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Consensus for Hegselmann–Krause type models with time variable time delays

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In this paper, we analyze a Hegselmann–Krause opinion formation model with time variable time delay and prove that if the influence function is always positive, then there is exponential convergence to consensus without requiring any smallness assumptions on the time delay function. The analysis is then extended to a model with distributed time delay.

KEYWORDS
consensus, multi-agent systems, time delays

MSC CLASSIFICATION
34D05, 91D10, 34K20

1 | INTRODUCTION

Multiagent systems have attracted, in recent years, the attention of many researchers in several scientific disciplines, such as biology [1, 2], economics [3, 4], robotics [5, 6], control theory [7–12], social sciences [13–17]. In particular, we mention the celebrated Hegselmann–Krause (HK) model [18],

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(t)(x_j(t) - x_i(t)), \qquad i = 1, \dots, N,$$

formulated to describe the opinion formation in a group of individuals (see previous works [19, 20] for the related PDE model), and its second-order version, that is, the Cucker–Smale (CS) model [2],

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= \sum_{j=1}^{N} a_{ij}(t)(v_j(t) - v_i(t)), \qquad i = 1, \dots, N, \end{aligned}$$

introduced to describe flocking phenomena. In both cases, the functions $a_{ij}(t)$ are called communication rates. A typical feature is the emergence of a collective behavior, namely, under quite general assumptions, solutions to such systems converge to consensus or, in the case of the CS model, to flocking.

It is also natural to consider in such models time delay effects. Time delays often appear in many phenomena from biology, social sciences, economics, physics, engineering, etc. Concerning multiagent systems, in particular opinion formation models, we have to take into account a certain time lag in the information propagation or in the reaction time. Of course, now, social media development greatly reduces the time delay size. However, small time delays are always present and therefore the mathematical models should take them into account.

The presence of a delay makes the models more difficult to deal with since a delay, even small, can destroy some geometric features typical of the undelayed models. In particular, for Hegselmann–Krause models with always positive symmetric interactions, it is easy to show that the system converges to consensus due to symmetry reasons. If we add a delay in such models, then the symmetry is broken and, in turn, the asymptotic analysis requires finer arguments.

On the other hand, despite mathematical difficulties to overcome, the presence of time delays, which naturally appear in applications, allows us to better describe the real features of the models.

In this paper, we deal with a Hegselmann–Krause opinion formation model with time variable time delay. The HK model has been originally introduced to describe bounded confidence interactions [18], namely, agents are influenced only by agents in a certain "confidence" radius. Here, as in recent papers (see, e.g., previous studies [11, 21]), we keep the same name to deal with strictly positive influence functions.

There is a wide literature concerning opinion formation models in the undelayed case (see, e.g., previous works [22–26]). The fractional Hegselmann–Krause model has been studied in Almeida et al. [27] while previous studies [28–30] deal with noisy models.

Multiagent systems with time delays have already been studied by some authors. Flocking results for the CS model with delay have been proved [31–37] in different settings, under a smallness assumption on the time delay size. We mention also Dong et al. [38] for the analysis of a thermomechanical CS model with delay.

Concerning the Hegselmann–Krause model for opinion formation, convergence to consensus results have been proved in presence of small time delays in previous works [39–41]. More recently, in Haskovec [42], a consensus result is proved, in the case of a constant time delay, without requiring any upper bound on the time delay. With respect to Haskovec [42], we consider time variable time delays instead of constant delays. This requires a finer analysis of the system. Furthermore, also in the constant case, we improve the convergence to consensus result. Indeed, here, we prove exponential convergence to consensus while there only asymptotic convergence is proved. We mention also Lu et al. [21] for a consensus result without any smallness assumptions on the time delay size but in the particular case of constant interaction coefficients. Finally, a flocking result has been recently obtained in Rodriguez Cartabia [43] for a CS model with constant time delay without any restrictions on the time delay size, applying a step by step procedure. We mention also Pignotti and Reche Vallejo [44] for a flocking result without smallness assumption on the time delay related to a CS model with leadership. Here, we extend the argument of Rodriguez Cartabia [43] to the Hegselmann–Krause opinion formation model in the case of a time-dependent time delay. We are also able to consider a more general influence function with respect to previous literature on HK models, without requiring any monotonicity assumptions.

We also consider the continuity type equation obtained as mean-field limit of the particle model, when the number N of the agents tends to infinity. Indeed, since the constants appearing in the exponential consensus estimate for the discrete model are independent of the number of the agents, we can extend the consensus result to the related PDE model.

Moreover, we extend the results obtained for the Hegselmann–Krause model with time variable time delay to a model with distributed time delay, namely each agent is influenced by other agents' opinions in a certain time interval (cf., previous works [35, 40]). Also in this case, we obtain an exponential consensus estimate without any restrictions on the time delays sizes. This extends and improves the analysis in Paolucci [40] where a consensus estimate has been obtained, subject to a smallness assumption on the time delay size. Moreover, here, as for the pointwise time delay case, we do not require any monotonicity properties on the influence function ψ that is assumed only continuous and bounded.

Even in the distributed case, since the constants in the consensus estimate are independent of the number of the agents, one can extend the consensus theorem to the related PDE model.

The rest of the paper is organized as follows. In Section 2, we give some preliminary results based on continuity arguments and Gronwall's inequality. In Section 3, we prove our consensus result for the particle model with pointwise time variable time delay, while in Section 4, we formulate its extension to the related continuity type equation. Section 5 is devoted to the analysis of the HK model with distributed time delay (5.1). Finally, in Section 6, we give some numerical tests illustrating our theoretical results.

2 | PRELIMINARIES

Consider a finite set of $N \in \mathbb{N}$ particles, with $N \geq 2$. Let $x_i(t) \in \mathbb{R}^d$ be the opinion of the i-th particle at time t. We shall denote with $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the usual norm and scalar product on $\mathbb{R}^d$, respectively. The interactions between the elements of the system are described by the following Hegselmann–Krause type model with variable time delays:

$$\frac{d}{dt}x_i(t) = \sum_{j:j\neq i} a_{ij}(t)(x_j(t-\tau(t)) - x_i(t)), \quad t > 0, \forall i = 1, \dots, N, \tag{2.1}$$

with weights a_{ij} of the form

$$a_{ij}(t) := \frac{1}{N-1}\psi(x_i(t), x_j(t-\tau(t))), \quad \forall t > 0,\ \forall i, j = 1, \dots, N, \tag{2.2}$$

where $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a positive function, and the time delay $\tau : [0, \infty) \to [0, \infty)$ is a continuous function satisfying

$$0 \leq \tau(t) \leq \bar{\tau}, \quad \forall t \geq 0, \tag{2.3}$$

for some positive constant $\bar{\tau}$. The initial conditions

$$x_i(s) = x_i^0(s), \quad \forall s \in [-\bar{\tau}, 0], \forall i = 1, \dots, N, \tag{2.4}$$

are assumed to be continuous functions.

The influence function ψ is assumed to be continuous. Moreover, we assume that it is bounded, and we denote

$$K := \|\psi\|_\infty.$$

For existence results for the above model, we refer to the literature [45, 46]. Here, we concentrate on the asymptotic behavior of solutions.

Let $\{x_i\}_{i=1,\dots,N}$ be solution to (2.1) under the initial conditions (2.4).

Definition 2.1. For each $t \geq -\bar{\tau}$, we define the diameter $d(\cdot)$ of the solution as

$$d(t) := \max_{i,j=1,\dots,N} |x_i(t) - x_j(t)|. \tag{2.5}$$

Definition 2.2. We say that a solution $\{x_i\}_{i=1,\dots,N}$ to system (2.1) converges to *consensus* if

$$\lim_{t\to+\infty} d(t) = 0.$$

Before to prove our convergence to consensus result, we give some preliminary lemmas.

The following results generalize and extend the ones developed in Rodriguez Cartabia [43] in the case of a Cucker–Smale model with constant time delay. In particular, to deal with time-dependent time delays, in the next lemma, we combine arguments from Rodriguez Cartabia [43] with a continuity argument used in Choi et al. [39] for a HK model with time-dependent time delay.

Lemma 2.1. *For each $v \in \mathbb{R}^d$ and $T \geq 0$, we have that*

$$\min_{j=1,\dots,N}\ \min_{s\in[T-\bar{\tau},T]} \langle x_j(s), v\rangle \leq \langle x_i(t), v\rangle \leq \max_{j=1,\dots,N}\ \max_{s\in[T-\bar{\tau},T]} \langle x_j(s), v\rangle, \tag{2.6}$$

for all $t \geq T - \bar{\tau}$ and $i = 1, \dots, N$.

