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Frequency noise of laser gyros: supplement

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This note provides additional information to supplement the study of Ref. [1]. Specifically, it presents a comprehensive derivation of the power spectra for the amplitude and phase fluctuations of the output radiation emitted by the two counter-propagating modes of a laser gyro in the phase-locked regime. The derivation involves solving the linearized equations for the quantum operators that describe the laser dynamics, supplemented with the appropriate quantum noise terms.

In this note, we present additional findings to complement the results discussed in [1]. We will focus first on the situation in which locking is caused by the coupling induced by back-reflection from a grating generated in the saturated gain medium and occurs with frequency difference Ω_0 between the two modes. Specifically, we investigate the influence of quantum noise introduced by the gain coupling of the two counterpropagating modes. Our analysis reveals that this coupling contributes a minor term to the noise spectra, on the order of the ratio κ_g/γ between the mode coupling coefficient and the photon lifetime. Additionally, we provide explicit expressions for the power spectra of the amplitude and phase fluctuations of the *output* radiation emitted by the two counterpropagating modes of a laser gyro operating in the phase-locked regime, as well as the spectrum of their correlations. The spectra of the output radiation naturally incorporate the impact of shot-noise arising from the detection process. To find the output spectra, we solve the linearized equations that describe the dynamics of the two counterpropagating modes, using the operators that characterize laser behavior and incorporating the necessary noise operators to preserve the commutation relations.

In the concluding section, we also outline the derivation of the equations describing the fluctuations of the output fields in a scenario where locking occurs due to passive reflections. This situation can relate to two possible scenarios. The first scenario involves reflections from a time-dependent index grating, resulting in non-degenerate frequency locking. The second scenario involves reflections from static intracavity elements, causing undesired locking of the two modes at the same frequency.

1. LOCKING BY DYNAMIC GAIN MODULATION

We will use the annihilation operators $\mathbf{a}(t)$ and $\mathbf{b}(t)$ to represent the amplitudes of the two modes centered at frequency $\omega_0 + \Omega_0/2$ and $\omega_0 - \Omega_0/2$, where ω_0 is the optical frequency. These modes correspond to the primed operators $\mathbf{a}'(t)$ and $\mathbf{b}'(t)$ used in the main text. In addition, we redefine $\mathbf{s}_a(t)e^{i\Omega_0 t/2} \mapsto \mathbf{s}_a(t)$, $\mathbf{s}_b(t)e^{-i\Omega_0 t/2} \mapsto \mathbf{s}_b(t)$, $\mathbf{s}_a^{(-)}(t)e^{i\Omega_0 t/2} \mapsto \mathbf{s}_a^{(-)}(t)$ and $\mathbf{s}_b^{(-)}(t)e^{-i\Omega_0 t/2} \mapsto \mathbf{s}_b^{(-)}(t)$, with the new noise operators having the same statistical properties of the original ones. The equations for the amplitude of the two modes are then [1]

$$\frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} = \kappa_g \mathbf{b}(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{a}(t) - ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}_a^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_a(t), \quad (1)$$

$$\frac{\mathrm{d}\mathbf{b}(t)}{\mathrm{d}t} = \kappa_g^* \mathbf{a}(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{b}(t) - ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}_b^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_b(t). \quad (2)$$

These equations can be simplified by setting $\kappa_g = |\kappa_g|e^{i\varphi_g}$ and defining $\mathbf{b}(t) = \mathbf{b}'(t)e^{-i\varphi_g/2}$ and $\mathbf{a}(t) = \mathbf{a}'(t)e^{i\varphi_g/2}$. In terms of the new phase shifted fields, the coupling coefficient is real and positive, so that Eqs. (1) and (2) can be rewritten, dropping the primes to simplify the notation, as

$$\frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} = \kappa_g \mathbf{b}(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{a}(t) - ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}_a^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_a(t), \quad (3)$$

$$\frac{\mathrm{d}\mathbf{b}(t)}{\mathrm{d}t} = \kappa_g \mathbf{a}(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{b}(t) - ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}_b^{(-)}(t) + \sqrt{\gamma}\,\mathbf{s}_b(t). \quad (4)$$

The coupling of the two modes through a gain grating is not hermitian. As a consequence, the spatial components of the material polarization that couple with the two counterpropagating modes mix, resulting in a statistical dependence of the corresponding noise terms $\mathbf{s}_a^{(-)}$ and $\mathbf{s}_b^{(-)}$, which in absence of coupling are independent. The mixing of the noise terms can be quantified if we require the preservation of the commutation rules $[\mathbf{a},\mathbf{b}^\dagger]=0$, which express the independence of the two modes. This requirement is satisfied if

$$\frac{2Ng^2}{\Gamma}[\mathbf{s}_a^{(-)}(t),\mathbf{s}_b^{(-)\dagger}(t')]=-2\kappa_g\delta(t-t'). \quad (5)$$

The commutation relations alone do not fully specify the correlations of the noise operators. However, we notice that the noise sources of the material polarization are creation operators so that $\mathbf{s}_a^{(-)}$ and $\mathbf{s}_b^{(-)}$ when applied on the left, and $\mathbf{s}_a^{(-)\dagger}$ and $\mathbf{s}_b^{(-)\dagger}$ when applied on the right, to a state representing a fully inverted gain medium should give zero. These conditions, combined with the commutation relations (5), give

$$\langle \mathbf{s}_a^{(-)}(t)\mathbf{s}_b^{(-)\dagger}(t')\rangle = 0, \quad (6)$$

$$\langle \mathbf{s}_b^{(-)\dagger}(t')\mathbf{s}_a^{(-)}(t)\rangle = \frac{\Gamma\kappa_g}{Ng^2}\delta(t-t'), \quad (7)$$

$$\langle \mathbf{s}_a^{(-)}(t)\mathbf{s}_b^{(-)}(t')\rangle = 0, \quad (8)$$

$$\langle \mathbf{s}_a^{(-)\dagger}(t)\mathbf{s}_b^{(-)\dagger}(t')\rangle = 0. \quad (9)$$

The equation for the carrier number is [1]

$$\begin{aligned}\frac{\mathrm{d}\mathbf{n}(t)}{\mathrm{d}t} &= R-\frac{\mathbf{n}(t)}{\tau}-\frac{4g^2}{\Gamma}\mathbf{n}(t)\left[\mathbf{a}^\dagger(t)\mathbf{a}(t)+\mathbf{b}^\dagger(t)\mathbf{b}(t)\right]\\ &+i2g\left(\frac{2N}{\Gamma}\right)^{1/2}\left[\mathbf{a}^\dagger(t)\mathbf{s}_a^{(-)}(t)+\mathbf{b}^\dagger(t)\mathbf{s}_b^{(-)}(t)\right.\\ &\left.-\mathbf{s}_a^{(-)\dagger}(t)\mathbf{a}(t)-\mathbf{s}_b^{(-)\dagger}(t)\mathbf{b}(t)\right]. \end{aligned}\quad (10)$$

We did not include in the equation for the carrier number the term

$$\frac{\mathrm{d}\Delta\mathbf{n}(t)}{\mathrm{d}t}=-\frac{4g^2}{\Gamma}\mathbf{n}(t)\left[\mathbf{a}^\dagger(t)\mathbf{b}(t)e^{i\Omega_0 t}+\mathbf{b}^\dagger(t)\mathbf{a}(t)e^{-i\Omega_0 t}\right], \quad (11)$$

responsible for the coupling between the two modes because $\Delta\mathbf{n}(t)$ has been implicitly considered in Eqs. (1) and (2) through the coefficient κ_g. In the analysis that follows, we will take into consideration the saturation of this term and its influence on the laser dynamics using the fundamental property that in the laser cavity each photon is generated through the decay of a single carrier.

