called the *supertranspose* and the embedding is given by the monomorphism φ from A to $M_{2p}(F)$ with elementary grading given by $(g|\operatorname{Rev} g)$ and superinvolution $\overline{\vartheta}_{2p}$:

$$(a + cb, a' + cb') \longmapsto \begin{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} a' & b' \\ b' & a' \end{pmatrix}^{\overline{\vartheta}_p} \end{pmatrix}.$$

For each type of simple $\diamond$-algebras, any $a \in A$ gives rise to a pair of $p \times p$ matrices (a', a'') such that

$$\varphi(a) = \begin{pmatrix} a' & 0 \\ 0 & a'' \end{pmatrix} \in M_{2p}(F)$$

for the appropriate p, and in each case φ turns out to be a $\diamond$-embedding.

Now let $(B_1, \dots, B_q)$ be a sequence of simple $\diamond$-algebras, each B_u with size p_u and grading word g_u, chosen for the embedding $B_u \hookrightarrow M_{2p_u}(F)$. Note that the *size* of B_u is

$$p_u := \begin{cases} k_u + 2s_u & \text{if } B_u = M_{k_u, 2s_u}(F), \\ k_u + l_u & \text{if } B_u = M_{k_u, l_u}(F) \oplus M_{k_u, l_u}(F)^{sop}, \\ 2k_u & \text{if } B_u = M_{k_u, k_u}(F) \text{ or } B_u = Q(k_u) \oplus Q(k_u)^{sop}. \end{cases}$$

Then consider the semisimple $\diamond$-algebra $B = B_1 \oplus \cdots \oplus B_q$ and set the size to

$$p := p_1 + \cdots + p_q.$$

Set $g := (g_1|g_2|\cdots|g_q) \in \mathbb{Z}_2^p$, so that the algebra $M_{2p}(F)$ has grading induced by $\tilde{g}$ and it is endowed with the standard supertranspose trp. Then the map $\varphi\colon B \to M_{2p}(F)$ defined by

$$\varphi(a_1 \dots, a_q) = \varphi\left(\begin{pmatrix} a_1' & 0 \\ 0 & a_1'' \end{pmatrix}, \dots, \begin{pmatrix} a_q' & 0 \\ 0 & a_q'' \end{pmatrix}\right) = \begin{pmatrix} a_1' & & & & & \\ & \ddots & & & & \\ & & a_q' & & & \\ & & & a_q'' & & \\ & & & & \ddots & \\ & & & & & a_1'' \end{pmatrix} \tag{4.2}$$

is a $\diamond$-embedding of the semisimple $\diamond$-algebra B into $M_{2p}(F)$. We say that $(M_{2p}(F), g)$ is the simple *enveloping $\diamond$-algebra* of the sequence $(B_1, \dots, B_q)$.

Now, for $i, j \in \{1, \dots, q\}$, let

$$U_{ij} := e_{ij} \otimes M_{p_i \times p_j}(F) \subseteq M_{2p}(F) \tag{4.3}$$

and denote its trp-image by $\overline{U_{ij}} := U_{ij}^{trp} \subseteq M_{2p}(F)$.

Definition 4.20. *If $(B_1, \dots, B_q)$ is a sequence of finite-dimensional simple $\diamond$-algebras, let $M_{2p}(F)$ be its simple enveloping $\diamond$-algebra and define*

$$UT^{\diamond}(B_1, \dots, B_q) := \varphi(B) \oplus \bigoplus_{1 \le i < j \le q} (U_{ij} \oplus \overline{U_{ij}}) \subseteq U$$

where $U := \bigoplus_{1 \le i \le j \le q} (U_{ij} \oplus \overline{U_{ij}}) \subseteq M_{2p}(F)$.

Note that the Jacobson radical of $UT^{\diamond}(B_1, \dots, B_q)$ is

$$J := \bigoplus_{1 \le i < j \le q} (U_{ij} \oplus \overline{U_{ij}}).$$

Finally, for any $\alpha \in \mathbb{Z}_2^p$, $UT_{\alpha}^{\diamond}(B_1, \dots, B_q)$ is the $\diamond$-algebra $UT^{\diamond}(B_1, \dots, B_q)$ with grading induced by $\tilde{\alpha} + \tilde{g}$. We will call any algebra of this kind a *triangular algebra*.

We are in a position to state the following result.

