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Constant speed random particles spontaneously confined on the surface of an expanding sphere

Maurizio Serva

Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università dell'Aquila, L'Aquila, Italy
E-mail: serva@univaq.it

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Abstract. The particles that we describe here can only move at the speed of light c in three-dimensional space. The velocity, which randomly but continuously changes direction, can be represented as a point on the surface of a sphere of constant radius c, and its trajectories may only connect points of this variety. The Wiener process that we use to describe the velocity dynamics on the surface of the sphere is anisotropic since the infinitesimal variation of the velocity is not only always orthogonal to the velocity itself (which guarantees a constant speed), but also to the position. This choice for the infinitesimal variation of the velocity is the one that best slows down the diffusion of particles in space by random motion at the speed of light. As a result of these dynamics, the position of the particles spontaneously remain confined on the surface of an expanding sphere whose radius increases, for large times, as the square root of time.

Keywords: Wiener processes, constant speed, Itô calculus, relativity

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Contents

1. Introduction

Research on relativistic stochastic processes has a long tradition and it has sprouted in a large number of results (see for example [1–10]). The particle dynamics are typically modeled by assuming that the velocity evolves according to some Langevin process modified in order that the speed is never superluminal.

There is a direct physical rationale for research in this field since relativity must be taken into account when the speeds involved in diffusion are comparable to the speed of light as, for example, in astronomical phenomena. There is also an indirect mathematical motivation related to the possibility of extending the Feynman–Kac formula to the relativistic quantum equations; this can be done, in principle, by means of the backward Kolmogorov equation for the processes.

However, ultra-relativistic stochastic models of light-speed particles have received less attention. This class of processes, which are mathematically interesting in their own right, could be suitable for modeling the Brownian motion of constant-speed particles (not necessarily the speed of light) and for providing new probabilistic tools for the solution of the relativistic equations of quantum mechanics (in this case, the speed is necessarily that of light).

Light-speed particles in 1+1 space-time dimension have only two alternatives: to move in one direction or the opposite with the same speed, with the only option being

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randomly switching from one to the other of two possible velocities. This stochastic process was considered in 1956 by Kac, who proved that the associated probability density satisfies the telegrapher's equation [11–13].

About thirty years later, Kac *et al* noticed that the telegrapher equation could be associated both with the Dirac equation in 1+1 dimension (first-order formulation) and with the Klein–Gordon equation also in 1+1 dimension (second-order formulation). Using this equivalence, they were able to give a probabilistic solution (by the backward Kolmogorov equation) to these fundamental quantum equations [14]. This result was later refined and extended in [15, 16].

Indeed, the process considered in [11, 14–16] is part of a larger class; in fact, by Lorentz boosts, new processes can be obtained with particles moving at the speed of light (a simple consequence of the fact that a light-speed particle in an inertial frame is also light-speed in any other inertial frame). The processes of this larger class have in general an unbalanced probability rate of velocity inversion, i.e the inversions from right to left occur with a different probability rate to those from left to right, and as a consequence, the particle may have a non-vanishing average velocity.

The class of these one-dimensional light-speed processes was further extended by considering inversion rates that not only depend on the sign of the velocity, but also on the position and time. This extension gave the possibility to reformulate the quantum mechanics of a relativistic particle in terms of stochastic processes [17] in the spirit of Nelson's stochastic mechanics [18].

The weak point of all these results, i.e. the original results of Kac from 1956 and all the derived ones, is that they only work for particles in the 1+1 space-time dimension, and a direct extension to higher dimensions has proven to be complicated.

In very recent research [19, 20], progress has been made concerning the 3+1 space-time dimensional Dirac equation. The authors followed a Bohm-type approach [21, 22], in which the particle randomly switches between two possible velocities, each of which is associated with one of the two values of a dichotomous variable. Both the probability rate of a switch and the two velocities depend on position and time, and they are determined by the wave function.

We also tried to take a step in the same direction [13, 23, 24] following a different approach. We considered a family of processes that generalizes the Kac approach to the 3+1 space-time dimension case, assuming that a particle only moves at the speed of light c and that the velocity may take any value compatible with this speed. This implies that the velocity can be represented as a point on the surface of a sphere of radius c. We also assumed that the velocity performs an isotropic Wiener process on this surface; this means that the infinitesimal variation of the velocity can take with equal probability all possible directions orthogonal to the velocity itself. The behavior of the position $\mathbf{x}(t)$ is diffusive; in fact, for large times one has the expected value $E[|\mathbf{x}(t)|^2] \sim t$.

Since our particles live in a relativistic world, together with the isotropic 'rest frame' process of the velocity on the sphere surface, we also considered the whole family of anisotropic processes that result from Lorentz boosts [23] for which, nevertheless, the velocity continues to be confined to the sphere surface (light-speed particles are still light-speed particles under a Lorentz boost).

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Here, we consider particles in 3+1 space-time dimensions, which always move at the speed of light and which randomly but continuously change their velocity according to a Wiener process. However, in the present case, the rest frame Wiener process on the spherical surface is not isotropic since the infinitesimal variation of the velocity is not only always orthogonal to the velocity (which guarantees a constant speed), but also to the position. An unexpected result of these dynamics is that the position of the particles spontaneously remains confined on the surface of an expanding sphere. This means that the particles move at the speed of light, but their random movement takes place entirely on the surface of the expanding sphere.

To avoid confusion, let us highlight that both the velocity and position are confined on the surface of a sphere. For the velocity, the sphere has a constant radius c, and this is a simple consequence of the hypothesis that the speed is constant. For the position, however, the sphere has a radius that increases over time, and this is quite a surprising consequence of the dynamics. Moreover, for large times, this radius goes as the square root of time; therefore, it can be said that the particles are subject to diffusion.