If we define

$$\Delta g=-\gamma+\frac{2g^2}{\Gamma}n_0, \quad (12)$$

where we have set $\mathbf{n}=n_0+\delta\mathbf{n}$ with n_0 is the steady state value of $\mathbf{n}$, the condition for steady state of Eqs. (3) and (4) is

$$\kappa_g|b_0|e^{i\Delta\varphi}+\frac{\Delta g}{2}|a_0| = 0, \quad (13)$$

$$\kappa_g|a_0|e^{-i\Delta\varphi}+\frac{\Delta g}{2}|b_0| = 0, \quad (14)$$

where we defined $a_0=|a_0|e^{i\varphi_a}$, $b_0=|b_0|e^{i\varphi_b}$, and $\Delta\varphi=\varphi_a-\varphi_b$. Steady state is achieved for the two modes with equal amplitudes $|a_0|=|b_0|$, for $\kappa_g\sin(\Delta\varphi)=0$ and for $\Delta g=-2\kappa_g\cos(\Delta\varphi)$. Of the two possible solutions $\Delta\varphi=0$ and $\Delta\varphi=\pi$, only the one with $\Delta g<0$ is stable. In the following, we will assume without loss of generality that the phase reference for the two modes is chosen such that a_0 and b_0 are real, so that $\varphi_a=\varphi_b=0$.

Using $\gamma=2g^2n_0/\Gamma-\Delta g$ and assuming full inversion $n_0\simeq N$ and that at steady state $\kappa_g=-\Delta g/2$, and defining $\mathbf{a}=a_0+\delta\mathbf{a}$ and $\mathbf{b}=b_0+\delta\mathbf{b}$, the equations for the displacements of the mode amplitudes become

$$\begin{aligned}\frac{\mathrm{d}\delta\mathbf{a}(t)}{\mathrm{d}t} &= \kappa_g\delta\mathbf{b}(t)+\frac{\Delta g}{2}\delta\mathbf{a}(t)+\frac{g^2}{\Gamma}a_0\delta\mathbf{n}(t)\\ &-i\sqrt{\gamma-2\kappa_g}\,\mathbf{s}_a^{(-)}(t)+\sqrt{\gamma}\,\mathbf{s}_a(t),\end{aligned}\quad (15)$$

$$\begin{aligned}\frac{\mathrm{d}\delta\mathbf{b}(t)}{\mathrm{d}t} &= \kappa_g\delta\mathbf{a}(t)+\frac{\Delta g}{2}\delta\mathbf{b}(t)+\frac{g^2}{\Gamma}b_0\delta\mathbf{n}(t)\\ &-i\sqrt{\gamma-2\kappa_g}\,\mathbf{s}_b^{(-)}(t)+\sqrt{\gamma}\,\mathbf{s}_b(t).\end{aligned}\quad (16)$$

The correlation of the noise sources for the gain material are given by Eqs. (6)–(9) where, using $2g^2N/\Gamma=\gamma+\Delta g=\gamma-2\kappa_g$, Eq. (7) becomes

$$\langle \mathbf{s}_b^{(-)\dagger}(t)\mathbf{s}_a^{(-)}(t')\rangle=\frac{2\kappa_g}{\gamma-2\kappa_g}\delta(t-t'). \quad (17)$$

Using that at steady state $a_0=b_0$ and defining the two uncoupled eigenmodes of the system, also known as supermodes,

$$\mathbf{c}_+(t) = \frac{\delta\mathbf{a}(t)+\delta\mathbf{b}(t)}{\sqrt{2}}, \quad (18)$$

$$\mathbf{c}_-(t) = \frac{\delta\mathbf{a}(t)-\delta\mathbf{b}(t)}{\sqrt{2}}, \quad (19)$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}\mathbf{c}_+(t)}{\mathrm{d}t} &= \frac{g^2}{\sqrt{2}\,\Gamma}(a_0+b_0)\,\delta\mathbf{n}(t)\\ &+\sqrt{\gamma/2}\,[\mathbf{s}_a(t)+\mathbf{s}_b(t)]\\ &-i\sqrt{(\gamma-2\kappa_g)/2}\left[\mathbf{s}_a^{(-)}(t)+\mathbf{s}_b^{(-)}(t)\right],\end{aligned}\quad (20)$$

$$\begin{aligned}\frac{\mathrm{d}\mathbf{c}_-(t)}{\mathrm{d}t} &= -2\kappa_g\mathbf{c}_-(t)+\sqrt{\gamma/2}\,[\mathbf{s}_a(t)-\mathbf{s}_b(t)]\\ &-i\sqrt{(\gamma-2\kappa_g)/2}\left[\mathbf{s}_a^{(-)}(t)-\mathbf{s}_b^{(-)}(t)\right].\end{aligned}\quad (21)$$

Let us define now the noise operators

$$\mathbf{s}_+(t) = \frac{\mathbf{s}_a(t)+\mathbf{s}_b(t)}{\sqrt{2}}, \quad (22)$$

$$\mathbf{s}_-(t) = \frac{\mathbf{s}_a(t)-\mathbf{s}_b(t)}{\sqrt{2}}, \quad (23)$$

$$\mathbf{s}_+^{(-)}(t) = \sqrt{\frac{\gamma-2\kappa_g}{\gamma}}\,\frac{\mathbf{s}_a^{(-)}(t)+\mathbf{s}_b^{(-)}(t)}{\sqrt{2}}, \quad (24)$$

$$\mathbf{s}_-^{(-)}(t) = \sqrt{\frac{\gamma-2\kappa_g}{\gamma-4\kappa_g}}\,\frac{\mathbf{s}_a^{(-)}(t)-\mathbf{s}_b^{(-)}(t)}{\sqrt{2}}. \quad (25)$$

It may easily be verified that the new noise operators are independent

$$\mathbf{s}_\pm^{(-)}(t)\mathbf{s}_\mp^{(-)}(t)=0, \quad (26)$$

$$\mathbf{s}_\pm(t)\mathbf{s}_\mp^{(-)\dagger}(t)=\mathbf{s}_\pm^{(-)\dagger}(t)\mathbf{s}_\mp(t)=0, \quad (27)$$

and have, for $\gamma>2\kappa_g$, the same commutation relations of the equivalent uncoupled operators with full inversion $\langle\sigma_3\rangle=1$,

$$\mathbf{s}_\pm^{(-)\dagger}(t)\mathbf{s}_\pm^{(-)}(t')=\delta(t-t'), \quad (28)$$

$$\mathbf{s}_{\pm}^{(-)}(t)\mathbf{s}_{\pm}^{(-)\dagger}(t)=\mathbf{s}_{\pm}^{(-)}(t)\mathbf{s}_{\pm}^{(-)}(t)=\mathbf{s}_{\pm}^{(-)\dagger}(t)\mathbf{s}_{\pm}^{(-)\dagger}(t)=0. \quad (29)$$

The commutation relations of $\mathbf{c}_{\pm}(t)$ are $[\mathbf{c}_{\pm}(t),\mathbf{c}_{\pm}^{\dagger}(t)]=1$.