Theorem 4.21. *Let A be a finite-dimensional $\diamond$-algebra of $\diamond$-exponent d. Then there exist finite-dimensional simple $\diamond$-algebras $B_1, \dots, B_m$ where $\dim_F B_i = p_i$ and an element $g \in \mathbb{Z}_2^p$, where $p = p_1 + \dots + p_m$, such that $\dim_F(B_1 \oplus \dots \oplus B_m) = d$ and*

$$UT_g^{\diamond}(B_1, \dots, B_m) \in \mathrm{var}^{\diamond}(A).$$

Remark 4.22. Finally, we observe that if $\dim_F(B_1 \oplus \dots \oplus B_m) = d$, then

$$\exp^{\diamond}(UT_g^{\diamond}(B_1, \dots, B_m)) = d.$$

4.4.2 On the central $\diamond$-exponent

In this subsection we aim to prove the existence of the central $\diamond$-exponent $\exp^{\diamond,z}(A)$ for a finite-dimensional $\diamond$-algebra A. To this end, we will need to examine the possible values of $\exp^{\diamond}(A)$. We start by considering the situations $\exp^\diamond(A) \in \{0, 1, 3, d\}$, where $d \geq 5$. In the next subsection, we treat the cases $\exp^\diamond(A) = 2$ and $\exp^\diamond(A) = 4$.

• $\mathbf{exp^\diamond(A) = 0}$: in this case A is nilpotent and clearly we have that $\exp^{\diamond,z}(A) = 0$.

• $\mathbf{exp^\diamond(A) = 1}$: there are two possibilities.

If $c_n^{\diamond,z}(A) = 0$ for some n, then we have that $c_k^{\diamond,z}(A) = 0$ for all $k \geq n$ and so $\exp^{\diamond,z}(A) = 0$.

If $c_n^{\diamond,z}(A) \neq 0$ for all n, then, according to Remark 4.18, there exist constants C_2, t_2 such that, for all n,

$$1 \leq c_n^{\diamond,z}(A) \leq C_2 n^{t_2}.$$

Hence $\exp^{\diamond,z}(A) = 1$.

• $\mathbf{exp^\diamond(A) > 1}$.

First we give an example, highlighting the important role that triangular algebras play in this situation.

Example 4.23. Let us consider the $\diamond$-algebra $A = B_1 \oplus B_2 + J$ such that $B_1 J B_2 \neq 0$, where

- $B_1 = M_{2,0}(F) \oplus M_{2,0}(F)^{sop}$ with exchange superinvolution;
- $B_2 = M_{1,1}(F)$ with transpose superinvolution.

Notice that $\exp^\diamond(B_1) = 8$ and, by [25], $\exp^\diamond(B_2) = 4$. Since $\dim_F(B_1 \oplus B_2) = 12$, it follows that

$$\exp^\diamond(A) = 12.$$

Now, we want to construct $UT_g^\diamond(B_1, B_2) \in \mathrm{var}^\diamond(A)$, by following the steps explained in the previous subsection. We have $p_1 = 2$, $p_2 = 2$, $p = p_1 + p_2 = 4$ and $2p = 8$. Hence

$$\tilde{U} = UT_g^\diamond(B_1, B_2) := \begin{pmatrix} a & b & u_{11} & u_{12} & & & & \\ c & d & u_{21} & u_{22} & & & & \\ & & j & k & & & & \\ & & l & m & & & & \\ & & & & j & -k & \overline{u}_{11} & \overline{u}_{12} \\ & & & & -l & m & \overline{u}_{21} & \overline{u}_{22} \\ & & & & & & i & h \\ & & & & & & f & e \end{pmatrix}$$

where $g = (0^3, 1^2, 0^3)$, $\overline{u}_{rt} = u_{rt}^{\vartheta}$ for $r, t = 1, 2$ and according to the notation established in (4.3),

$$J(\tilde{U}) = U_{12} \oplus \overline{U_{12}}.$$

Our next goal is to show that $J(\tilde{U}) \cap Z(\tilde{U}) = \{0\}$. Consider $j \in J(\tilde{U}) \cap Z(\tilde{U})$. So $j \in U_{12} \cup \overline{U_{12}}$. Moreover, since $j \in Z(\tilde{U})$, we must have $a_1 j = j a_1$ and $a_2 j = j a_2$, where $a_1 := \psi(1_{B_1})$ and $a_2 := \psi(1_{B_2})$. So,

$$a_1 = e_{11} + e_{22} + e_{77} + e_{88} \text{ and } a_2 = e_{33} + e_{44} + e_{55} + e_{66}.$$

An easy computation shows that

$$a_1 j = u_{11}e_{13} + u_{12}e_{14} + u_{21}e_{23} + u_{22}e_{24},$$
$$j a_1 = \overline{u}_{11}e_{57} + \overline{u}_{12}e_{58} + \overline{u}_{21}e_{67} + \overline{u}_{22}e_{68}.$$

Since $a_1 j = j a_1$, it follows that $u_{rt} = \bar{u}_{rt} = 0$, for all $r, t = 1, 2$, and so $j = 0$, as desired.

Observe that, if f is a central multilinear $\diamond$-polynomial, we may assume without loss of generality that every variable of f appears in every monomial of f. In fact if $f = \sum f_i$ is a sum of monomials f_i, where distinct sets of variables appear in each f_i, then each monomial f_i is also a central $\diamond$-polynomial. In order to prove the last statement, fix an index k and take an arbitrary evaluation $\mu(f_k)$. Now define the evaluation μ' of f such that $\mu'(f_k) = \mu(f_k)$ and all variables not appearing in f_k are evaluated in zero. Since f is a central $\diamond$-polynomial, we have that $\mu'(f) = \sum \mu'(f_i) = \mu'(f_k) = \mu(f_k)$ is a central element; thus f_k is also a central $\diamond$-polynomial.