2. Stochastic equations

The particle velocity is $c\mathbf{n}(t)$, where $\mathbf{n}(t)$ is a unitary vector so that the speed always equals c. The stochastic equations we consider are

$$\begin{aligned} \mathrm{d}\mathbf{x}(t) &= c\mathbf{n}(t)\,\mathrm{d}t, \\ \mathrm{d}\mathbf{n}(t) &= -\frac{\omega^2}{2}\mathbf{n}(t)\,\mathrm{d}t + \omega\mathbf{n}_\perp(t)\,\mathrm{d}w(t), \end{aligned} \tag{1}$$

where, following Itô, $\mathrm{d}\mathbf{x}(t) = \mathbf{x}(t+\mathrm{d}t) - \mathbf{x}(t)$ and $\mathrm{d}\mathbf{n}(t) = \mathbf{n}(t+\mathrm{d}t) - \mathbf{n}(t)$. The diffusion coefficient ω has the dimension of the inverse of the square root of time, and $\mathrm{d}w(t) = w(t+\mathrm{d}t) - w(t)$ is a standard Wiener increment, i.e. $E[\mathrm{d}w(t)] = 0$, $E[(\mathrm{d}w(t))^2] = \mathrm{d}t$. Moreover, we assume that $\mathbf{n}_\perp(t)$ is a unitary vector orthogonal both to the velocity and the position; more precisely, we assume

$$\mathbf{n}_\perp(t) = \frac{\mathbf{x}(t)\times\mathbf{n}(t)}{|\mathbf{x}(t)\times\mathbf{n}(t)|} \tag{2}$$

when $\mathbf{x}(t)\times\mathbf{n}(t) \neq 0$, and we assume that $\mathbf{n}_\perp(t)$ is any unitary vector orthogonal to $\mathbf{n}(t)$ otherwise.

It can be verified that $|\mathbf{n}(t)| = 1$ at any time if $|\mathbf{n}(0)| = 1$; in fact, following Itô, one has

$$\mathrm{d}|\mathbf{n}(t)|^2 = 2\mathbf{n}(t)\cdot\mathrm{d}\mathbf{n}(t) + \omega^2|\mathbf{n}_\perp(t)|^2\mathrm{d}t = -\omega^2\left(|\mathbf{n}(t)|^2 - 1\right)\mathrm{d}t, \tag{3}$$

where for the second equality we took into account that $\mathbf{n}_\perp(t)\cdot\mathbf{n}(t) = 0$ and $|\mathbf{n}_\perp(t)|^2 = 1$. It is easy to verify that the solution of this equation is $|\mathbf{n}(t)| = 1$ at any time if $|\mathbf{n}(0)| = 1$. Therefore, $\mathbf{n}(t)$ can be represented as a point on a sphere of unitary radius, i.e. the velocity $c\mathbf{n}(t)$ can be represented as a point on a sphere of radius c.

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We assume that the particle is initially at the origin. We will show that $\mathbf{x}(t)\times\mathbf{n}(t)$ equals zero only at the initial time $t=0$ where $\mathbf{x}\,(0)=0$; on the contrary, at any later time $t>0$, not only $\mathbf{x}(t)\neq 0$ but $\mathbf{x}(t)$ and $\mathbf{n}(t)$ are not parallel.

These equations can eventually be solved assuming without loss of generality that $\mathbf{n}(0)=\mathbf{k}=(0,0,1)$. Given that $\mathbf{x}(0)\times\mathbf{n}(0)=0$, the unitary vector $\mathbf{n}_\perp(0)$ can arbitrarily be chosen orthogonally to $\mathbf{n}(0)=\mathbf{k}$. We can take, for example, $\mathbf{n}_\perp(0)=\mathbf{i}=(1,0,0)$; however, we will see that the probability density for the position and velocity at any time $t>0$ is not affected by the choice of the value of $\mathbf{n}_\perp(0)$.

3. Deterministic equations and solutions

We consider here two relevant derived variables whose equations are deterministic and whose values can be explicitly computed at any time t. Let us define

$$\begin{aligned}\xi(t)&=\mathbf{x}(t)\cdot\mathbf{n}(t),\\ \rho(t)&=|\mathbf{x}(t)|,\end{aligned}\tag{4}$$

we easily derive from equations (1)

$$\begin{aligned}\mathrm{d}\xi(t)&=c\mathrm{d}t-\frac{\omega^2}{2}\xi(t)\,\mathrm{d}t,\\ \mathrm{d}\rho^2(t)&=2c\xi(t)\,\mathrm{d}t,\end{aligned}\tag{5}$$

which, surprisingly, are deterministic equations, at variance with the corresponding equations of the isotropic model considered in [13, 23, 24]. System (5) can be easily solved, obtaining

$$\begin{aligned}\xi(t)&=\frac{2c}{\omega^2}\left(1-e^{-\frac{\omega^2}{2}t}\right),\\ \rho(t)&=\frac{2c}{\omega^2}\left(\omega^2 t-2+2e^{-\frac{\omega^2}{2}t}\right)^{\frac{1}{2}},\end{aligned}\tag{6}$$

where, given that the system is at the origin at time zero ($\mathbf{x}(0)=0$), we have used the initial conditions $\xi(0)=0$, $\rho(0)=0$. Notice that the initial values for these equations are independent of the initial values $\mathbf{n}(0)$ and $\mathbf{n}_\perp(0)$.

The second of the above equalities implies that the particle remains confined to the surface of an expanding sphere of radius $\rho(t)$, while the first equality implies that the cosine of the angle between the position $\mathbf{x}(t)$ and the velocity $c\mathbf{n}(t)$ is $\frac{\mathbf{x}(t)\cdot n(t)}{|\mathbf{x}(t)|}=\frac{\xi(t)}{\rho(t)}$.

