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Multi-species PDE systems with nonlocal interactions and small inertia

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Dottoranda
Valeria Iorio

Coordinatore del corso
Prof. Davide Gabrielli

Tutor
Prof. Marco Di Francesco

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Abstract

The present PhD thesis deals with the study of aggregation PDE systems with many species coupled through nonlocal interaction and considering inertial effects.

We first study a multi-dimensional system, considering both smooth and singular self-interaction potentials and requiring smooth assumptions on cross-interaction potentials. We provide existence and uniqueness results of measure solutions considering initial data in a Wasserstein space of probability measures. Then, we investigate the small inertia limits for both the smooth and singular case, proving convergence results towards the corresponding macroscopic first order systems. These results extend to the many species case previous results by Fetecau-Sun and Choi-Jeong.

We construct an upwind finite volume scheme for a kinetic system with two species. Here, the inertia term is not considered, and we require smooth assumptions on interaction potentials. A convergence result for the scheme is provided, without any restriction on the mesh size. This result is inspired by previous result by Filbet with minor modifications and a slight improvement of the rate of convergence.

Furthermore, we study a one-dimensional macroscopic system for two species coupled through nonlocal interactions, with an additional damping parameter. This system describes the dynamics of interacting particles; in case of collisions a sticky particles condition is adopted. We prove existence and uniqueness of measure solutions by using optimal transportation theory and taking initial data in a space of probability measures with finite second moments. A large-time large-damping result is obtained, proving the convergence towards the corresponding first order system. Finally, we investigate the case with Newtonian potentials for the self-interaction terms, with additional confining external potentials. For the latter case, we prove existence of solutions and a large time collapse result, showing the convergence towards Dirac delta solutions. The results are complemented with numerical simulations. Previous results on this problem only dealt with the one species case, see Brenier et al. for the existence of sticky particles and Carrillo, Choi and Tse for the large damping limit. We stress that the technique we use in the large damping limit is totally new.

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Chapter 1

Introduction

1.1 Particle and continuum models

Nonlocal aggregation models describe phenomena related to many domains of sciences, such as biology, robotics and space missions, artificial intelligence, social sciences, and traffic and pedestrian flows. In particular, for biological applications, the fundamental motivation is to understand and investigate the formation of many spectacular groups observed in nature, such as swarms, schools of fishes, and flocks of birds.

This kind of models can be studied by adopting two main points of view: *individual-based*, where one considers the dynamics of each particle in the group, and *partial differential equations*, describing the evolution of a density of individuals.

The use of (integro-)partial differential equations in this context has become very popular in order to analyse the evolution of a population density $\rho(t,x)$ subject to a space-dependent nonlocal interaction force W. A simple example of said models is the following

$$\frac{\partial \rho}{\partial t} = \operatorname{div}(\rho \nabla W * \rho). \tag{1.1}$$

In equation (1.1), x is a spatial variable typically ranging in $\mathbb{R}^d$, $t \geq 0$ is the time variable, $W = W(x)$ is a given interaction potential and $*$ denotes the spatial convolution. The potential W typically accounts for attractive or repulsive drift among individuals. Equation (1.1) is an example of *macroscopic model*, in which the population is treated in its entirety.

On the other hand, *discrete modelling* considers the evolution of each particle in the group. The "particle counterpart" of equation (1.1) is

$$\dot{x}_i = -\frac{1}{N} \sum_{j=1}^{N} \nabla W(x_i - x_j), \tag{1.2}$$

$i = 1, \ldots, N$, where N is the number of particles, x_i are the positions of the particles, and W is the same interaction potential as in (1.1). Observing the formations of aggregation phenomena occurring in nature, the particle description typically includes three mechanisms in three zones: a short-range repulsion zone, a long-range attraction zone and an alignment or orientation zone. These models are called *three-zone models*. System (1.2) is an example of *microscopic model*, in which the evolution of each particle is influenced by the interaction with the other particles.

First order models are often "too restrictive" in many cases, and a *second order* approach is more appropriate to describe the dynamics since it takes into account *inertial* *effects*. At least formally, a similar split-up between macroscopic and microscopic models holds also if we deal with second order systems, namely a second order PDEs system describing the evolution of a population density $\rho(t, x)$ under the influence of a nonlocal interaction force $W(x)$ can be written as

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}(\rho v) = 0\,, \\ \dfrac{\partial}{\partial t}(\rho v) + \mathrm{div}(\rho v \otimes v) = -\rho \nabla W * \rho\,, \end{cases} \tag{1.3}$$

and the corresponding individual-based system is given by the second order particle system

$$\begin{cases} \dot{x}_i = v_i\,, \\ \dot{v}_i = -\dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \nabla W(x_i - x_j)\,. \end{cases} \tag{1.4}$$

In system (1.3), $v = v(x, t)$ is the Eulerian velocity of the population. Moreover, the first equation in (1.3) is the continuity equation and describes the local conservation of mass, while the second equation describes the balance of momentum.

Together with the modeling from the two points of view described above, recently an increasing attention has been devoted to the kinetic description in this context. If we set $x \in \mathbb{R}^d$ as the position and $v \in \mathbb{R}^d$ as the velocity, a kinetic equation studies the evolution of $f(t, x, v)$, that is the probability measure of individuals at position x, with velocity v at time $t \geq 0$. The kinetic equation associated to particle system (1.4) is

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot ((\nabla_x W * \rho) f)\,, \tag{1.5}$$

where $\rho(t, x)$ is the macroscopic population density, i.e.,

$$\rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v)\, dv.$$

Equation (1.5) is an example of *mesoscopic model*, in which a statistical description of the interacting agent system is presented.

In recent years, the attention of many researchers in this field turned to systems with *many species*, motivated for example by opinion formation models, pedestrian movements, and other aggregation phenomena in biology. The second order modelling approach via (1.3) and (1.4) and the kinetic approach via (1.5) seem to be more useful since the inertial effects, sometimes referred in these contexts as “persistence” effects, do play a role in the model’s dynamics.