Inserting the new noise operators and using $2g^2n_0/\Gamma=\gamma-2\kappa_g$, Eqs. (20) and (21) become

$$\frac{\mathrm{d}\mathbf{c}_+(t)}{\mathrm{d}t} = \frac{1}{2}\left(\gamma-2\kappa_g\right)\frac{a_0+b_0}{\sqrt{2}}\frac{\delta\mathbf{n}(t)}{n_0} +\sqrt{\gamma}\left[-i\mathbf{s}_+^{(-)}(t)+\mathbf{s}_+(t)\right], \quad (30)$$

$$\frac{\mathrm{d}\mathbf{c}_-(t)}{\mathrm{d}t} = -2\kappa_g\mathbf{c}_-(t)-i\sqrt{\gamma-4\kappa_g}\,\mathbf{s}_-^{(-)}(t)+\sqrt{\gamma}\,\mathbf{s}_-(t). \quad (31)$$

Equation (31) shows the presence of a restoring force for the difference of the amplitudes of the two modes proportional to the coupling coefficient κ_g. This is a manifestation of the locking between the two modes.

To get to a closed form of Eq. (30), let us expand Eq. (10) to first order, obtaining the equation for the fluctuations of the carriers as

$$\frac{\mathrm{d}\delta\mathbf{n}(t)}{\mathrm{d}t} = -\frac{\delta\mathbf{n}(t)}{\tau}-\frac{4g^2}{\Gamma}(a_0^2+b_0^2)\delta\mathbf{n}(t) -\frac{8g^2}{\Gamma}n_0\left[a_0\delta\mathbf{a}_1(t)+b_0\delta\mathbf{b}_1(t)\right] -4g\left(\frac{2N}{\Gamma}\right)^{1/2}\left[a_0\mathbf{s}_{a,2}^{(-)}(t)+b_0\mathbf{s}_{b,2}^{(-)}(t)\right]. \quad (32)$$

This equation however does not include the effect of the depletion of the carriers that generate the gain grating, whose dynamics is described by Eq. (11). Instead of constructing a model to describe the formation of the gain grating and its interaction with the two counterpropagating modes, which would necessitate making assumptions about the complex physics of the laser that are challenging to evaluate, like for instance the carrier diffusion attenuating the grating amplitude, we choose to introduce a term that account for this effect without a formal derivation, relying on the principle that each photon is generated through the decay of a single carrier. To this aim, we notice that the coupling induced by the gain grating produces a rate of photon production

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{a}^\dagger\mathbf{a}+\mathbf{b}^\dagger\mathbf{b}\right)_{\text{coupling}}=2\kappa_g\left(\mathbf{b}^\dagger\mathbf{a}+\mathbf{a}^\dagger\mathbf{b}\right), \quad (33)$$

and therefore the change of carrier number caused by fluctuations of $\mathbf{a}$ and $\mathbf{b}$ is

$$\left(\frac{\mathrm{d}\delta\mathbf{n}}{\mathrm{d}t}\right)_{\text{coupling}}=-4\kappa_g\left[a_0\left(\delta\mathbf{b}+\delta\mathbf{b}^\dagger\right)+b_0\left(\delta\mathbf{a}+\delta\mathbf{a}^\dagger\right)\right], \quad (34)$$

where we used that a_0 and b_0 are real. Equation (32) supplemented with the coupling term (34) becomes

$$\frac{\mathrm{d}\delta\mathbf{n}(t)}{\mathrm{d}t} = -\frac{\delta\mathbf{n}(t)}{\tau}-\frac{4g^2}{\Gamma}(a_0^2+b_0^2)\delta\mathbf{n}(t) -\frac{8g^2}{\Gamma}n_0\left[a_0\delta\mathbf{a}_1(t)+b_0\delta\mathbf{b}_1(t)\right] -8\kappa_g\left[a_0\delta\mathbf{b}_1(t)+b_0\delta\mathbf{a}_1(t)\right] -4g\left(\frac{2N}{\Gamma}\right)^{1/2}\left[a_0\mathbf{s}_{a,2}^{(-)}(t)+b_0\mathbf{s}_{b,2}^{(-)}(t)\right]. \quad (35)$$

Using now once again our assumption of full inversion $n_0=N$, we can replace $2g^2N/\Gamma=\gamma-2\kappa_g$ and therefore

$$\frac{\mathrm{d}\delta\mathbf{n}(t)}{\mathrm{d}t} = -\frac{\delta\mathbf{n}(t)}{\tau}-2(\gamma-2\kappa_g)(a_0^2+b_0^2)\frac{\delta\mathbf{n}(t)}{n_0} -4(\gamma-2\kappa_g)\left[a_0\delta\mathbf{a}_1(t)+b_0\delta\mathbf{b}_1(t)\right] -8\kappa_g\left[a_0\delta\mathbf{b}_1(t)+b_0\delta\mathbf{a}_1(t)\right] -4\sqrt{\gamma-2\kappa_g}\left[a_0\mathbf{s}_{a,2}^{(-)}(t)+b_0\mathbf{s}_{b,2}^{(-)}(t)\right]. \quad (36)$$

Assuming strong saturation and neglecting spontaneous emission compared to stimulated emission, we may assume that the carriers adiabatically follow the field fluctuations, so that we obtain

$$\frac{\delta\mathbf{n}(t)}{n_0} = -\frac{2}{a_0^2+b_0^2}\left[a_0\delta\mathbf{a}_1(t)+b_0\delta\mathbf{b}_1(t)\right] -\frac{4\kappa_g}{(\gamma-2\kappa_g)(a_0^2+b_0^2)}\left[b_0\delta\mathbf{a}_1(t)+a_0\delta\mathbf{b}_1(t)\right] -\frac{2}{\sqrt{\gamma-2\kappa_g}(a_0^2+b_0^2)}\left[a_0\mathbf{s}_{a,2}^{(-)}(t)+b_0\mathbf{s}_{b,2}^{(-)}(t)\right]. \quad (37)$$

Entering this expression into Eq. (30) yields

$$\frac{\mathrm{d}\mathbf{c}_+(t)}{\mathrm{d}t} = -\frac{(\gamma-2\kappa_g)(a_0+b_0)}{\sqrt{2}(a_0^2+b_0^2)}\left[a_0\delta\mathbf{a}_1(t)+b_0\delta\mathbf{b}_1(t)\right] -\frac{2\kappa_g(a_0+b_0)}{\sqrt{2}(a_0^2+b_0^2)}\left[b_0\delta\mathbf{a}_1(t)+a_0\delta\mathbf{b}_1(t)\right] -\frac{\sqrt{(\gamma-2\kappa_g)}(a_0+b_0)}{\sqrt{2}(a_0^2+b_0^2)}\left[a_0\mathbf{s}_{a,2}^{(-)}(t)+b_0\mathbf{s}_{b,2}^{(-)}(t)\right] -i\sqrt{\gamma}\mathbf{s}_+^{(-)}(t)+\sqrt{\gamma}\,\mathbf{s}_+(t). \quad (38)$$

Using the steady state condition $a_0=b_0$ we obtain

$$\frac{\mathrm{d}\mathbf{c}_+(t)}{\mathrm{d}t} = -\gamma\mathbf{c}_{+,1}(t) -\sqrt{\gamma}\left[\mathbf{s}_{+,2}^{(-)}(t)+i\mathbf{s}_+^{(-)}(t)-\mathbf{s}_+(t)\right]. \quad (39)$$

Equation (39) shows that the evolution equation of the supermode sum of the two counterpropagating modes is equal to that of an uncoupled single mode with cavity loss (hence, cavity Q factor) equal to the cavity loss of the two coupled laser modes.