Moreover, since both $\xi(t)$ and $\rho(t)$ are strictly positive for $t>0$, the particle is at the origin only at time zero, and the angle between the position and velocity never vanishes for positive times. This means that in the second equation of system (1) we can use $\mathbf{n}_\perp(t)$ as defined in (2) at any positive time.

Notice that for large times the radius grows as $\rho(t)\sim\frac{2c}{\omega}\sqrt{t}$, which means a diffusive-like behavior $|\mathbf{x}(t)|^2\sim\frac{4c^2}{\omega^2}t$. Also notice that the radial component of the velocity

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$\frac{\mathbf{x}(t)\cdot c\,n(t)}{|\mathbf{x}(t)|} = \frac{c\xi(t)}{\rho(t)}$ asymptotically behaves as $\frac{c}{\omega\sqrt{t}}$, which means that for large times the position $\mathbf{x}(t)$ and the velocity $c\mathbf{n}(t)$ tend to be orthogonal.

Let us remark that the following equality also holds:

$$|\mathbf{x}(t) \times \mathbf{n}(t)| = \left[\rho^2(t) - \xi^2(t)\right]^{\frac{1}{2}} = \xi_+(t), \tag{7}$$

where the second equality is a definition and the first is a consequence of the geometrical relation $|\mathbf{x}\times\mathbf{n}|^2 = |\mathbf{x}|^2 - (\mathbf{x}\cdot\mathbf{n})^2$. The explicit expression of $\xi_+(t)$ is

$$\xi_+(t) = \frac{2c}{\omega^2}\left(\omega^2 t - 3 + 4e^{-\frac{\omega^2}{2}t} - e^{-\omega^2 t}\right)^{\frac{1}{2}}. \tag{8}$$

The reason why we used the symbol $\xi_+(t)$ will be made clear in the next section. The equality (7) also allows us to write

$$\mathbf{n}_\perp(t) = \frac{\mathbf{x}(t)\times\mathbf{n}(t)}{\xi_+(t)}, \tag{9}$$

which can optionally be used in the second equation in (1) in place of (2).

4. Moving reference frame and relative position

Given that $\mathbf{n}(t)$ changes over time, the only certainty is that the particle position will be on the sphere surface of radius $\rho(t)$, but the distribution on it will depend on the initial conditions. If the initial distribution of the velocity $c\mathbf{n}(0)$ is uniform in all directions, then both the distribution of the velocity $c\mathbf{n}(t)$ on the surface of the sphere of radius c and the distribution of the position $\mathbf{x}(t)$ on the sphere of radius $\rho(t)$ will also be uniform at any positive time. This fact is independent of the initial value or distribution of $\mathbf{n}_\perp(0)$ since, as we will see later, $\mathbf{n}_\perp(0)$ does not influence the future evolution.

Where and with what probability $c\mathbf{n}(t)$ and $\mathbf{x}(t)$ can be found on the respective surfaces when the initial condition $\mathbf{n}(0)$ has a specific value is a problem that needs further investigation.

First of all, notice that the component of $\mathbf{x}(t)$ in the direction of $\mathbf{n}_\perp(t)$ vanishes at any positive time, i.e.

$$\mathbf{x}(t)\cdot\mathbf{n}_\perp(t) = 0; \tag{10}$$

in fact, this equality immediately follows from definition (2).

The third component of the triad of mutually orthogonal unitary vectors (after $\mathbf{n}$ and $\mathbf{n}_\perp$) can be written as

$$\mathbf{n}_+(t) = \mathbf{n}(t)\times\mathbf{n}_\perp(t) = \frac{\mathbf{n}(t)\times\left(\mathbf{x}(t)\right)\times\mathbf{n}(t)\big)}{|\mathbf{x}(t)\times\mathbf{n}(t)|}, \tag{11}$$

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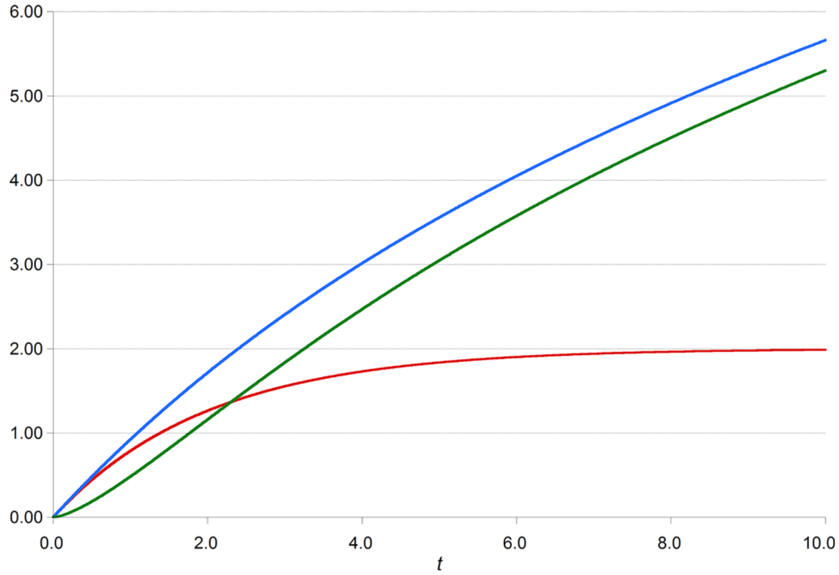

Figure 1. In the figure we plot $\xi(t)$ (red), $\xi_+(t)$ (green) and $\rho(t) = (\xi^2(t) + \xi_+^2(t))^{\frac{1}{2}}$ (blue) as functions of time. The choice of parameters is $\omega = 1$ and $c = 1$. $\xi(t)$ is asymptotically constant, while $\xi_+(t)$ and $\rho(t)$ behave asymptotically as the square root of time.