Proof. Let $T \geq 0$ be fixed. First of all, we note that the inequalities in (2.6) are satisfied for every $t \in [T-\bar{\tau}, T]$.

Now, given a vector $v \in \mathbb{R}^d$, we set

$$M_T = \max_{j=1,\dots,N}\ \max_{s\in[T-\bar{\tau},T]} \langle x_j(s), v\rangle.$$

For all $\epsilon > 0$, let us define

$$K^\epsilon := \left\{ t > T : \max_{i=1,\dots,N} \langle x_i(s), v\rangle < M_T + \epsilon, \forall s \in [T, t) \right\}.$$

By continuity, we have that $K^{\epsilon} \neq \emptyset$. Thus, denoted with

$$S^{\epsilon} := \sup K^{\epsilon},$$

it holds that $S^{\epsilon} > T$.

We claim that $S^{\epsilon} = +\infty$. Indeed, suppose by contradiction that $S^{\epsilon} < +\infty$. Note that by definition of S^{ϵ}, it turns out that

$$\max_{i=1,\dots,N} \langle x_i(t), v\rangle < M_T + \epsilon, \quad \forall t \in (T, S^{\epsilon}), \tag{2.7}$$

and

$$\lim_{t \to S^{\epsilon -}} \max_{i=1,\dots,N} \langle x_i(t), v\rangle = M_T + \epsilon. \tag{2.8}$$

For all $i = 1, \dots, N$ and $t \in (T, S^{\epsilon})$, we compute

$$\frac{d}{dt}\langle x_i(t), v\rangle = \frac{1}{N-1}\sum_{j:j\neq i} \psi(x_i(t), x_j(t-\tau(t)))\langle x_j(t-\tau(t)) - x_i(t), v\rangle.$$

Notice that being $t \in (T, S^{\epsilon})$, then $t - \tau(t) \in (T - \bar{\tau}, S^{\epsilon})$ and

$$\langle x_j(t-\tau(t)), v\rangle < M_T + \epsilon, \quad \forall j = 1, \dots, N. \tag{2.9}$$

Moreover, (2.7) implies that

$$\langle x_i(t), v\rangle < M_T + \epsilon,$$

so that

$$M_T + \epsilon - \langle x_i(t), v\rangle \geq 0.$$

Combining this last fact with (2.9), we can write

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t), v\rangle &\leq \frac{1}{N-1}\sum_{j:j\neq i} \psi(x_i(t), x_j(t-\tau(t)))(M_T + \epsilon - \langle x_i(t), v\rangle) \\
&\leq K(M_T + \epsilon - \langle x_i(t), v\rangle), \quad \forall t \in (T, S^{\epsilon}).
\end{aligned}$$

Then, from Gronwall's inequality, we get

$$\begin{aligned}
\langle x_i(t), v\rangle &\leq e^{-K(t-T)}\langle x_i(T), v\rangle + K(M_T + \epsilon)\int_T^t e^{-K(t-s)}ds \\
&= e^{-K(t-T)}\langle x_i(T), v\rangle + (M_T + \epsilon)e^{-Kt}\left(e^{Kt} - e^{KT}\right) \\
&= e^{-K(t-T)}\langle x_i(T), v\rangle + (M_T + \epsilon)\left(1 - e^{-K(t-T)}\right) \\
&\leq e^{-K(t-T)}M_T + M_T + \epsilon - M_T e^{-K(t-T)} - \epsilon e^{-K(t-T)} \\
&= M_T + \epsilon - \epsilon e^{-K(t-T)} \\
&\leq M_T + \epsilon - \epsilon e^{-K(S^{\epsilon} - T)},
\end{aligned}$$

for all $t \in (T, S^{\epsilon})$. We have to proved that $\forall i = 1, \dots, N$,

$$\langle x_i(t), v\rangle \leq M_T + \epsilon - \epsilon e^{-K(S^{\epsilon} - T)}, \quad \forall t \in (T, S^{\epsilon}).$$

Thus, we get

$$\max_{i=1,\dots,N} \langle x_i(t), v\rangle \leq M_T + \epsilon - \epsilon e^{-K(S^{\epsilon} - T)}, \quad \forall t \in (T, S^{\epsilon}). \tag{2.10}$$

Letting $t \to S^{\epsilon -}$ in (2.10), from (2.8), we have that

$$M_T + \epsilon \leq M_T + \epsilon - \epsilon e^{-K(S^{\epsilon} - T)} < M_T + \epsilon,$$

which is a contradiction. Thus, $S^\epsilon = +\infty$, which means that

$$\max_{i=1,\dots,N} \langle x_i(t), v\rangle < M_T + \epsilon, \quad \forall t > T.$$

From the arbitrariness of ϵ, we can conclude that

$$\max_{i=1,\dots,N} \langle x_i(t), v\rangle \le M_T, \quad \forall t > T,$$

from which

$$\langle x_i(t), v\rangle \le M_T, \quad \forall t > T, \forall i = 1, \dots, N,$$

which proves the second inequality in (2.6). Now, to prove the other inequality, let $v \in \mathbb{R}^d$ and define

$$m_T = \min_{j=1,\dots,N} \min_{s\in[T-\bar{\tau},T]} \langle x_j(s), v\rangle.$$

Then, for all $i = 1, \dots, N$ and $t > T$, by applying the second inequality in (2.6) to the vector $-v \in \mathbb{R}^d$,, we get

$$\begin{aligned} -\langle x_j(s), v\rangle = \langle x_i(t), -v\rangle &\le \max_{j=1,\dots,N} \max_{s\in[T-\bar{\tau},T]} \langle x_j(s), -v\rangle \\ &= -\min_{j=1,\dots,N} \min_{s\in[T-\bar{\tau},T]} \langle x_j(s), v\rangle = -m_T, \end{aligned}$$

from which

$$\langle x_j(s), v\rangle \ge m_T.$$

Thus, also the first inequality in (2.6) is fulfilled. □

We now introduce some notation.

Definition 2.3. We define

$$D_0 = \max_{i,j=1,\dots,N} \max_{s,t\in[-\bar{\tau},0]} |x_i(s) - x_j(t)|,$$

and, in general, we define the sequence

$$D_n := \max_{i,j=1,\dots,N} \max_{s,t\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} |x_i(s) - x_j(t)|, \ \forall n \in \mathbb{N}. \tag{2.11}$$

Let us denote with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Lemma 2.2. *For each $n \in \mathbb{N}_0$ and $i, j = 1, \dots, N$, we get*

$$|x_i(s) - x_j(t)| \le D_n, \quad \forall s, t \ge n\bar{\tau} - \bar{\tau}. \tag{2.12}$$

Proof. Fix $n \in \mathbb{N}_0$ and $i, j = 1, \dots, N$. Given $s, t \ge n\bar{\tau} - \bar{\tau}$, if $|x_i(s) - x_j(t)| = 0$ then of course $D_n \ge 0 = |x_i(s) - x_j(t)|$. Thus, we can assume $|x_i(s) - x_j(t)| > 0$, and we set

$$v = \frac{x_i(s) - x_j(t)}{|x_i(s) - x_j(t)|}.$$

It turns out that v is a unit vector, and by using (2.6) with $T = n\bar{\tau}$ and the Cauchy–Schwarz inequality, we can write

$$\begin{aligned} |x_i(s) - x_j(t)| &= \langle x_i(s) - x_j(t), v\rangle = \langle x_i(s), v\rangle - \langle x_j(t), v\rangle \\ &\le \max_{l=1,\dots,N} \max_{r\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} \langle x_l(r), v\rangle - \min_{l=1,\dots,N} \min_{r\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} \langle x_l(r), v\rangle \\ &\le \max_{l,k=1,\dots,N} \max_{r,\sigma\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} \langle x_l(r) - x_k(\sigma), v\rangle \\ &\le \max_{l,k=1,\dots,N} \max_{r,\sigma\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} |x_l(r) - x_k(\sigma)||v| = D_n, \end{aligned}$$

which proves (2.12).

Remark 2.3. Let us note that from (2.12), in particular, it follows that

$$|x_i(s) - x_j(t)| \leq D_0, \ \forall s, t \geq -\bar{\tau}. \tag{2.13}$$

Moreover, for the sequence $\{D_n\}_n$ defined in (2.11), it holds

$$D_{n+1} \leq D_n, \ \forall n \in \mathbb{N}_0. \tag{2.14}$$

With an analogous argument, one can find a bound on $|x_i(t)|$, uniform with respect to t and $i = 1, \ldots, N$. Indeed, we have the following lemma.

