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Research Paper

Codimensions of algebras with pseudoautomorphism and their exponential growth ☆

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Let F be a fixed field of characteristic zero containing an element i such that $i^2 = -1$. In this paper we consider finite dimensional superalgebras over F endowed with a pseudoautomorphism p and we investigate the asymptotic behavior of the corresponding sequence of p-codimensions $c_n^p(A)$, $n = 1, 2, \ldots$. First we give a positive answer to a conjecture of Amitsur in this setting: the p-exponent $\exp^p(A) = \lim_{n\to\infty} \sqrt[n]{c_n^p(A)}$ always exists and it is an integer. In the final part we characterize the algebras whose exponential growth is bounded by 2.

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1. Introduction

Let F be a fixed field of characteristic zero containing an element i such that $i^2 = -1$ and consider an associative superalgebra $A = A_0 \oplus A_1$ over F. A graded linear map p is a pseudoautomorphism on A if, for any homogeneous elements $a, b \in A_0 \cup A_1$,

$$p^2(a) = (-1)^{|a|}a \qquad \text{and} \qquad p(ab) = (-1)^{|a||b|}p(a)p(b).$$

In [9], the author proved that pseudoautomorphisms represent the connection link between graded involutions, superinvolutions and pseudoinvolutions; such maps play a prominent role in the setting of Lie and Jordan superalgebras (see for instance [14–16]) and they have been extensively studied within the theory of polynomial identities (see [1,12] and the references therein).

It is well-known that the study of the polynomial identities satisfied by an ordinary algebra A (with no additional structure) is equivalent to the study of the multilinear ones and an effective way to measure such identities is through the sequence of codimensions $c_n(A)$, $n = 1, 2, \ldots$, of A. Recall that if P_n is the space of multilinear polynomials in the non-commuting variables $x_1, \ldots, x_n$ and $\mathrm{Id}(A)$ is the ideal of identities of A, then $c_n(A) = \dim P_n/(P_n \cap \mathrm{Id}(A))$. The asymptotic behavior of this sequence has been extensively studied leading to classification results of several varieties of algebras. The key result in this area says that the sequence of codimensions of an algebra satisfying a non-trivial polynomial identity is exponentially bounded ([17]) and its exponential rate of growth is an integer ([3,4]).

Let A be a superalgebra with pseudoautomorphism. In this paper we study the p-polynomial identities satisfied by A and we investigate the asymptotic behavior of the corresponding sequence $c_n^p(A)$ of p-codimensions. Notice that such a sequence is bounded from above by $4^n n!$. Nevertheless when A satisfies an ordinary (non trivial) identity, $c_n^p(A)$ is exponentially bounded (see [6]).

Now assume that A has finite dimension over the field F. In the first part of the paper we determine the exponential rate of growth of the sequence of p-codimensions, showing that

$$\exp^p(A) = \lim_{n\to\infty} \sqrt[n]{c_n^p(A)}$$

exists and it is a non-negative integer, called the p-exponent of A. Moreover $\exp^p(A)$ can be explicitly computed and it turns out to be equal to the dimension of a suitable semisimple p-subalgebra of A.

The last part of the paper is devoted to the characterization of those algebras whose p-exponent is bounded by 2 (see also [2,5,8,11]). If the p-exponent of an algebra A is bounded by 1, it is equivalent to say that the p-codimensions are polynomially bounded and that the variety generated by A does not contain the group algebra of $\mathbb{Z}_2$ and the algebra of 2×2 upper triangular matrices with suitable pseudoautomorphisms (see [6]). These are the only algebras generating minimal varieties of p-exponent 2.

Finally, new results concerning p-algebras generating varieties of minimal p-exponent > 2 will be obtained.

It is important to highlight that the starting point in the proof of all results of this paper is the Wedderburn-Malcev decomposition of a finite dimensional p-algebra based on the classification of the simple ones given in [10].

2. Superalgebras with pseudoautomorphism

Throughout this paper F will denote a field of characteristic zero containing an element i such that $i^2 = -1$ and $A = A_0 \oplus A_1$ an associative superalgebra (an algebra graded by $\mathbb{Z}_2$, the cyclic group of order 2) over F endowed with a pseudoautomorphism p. We say that a linear map $p\colon A \to A$ is a pseudoautomorphism if it preserves the grading (graded map) and for any elements $a, b \in A_0 \cup A_1$,

$$p^2(a) = (-1)^{|a|}a \qquad \text{and} \qquad p(ab) = (-1)^{|a||b|}p(a)p(b).$$

Here $|c| = 0$ or 1 denotes the homogeneous degree of $c \in A_0 \cup A_1$.

If A is a superalgebra with pseudoautomorphism we shall call it simply a p-algebra.

In case A is a finite dimensional algebra, its structure is known from a generalization of Wedderburn-Malcev's theorem proved in [6, Theorem 3]. Before stating it, recall that an ideal (subalgebra) I of A is a p-ideal (subalgebra) of A if it is a graded ideal (subalgebra) and $p(I) = I$. The algebra A is a simple p-algebra if $A^2 \neq 0$ and A has no non-trivial p-ideals.

Theorem 1. *Let A be a finite dimensional p-algebra. Then there exists a semisimple p-subalgebra B such that*

$$A = B + J = B_1 \oplus \cdots \oplus B_k + J,$$

where J, the Jacobson radical of A, is a p-ideal of A and $B_1, \ldots, B_k$ are simple p-algebras.

Since the classification of the simple p-algebras is known, the above result can be further refined.