Utilizing Eqs. (31) and (39) to find $\mathrm{d}[\mathbf{c}_{\pm}(t),\mathbf{c}_{\pm}^{\dagger}(t)]$ and after using the averages in Eqs. (28) and (29), it is easy to verify that $\mathrm{d}[\mathbf{c}_{\pm}(t),\mathbf{c}_{\pm}^{\dagger}(t)]=0$ hence that Eqs. (31) and (39) preserve the commutation rules. The preservation of the commutation rules of the two independent and orthogonal supermodes insures that modes obtained by unitary transformations, including the original counterpropagating modes, also preserve the same commutation rules hence do not violate any of the minimum uncertainties relations related to those commutation rules.

Defining the two quadratures for the operators associated to the supermodes as $\mathbf{c}_1=(\delta\mathbf{c}+\delta\mathbf{c}^\dagger)/2$ and $\mathbf{c}_2=(\delta\mathbf{c}-\delta\mathbf{c}^\dagger)/(2i)$, we obtain from Eqs. (31) and (39)

$$\frac{\mathrm{d}\mathbf{c}_{-,1}(t)}{\mathrm{d}t} = -2\kappa_g\mathbf{c}_{-,1}(t)+\sqrt{\gamma-4\kappa_g}\mathbf{s}_{-,2}^{(-)}(t)+\sqrt{\gamma}\mathbf{s}_{-,1}(t), \quad (40)$$

$$\frac{\mathrm{d}\mathbf{c}_{-,2}(t)}{\mathrm{d}t} = -2\kappa_g\mathbf{c}_{-,2}(t)-\sqrt{\gamma-4\kappa_g}\mathbf{s}_{-,1}^{(-)}(t)+\sqrt{\gamma}\mathbf{s}_{-,2}(t), \quad (41)$$

$$\frac{d\mathbf{c}_{+,1}(t)}{dt} = -\gamma\mathbf{c}_{+,1}(t) + \sqrt{\gamma}\,\mathbf{s}_{+,1}(t), \tag{42}$$

$$\frac{d\mathbf{c}_{+,2}(t)}{dt} = \sqrt{\gamma}\left[\mathbf{s}_{+,2}(t) - \mathbf{s}_{+,1}^{(-)}(t)\right]. \tag{43}$$

Defining the Fourier transform as

$$\mathbf{c}(\omega) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\exp(-i\omega t)\mathbf{c}(t), \tag{44}$$

we may readily solve Eqs. (41), (40), (42) and (43) in the Fourier domain as

$$\mathbf{c}_{-,1}(\omega) = \frac{\sqrt{\gamma}\mathbf{s}_{-,1}(\omega) + \sqrt{\gamma - 4\kappa_g}\mathbf{s}_{-,2}^{(-)}(\omega)}{-i\omega + 2\kappa_g}, \tag{45}$$

$$\mathbf{c}_{-,2}(\omega) = \frac{\sqrt{\gamma}\mathbf{s}_{-,2}(\omega) - \sqrt{\gamma - 4\kappa_g}\mathbf{s}_{-,1}^{(-)}(\omega)}{-i\omega + 2\kappa_g}, \tag{46}$$

$$\mathbf{c}_{+,1}(\omega) = \frac{\sqrt{\gamma}}{-i\omega + \gamma}\,\mathbf{s}_{+,1}(\omega), \tag{47}$$

$$\mathbf{c}_{+,2}(\omega) = -\frac{\sqrt{\gamma}}{i\omega}\left[\mathbf{s}_{+,2}(\omega) - \mathbf{s}_{+,1}^{(-)}(\omega)\right]. \tag{48}$$

Let us first analyze the fluctuations of the phases of the intracavity modes, which are the quantities analyzed in the main text. Being the output radiation in a vacuum state, we have

$$\langle \mathbf{s}_{\pm,i}(t)\mathbf{s}_{\pm,i}(t')\rangle = \frac{1}{4}\delta(t-t'), \quad i = 1,2. \tag{49}$$

In addition, Eqs. (26)–(29) imply

$$\langle \mathbf{s}_{\pm,i}^{(-)}(t)\mathbf{s}_{\pm,i}^{(-)}(t')\rangle = \frac{1}{4}\delta(t-t'), \quad i = 1,2, \tag{50}$$

so that Eq. (45) yields

$$\langle \mathbf{c}_{-,1}(\omega)\mathbf{c}_{-,1}^{\dagger}(\omega')\rangle = \frac{\gamma - 4\kappa_g}{2\,(\omega^2 + 4\kappa_g^2)}2\pi\delta(\omega-\omega'). \tag{51}$$

The fluctuations of the difference of the phases of the emitted radiation are the difference between the fluctuations of the in-quadrature components of the intracavity mode amplitude divided by the average mode amplitude $a_0 = b_0 = \sqrt{P/(\gamma\hbar\omega_0)}$ where P is the average output power per mode, that is $\Delta\boldsymbol{\varphi} = \sqrt{\gamma\hbar\omega/P}\,[\sqrt{2}\mathbf{c}_{-,1}(\omega)]$ so that the spectrum of the fluctuations of the phase difference is

$$\langle \Delta\boldsymbol{\varphi}(\omega)\Delta\boldsymbol{\varphi}^{\dagger}(\omega')\rangle = \frac{\hbar\omega_0\gamma(\gamma-4\kappa_g)}{P(\omega^2+4\kappa_g^2)}2\pi\delta(\omega-\omega'), \tag{52}$$

which is the result given in the main text [1], with a small correction of the order of κ_g/γ arising from the fact that the analysis presented here accounts for the non-hermiticity of the mode coupling. Notice that the term $2\pi\delta(\omega-\omega')$ appearing here and in all other spectra is removed by integration over frequency $f' = \omega'/(2\pi)$. This procedure returns for any given spectrum $\langle \mathbf{x}(\omega)\mathbf{x}(\omega')\rangle$ the Fourier transform of $\langle \mathbf{x}(t)\mathbf{x}(0)\rangle$, that is, if $\mathbf{x}(t)$ is a stationary process, the power spectrum of $\mathbf{x}(t)$.

