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Frequency noise of laser gyros

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Laser gyros are powerful tools used to test the predictions of the general theory of relativity. The precision of a measurement of the rotation rate with a laser gyro is limited by the frequency noise of the beat between two counterpropagating modes of a ring laser. The frequency noise of a single mode of a laser is limited by quantum mechanical constraints because it is related to the maximum precision with which the phase of a coherent state can be measured. If two modes are not correlated, the variance of the fluctuations of the difference of their frequencies is the sum of the variance of the frequency noise of the two modes. If the two modes are correlated, this result does not hold any longer. In this paper, we show that a laser gyro has mechanisms capable of dynamically locking the two modes together without forcing them to the same frequency. The lock of modes decouples the noise of the beat note from the frequency noise of the individual modes, thus allowing the realization of sub-shot noise laser gyros.

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1. INTRODUCTION

Laser gyros are a powerful tool used to test general relativity predictions [1–3]. They enable a precise measurement of the rotation rate by measuring the beat of two counterpropagating modes of a ring laser. The basic idea is that rotation breaks the symmetry between conterpropagating modes, and the frequency difference between the two modes is proportional to the rotation rate of the laser gyro. The precision of the measurement depends on the frequency stability of the beat note obtained by detecting the intensity of a coherent combination of the two modes. If the modes are independent and of equal power, the variance of the frequency noise of the beat is twice the variance of the frequency noise of each individual mode [4–7]. The frequency noise of each individual mode originates from constraints dictated by quantum mechanics and in particular from the precision of a measurement of the phase of a coherent state [4]. The physical mechanisms that make the laser radiation compliant with these constraints come for one half from the quantum noise of the active medium and the other half from the vacuum fluctuations entering from the output port of the laser [8]. Such noise sources are responsible for the phase and frequency noise of the laser, and for the non-zero linewidth of the emitted radiation.

In this paper, we show that under proper conditions, the two counterpropagating modes of a laser gyro can lock together while still maintaining a different frequency. When these conditions are fulfilled, the noise of the frequency of the beat note decouples to the noise of the individual modes. This result can be understood by the analogy with mode-locked lasers. In passively mode-locked lasers, the locking mechanism is associated with pulsed operation. The linewidth of the single line of the spectrum of the emitted radiation is Lorentzian, but the frequency fluctuations are strongly correlated, to the extent that the spectral purity of the beat note between the spectral lines of the emitted frequency comb [9] has been exploited for the realization of very accurate clockworks [10]. In ring lasers, locking of counterpropagating modes is the result of reflections. When reflections occur from static cavity elements such as cavity mirrors, the two modes locks at the same frequency. When reflections come from the slowly moving grating, generated in a nonlinear medium with slow response by the beat of the two counterpropagating modes themselves, they tend to stabilize the difference frequency of the two modes. We speculate that this mechanism is at work in the best-performing laser gyros operating around the world, when spurious reflections from static cavity elements are minimized, and that may in particular explain the observation of sub-shot noise performance of the GINGERino laser gyro that appeared in the literature recently [11–13].

One may use these results for investigating the possibility of alternate laser design, where suitable nonlinear elements are inserted in the laser cavity to stabilize the mode beat. Our findings pave the way for the realization of sub-shot noise laser gyros of unprecedented accuracy for ultra-precise testing of the predictions of general relativity.

2. SINGLE-MODE CASE

Following the analysis of Yamamoto and Haus [8], let us consider first a single mode of an empty cavity $\mathbf{a}(t)$ with bosonic commutation relations

$$[\mathbf{a}(t), \mathbf{a}^{\dagger}(t)] = 1, \quad (1)$$

coupled to an outside optical *wave* $\mathbf{s}_a(t)$ with commutation relations

$$[\mathbf{s}_a(t), \mathbf{s}_a^{\dagger}(t')] = \delta(t - t'). \quad (2)$$

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The wave reflected from the cavity is given by [8,14,15]

$$\mathbf{r}_a(t) = -\mathbf{s}_a(t) + \sqrt{\gamma}\mathbf{a}(t). \quad (3)$$

The temporal evolution of the mode $\mathbf{a}(t)$ is described by the differential equation

$$\frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} = -\frac{\gamma}{2}\mathbf{a}(t) + \sqrt{\gamma}\mathbf{s}_a(t). \quad (4)$$

Assuming that the outside wave is incident upon the cavity from a time much longer than $1/\gamma$, the solution of Eq. (3) is

$$a(t) = \sqrt{\gamma}\int_{-\infty}^{t} \mathrm{d}u \exp\left[-\frac{\gamma}{2}(t-u)\right]\mathbf{s}_a(u), \quad (5)$$

so that the two-time commutation relations of $\mathbf{r}_a(t)$ are

$$[\mathbf{a}(t), \mathbf{a}^\dagger(t')] = \gamma \exp\left[-\frac{\gamma}{2}(t+t')\right]\int_{-\infty}^{t}\mathrm{d}u\int_{-\infty}^{t'}\mathrm{d}u' \times \exp\left[\frac{\gamma}{2}(u+u')\right][\mathbf{s}_a(u), \mathbf{s}_a^\dagger(u')], \quad (6)$$

that is

$$\left[\boldsymbol{a}(t), \boldsymbol{a}^\dagger(t')\right] = \exp\left(-\frac{\gamma}{2}\left|t-t'\right|\right), \quad (7)$$

consistent with the bosonic commutation rule (1) for $t = t'$.

The commutation relations of the reflected wave $\mathbf{r}_a(t)$ are

$$[\mathbf{r}_a(t), \mathbf{r}_a^\dagger(t')] = [\mathbf{s}_a(t), \mathbf{s}_a^\dagger(t')] + \gamma[\mathbf{a}(t), \mathbf{a}^\dagger(t')] - \sqrt{\gamma}\left([\mathbf{a}(t), \mathbf{s}_a^\dagger(t')] + [\mathbf{s}_a(t), \mathbf{a}^\dagger(t')]\right). \quad (8)$$

Being

$$[\mathbf{a}(t), \mathbf{s}_a^\dagger(t')] = \sqrt{\gamma}\int_{-\infty}^{t}\mathrm{d}u \exp\left[-\frac{\gamma}{2}(t-u)\right][\mathbf{s}_a(u), \mathbf{s}_a^\dagger(t')], \quad (9)$$

that is

$$[\mathbf{a}(t), \mathbf{s}_a^\dagger(t')] = \exp\left[-\frac{\gamma}{2}(t-t')\right]u(t-t'), \quad (10)$$

and also

$$[\mathbf{s}_a(t), \mathbf{a}^\dagger(t')] = \exp\left[-\frac{\gamma}{2}(t'-t)\right]u(t'-t), \quad (11)$$

where $u(t) = 1$ for $t > 0$, $u(t) = 0$ for $t < 0$, and $u(0) = 1/2$, so that we obtain

$$[\mathbf{r}_a(t), \mathbf{r}_a^\dagger(t')] = [\mathbf{s}_a(t), \mathbf{s}_a^\dagger(t)], \quad (12)$$

and hence that the commutation relation of the output optical wave was the same of the input wave, as it should be.