To this end, consider the following simple superalgebras:

- $Q(n) = M_n(F) \oplus cM_n(F)$, where $M_n(F)$ is the algebra of $n \times n$ matrices over F and $c^2 = 1$;
- $M_{k,h}(F)$, the algebra of $n \times n$ matrices, $n = k + h$, $k \geq h \geq 0$, with the following $\mathbb{Z}_2$-grading

$$M_{k,h}(F) = \left\{ \begin{pmatrix} K & 0 \\ 0 & H \end{pmatrix} \mid K \in M_k(F),\ H \in M_h(F) \right\}$$

$$\oplus \left\{ \begin{pmatrix} 0 & R \\ S & 0 \end{pmatrix} \mid R \in M_{k\times h}(F),\ S \in M_{h\times k}(F) \right\}.$$

Given a superalgebra B, let $\bar{B}$ denote the superalgebra with the same graded vector space structure as B and product $\circ$ given on homogeneous elements a, b by the formula

$$a \circ b := (-1)^{|a||b|}ab.$$

Two p-algebras (A, p) and (A', p') are isomorphic if there exists an isomorphism of superalgebras $\tau\colon A \to A'$ such that $\tau(p(a)) = p'(\tau(a))$, for any $a \in A$.

Now we are ready to state the classification theorem of the finite dimensional simple p-algebras (see [10]).

Theorem 2. *Assume that the field F is also algebraically closed. A finite dimensional p-simple superalgebra A over F is isomorphic to one of the following:*

(1) $M_{k,h}(F)$ *endowed with the pseudoautomorphism*

$$p\left(\begin{pmatrix} K & R \\ S & H \end{pmatrix}\right) = \begin{pmatrix} PKP & \pm iPRQ \\ \pm iQSP & QHQ \end{pmatrix},$$

$P = \begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{pmatrix}, Q = \begin{pmatrix} I_{h_1} & 0 \\ 0 & -I_{h_2} \end{pmatrix}$, *$I_j$'s are identity matrices,* $k = k_1 + k_2$, $h = h_1 + h_2$, $k_1 \geq k_2$, $h_1 \geq h_2$.

(2) $M_{k,k}(F)$ *endowed with the pseudoautomorphism p given by* $p\left(\begin{pmatrix} K & R \\ S & H \end{pmatrix}\right) = \begin{pmatrix} H & iS \\ iR & K \end{pmatrix}$.

(3) $M_{k,h}(F) \oplus \overline{M_{k,h}(F)}$ *with the pseudoautomorphism pex given by* $pex(a, b) = \left((-1)^{|(a,b)|}b, a\right)$.

(4) $Q(n) \oplus \overline{Q(n)}$, *with the pseudoautomorphism pex defined above.*

(5) $Q(n)$ *endowed with the following pseudoautomorphism*

$$p(a + cb) = f(a) \pm icf(b),$$

where f is an automorphism of order ≤ 2 *on* $M_n(F)$.

3. The p-exponent

Now we are interested in studying the p-algebras in the context of the theory of polynomial identities.

If A is a p-algebra, since char $F = 0$ and there exists $i \in F$ such that $i^2 = -1$, we can write

$$A = A_0^+ \oplus A_0^- \oplus A_1^i \oplus A_1^{-i}.$$

Here $A_0^+ = \{a \in A_0 \mid p(a) = a\}$ and $A_0^- = \{a \in A_0 \mid p(a) = -a\}$ are the sets of symmetric and skew elements of A_0, respectively, and $A_1^i = \{a \in A_1 \mid p(a) = ia\}$ and $A_1^{-i} = \{a \in A_1 \mid p(a) = -ia\}$ are the sets of the so-called i-symmetric and i-skew elements of A_1, respectively.

One can define in a natural way a pseudoautomorphism on the free associative algebra $F\langle X\rangle$ on a countable set X over F. We write X as the union of two disjoint infinite sets Y and Z, requiring that their elements are of homogeneous degree 0 and 1, respectively. Then each set is written as the disjoint union of two other infinite sets of symmetric and skew elements and of i-symmetric and i-skew elements, respectively. The free superalgebra with pseudoautomorphism is denoted $F\langle Y \cup Z, p\rangle$ and we write

$$F\langle Y \cup Z, p\rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \dots\rangle,$$

where y_i^+ stands for a (even) symmetric variable, y_i^- for a (even) skew variable, z_i^+ for a (odd) i-symmetric variable and z_i^- for a (odd) i-skew variable.

A polynomial $f \in F\langle Y \cup Z, p\rangle$ is a p-polynomial identity of A (or simply a p-identity), and we write $f \equiv 0$, if it vanishes for all substitutions $y^\pm \mapsto a^\pm \in A_0^\pm$, $z^\pm \mapsto b^\pm \in A_1^{\pm i}$. Let $\mathrm{Id}^p(A)$ denote the set of all p-identities of A. Clearly it is an ideal of $F\langle Y \cup Z, p\rangle$ invariant under all endomorphisms of the free superalgebra commuting with the pseudoautomorphism p. As in the ordinary case, it is easily seen that in characteristic zero, every p-identity is equivalent to a system of multilinear p-identities. Hence if we denote by

$$P_n^p = \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n,\ w_i \in \{y_i^+, y_i^-, z_i^+, z_i^-\},\ i = 1, \dots, n\}$$

the space of multilinear polynomials of degree n in the variables y_i^+, y_i^-, z_i^+, z_i^-, $i = 1, \dots, n$, the study of $\mathrm{Id}^p(A)$ is equivalent to the study of $P_n^p \cap \mathrm{Id}^p(A)$, for all $n \geq 1$. The n-th p-codimension of A is the non-negative integer

$$c_n^p(A) = \dim_F \frac{P_n^p}{P_n^p \cap \mathrm{Id}^p(A)},\ n \geq 1.$$

If A satisfies an ordinary polynomial identity, it was proved that $c_n^p(A)$, $n = 1, 2, \dots$, is exponentially bounded ([6]).

The first aim of this paper is to determine the exponential rate of growth of the sequence of p-codimensions of a finite dimensional p-algebra. We start with the following definition.

Definition 3. Let $A = B_1 \oplus \cdots \oplus B_k + J$ be a finite dimensional p-algebra and let $C_1, \dots, C_h$ be distinct simple p-subalgebras of A from $\{B_1, \dots, B_k\}$. The p-algebra $C = C_1 \oplus \cdots \oplus C_h$ is called p-admissible if

$$C_1 J \cdots JC_{h-1} JC_h \neq 0.$$

Now we can define the following integer.