The amplitude of the emitted radiation is given by [2]

$$\mathbf{r}_a(t) = -\mathbf{s}_a(t) + \sqrt{\gamma}\,\mathbf{a}(t), \tag{53}$$

$$\mathbf{r}_b(t) = -\mathbf{s}_b(t) + \sqrt{\gamma}\,\mathbf{b}(t), \tag{54}$$

so that defining

$$\mathbf{r}_{\pm}(t) = \frac{1}{\sqrt{2}}\left[\mathbf{r}_a(t) \pm \mathbf{r}_b(t)\right], \tag{55}$$

we obtain for the fluctuations

$$\delta\mathbf{r}_{+}(t) = -\mathbf{s}_{+}(t) + \sqrt{\gamma}\,\mathbf{c}_{+}(t), \tag{56}$$

$$\delta\mathbf{r}_{-}(t) = -\mathbf{s}_{-}(t) + \sqrt{\gamma}\,\mathbf{c}_{-}(t). \tag{57}$$

The quadratures of the emitted radiation are readily obtained by inserting Eqs. (45)–(48) into Eqs. (56) and (57), so as to obtain

$$\delta\mathbf{r}_{-,1}(\omega) = \frac{\gamma\,\mathbf{s}_{-,1}(\omega) + \sqrt{\gamma(\gamma-4\kappa_g)}\,\mathbf{s}_{-,2}^{(-)}(\omega)}{-i\omega + 2\kappa_g} - \mathbf{s}_{-,1}(\omega), \tag{58}$$

$$\delta\mathbf{r}_{-,2}(\omega) = \frac{\gamma\,\mathbf{s}_{-,2}(\omega) - \sqrt{\gamma(\gamma-4\kappa_g)}\,\mathbf{s}_{-,1}^{(-)}(\omega)}{-i\omega + 2\kappa_g} - \mathbf{s}_{-,2}(\omega), \tag{59}$$

$$\delta\mathbf{r}_{+,1}(\omega) = \frac{i\omega}{-i\omega+\gamma}\,\mathbf{s}_{+,1}(\omega), \tag{60}$$

$$\delta\mathbf{r}_{+,2}(\omega) = -\frac{\gamma}{i\omega}\left[\mathbf{s}_{+,2}(\omega) - \mathbf{s}_{+,1}^{(-)}(\omega)\right] - \mathbf{s}_{+,2}(\omega). \tag{61}$$

As a consistency check, using that

$$[\mathbf{s}_{\pm,1}(\omega),\mathbf{s}_{\pm,1}^{\dagger}(\omega')] = [\mathbf{s}_{\pm,2}(\omega),\mathbf{s}_{\pm,2}^{\dagger}(\omega')] = 0, \tag{62}$$

$$[\mathbf{s}_{\pm,1}(\omega),\mathbf{s}_{\pm,2}^{\dagger}(\omega')] = \frac{1}{4}\,[2\pi\delta(\omega-\omega')], \tag{63}$$

$$[\mathbf{s}_{\pm,1}^{(-)},\mathbf{s}_{\pm,1}^{(-)\dagger}(\omega')] = [\mathbf{s}_{\pm,2}^{(-)},\mathbf{s}_{\pm,2}^{(-)\dagger}(\omega')] = 0, \tag{64}$$

$$[\mathbf{s}_{\pm,1}^{(-)},\mathbf{s}_{\pm,2}^{(-)\dagger}(\omega')] = \frac{1}{4}\,[2\pi\delta(\omega-\omega')], \tag{65}$$

one may show right away that

$$[\delta\mathbf{r}_{\pm,1}(\omega),\delta\mathbf{r}_{\pm,1}^{\dagger}(\omega)] = [\delta\mathbf{r}_{\pm,2}(\omega),\delta\mathbf{r}_{\pm,2}^{\dagger}(\omega)] = 0, \tag{66}$$

$$[\delta\mathbf{r}_{\pm,1}(\omega),\delta\mathbf{r}_{\pm,2}^{\dagger}(\omega')] = \frac{1}{4}\,[2\pi\delta(\omega-\omega')], \tag{67}$$

so that the above equations correctly describe supermodes that are independent waves with bosonic commutation rules.

Using once again the correlation functions of the noise terms (62)–(65) we obtain

$$\langle \delta\mathbf{r}_{-,2}(\omega)\delta\mathbf{r}_{-,2}^{\dagger}(\omega')\rangle = \frac{1}{4}\left[\frac{2\gamma(\gamma-4\kappa_g)}{\omega^2+4\kappa_g^2}+1\right]2\pi\delta(\omega-\omega'), \tag{68}$$

$$\langle \delta\mathbf{r}_{+,2}(\omega)\delta\mathbf{r}_{+,2}^{\dagger}(\omega')\rangle = \frac{1}{4}\left(\frac{2\gamma^2}{\omega^2}+1\right)2\pi\delta(\omega-\omega'). \tag{69}$$

The fluctuations of the difference of the phases of the emitted radiation $\Delta\boldsymbol{\varphi}_{\text{out}}(\omega)$ is the difference of the fluctuations of the in-quadrature components divided by the amplitude of the output per mode in photon units $\sqrt{P/(\hbar\omega_0)}$, that is $\Delta\boldsymbol{\varphi}_{\text{out}} = \sqrt{\hbar\omega_0/P}\,[\sqrt{2}\,\delta\mathbf{r}_{-,2}]$ so that the spectrum of the phase difference is $\langle \Delta\boldsymbol{\varphi}_{\text{out}}(\omega)\Delta\boldsymbol{\varphi}_{\text{out}}^{\dagger}(\omega')\rangle = 2(\hbar\omega_0/P)\langle \delta\mathbf{r}_{-,2}(\omega)\delta\mathbf{r}_{-,2}^{\dagger}(\omega')\rangle$, that is

$$\langle \Delta\boldsymbol{\varphi}_{\text{out}}(\omega)\Delta\boldsymbol{\varphi}_{\text{out}}^{\dagger}(\omega')\rangle = \frac{\hbar\omega_0}{2P}\left[\frac{2\gamma(\gamma-4\kappa_g)}{\omega^2+4\kappa_g^2}+1\right]2\pi\delta(\omega-\omega'). \tag{70}$$

The spectra of the phase fluctuations of the beat between the intracavity fields Eq. (52) and that of the output waves Eq. (70) differ primarily in the region $\omega \gg \gamma$, where the spectrum of the emitted radiation follows the phase fluctuations of the vacuum reflected from the cavity and the variance of the phase fluctuations of the beat are the sum of the variances of the phase fluctuations of two coherent states. In semiclassical terms, the is the manifestation of the shot-noise of the detection.