Lemma 2.4. *For every $i = 1, \ldots, N$,, we have that*

$$|x_i(t)| \leq M^0, \quad \forall t \geq -\bar{\tau}, \tag{2.15}$$

where

$$M^0 := \max_{i=1,\ldots,N} \max_{s \in [-\bar{\tau}, 0]} |x_i(s)|.$$

Proof. Given $i = 1, \ldots, N$ and $t \geq -\bar{\tau}$, if $|x_i(t)| = 0$ then trivially $M^0 \geq 0 = |x_i(t)|$. On the contrary, if $|x_i(t)| > 0$, we define

$$v = \frac{x_i(t)}{|x_i(t)|},$$

which is a unit vector for which we can write

$$|x_i(t)| = \langle x_i(t), v \rangle.$$

Then, by applying (2.6) for $T = 0$ and by using the Cauchy–Schwarz inequality, we get

$$\begin{aligned} |x_i(t)| &\leq \max_{j=1,\ldots,N} \max_{s \in [-\bar{\tau}, 0]} \langle x_j(s), v \rangle \leq \max_{j=1,\ldots,N} \max_{s \in [-\bar{\tau}, 0]} |x_j(s)||v| \\ &= \max_{j=1,\ldots,N} \max_{s \in [-\bar{\tau}, 0]} |x_j(s)| = M^0, \end{aligned}$$

which proves (2.15). □

Remark 2.5. From the estimate (2.15), since the influence function ψ is continuous, we deduce that

$$\psi(x_i(t), x_j(t - \tau(t))) \geq \psi_0 := \min_{|y|, |z| \leq M^0} \psi(y, z) > 0, \tag{2.16}$$

for all $t \geq 0$, for all $i, j = 1, \ldots, N$.

The following lemma extends and improves an analogous result in Rodriguez Cartabia [43] for Cucker–Smale model with constant time delay. The presence of a time variable time delay requires a more careful analysis since now, for any time t, we just know that $t - \tau(t) \leq t$.

Lemma 2.6. *For all $i, j = 1, \ldots, N$, unit vector $v \in \mathbb{R}^d$ and $n \in \mathbb{N}_0$, we have that*

$$\langle x_i(t) - x_j(t), v \rangle \leq e^{-K(t-t_0)} \langle x_i(t_0) - x_j(t_0), v \rangle + \left(1 - e^{-K(t-t_0)}\right) D_n, \tag{2.17}$$

for all $t \geq t_0 \geq n\bar{\tau}$, where D_n is as in (2.11). Moreover, for all $n \in \mathbb{N}_0$, we get

$$D_{n+1} \leq e^{-K\bar{\tau}} d(n\bar{\tau}) + (1 - e^{-K\bar{\tau}}) D_n, \tag{2.18}$$

where $d(\cdot)$ is the diameter defined in (2.5).

Proof. Fix $n \in \mathbb{N}_0$ and $v \in \mathbb{R}^d$ such that $|v| = 1$. We set

$$M_n = \max_{i=1,\dots,N} \max_{t\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} \langle x_i(t), v\rangle,$$

$$m_n = \min_{i=1,\dots,N} \min_{t\in[n\bar{\tau}-\bar{\tau},n\bar{\tau}]} \langle x_i(t), v\rangle.$$

Then, it is easy to see that $M_n - m_n \le D_n$. Now, for all $i = 1, \dots, N$ and $t \ge t_0 \ge n\bar{\tau}$, we have that

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t), v\rangle &= \sum_{j:j\neq i} a_{ij}(t)\langle x_j(t-\tau(t)) - x_i(t), v\rangle \\
&= \frac{1}{N-1}\sum_{j:j\neq i} \psi(x_i(t), x_j(t-\tau(t)))(\langle x_j(t-\tau(t)), v\rangle - \langle x_i(t), v\rangle) \\
&\le \frac{1}{N-1}\sum_{j:j\neq i} \psi(x_i(t), x_j(t-\tau(t)))(M_n - \langle x_i(t), v\rangle).
\end{aligned}$$

Note that being $t \ge n\bar{\tau}$, $\langle x_i(t), v\rangle \le M_n$ from (2.6). Therefore, we have that $M_n - \langle x_i(t), v\rangle \ge 0$, and we can write

$$\frac{d}{dt}\langle x_i(t), v\rangle \le \frac{1}{N-1}K\sum_{j:j\neq i}(M_n - \langle x_i(t), v\rangle) = K(M_n - \langle x_i(t), v\rangle).$$

Thus, from the Gronwall's inequality, it comes that

$$\begin{aligned}
\langle x_i(t), v\rangle &\le e^{-K(t-t_0)}\langle x_i(t_0), v\rangle + \int_{t_0}^{t} KM_n e^{-K(t-t_0)+K(s-t_0)}ds \\
&= e^{-K(t-t_0)}\langle x_i(t_0), v\rangle + e^{-K(t-t_0)}M_n(e^{K(t-t_0)} - 1),
\end{aligned}$$

that is,

$$\langle x_i(t), v\rangle \le e^{-K(t-t_0)}\langle x_i(t_0), v\rangle + \left(1 - e^{-K(t-t_0)}\right) M_n. \tag{2.19}$$

On the other hand, for all $i = 1, \dots, N$ and $t \ge t_0 \ge n\bar{\tau}$, it holds that

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t), v\rangle &= \frac{1}{N-1}\sum_{i:j\neq i} \psi(x_i(t), x_j(t-\tau(t)))\left(\langle x_j(t-\tau(t)), v\rangle - \langle x_i(t), v\rangle\right) \\
&\ge \frac{1}{N-1}\sum_{j:j\neq i} \psi(x_i(t), x_j(t-\tau(t)))\left(m_n - \langle x_i(t), v\rangle\right).
\end{aligned}$$

Note that from (2.6), $\langle x_i(t), v\rangle \ge m_n$ since $t \ge n\bar{\tau}$. Thus, $m_n - \langle x_i(t), v\rangle \le 0$, and by recalling that ψ is bounded, we get

$$\frac{d}{dt}\langle x_i(t), v\rangle \ge K(m_n - \langle x_i(t), v\rangle).$$

Hence, by using Gronwall's inequality, it turns out that

$$\langle x_i(t), v\rangle \ge e^{-K(t-t_0)}\langle x_i(t_0), v\rangle + \left(1 - e^{-K(t-t_0)}\right) m_n. \tag{2.20}$$

Therefore, for all $i, j = 1, \dots, N$ and $t \ge t_0 \ge n\bar{\tau}$, by using (2.19) and (2.20) and by recalling that $M_n - m_n \le D_n$, we finally get

$$\begin{aligned}
\langle x_i(t) - x_j(t), v\rangle &= \langle x_i(t), v\rangle - \langle x_j(t), v\rangle \\
&\le e^{-K(t-t_0)}\langle x_i(t_0), v\rangle + \left(1 - e^{-K(t-t_0)}\right) M_n \\
&\quad - e^{-K(t-t_0)}\langle x_j(t_0), v\rangle - \left(1 - e^{-K(t-t_0)}\right) m_n \\
&= e^{-K(t-t_0)}\langle x_i(t_0) - x_j(t_0), v\rangle + \left(1 - e^{-K(t-t_0)}\right)(M_n - m_n) \\
&\le e^{-K(t-t_0)}\langle x_i(t_0) - x_j(t_0), v\rangle + \left(1 - e^{-K(t-t_0)}\right) D_n,
\end{aligned}$$

that is, (2.17) holds true.

Now, we prove (2.18). Given $n \in \mathbb{N}_0$, let $i,j = 1, \dots, N$ and $s,t \in [n\bar{\tau}, n\bar{\tau}+\bar{\tau}]$ be such that $D_{n+1} = |x_i(s) - x_j(t)|$. Note that if $|x_i(s) - x_j(t)| = 0$, then obviously

$$0 = D_{n+1} \le e^{-K\bar{\tau}} d(n\bar{\tau}) + (1 - e^{-K\bar{\tau}}) D_n.$$

So we can assume $|x_i(s) - x_j(t)| > 0$. Let us define the unit vector

$$v = \frac{x_i(s) - x_j(t)}{|x_i(s) - x_j(t)|}.$$

Hence, we can write

$$D_{n+1} = \langle x_i(s) - x_j(t), v\rangle = \langle x_i(s), v\rangle - \langle x_j(t), v\rangle.$$

Now, by using (2.19) with $t_0 = n\bar{\tau}$, we have that

$$\begin{aligned}\langle x_i(s), v\rangle &\le e^{-K(s-n\bar{\tau})}\langle x_i(n\bar{\tau}), v\rangle + (1 - e^{-K(s-n\bar{\tau})})M_n \\ &= e^{-K(s-n\bar{\tau})}(\langle x_i(n\bar{\tau}), v\rangle - M_n) + M_n.\end{aligned}$$