where, again, the second equality immediately follows from definition (2). Given the geometrical relation $\mathbf{n} \times (\mathbf{x} \times \mathbf{n}) = \mathbf{x} - \mathbf{n}(\mathbf{n} \cdot \mathbf{x})$, one can write

$$\mathbf{n}_+ (t) = \frac{\mathbf{x}(t) - (\mathbf{n}(t) \cdot \mathbf{x}(t))\, \mathbf{n}(t)}{|\mathbf{x}(t) \times \mathbf{n}(t)|} = \frac{\mathbf{x}(t) - \xi(t)\, \mathbf{n}(t)}{\xi_+(t)}, \tag{12}$$

then, the component of $\mathbf{x}(t)$ in the direction of $\mathbf{n}_+$ is

$$\mathbf{x}(t) \cdot \mathbf{n}_+ (t) = \frac{\rho^2(t) - \xi^2(t)}{\xi_+(t)} = \xi_+(t), \tag{13}$$

which explains the choice of the symbol $\xi_+(t)$ in definition (7).

In conclusion, according to (12), the position of the particle with respect to the moving reference frame $\mathbf{n}(t), \mathbf{n}_\perp(t), \mathbf{n}_+(t)$ is

$$\mathbf{x}(t) = \xi(t)\, \mathbf{n}(t) + \xi_+(t)\, \mathbf{n}_+(t), \tag{14}$$

which is coherent with $|\mathbf{x}(t)|^2 = \xi^2(t) + \xi_+^2(t) = \rho^2(t)$. The position $\mathbf{x}(t)$ has no component along $\mathbf{n}_\perp(t)$, and the three vectors $\mathbf{n}(t)$, $\mathbf{x}(t)$ and $\mathbf{n}_+(t)$ lie on the same plane orthogonal to $\mathbf{n}_\perp(t)$.

The three quantities $\xi(t)$, $\xi_+(t)$ and $\rho(t)$ are plotted in figure 1, where the choice of parameters is $\omega = 1$ and $c = 1$. Notice that $\xi(t)$, which is the component of $\mathbf{x}(t)$ parallel to the velocity, tends to a constant for large times while the orthogonal component $\xi_+(t)$ grows as the square root of time as well as the radius $\rho(t)$.

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5. Averages of position and velocity

It is easy to compute the averages of velocity and position. From the second of the equations in (1), taking the averages, we have

$$\mathrm{d}E\left[\mathbf{n}(t)\right] = -\frac{\omega^2}{2}E\left[\mathbf{n}(t)\right]\mathrm{d}t, \tag{15}$$

which implies

$$E\left[\mathbf{n}(t)\right] = e^{-\frac{\omega^2}{2}t}\mathbf{n}(0), \tag{16}$$

where, in this case, we can choose $\mathbf{n}(0) = \mathbf{k} = (0,0,1)$ without loss of generality.

Moreover, from the first of the equations in (1) and from the initial condition $\mathbf{x}(0) = 0$, we have

$$\mathbf{x}(t) = c\int_0^t \mathbf{n}(s)\,\mathrm{d}s, \tag{17}$$

which implies

$$E\left[\mathbf{x}(t)\right] = c\int_0^t E\left[\mathbf{n}(s)\right]\mathrm{d}s = \frac{2c}{\omega^2}\left(1 - e^{-\frac{\omega^2}{2}t}\right)\mathbf{n}(0). \tag{18}$$

The average (16) entails that in the long run $E[\mathbf{n}(t)] \to 0$, which suggests that $\mathbf{n}(t)$ tends to be uniformly distributed on the surface of the unitary radius sphere.

Also, $\mathbf{x}(t)$ tends to be uniformly distributed on the surface of the sphere of radius $\rho(t)$. The fact that $E[\mathbf{x}(t)] \to \frac{2c}{\omega^2}\mathbf{n}(0)$ is misleading, since two phenomena contribute to this result: the tendency to uniform distribution and the expansion of the sphere.

In order to isolate the first phenomenon, it is sufficient to consider the unitary vector $\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}$, which also lives on the surface of a unitary radius sphere, and whose average tends to zero in the long run. In fact,

$$E\left[\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}\right] = \frac{E\left[\mathbf{x}(t)\right]}{\rho(t)} = \frac{\left(1 - e^{-\frac{\omega^2}{2}t}\right)}{\left(\omega^2 t - 2 + 2e^{-\frac{\omega^2}{2}t}\right)^{\frac{1}{2}}}\mathbf{n}(0), \tag{19}$$

which implies $E\left[\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}\right] \simeq \frac{1}{\omega\sqrt{t}}\mathbf{n}(0)$ for large times.

We underline that both the averages of $\mathbf{n}(t)$ and $\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}$ are independent of the initial value $\mathbf{n}_\perp(0) = \mathbf{i}$, which, indeed, is immediately forgotten after $t = 0$. Moreover, both the averages are parallel to $\mathbf{n}(0) = \mathbf{k}$; indeed, we will see that the probability densities of $\mathbf{n}(t)$ and $\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}$ are invariant for rotations around the vertical axis.