Let us now assume that a gain medium is inserted into the cavity (see Fig. 1), which we represent as a statistical mixture of N two-level atoms. Let us define the operators

$$\boldsymbol{\sigma}_- = \sum_{i=1}^{N}\frac{1}{N}(|1\rangle\langle 2|)_i, \quad (13)$$

where $|1\rangle$ and $|2\rangle$ are the two levels, and

$$\boldsymbol{\sigma}_3 = \sum_{i=1}^{N}\frac{1}{N}(|2\rangle\langle 2| - |1\rangle\langle 1|)_i. \quad (14)$$

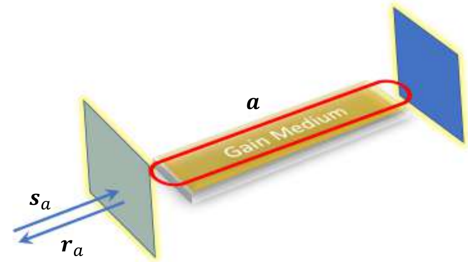

Fig. 1. Representation of the laser cavity. The front mirror is a partially reflecting mirror with power reflectivity R, such that $\gamma = (1-R)/\tau_{rt}$, where τ_{rt} is the cavity roundtrip time, whereas the backward mirror is fully reflecting.

It is easy to show that $\boldsymbol{\sigma}_-$ and $\boldsymbol{\sigma}_3$ obey the commutation relations

$$[\boldsymbol{\sigma}_-, \boldsymbol{\sigma}_-^\dagger] = -\frac{\boldsymbol{\sigma}_3}{N}, \quad (15)$$

and the anti-commutation

$$\{\boldsymbol{\sigma}_-, \boldsymbol{\sigma}_-^\dagger\} = \frac{1}{N}. \quad (16)$$

The spontaneous decay of $\boldsymbol{\sigma}_3$ is described by

$$\frac{\mathrm{d}\boldsymbol{\sigma}_-(t)}{\mathrm{d}t} = -\Gamma\boldsymbol{\sigma}_-(t) + \left(\frac{2\Gamma}{N}\right)^{1/2}\mathbf{s}^{(-)}(t), \quad (17)$$

where a noise source $\mathbf{s}^{(-)}(t)$ with commutation relation

$$[\mathbf{s}^{(-)}(t), \mathbf{s}^{(-)\dagger}(t')] = -\boldsymbol{\sigma}_3(t)\delta(t-t'), \quad (18)$$

and anti-commutation

$$\{\mathbf{s}^{(-)}(t), \mathbf{s}^{(-)\dagger}(t')\} = \delta(t-t') \quad (19)$$

is required to preserve the commutation and anti-commutation relations, as it may be verified for the commutator (and similarly for the anti-commutator) by calculating $\mathrm{d}[\boldsymbol{\sigma}_3(t), \boldsymbol{\sigma}_3^\dagger(t)]$ and using that

$$[\mathbf{s}^{(-)}(t)\mathrm{d}t, \mathbf{s}^{(-)\dagger}(t)\mathrm{d}t] = -\boldsymbol{\sigma}_3(t)\mathrm{d}t. \quad (20)$$

Being $\boldsymbol{\sigma}_-(t)^2 = 0$, we also have $\mathbf{s}^{(-)}(t)^2 = \mathbf{s}^{(-)\dagger}(t)^2 = 0$, and this completes the characterization of the noise operator. If the active medium is placed into the cavity that we described above, the coupling with the cavity mode is described by the equation for $\mathbf{a}(t)$:

$$\frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} = -\frac{\gamma}{2}\mathbf{a}(t) - igN\boldsymbol{\sigma}_-(t) + \sqrt{\gamma}\mathbf{s}_a(t), \quad (21)$$

where g is a coefficient proportional to the gain, and by the equation for $\boldsymbol{\sigma}_-$:

$$\frac{\mathrm{d}\boldsymbol{\sigma}_-(t)}{\mathrm{d}t} = -\Gamma\boldsymbol{\sigma}_-(t) + ig\boldsymbol{\sigma}_3(t)\mathbf{a}(t) + \left(\frac{2\Gamma}{N}\right)^{1/2}\mathbf{s}^{(-)}(t). \quad (22)$$

In the presence of optical pumping with pumping rate R_p, the equation for the population inversion $\mathbf{n}(t) = N\boldsymbol{\sigma}_3(t)$ is

$$\frac{\mathrm{d}\mathbf{n}(t)}{\mathrm{d}t} = R_p - \frac{\mathbf{n}}{\tau} + i2gN\left[\mathbf{a}^\dagger(t)\boldsymbol{\sigma}_-(t) - \boldsymbol{\sigma}_-^\dagger(t)\mathbf{a}(t)\right], \quad (23)$$

where τ is the spontaneous carrier lifetime.

Assuming $\Gamma \gg 1/\tau$, we may neglect $\mathrm{d}\boldsymbol{\sigma}_-(t)/\mathrm{d}t$ in Eq. (22) compared to $-\Gamma\boldsymbol{\sigma}_-(t)$. This procedure yields

$$\boldsymbol{\sigma}_-(t) = i\frac{g}{N\Gamma}\mathbf{n}(t)\mathbf{a}(t) + \left(\frac{2}{N\Gamma}\right)^{1/2}\mathbf{s}^{(-)}(t), \tag{24}$$

and this identity once inserted into the equation for $\mathbf{n}(t)$ permits to adiabatically eliminate $\boldsymbol{\sigma}_-(t)$ in Eqs. (21) and (23), which become

$$\frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} = \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{a}(t) - ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}^{(-)}(t) + \sqrt{\gamma}\mathbf{s}_a(t), \tag{25}$$

$$\frac{\mathrm{d}\mathbf{n}(t)}{\mathrm{d}t} = R_p - \frac{\mathbf{n}(t)}{\tau} - \frac{4g^2}{\Gamma}\mathbf{n}(t)\mathbf{a}^\dagger(t)\mathbf{a}(t) + i2g\left(\frac{2N}{\Gamma}\right)^{1/2}\left[\mathbf{a}^\dagger(t)\mathbf{s}^{(-)}(t) - \mathbf{s}^{(-)\dagger}(t)\mathbf{a}(t)\right]. \tag{26}$$

The commutation relations of the noise term in Eq. (25)

$$\mathbf{S}_a(t) = -ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}^{(-)}(t) + \sqrt{\gamma}\mathbf{s}_a(t) \tag{27}$$

is

$$[\mathbf{S}_a(t), \mathbf{S}_a^\dagger(t')] = 2\left(\frac{\gamma}{2} - \frac{g^2}{\Gamma}\mathbf{n}\right)\delta(t-t'). \tag{28}$$

Using

$$\mathrm{d}[\mathbf{a}(t), \mathbf{a}^\dagger(t)] = [\mathrm{d}\mathbf{a}(t), \mathbf{a}^\dagger(t)] + [\mathbf{a}(t), \mathrm{d}\mathbf{a}^\dagger(t)] + [\mathrm{d}\mathbf{a}(t), \mathrm{d}\mathbf{a}^\dagger(t)], \tag{29}$$

and

$$[\mathrm{d}\mathbf{a}(t), \mathrm{d}\mathbf{a}^\dagger(t)] = [\mathbf{S}_a(t)\mathrm{d}t, \mathbf{S}_a^\dagger(t)\mathrm{d}t] = 2\left[\frac{\gamma}{2} - \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathrm{d}t, \tag{30}$$

we may show that the commutation relations Eq. (28) imply $\mathrm{d}[\mathbf{a}(t), \mathbf{a}^\dagger(t)] = 0$, thus ensuring the preservation of the commutation relations for $\mathbf{a}(t)$, also in the presence of the interaction with the gain medium.