Thus, since $s \le n\bar{\tau} + \bar{\tau}$ and $\langle x_i(n\bar{\tau}), v\rangle - M_n \le 0$ from (2.6), we get

$$\begin{aligned}\langle x_i(s), v\rangle &\le e^{-K\bar{\tau}}(\langle x_i(n\bar{\tau}), v\rangle - M_n) + M_n \\ &\le e^{-K\bar{\tau}}\langle x_i(n\bar{\tau}), v\rangle + (1 - e^{-K\bar{\tau}})M_n.\end{aligned} \tag{2.21}$$

Similarly, by taking into account (2.6) and (2.20), we have that

$$\langle x_j(t), v\rangle \ge e^{-K\bar{\tau}}\langle x_j(n\bar{\tau}), v\rangle + (1 - e^{-K\bar{\tau}})m_n. \tag{2.22}$$

Therefore, combining (2.21) and (2.22), we can write

$$\begin{aligned}D_{n+1} &\le e^{-K\bar{\tau}}\langle x_i(n\bar{\tau}), v\rangle + (1 - e^{-K\bar{\tau}})M_n - e^{-K\bar{\tau}}\langle x_j(n\bar{\tau}), v\rangle - (1 - e^{-K\bar{\tau}})m_n \\ &= e^{-K\bar{\tau}}\langle x_i(n\bar{\tau}) - x_j(n\bar{\tau}), v\rangle + (1 - e^{-K\bar{\tau}})(M_n - m_n).\end{aligned}$$

Then, by recalling that $M_n - m_n \le D_n$ and by using the Cauchy–Schwarz inequality, we can conclude that

$$\begin{aligned}D_{n+1} &\le e^{-K\bar{\tau}}|x_i(n\bar{\tau}) - x_j(n\bar{\tau})||v| + (1 - e^{-K\bar{\tau}})D_n \\ &\le e^{-K\bar{\tau}} d(n\bar{\tau}) + (1 - e^{-K\bar{\tau}})D_n,\end{aligned}$$

since $|v| = 1$ and from the definition of $d(\cdot)$. □

Lemma 2.7. *There exists a constant $C \in (0,1)$, independent of $N \in \mathbb{N}$, such that*

$$d(n\bar{\tau}) \le C D_{n-2}, \tag{2.23}$$

for all $n \ge 2$, where $d(\cdot)$ is the diameter defined in (2.5) and the sequence $\{D_n\}_n$ is as in (2.11).

Proof. If $d(n\bar{\tau}) = 0$, then of course inequality (2.23) holds for any constant $C \in (0,1)$. So suppose $d(n\bar{\tau}) > 0$. Let $i,j = 1, \dots, N$ be such that $d(n\bar{\tau}) = |x_i(n\bar{\tau}) - x_j(n\bar{\tau})|$. We set

$$v = \frac{x_i(n\bar{\tau}) - x_j(n\bar{\tau})}{|x_i(n\bar{\tau}) - x_j(n\bar{\tau})|}.$$

Then, v is a unit vector for which we can write

$$d(n\bar{\tau}) = \langle x_i(n\bar{\tau}) - x_j(n\bar{\tau}), v\rangle.$$

Let us define

$$M_{n-1} = \max_{l=1,\dots,N} \max_{s\in[n\bar{\tau}-2\bar{\tau},n\bar{\tau}-\bar{\tau}]} \langle x_l(s), v\rangle,$$

$$m_{n-1} = \min_{l=1,\dots,N} \min_{s\in[n\bar{\tau}-2\bar{\tau},n\bar{\tau}-\bar{\tau}]} \langle x_l(s), v\rangle.$$

So $M_{n-1} - m_{n-1} \le D_{n-1}$. Now, we distinguish two different situations.

Case I. Assume that there exists $t_0 \in [n\bar{\tau} - 2\bar{\tau}, n\bar{\tau}]$ such that

$$\langle x_i(t_0) - x_j(t_0), v\rangle < 0.$$

Then, from (2.17) with $n\bar{\tau} \ge t_0 \ge n\bar{\tau} - 2\bar{\tau}$, we have that

$$d(n\bar{\tau}) \le e^{-K(n\bar{\tau}-t_0)}\langle x_i(t_0) - x_j(t_0), v\rangle + (1 - e^{-K(n\bar{\tau}-t_0)})D_{n-2}$$

$$\le (1 - e^{-K(n\bar{\tau}-t_0)})D_{n-2} \le (1 - e^{-2K\bar{\tau}})D_{n-2}.$$

Case II. Suppose that

$$\langle x_i(t) - x_j(t), v\rangle \ge 0, \quad \forall t \in [n\bar{\tau} - 2\bar{\tau}, n\bar{\tau}]. \tag{2.24}$$

Then, for every $t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]$, we have that

$$\begin{aligned}
\frac{d}{dt}\langle x_i(t) - x_j(t), v\rangle &= \frac{1}{N-1}\sum_{l:l\neq i}\psi(x_i(t), x_l(t-\tau(t)))\langle x_l(t-\tau(t)) - x_i(t), v\rangle \\
&\quad - \frac{1}{N-1}\sum_{l:l\neq j}\psi(x_i(t), x_l(t-\tau(t)))\langle x_l(t-\tau(t)) - x_j(t), v\rangle \\
&= \frac{1}{N-1}\sum_{l:l\neq i}\psi(x_i(t), x_l(t-\tau(t)))(\langle x_l(t-\tau(t)), v\rangle - M_{n-1} + M_{n-1} - \langle x_i(t), v\rangle) \\
&\quad + \frac{1}{N-1}\sum_{l:l\neq j}\psi(x_i(t), x_l(t-\tau(t)))(\langle x_j(t), v\rangle - m_{n-1} + m_{n-1} - \langle x_l(t-\tau(t)), v\rangle) := S_1 + S_2.
\end{aligned}$$

Now, being $t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]$, it holds that $t - \tau(t) \in [n\bar{\tau} - 2\bar{\tau}, n\bar{\tau}]$. Therefore, both $t, t - \tau(t) \ge n\bar{\tau} - 2\bar{\tau}$ and from (2.6), we have that

$$m_{n-1} \le \langle x_k(t), v\rangle \le M_{n-1},\ \ m_{n-1} \le \langle x_k(t-\tau(t)), v\rangle \le M_{n-1}, \quad \forall k = 1, \dots, N. \tag{2.25}$$

Therefore, using (2.15), we get

$$\begin{aligned}
S_1 &= \frac{1}{N-1}\sum_{l:l\neq i}\psi(x_i(t), x_l(t-\tau(t)))(\langle x_l(t-\tau(t)), v\rangle - M_{n-1}) \\
&\quad + \frac{1}{N-1}\sum_{l:l\neq i}\psi(x_i(t), x_l(t-\tau(t)))(M_{n-1} - \langle x_i(t), v\rangle) \\
&\le \frac{1}{N-1}\psi_0\sum_{l:l\neq i}(\langle x_l(t-\tau(t)), v\rangle - M_{n-1}) + K(M_{n-1} - \langle x_i(t), v\rangle),
\end{aligned}$$

and

$$\begin{aligned}
S_2 &= \frac{1}{N-1}\sum_{l:l\neq j}\psi(x_i(t), x_l(t-\tau(t)))(\langle x_j(t), v\rangle - m_{n-1}) \\
&\quad + \frac{1}{N-1}\sum_{l:l\neq j}\psi(x_i(t), x_l(t-\tau(t)))(m_{n-1} - \langle x_l(t-\tau(t)), v\rangle) \\
&\le K(\langle x_j(t), v\rangle - m_{n-1}) + \frac{1}{N-1}\psi_0\sum_{l:l\neq j}(m_{n-1} - \langle x_l(t-\tau(t)), v\rangle).
\end{aligned}$$

Combining this last fact with (2.25), it comes that

$$\begin{aligned}\frac{d}{dt}\langle x_i(t)-x_j(t),v\rangle &\le K(M_{n-1}-m_{n-1}-\langle x_i(t)-x_j(t),v\rangle)\\&\quad+\frac{1}{N-1}\psi_0\sum_{l:l\neq i,j}(\langle x_l(t-\tau(t)),v\rangle-M_{n-1}+m_{n-1}-\langle x_l(t-\tau(t)),v\rangle)\\&\quad+\frac{1}{N-1}\psi_0(\langle x_j(t-\tau(t)),v\rangle-M_{n-1}+m_{n-1}-\langle x_i(t-\tau(t)),v\rangle)\\&=K(M_{n-1}-m_{n-1})-K\langle x_i(t)-x_j(t),v\rangle+\frac{N-2}{N-1}\psi_0(-M_{n-1}+m_{n-1})\\&\quad+\frac{1}{N-1}\psi_0(\langle x_j(t-\tau(t)),v\rangle-M_{n-1}+m_{n-1}-\langle x_i(t-\tau(t)),v\rangle).\end{aligned}$$