If we define $\boldsymbol{\varphi}_a = \delta\mathbf{r}_{a,2}/a_0$ and $\boldsymbol{\varphi}_b = \delta\mathbf{r}_{b,2}/b_0$ as the deviation of the phases of the emitted radiation from the steady state and use that $a_0 = b_0$ we obtain $\langle\boldsymbol{\varphi}_a(\omega)\boldsymbol{\varphi}_a^\dagger(\omega')\rangle = \langle\boldsymbol{\varphi}_b(\omega)\boldsymbol{\varphi}_b^\dagger(\omega')\rangle$ with

$$\langle\boldsymbol{\varphi}_a(\omega)\boldsymbol{\varphi}_a^\dagger(\omega')\rangle = \frac{\hbar\omega_0}{2P}\left[\frac{\gamma^2}{\omega^2} + \frac{\gamma(\gamma-4\kappa_g)}{\omega^2+4\kappa_g^2} + 1\right]2\pi\delta(\omega-\omega'), \tag{71}$$

$$\langle\boldsymbol{\varphi}_a(\omega)\boldsymbol{\varphi}_b^\dagger(\omega')\rangle = \frac{\hbar\omega_0}{2P}\left[\frac{\gamma^2}{\omega^2} - \frac{\gamma(\gamma-4\kappa_g)}{\omega^2+4\kappa_g^2}\right]2\pi\delta(\omega-\omega'). \tag{72}$$

Three spectral regions are present. In the locking region $|\omega| \ll 2\kappa_g$, the phase fluctuations of the two modes are fully correlated with $\langle\boldsymbol{\varphi}_a(\omega)\boldsymbol{\varphi}_a^\dagger(\omega')\rangle \simeq \langle\boldsymbol{\varphi}_a(\omega)\boldsymbol{\varphi}_b^\dagger(\omega')\rangle$. In this spectral region, the variance of the phase fluctuations of each mode is one half of the free-running phase fluctuations of independent modes with the same output power and, similarly to the mode-locking case [3, 4], equal to the phase fluctuations of a single mode whose power is equal to the total power emitted by the laser. For $2\kappa_g < |\omega| < \gamma$, the two modes are unlocked and the phase fluctuations are the same of two free running modes of a laser which follow the Schawlow–Townes formula. For $|\omega| \gg \gamma$ the phase fluctuations are those of a radiation in a coherent state, as expected because they are the shot-noise fluctuations of the vacuum field reflected by the laser cavity outside its frequency cutoff. The expressions of the frequency noise spectra of the mode beat and of the two counterpropagating mode can be readily obtained multiplying by ω^2 the corresponding phase noise spectra.

Let us now analyze the amplitude fluctuations. We have

$$\langle\delta\mathbf{r}_{+,1}(\omega)\delta\mathbf{r}_{+,1}^\dagger(\omega')\rangle = \frac{\omega^2}{4(\omega^2+\gamma^2)}2\pi\delta(\omega-\omega'), \tag{73}$$

$$\langle\delta\mathbf{r}_{-,1}(\omega)\delta\mathbf{r}_{-,1}^\dagger(\omega')\rangle = \frac{1}{4}\left[\frac{2\gamma(\gamma-4\kappa_g)}{\omega^2+4\kappa_g^2} + 1\right]2\pi\delta(\omega-\omega'), \tag{74}$$

and consequently $\langle\delta\mathbf{r}_{a,1}(\omega)\delta\mathbf{r}_{a,1}^\dagger(\omega')\rangle = \langle\delta\mathbf{r}_{b,1}(\omega)\delta\mathbf{r}_{b,1}^\dagger(\omega')\rangle$ and

$$\langle\delta\mathbf{r}_{a,1}(\omega)\delta\mathbf{r}_{a,1}^\dagger(\omega')\rangle = \frac{1}{8}\left[\frac{2\gamma(\gamma-4\kappa_g)}{\omega^2+4\kappa_g^2} - \frac{\gamma^2}{\omega^2+\gamma^2} + 2\right] 2\pi\delta(\omega-\omega'), \tag{75}$$

$$\langle\delta\mathbf{r}_{a,1}(\omega)\delta\mathbf{r}_{b,1}^\dagger(\omega')\rangle = \frac{1}{8}\left[\frac{2\gamma(\gamma-4\kappa_g)}{\omega^2+4\kappa_g^2} + \frac{\gamma^2}{\omega^2+\gamma^2}\right] 2\pi\delta(\omega-\omega'). \tag{76}$$

For $|\omega| \ll \gamma$, similarly to the amplitude squeezing of the radiation emitted from the laser when pump fluctuations are suppressed [5, 6], the fluctuations of the sum of the amplitudes of the two modes (the fluctuations of the amplitude of the supermode) are below the quantum noise limit (sub-Poissonian) and zero at $\omega = 0$. The amplitudes of the two modes are locked, with a finite variance, for $\omega \ll 2\kappa_g$, and their fluctuations are correlated. For $2\kappa_g < |\omega| < \gamma$, the two modes are unlocked and their amplitudes experience partition noise, while the fluctuations of the sum of their amplitude are still suppressed. For $|\omega| \gg \gamma$, above the cutoff introduced by the laser cavity, the amplitude fluctuations are those of a radiation in a coherent state, because they are those of the vacuum state reflected from the cavity.

It is interesting to discuss the autocorrelation function of the phase fluctuations of the beat of the output fields. Let us suppose that the measurement is performed with a finite bandwidth B, by assuming an ideal square low-pass filter of bandwidth B with a flat unit response for $|\omega|/(2\pi) \leq B/2$ and zero outside. This situation describes, for instance, an ideal measurement with a sampling period $T_{\text{sampling}} = 1/B$. Then, integration over ω' in the two-dimensional inverse Fourier transform of Eq. (70) produces a result that depends only on $T = t' - t$. For $B \gg 2\kappa_g$, we may neglect the effect of frequency filtering on the first term by approximating, in the convolution with this term, the sinc generated by the spectral filtering with a Dirac delta function. After doing so, another inverse Fourier transformation with respect to ω produces

$$\langle\Delta\boldsymbol{\varphi}_{\text{out}}(t+T)\Delta\boldsymbol{\varphi}_{\text{out}}(t)\rangle = \frac{\hbar\omega_0}{2P}\left[\frac{\gamma(\gamma-4\kappa_g)}{2\kappa_g}\exp\left(-2\kappa_g T\right) + B\,\frac{\sin(\pi B T)}{\pi B T}\right]. \tag{77}$$

The sinc appearing in this expression represents the effect of the filtered vacuum noise reflected from the laser cavity or, in a semiclassical language, the shot-noise of the detection. Using now Eq. (77) in the expression for the Allan variance in terms of the time autocorrelation function

$$\sigma_T^2 = \frac{1}{T^2}\left[3\left\langle\Delta\boldsymbol{\varphi}_{\text{out}}(t)^2\right\rangle - 4\left\langle\Delta\boldsymbol{\varphi}_{\text{out}}(t+T)\Delta\boldsymbol{\varphi}_{\text{out}}(t)\right\rangle + \left\langle\Delta\boldsymbol{\varphi}_{\text{out}}(t+2T)\Delta\boldsymbol{\varphi}_{\text{out}}(t)\right\rangle\right], \tag{78}$$

and assuming that T is a multiple of the sampling period if the filtering is the effect of sampling, or in general that $T \gg 1/B$, we obtain

$$\sigma_T^2 = \frac{\hbar\omega_0}{2PT^2}\left[\frac{\gamma(\gamma-4\kappa_g)}{2\kappa_g}\left[3 - 4\exp\left(-2\kappa_g T\right) + \exp\left(-4\kappa_g T\right)\right] + 3B\right]. \tag{79}$$

For $\kappa_g T \gg 1$ we have

$$\sigma_T^2 = \frac{3\hbar\omega_0}{2PT^2}\left[\frac{\gamma(\gamma-4\kappa_g)}{2\kappa_g} + B\right]. \tag{80}$$

The term proportional to B is the effect for long T of the high frequency portion of the vacuum noise fluctuations reflected by the laser and coherently added to the emitted light beams. This contribution is negligible for $B \ll \gamma(\gamma-4\kappa_g)/(2\kappa_g)$. Equations (79) and (80) are the same expressions given in the main text, with the addition of the shot-noise contribution and with a small correction arising from the interference between the emitted radiation and the vacuum field reflected from the cavity.