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6. More on equations and averages

Let us write the stochastic equation for the variable $\mathbf{x}(t)\times\mathbf{n}(t)$:

$$\mathrm{d}\left[\mathbf{x}(t)\times\mathbf{n}(t)\right] = -\frac{\omega^2}{2}\left[\mathbf{x}(t)\times\mathbf{n}(t)\right]\mathrm{d}t + \frac{\omega}{\xi_+(t)}\left[\mathbf{x}(t)\times(\mathbf{x}(t)\times\mathbf{n}(t))\right]\mathrm{d}w(t), \tag{20}$$

which, given the geometrical relation $\mathbf{x}\times(\mathbf{x}\times\mathbf{n}) = \mathbf{x}(\mathbf{n}\cdot\mathbf{x}) - \mathbf{n}|\mathbf{x}|^2$ becomes

$$\mathrm{d}\left[\mathbf{x}(t)\times\mathbf{n}(t)\right] = -\frac{\omega^2}{2}\left[\mathbf{x}(t)\times\mathbf{n}(t)\right]\mathrm{d}t + \frac{\omega}{\xi_+(t)}\left[\xi(t)\mathbf{x}(t) - \rho^2(t)\mathbf{n}(t)\right]\mathrm{d}w(t). \tag{21}$$

Then, taking the average, we obtain

$$\mathrm{d}E\left[\mathbf{x}(t)\times\mathbf{n}(t)\right] = -\frac{\omega^2}{2}E\left[\mathbf{x}(t)\times\mathbf{n}(t)\right]\mathrm{d}t, \tag{22}$$

which implies

$$E\left[\mathbf{x}(t)\times\mathbf{n}(t)\right] = e^{-\frac{\omega^2}{2}t}E\left[\mathbf{x}(0)\times\mathbf{n}(0)\right] = 0, \tag{23}$$

where the last equality is the immediate consequence of $\mathbf{x}(0)=0$. In turn, equality (9), together with the fact that $\xi_+(t)$ is strictly positive when $t>0$, implies

$$E\left[\mathbf{n}_\perp(t)\right] = \frac{E\left[\mathbf{x}(t)\times\mathbf{n}(t)\right]}{\xi_+(t)} = 0 \tag{24}$$

for any positive time. The choice $\mathbf{n}_\perp(0)=\mathbf{i}$ (as well as any other equivalent choice) is, therefore, irrelevant for the average of $\mathbf{n}_\perp(t)$ at $t>0$. Indeed, as we will see, the value chosen for $\mathbf{n}_\perp(0)$ is irrelevant for the probability density of $\mathbf{n}_\perp(t)$ at any positive time.

Finally, the average of $\mathbf{n}_+(t)$ can be computed using equation (14):

$$E\left[\mathbf{n}_+(t)\right] = \frac{E\left[\mathbf{x}(t)\right] - E\left[\mathbf{n}(t)\right]\xi(t)}{\xi_+(t)} = \frac{\left(1-e^{-\frac{\omega^2}{2}t}\right)^2}{\left(\omega^2 t - 3 + 4e^{-\frac{\omega^2}{2}t} - e^{-\omega^2 t}\right)^{\frac{1}{2}}}\mathbf{n}(0), \tag{25}$$

where for the second equality we have used the averages previously computed. Notice that for large times one has $E[\mathbf{n}_+(t)] \simeq \frac{1}{\omega\sqrt{t}}\mathbf{n}(0)$.

In conclusion, the averages of $\mathbf{n}(t)$, $\mathbf{n}_+(t)$ and $\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}$ only have non-vanishing components $E[\mathbf{n}(t)]\cdot\mathbf{n}(0)$, $E[\mathbf{n}_+(t)]\cdot\mathbf{n}(0)$ and $E\left[\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}\right]\cdot\mathbf{n}(0)$ in the direction of $\mathbf{n}(0)=\mathbf{k}$, which are plotted in figure 2. On the contrary, the average of $\mathbf{n}_\perp(t)$ equals zero at any positive time, while at time zero one has $\mathbf{n}_\perp(0)=\mathbf{i}$. These results suggest that for positive times

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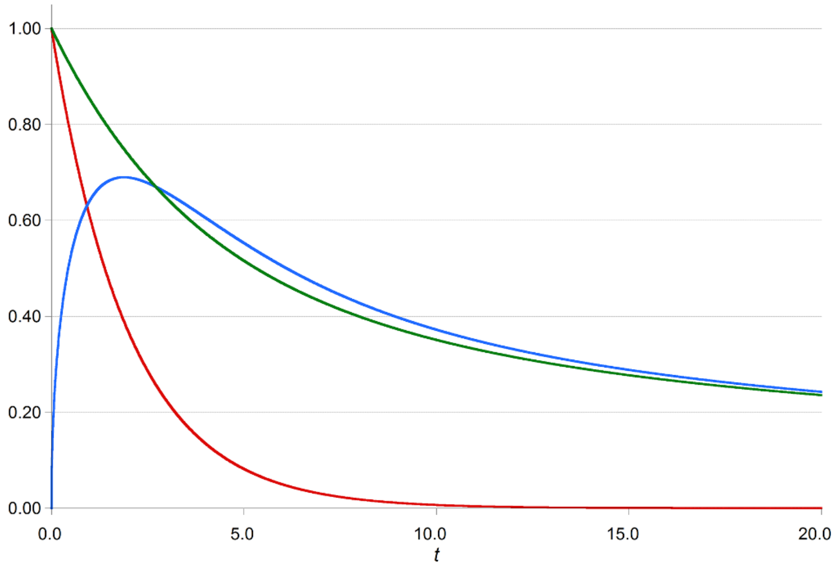

Figure 2. The averages of $\mathbf{n}(t)$, $\mathbf{n}_+(t)$ and $\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}$ only have non-vanishing components in the direction of $\mathbf{n}(0)$, while the average of $\mathbf{n}_\perp(t)$ equals zero. In the figure we plot $E[\mathbf{n}(t)]\cdot\mathbf{n}(0)$ (red), $E[\mathbf{n}_+(t)]\cdot\mathbf{n}(0)$ (blue) and $E\left[\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}\right]\cdot\mathbf{n}(0)$ (green). The choice of parameters is $\omega=1$ and $c=1$. Notice that the average of $\mathbf{n}(t)$ decays exponentially in time while the averages of $\mathbf{n}_+(t)$ and $\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}$ decay as $\frac{1}{\sqrt{t}}$.

the probability densities of $\mathbf{n}(t)$, $\mathbf{n}_+(t)$, $\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}$ and also $\mathbf{n}_\perp(t)$ are invariant for rotations around the vertical axis despite the asymmetric initial conditions due to the choice of $\mathbf{n}_\perp(0)$. This fact will be confirmed in the following section by numerical simulations.