In this PhD thesis, we will focus on multi-species second order systems and in particular on macroscopic and kinetic models with nonlocal interactions.

In this introductory Chapter we will describe the problems we address in this thesis and the tools and techniques adopted to investigate them.

1.2 Small inertia limits

One of the main goals in this PhD thesis is to study the behaviour of solutions of our systems in relation to the persistence effects.

In Chapter 2, we extend to many species the particle model proposed in the paper [10] by Bodnar and Velazquez, that, in Newton's law form, is

$$\begin{cases} \varepsilon \dfrac{d^2 x_i}{dt^2} + \dfrac{dx_i}{dt} = F_i \,, \\ F_i = -\dfrac{1}{N} \displaystyle\sum_{j \neq i} \nabla_{x_i} W(x_i - x_j) \,, \end{cases} \tag{1.6}$$

$i = 1, \dots, N$, where x_i are the particle locations and W is a nonlocal interaction kernel, and $\varepsilon > 0$ is the *inertia* parameter. From a biological point of view, (1.6) takes into account a small response time of individuals. This occurs since when two agents interact, their "reactions" are not immediate, but a little time to replay is necessary. Formally, we see that by sending $\varepsilon \to 0$, we derive the first order model (1.2).

In Chapters 2 and 3 we will obtain rigorously small inertia limits for multi-species systems, considering both smooth potentials and singular self-potentials.

A relevant phenomenon we consider in this thesis is the *damping*, see the first order term in the first equation of (1.6). The macroscopic system (1.3) can be seen as a nonlocal version of compressible gas-dynamics, where in place of the pressure term $-\nabla p(\rho)$, a nonlocal interaction force $-\rho \nabla W * \rho$ is considered. Including also a friction term $\sigma > 0$ and an external force $-\nabla V$, the full model with pressure can be written as

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0 \,, \\ \dfrac{\partial}{\partial t}(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla p(\rho) = -\rho \nabla W * \rho - \rho \nabla V - \sigma \rho v. \end{cases} \tag{1.7}$$

At least on a formal level, an equation of the form

$$\frac{\partial \rho}{\partial t} = \operatorname{div}(\nabla p(\rho) + \rho \nabla (V + W * \rho))$$

can be obtained by rescaling time in (1.7) as $t = \sigma \tau$ and sending $\sigma \to +\infty$. This limit regime is called *overdamped limit* or *large friction limit.*

In this PhD thesis we will deal with pressure-less systems, namely $p = 0$.

In Chapter 5, starting from the model investigated in [12] that we briefly describe in Section 1.3, we extend said model to the two-species case, considering in addition a damping effect. We want to underline here that, after said time rescaling, the large damping effect can be seen as a small inertia effect, since a parameter $\sigma > 0$ equal to the reciprocal of ε will be sent to $+\infty$, in a regime that is equivalent to $\varepsilon \to 0$.

1.3 Sticky particle dynamics

As said, in this PhD thesis we consider pressure-less systems. From the macroscopic point of view, this implies that the population density $\rho(t, x)$ in (1.3) is not forced to be absolutely continuous with respect to the Lebesgue measure. Therefore, "particle" solutions in the spirit of (1.4) are allowed. When two particles collide, a standard way to continue the solution after collision is the so-called *sticky particle condition*, which forces particles to stay attached to each other after collision, with a post-collisional velocity that is uniquely determined by the conservation of momentum.

In this Section we briefly introduce the *sticky particles dynamics*, since we will adopt it in Chapter 5, where we will focus on a one-dimensional two-species pressure-less Euler system with nonlocal interactions. In particular, we consider here the to one-species case investigated in [12]. Here the authors study a model in dimension $d = 1$ for the evolution of a sticky particle system via adhesion dynamic and the macroscopic one-dimensional system considered is

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho v) = 0, \\ \dfrac{\partial}{\partial t}(\rho v) + \dfrac{\partial}{\partial x}(\rho v^2) = f[\rho], \end{cases} \tag{1.8}$$

in $[0, +\infty] \times \mathbb{R}$, with initial datum $(\rho, v)(t = 0) = (\rho_0, v_0)$.

In system (1.8), $\rho(t, \cdot) \in \mathcal{P}_2(\mathbb{R})$ is the population density, $v(t, \cdot) \in L^2(\mathbb{R}, \rho(t, \cdot))$ its Eulerian velocity and the map $f : \mathcal{P}_2(\mathbb{R}) \to \mathcal{M}(\mathbb{R})$ describes the force field. $\mathcal{P}_2(\mathbb{R})$ is the set of probability measures on $\mathbb{R}$ with finite second moment and $\mathcal{M}(\mathbb{R})$ is the space of signed Borel measures with finite total variation. The form of the force field considered in [12] is

$$f[\rho] = -\rho \partial_x \Big(V(x) + \int_{\mathbb{R}} W(x - y)\, d\rho(y) \Big), \tag{1.9}$$

and suitable assumptions on V and W are prescribed.

The sticky particle dynamics implies that particles stick together when collisions occur and then they are not allowed to split any longer. This can be formulated by considering the discrete measures

$$\rho^N(t, \cdot) := \sum_{i=1}^{N} m_i \delta_{x_i(t)}, \qquad (\rho v)^N(t, \cdot) := \sum_{i=1}^{N} m_i v_i(t) \delta_{x_i(t)}, \tag{1.10}$$

concentrated in a finite set of N particles with positions $x_i(t)$, masses m_i, velocities $v_i(t)$, $i = 1, \dots, N$, with ordered locations

$$x_1(t) \le x_2(t) \le \ldots \le x_{N-1}(t) \le x_N(t).$$

We also set $x(t) = (x_1(t), \dots, x_N(t))$, $v(t) = (v_1(t), \dots, v_N(t))$ belonging to $\mathbb{R}^N$, and $m = (m_1, \dots, m_N)$ belonging to $[0, +\infty)^N$. In this framework, the force field applied to the discrete measure ρ^N is given by

$$f[\rho^N](x) = \int_{\mathbb{R}} a(x)\, d\rho^N(x) = \frac{1}{N} \sum_{i=1}^{N} m_i a(x_i),$$

with

$$a(x) = -V'(x) - \sum_{k=1}^{N} m_k W'(x - x_k).$$

The above force field is well-defined on collisions only if the interaction potential W is $\mathcal{C}^1(\mathbb{R})$. The pairs (x_i, v_i) in (1.10) solve the ordinary differential equations system

$$\dot{x}_i(t) = v_i(t), \qquad \dot{v}_i(t) = a(x_i(t)),$$

equipped with initial data, between two consecutive particle collisions.