Let us now linearize Eqs. (25) and (26) around the steady state by setting

$$\mathbf{a}(t) = a_0 + \delta\mathbf{a}(t), \tag{31}$$

$$\mathbf{n}(t) = n_0 + \delta\mathbf{n}(t), \tag{32}$$

with a_0 and n_0 c-numbers. The commutation relations for $\delta\mathbf{a}(t)$ are equal to the commutation relations for $\mathbf{a}(t)$. The steady state value of the population inversion is

$$n_0 = \frac{\gamma\Gamma}{2g^2}, \tag{33}$$

so that linearization of Eqs. (25) and (26) yields

$$\frac{\mathrm{d}\delta\mathbf{a}(t)}{\mathrm{d}t} = \frac{g^2}{\Gamma}a_0\delta\mathbf{n}(t) - ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}^{(-)}(t) + \sqrt{\gamma}\mathbf{s}_a(t), \tag{34}$$

$$\frac{\mathrm{d}\delta\boldsymbol{n}(t)}{\mathrm{d}t} = -\frac{\delta\boldsymbol{n}(t)}{\tau} - \frac{4g^2}{\Gamma}a_0^2\delta\boldsymbol{n}(t) - \frac{4g^2}{\Gamma}n_0a_0\left[\delta\boldsymbol{a}(t) + \delta\boldsymbol{a}^\dagger(t)\right] + i2g\left(\frac{2N}{\Gamma}\right)^{1/2}a_0\left[\boldsymbol{s}^{(-)}(t) - \boldsymbol{s}^{(-)\dagger}(t)\right], \tag{35}$$

where we assumed a_0 as real implying the definition of a phase reference for the field.

Adiabatic elimination of the population inversion in the high-gain regime in which $1/\tau \ll 4g^2a_0^2/\Gamma$ gives

$$\delta\mathbf{n}(t) = -\frac{n_0}{a_0}\left[\delta\mathbf{a}(t) + \delta\mathbf{a}^\dagger(t)\right] + i\frac{\Gamma}{2g}\left(\frac{2N}{\Gamma}\right)^{1/2}\frac{1}{a_0}\left[\mathbf{s}^{(-)}(t) - \mathbf{s}^{(-)\dagger}(t)\right]. \tag{36}$$

This equation, inserted into Eq. (34) gives

$$\frac{\mathrm{d}\delta\mathbf{a}(t)}{\mathrm{d}t} = -\gamma\frac{\delta\mathbf{a}(t) + \delta\mathbf{a}^\dagger(t)}{2} + \sqrt{\gamma}\mathbf{s}_a(t) - ig\left(\frac{2N}{\Gamma}\right)^{1/2}\frac{\mathbf{s}^{(-)}(t) + \mathbf{s}^{(-)\dagger}(t)}{2}. \tag{37}$$

With strong pumping, the medium is fully inverted so that $n_0 \simeq N$, and using Eq. (33), we obtain $\gamma = 2n_0g^2/\Gamma \simeq 2Ng^2/\Gamma$. Therefore,

$$\frac{\mathrm{d}\delta\mathbf{a}(t)}{\mathrm{d}t} = -\gamma\frac{\delta\mathbf{a}(t) + \delta\mathbf{a}^\dagger(t)}{2} + \sqrt{\gamma}\mathbf{s}_a(t) - i\sqrt{\gamma}\frac{\mathbf{s}^{(-)}(t) + \mathbf{s}^{(-)\dagger}(t)}{2}. \tag{38}$$

The equations for the in-phase component $\delta\mathbf{a}_1(t) = [\delta\mathbf{a}(t) + \delta\mathbf{a}^\dagger(t)]/2$ and the in-quadrature component $\delta\mathbf{a}_2(t) = [\delta\mathbf{a}(t) - \delta\mathbf{a}^\dagger(t)]/(2i)$ are

$$\frac{\mathrm{d}\delta\mathbf{a}_1(t)}{\mathrm{d}t} = -\gamma\delta\mathbf{a}_1(t) + \sqrt{\gamma}\mathbf{s}_{a,1}(t), \tag{39}$$

and

$$\frac{\mathrm{d}\delta\mathbf{a}_2(t)}{\mathrm{d}t} = \sqrt{\gamma}\left[\mathbf{s}_2(t) - \mathbf{s}_1^{(-)}(t)\right], \tag{40}$$

where $\mathbf{s}_1^{(-)}(t) = [\mathbf{s}^{(-)}(t) + \mathbf{s}^{(-)\dagger}(t)]/2$, $\mathbf{s}_2^{(-)}(t) = [\mathbf{s}^{(-)}(t) - \mathbf{s}^{(-)\dagger}(t)]/(2i)$, $\mathbf{s}_{a,1}(t) = [\mathbf{s}_a(t) + \mathbf{s}_a^\dagger(t)]/2$, and $\mathbf{s}_{a,2}(t) = [\mathbf{s}_a(t) - \mathbf{s}_a^\dagger(t)]/(2i)$.

Solving the equation for the in-phase component Eq. (39) in the Fourier domain, we obtain

$$\delta\mathbf{a}_1(\omega) = \frac{\sqrt{\gamma}\mathbf{s}_{a,1}(\omega)}{-i\omega + \gamma}, \tag{41}$$

which inserted into the equation for the fluctuations of $\mathbf{r}_{a,1}(\omega)$ given by Eq. (3) yields

$$\delta\mathbf{r}_{a,1}(\omega) = \frac{i\omega\gamma}{-i\omega + \gamma}\mathbf{s}_{a,1}(\omega). \tag{42}$$

For $\omega \ll \gamma$, we have $\delta \boldsymbol{r}_{a,1}(\omega) \simeq 0$ [8,15], whereas for $\omega \gg \gamma$, we have $\delta \mathbf{r}_{a,1}(\omega) = -\mathbf{s}_{a,1}(\omega)$, so that in this regime, the incoming vacuum fluctuations are reflected from the cavity with a π-phase shift, producing a coherent state at output.