Therefore, since from (2.24) $\langle x_i(t-\tau(t))-x_j(t-\tau(t)),v\rangle\ge 0$, we get

$$\begin{aligned}\frac{d}{dt}\langle x_i(t)-x_j(t),v\rangle &\le K(M_{n-1}-m_{n-1})-K\langle x_i(t)-x_j(t),v\rangle\\&\quad+\frac{N-2}{N-1}\psi_0(-M_{n-1}+m_{n-1})+\frac{1}{N-1}\psi_0(-M_{n-1}+m_{n-1})\\&\quad-\frac{1}{N-1}\psi_0\langle x_i(t-\tau(t))-x_j(t-\tau(t)),v\rangle\\&\le K(M_{n-1}-m_{n-1})-K\langle x_i(t)-x_j(t),v\rangle+\psi_0(-M_{n-1}+m_{n-1})\\&=(K-\psi_0)\,(M_{n-1}-m_{n-1})-K\langle x_i(t)-x_j(t),v\rangle.\end{aligned}$$

Hence, from Gronwall's inequality, it comes that

$$\begin{aligned}\langle x_i(t)-x_j(t),v\rangle &\le e^{-K(t-n\bar\tau+\bar\tau)}\langle x_i(n\bar\tau-\bar\tau)-x_j(n\bar\tau-\bar\tau),v\rangle\\&\quad+(K-\psi_0)\,(M_{n-1}-m_{n-1})\int_{n\bar\tau-\bar\tau}^{t}e^{-K(t-s)}ds,\end{aligned}$$

for all $t\in[n\bar\tau-\bar\tau,n\bar\tau]$. In particular, for $t=n\bar\tau$ it comes that

$$\begin{aligned}d(n\bar\tau)&\le e^{-K\bar\tau}\langle x_i(n\bar\tau-\bar\tau)-x_j(n\bar\tau-\bar\tau),v\rangle+\frac{K-\psi_0}{K}(M_{n-1}-m_{n-1})(1-e^{-K\bar\tau})\\&\le e^{-K\bar\tau}|x_i(n\bar\tau-\bar\tau)-x_j(n\bar\tau-\bar\tau)||v|+\frac{K-\psi_0}{K}(M_{n-1}-m_{n-1})(1-e^{-K\bar\tau})\\&\le e^{-K\bar\tau}d(n\bar\tau-\bar\tau)+\frac{K-\psi_0}{K}(M_{n-1}-m_{n-1})(1-e^{-K\bar\tau}),\end{aligned}$$

since $|v|=1$ and from the definition (2.5) of $d(\cdot)$. Then, by recalling that $M_{n-1}-m_{n-1}\le D_{n-1}$, we get

$$d(n\bar\tau)\le e^{-K\bar\tau}d(n\bar\tau-\bar\tau)+\frac{K-\psi_0}{K}D_{n-1}(1-e^{-K\bar\tau}).$$

Finally, by using (2.12) and (2.14), we have that

$$\begin{aligned}d(n\bar\tau)&\le e^{-K\bar\tau}D_n+\frac{K-\psi_0}{K}D_{n-1}(1-e^{-K\bar\tau})\\&\le e^{-K\bar\tau}D_{n-2}+\frac{K-\psi_0}{K}D_{n-2}(1-e^{-K\bar\tau})\\&=\left[1-\frac{\psi_0}{K}(1-e^{-K\bar\tau})\right]D_{n-2}.\end{aligned}\tag{2.26}$$

Now, we set

$$C=\max\left\{1-e^{-2K\bar\tau},1-\frac{\psi_0}{K}(1-e^{-K\bar\tau})\right\}\in(0,1).\tag{2.27}$$

Then, taking into account (2.26), we can conclude that C is the constant for which inequality (2.23) holds. □

3 | EXPONENTIAL CONSENSUS ESTIMATE

We are ready to prove our convergence to consensus result.

Theorem 3.1. *Assume that* $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *is a positive, bounded, continuous function. Moreover, let* $x_i^0 : [-\bar{\tau}, 0] \to \mathbb{R}^d$ *be a continuous function, for any* $i = 1, \dots, N$. *Then, for every solution* $\{x_i\}_{i=1,\dots,N}$ *to (2.1) under the initial conditions (2.4), the diameter* $d(\cdot)$ *satisfies the exponential decay estimate*

$$d(t) \leq \left(\max_{i,j=1,\dots,N} \max_{r,s\in[-\bar{\tau},0]} |x_i(r) - x_j(s)| \right) e^{-\gamma(t-2\bar{\tau})}, \ \forall t \geq 0, \tag{3.1}$$

for a suitable positive constant γ *independent of* N.

Proof. Let $\{x_i\}_{i=1,\dots,N}$ be solution to (2.1) and (2.4). We claim that

$$D_{n+1} \leq \tilde{C} D_{n-2}, \quad \forall n \geq 2, \tag{3.2}$$

for some constant $\tilde{C} \in (0, 1)$. Indeed, given $n \geq 2$, from (2.14), (2.18), and (2.23), we have that

$$\begin{aligned} D_{n+1} &\leq e^{-K\bar{\tau}} d(n\bar{\tau}) + (1 - e^{-K\bar{\tau}}) D_n \\ &\leq e^{-K\bar{\tau}} C D_{n-2} + (1 - e^{-K\bar{\tau}}) D_n \\ &\leq e^{-K\bar{\tau}} C D_{n-2} + (1 - e^{-K\bar{\tau}}) D_{n-2} \\ &\leq (1 - e^{-K\bar{\tau}}(1 - C)) D_{n-2}, \end{aligned}$$

where the constant C is defined in (2.27). So setting

$$\tilde{C} = 1 - e^{-K\bar{\tau}}(1 - C),$$

we can conclude that $\tilde{C} \in (0, 1)$ is the constant for which (3.2) holds true.

This implies that

$$D_{3n} \leq \tilde{C}^n D_0, \ \forall n \geq 1. \tag{3.3}$$

Indeed, by induction, if $n = 1$, we know from (3.2) that

$$D_3 \leq \tilde{C} D_0.$$

So assume that (3.3) holds for $n \geq 1$ and we prove it for $n + 1$. By using again (3.2) and from the induction hypothesis, it comes that

$$D_{3(n+1)} \leq \tilde{C} D_{3n} \leq \tilde{C}\tilde{C}^n D_0 = \tilde{C}^{n+1} D_0,$$

that is, (3.3) is fulfilled.

Notice that (3.3) can be rewritten as

$$D_{3n} \leq e^{-3n\gamma\bar{\tau}} D_0, \quad \forall n \in \mathbb{N}_0, \tag{3.4}$$

with

$$\gamma = \frac{1}{3\bar{\tau}} \ln \left(\frac{1}{\tilde{C}} \right).$$

Now, fix $i, j = 1, \dots, N$ and $t \geq 0$. Then, $t \in [3n\bar{\tau} - \bar{\tau}, 3n\bar{\tau} + 2\bar{\tau}]$, for some $n \in \mathbb{N}_0$. Therefore, by using (2.12) and (3.4), it turns out that

$$|x_i(t) - x_j(t)| \leq D_{3n} \leq e^{-3n\gamma\bar{\tau}} D_0.$$

Thus, being $t \leq 3n\bar{\tau} + 2\bar{\tau}$, then $-3n\bar{\tau} \leq -t + 2\bar{\tau}$, and we get

$$|x_i(t) - x_j(t)| \leq e^{-\gamma(t-2\bar{\tau})} D_0.$$

Therefore,

$$d(t) \le e^{-\gamma(t-2\bar{\tau})} D_0, \;\; \forall t \ge 0,$$

and (3.1) is proved. □

Remark 3.2. Let us note that Theorem 3.1 holds, in particular, for weights a_{ij} of the form

$$a_{ij}(t) := \frac{1}{N-1} \tilde{\psi}(|x_i(t) - x_j(t-\tau(t))|), \;\; \forall t > 0, \forall i, j = 1, \dots, N,$$

where $\tilde{\psi} : \mathbb{R} \to \mathbb{R}$ is a positive, bounded, and continuous function. In this case, we can estimate from below $\tilde{\psi}$ in terms of the distance between the initial opinions, namely,

$$\tilde{\psi}(|x_i(t) - x_j(t-\tau(t))|) \ge \tilde{\psi}_0 := \min_{s \in [0, D_0]} \tilde{\psi}(s) > 0, \;\; \forall t \ge 0,$$

where D_0 is as in Definition 2.3. Then, the proof of Theorem 3.1 follows with the same arguments we have employed in the general case of weights of the type (2.2).