When a collision between two particles of masses m_k and m_{k+1} occurs at a time $t > 0$, the velocities of each of them are updated to

$$v_k(t+) = v_{k+1}(t+) = \frac{m_k v_k(t-) + m_{k+1} v_{k+1}(t-)}{m_k + m_{k+1}},$$

so that the momentum is conserved during the collision. If more then two particles stick together, collisions can be treated in a similar way. Then, one can observe that the measures ρ^N and $(\rho v)^N$ solve system (1.8). After a collision, the particles can be relabelled such that system (1.10) is still satisfied with the number of particles reduced because of the collision.

Since we are in the one-dimensional case, we consider ordered particles and this ordering has to be preserved. Let us introduce the closed convex cone

$$\mathbb{K}^N := \{x \in \mathbb{R}^N \,:\, x_1 \leq x_2 \leq \ldots \leq x_N\},$$

and assume that $x(t) \in \mathbb{K}^N$ for all $t \geq 0$. This implies that only consecutive particles may collide, so that we define

$$\Omega_x := \{j \,:\, x_j = x_{j+1},\, j = 1, \ldots, N\}.$$

Notice that the adhesion dynamics implies that the sets $\Omega_{x(t)}$ are non-decreasing in time. When the vector $x(t)$ touches the boundary of $\mathbb{K}^N$, namely

$$\partial\mathbb{K}^N = \{x \in \mathbb{K}^N \,:\, \Omega_x \neq \emptyset\},$$

an instantaneous force changes its velocity such that it still belongs to $\mathbb{K}^N$. Thus, introducing the normal cone of $\mathbb{K}^N$,

$$N_x\mathbb{K}^N := \{l \in \mathbb{R}^N \,:\, l \cdot (y - x) \leq 0 \text{ for all } y \in \mathbb{K}^N\},$$

one obtains that the said instantaneous force belongs to $N_x\mathbb{K}^N$.

Therefore, one can investigate the second order differential inclusion (cf. [14])

$$\dot{x} = v, \qquad \dot{v} + N_x\mathbb{K}^N \ni a(x),$$

in $[0, +\infty[$. The system above can be rephrased in terms of sub-differential inclusions in Lagrangian framework, that is strictly related to the Eulerian description, as we do in Chapter 5.

1.4 The finite volume method

Another gol of this PhD thesis is to investigate a finite volume method for a two-species system with nonlocal interactions at kinetic level, see Chapter 4. The development of these kind of numerical methods has become very important since they preserve the conservative properties of the PDEs. In order to give an idea of the method and in particular of what we will perform, let us give a brief introduction, [54]. Consider the scalar advection equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0\,, \tag{1.11}$$

where $u(x,t) \in \mathbb{R}^m$ is the unknown. Linear hyperbolic equations are quite easy to study, since the initial value problem is well posed for them and the solution is regular as the initial data for any time. By discretizing both in space and in time, it is possible to construct a stable scheme. Let $(x_j)_j$ be finite ordered equidistant points in $\mathbb{R}$ and T the final time and set $\Delta x = x_{j+1} - x_j$ and $\Delta t = T/N_T$ the space and time steps respectively, for some $N_T \in \mathbb{N}$. The *upwind* scheme uses a first order approximation of the space derivative, and the upwinding time discretization follows by discretizing the space derivatives as

$$\left.\frac{\partial u}{\partial x}\right|_{x_j} \approx \begin{cases} \dfrac{u_j - u_{j-1}}{\Delta x}, & \text{if } c \geq 0\,, \\ \dfrac{u_{j+1} - u_j}{\Delta x}, & \text{if } c < 0\,. \end{cases}$$

Thus, by explicit Euler and first order upwind space discretization, the scheme in compact form is

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(c^+(u_j^n - u_{j-1}^n) - c^-(u_{j+1}^n - u_j^n)),$$

where $c^+ = \max\{c, 0\}$, $c^- = -\min\{c, 0\}$ and $n \in \{0, \ldots, N_T - 1\}$. From now on, assume $c > 0$. In order to study the consistency of the upwind scheme, we apply the discrete operator to the exact solution of (1.11) obtaining

$$\mathcal{L}_\Delta u(x,t) = \frac{u(x, t + \Delta t) - u(x,t)}{\Delta t} + c\frac{u(x,t) - u(x - \Delta x, t)}{\Delta x}.$$

By considering in the previous equation the Taylor expansion of $u(x,t)$ both in space and in time, we get

$$\mathcal{L}_\Delta u(x,t) = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x, t + \tau) - \frac{c\Delta x}{2}\frac{\partial^2 u}{\partial x^2}(x - \xi, t),$$

with $\tau \in [0, \Delta t]$ and $\xi \in [0, \Delta x]$. If u satisfies equation (1.11), then the consistency error is

$$d(x,t) = \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x, t + \tau) - \frac{c\Delta x}{2}\frac{\partial^2 u}{\partial x^2}(x - \xi, t) = \mathcal{O}(\Delta x, \Delta t).$$

A scheme is defined *consistent* if $d(x,t) \to 0$ as $\Delta x \to 0$ and $\Delta t \to 0$. Here we see that the scheme is consistent to the first order in both Δt and Δx.