Now, let f be a multilinear central $\diamond$-polynomial. As $J(\tilde{U}) \cap Z(\tilde{U}) = \{0\}$ and $B_1 \cap B_2 = \{0\}$, then we have to evaluate all the variables of f in the same simple component, either B_1 or B_2. It follows that any evaluation λ of f is such that $\lambda(f) = \alpha a_1$ or $\lambda(f) = \beta a_2$, for some $\alpha, \beta \in F$. Taking $m \in U_{12}$, it follows that $\alpha a_1 m = \alpha m = m \alpha a_1 = 0$. Hence $\alpha = 0$. Analogously one gets $\beta = 0$. In conclusion

$$c_n^{\diamond,z}(\tilde{U}) = c_n^{\diamond}(\tilde{U}).$$

Since $\tilde{U} \in \mathrm{var}^{\diamond}(A)$, according to [25], there exist some constants $Q_1 > 0$, t_1 such that, for all n,

$$c_n^{\diamond,z}(A) \geq c_n^{\diamond,z}(\tilde{U}) = c_n^{\diamond}(\tilde{U}) \geq Q_1 n^{t_1} 12^n.$$

Taking into account Remark 4.18, we can conclude that $\exp^{\diamond,z}(A) = 12$.

We see that the Jacobson radical of $UT_g^{\diamond}(B_1, \ldots, B_m)$ has trivial intersection with its center.

Lemma 4.24. $Z(UT_g^{\diamond}(B_1, \ldots, B_m)) \cap J(UT_g^{\diamond}(B_1, \ldots, B_m)) = \{0\}$.

Proof. It is sufficient to follow step by step the proof of [42, Proposition 4.1], with the necessary changes. ■

Using the same arguments of [42, Proposition 4.2], by Lemma 4.24, we get the following result.

Proposition 4.25. *If $m \geq 2$, then $UT_g^{\diamond}(B_1, \ldots, B_m)$ does not have proper central $\diamond$-polynomials.*

Now we are ready to prove the following result.

Corollary 4.26. *Considering $\tilde{U} = UT_g^{\diamond}(B_1, \ldots, B_m)$ with $m \geq 2$, we have $c_n^{\diamond,z}(\tilde{U}) = c_n^{\diamond}(\tilde{U})$. Moreover, if A is a finite-dimensional $\diamond$-algebra such that*

- $\exp^{\diamond}(A) = d$ *and*
- $\tilde{U} \in \mathrm{var}^{\diamond}(A)$, *with* $\dim_F(B_1 \oplus \cdots \oplus B_m) = d$,

then $\exp^{\diamond,z}(A) = d$.

Proof. As observed in Section 4, we have that $\exp^\diamond(\tilde{U}) = d$ and $\tilde{U} \in \mathrm{var}^\diamond(A)$. By Proposition 4.25, we get

$$c_n^{\diamond,z}(A) \geq c_n^{\diamond,z}(\tilde{U}) = c_n^\diamond(\tilde{U}) \geq Q_1 n^{r_1} d^n,$$

for some constants Q_1, r_1. Using Remark 4.18, this implies that $\exp^{\diamond,z}(A) = d$. ■

• $\exp^\diamond(A) = 3$.

Observe that there is no 3-dimensional simple $\diamond$-algebra. Hence, by Theorem 4.21, at least one of the triangular algebras $UT_g^\diamond(F,F,F)$ or $UT_g^\diamond(B,F)$, where $\dim_F B = 2$, belongs to $\mathrm{var}^\diamond(A)$. Then, as a direct consequence of Corollary 4.26, we conclude that $\exp^{\diamond,z}(A) = \exp^\diamond(A) = 3$.

• $\exp^\diamond(A) \geq 5$.

Our main goal is to prove that, in this situation, the central $\diamond$-exponent exists and it has the same value as $\exp^\diamond(A)$. To this end, we need a technical lemma.

Lemma 4.27. *Let B be a finite-dimensional simple $\diamond$-algebra such that $\dim_F B \geq 5$. Then there exists an element $b \in B$ such that, for all non-zero $a \in Z(B)$, we have that $ab \notin Z(B)$.*

Proof. According to Theorem 4.6, we have to consider the following possibilities.

- $B \cong M_{k,l}(F)$, with $k + l \geq 2$.

  The element we are looking for is $b = e_{1,k+l} + I_{k+l}$. In fact, taking $a = \delta I_{k+l} \in Z(B)$, we get

  $$ab = \delta(I_{k+l} + e_{1,k+l}) \notin Z(B).$$

- $B \cong M_{k,l}(F) \oplus M_{k,l}^{sop}(F)$, with $k \geq 2$.

  The element we are looking for is $b = (I_{k+l} + e_{1,k+l}, I_{k+l} + e_{1,k+l})$.