Definition 4. Let A be a finite dimensional p-algebra. We set

$$d = d(A) := \max \{\dim_F C : \ C \text{ is an admissible p-subalgebra of } A\}.$$

The role of such an integer in the description of the asymptotic behavior of the p-codimensions is explained in the following result.

Theorem 5. *Let A be a finite dimensional p-algebra over F and consider the integer d of Definition 4. Then there exist constants $a_1 > 0$ and a_2, b_1, b_2 such that*

$$a_1 n^{b_1} d^n \leq c_n^p(A) \leq a_2 n^{b_2} d^n.$$

Hence the p-exponent of A, $\exp^p(A) = \lim_{n\to\infty} \sqrt[n]{c_n^p(A)}$ exists and it is a non-negative integer.

Proof. Since the p-codimensions do not change by extending the ground field, we may assume that the field F is algebraically closed. Now the result can be proved, with the necessary changes, by following word by word the proof given in [7] in the setting of superalgebras with superinvolution. □

As an immediate consequence we get the following.

Corollary 6. *Under the hypotheses of Theorem 5, the sequence $c_n^p(A)$, $n = 1, 2, \ldots$, either is polynomially bounded (i.e., $\exp^p(A) \leq 1$) or it grows exponentially (i.e., $\exp^p(A) \geq 2$).*

In [6], the second author described the varieties of p-algebras of polynomial growth by giving a finite list of p-algebras to be excluded from the variety. Recall that the growth of a variety $\mathcal{V}$ of p-algebras is defined as the growth of the sequence of p-codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \text{var}^p(A)$. Then we say that $\mathcal{V}$ has polynomial growth if $c_n^p(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n^p(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

Let $F \oplus F$ be the two-dimensional commutative algebra. We can see it as a superalgebra with trivial grading or with the natural grading

$$F \oplus F = F(1, 1) \oplus F(1, -1).$$

We consider the following three superalgebras with pseudoautomorphism:

- D, the algebra $F \oplus F$ with trivial grading and pseudoautomorphism $ex(a, b) = (b, a)$.

- D^i, the algebra $F \oplus F$ with natural grading and pseudoautomorphism $ex_i(a,b) = i^{|(a,b)|}(b,a)$.
- D^{-i}, the algebra $F \oplus F$ with natural grading and pseudoautomorphism $ex_{-i}(a,b) = (-i)^{|(a,b)|}(b,a)$.

Now consider the algebra $UT_2(F) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a,b,c \in F \right\}$ of 2×2 upper-triangular matrices. We can see it as a superalgebra with trivial grading or with the natural grading

$$UT_2(F) = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \mid a,c \in F \right\} \oplus \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in F \right\}.$$

We consider the following four superalgebras with pseudoautomorphism:

- UT_2, the algebra $UT_2(F)$ with trivial grading and trivial pseudoautomorphism.
- UT_2^-, the algebra $UT_2(F)$ with trivial grading and pseudoautomorphism: $p\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$.
- UT_2^i, the algebra $UT_2(F)$ with natural grading and pseudoautomorphism: $p\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & ib \\ 0 & c \end{pmatrix}$.
- UT_2^{-i}, the algebra $UT_2(F)$ with natural grading and pseudoautomorphism: $p\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -ib \\ 0 & c \end{pmatrix}$.

These p-algebras characterize the varieties of polynomial growth as stated in the following result ([6, Theorem 9]).

Theorem 7. *Let A be a finite dimensional p-algebra over F. The sequence $c_n^p(A)$, $n = 1, 2, \ldots$, is polynomially bounded (i.e., $\exp^p(A) \leq 1$) if and only if UT_2, UT_2^-, UT_2^i, UT_2^{-i}, D, D^i, $D^{-i} \notin \mathrm{var}^p(A)$.*

Given two p-algebras A and B we say that they are equivalent if $\mathrm{Id}^p(A) = \mathrm{Id}^p(B)$.

Corollary 8. *[6] The algebras UT_2, UT_2^-, UT_2^i, UT_2^{-i}, D, D^i, D^{-i} are the only finite dimensional p-algebras, up to equivalence, generating varieties of almost polynomial growth.*

Now we recall that a variety $\mathcal{V}$ of p-algebras is minimal with respect to the p-exponent if for any proper subvariety $\mathcal{U}$, we have that $\exp^p(\mathcal{V}) > \exp^p(\mathcal{U})$. Here the p-exponent of a variety is the p-exponent of a generating algebra. Since the above algebras have p-exponent equal to 2, by using this definition we get the following result.

Corollary 9. *The algebras UT_2, UT_2^-, UT_2^i, UT_2^{-i}, D, D^i, D^{-i} are the only finite dimensional p-algebras, up to equivalence, generating minimal varieties of p-exponent 2.*

4. Characterizing algebras with p-exponent bounded by 2

We start this section by constructing a suitable finite list of p-algebras generating varieties of p-exponent ≥ 2.

Let us consider the following simple p-algebras:

- C_1, the superalgebra $M_{2,0}(F)$ with trivial pseudoautomorphism id;
- C_2, the superalgebra $M_{2,0}(F)$ with pseudoautomorphism p given by $p\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$;
- C_3, the superalgebra $M_{1,1}(F)$ with pseudoautomorphism p given by $p\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & ib \\ ic & d \end{pmatrix}$;
- C_4, the superalgebra $M_{1,1}(F)$ with pseudoautomorphism p given by $p\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -ib \\ -ic & d \end{pmatrix}$;
- C_5, the superalgebra $M_{1,1}(F)$ with pseudoautomorphism p given by $p\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} d & ic \\ ib & a \end{pmatrix}$;
- C_6, the superalgebra $Q(1) \oplus \overline{Q(1)}$ with the pseudoautomorphism $pex(a,b) = \big((-1)^{|(a,b)|}b, a\big)$.

By Theorem 5 we get the following result.

Remark 10. For any $i = 1, \ldots, 6$, $\exp^p(C_i) = 4$.