Concerning the *stability* of the scheme, we can check it assuming periodic boundary conditions and considering a solution of the form

$$u_j^n = \rho^n e^{ij\xi}\,.$$

Thus, we get

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}c(u_j^n - u_{j-1}^n)\,.$$

After some computations, one can see that the amplification factor ρ is such that $|\rho|^2 < 1$ if and only if $c\frac{\Delta t}{\Delta x} < 1$. This stability condition is a particular case of a more general condition known as CFL condition. In theory, the CFL number relates the speed of which information propagates in the mesh to the time step size.

In order to be able to construct the scheme proposed in Chapter 4, let us now switch to a hyperbolic quasilinear equation. Consider

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0\,, \tag{1.12}$$

where $u(x,t) \in \mathbb{R}^m$ is the unknown, and $f : \mathbb{R}^m \to \mathbb{R}^m$ is a given smooth function. The initial value problem for these kind of equations is well posed locally in time, and, generally, the solution loses the regularity of the initial data in finite time. Thus one needs to consider weak solutions that, assuming smooth initial data, seem piecewise smooth functions containing jump discontinuities. If we want a similar propagation speed of discontinuities at discrete level, then the numerical scheme has to be a *conservative scheme*.

In order to derive the scheme, let us construct a *numerical mesh*: we divide the space by using $N+1$ ordered equidistant points x_j and consider N cells $C_j = [x_{j-1/2}, x_{j+1/2}]$ with centers x_j, as $j = 1, \dots, N$. Integrating equation (1.12) over a cell C_j and dividing by $\Delta x = x_{j+1/2} - x_{j-1/2}$, we obtain

$$\frac{d\bar{u}_j}{dt} + \frac{1}{\Delta x}[f(u(x_{j+1/2}, t)) - f(u(x_{j-1/2}, t))] = 0\,,$$

where $\bar{u}_j$ denotes the cell average, i.e.,

$$\bar{u}_j(t) := \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t)\,dx.$$

Thus the numerical scheme should be

$$\frac{d\bar{u}_j}{dt} = -\frac{1}{\Delta x}[F_{j+1/2} - F_{j-1/2}],$$

with $F_{j+1/2}$ the numerical flux on the right edge of the cell C_j. Finally, if we take the numerical flux $F_{j+1/2}$ as a function of the cell averages, in a simple case $\bar{u}_j$ and $\bar{u}_{j+1}$, we arrive at a *semidiscrete* scheme.

A *fully discrete* scheme can be obtained by integrating equation (1.12) in a time-space cell, getting

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} [f(u(x_{j+1/2}, t)) - f(u(x_{j-1/2}, t))]\,dt\,,$$

with $t^n := n\Delta t$, where Δt is the time step. This suggests a numerical scheme of the form

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x}(F_{j+1/2}^n - F_{j-1/2}^n)\,, \tag{1.13}$$

where the numerical flux $F_{j+1/2}^n$ approximates the time average of f along the edge of the cell average.

An example of a fully discrete scheme is the upwind scheme we saw above.

This kind of scheme is conservative. Indeed, if $x \in [a,b]$ and we assume a periodic boundary condition on $[a,b]$, i.e., $u(a,t) = u(b,t)$, then, at continuous level,

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = 0\,.$$

Now, if we adopt the fully discrete scheme constructed above, this property is still satisfied. Indeed, summing scheme (1.13) over $j \in \{1, \dots, N\}$, we get

$$\sum_{j=1}^{N} \bar{u}_j^{n+1} = \sum_{j=1}^{N} \bar{u}_j^n$$

since we deal with a telescopic sum because of the periodicity.

In Chapter 4 we will construct a upwind scheme for a system of two kinetic equations that are coupled with nonlocal interactions terms.

1.5 Optimal transportation theory

In the next two Sections we present some notions of *optimal transportation theory* and *Wasserstein distances*, [2, 59, 60]. We will use them in Chapters 2, 3 and 5.

Optimal transportation theory studies how to transfer a given mass distribution from one configuration to another in an optimal way. The two configurations must have the same mass.

Let (X, μ) and (Y, ν) be two probability spaces. Let $T : X \to Y$ be a measurable map. The measure $\nu = T\#\mu$, called *push-forward of* μ *through* T, is defined by

$$T\#\mu(A) = \mu(T^{-1}(A)),$$

for all Borel subsets $A \subset Y$, with $T^{-1}(A) = \{x \in X \;:\; T(x) \in A\}$. This measure is characterized by

$$\int_Y f\, dT\#\mu = \int_X f \circ T\, d\mu,$$

for all Borel function $f : Y \to \mathbb{R}$.

We now set a non-negative measurable *cost function* $c : X \times Y \to \mathbb{R}$. It can be interpreted as the work needed to transfer one unit of mass from location $x \in X$ to location $y \in Y$.

1.5.1 Monge and Kantorovich formulations of the optimal transportation problem

Monge's optimal transportation problem consists in minimizing the functional

$$I[T] = \int_X c(x, T(x))\, d\mu(x)$$

over the set of *transport maps* T from μ to ν, i.e., all maps T such that $T\#\mu = \nu$.

This formulation requires that mass cannot be separated, which means that to each location x corresponds a unique destination y.

A generalization of the Monge's formulation is defined as follows. We consider a probability measure defined on the product space $X \times Y$. Informally, a transference plan γ is such that all the mass in the point x coincides with $d\mu(x)$, and all the mass moved to y coincides with $d\nu(y)$. Moreover, the mass in x is sent to location y with probability given by $\gamma(x, y)$, and a priori some mass in x can be transported to many destinations y's, namely it may be split in many parts.

Rigorously, we define a *transference plan with marginals* μ *and* ν as a probability measure γ on the product space $X \times Y$ satisfying

$$\gamma[A \times Y] = \mu[A], \qquad \gamma[X \times B] = \nu[B], \tag{1.14}$$

for all measurable subsets $A \subset X$, $B \subset Y$. Equivalently, we require that

$$\pi^1\#\gamma = \mu, \qquad \pi^2\#\gamma = \nu,$$

where π^i is the projection operator on the i-th component of the product space. In other words, this means

$$\int_Y d\gamma(x, y) = d\mu(x), \qquad \int_X d\gamma(x, y) = d\nu(y).$$

More precisely, this is equivalent to have

$$\int_{X\times Y}[\varphi(x)+\psi(y)]\,d\gamma(x,y)=\int_X\varphi(x)\,d\mu(x)+\int_Y\psi(y)\,d\nu(y),$$

for all $(\varphi,\psi)\in L^1(d\mu)\times L^1(d\nu)$. Set

$$\Pi(\mu,\nu):=\{\gamma:X\times Y\to\mathbb{R}\ :\ (1.14)\text{ holds}\}.$$

Notice that $\Pi(\mu,\nu)\neq\emptyset$, since $\mu\otimes\nu\in\Pi(\mu,\nu)$.