4 | THE CONTINUUM HK MODEL WITH POINTWISE TIME DELAY

In this section, we consider the continuum model obtained as the mean-field limit of the particle system when $N \to \infty$. Let $\mathcal{M}(\mathbb{R}^d)$ be the set of probability measures on the space $\mathbb{R}^d$. Then, the continuum model associated with the particle system (2.1) is given by

$$\partial_t \mu_t + \text{div } (F[\mu_{t-\tau(t)}]\mu_t) = 0, \;\; t > 0, \mu_s = g_s, \;\; x \in \mathbb{R}^d, \;\; s \in [-\bar{\tau}, 0], \tag{4.1}$$

where the velocity field F is defined as

$$F[\mu_{t-\tau(t)}](x) = \int_{\mathbb{R}^d} \psi(x, y)(y - x)\, d\mu_{t-\tau(t)}(y), \tag{4.2}$$

and $g_s \in C([-\bar{\tau}, 0]; \mathcal{M}(\mathbb{R}^d))$.

We assume that the potential $\psi(\cdot, \cdot)$ in (4.2) is Lipschitz continuous, namely there exists $L > 0$ such that for any $(x, y), (x', y') \in \mathbb{R}^{2d}$, it holds

$$|\psi(x, y) - \psi(x', y')| \le L(|y - y'| + |x - x'|).$$

Definition 4.1. Let $T > 0$. We say that $\mu_t \in C([0, T); \mathcal{M}(\mathbb{R}^d))$ is a measure-valued solution to (4.1) on the time interval $[0, T)$ if for all $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, T))$, we have

$$\int_0^T \int_{\mathbb{R}^d} \left(\partial_t \varphi + F[\mu_{t-\tau(t)}](x) \cdot \nabla_x \varphi\right) d\mu_t(x) dt + \int_{\mathbb{R}^d} \varphi(x, 0) dg_0(x) = 0. \tag{4.3}$$

Before stating the consensus result for solutions to model (4.1), we recall some basic tools on probability spaces and measures.

Definition 4.2. Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ be two probability measures on $\mathbb{R}^d$. We define the 1-Wasserstein distance between μ and ν as

$$d_1(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| d\pi(x, y),$$

where $\Pi(\mu, \nu)$ is the space of all couplings for μ and ν, namely all those probability measures on $\mathbb{R}^{2d}$ having as marginals μ and ν:

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x) d\pi(x, y) = \int_{\mathbb{R}^d} \varphi(x) d\mu(x), \;\; \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(y) d\pi(x, y) = \int_{\mathbb{R}^d} \varphi(y) d\nu(y),$$

for all $\varphi \in C_b(\mathbb{R}^d)$.

Let us introduce the space $\mathcal{P}_1$ of all probability measures with finite first-order moment. It is well-known that $(\mathcal{P}_1(\mathbb{R}^d), d_1(\cdot, \cdot))$ is a complete metric space.

Now, we define the position diameter for a compactly supported measure $g \in \mathcal{P}_1(\mathbb{R}^d)$ as follows:

$$d_X[g] := \text{diam}(supp\ g).$$

Since the consensus result for the particle model (2.1) holds without any upper bounds on the time delay $\tau(\cdot)$, one can improve the consensus theorem for the PDE model (4.1) obtained in Choi et al. [39] removing the smallness assumption on the time delay $\tau(t)$. We omit the proof since, once we have the result for the particle system (2.1), the consensus estimate for the continuum model is obtained with arguments analogous to the ones in Choi et al. [39] and Paolucci and Pignotti [47].

Theorem 4.1. *Let $\mu_t \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))$ be a measure-valued solution to (4.1) with compactly supported initial datum $g_s \in C([-\bar{\tau},0];\mathcal{P}_1(\mathbb{R}^d))$ and let F as in (4.2). Then, there exists a constant $C>0$ such that*

$$d_X(\mu_t) \le \left(\max_{s\in[-\bar{\tau},0]} d_X(g_s)\right) e^{-Ct},$$

for all $t \ge 0$.

5 | DISTRIBUTED TIME DELAY

Now, we extend the results obtained for the Hegselmann–Krause model with a pointwise time delay to a model with distributed time delay. In particular, we consider the system

$$\frac{d}{dt}x_i(t) = \frac{1}{h(t)}\sum_{j:j\neq i}\int_{t-\tau_2(t)}^{t-\tau_1(t)} \alpha(t-s)a_{ij}(t;s)(x_j(s)-x_i(t))ds, \quad t>0, \quad \forall i=1,\dots,N, \tag{5.1}$$

where the time delays $\tau_1 : [0,\infty) \to [0,\infty)$, $\tau_2 : [0,\infty) \to [0,\infty)$ are continuous functions satisfying

$$0 \le \tau_1(t) < \tau_2(t) \le \bar{\tau}, \ \forall t \ge 0, \tag{5.2}$$

for some positive constant $\bar{\tau}$.

The communication rates $a_{ij}(t;s)$ are of the form

$$a_{ij}(t;s) := \frac{1}{N-1}\psi(x_i(t),x_j(s)), \quad \forall t\ge 0, \quad \forall i,j=1,\dots,N, \tag{5.3}$$

where $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a positive function.

Moreover, $\alpha : [0,\bar{\tau}] \to (0,+\infty)$ is a continuous weight function and

$$h(t) := \int_{\tau_1(t)}^{\tau_2(t)} \alpha(s)ds, \quad \forall t \ge 0. \tag{5.4}$$

Note that since we assume $\tau_1(t) < \tau_2(t)$ and $\alpha(t)>0$, $\forall t \ge 0$, then the function $h(t)$ is always positive.

The initial conditions

$$x_i(s) = x_i^0(s), \quad \forall s \in [-\bar{\tau},0], \quad \forall i=1,\dots,N,$$

are assumed to be continuous functions. Moreover, the influence function ψ is assumed to be continuous and bounded, and let us denote $K := \|\psi\|_\infty$. Here, we study the asymptotic behavior of solutions to system (5.1). As in Section 2, one can prove the following crucial lemma.

Lemma 5.1. *Let $\{x_i\}_{i=1,\dots,N}$ be a solution to system (5.1) with continuous initial conditions. Then, for each $v \in \mathbb{R}^d$ and $T \ge 0$, we have that*

$$\min_{j=1,\dots,N}\min_{s\in[T-\bar{\tau},T]}\langle x_j(s),v\rangle \le \langle x_i(t),v\rangle \le \max_{j=1,\dots,N}\max_{s\in[T-\bar{\tau},T]}\langle x_j(s),v\rangle, \tag{5.5}$$

for all $t \ge T-\bar{\tau}$ and $i=1,\dots,N$.

Proof. First of all, we note that for each $v \in \mathbb{R}^d$ and $T \geq 0$, the inequalities in the statement are satisfied for every $t \in [T-\bar{\tau}, T]$.

Now, fix $T \geq 0$, a vector $v \in \mathbb{R}^d$ and a positive constant ϵ. Define the constant M_T and the set K^ϵ as in the proof of Lemma 2.1. Then, denoted as before $S^\epsilon := \sup K^\epsilon$, it holds that $S^\epsilon > T$.

We claim that $S^\epsilon = +\infty$. Indeed, suppose by contradiction that $S^\epsilon < +\infty$. Note that by definition of S^ϵ it turns out that

$$\max_{i=1,\dots,N} \langle x_i(t), v\rangle < M_T + \epsilon, \quad \forall t \in (T, S^\epsilon), \tag{5.6}$$

and

$$\lim_{t\to S^{\epsilon-}} \max_{i=1,\dots,N} \langle x_i(t), v\rangle = M_T + \epsilon. \tag{5.7}$$

For all $i = 1, \dots, N$ and $t \in (T, S^\epsilon)$, we compute

$$\begin{aligned}\frac{d}{dt}\langle x_i(t), v\rangle &= \frac{1}{h(t)} \sum_{j:j\neq i} \int_{t-\tau_2(t)}^{t-\tau_1(t)} \alpha(t-s) a_{ij}(t;s) \langle x_j(s) - x_i(t), v\rangle ds \\ &= \frac{1}{N-1}\frac{1}{h(t)} \sum_{j:j\neq i} \int_{t-\tau_2(t)}^{t-\tau_1(t)} \alpha(t-s)\psi(x_i(t), x_j(s))(\langle x_j(s), v\rangle - \langle x_i(t), v\rangle) ds.\end{aligned}$$