The above p-algebras allow us to prove the following lemma.

Lemma 11. *Let B be a simple p-algebra with $\dim_F B \geq 4$. Then $C_j \in \mathrm{var}^p(B)$, for some $j \in \{1, \ldots, 6\}$.*

Proof. We shall prove the lemma by constructing a p-subalgebra of B isomorphic to C_j for some $j \in \{1, \ldots, 6\}$.

Case 1. $B = (M_{k,h}(F), p)$ with $p\left(\begin{pmatrix} K & R \\ S & H \end{pmatrix}\right) = \begin{pmatrix} PKP & \pm iPRQ \\ \pm iQSP & QHQ \end{pmatrix}$, $P = \begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{pmatrix}$, $Q = \begin{pmatrix} I_{h_1} & 0 \\ 0 & -I_{h_2} \end{pmatrix}$.

Suppose first that $h = 0$. This means that the superalgebra B has trivial grading and the pseudoautomorphism p is just a graded automorphism. If $k_2 = 0$, p is the identity map and it is immediate to see that the p-subalgebra of B generated by the elements $a_1 = e_{1,1}$, $a_2 = e_{1,k_1}$, $a_3 = e_{k_1,1}$, $a_4 = e_{k_1,k_1}$ is isomorphic to C_1. Now assume $k_2 > 0$. We easily get that the p-subalgebra C' of B generated by the elements $a_1 = e_{1,1}$, $a_2 = e_{k_1+1,k_1+1}$, $a_3 = e_{1,k_1+1}$, $a_4 = e_{k_1+1,1}$ is isomorphic to C_2 through the isomorphism $f\colon C' \to C_2$, given by

$$f(a_1) = e_{1,1}, \quad f(a_2) = e_{2,2}, \quad f(a_3) = e_{1,2}, \quad f(a_4) = e_{2,1}.$$

We are left to deal with the case $h > 0$. Let C' be the p-subalgebra of B generated by the elements $a_1 = e_{1,1}$, $a_2 = e_{k+1,k+1}$, $a_3 = e_{1,k+1}$, $a_4 = e_{k+1,1}$. The linear map f given by

$$f(a_1) = e_{1,1}, \quad f(a_2) = e_{2,2}, \quad f(a_3) = e_{1,2}, \quad f(a_4) = e_{2,1},$$

is an isomorphism of p-algebras between C' and C_3 or C_4, according to the sign $(\pm i)$ of the pseudoautomorphism p.

Case 2. $B = (M_{k,k}(F), p)$ with $p\left(\begin{pmatrix} K & R \\ S & H \end{pmatrix}\right) = \begin{pmatrix} H & iS \\ iR & K \end{pmatrix}$.

Let C' be the p-subalgebra of B generated by $a_1 = e_{1,1}$, $a_2 = e_{k+1,k+1}$, $a_3 = e_{1,k+1}$, $a_4 = e_{k+1,1}$. Hence we obtain an isomorphism of p-algebras between C' and C_5 via the linear map $f\colon C' \to C_5$ given by

$$f(a_1) = e_{1,1}, \quad f(a_2) = e_{2,2}, \quad f(a_3) = e_{1,2}, \quad f(a_4) = e_{2,1}.$$

Case 3. $B = \left(M_{k,h}(F) \oplus \overline{M_{k,h}(F)}, pex\right)$, $k \geq h \geq 0$.

Suppose first $k = h = 1$. The p-subalgebra C' generated by the elements $a_1 = (e_{1,1} + e_{2,2}, 0)$, $a_2 = (e_{1,2} + e_{2,1}, 0)$, $a_3 = (0, e_{1,1} + e_{2,2})$, $a_4 = (0, e_{1,2} + e_{2,1})$ is isomorphic to $C_6 = (F \oplus cF) \oplus \overline{(F \oplus cF)}$ through the isomorphism of p-algebras: $f\colon C' \to C_6$, given by

$$f(a_1) = (1, 0), \quad f(a_2) = (c1, 0), \quad f(a_3) = (0, 1), \quad f(a_4) = (0, c1).$$

Now assume $k > 1$. In this case, we get that the p-subalgebra of B generated by $a_1 = (e_{1,1}, e_{1,1})$, $a_2 = (e_{2,2}, e_{2,2})$, $a_3 = (e_{1,2}, e_{1,2})$, $a_4 = (e_{2,1}, e_{2,1})$ is isomorphic to C_1.

Case 4. $B = \left(Q(n) \oplus \overline{Q(n)}, pex\right)$.

If $n = 1$, then $B = C_6$ and there is nothing to prove. Now, let $n > 1$. Then C_1 is isomorphic to the p-subalgebra of B generated by the elements $a_1 = (e_{1,1}, e_{1,1})$, $a_2 = (e_{2,2}, e_{2,2})$, $a_3 = (e_{1,2}, e_{1,2})$ and $a_4 = (e_{2,1}, e_{2,1})$.

Case 5. $B = Q(n)$ with pseudoautomorphism $p(a + cb) = f(a) \pm icf(b)$, f automorphism of order ≤ 2 on $M_n(F)$.