Kantorovich's formulation consists in minimizing the functional

$$I[\gamma]=\int_{X\times Y}c(x,y)\,d\gamma(x,y),$$

with $\gamma\in\Pi(\mu,\nu)$. If γ is a transference plan, $I[\gamma]$ is called *total transportation cost* associated to γ. The *optimal transportation cost* between μ and ν is defined by

$$\mathcal{T}_c(\mu,\nu):=\inf_{\gamma\in\Pi(\mu,\nu)}I[\gamma].$$

The transference plans such that $I[\gamma]=\mathcal{T}_c(\mu,\nu)$ are called *optimal transportation maps.*

We can observe that this problem is a generalization of the Monge's problem. Indeed, if $T:X\to Y$ is a transport map, one can consider the transference plan $\gamma_T:=(\mathrm{id}\times T)\#\mu$. In particular, we have that

$$d\gamma_T(x,y)=d\mu(x)\delta[y=T(x)],$$

and satisfies what follows: for any non-negative measurable map ζ on $X\times Y$,

$$\int_{X\times Y}\zeta(x,y)\,d\gamma_T(x,y)=\int_X\zeta(x,T(x))\,d\mu(x).$$

Therefore, the *total transportation cost* in Monge's formulation is defined by

$$I[T]=\int_X c(x,T(x))\,d\mu(x).$$

We have that γ_T belongs to $\Pi(\mu,\nu)$ if

$$\int_X[\varphi(x)+\psi\circ T(x)]\,d\mu(x)=\int_X\varphi(x)\,d\mu(x)+\int_Y\psi(y)\,d\nu(y),$$

that is

$$\int_X(\psi\circ T)\,d\mu=\int_Y\psi\,d\nu.$$

Thus, for all $\psi\in L^1(d\nu)$, the measurable function $\psi\circ T$ should belong to $L^1(d\mu)$ and the values of the two integrals above should be equal. Equivalently, γ_T belongs to $\Pi(\mu,\nu)$ if

$$\nu(B)=\mu[T^{-1}(B)],$$

for all measurable subsets $B\subset Y$, that can be rewritten as

$$\nu=T\#\mu.$$

This shows that Kantorovich's formulation generalizes Monge's formulation.

The main advantage in the Kantorovich's approach is that a minimizer γ of $I[\gamma]$ in $\Pi(\mu,\nu)$ always exists, as we prove in the next Proposition, see [1, 2, 59, 60]. First, we state the Prokhorov Theorem.

Theorem 1.1 (Prokhorov Theorem)**.** *Let X be a complete and separable metric space. A subset $\mathcal{K} \subset \mathcal{P}(X)$ is tight, that is for all $\varepsilon > 0$ there exists K_ε compact in X such that*

$$\mu(X \setminus K_\varepsilon) \leq \varepsilon$$

for all $\mu \in \mathcal{K}$, if and only if then $\mathcal{K}$ is relatively compact in $\mathcal{P}(X)$.

Proposition 1.1 (Existence of optimal plan)**.** *Let X and Y be two complete and separable metric spaces and let $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ be two probability measures. Let c be a lower semi-continuous cost function. Then there exists a transference map $\gamma \in \Pi(\mu, \nu)$ minimizing $I[\gamma]$.*

Proof. We claim that $\Pi(\mu, \nu)$ is weakly closed. Let $\delta > 0$ and let $K \subset X$ and $L \subset Y$ be such that

$$\mu[X \setminus X] \leq \delta, \qquad \nu[Y \setminus L] \leq \delta.$$

For any $\gamma \in \Pi(\mu, \nu)$, we have

$$\gamma[(X \times Y) \setminus (K \times L)] \leq \gamma[(X \times (Y \setminus L)] + \gamma[(X \setminus K) \times L] \leq 2\delta.$$

Therefore the set $\Pi(\mu, \nu)$ is tight, and, by Prokhorov Theorem, it is relatively compact with respect to the weak topology. Furthermore, since the conditions defining $\Pi(\mu, \nu)$ are continuous with respect to the weak topology, then $\Pi(\mu, \nu)$ is weakly closed.
Now, let $(\gamma_k)_{k\in\mathbb{N}}$ be a minimizing sequence and let $\gamma_* \in \Pi(\mu, \nu)$ be a limit point of the sequence. Since, c is a lower semi-continuous and non-negative function, there is a sequence $(c_\ell)_{\ell \in N}$ of non-decreasing bounded Lipschitz functions such that c is its pointwise supremum. Then, by using the monotone convergence Theorem, we derive

$$\begin{aligned}
\int_{X\times Y} c(x,y) d\gamma_*(x,y) &= \lim_\ell \int_{X\times Y} c_\ell(x,y) d\gamma_*(x,y) \\
&\leq \lim_\ell \limsup_k \int_{X\times Y} c_\ell(x,y)\, d\gamma_k(x,y) \\
&\leq \limsup_k \int_{X\times Y} c(x,y) d\gamma_k(x,y) \\
&= \inf I[\gamma],
\end{aligned}$$

that concludes the proof. □

1.6 Wasserstein distances

Let (X, d) be a metric space. Let $\mathcal{P}_p(X)$ be the set of probability measures on X with finite moment of order p, for $p \geq 0$, i.e., $\mu \in \mathcal{P}_p(X)$ if for an arbitrary $y \in X$,

$$\int_X d(x,y)^p \, d\mu(x) < +\infty.$$

This definition does not depend on the choice of the point y, thus the property $\mu \in \mathcal{P}_p(X)$ is well-posed.