7. Numerical solutions

Without loss of generality, we assume from now that $\omega=1$ and $c=1$. The discrete version of the second equation in (1), in its more intuitive form, appears as

$$\mathbf{n}(t+\Delta t)=\left(1-\frac{\Delta t}{2}\right)\mathbf{n}(t)+\Delta w(t)\,\mathbf{n}_\perp(t), \tag{26}$$

where $\mathbf{n}_\perp(t)$ is given by (9) and (8) or directly by (2). As should be clear by the notation, $t=m\Delta t$; moreover, the Wiener increments $\Delta w(t)=w(t+\Delta t)-w(t)$ are independent Gaussian variables with vanishing average and variance Δt.

The above equation preserves the unitarity of $\mathbf{n}(t)$ only in the continuous limit $\Delta t\to 0$. In order to guarantee the unitarity of the velocity $\mathbf{n}(t)$ at each time step, even for a finite Δt, we prefer to replace (26) with

$$\mathbf{n}(t+\Delta t)=\left(1-(\Delta w(t))^2\right)^{\frac{1}{2}}\,\mathbf{n}(t)+\Delta w(t)\,\mathbf{n}_\perp(t). \tag{27}$$

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In the limit of small Δt, equations (26) and (27) become identical and both coincide with the second equation in (1). Equation (27) holds for $t>0$, while at $t=0$ the equation is

$$\mathbf{n}(\Delta t)=\left(1-(\Delta w(0))^2\right)^{\frac{1}{2}}\mathbf{k}+\Delta w(0)\,\mathbf{i}, \tag{28}$$

where we have assumed, without loss of generality, that $\mathbf{n}(0)=\mathbf{k}=(0,0,1)$ and $\mathbf{n}_\perp(0)=\mathbf{i}=(1,0,0)$.

Finally, the discrete version of the first equation in (1) is

$$\mathbf{x}(t+\Delta t)=\mathbf{x}(t)+\mathbf{n}(t)\,\Delta t, \tag{29}$$

which holds for any time $t\geqslant 0$ and with initial condition $\mathbf{x}(0)=0$.

The numerical solutions consist of $5\cdot 10^5$ realizations of the above equations up to time $t=30$. The size of a time step is $\Delta t=10^{-4}$; therefore, the number of steps in each realization is $3\cdot 10^5$.

8. Output of simulations

Let us first consider the velocity $\mathbf{n}(t)$ (in this section we still assume $c=1$ and $\omega=1$), whose components can be expressed in spherical coordinates as $n_3(t)=\cos(\theta_n(t))$, $n_1(t)=\sin(\theta_n(t))\cos(\phi_n(t))$ and $n_2(t)=\sin(\theta_n(t))\sin(\phi_n(t))$, where $\theta_n(t)$ and $\phi_n(t)$ are the corresponding latitude and longitude.

We numerically compute the distributions of $n_1(t)$, $n_2(t)$, $n_3(t)=\cos(\theta_n(t))$ and $\frac{\phi_n(t)}{\pi}$, which are plotted in figure 3 at different times from $t=0.1$ to $t=10$. All four of these variables have distributions supported on the interval $[-1,1]$. At all times, the distribution of $\frac{\phi_n(t)}{\pi}$ is uniform, meaning that the distribution of $\mathbf{n}(t)$ on the unitary sphere is invariant for rotations around the vertical axis. Interestingly, there is no memory of the initial time asymmetry due to the choice $\mathbf{n}_\perp(0)=\mathbf{i}$, as is also shown by the fact that the distributions of $n_1(t)$ and $n_2(t)$ are identical and symmetric. On the other hand, the distribution of $n_3(t)=\cos(\theta_n(t))$ is asymmetric, being more centered on positive values (at $t=0$ it is completely centered at the position $n_3=1$, which corresponds to the initial value $\mathbf{n}(0)=\mathbf{k}$).

The distributions of $n_1(t)$, $n_2(t)$ and $n_3(t)$ become uniform in about 10 s, meaning that at the same time the distribution of the velocity $\mathbf{n}(t)$ becomes uniform on the surface of the sphere. The averages of $n_1(t)$ and $n_2(t)$, given the symmetry of their distributions, vanish at all times, at variance with the average of $n_3(t)$. This is exactly what it is shown in (16).

Then, we consider the normalized position $\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}$, whose components can be expressed in spherical coordinates as $\frac{x_3(t)}{|\mathbf{x}(t)|}=\cos(\theta_x(t))$, $\frac{x_1(t)}{|\mathbf{x}(t)|}=\sin(\theta_x(t))\cos(\phi_x(t))$ and $\frac{x_2(t)}{|\mathbf{x}(t)|}=\sin(\theta_x(t))\sin(\phi_x(t))$, where $\theta_x(t)$ and $\phi_x(t)$ are the corresponding latitude and longitude. The distributions of these quantities are shown in figure 4 at different times from $t=0.5$ to $t=30$. The qualitative behavior is similar to that of figure 3, including the fact