Since $\dim_F B \geq 4$, we have that $n \geq 2$. Hence we can consider the p-subalgebra C' of B generated by $a_1 = e_{1,1}$, $a_2 = e_{2,2}$, $a_3 = e_{1,2}$ and $a_4 = e_{2,1}$. C' has trivial grading and induced pseudoautomorphism p. Since f is an order 2 automorphism of $M_n(F)$, it is well-known that it acts trivially on $M_2(F)$ or

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

According to the action of f, we get a p-algebras isomorphism between C' and C_1 or C_2 and we are done. □

Next we need to consider some suitable $\mathbb{Z}_2$-gradings and pseudoautomorphisms on the algebra UT_3 of 3×3 upper triangular matrices. Recall that an arbitrary triple (g_1, g_2, g_3) of elements of $\mathbb{Z}_2$ defines an elementary $\mathbb{Z}_2$-grading on UT_3 by setting:

$$(UT_3)_0 = \text{span}\{e_{i,j} \mid g_i + g_j = 0 \ (\text{mod } 2)\} \text{ and } (UT_3)_1 = \text{span}\{e_{i,j} \mid g_i + g_j = 1 \ (\text{mod } 2)\}.$$

On UT_3 we can define the following automorphisms (of order ≤ 2):

$$id\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\right) = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}, \qquad \varphi_1\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\right) = \begin{pmatrix} a & -b & -c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix},$$

$$\varphi_2\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\right) = \begin{pmatrix} a & b & -c \\ 0 & d & -e \\ 0 & 0 & f \end{pmatrix}, \qquad \varphi_3\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\right) = \begin{pmatrix} a & -b & c \\ 0 & d & -e \\ 0 & 0 & f \end{pmatrix}.$$

If UT_3 is endowed with trivial grading, the above automorphisms can be seen as pseudoautomorphisms.

Given any superalgebra $A = A_0 \oplus A_1$, one can consider the following pseudoautomorphism (recall that $i^2 = -1$)

$$\begin{aligned} p\colon A_0 \oplus A_1 &\to A_0 \oplus A_1 \\ a_0 + a_1 &\mapsto a_0 + ia_1. \end{aligned}$$

Notice that in case the superalgebra has trivial grading, the pseudoautomorphism p is actually the identity map.

According to the result of [13], it is not difficult to see that the composition between p and a graded automorphism of order ≤ 2 on UT_3 is a pseudoautomorphism of UT_3. Hence we have the following p-algebras:

- C_7, with trivial grading and trivial pseudoautomorphism id;
- C_8, with trivial grading and pseudoautomorphism φ_1;
- C_9, with trivial grading and pseudoautomorphism φ_2;

- C_{10}, with trivial grading and pseudoautomorphism φ_3;
- C_{11}, with grading induced by $(0,0,1)$ and pseudoautomorphism p;
- C_{12}, with grading induced by $(0,0,1)$ and pseudoautomorphism $p \circ \varphi_1$;
- C_{13}, with grading induced by $(0,0,1)$ and pseudoautomorphism $p \circ \varphi_2$;
- C_{14}, with grading induced by $(0,0,1)$ and pseudoautomorphism $p \circ \varphi_3$;
- C_{15}, with grading induced by $(0,1,1)$ and pseudoautomorphism p;
- C_{16}, with grading induced by $(0,1,1)$ and pseudoautomorphism $p \circ \varphi_1$;
- C_{17}, with grading induced by $(0,1,1)$ and pseudoautomorphism $p \circ \varphi_2$;
- C_{18}, with grading induced by $(0,1,1)$ and pseudoautomorphism $p \circ \varphi_3$;
- C_{19}, with grading induced by $(0,1,0)$ and pseudoautomorphism p;
- C_{20}, with grading induced by $(0,1,0)$ and pseudoautomorphism $p \circ \varphi_1$;
- C_{21}, with grading induced by $(0,1,0)$ and pseudoautomorphism $p \circ \varphi_2$;
- C_{22}, with grading induced by $(0,1,0)$ and pseudoautomorphism $p \circ \varphi_3$.

Remark 12. For $j = 7, \ldots, 22$, we have that $\exp^p(C_j) = 3$.

Proof. All the p-algebras C_j have the same Wedderburn-Malcev decomposition:

$$C_j = A_1 \oplus A_2 \oplus A_3 + J,$$

where $A_1 = Fe_{1,1}$, $A_2 = Fe_{2,2}$, $A_3 = Fe_{3,3}$ and $J = Fe_{1,2} \oplus Fe_{1,3} \oplus Fe_{2,3}$. Since $A_1JA_2JA_3 \neq 0$, $A_1 \oplus A_2 \oplus A_3$ is a maximal dimensional p-admissible subalgebra and the result follows by Theorem 5. □

The above p-algebras allow us to prove the following lemma.

Lemma 13. *Let $A = B_1 \oplus \cdots \oplus B_k + J$ be a finite dimensional p-algebra over an algebraically closed field F of characteristic zero. If there exist three distinct simple components $B_{i_1} \cong B_{i_2} \cong B_{i_3} \cong F$ such that $B_{i_1}JB_{i_2}JB_{i_3} \neq 0$, then $C_j \in \mathrm{var}^p(A)$, for some $j \in \{7, \ldots, 22\}$.*

Proof. Let e_1, e_2, e_3 be the unit elements of $B_{i_1}, B_{i_2}, B_{i_3}$, respectively. Then $e_l^2 = e_l^p = e_l \in (B_{i_l})_0$ and $e_r e_s = \delta_{rs} e_r$, for $r, s, l = 1, 2, 3$. Since $B_{i_1}JB_{i_2}JB_{i_3} \neq 0$ then $e_1Je_2Je_3 \neq 0$. So we may assume that there exist homogeneous (symmetric, skew, i-symmetric or i-skew) elements $j, j' \in J$ such that

$$e_1 j e_2 j' e_3 \neq 0.$$

Consider the p-subalgebra U of A linearly generated by

$$e_1, \quad e_2, \quad e_3, \quad e_1 j e_2, \quad e_2 j' e_3, \quad e_1 j e_2 j' e_3.$$

The linear map $f\colon U \to UT_3$, defined by

$$f(e_1) = e_{1,1}, \quad f(e_2) = e_{2,2}, \quad f(e_3) = e_{3,3}, \quad f(e_1 j e_2) = e_{1,2},$$
$$f(e_2 j' e_3) = e_{2,3}, \quad f(e_1 j e_2 j' e_3) = e_{1,3},$$

is an isomorphism of algebras. Now, by taking into account the homogeneous degrees of j and j' and their symmetry with respect to the pseudoautomorphism p, we get an isomorphism of p-algebras between U and C_j, for some $j = 7, \dots, 22$. $\square$