- $B \cong Q(k) \oplus Q(k)^{sop}$, $k \geq 2$.

  In this case, we can consider $b = (I_k + e_{1,k}, I_k + e_{k,1})$.

■

Now we are in a position to prove the main result of this section.

Theorem 4.28 (Giordani, Ioppolo, dos Santos, Vieira). *Let A be a finite-dimensional $\diamond$-algebra. If $\exp^\diamond(A) = d \geq 5$, then there exist non-zero constants Q_1, C_2, r_1, t_2, such that, for n large enough,*

$$Q_1 n^{r_1} d^n \leq c_n^{\diamond,z}(A) \leq C_2 n^{t_2} d^n.$$

Proof. According to Remark 4.18, we already have the desired upper bound. In order to complete the proof, we need to find a suitable lower bound as well.

Let $\tilde{U} = UT_g^\diamond(B_1, \ldots, B_m)$ be the $\diamond$-algebra obtained according to Theorem 4.21. Since $\tilde{U} \in \mathrm{var}^\diamond(A)$, if $m \geq 2$, we are done by Corollary 4.26.

Now assume $m = 1$. Hence $\tilde{U} = UT_g^\diamond(B_1) \cong B_1$, that is, $\tilde{U}$ is a simple $\diamond$-algebra with dimension $d \geq 5$ and $\exp^\diamond(\tilde{U}) = d$. Recall that, since $\tilde{U} \in \mathrm{var}^\diamond(A)$, we have that $\mathrm{Id}^{\diamond,z}(A) \subseteq \mathrm{Id}^{\diamond,z}(\tilde{U})$, and so $c_n^{\diamond,z}(A) \geq c_n^{\diamond,z}(\tilde{U})$. Thus we are left to find a lower bound for $c_n^{\diamond,z}(\tilde{U})$. Fix n and write $c_n^\diamond(\tilde{U}) := N$.

This means that there exist $f_1, \dots, f_N \in P_n^\diamond$ that are linearly independent $\diamond$-polynomials modulo $\mathrm{Id}^\diamond(\tilde{U})$.

We wish to prove that, in this situation, $c_{n+1}^{\diamond,z}(\tilde{U}) \geq c_n^\diamond(\tilde{U})$. Consider the multilinear $\diamond$-polynomial

$$w(y_{n+1}^+, y_{n+1}^-, z_{n+1}^+, z_{n+1}^-) = y_{n+1}^+ + y_{n+1}^- + z_{n+1}^+ + z_{n+1}^-.$$

It is clear that, for any element $u \in \tilde{U}$, there must be an evaluation μ' such that $\mu'(w) = u$. We will show that $f_1 w, \dots, f_N w$ are linearly independent modulo $\mathrm{Id}^{\diamond,z}(\tilde{U})$.

Fix $\lambda_1, \dots, \lambda_N$ not all zero scalars and consider the $\diamond$-polynomial $f = \lambda_1 f_1 + \cdots + \lambda_N f_N$. Since f is not an identity of $\tilde{U}$, there exists an evaluation μ'' of f such that $\mu''(f) \neq 0$. We wish to show that fw is not a central $\diamond$-polynomial of $\tilde{U}$. To this end, consider the evaluation μ of fw such that $\mu(fw) = \mu''(f)\mu'(w)$, where

$$\mu'(w) = \begin{cases} 1_{\tilde{U}} & \text{if } \mu''(f) \notin Z(\tilde{U}) \\ b \in \tilde{U} & \text{otherwise} \end{cases}$$

where $b \in \tilde{U}$ is the element given in Lemma 4.27.

In both cases, the result of the evaluation $\mu(fw)$ does not belong to the center of $\tilde{U}$ and so we conclude that fw is not central. A similar reasoning proves that each $\diamond$-polynomial f_i is also not central. Thus we get that $f_1 w, \dots, f_N w$ are linearly independent modulo $\mathrm{Id}^{\diamond,z}(\tilde{U})$. This means that $c_{n+1}^{\diamond,z}(\tilde{U}) \geq N = c_n^\diamond(\tilde{U})$. Since $c_n^\diamond(\tilde{U}) \geq Q_1 n^{r_1} d^n$, for some constants Q_1, r_1, we are sure that, for n large enough, we have $c_n^{\diamond,z}(\tilde{U}) \geq Q_1 n^{r_1} d^n$. The proof is complete. ■

As a consequence we immediately get the following corollary.

Corollary 4.29. *If A is a finite-dimensional $\diamond$-algebra and* $\exp^\diamond(A) \geq 5$, *then* $\exp^{\diamond,z}(A)$ *exists and*

$$\exp^{\diamond,z}(A) = \exp^\diamond(A).$$

4.4.3 Particular cases

In this subsection we shall treat the remaining possibilities for the $\diamond$-exponent, that is $\exp^\diamond(A) \in \{2, 4\}$.

• $\mathbf{exp^\diamond(A) = 2}$.