Using Eq. (18), and being $\langle \sigma_3 \rangle = 1$ for full inversion, we obtain

$$\langle \mathbf{s}_i^{(-)}(t)\mathbf{s}_i^{(-)}(t') \rangle = \frac{1}{4}\delta(t-t'), \quad i = 1, 2, \tag{43}$$

and using Eq. (3)

$$\langle \mathbf{s}_i(t)\mathbf{s}_i(t') \rangle = \frac{1}{4}\delta(t-t'), \quad i = 1, 2. \tag{44}$$

Equation (40) shows that the diffusion coefficient for the in-quadrature fluctuations is equal to $\gamma/2$, so that the diffusion coefficient for the phase fluctuations, defined as

$$\Delta \boldsymbol{\varphi} = \frac{\delta \mathbf{a}_2(t)}{a_0} \tag{45}$$

is

$$D_\varphi = \frac{\gamma}{2a_0^2}, \tag{46}$$

so that the laser line-width is

$$\Delta \nu = \frac{D_\varphi}{2\pi} = \frac{\gamma}{4\pi a_0^2}. \tag{47}$$

If we use the expression for the output power of the laser $P = \gamma a_0^2 \hbar \omega_0$, we obtain the well-known Schawlow–Townes linewidth formula:

$$\Delta \nu = \frac{\gamma^2 \hbar \omega_0}{4\pi P}. \tag{48}$$

The uncertainty of a frequency measurement over a time T is

$$\boldsymbol{\omega}_{\text{meas}} T = \omega_0 T + \Delta \boldsymbol{\varphi}(t+T) - \Delta \boldsymbol{\varphi}(t), \tag{49}$$

so that, using $\langle [\Delta \boldsymbol{\varphi}(t+T) - \Delta \boldsymbol{\varphi}(t)]^2 \rangle = D_\varphi T$, we obtain

$$\Delta \omega_{\text{meas}}^2 = \frac{\langle [\Delta \boldsymbol{\varphi}(t+T) - \Delta \boldsymbol{\varphi}(t)]^2 \rangle}{T^2} = \frac{\gamma}{2a_0^2 T}, \tag{50}$$

where we defined the uncertainty of the frequency measurement as $\Delta\omega_{\text{meas}} = (\langle \Delta \boldsymbol{\omega}_{\text{meas}} \rangle^2)^{1/2}$. Equation (50) can be interpreted in simple physical terms. The variance of a phase measurement on a coherent state of amplitude a_0 is $\Delta \varphi_{\text{coh}}^2 = 1/(4a_0^2)$. Nyquist criterion states that the number of independent measurements that can be performed over a time T on a signal of correlation time $1/\gamma$ [see Eq. (42)] is $N_{\text{meas}} = (2T)/(1/\gamma)$, so that the variance of the frequency measurement is $\Delta \omega_{\text{coh}}^2 = (\Delta \varphi_{\text{coh}}^2 / T^2)/N_{\text{meas}}$, which returns Eq. (50) [4].

Using Eq. (50), the relation that links a_0^2 to the output power of the laser P, namely $a_0^2 = P/(\gamma \hbar \omega_0)$, we obtain

$$\Delta \omega_{\text{meas}} = \frac{\omega_0}{Q}\sqrt{\frac{\hbar \omega_0}{2PT}}, \tag{51}$$

where we defined the cavity quality factor as $Q = \omega_0/\gamma$.

3. LASER GYRO: A TWO-MODE CASE

While the laser linewidth and the precision of a measurement of the frequency of a single laser mode are prone to strong quantum mechanical constraints, the frequency difference of two modes is not. Of course, if the two modes are independent, the variance of the fluctuations of the difference frequency is the sum of the variances of the individual modes. Different is the case of correlated modes. The case of the beat of two modes of a mode-locked laser is an example where the beat of two modes has precision orders of a magnitude larger than the precision of each individual mode frequency [9]. This property enables the transfer down to microwave frequencies of extremely stable optical oscillations and vice versa [10]. It is therefore worth investigating whether there are any active locking mechanisms (or can be induced by a suitable design) in laser gyros.

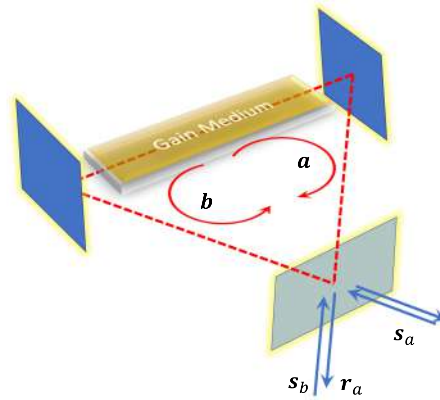

Fig. 2. Representation of the ring laser cavity. The front mirror is a partially reflecting mirror with power reflectivity R, such that $\gamma = (1-R)/\tau_{\text{rt}}$, where τ_{rt} is the cavity roundtrip time, whereas other two mirrors are fully reflecting.

Let us consider a ring laser with two counterpropagating modes (see Fig. 2), one forward-propagating, centered at frequency $\omega_0 + \Omega_0/2$:

$$\frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} = -i\frac{\Omega_0}{2}\mathbf{a}(t) - \frac{\gamma}{2}\mathbf{a}(t) - igN[\boldsymbol{\sigma}_-(t)]_a + \sqrt{\gamma}\mathbf{s}_a(t), \tag{52}$$

and the other backward propagating centered at frequency $\omega_0 - \Omega_0/2$:

$$\frac{\mathrm{d}\mathbf{b}(t)}{\mathrm{d}t} = i\frac{\Omega_0}{2}\mathbf{b}(t) - \frac{\gamma}{2}\mathbf{b}(t) - igN[\boldsymbol{\sigma}_-(t)]_b + \sqrt{\gamma}\mathbf{s}_b(t). \tag{53}$$

Here $[\boldsymbol{\sigma}_-(t)]_a$ and $[\boldsymbol{\sigma}_-(t)]_b$ are the (suitably normalized) spatial Fourier components of $\boldsymbol{\sigma}_-$ proportional to $\exp(ikz)$ and $\exp(-ikz)$ that couple with the forward and backward propagating waves. The equation for $\boldsymbol{\sigma}_-$ becomes

$$\frac{\mathrm{d}\boldsymbol{\sigma}_-(t)}{\mathrm{d}t} = -\Gamma\boldsymbol{\sigma}_-(t) + ig\boldsymbol{\sigma}_3(t)(\mathbf{a}(t)+\mathbf{b}(t)) + \left(\frac{2\Gamma}{N}\right)^{1/2}\mathbf{s}^{(-)}(t). \tag{54}$$

In the presence of optical pumping with pumping rate R_p, the equation for the population inversion $\mathbf{n}(t) = N\boldsymbol{\sigma}_3(t)$ is

$$\frac{\mathrm{d}\mathbf{n}(t)}{\mathrm{d}t} = R_p - \frac{\mathbf{n}}{\tau} + i2gN[(\mathbf{a}^\dagger(t)+\mathbf{b}^\dagger(t))\boldsymbol{\sigma}_-(t) - \boldsymbol{\sigma}_-^\dagger(t)(\mathbf{a}(t)+\mathbf{b}(t))], \tag{55}$$

where τ is the spontaneous lifetime.