Let now consider the cost function $c(x, y) = d(x, y)^p$, with $p \in [1, \infty)$. Given two probability measures μ, ν on the metric space (X, d), the *Monge-Kantorovich distance of order* p, or *Monge-Kantorovich distance with exponent* p between μ and ν is

$$W_p(\mu, \nu) = \left(\inf_{\gamma \in \Pi(\mu,\nu)} \int_{X\times X} d(x,y)^p \, d\gamma(x,y) \right)^{1/p},$$

where $\Pi(\mu, \nu)$ is the class of transference maps between μ and ν. The Wasserstein distance W_p defines a metric on $\mathcal{P}_p(X)$.

The Monge-Kantorovich distance of order 2, namely W_2, is called *quadratic Wasserstein distance*. The Monge-Kantorovich distance with exponent 1, i.e., W_1, is said *Monge-Kantorovich-Rubinstein distance*.

Another notion of distance we will deal with in Chapter 3 is the one of *bounded Lipschitz distance* d_{BL} for probability measures defined by, see [19],

$$d_{\mathrm{BL}}(\mu, \nu) := \sup \left\{ \left| \int_{\mathbb{R}^d} \phi \, d\mu - \int_{\mathbb{R}^d} \phi \, d\nu \right| \quad : \quad \|\phi\|_{L^\infty(\mathbb{R}^d)} \leq 1, \right.$$
$$\left. \|\phi\|_{\mathrm{Lip}} := \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|} \leq 1 \right\}.$$

Note that the bounded Lipschitz distance and the 1-Wasserstein distance are equivalent in the set of probability measure with finite first moment.

In this PhD thesis we will consider models with many species, therefore we will work in product spaces, and we will introduce suitable notions of distances on these spaces.

In the next Proposition we state a property on the convergence of measures, [59].

Proposition 1.2. *Let (X, d) be a complete and separable metric space. Let $p \in (0, \infty)$, let μ_n be a sequence of probability measures in $\mathcal{P}_p(X)$, and let $\mu \in \mathcal{P}_p(X)$. Then, the following are equivalent:*

1. *$W_p(\mu_n, \mu) \to 0$ as $n \to \infty$.*
2. *$\mu_n \to \mu$ in the weak sense as $n \to \infty$ and the following tightness condition holds: for any $x_0 \in X$,*

$$\lim_{R \to \infty} \limsup_{k \to \infty} \int_{d(x_0, x) \geq R} d(x_0, x)^p d\mu_n = 0.$$

3. *$\mu_n \to \mu$ in the weak sense as $n \to \infty$, and there is convergence of the moment of order p, i.e., for any $x_0 \in X$,*

$$\int_{X \times X} d(x_0, x)^p \, d\mu_n(x) \to \int_{X \times X} d(x_0, x)^p \, d\mu(x),$$

as $n \to \infty$.

1.6.1 One-dimensional case

In this Subsection we deal with $d = 1$ and $p = 2$, i.e., we consider the 2-Wasserstein distance on $\mathbb{R}$. In this case, there exists a unique optimal plan $\gamma \in \Pi_o(\mu, \nu)$ for which the infimum in

$$W_2^2(\mu, \nu) = \inf_{\gamma \in \Pi(\mu, \nu)} \left\{ \iint_{\mathbb{R} \times \mathbb{R}} |x - y|^2 \, d\gamma(x, y) \right\}$$

is attained, with $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$, [59]. This measure γ can be characterised by the monotone rearrangements or pseudo-inverses of μ and ν: given $\mu \in \mathcal{P}(\mathbb{R})$, its monotone rearrangement is

$$X_\mu(m) := \inf\{x \, : \, M_\mu(x) > m\} \qquad \text{for all } m \in \Omega,$$

where $\Omega \coloneqq (0,1)$ and M_μ is the cumulative distribution of the measure μ, i.e.,

$$M_\mu(x) \coloneqq \mu\big((-\infty, x]\big) \qquad \text{for all } x \in \mathbb{R}.$$

The map X_μ is right-continuous and non-decreasing and satisfies, by denoting the one-dimensional Lebesgue measure on Ω by $\mathfrak{m}$,

$$(X_\mu)\#\mathfrak{m} = \mu, \qquad \int_{\mathbb{R}} \zeta(x)\,\mu(dx) = \int_\Omega \zeta\big(X_\mu(m)\big)\,dm,$$

for all Borel maps $\zeta : \mathbb{R} \to \mathbb{R}$. In particular, $\mu \in \mathcal{P}_2(\mathbb{R})$ if and only if $X_\mu \in L^2(\Omega)$. Moreover, the joint map $X_{\mu,\nu} : \Omega \to \mathbb{R} \times \mathbb{R}$ defined by $X_{\mu,\nu}(m) \coloneqq \big(X_\mu(m), X_\nu(m)\big)$ characterises the optimal transportation plan $\gamma \in \Pi_o(\mu,\nu)$ by the formula

$$\gamma = \big(X_{\mu,\nu}\big)\#\mathfrak{m},$$

according to which

$$W_2^2(\mu,\nu) = \int_\Omega |X_\mu(m) - X_\nu(m)|^2\,dm,$$

thus the 2-Wasserstein distance can be reformulated in terms of pseudo-inverses. Furthermore, introducing the closed convex set of non-decreasing functions in the Hilbert space $L^2(\Omega)$, i.e.,

$$\mathcal{K} \coloneqq \{X \in L^2(\Omega) \;:\; X \text{ is non-decreasing}\}, \tag{1.15}$$

and observing that there is always a right-continuous representative due to monotonicity, the map

$$\Psi : \mathcal{P}_2(\mathbb{R}) \ni \mu \mapsto X_\mu \in \mathcal{K} \tag{1.16}$$

is a distance-preserving bijection between the space of probability measures with finite second moment $\mathcal{P}_2(\mathbb{R})$ and the convex cone $\mathcal{K}$ of non-decreasing $L^2(\Omega)$-functions. By using this bijection, we can rewrite the Eulerian system in terms of pseudo-inverses.