Assume first that $UT_g^\diamond(F, F) \in \mathrm{var}^\diamond(A)$. By Corollary 4.26, we immediately get $\exp^{\diamond,z}(A) = 2$.

On the other hand, by Theorem 4.21, there exists $B_i \in \mathrm{var}^\diamond(A)$, where B_i is a simple $\diamond$-algebra with $\exp^\diamond(B_i) = 2$. According to Theorem 4.6, up to isomorphism, we have that $B_i \cong F \oplus F$ with trivial grading and exchange superinvolution. Since the grading is trivial, the exchange superinvolution can be seen as a (graded) involution and so, as in [42], we obtain the following result.

Proposition 4.30 (Giordani, Ioppolo, dos Santos, Vieira). *Suppose that A is a finite-dimensional $\diamond$-algebra, where* $\exp^\diamond(A) = 2$ *and* $UT_g^\diamond(F, F) \notin \mathrm{var}^\diamond(A)$, *then* $\exp^{\diamond,z}(A)$ *exists and* $\exp^{\diamond,z}(A) \in \{0, 1, 2\}$.

In conclusion, we have the following corollary.

Corollary 4.31. *If A is a finite-dimensional $\diamond$-algebra such that* $\exp^{\diamond}(A) = 2$, *then* $\exp^{\diamond,z}(A)$ *exists and* $\exp^{\diamond,z}(A) \in \{0,1,2\}$.

• $\mathbf{exp^{\diamond}(A) = 4}$.

By Corollary 4.26, if $UT_g^{\diamond}(B_1,\dots,B_m) \in \mathrm{var}^{\diamond}(A)$, $g \in \mathbb{Z}_2^p$, $p = p_1 + \cdots + p_m$ with $\dim_F(B_1 \oplus \cdots \oplus B_m) = 4$ and $m \geq 2$, then

$$\exp^{\diamond,z}(A) = 4.$$

Otherwise, we have that $m = 1$. Consider $A = B_1 \oplus \cdots \oplus B_m + J(A)$. By Theorem 4.21, there exists $B_i \in \mathrm{var}^{\diamond}(A)$, where B_i is a simple $\diamond$-algebra with $\dim_F(B_i) = \exp^{\diamond}(B_i) = 4$. According to Theorem 4.6, it follows that B_i must be isomorphic to one of the following simple $\diamond$-algebras:

- $M_{2,0}(F)$ with trivial grading and orthosymplectic superinvolution;
- $M_{1,1}(F)$ with transpose superinvolution;
- $(F + cF) \oplus (F + cF)^{sop}$, with grading $(F \oplus F, cF \oplus cF)$ and exchange superinvolution.

In the first two cases, we can find suitable elements as those given in Lemma 4.27. Consequently, applying similar arguments as those used in the proof of [42, Theorem 4.5], it is possible to find a lower bound for the central $\diamond$-graded codimensions of A and conclude that $\exp^{\diamond,z}(A) = 4$.

From now on, we will focus our attention on the third situation.

Remark 4.32. The simple $\diamond$-algebra $B = (F + cF) \oplus (F + cF)^{sop}$ has 3 distinct homogeneous elements x_1, x_2 and x_3 such that $x_1^2 = (1,1)$ and $x_i^2 = (1,-1)$, with $i = 2,3$. In fact, it is enough to consider

- $x_1 := (1,-1)$, a skew element of homogeneous degree 0;
- $x_2 := (c1, c1)$, a symmetric element of homogeneous degree 1;
- $x_3 := (c1, -c1)$, a skew element of homogeneous degree 1.

Again we will follow a similar reasoning as the one presented in [42]. First consider the case $r = 1$, that is B_1 is the single simple component of the maximal semisimple subalgebra of A which is isomorphic to $(F + cF) \oplus (F + cF)^{sop}$.

Proposition 4.33 (Giordani, Ioppolo, dos Santos, Vieira). *Suppose that* $A = B_1 + J(A)$, *where* $B_1 \cong (F + cF) \oplus (F + cF)^{sop}$, *then* $\exp^{\diamond,z}(A)$ *exists and it is either* 0 *or* 4.

Proof. Assume first that for all $b \in A$, $1_{B_1} b \in Z(A)$. Then for n large enough, $c_n^{\diamond,z}(A) = 0$ and so

$$\exp^{\diamond,z}(A) = 0.$$

Now assume there exists an element $b \in A$ such that $1_{B_1} b \notin Z(A)$, and we may assume it to be homogeneous without loss of generality. In order to show the existence of the central $\diamond$-graded exponent we will find a suitable lower bound for $c_n^{\diamond,z}(A)$.