Adiabatic elimination of $\boldsymbol{\sigma}_-(t)$ in Eq. (54) gives

$$\boldsymbol{\sigma}_-(t) = i\frac{g}{N\Gamma}\mathbf{n}(t)(\mathbf{a}(t)+\mathbf{b}(t)) + \left(\frac{2}{N\Gamma}\right)^{1/2}\mathbf{s}^{(-)}(t), \quad (56)$$

that is, the expected linear dependence of the medium polarization on the optical field. Inserting Eq. (56) into Eqs. (52) and (53) and projecting $\boldsymbol{\sigma}_-(t)$ over the two counterpropagating modes gives

$$\frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} = -i\frac{\Omega_0}{2}\mathbf{a}(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{a}(t) - ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}_a^{(-)}(t) + \sqrt{\gamma}\mathbf{s}_a(t), \quad (57)$$

$$\frac{\mathrm{d}\mathbf{b}(t)}{\mathrm{d}t} = i\frac{\Omega_0}{2}\mathbf{b}(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{b}(t) - ig\left(\frac{2N}{\Gamma}\right)^{1/2}\mathbf{s}_b^{(-)}(t) + \sqrt{\gamma}\mathbf{s}_b(t). \quad (58)$$

Here, $\mathbf{s}_{a,b}^{(-)}(t)$ are the result of the projection of the noise term $\mathbf{s}^{(-)}(t)$ over the spatial mode profile $\exp(ikz)$ and $\exp(-ikz)$. Local multiplication by $\exp(\pm ikz)$ generates two independent noise terms with the same commutation properties of $\mathbf{s}^{(-)}(t)$. As a check, it may be verified that, if $\mathbf{s}_{a,b}^{(-)}(t)$ obey the commutation rule (18), Eqs. (57) and (58) preserve the bosonic commutation rules of the two modes. Entering Eq. (56) into Eq. (55) and expanding the product of the mode amplitudes yields

$$\frac{\mathrm{d}\mathbf{n}(t)}{\mathrm{d}t} = R_p - \frac{\mathbf{n}(t)}{\tau} - \frac{4g^2}{\Gamma}\mathbf{n}(t)[\mathbf{a}^\dagger(t)\mathbf{a}(t) + \mathbf{b}^\dagger(t)\mathbf{b}(t) + \mathbf{a}^\dagger(t)\mathbf{b}(t) + \mathbf{b}^\dagger(t)\mathbf{a}(t)] + i2g\left(\frac{2N}{\Gamma}\right)^{1/2}[(\mathbf{a}^\dagger(t)+\mathbf{b}^\dagger(t))\mathbf{s}^{(-)}(t) - \mathbf{s}^{(-)\dagger}(t)(\mathbf{a}(t)+\mathbf{b}(t))]. \quad (59)$$

Being $\Omega \ll 1/\tau$, we may assume that $\mathbf{n}(t)$ adiabatically follows the modulation frequency, so that the steady state of $\mathbf{n}$ is

$$n(t) = \frac{R_p}{1/\tau + (4g^2/\Gamma)\left(|a_0|^2 + |b_0|^2 + a_0^* b_0 e^{i\Omega_0 t} + a_0 b_0^* e^{-i\Omega_0 t}\right)}. \quad (60)$$

The terms $a_0^* b_0$ and $a_0 b_0^*$ account for a gain grating generated by the beat of the two counterpropagating modes over the gain medium. The nature of this grating may be understood by considering that the two counterpropagating modes collide over the active medium and generate the intensity pattern:

$$I(z,t) = |A\exp(-i\Omega_0 t/2 + ikz) + B\exp(i\Omega_0 t/2 - ikz)|^2, \quad (61)$$

where A and B are the amplitudes of the forward and backward propagating modes at the position of the gain medium. Expanding the expression of the intensity, we obtain

$$I(z,t) = |A|^2 + |B|^2 + AB^*\exp(-i\Omega_0 t + 2ikz) + A^*B\exp(i\Omega_0 t - 2ikz). \quad (62)$$

The grating moves at the speed $\Omega_0/(2k) = (f_1 - f_2)\lambda/2$, in the GINGERino case [16,17] about 89 microns per second. In a gas laser, the amplitude of the grating tends to be attenuated by diffusion, so that we may expand the above expression to first order:

$$n(t) = n_0\left(1 - \frac{a_0^* b_0 e^{i\Omega_0 t} + a_0 b_0^* e^{-i\Omega_0 t}}{|a_0|^2 + |b_0|^2}\right), \quad (63)$$

where we assumed high saturation so that $(4g^2/\Gamma)(|a_0|^2 + |b_0|^2) \gg 1/\tau$ and defined

$$n_0 = \frac{R_p\Gamma}{4g^2\left(|a_0|^2 + |b_0|^2\right)}. \quad (64)$$

Similarly to the single mode case, the phase fluctuations are independent of the fluctuations of the carrier, so that $\mathbf{n}(t)$ can be replaced by its steady state value n_0:

$$\frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} = -i\frac{\Omega_0}{2}\mathbf{a}(t) + \kappa_g\mathbf{b}(t)e^{-i\Omega_0 t} + \left(-\frac{\gamma}{2} + \frac{g^2}{\Gamma}n_0\right)\mathbf{a}(t) - i\sqrt{\gamma}\mathbf{s}_a^{(-)}(t) + \sqrt{\gamma}\mathbf{s}_a(t), \quad (65)$$

$$\frac{\mathrm{d}\boldsymbol{b}(t)}{\mathrm{d}t} = i\frac{\Omega_0}{2}\boldsymbol{b}(t) + \kappa_g^*\boldsymbol{a}(t)e^{i\Omega_0 t} + \left(-\frac{\gamma}{2} + \frac{g^2}{\Gamma}n_0\right)\boldsymbol{b}(t) - i\sqrt{\gamma}\boldsymbol{s}_b^{(-)}(t) + \sqrt{\gamma}\boldsymbol{s}_b(t), \quad (66)$$

where we used $\gamma = 2g^2 n_0/\Gamma$ and assumed full inversion so that $N \simeq n_0$, and we defined

$$\kappa_g = -\frac{g^2 n_0}{\Gamma}\frac{\xi a_0 b_0^*}{|a_0|^2 + |b_0|^2}. \quad (67)$$

Here, we have introduced a factor $\xi < 1$ to account for the reduction of the grating amplitude caused by diffusion of the active atoms and by the fact that in HeNe lasers, the presence of different isotopes of Ne results in that different sets of atoms couple with uneven strengths to the two countepropagating waves. The term κ_g, proportional to $a_0 b_0^*$, couples the backward propagating mode to the forward propagating mode, because the spatial modulation proportional to $\exp(2ikz)$ promotes phase matching between the backward propagating wave, with spatial dependence $\exp(-ikz)$, and the forward propagating wave with spatial dependence $\exp(ikz)$. By a similar mechanism, the term κ_g^*, proportional to $a_0^* b_0$, couples the forward propagating mode to the backward propagating mode.