Notice that being $t \in (T, S^\epsilon)$, then $t-\tau_2(t), t-\tau_1(t) \in (T-\bar{\tau}, S^\epsilon)$ and

$$\langle x_j(s), v\rangle < M_T + \epsilon, \quad \forall s \in [t-\tau_2(t), t-\tau_1(t)], \forall j = 1, \dots, N. \tag{5.8}$$

Moreover, (5.6) implies that

$$\langle x_i(t), v\rangle < M_T + \epsilon,$$

so that

$$M_T + \epsilon - \langle x_i(t), v\rangle \geq 0.$$

Combining this last fact with (5.8) and by recalling of (5.4), we can write

$$\begin{aligned}\frac{d}{dt}\langle x_i(t), v\rangle &\leq \frac{1}{N-1}\frac{1}{h(t)} \sum_{j:j\neq i} \int_{t-\tau_2(t)}^{t-\tau_1(t)} \alpha(t-s)\psi(x_i(t), x_j(s))(M_T + \epsilon - \langle x_i(t), v\rangle) ds \\ &\leq \frac{K}{N-1}\frac{1}{h(t)} (M_T + \epsilon - \langle x_i(t), v\rangle) \sum_{j:j\neq i} \int_{t-\tau_2(t)}^{t-\tau_1(t)} \alpha(t-s) ds \\ &= K\frac{1}{h(t)}(M_T + \epsilon - \langle x_i(t), v\rangle) \int_{t-\tau_2(t)}^{t-\tau_1(t)} \alpha(t-s) ds \\ &= K(M_T + \epsilon - \langle x_i(t), v\rangle),\end{aligned}$$

for all $t \in (T, S^\epsilon)$. Then, Gronwall's Lemma allows us to conclude the proof of the second inequality arguing analogously to the proof of Lemma 2.1. Also, the proof of the first inequality is obtained similarly with respect to the pointwise time delay case. We omit the details. □

As before, one can define the quantities D_n, $n \in \mathbb{N}_0$, and prove the analogous, for solutions to the model with distributed time delay (5.1), of the lemmas in Section 2. Then, the following exponential convergence to consensus holds.

Theorem 5.2. *Assume that $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a positive, bounded, continuous function. Then, every solution $\{x_i\}_{i=1,\dots,N}$ to (5.1), with continuous initial conditions $x_i^0 : [-\bar{\tau}, 0] \to \mathbb{R}^d$, satisfies the exponential decay estimate*

$$d(t) \leq \left(\max_{i,j=1,\dots,N} \max_{r,s\in[-\bar{\tau},0]} |x_i(r) - x_j(s)| \right) e^{-\gamma(t-2\bar{\tau})}, \quad \forall t \geq 0,$$

for a suitable positive constant γ independent of N.

Remark 5.3. Let us note that Theorem 5.2 holds, in particular, for weights a_{ij} of the form

$$a_{ij}(t;s) := \frac{1}{N-1}\tilde{\psi}(|x_i(t) - x_j(s)|), \quad \forall t > 0, \forall i, j = 1, \dots, N,$$

where $\tilde{\psi} : \mathbb{R} \to \mathbb{R}$ is a positive, bounded, and continuous function. In this case, one can bound from below the influence function $\tilde{\psi}$ as explained in Remark 3.2.

The related PDE model is now

$$\begin{aligned} &\partial_t \mu_t + \operatorname{div}\left(\frac{1}{h(t)} \int_{t-\tau_2(t)}^{t-\tau_1(t)} \alpha(t-s) F[\mu_s] ds\, \mu_t\right) = 0, \quad t > 0, \\ &\mu_s = g_s, \quad x \in \mathbb{R}^d, \quad s \in [-\bar{\tau}, 0], \end{aligned} \tag{5.9}$$

where the velocity field F is given by

$$F[\mu_s](x) = \int_{\mathbb{R}^d} \psi(x, y)(y - x)\, d\mu_s(y), \tag{5.10}$$

and $g_s \in C([-\bar{\tau}, 0]; \mathcal{M}(\mathbb{R}^d))$.

As before, we assume that the potential $\psi(\cdot, \cdot)$ in (5.10) is also Lipschitz continuous with respect to the two arguments.

Definition 5.1. Let $T > 0$. We say that $\mu_t \in C([0, T); \mathcal{M}(\mathbb{R}^d))$ is a measure-valued solution to (5.9) on the time interval $[0, T)$ if for all $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, T))$, we have

$$\int_0^T \int_{\mathbb{R}^d} \left(\partial_t \varphi + \frac{1}{h(t)} \int_{t-\tau_2(t)}^{t-\tau_1(t)} \alpha(t-s) F[\mu_s](x) ds \cdot \nabla_x \varphi\right) d\mu_t(x) dt + \int_{\mathbb{R}^d} \varphi(x, 0) dg_0(x) = 0.$$

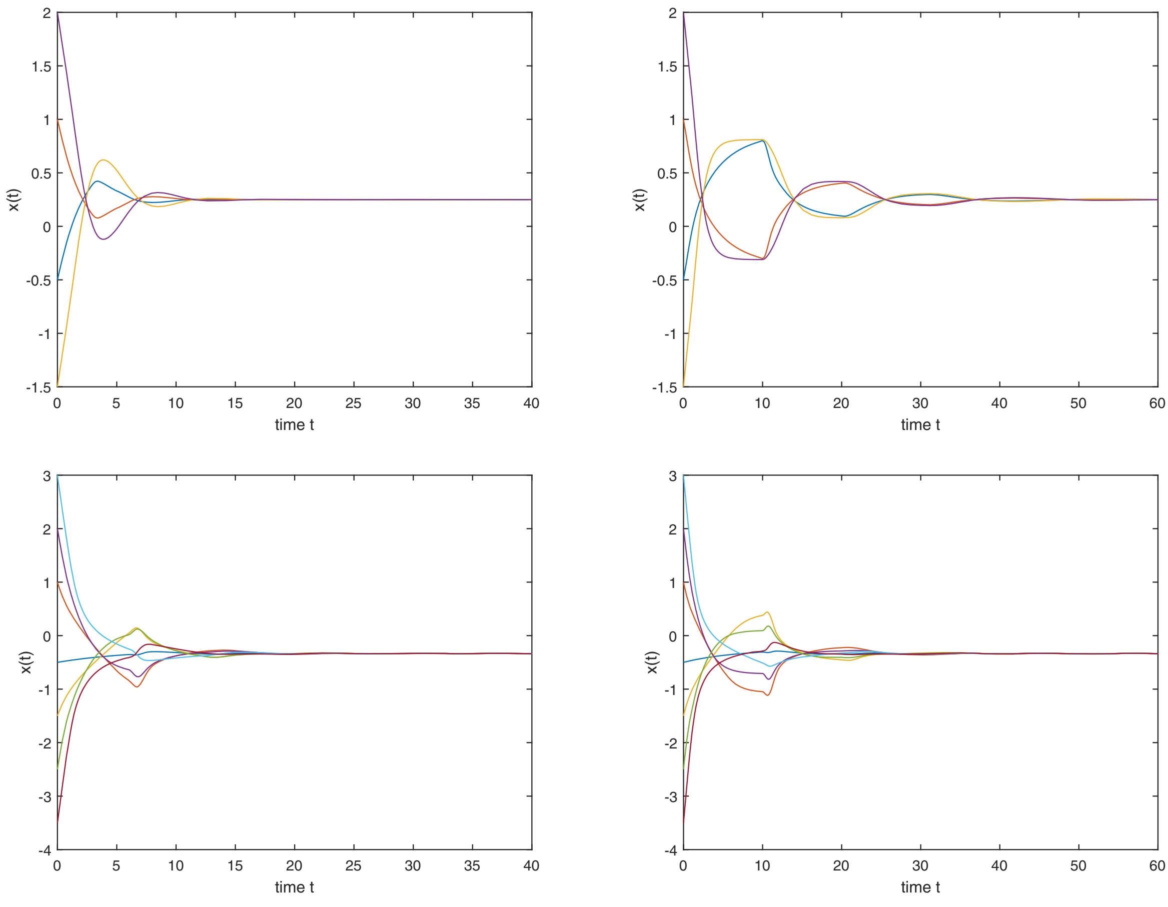

FIGURE 1 Communication rates (6.1): time evolution of solutions with different time delays and number N of agents; $\tau = 3, N = 4$ (top left), $\tau = 10, N = 4$ (top right), $\tau = 3, N = 7$ (bottom left), $\tau = 10, N = 7$ (bottom right). [Colour figure can be viewed at wileyonlinelibrary.com]

Since the consensus result for the particle model (5.1) holds without any upper bounds on the time delays $\tau_1(\cdot), \tau_2(\cdot)$, one can improve the consensus theorem for the PDE model (5.9) of Paolucci [40]. Indeed, in Paolucci [40], where the author concentrates in the case $\tau_1(t) \equiv 0$, the consensus estimate is obtained under a smallness condition on the time delay. The proof is analogous, then we omit it.