Let us fix $n \geq 4$ and for $0 \leq i \leq n-2$ with $n-i-2$ even, consider the following set of monomials

$$Q_i := \{y_{q_1}^{+} \cdots y_{q_i}^{+} y_{r_1}^{-} \cdots y_{r_j}^{-} z_{s_1}^{+} \cdots z_{s_k}^{+} z_{t_1}^{-} \cdots z_{t_l}^{-} w'_{n-1} w_n\}$$

- w_n is a variable of the same homogeneous degree as the element b;
- j, k, l run through all combinations of even non-negative integers such that

$$j + k + l = n - i - 2;$$

- $w'_{n-1} = y^+_{n-1}$ if $k + l \equiv 0 \pmod 4$ or $w'_{n-1} = z^+_{n-1}$ otherwise;
- $q_1 < \cdots < q_i$, $r_1 < \cdots < r_j$, $s_i < \cdots < s_k$, $t_1 < \cdots < t_l$;
- $\{q_1, \ldots, q_i\} \cup \{r_1, \ldots, r_j\} \cup \{s_1, \ldots, s_k\} \cup \{t_1, \ldots, t_l\} = \{1, \ldots, n-2\}$.

Also we consider $Q := \bigcup\limits_{n-i-2 \text{ even}} Q_i$.

By following the same steps as in [42], we can conclude that $|Q| \geq 4^{n-4}$. In fact,

$$|Q| = 2^{2n-7} - 2^{n-4} \geq 4^{n-4}, \text{ if } n \text{ is odd},$$
$$|Q| = 2^{2n-7} + 2^{n-3} \geq 4^{n-4}, \text{ if } n \text{ is even}.$$

We are left to show that the elements of Q are not central and linearly independent modulo $\mathrm{Id}^{\diamond,z}(A)$.

Recall that x_1, x_2 and x_3 are the elements given in Remark 4.32. We will consider an evaluation λ such that w_n goes in the element b and the remaining variables will be evaluated in 1_{B_1}, x_1, x_2 and x_3 according to their corresponding homogeneous degrees. It follows that

$$\lambda\left(y^+_{q_1} \cdots z^-_{t_l} w'_{n-1} w_n\right) = \begin{cases} 1_{B_1} 1_{B_1} b = 1_{B_1} b \notin Z(A) & \text{if } k + l \equiv 0 \pmod 4, \\ x_1 x_1 b = 1_{B_1} b \notin Z(A) & \text{if } k + l \not\equiv 0 \pmod 4. \end{cases}$$

In conclusion, all the elements of Q are not central. The proof that they are linearly independent is identical to the one presented in [42].

By putting all this together, for some constants C_2 and t_2, the following inequality holds for all $n \geq 4$:

$$4^{n-4} \leq c_n^{\diamond,z}(A) \leq C_2 n^{t_2} 4^n.$$

It follows that $\exp^{\diamond,z}(A) = 4$ and the proof is complete. ■

According to the above proposition, we can apply the same steps as in [42], to obtain the following result.

Proposition 4.34 (Giordani, Ioppolo, dos Santos, Vieira). *Let $A = B + J(A)$ with $\exp^{\diamond}(A) = 4$, where $B = B_1 \oplus \cdots \oplus B_m$, $m \geq 2$, and $B_i \cong (F + cF) \oplus (F + cF)^{sop}$, for some $i \in \{1, \ldots, m\}$. Then $\exp^{\diamond,z}(A)$ exists and $\exp^{\diamond,z}(A) \in \{0, 1, 2, 3, 4\}$.*

In the next corollary we summarize all the above results.

Corollary 4.35 (Giordani, Ioppolo, dos Santos, Vieira). *If A is a finite-dimensional $\diamond$-algebra such that $\exp^{\diamond}(A) = 4$ then $\exp^{\diamond,z}(A)$ exists and $\exp^{\diamond,z}(A) \in \{0, 1, 2, 3, 4\}$.*

Finally, we provide an example that illustrates what can happen when the hypotheses of Proposition 4.33 are not satisfied.

Example 4.36. Let us consider the $\diamond$-algebra $A = A_1 \oplus A_2$, where

- $A_1 = (F + cF) \oplus (F + cF)^{sop}$;
- $A_2 = UT_g^{\diamond}(F, F, F)$, with $g = (0^3)$.

Notice that A_1 is a commutative algebra, $\exp^{\diamond}(A_1) = 4$ and $\exp^{\diamond}(A_2) = 3$. Thus $\exp^{\diamond}(A) = 4$.

Now let us compute the central $\diamond$-exponent of A. Since A_1 is a commutative algebra, any polynomial with no constant term is central for A_1. This implies that

$$\mathrm{Id}^{\diamond,z}(A) = \mathrm{Id}^{\diamond,z}(A_1) \cap \mathrm{Id}^{\diamond,z}(A_2) = \mathrm{Id}^{\diamond,z}(A_2).$$

So $c_n^{\diamond,z}(A) = c_n^{\diamond,z}(A_2)$. Moreover, we can apply Proposition 4.25 to A_2 and conclude that it has no proper central polynomials. Then $c_n^{\diamond,z}(A) = c_n^{\diamond,z}(A_2) = c_n^{\diamond}(A_2)$. This implies that

$$\exp^{\diamond,z}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{\diamond,z}(A)} = \lim_{n\to\infty} \sqrt[n]{c_n^{\diamond}(A_2)} = \exp^{\diamond}(A_2) = 3.$$

We have proved that $\exp^{\diamond}(A) = 4$ whereas $\exp^{\diamond,z}(A) = 3$.