A potentially relevant additional effect may also arise because the Kramers–Kronig relations dictate that a dynamic gain change with an asymmetric spectrum always generates a dynamic index change. This phenomenon is similar to the effect that leads to the Henry's α factor in semiconductor lasers [18]. Consequently, in addition to the coupling directly induced by gain modulation, the resulting index grating can further enhance the coupling between the two counterpropagating waves, by a mechanism similar to that employed in index-coupled distributed feedback lasers [19]. The effect of a dynamic index grating is essentially to change the value of the coupling constant κ_g into $(1+\alpha)\kappa_g$, with α being a phenomenological constant. A detailed quantum analysis of the case in which coupling is caused by a dynamic index grating is reported in Supplement 1.

Reflections may also occur from various optical elements in the optical cavity, primarily from cavity mirrors. In this case, however, reflections do not change the frequency of the field. Including this process into Eqs. (65) and (66) by an extra backscattering coefficient κ_m, they become

$$\frac{\mathrm{d}\mathbf{a}(t)}{\mathrm{d}t} = -i\frac{\Omega_0}{2}\mathbf{a}(t) + \left(\kappa_g e^{-i\Omega_0 t} + \kappa_m\right)\mathbf{b}(t) + \left(-\frac{\gamma}{2} + \frac{g^2}{\Gamma}n_0\right)\mathbf{a}(t) - i\sqrt{\gamma}\mathbf{s}_a^{(-)}(t) + \sqrt{\gamma}\mathbf{s}_a(t), \tag{68}$$

$$\frac{\mathrm{d}\mathbf{b}(t)}{\mathrm{d}t} = i\frac{\Omega_0}{2}\mathbf{b}(t) + \left(\kappa_g^* e^{i\Omega_0 t} + \kappa_m^*\right)\mathbf{a}(t) + \left(-\frac{\gamma}{2} + \frac{g^2}{\Gamma}n_0\right)\mathbf{b}(t) - i\sqrt{\gamma}\mathbf{s}_b^{(-)}(t) + \sqrt{\gamma}\mathbf{s}_b(t). \tag{69}$$

The coupling between the two modes is non-hermitian, so that Eqs. (68) and (69) need to be modified to preserve the commutation relations. The following analysis is therefore valid for small κ_g and κ_m. A rigorous quantum analysis of the coupling of the two counterpropagating modes is reported in Supplement 1.

These equations are linear in the fields. Thus, a meaningful analysis can be performed assuming classical fields, with noise sources whose strength are dictated by quantum mechanics. Considering only the deterministic part, and defining $\Delta g = -\gamma + 2g^2 n_0/\Gamma$, $a_0 = |a_0|\exp(i\varphi_a)$, $b_0 = |b_0|\exp(i\varphi_b)$, $\kappa_g = |\kappa_g|\exp(i\varphi_g)$, and $\kappa_g = |\kappa_m|\exp(i\varphi_m)$, we obtain:

$$\frac{\mathrm{d}\varphi_a}{\mathrm{d}t} = -\frac{\Omega_0}{2} + \frac{|b_0|}{|a_0|}[|\kappa_g|\sin(\varphi_b - \varphi_a + \varphi_g - \Omega_0 t) + |\kappa_m|\sin(\varphi_b - \varphi_a + \varphi_m)], \tag{70}$$

$$\frac{\mathrm{d}\varphi_b}{\mathrm{d}t} = \frac{\Omega_0}{2} - \frac{|a_0|}{|b_0|}[|\kappa_g|\sin(\varphi_b - \varphi_a + \varphi_g - \Omega_0 t) + |\kappa_m|\sin(\varphi_b - \varphi_a + \varphi_m)]. \tag{71}$$

We also have

$$\frac{\mathrm{d}|a_0|}{\mathrm{d}t} = \frac{\Delta g}{2}|a_0| + |b_0|[|\kappa_g|\cos(\varphi_b - \varphi_a + \varphi_g - \Omega_0 t) + |\kappa_m|\cos(\varphi_b - \varphi_a + \varphi_m)], \tag{72}$$

$$\frac{\mathrm{d}|b_0|}{\mathrm{d}t} = \frac{\Delta g}{2}|b_0| + |a_0|[|\kappa_g|\cos(\varphi_b - \varphi_a + \varphi_g - \Omega_0 t) + |\kappa_m|\cos(\varphi_b - \varphi_a + \varphi_m)]. \tag{73}$$

These equations admit stable stationary solutions for $|a_0|$ and $|b_0|$ only when either $|\kappa_g|$ or $|\kappa_m|$ is predominant, so that the other can be neglected. Let us consider these two cases separately.

A. Scattering Due to Mirrors Is Predominant

This case corresponds to $\kappa_g = 0$. In this case, by defining $\Delta\varphi = \varphi_a - \varphi_b - \varphi_m$, we obtain

$$\frac{\mathrm{d}\Delta\varphi}{\mathrm{d}t} = \Omega_0 - |\kappa_m|\left(\frac{|b_0|}{|a_0|} + \frac{|a_0|}{|b_0|}\right)\sin(\Delta\varphi), \tag{74}$$

and also

$$\frac{\mathrm{d}|a_0|}{\mathrm{d}t} = \frac{\Delta g}{2}|a_0| + |\kappa_m||b_0|\cos(\Delta\varphi), \tag{75}$$

$$\frac{\mathrm{d}|b_0|}{\mathrm{d}t} = \frac{\Delta g}{2}|b_0| + |\kappa_m||a_0|\cos(\Delta\varphi). \tag{76}$$

Of course, if $|\kappa_m|$ is negligible $\Delta\varphi = \Omega_0 t$. However, two steady-state solutions with a time-independent value of $\Delta\varphi$ exist if $\Omega_0 \le 2|\kappa_m|$. This steady state corresponds to two counterpropagating modes with the same frequency and locked phase, and is achieved for $|a_0| = |b_0|$, $\Delta g = -2|\kappa_m|\cos(\Delta\varphi)$, and for values of Ω_0 such that

$$\Omega_0 = 2|\kappa_m|\sin(\Delta\varphi). \tag{77}$$

Of the two solutions, only that with $\Delta g = -2|\kappa_m|\cos(\Delta\varphi) < 0$ is stable. The maximum value of Ω_0 compatible with this steady state solution is $\Omega_{\text{lock-in}} = 2|\kappa_m|$.

Locking at a zero difference frequency should be avoided in the proper operation of a laser gyro. The value of $|\kappa_m|$ can be estimated from the frequency $f_{\text{lock-in}} = \Omega_{\text{lock-in}}/(2\pi)$ reported for operating laser gyros in Table II of ref. [3], which ranges from 8 to 240 mHz.