2. LOCKING BY PASSIVE REFLECTIONS

The mode coupling mechanism in a laser gyro bears similarities to the two different coupling mechanisms that operate in distributed feedback (DFB) lasers. In these lasers, photon confinement is achieved through index coupling, gain coupling, or a combination of both [7]. A design based on gain coupling was initially suggested during the early development of semiconductor DFB lasers as a means to prevent mode degeneracy at the edge of the stop band, a problem commonly encountered in conventional index coupled DFB lasers. However, index coupling has emerged as the preferred solution for commercial DFB lasers currently in use. This is primarily due to the successful resolution of mode degeneracy in the so-called quarter wave shifted DFB lasers, employing a quarter wave shift of the index modulation at the center of the grating, which generates a narrow-band mode at the center of the stop band. Gain coupling involves creating photon confinement by periodically modulating the gain. This physics is akin to the operation of a laser gyro with a gain grating, the main difference lying in the slow temporal modulation of the grating, resulting in the coupling of two non-degenerate frequencies. Since the coupling mechanism involves material gain, the quantum mechanical noise operators affected by the coupling are those relevant to the material polarization. Index coupling, on the other hand, achieves photon confinement by backscattering from an index grating, which establish an effective optical cavity by constructing interference of multiple reflections. This scenario is similar to the case where degenerate coupling is induced by passive reflections in laser gyros. Ideally, no gain or scattering loss is involved in the process.

In a laser gyro, non-degenerate locking may also result from index coupling when it is caused by a time-dependent refractive index modulation induced by the mode beat. The mechanism for the generation of the index modulation may be either Kerr effect within a dielectric slab inserted in the laser cavity, or the attendant index modulation associated to gain modulation within the gain medium itself. In the latter case, the index modulation arises from gain-index coupling governed by the Kramers-Kronig relations in a gain medium with asymmetric spectral profile, similar to the mechanism underlying the Henry's linewidth enhancement factor in semiconductor lasers [8]. Equivalent to index coupling is the case in which locking occurs because of undesired passive reflections in the laser cavity, with the difference that in this case locking occurs at degenerate frequency and it is of course an undesired process.

A quantum mechanical analysis of the cases in which locking occurs because of passive reflections (either time-dependent or time-independent) in the laser cavity is more straightforward than that in which coupling is induced by gain grating, and must account for two effects. One is the reduction of the cavity loss, which becomes $\gamma - 2\kappa_m$ and produces a reduction of the coupling between the internal laser modes and the vacuum fluctuations impinging upon the laser cavity. The second is the coupling between the vacuum noise of the two counterpropagating modes caused by the internal reflections. This coupling, similarly to the coupling of the polarization noise in the case of the coupling by the gain grating, correlates the noise sources associated to the coupling with the outside vacuum fluctuations. The quantum noise operators associated to material polarization remain uncorrelated like in the case of absence of coupling. An outline of the derivation of the main equations that describe the fluctuations of the output fields is reported below. The mode amplitudes $\mathbf{a}(t)$ and $\mathbf{b}(t)$ should be intended as centered at $\omega_0 + \Omega_0/2$ and $\omega_0 - \Omega_0/2$ when locking occurs because of a time-dependent index grating, or both around ω_0 when it occurs because of undesired static reflections.

Let us start to note that, to preserve the commutation relations $[\mathbf{a}, \mathbf{b}^\dagger] = 0$, we require that $[\mathbf{s}_a(t), \mathbf{s}_b^\dagger(t')] = -(2\kappa_m/\gamma)\delta(t-t')$, equivalent to $\mathbf{s}_a(t)\mathbf{s}_b^\dagger(t') = -(2\kappa_m/\gamma)\delta(t-t')$ and $\mathbf{s}_b^\dagger(t')\mathbf{s}_a(t) = 0$ being the modes $\mathbf{a}$ and $\mathbf{b}$ in a vacuum state. Equations (15), (16), (20) and (21) stay the same with κ_g replaced by κ_m. If we then define, similarly to Eqs. (22)–(25), new independent noise terms

$$\mathbf{s}_+(t) = \sqrt{\frac{\gamma}{\gamma - 2\kappa_m}}\,\frac{\mathbf{s}_a(t) + \mathbf{s}_b(t)}{\sqrt{2}}, \tag{81}$$

$$\mathbf{s}_-(t) = \sqrt{\frac{\gamma}{\gamma + 2\kappa_m}}\,\frac{\mathbf{s}_a(t) - \mathbf{s}_b(t)}{\sqrt{2}}, \tag{82}$$

$$\mathbf{s}_+^{(-)}(t) = \frac{\mathbf{s}_a^{(-)}(t) + \mathbf{s}_b^{(-)}(t)}{\sqrt{2}}, \tag{83}$$

$$\mathbf{s}_-^{(-)}(t) = \frac{\mathbf{s}_a^{(-)}(t) - \mathbf{s}_b^{(-)}(t)}{\sqrt{2}}, \tag{84}$$

we obtain from Eq. (21)

$$\frac{\mathrm{d}\mathbf{c}_-(t)}{\mathrm{d}t} = -2\kappa_m\mathbf{c}_-(t) - i\sqrt{\gamma - 2\kappa_m}\,\mathbf{s}_-^{(-)}(t) + \sqrt{\gamma + 2\kappa_m}\,\mathbf{s}_-(t). \tag{85}$$

Interestingly, the equation for the fluctuations of the carrier in this case follows rigorously Eq. (32), which of course does not necessitate any additional term to account for the effect of the carrier depletion induced by gain grating. This is because the apparent gain induced by the index grating is the effect of the reduction of the cavity loss caused by the extra reflection from the grating which adds to that of the cavity mirrors, and is balanced by a modification of the noise produced by the coupling with the outside vacuum fluctuations. In the gain modulation case, the extra reflection is caused by gain modulation, and its effect is accounted for by a reduction of the noise due to the material polarization. The carrier fluctuations are then given by Eq. (37) without the correction term

$$\begin{aligned}\frac{\delta\mathbf{n}(t)}{n_0} &= -\frac{2}{a_0^2 + b_0^2}\left[a_0\delta\mathbf{a}_1(t) + b_0\delta\mathbf{b}_1(t)\right] \\ &\quad - \frac{2}{\sqrt{\gamma - 2\kappa_m}(a_0^2 + b_0^2)}\left[a_0\mathbf{s}_{a,2}^{(-)}(t) + b_0\mathbf{s}_{b,2}^{(-)}(t)\right],\end{aligned} \tag{86}$$

so that entering this expression into Eq. (20) with $a_0 = b_0$ and $2g^2 n_0/\Gamma = \gamma - 2\kappa_m$ gives

$$\begin{aligned}\frac{\mathrm{d}\mathbf{c}_+(t)}{\mathrm{d}t} &= -(\gamma - 2\kappa_m)\mathbf{c}_{+,1}(t) \\ &\quad - \sqrt{\gamma - 2\kappa_m}\left[\mathbf{s}_{+,2}^{(-)}(t) + i\mathbf{s}_+^{(-)}(t) - \mathbf{s}_+(t)\right].\end{aligned} \tag{87}$$