Let us consider the algebra

$$M = \left\{ \begin{pmatrix} a+\alpha a' & e+\alpha e' & 0 & 0 \\ 0 & b+\alpha b' & 0 & 0 \\ 0 & 0 & c+\alpha c' & 0 \\ 0 & 0 & f+\alpha f' & d+\alpha d' \end{pmatrix} \mid \right.$$
$$\left. a, a', b, b', c, c', d, d', e, e', f, f' \in F, \ \alpha^2 = 1 \right\}$$

and the automorphism $\dagger$ on it given by

$$\dagger \left(\begin{pmatrix} a+\alpha a' & e+\alpha e' & 0 & 0 \\ 0 & b+\alpha b' & 0 & 0 \\ 0 & 0 & c+\alpha c' & 0 \\ 0 & 0 & f+\alpha f' & d+\alpha d' \end{pmatrix} \right)$$
$$= \begin{pmatrix} d+\alpha d' & f+\alpha f' & 0 & 0 \\ 0 & c+\alpha c' & 0 & 0 \\ 0 & 0 & b+\alpha b' & 0 \\ 0 & 0 & e+\alpha e' & a+\alpha a' \end{pmatrix}.$$

We denote by M_1 the algebra M such that $a' = b' = c' = d' = e' = f' = 0$, endowed with trivial grading and pseudoautomorphism $\dagger$. Instead we use the symbol M_2 in case the grading is the elementary one induced by $(0, 1, 1, 0)$ and the pseudoautomorphism is $p \circ \dagger$. We need the following p-algebras:

- C_{23}, the subalgebra of M_1 with $b = c$;
- C_{24}, the subalgebra of M_1 with $a = d$;
- C_{25}, the subalgebra of M_2 with $b = c$;
- C_{26}, the subalgebra of M_2 with $a = d$.

Now notice that the algebra M can be seen as a superalgebra also with the following grading:

$$\left\{ \begin{pmatrix} a & e & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & f & d \end{pmatrix} \mid a, b, c, d, e, f \in F \right\}$$

$$\oplus \left\{ \begin{pmatrix} \alpha a' & \alpha e' & 0 & 0 \\ 0 & \alpha b' & 0 & 0 \\ 0 & 0 & \alpha c' & 0 \\ 0 & 0 & \alpha f' & \alpha d' \end{pmatrix} \mid a', b', c', d', e', f' \in F \right\}.$$

We denote by M_i the superalgebra M with the pseudoautomorphism ρ_i given by

$$\rho_i \left(\begin{pmatrix} a+\alpha a' & e+\alpha e' & 0 & 0 \\ 0 & b+\alpha b' & 0 & 0 \\ 0 & 0 & c+\alpha c' & 0 \\ 0 & 0 & f+\alpha f' & d+\alpha d' \end{pmatrix} \right) = \begin{pmatrix} d+i\alpha d' & f+i\alpha f' & 0 & 0 \\ 0 & c+i\alpha c' & 0 & 0 \\ 0 & 0 & b+i\alpha b' & 0 \\ 0 & 0 & e+i\alpha e' & a+i\alpha a' \end{pmatrix}.$$

Analogously, M_{-i} denote the superalgebra M with pseudoautomorphism ρ_{-i} defined as ρ_i but with $-i$ instead of i.

The last p-algebras we have to consider are the following:

- C_{27} is the subalgebra of M_i with $a = d$, $b = c$, $a' = d'$ and $c' = b' = 0$;
- C_{28} is the subalgebra of M_{-i} with $a = d$, $b = c$, $a' = d'$ and $c' = b' = 0$;
- C_{29} is the subalgebra of M_i with $a = d$, $b = c$, $b' = c'$ and $a' = d' = 0$;
- C_{30} is the subalgebra of M_{-i} with $a = d$, $b = c$, $b' = c'$ and $a' = d' = 0$.

Using the same approach as in Remark 12, we get the following result.

Remark 14. For $j = 23, \ldots, 30$, we have that $\exp^p(C_j) = 3$.

Now we can prove the following lemma.

Lemma 15. *Assume that the field F is also algebraically closed. Let $A = B_1 \oplus \cdots \oplus B_k + J$ be a finite dimensional p-algebra over F such that $B_l J B_m \neq 0$ with $(B_l, B_m) \in \{(F, D), (D, F), (F, D^i), (D^i, F), (F, D^{-i}), (D^{-i}, F)\}$, $l \neq m$. Then $C_j \in \mathrm{var}^p(A)$, for some $j \in \{23, \ldots, 30\}$.*

Proof. Suppose first that $(B_l, B_m) = (D, F)$. Let $e_l = e_1 + e_2$ and $e_m = e_3$ be the unit elements of B_l and B_m, respectively. Clearly $p(e_1) = e_2$ and $p(e_3) = e_3$. Since $B_l J B_m \neq 0$, there exists a homogeneous element $j \in J$ such that

$$e_l j e_m = (e_1 + e_2) j e_3 \neq 0.$$

Without loss of generality, we may assume that $e_1 j e_3 \neq 0$. Let U be the p-algebra linearly generated by the elements

$$e_1, \quad e_2, \quad e_3, \quad e_1 j e_3, \quad e_2 p(j) e_3.$$

When the homogeneous degree of j is 0, the map f defined by

$$f(e_1) = e_{1,1}, \quad f(e_2) = e_{4,4}, \quad f(e_3) = e_{2,2}+e_{3,3}, \quad f(e_1 j e_3) = e_{1,2}, \quad f(e_2 p(j) e_3) = e_{4,3},$$

is an isomorphism of p-algebras between U and C_{23}. When the homogeneous degree of j is 1, the map f defined by

$$f(e_1) = e_{1,1}, \quad f(e_2) = e_{4,4}, \quad f(e_3) = e_{2,2}+e_{3,3}, \quad f(e_1 j e_3) = e_{1,2}, \quad f(e_2 p(j) e_3) = ie_{4,3},$$

is an isomorphism of p-algebras between U and C_{25}.

In a similar way, when $(B_l, B_m) = (F, D)$ we obtain that $C_{24}, C_{26} \in \mathrm{var}^p(A)$.