When the locking condition is established, the two modes of equal frequency produce a static standing grating in the gain medium, and the reflection from this grating further stabilize the locking state. The effect in a laser gyro of reflections from a standing index grating has been analyzed in [1].

B. Scattering Due to Gain Is Predominant

This case corresponds to set $\kappa_m = 0$ in Eqs. (68) and (69), and is more conveniently studied by frequency shifting $\mathbf{a}(t)$ by $-\Omega_0/2$ and $\mathbf{a}(t)$ by $\Omega_0/2$ by

$$\mathbf{a}'(t) = \mathbf{a}(t)\exp(i\Omega_0 t/2), \tag{78}$$

$$\mathbf{b}'(t) = \mathbf{b}(t)\exp(-i\Omega_0 t/2), \tag{79}$$

so that the transformed field $\mathbf{a}'(t)$ is centered at frequency $\omega_0 - \Omega_0/2$ and $\mathbf{b}'(t)$ around $\omega_0 + \Omega_0/2$. The new fields obey the following equations:

$$\frac{\mathrm{d}\mathbf{a}'(t)}{\mathrm{d}t} = \kappa_g\mathbf{b}'(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{a}'(t) + \left[-i\sqrt{\gamma}\mathbf{s}_a^{(-)}(t) + \sqrt{\gamma}\mathbf{s}_a(t)\right]e^{i\Omega_0 t/2}, \tag{80}$$

$$\frac{\mathrm{d}\mathbf{b}'(t)}{\mathrm{d}t} = \kappa_g^*\mathbf{a}'(t) + \left[-\frac{\gamma}{2} + \frac{g^2}{\Gamma}\mathbf{n}(t)\right]\mathbf{b}'(t) + \left[-i\sqrt{\gamma}\mathbf{s}_b^{(-)}(t) + \sqrt{\gamma}\mathbf{s}_b(t)\right]e^{-i\Omega_0 t/2}. \tag{81}$$

The transformations (78) and (79) allow us to define independent phase references for the two modes. Defining $\Delta\varphi' = \varphi_a' - \varphi_b' - \varphi_g$, where φ_a' and $-\varphi_b'$ are the phases of the frequency-shifted fields, corresponding to $\Delta\varphi' = \varphi_a - \varphi_b - \varphi_g + \Omega_0 t$ in terms of the phases of the original fields, we obtain

$$\frac{d\Delta\phi'}{dt} = -\left|\kappa_g\right|\left(\frac{|b_0|}{|a_0|} + \frac{|a_0|}{|b_0|}\right)\sin(\Delta\phi'), \tag{82}$$

and also

$$\frac{d|a_0|}{dt} = \frac{\Delta g}{2}|a_0| + |\kappa_g||b_0|\cos(\Delta\varphi), \quad (83)$$

$$\frac{d|b_0|}{dt} = \frac{\Delta g}{2}|b_0| + |\kappa_g||a_0|\cos(\Delta\varphi). \quad (84)$$

Steady state is achieved for $|a_0| = |b_0|$, $\Delta g = -2|\kappa_g|\cos(\Delta\varphi)$ and for values of $\Delta\varphi = 0$ and $\Delta\varphi = \pi$. Of the two solutions, only $\Delta\varphi = 0$ is stable because $\Delta g = -2|\kappa_g|\cos(\Delta\varphi) < 0$. This condition correspond to a locking of the two modes at a difference frequency Ω_0.

Linearization of Eq. (82) about the steady state $\Delta\varphi' = 0$ (and removing the prime for simplicity of notation) gives

$$\frac{d\Delta\varphi}{dt} = -2|\kappa|\Delta\varphi. \quad (85)$$

This equation can be extended to the quantum domain defining $\Delta\boldsymbol{\varphi} = \delta\mathbf{a}_2'/a_0 - \delta\boldsymbol{b}_2'/b_0$ and adding the proper noise terms as

$$\frac{d\Delta\boldsymbol{\varphi}}{dt} = -2|\kappa|\Delta\boldsymbol{\varphi} + \mathbf{s}_{\Delta\varphi}, \quad (86)$$

where

$$\mathbf{s}_{\Delta\varphi} = \frac{\sqrt{\gamma}}{a_0}\left[\mathbf{s}_{a,2}^{(-)}(t) + \mathbf{s}_{a,1}(t)\right]e^{i\Omega_0 t/2} - \frac{\sqrt{\gamma}}{b_0}\left[\mathbf{s}_{b,2}^{(-)}(t) + \mathbf{s}_{b,1}(t)\right]e^{-i\Omega_0 t/2}. \quad (87)$$

The frequency shift of the two independent white noise terms in the two lines of Eq. (87) has no effect on their statistical properties and can be neglected. Solution of Eq. (87) shows that $\Delta\varphi$ has a Lorentzian spectrum. The phase noise $\Delta\boldsymbol{\varphi}$ is a stationary process with power spectrum

$$W_{\Delta\varphi}(\omega) = \frac{\gamma^2\hbar\omega_0}{P(\omega^2 + 4|\kappa_g|^2)}, \quad (88)$$

corresponding to the following auto-correlation function of the phase fluctuations

$$\langle\Delta\varphi(t+\tau)\Delta\varphi(t)\rangle = \frac{\gamma^2\hbar\omega_0}{4P|\kappa_g|}\exp\left(-2|\kappa_g||\tau|\right). \quad (89)$$

Here, we assumed once again the full inversion $\langle\sigma_3\rangle = 1$. The power spectrum of the (angular) frequency fluctuations is therefore

$$W_{\Delta\omega_{\rm meas}}(\omega) = \frac{\gamma^2\hbar\omega_0}{P}\frac{\omega^2}{\omega^2 + 4|\kappa_g|^2}. \quad (90)$$

The uncertainty of a frequency measurement performed over a time T is

$$\boldsymbol{\omega}_{\rm meas}T = \omega_0 T + \Delta\boldsymbol{\varphi}(t+T) - \Delta\boldsymbol{\varphi}(t), \quad (91)$$

so that, considering that $\langle\Delta\varphi^2\rangle = 2\langle\varphi(t)\rangle - 2\langle\Delta\varphi(t+T)\Delta\varphi(t)\rangle$, the uncertainty in a frequency measurement defined like in Eq. (50), is

$$\Delta\omega_{\rm meas}^2 = \frac{\gamma^2\hbar\omega_0}{2PT^2|\kappa_g|}\left[1 - \exp\left(-2|\kappa_g|T\right)\right], \quad (92)$$

that is, using $\gamma = \omega_0/Q$,

$$\Delta\omega_{\rm meas} = \frac{\omega_0}{Q}\sqrt{\frac{\hbar\omega_0}{PT}\left[\frac{1 - \exp\left(-2|\kappa_g|T\right)}{2|\kappa_g|T}\right]}. \quad (93)$$