Theorem 5.4. *Let $\mu_t \in C([0,T]; \mathcal{P}_1(\mathbb{R}^d))$ be a measure-valued solution to (5.9) with compactly supported initial datum $g_s \in C([-\bar{\tau}, 0]; \mathcal{P}_1(\mathbb{R}^d))$ and let F as in (5.10). Then, there exists a constant $C > 0$ such that*

$$d_X(\mu_t) \leq \left(\max_{s\in[-\bar{\tau},0]} d_X(g_s) \right) e^{-Ct},$$

for all $t \geq 0$.

6 | NUMERICAL TESTS

In this section, we present some numerical tests for the particle system (2.1) with weights a_{ij} in (2.2) defined via functions

$$\psi(r, r') = \tilde{\psi}(|r - r'|),$$

always positive but nonmonotonic.

In particular, we consider an oscillatory function

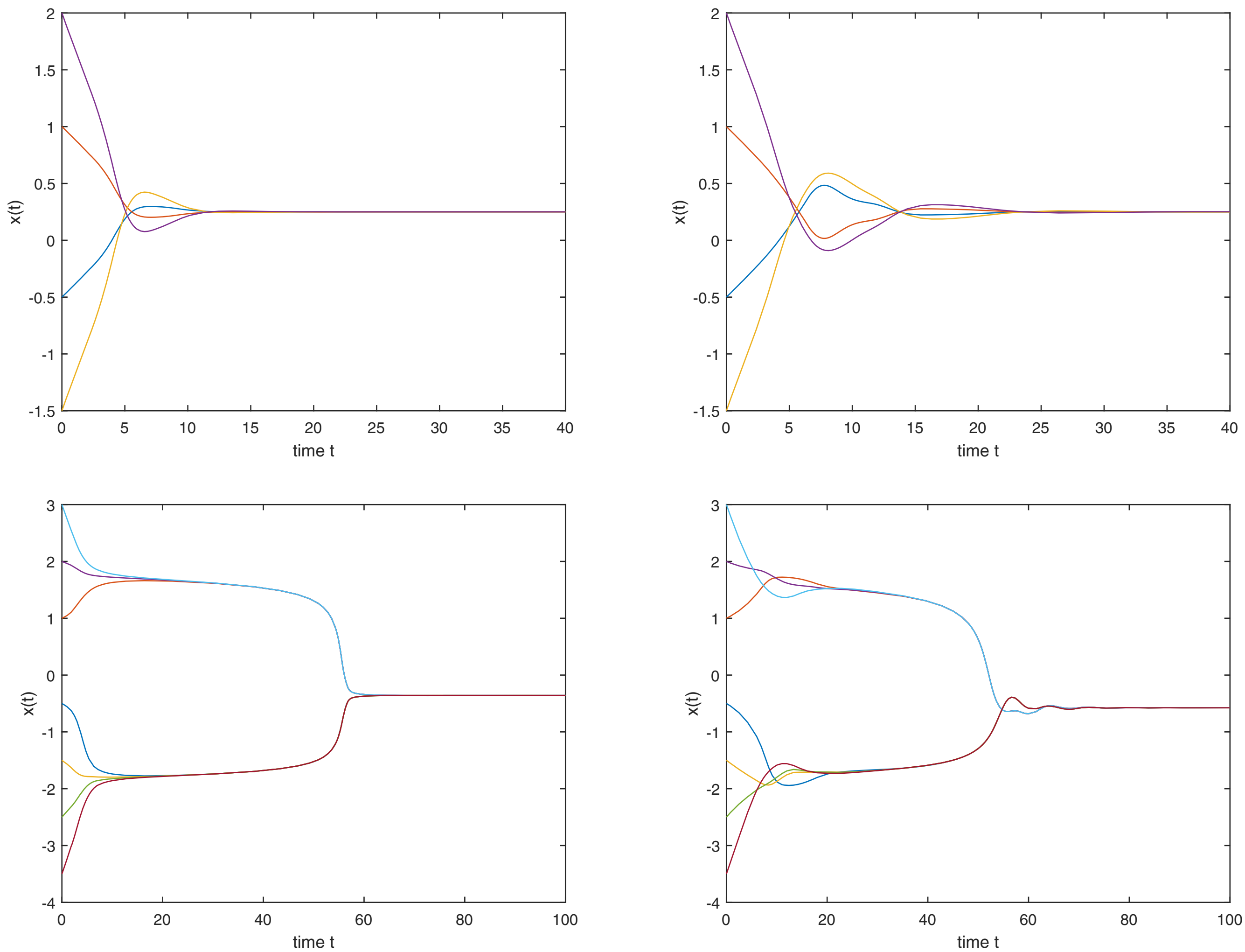

FIGURE 2 Communication rates (6.2): time evolution of solutions with different time delays and number N of agents; $\tau = 3, N = 4$ (top left), $\tau = 6, N = 4$ (top right), $\tau = 1, N = 7$ (bottom left), $\tau = 6, N = 7$ (bottom right). [Colour figure can be viewed at wileyonlinelibrary.com]

$$\tilde{\psi}(r) = \sin^2 r + \frac{1}{1+r^2}, \quad r \in [0, +\infty), \tag{6.1}$$

and a translated gaussian function like

$$\tilde{\psi}(r) = e^{-(r-1)^2}, \quad r \in [0, +\infty). \tag{6.2}$$

These are significant examples since, besides the more studied case with $\tilde{\psi}$ monotonic, it is important to consider some oscillatory behaviors in the agents' interaction or interactions which are more relevant when the distance between the agents is close to a certain value.

In Figure 1, we see the evolution of agents' opinions in the case of the interaction potential of an oscillatory type defined in (6.1), respectively for $N = 4$ (in the top) and $N = 7$ (in the bottom), considering time delays $\tau = 3$ and $\tau = 10$. We see that after an initial oscillatory behavior, the system tends toward consensus. In case of the larger time delay, in order to observe the consensus behavior we have to wait a larger time (we take the time $t \in [0, 60]$ in the case $\tau = 10$ while $t \in [0, 40]$ is enough for $\tau = 3$.). However, in agreement with previous theoretical analysis, the consensus among the agents is always achieved.

In Figure 2, we observe the opinions' evolution in the case of the potential function (6.2). We consider different time delays and, as in the previous case, $N = 4$ or $N = 7$. Also in such a case, we can see that the system converges to a consensus after an initial oscillatory behavior. In the case of a larger delay, the convergence to consensus can be observed after a larger time. In particular, in the case of $N = 7$ agents, we first observe the formation of two clusters. This is related to the form of the influence function.

These examples confirm our theoretical analysis concerning the fact that the consensus state is obtained independently of the time delay size. Moreover, this happens even for influence functions, like the ones defined in (6.1) and (6.2) nonmonotonic.

7 | CONCLUSIONS

In this paper, we studied Hegselmann–Krause opinion formation models with time variable time delays. In particular, we considered a pointwise time-dependent time delay and a distributed time delay. We considered always positive interaction coefficients,

$$a_{ij}(t) = \frac{1}{N-1}\psi(x_i(t), x_j(t-\tau(t))), \quad i, j = 1, \ldots, N,$$

but we removed classical monotonicity assumptions on the potential ψ that is not necessarily, now, in the form

$$\psi(r, r') = \psi(|r - r'|), \quad \forall r, r' \in \mathbb{R}^d.$$

Besides the more general form of the interaction strengths, a significant novelty, with respect to previous literature, is that we were able to prove convergence to consensus without requiring any smallness assumptions on the time delay functions. This was known (see Haskovec [42]) only in the case of a constant time delay (see also Rodriguez Cartabia [43] for the analysis of a Cucker–Smale model with constant time delay). Furthermore, in Haskovec [42], only asymptotic convergence to consensus was proved. Then, our exponential convergence result improves also the result for the constant time delay case.

Since our estimates for the ODE systems were independent of the number N of agents, we were able to extend the convergence theorems to the continuity equations obtained as the mean-field limit of the particle models, when N goes to infinity (cf., previous works [33, 39]).

As a further research direction, it would be interesting to consider the case in which the interaction between the agents, or between some pairs of agents, can degenerate, namely, it can become null, in some time intervals. In such a case, it is not guaranteed that the system tends toward a consensus state. However, one can look for sufficient conditions ensuring consensus or clusters formation. Our approach could give some insights in such a direction.

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CONFLICTS OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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