Here, it is important to highlight that the previous example is not an isolated case but represents a general behavior. In fact, following step by step the arguments employed in [42], it is possible to show that any $\diamond$-algebra A satisfying $\exp^{\diamond,z}(A) < \exp^{\diamond}(A)$ is a direct sum of a commutative $\diamond$-algebra A_1 with $\exp^{\diamond}(A) = \exp^{\diamond}(A_1)$ and a $\diamond$-algebra A_2 such that $\exp^{\diamond}(A_2) < \exp^{\diamond}(A)$.

Bibliography

[1] E. Aljadeff and A. Giambruno. Multialternating graded polynomials and growth of polynomial identities. *Proceedings of the American Mathematical Society*, 141:3055–3065, 2013.

[2] E. Aljadeff, A. Giambruno, and Y. Karasik. Polynomial identities with involution, superinvolutions and the grassmann envelope. *Proceedings of the American Mathematical Society*, 145(5):1843–1857, 2017.

[3] E. Aljadeff, A. Giambruno, and D. La Mattina. Graded polynomial identities and exponential growth. *Journal für die Reine und Angewandte Mathematik*, 650:83–100, 2011.

[4] Y. Bahturin, M. Tvalavadze, and T. Tvalavadze. Group gradings on superinvolution simple superalgebras. *Linear Algebra and its Applications*, 431(5):1054–1069, 2009.

[5] Y. Bahturin and M. Zaicev. Graded algebras and graded identities. *Lecture Notes in Pure and Applied Mathematics*, pages 101–140, 2003.

[6] F. Benanti, A. Giambruno, and M. Pipitone. Polynomial identities on superalgebras and exponential growth. *J. Algebra*, 269:422–438, 2003.

[7] E. Campedel, G. Giordani, and A. Ioppolo. Codimensions of algebras with pseudoautomorphism and their exponential growth. *Journal of Algebra*, 2025.

[8] O. M. Di Vincenzo and V. C. Nardozza. Minimal affine varieties of superalgebras with superinvolution: a characterization. *Linear Algebra and its Applications*, 698:514–536, 2024.

[9] O. M. Di Vincenzo and V. C. Nardozza. Minimal $*$-varieties and superinvolutions. *Israel Journal of Mathematics*, pages 311–346, 2025.

[10] A. Giambruno, A. Ioppolo, and D. La Mattina. Varieties of algebras with superinvolution of almost polynomial growth. *Algebras and Representation Theory*, 19:599–611, 2016.

[11] A. Giambruno, A. Ioppolo, and D. La Mattina. Superalgebras with involution or superinvolution and almost polynomial growth of the codimensions. *Algebras and Representation Theory*, 22(4):961–976, 2019.

[12] A. Giambruno and D. La Mattina. Graded polynomial identities and codimensions: Computing the exponential growth. *Advances in Mathematics*, 225:859–881, 2010.

[13] A. Giambruno and S. Mischchenko. Polynomial growth of the $*$-codimensions and young diagrams. *Comm. Algebra*, 29(1):277–284, 2001.

[14] A. Giambruno, S. Mischchenko, and M. V. Zaicev. Polynomial identities on superalgebras and almost polynomial growth. *Comm. Algebra*, 29(9):3787–3800, 2001.

[15] A. Giambruno and A. Regev. Wreath products and p.i. algebras. *J. Pure Appl. Algebra*, 35(2):133 – 149, 1985.

[16] A. Giambruno and M. V. Zaicev. A characterization of varieties of associative algebras of exponent two. *Serdica Math. J.*, 26:245–252, 2000.

[17] A. Giambruno and M. V. Zaicev. *Polynomial identities and asymptotic methods*, volume 122. American Mathematical Soc., 2005.

[18] A. Giambruno and M. V. Zaicev. Central polynomials and growth functions. *Israel J. Math.*, 226:15–28, 2018.

[19] A. Giambruno and M. V. Zaicev. Central polynomials of associative algebras and their growth. *Proc. Amer. Math. Soc. 147*, 3:909–919, 2019.

[20] G. Giordani. On superalgebras with pseudoautomorphism of polynomial codimension growth. *Communications in Algebra*, 52(10):4093–4104, 2024.

[21] G. Giordani, A. Ioppolo, A. A. P. dos Santos, and A. C. Vieira. On the central expponent of superalgebras with superinvolution. *Canadian Mathematical Bulletin*, published online 2025:1–13.

[22] C. Gomez-Ambrosi and I. P. Shestakov. On the lie structure of the skew-elements of a simple superalgebra with involution. *J. Algebra*, 208(1):43–71, 1998.

[23] A. Gordienko. Amitsur's conjecture for associative algebras with a generalized hopf action. *Journal of Pure and Applied Algebra*, 217(8):1395–1411, 2013.