In the limit $|\kappa_g|T \to 0$, we obtain

$$\Delta\omega_{\rm meas} = \frac{\omega_0}{Q}\sqrt{\frac{\hbar\omega_0}{PT}}, \quad |\kappa_g|T \to 0, \quad (94)$$

that is, the known result for independent modes and is $\sqrt{2}$ times bigger than the frequency uncertainty of a single mode given by Eq. (51) [4], whereas for $|\kappa_g|T \gg 1$, we have

$$\Delta\omega_{\rm meas} = \frac{\omega_0}{QT}\sqrt{\frac{\hbar\omega_0}{2P|\kappa_g|}}, \quad |\kappa_g|T \gg 1. \quad (95)$$

The Allan variance can be easily calculated from the autocorrelation function as

$$\sigma_T^2 = \frac{\gamma^2\hbar\omega_0}{4P|\kappa_g|T^2}\left[3 - 4\exp\left(-2|\kappa_g|T\right) + \exp\left(-4|\kappa_g|T\right)\right]. \quad (96)$$

For $|\kappa_g|T \to 0$, we obtain the Allan variance for unlocked modes, corresponding to white frequency noise

$$\sigma_T^2 = \frac{\gamma^2\hbar\omega_0}{PT}, \quad |\kappa_g|T \to 0, \quad (97)$$

whereas for $|\kappa_g|T \gg 1$, the Allan variance of white phase noise

$$\sigma_T^2 = \frac{3\gamma^2\hbar\omega_0}{4P|\kappa_g|T^2}, \quad |\kappa_g|T \gg 1. \quad (98)$$

The complete expressions of the Allan variances of the output of a laser gyro, which include the effect of the shot noise of the detection, are given in Supplement 1.

4. DISCUSSION

Our analysis assumed that the two counterpropagating modes share the same gain medium, implicitly neglecting spectral hole burning in the gain medium and considering only the saturation of the total number of inverted atoms. Indeed, in HeNe laser gyros, a 50:50 mixture of two isotopes of Ne, Ne^{20}, and Ne^{22} are used aiming to stabilize the two-mode operation [1,20,21]. With only one isotope of Ne present (resulting in a single gain curve), both counterpropagating modes would create a hole at the peak of the Doppler broadened-gain spectrum, corresponding to a component of velocity along the local laser axis that approaches zero. However, when two isotopes with different gain spectra are present, the peak of the composite gain spectrum shifts to an intermediate position between the two individual spectra. Consequently, the laser operating frequency deviates from the maximum of the gain curve of each of the two isotopes, and the atoms responsible for the gain at that frequency have opposite velocities for each of the two counterpropagating modes, with the maximum velocity difference occurring when the isotopic ratio is 50:50. With this choice, the two counterpropagating waves create holes in the gain spectrum with a maximum frequency separation, stabilizing the two-mode operation of the laser.

These considerations may lead us to conclude that a comprehensive description of the laser should consider independent gain media for the two counterpropagating modes. We believe, however, that this is an oversimplification. Indeed, out analysis relies on the existence of a gain grating in the laser gyro, and the aforementioned picture does not hinder the effectiveness of its generation. The coupling between the gain medium and the two counterpropagating modes is minimal at the nodes of the quasi-static standing wave, while it is maximal at the antinodes, a principle used to control spontaneous emission in micorocavities [22]. The resulting depletion of active atoms follows the modulation of the standing wave and generates a gain modulation, whose persistence is counteracted by diffusion. Once spectral holes are formed in the HeNe gain spectrum at a specific frequency for each counterpropagating mode, collisions act to restore a thermal distribution of the inverted atoms, thereby equalizing the gain compression across the entire gain bandwidth. This further justifies our assumption that the two counterpropagating modes share the same active medium.

Additionally, the dynamic index grating generated by a broadband gain modulation, which we discussed previously, may further contribute to the coupling between the two counterpropagating waves. This mechanism can be very effective because the gain modulations that are effective in generating the index grating are not only those at the lasing frequency, but also those generated at a shifted frequency by the counterpropagating waves.

5. CONCLUSION

In the absence of locking, the two modes fluctuate independently, and their phase difference undertakes free diffusion. The effect of the gain grating is to lock the relative phase of the two modes. While free diffusion of the individual modes is not affected, the relative phase diffusion is suppressed. Mathematically, this is the result of the presence of restoring force in the dynamical equation for the phase difference, which effectively suppresses the effect of the quantum noise on the phase difference between the two modes, thus stabilizing the difference frequency of the laser gyro.

This scenario is very similar to the mode-locked laser case [9], where the width of the individual lines of the frequency comb have a Lorentzian shape with the Schawlow–Townes linewidth corresponding to the total intracavity power, whereas the linewidth of the beat is delta-like if the repetition rate of the laser is locked to an external microwave source by a feedback loop acting on the cavity length [9]. This property is used in the realization of clockworks based on optical transitions using phase-stabilized mode-locked lasers [10].

In a conventional laser, the mode spacing is determined by the cavity geometry, namely, by the roundtrip time. In a laser gyro, the spacing between the two couterpropagating modes is determined by the cavity geometry and by the rotation rate of the gyro, which produces an effective roundtrip time difference between the two modes. In the absence of locking, in both cases, the instantaneous frequency difference between the two modes is affected by their independent phase diffusion.

The modes of a laser may lock together when the locked configuration requires lower energy compared to the unlocked one. This is the case of passively mode-locked lasers, where the locked configuration corresponds to a pulsed operation, with pulses energetically preferred because of the presence of a saturable absorbing action within the laser cavity. In the case of a laser gyro where the reflection from a dynamical gain (or index) grating occurs, the configuration in which the two modes are locked requires less gain because of the constructive interference with the component of the opposite propagating mode reflected from the gain grating. In the case of mode-locked lasers, the mode beat has a residual linewidth because frequency noise, also originated by the spontaneous emission and hence of quantum origin, couples to the pulse timing via the intracavity dispersion, inducing a timing jitter that perturbs the ideal periodicity of the pulse train [9]. If timing jitter is controlled, like in the case of active mode locking, the individual lines of the frequency comb have a linewidth that depends on the stability of the intracavity optical modulator.

In laser gyros where spurious reflections from the mirrors are minimized, dynamic locking of the two counterpropagating modes is caused by a dynamic gain grating that controls the fast fluctuations induced by the spontaneous emission. Like in passively mode-locked lasers, the locking does not prevent the possibility that the mode beat follows the dynamic change of the mode spacing, if this change occurs over a time scale longer than the lifetime of the grating, which is related to the excited state lifetime of the active medium. The locking mechanism may be responsible for the recently observed sub-shot-noise performance of the GINGERino laser gyro [11–13]. We may speculate that locking of non-degenerate modes may also be stabilized by a suitable design of the laser, adding for instance a slow saturable absorber into the laser cavity, or by a feedback loop with a long integration time acting on the cavity roundtrip time to stabilize the beat frequency between the two counterpropagating modes.

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Supplemental document. See Supplement 1 for supporting content.

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