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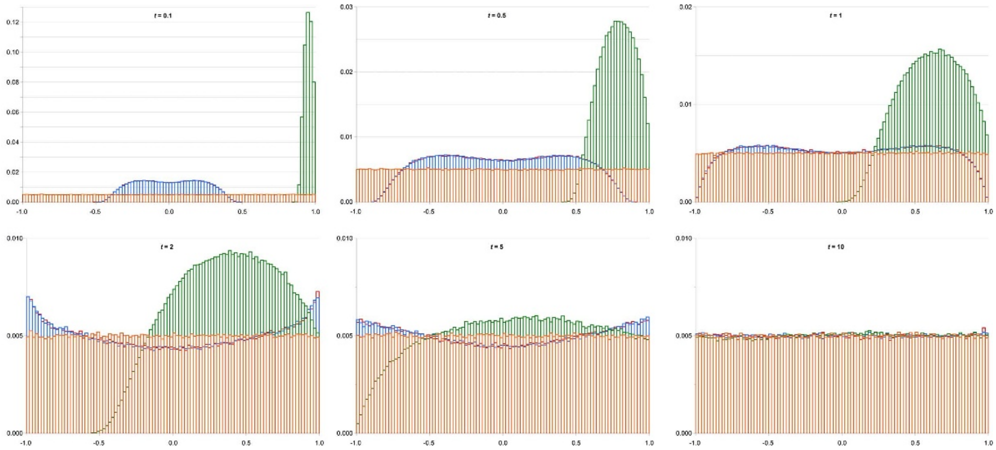

Figure 3. Distributions of $n_3(t) = \cos(\theta_n(t))$ (green), $n_1(t)$ (red), $n_2(t)$ (blue) and $\frac{\phi_n(t)}{\pi}$ (orange), where $\theta_n(t)$ and $\phi_n(t)$ are the relative latitude and longitude. At all times, the distribution of $\frac{\phi_n(t)}{\pi}$ is uniform, meaning that the distribution of $\mathbf{n}(t)$ on the surface of the sphere is invariant for rotations around the vertical axis. The distributions of $n_1(t)$ and $n_2(t)$ are identical and symmetric, while the distribution of $n_3(t) = \cos(\theta_n(t))$ is asymmetric, being more centered on positive values. These last three distributions become uniform in about 10 s, meaning that in the same time the distribution of the velocity $\mathbf{n}(t)$ becomes uniform on the surface of the sphere of unitary radius.

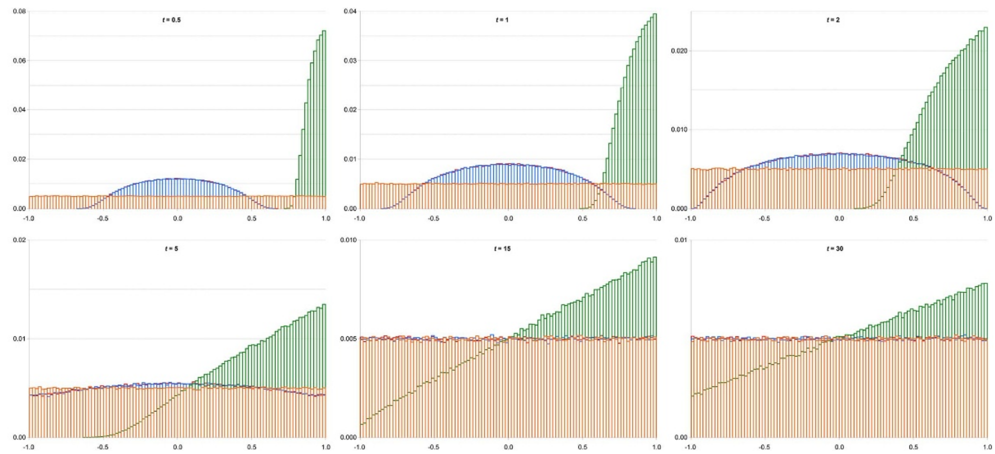

Figure 4. Distributions of $\frac{x_3(t)}{|\mathbf{x}(t)|} = \cos(\theta_x(t))$ (green), $\frac{x_1(t)}{|\mathbf{x}(t)|}$ (red), $\frac{x_2(t)}{|\mathbf{x}(t)|}$ (blue) and $\frac{\phi_x(t)}{\pi}$ (orange), where $\theta_x(t)$ and $\phi_x(t)$ are the relative latitude and longitude. The qualitative behavior is similar to that of figure 3, including the fact that the longitude is uniformly distributed at any time. The only difference is the much slower progression towards the uniform distribution for $\frac{x_3(t)}{|\mathbf{x}(t)|}$, which is not completed in 30 s, while for $n_3(t)$ it is already done in 10 s. This is because the relaxation of the distribution of $n_3(t)$ is exponential in time, while for $\frac{x_3(t)}{|\mathbf{x}(t)|}$ it goes as $\frac{1}{\sqrt{t}}$.

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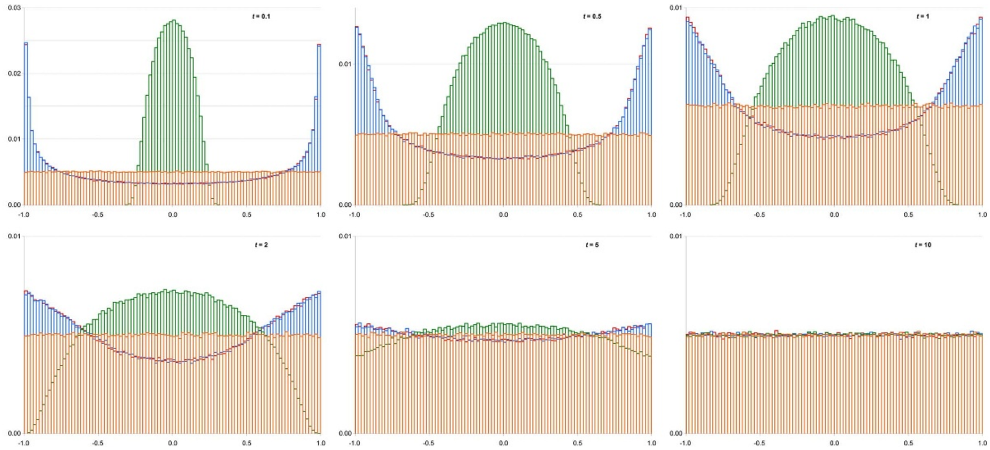