[24] A. Gordienko. Graded group actions and generalized h-actions compatible with gradings. *Linear Algebra and its Applications*, 682:96–121, 2024.

[25] A. Ioppolo. The exponent for superalgebras with superinvolution. *Linear Algebra Appl.*, 555:1–20, 2018.

[26] A. Ioppolo. Graded linear maps on superalgebras. *Journal of Algebra*, 605:377–393, 2022.

[27] A. Ioppolo and D. La Mattina. Classifying simple superalgebras with automorphism and pseudoautomorphism. *Journal of Algebra*, 649:1–11, 2024.

[28] A. Ioppolo and F. Martino. Classifying g-graded algebras of exponent two. *Israel J. Math.*, 229:341–356, 2019.

[29] A. Ioppolo and F. Martino. Gradings and graded linear maps on algebras. *Forum Mathematicum*, 2024.

[30] A. R. Kemer. T-ideals with power growth of the codimensions are specht. *Siberian Mathematicval Journal (in Russian)*, 19:54 – 69, 1978.

[31] A. R. Kemer. Varieties of finite rank. *Proc. 15-th All the Union Algebraic Conf., Krasnoyarsk (in Russian)*, 2:from 73, 1979.

[32] A. R. Kemer. Varieties and z_2-graded algebras. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(5):1042–1059, 1984.

[33] P. Koshlukov and D. La Mattina. Graded algebras with polynomial growth of their codimensions. *Journal of Algebra*, 434:115–137, 2015.

[34] D. Krakowski and A. Regev. The polynomial identities of the grassmann algebra. *Trans. Amer. Math. Soc.*, 181, 1973.

[35] D. La Mattina. Varieties of almost polynomial growth: classifying their subvarieties. *Manuscripta Math.*, 123, 2007.

[36] D. La Mattina, R. B. dos Santos, and A. C. Vieira. Proper central exponent of superalgebras with graded involution or superinvolution. *Math. Z.*, 2025.

[37] D. La Mattina, F. Martino, and C. Rizzo. Central polynomials of graded algebras: Capturing their exponential growth. *Journal of Algebra*, 600, 2022.

[38] Y. Malcev. A basis for the identities of the algebra of upper triangular matrices. *Algebra Logic*, 10:242–247, 1971.

[39] F. Martino and C. Rizzo. Growth of central polynomials of algebras with involution. *Transactions of the American Mathematical Society*, 375(1):429–453, 2022.

[40] M. L. Racine. Primitive superalgebras with superinvolution. *J. Algebra*, 206(2):588–614, 1998.

[41] A. Regev. Existence of identities in $a \otimes b$. *Isr. J. Math.*, 11:131–152, 1972.

[42] A. A. P. d. Santos, R. B. dos Santos, and A. C. Vieira. Central exponent of algebras with graded involution. *Int. J. Algebra Comput.*, 35(03):467–487, 2025.

[43] A. Valenti. The graded identities of upper triangular matrices of size two. *J. Pure Appl. Algebra*, 172:325–335, 2002.

Acknowledgements

"I open at the close"[1]

This is the short version of my acknowledgements. The full one will only be released in the printed version of this thesis, at the very end. Anyway, it is important for me to say some grateful words to the people who made all of this possible.

I start by acknowledging the University of L'Aquila and all the people who work here, making everything work well. I thank the DISIM and particularly all the professors and researchers I interacted with, mainly from the Algebra and Geometry Group, because I learned a lot from all of you and the opportunities you gave me.

Clearly, I want to sincerely thank my supervisor, Professor Antonio Ioppolo, for working with me and guiding me through this journey. Without you, we wouldn't be here today and this thesis wouldn't exist. Thank you for everything you've done for me, for the enormous and wonderful opportunities you've given me and your precious advice. I truly hope we will go on working together.

I also want to thank Professor Ana Cristina Viera from the Institute of Exact Sciences at the Federal University of Minas Gerais for welcoming me so warmly during my research visit to UFMG, for working with me, and for her kindness and attention. Muito obrigada, Professora!

Moreover, I would like to thank Dr. Carla Rizzo for welcoming me at Centre for Mathematics of the University of Coimbra, for working with me and giving me the opportunity to learn a lot during my stay in Portugal with her.

I thank GNSAGA Research group from INdAM for the fundings I received, which allowed me to take part in academic events and have great work and life experiences.

Moreover, I would like to thank the researchers and professors I have known from all around the world and with whom interacted during these three years. I also would like to thank some PhD colleagues for being with me during these years.

Sinecre acknowledgements go to my closest and dearest friends for staying always by my side and all the moments we shared. You are a special part of my life and I thank all of you for your important presence.

Lastly, but most importantly, I want to thank my whole family for the love and support they continuously offered me. I want to explicitly mention my parents Irene and Lorenzo and my sister Giulia. Without you, all of this would have remained just a dream and I will always be grateful for everything.

With lots of love,

Ginevra

[1] Harry Potter and the Deathly Hallows