1.7 Maximal monotone operators

In this Section we provide a brief introduction to maximal monotone operators theory developed by Brézis in [14], that we will apply in Chapter 5.

Let us start by defining such operator. Let H be a Hilbert set and $\mathcal{P}(H)$ its power set. Let $A : H \to \mathcal{P}(H)$ be a multivalued operator and $D(A) = \{x \in H \;:\; Ax \neq \emptyset\}$ its domain. We say the operator A is *monotone* if for all $x_1, x_2 \in D(A)$, and for all $y_1 \in Ax_1$, $y_2 \in Ax_2$,

$$(y_1 - y_2, x_1 - x_2) \geq 0.$$

For example, let φ be a convex and proper function on H, i.e., $\varphi : H \to]-\infty, +\infty]$ is such that $\varphi \not\equiv +\infty$ and $\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$, for all $x, y \in H$ and for all $t \in]0,1[$. Then the set $D(\varphi) = \{x \in H \;:\; \varphi(x) < +\infty\}$ is convex and the sub-differential $\partial\varphi$ defined as

$$y \in \partial\varphi(x) \iff \forall \xi \in H, \varphi(\xi) \geq \varphi(x) + (y, \xi - x)$$

is monotone in H.

A monotone operator A is maximal if there is no monotone operator that properly contains it. More precisely, A is a *maximal monotone operator* if and only if A is monotone and for all $x, y \in H$ such that $\eta \in A\xi$ and

$$(y - \eta, x - \xi) \geq 0 \quad \text{for all } \xi \in D(A),$$

then $y \in Ax$.

For instance, if a function φ is convex, proper and lower semi-continuous, then $\partial\varphi$ is a maximal monotone operator.

From basic convex analysis, we know that if A is a maximal monotone operator, then Ax is convex and closed for every $x \in D(A)$, and such sets admit a unique element with minimal norm. Setting $A^\circ x \coloneqq \mathsf{P}_{Ax}0$ the projection of $0 \in H$ onto $Ax \subset H$, where $\mathsf{P}_{Ax} : H \to Ax$ denotes the *projection* on Ax, then $A^\circ x$ is the unique element with minimal norm in Ax.

The theory we apply in this thesis, in particular in Chapter 5, is developed in [14, Chapter 3] and is devoted to solve an equation of the type

$$\frac{du}{dt} + Au \ni f, \quad u(0) = u_0, \tag{1.17}$$

where A is a multivalued operator on H and $f \in L^1((0,T); H)$.

We now introduce the notions of strong and weak solution and some results.

We say that u is a *strong solution* to (1.17) if $u \in \mathcal{C}([0,T], H)$ is absolutely continuous on the compact sets of $]0,T[$, differentiable almost everywhere on $]0,T[$ and satisfies

$$u(t) \in D(A), \quad \text{and} \quad \frac{du}{dt}(t) + Au(t) \ni f(t)$$

almost everywhere on $]0,T[$.

A function $u \in \mathcal{C}([0,T]; H)$ is said to be a *weak solution* to (1.17) if there exists two sequences $f_n \in L^1((0,T); H)$ and $u_n \in \mathcal{C}([0,T], H)$ such that u_n is a strong solution to $\frac{du_n}{dt} + Au_n \ni f_n$, $f_n \to f$ in $L^1((0,T); H)$ and $u_n \to u$ uniformly on $[0,T]$.

Next we gather some results from [14] that we mention in Chapter 5. In particular, we provide the proofs of Lemma A.5, Theorem 3.5, and Theorem 3.17 in [14].

Lemma 1.1. *Let $m \in L^1((0,T); \mathbb{R})$ be such that $m \geq 0$ almost everywhere on $]0,T[$, and $a \geq 0$ a constant. If $\phi : [0,T] \to \mathbb{R}$ is a continuous function satisfying*

$$\frac{1}{2}\phi^2(t) \leq \frac{1}{2}a^2 + \int_0^t m(s)\phi(s)\, ds$$

for all $t \in [0,T]$, then

$$|\phi(t)| \leq a + \int_0^t m(s)\, ds$$

for all $t \in [0,T]$.

Proof. Let $\varepsilon > 0$ and set

$$\psi_\varepsilon(t) = \frac{1}{2}(a+\varepsilon)^2 + \int_0^t m(s)\phi(s)\, ds.$$

Thus,

$$\frac{d\psi_\varepsilon}{dt}(t) = m(t)\psi(t)$$

almost everywhere on $]0,T[$, and, by using the assumption,

$$\frac{1}{2}\phi^2(t) \le \psi_0(t) \le \psi_\varepsilon(t)$$

for $t \in [0,T]$. Therefore, we get

$$\frac{d\psi_\varepsilon}{dt}(t) \le m(t)\sqrt{2}\sqrt{\psi_\varepsilon(t)}.$$

Now, we have that $\psi_\varepsilon(t) \ge \frac{1}{2}\varepsilon^2$ for $t \in [0,T]$, then the map $t \mapsto \psi_\varepsilon(t)$ is absolutely continuous and since $\frac{d}{dt}\sqrt{\psi_\varepsilon(t)} = \frac{1}{2\sqrt{\psi_\varepsilon(t)}}\frac{d\psi_\varepsilon}{dt}(t)$ almost everywhere, we derive

$$\frac{d}{dt}\sqrt{\psi_\varepsilon(t)} \le \frac{1}{\sqrt{2}}m(t)$$

almost everywhere on $]0,T[$. It follows that

$$\sqrt{\psi_\varepsilon(t)} \le \sqrt{\psi_\varepsilon(0)} + \frac{1}{\sqrt{2}}\int_0^t m(s)\,ds.$$