Suppose now that $(B_l, B_m) = (D^i, F)$. Let $e_l = e_1$ and $e_m = e_2$ be the unit elements of B_l and B_m, respectively. Clearly $p(e_1) = e_1$ and $p(e_2) = e_2$. Since $B_l J B_m \neq 0$, there exists a homogeneous element $j \in J$ such that

$$e_l j e_m = e_1 j e_2 \neq 0.$$

Let U be the p-algebra linearly generated by the elements

$$e_1, \quad e_2, \quad ce_1, \quad e_1 j e_2, \quad e_1 p(j) e_2, \quad ce_1 j e_2, \quad ce_1 p(j) e_2.$$

Then U is isomorphic to C_{27} as p-algebras, via f defined by

$$f(e_1) = e_{1,1} + e_{4,4}, \quad f(e_2) = e_{2,2} + e_{3,3}, \quad f(ce_1) = c(e_{1,1} + e_{4,4}), \quad f(e_1 j e_2) = e_{1,2},$$
$$f(e_1 p(j) e_2) = e_{4,3}, \quad f(ce_1 j e_2) = ce_{1,2}, \quad f(ce_1 p(j) e_2) = ce_{4,3},$$

when the homogeneous degree of j is 0, and f defined by

$$f(e_1) = e_{1,1} + e_{4,4}, \quad f(e_2) = e_{2,2} + e_{3,3}, \quad f(ce_1) = c(e_{1,1} + e_{4,4}), \quad f(e_1 j e_2) = ce_{1,2},$$
$$f(e_1 p(j) e_2) = ice_{4,3}, \quad f(ce_1 j e_2) = e_{1,2}, \quad f(ce_1 p(j) e_2) = ie_{4,3},$$

when the homogeneous degree of j is 1.

In a similar way, when $(B_l, B_m) = (D^{-i}, F)$, (F, D^i) or (F, D^{-i}), we obtain that C_{28}, C_{29} or $C_{30} \in \mathrm{var}^p(A)$, respectively. □

The following proposition proves that the list of p-algebras $C_1, \ldots, C_{30}$ cannot be reduced.

Proposition 16. *For all $l, j \in \{1, \ldots, 30\}$, $l \neq j$, $\mathrm{Id}^p(C_l) \nsubseteq \mathrm{Id}^p(C_j)$.*

Proof. By the classification of pseudoautomorphisms on UT_n given in [13], we have that the p-algebras C_i, $i = 7, \ldots, 22$ are not pairwise equivalent. Now, let y denote an even variable, z an odd one and x any variable. The proof is completed by putting together the following facts.

- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 1, 3, 4, 7, 11, 13, 15, 16, 19, 22$, $j \neq l$: in fact $y^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 1, 2, 7, 8, 9, 10, 23, 24$, $j \neq l$: in fact $z \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 3$, $j = 4$: in fact $z^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 4$, $j = 3$: in fact $z^+ \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 2, \dots, 6, 12, 14, 17, 18, 20, 21, 25, 26$, $j \neq l$: in fact $[y_1^+, y_2^+] \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 1, \dots, 5$, $j = 7, \dots, 22$: in fact $\big[[x_1, x_2]^2, x_3\big] \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 5$, $j = 2, 3, 4, 25, 26$: in fact $[y^+, x] \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 5$, $j = 6$: in fact $y^- z^- + z^- y^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 6$, $j = 1, \dots, 5, 7 \dots, 30$: in fact $[x_1, x_2] \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 7, \dots, 30$, $j = 1, \dots, 6$: in fact $\exp^p(C_l) < \exp^p(C_j)$.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 6, 23, \dots, 30$, $j = 1, \dots, 5, 7, \dots, 22$: in fact $[x_1, x_2][x_3, x_4] \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 7, \dots, 22$, $j = 23, \dots, 26$: in fact $y_1^- y_2^- y_3^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 23$, $j = 24$: in fact $[y_1^-, y_2^-] y_3^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 24$, $j = 23$: in fact $y_3^- [y_1^-, y_2^-] \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 25$, $j = 26$: in fact $z^+ y^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 26$, $j = 25$: in fact $y^- z^+ \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 27, \dots, 30$, $j = 23, \dots, 26$: in fact $y_1^- y_2^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 28, 30$, $j = 27, 29$: in fact $z_1^+ z_2^+ \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 27, 29$, $j = 28, 30$: in fact $z_1^- z_2^- \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 27, 30$, $j = 28, 29$: in fact $z^- z^+ \equiv 0$ on C_l but not on C_j.
- $\mathrm{Id}^p(C_l) \not\subseteq \mathrm{Id}^p(C_j)$, $l = 28, 29$, $j = 27, 30$: in fact $z^+ z^- \equiv 0$ on C_l but not on C_j. □

Now we are in a position to characterize the p-algebras A with $\exp^p(A) \leq 2$.

Theorem 17. *Let A be a finite dimensional p-algebra over F. Then $\exp^p(A) \leq 2$ if and only if $C_j \notin \mathrm{var}^p(A)$, for any $j \in \{1, \dots, 30\}$.*

Proof. Since we are dealing with p-codimensions that do not change by extending the base field, in what follows we may assume that the field F is algebraically closed.

First let $\exp^p(A) \leq 2$. Since $\exp^p(C_i) > 2$, by Remarks 10, 12 and 14, we get $C_j \notin \mathrm{var}^p(A)$, $j \in \{1, \dots, 30\}$.

Conversely, let $C_j \notin \mathrm{var}^p(A)$, for any $j \in \{1, \dots, 30\}$. Hence by Theorem 1 we can write $A = B_1 \oplus \cdots \oplus B_m + J$, where the B_j's are simple p-algebras isomorphic to those ones given in Theorem 2. Since $C_1, \dots, C_6 \notin \mathrm{var}^p(A)$, according to Lemma 11, we have that $\dim_F B_l < 4$, for any l.