Figure 5. Distributions of $\hat{n}_3(t) = \cos(\theta_{\hat{n}}(t))$ (green), $\hat{n}_1(t)$ (red), $\hat{n}_2(t)$ (blue) and $\frac{\phi_{\hat{n}}(t)}{\pi}$ (orange), where $\mathbf{n}_\perp(t) = (\hat{n}_1(t), \hat{n}_2(t), \hat{n}_3(t))$ and where $\theta_{\hat{n}}(t)$ and $\phi_{\hat{n}}(t)$ are the relative latitude and longitude. Again, the longitude is uniformly distributed at any positive time, although it is not at time zero ($\mathbf{n}_\perp(0) = \mathbf{i}$ implies $\phi_{\hat{n}} = 0$). The distributions of the variables $\hat{n}_3(t) = \cos(\theta_{\hat{n}}(t))$, $\hat{n}_1(t)$ and $\hat{n}_2(t)$ are symmetric at all positive times. Moreover, the distributions of $\hat{n}_1(t)$ and $\hat{n}_2(t)$ are identical at any positive time, although at time zero they are different ($\mathbf{n}_\perp(0) = \mathbf{i}$ implies $\hat{n}_1(0) = 1$ and $\hat{n}_2(0) = 0$). All distributions are uniform after 10 s.

that the longitude is uniformly distributed at all times, meaning that the distribution of $\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}$ on the unitary sphere is invariant for rotations around the vertical axis.

The only difference is the much slower progression toward the uniform distribution for $\frac{x_3(t)}{|\mathbf{x}(t)|}$, which is not completed in 30 s, while for $n_3(t)$ it is already done in about 10 s. The reason is that for $n_3(t)$ the relaxation is exponential, while for $\frac{x_3(t)}{|\mathbf{x}(t)|}$ it goes as $\frac{1}{\sqrt{t}}$. This can also be appreciated by comparing the average of $\mathbf{n}(t)$ in formula (16) and the average of $\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}$ in formula (19); the first decays exponentially in time, while the second goes as $\frac{1}{\sqrt{t}}$.

Finally, we consider the unitary vector $\mathbf{n}_\perp(t)$, whose components can be expressed in spherical coordinates as $\hat{n}_3(t) = \cos(\theta_{\hat{n}}(t))$, $\hat{n}_1(t) = \sin(\theta_{\hat{n}}(t))\cos(\phi_{\hat{n}}(t))$ and $\hat{n}_2(t) = \sin(\theta_{\hat{n}}(t))\sin(\phi_{\hat{n}}(t))$, where $\theta_{\hat{n}}(t)$ and $\phi_{\hat{n}}(t)$ are the corresponding latitude and longitude. These quantities are plotted in figure 5 at different times from $t = 0.1$ to $t = 10$. Once more, the longitude is uniformly distributed at any positive time, although it is not at time zero ($\mathbf{n}_\perp(0) = \mathbf{i}$ implies $\phi_{\hat{n}}(0) = 0$). This means that the distribution of $\mathbf{n}_\perp(t)$ on the unitary sphere is invariant for rotations around the vertical axis at positive times, although $\mathbf{n}_\perp(0) = \mathbf{i}$.

The distributions of the variables $\hat{n}_3(t) = \cos(\theta_{\hat{n}}(t))$, $\hat{n}_1(t)$ and $\hat{n}_2(t)$ are symmetric at any positive time, which explains why the average of $\mathbf{n}_\perp(t)$ vanishes (see (24)). As in figure 3, after 10 s, all the distributions are uniform, meaning that $\mathbf{n}_\perp(t)$ becomes uniformly distributed on the surface of a unitary sphere.

9. Conclusions and outlook

The equations in (1) describe light-speed particles that spontaneously remain confined on the surface of an expanding sphere of radius $\rho(t)$. For large times, $\rho(t)$ grows as the square root of time, so it can be said that the particles are subject to diffusion.

If the initial distribution of the velocity $c\mathbf{n}(0)$ is uniform in all directions, the distribution of the velocity $c\mathbf{n}(t)$ on the surface of the sphere of radius c and the distribution of the position $\mathbf{x}(t)$ on the sphere of radius $\rho(t)$ will be also uniform at any positive time. If, on the contrary, one chooses a specific value for the initial velocity (for example $\mathbf{n}(0) = \mathbf{k}$), the velocity $c\mathbf{n}(t)$ and the position $\mathbf{x}(t)$ will be still on the surfaces of the respective spheres, but while their distributions will be invariant for rotations around the vertical axis, they will be more centered around the north pole. The full symmetry is recovered only at large times.

We call the process described here the 'rest frame' process. Since our particles live in a relativistic world, it would be interesting to also consider and describe the whole family of processes that result from this rest frame process by Lorentz boosts. For all of them, obviously, the velocity is still bounded to the surface of a sphere of radius c (light-speed particles are still light-speed particles under Lorentz boost), but further asymmetry should appear in the stochastic equations so that the position of the particles will be confined on a variety different from the surface of a sphere of radius $\rho(t)$.

Even more interesting would be to find an exploitable connection between the backward Kolmogorov equation of this process and the Dirac equation, thus extending the analogy between the Schrödinger equation and the heat equation to the relativistic realm. This does not seem to be an easy task, but work in this direction is ongoing, with few but not vanishing chances of achieving the goal.

Data availability statement

The numerical methods employed to generate the data in this study are described in the paper and they can be easily reproduced. The code and/or the data are available from the corresponding author upon reasonable request.

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Conflict of interest

The author has no competing interests to declare.

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