It is straightforward to check, with the same procedure followed in the non-degenerate case, that the above equations preserve the commutation rules. The output fields of the two modes $\mathbf{a}(t)$ and $\mathbf{b}(t)$ do not obey in this case Eqs. (53) and (54), because $\mathbf{s}_a(t)$ and $\mathbf{s}_b(t)$ are not independent bosonic waves. On the other hand, the noise terms $\mathbf{s}_+(t)$ and $\mathbf{s}_-(t)$ associated to the two supermodes $\mathbf{c}_+(t)$ and $\mathbf{c}_-(t)$, are independent bosonic waves, so that two relations similar to Eqs. (53) and (54) can be established for the two supermodes. The coupling with the outside vacuum field of the two supermodes is quantified by the coefficients of $\mathbf{s}_+(t)$ and $\mathbf{s}_-(t)$ in the dynamical equations for $\mathbf{c}_+(t)$ and $\mathbf{c}_-(t)$, Eqs. (87) and (85). Being $\sqrt{\gamma - 2\kappa_m}$ the coefficient of $\mathbf{s}_+(t)$ in Eq. (87), and $\sqrt{\gamma + 2\kappa_m}$ the coefficient of $\mathbf{s}_-(t)$ in Eq. (85), power conservation requires that these equations are

$$\mathbf{r}_- = -\mathbf{s}_- + \sqrt{\gamma + 2\kappa_m}\,\mathbf{c}_-, \tag{88}$$

$$\mathbf{r}_+ = -\mathbf{s}_+ + \sqrt{\gamma - 2\kappa_m}\,\mathbf{c}_+. \tag{89}$$

It may be verified that these equalities ensure that the commutation relations of the reflected supermode fields are given by $[\mathbf{r}_\pm(t), \mathbf{r}_\pm^\dagger(t')] = \delta(t-t')/4$ at any time. Equations (87) and (89) show that the output field $\mathbf{r}_+$ of the supermode that represents the sum of two counterpropagating modes is the same of that of an uncoupled single mode with effective cavity loss $\gamma - 2\kappa_m$. Being $a_0 = b_0$, the average reflected power on the supermode difference is zero, that is $\langle \mathbf{r}_-^\dagger \mathbf{r}_- \rangle = 0$, and the average power of the two modes is $P = \hbar\omega_0 \langle \mathbf{r}_+^\dagger \mathbf{r}_+ \rangle = \hbar\omega_0(\gamma - 2\kappa_m)a_0^2$.

The spectra and correlation functions have limited practical relevance when modes lock with degenerate frequencies, but they are relevant when locking occurs with non-degenerate frequencies due to a time-dependent refractive index modulation. In this case, the expressions for the spectra and correlation functions are very similar to the non-degenerate case, and they can be obtained straightforwardly by following the same steps used to derive them in the case of gain modulation. The main results are

$$\langle \delta\mathbf{r}_{-,2}(\omega)\delta\mathbf{r}_{-,2}^\dagger(\omega') \rangle = \frac{1}{4}\left[\frac{2(\gamma^2 - 4\kappa_m^2)}{\omega^2 + 4\kappa_m^2} + 1\right] 2\pi\delta(\omega - \omega'), \tag{90}$$

$$\langle \delta\mathbf{r}_{+,2}(\omega)\delta\mathbf{r}_{+,2}^\dagger(\omega') \rangle = \frac{1}{4}\left[\frac{2(\gamma - 2\kappa_m)^2}{\omega^2} + 1\right] 2\pi\delta(\omega - \omega'). \tag{91}$$

The spectrum of the phase difference is $\langle \Delta\boldsymbol{\varphi}_{\text{out}}(\omega)\Delta\boldsymbol{\varphi}_{\text{out}}^\dagger(\omega') \rangle = 2(\hbar\omega_0/P)\langle \delta\mathbf{r}_{-,2}(\omega)\delta\mathbf{r}_{-,2}^\dagger(\omega') \rangle$, so that

$$\langle \Delta\boldsymbol{\varphi}_{\text{out}}(\omega)\Delta\boldsymbol{\varphi}_{\text{out}}^\dagger(\omega') \rangle = \frac{\hbar\omega_0}{2P}\left[\frac{2(\gamma^2 - 4\kappa_m^2)}{\omega^2 + 4\kappa_m^2} + 1\right] 2\pi\delta(\omega - \omega'). \tag{92}$$

The second term in this expression (proportional to the $+1$ in the square brackets) represents the impact of broadband vacuum noise fluctuations that are reflected by the cavity. In semiclassical terms, it corresponds to the shot noise of the detection process. On the other hand, the first term represents the contribution of phase fluctuations in the radiation emitted from the laser gyro. When $\gamma = 2\kappa_m$, this term becomes zero. This particular case lies outside the framework of our theory, which assumes a weak coupling between the two modes. Indeed, the emitted power is connected to the amplitude of the intracavity mode through the equation $P = \hbar\omega_0 \langle \mathbf{r}_+^\dagger \mathbf{r}_+ \rangle = \hbar\omega_0(\gamma - 2\kappa_m)a_0^2$. Therefore, if $\gamma = 2\kappa_m$, the coupling strength is so high that the output power of the laser gyro is zero, unless the amplitude of the intracavity mode a_0 is infinite.

3. DISCUSSION

Let us now discuss the independence of photon emission in the two counterpropagating modes. In a laser, the random nature of emission is attributed to two sources of noise. One source is associated with the output vacuum noise radiation leaking through the partially reflecting coupling mirrors, while the other is related to the inverted medium responsible for the gain. Each noise source contributes equally to the total noise. The very existence of these noise sources is related to the need of preserving the commutation rules of the emitted radiation of the laser.

When two modes are orthogonal, such as in free running lasers, they are independent, and so are the photon emission on the two modes. This is reflected by the fact that the quantum noise associated with each mode are uncorrelated. Conversely, when two modes are locked, they are no longer orthogonal, and the noise sources of the two modes need to exhibit a nonzero correlation. This is because the quantum operators representing the mode amplitudes must commute at all times, being this the mathematical expression of the property that it is always possible to measure the two modes independently, and the correlation of the noise sources is required to preserve the commutativity of the counterpropagating mode amplitudes at all times. A correlation of the noise sources has the physical meaning that the photon emission on the two counterpropagating modes is correlated. Nevertheless, it is important to note that the correlation between the noise sources has a negligible impact on the laser noise properties. The primary mechanism responsible for the reduction of the laser frequency noise stems from the semi-classical laser dynamics, which effectively locks the phase of the two modes.

As a final remark, the above analysis, for both locking mechanisms, can be straightforwardly extended to incomplete inversion by using the noise operators related to polarization in their general form $\mathbf{s}^{(-)}(t)\mathbf{s}^{(-)\dagger}(t') = (1/2)(1 - \boldsymbol{\sigma}_3)\delta(t-t')$ and $\mathbf{s}^{(-)\dagger}(t)\mathbf{s}^{(-)}(t') = (1/2)(1 + \boldsymbol{\sigma}_3)\delta(t-t')$, where $\boldsymbol{\sigma}_3$ is the inversion operator.

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