Suppose by contradiction that $\exp^p(A) > 2$. Then by Theorem 5, one of the following possibilities occurs:

1. there exist distinct $B_{i_1}, B_{i_2}, B_{i_3}$ such that $B_{i_1}JB_{i_2}JB_{i_3} \neq 0$ and $B_{i_1} \cong B_{i_2} \cong B_{i_3} \cong F$,
2. for some $i_1 \neq i_2$, $B_{i_1}JB_{i_2} \neq 0$ and $B_{i_1} \cong F$ and $B_{i_2} \cong D$ or D^i or D^{-i},
3. for some $i_1 \neq i_2$, $B_{i_1}JB_{i_2} \neq 0$ and $B_{i_1} \cong D$ or D^i or D^{-i} and $B_{i_2} \cong F$.

Notice that when we are in the situation $B_{i_1}JB_{i_2} \neq 0$ with $B_{i_1} \cong D$ or D^i or D^{-i} and $B_{i_2} \cong D$ or D^i or D^{-i}, it easily follows that one of the last two cases occurs.

If 1. holds, then, by Lemma 13, $C_j \in \operatorname{var}^p(A)$, for some $j \in \{7, \dots, 22\}$, a contradiction. We reach a contradiction also in all the other cases, since by Lemma 15, we should have that $C_j \in \operatorname{var}^p(A)$, for some $j \in \{23, \dots, 30\}$. □

In light of Theorems 7 and 17, we get the characterization of p-algebras with p-exponent equal to two.

Corollary 18. *Let A be a finite dimensional p-algebra over F. Then $\exp^p(A) = 2$ if and only if*

- *$C_j \notin \operatorname{var}^p(A)$, for all $j \in \{1, \dots, 30\}$ and*
- *either UT_2 or UT_2^- or UT_2^i or UT_2^{-i} or D or D^i or $D^{-i} \in \operatorname{var}^p(A)$.*

Now let us slightly change the definition of minimal varieties given at the end of Section 3: a variety $\mathcal{V}$ of p-algebras is minimal with respect to the p-exponent if for any proper subvariety $\mathcal{U}$, generated by a finite dimensional p-algebra, we have that $\exp^p(\mathcal{V}) > \exp^p(\mathcal{U})$. By using this definition we get the following.

Corollary 19.

1. *The p-algebras C_j, $j = 1, \dots, 6$, generate minimal varieties of p-exponent 4.*
2. *The p-algebras C_j, $j = 7, \dots, 30$, are the only finite dimensional algebras, up to equivalence, generating minimal varieties of p-exponent 3.*

Proof. We prove just item 2. (the proof of 1. is similar).

Let $\mathcal{V}$ be a proper subvariety of $\operatorname{var}^p(C_j)$, $j = 7, \dots, 30$. Clearly $C_j \notin \mathcal{V}$. Also, by Proposition 16, we get that $C_l \notin \mathcal{V}$, for any $l = 1, \dots, 30$. Then, from Theorem 17, $\exp^p(\mathcal{V}) \leq 2$ and we are done.

Now suppose that there exists a minimal variety $\mathcal{U}$ of p-exponent 3 which is not generated by any of the algebras in 2. Since $\mathcal{U}$ is minimal and its p-exponent is 3, $C_j \notin \mathcal{U}$, for any j. Then by Theorem 17 we should have $\exp^p(\mathcal{U}) \leq 2$, a contradiction. □

Data availability

No data was used for the research described in the article.

References

[1] L. Centrone, A. Estrada, A. Ioppolo, On PI-algebras with additional structures: rationality of Hilbert series and Specht's problem, J. Algebra 592 (2022) 300–356.
[2] A. Giambruno, A. Ioppolo, D. La Mattina, Trace codimensions of algebras and their exponential growth, Isr. J. Math. 254 (1) (2023) 431–459.
[3] A. Giambruno, M. Zaicev, On codimension growth of finitely generated associative algebras, Adv. Math. 140 (1998) 145–155.
[4] A. Giambruno, M. Zaicev, Exponential codimension growth of PI-algebras: an exact estimate, Adv. Math. 142 (1999) 221–243.
[5] A. Giambruno, M. Zaicev, A characterization of varieties of associative algebras of exponent two, Serdica Math. J. 26 (2000) 245–252.
[6] G. Giordani, On superalgebras with pseudoautomorphism of polynomial codimension growth, Commun. Algebra 52 (2024) 4093–4104.
[7] A. Ioppolo, The exponent for superalgebras with superinvolution, Linear Algebra Appl. 555 (2018) 1–20.
[8] A. Ioppolo, A characterization of superalgebras with pseudoinvolution of exponent 2, Algebr. Represent. Theory 24 (6) (2021) 1415–1429.
[9] A. Ioppolo, Graded linear maps on superalgebras, J. Algebra 605 (2022) 377–393.
[10] A. Ioppolo, D. La Mattina, Classifying simple superalgebras with automorphism and pseudoautomorphism, J. Algebra 649 (2024) 1–11.
[11] A. Ioppolo, F. Martino, Classifying G-graded algebras of exponent two, Isr. J. Math. 229 (2019) 341–356.
[12] A. Ioppolo, F. Martino, Varieties of algebras with pseudoinvolution: codimensions, cocharacters and colengths, J. Pure Appl. Algebra 226 (5) (2022) 106920.
[13] A. Ioppolo, F. Martino, Gradings and graded linear maps on algebras, Forum Math. (2024), https://doi.org/10.1515/forum-2024-0098, in press.
[14] V.G. Kac, Lie superalgebras, Adv. Math. 26 (1) (1977) 8–96.
[15] C. Martinez, E. Zelmanov, Representation theory of Jordan superalgebras. I, Trans. Am. Math. Soc. 362 (2) (2010) 815–846.
[16] M.L. Racine, E.I. Zelmanov, Simple Jordan superalgebras with semisimple even part, J. Algebra 270 (2) (2003) 374–444.
[17] A. Regev, Existence of identities in $A \otimes B$, Isr. J. Math. 11 (1972) 131–152.