Then, we get

$$|\phi(t)| \le \sqrt{2}\sqrt{\psi_\varepsilon(t)} \le \sqrt{2}\sqrt{\psi_\varepsilon(0)} + \int_0^t m(s)\,ds = a + \varepsilon + \int_0^t m(s)\,ds,$$

for all $t \in [0,T]$ and all $\varepsilon > 0$. This concludes the proof. □

Lemma 1.2. *Let A be a monotone operator, $f, g \in L^1((0,T);H)$, and u and v weak solutions to inclusions*

$$\frac{du}{dt} + Au \ni f \qquad \text{and} \qquad \frac{dv}{dt} + Av \ni g,$$

respectively. Then, for all $0 \le s \le t \le T$,

$$|u(t) - v(t)| \le |u(s) - v(s)| + \int_s^t |f(\sigma) - g(\sigma)|\,d\sigma,$$

and for all $0 \le s \le t \le T$, and for every $x, y \in A$,

$$(u(t) - u(s), u(s) - x) \le \frac{1}{2}|u(t) - x|^2 - \frac{1}{2}|u(s) - x|^2 \le \int_s^t (f(\sigma) - y, u(\sigma) - x)\,d\sigma.$$

Proof. Since this estimate is stable by passing to limit in $\mathcal{C}([0,T];H)$, we can assume that u and v are strong solutions. Thus, by using the monotonicity of A, we get

$$\frac{1}{2}\frac{d}{dt}|u(t) - v(t)|^2 = \left(\frac{du}{dt}(t) - \frac{dv}{dt}(t), u(t) - v(t)\right) \le (f(t) - g(t), u(t) - v(t)).$$

Since $|u(t) - v(t)|^2$ is absolutely continuous on the compact sets of $]0,T[$ and continuous on $[0,T]$, by integrating on $]s,t[$ we obtain

$$\frac{1}{2}|u(t) - v(t)|^2 - \frac{1}{2}|u(s) - v(s)|^2 \le \int_s^t (f(\sigma) - g(\sigma), u(\sigma) - v(\sigma))\,d\sigma.$$

Then, by applying Lemma 1.1, we derive the first inequality. The second inequality follows by considering $g \equiv y$ and $v \equiv x$. □

Theorem 1.2. *Let A be a maximal monotone operator on H, $f \in L^1((0,T);H)$ and $u \in \mathcal{C}([0,T];H)$ a weak solution to (1.17). Assume that t_0 is a Lebesgue point to the right of f and set $f(t_0+0) = \lim_{h\to 0} \frac{1}{h}\int_{t_0}^{t_0+h} f(s)\,ds$. Then, the following properties are equivalent:*

1. *$u(t_0) \in D(A)$.*
2. *$\liminf_{h\to 0} \frac{1}{h}|u(t_0+h)-u(t_0)| < +\infty$.*
3. *u is right differentiable in t_0. In this case*

$$\frac{d^+u}{dt}(t_0) = (f(t_0+0) - Au(t_0))^\circ.$$

Proof. Clearly *3.* implies *2.*. Let us prove that *2.* implies *1.*. Let $x, y \in A$, and set $\alpha := \liminf_{h\to 0} \frac{1}{h}|u(t_0+h)-u(t_0)|$. We have that

$$\frac{1}{h}\int_{t_0}^{t_0+h} (f(\sigma)-y, u(\sigma)-x)\,d\sigma \rightharpoonup (f(t_0+0)-y, u(t_0)-x)$$

as $h \to 0$. Moreover, by Lemma 1.2, we get

$$\left(\frac{u(t_0+h)-u(t_0)}{h}, u(t_0)-x\right) \le \frac{1}{h}\int_{t_0}^{t_0+h} (f(\sigma)-y, u(\sigma)-x)\,d\sigma.$$

Thus, by passing to the limit as $h \to 0$,

$$(\alpha, u(t_0)-x) \le (f(t_0+0)-y, u(t_0)-x),$$

for all $x, y \in A$. Since A is a maximal monotone operator, then $u(t_0) \in D(A)$. Furthermore, we get that $f(t_0+0)-\alpha \in Au(t_0)$. Now we prove that *1.* implies *3.*. Let $u(t_0) \in D(A)$. By applying Lemma 1.2 with $g(t) \equiv f(t_0+0) - (f(t_0+0)-Au(t_0))^\circ$ and $v(t) \equiv u(t_0)$, we get

$$|u(t_0+h)-u(t_0)| \le \int_{t_0}^{t_0+h} |f(\sigma) - f(t_0+0) + (f(t_0+0)-Au(t_0))^\circ|\,d\sigma,$$

and therefore

$$\limsup_{h\to 0} \frac{1}{h}|u(t_0+h)-u(t_0)| \le |(f(t_0+0)-Au(t_0))^\circ|.$$

From the previous point, we know that $f(t_0+0)-\alpha \in A(u(t_0))$. Thus $\alpha = (f(t_0+0) - Au(t_0))^\circ$ and, therefore, u is right differentiable in t_0 and

$$\frac{d^+u}{dt}(t_0) = (f(t_0+0) - Au(t_0))^\circ,$$

that proves the statement. □

The last result we prove involves a maximal monotone operator perturbed by a Lipschitz operator.

Theorem 1.3. *Let A be a maximal monotone operator, $\omega > 0$, $f \in L^1((0,T);H)$ and $u_0 \in \overline{D(A)}$. Then there exists a unique weak solution to*

$$\frac{du}{dt} + Au - \omega u \ni f, \qquad u(0) = u_0. \tag{1.18}$$

Proof. Let us begin by proving the uniqueness. Let u and v be two solutions to (1.18). By Lemma 1.2 we get

$$|u(t) - v(t)| \leq |u(s) - v(s)| + \omega \int_s^t |u(\tau) - v(\tau)|\, d\tau,$$

for all $0 \leq s \leq t \leq T$. Then, by Grönwall's inequality we derive

$$|u(t) - v(t)| \leq e^{\omega t}|u(0) - v(0)| = 0,$$

which proves uniqueness. Now, consider the iterative sequence defined by $u_0(t) \equiv u_0$, and u_{n+1} is the weak solution to

$$\frac{du_{n+1}}{dt} + Au_{n+1} \ni f + \omega u_{n+1}, \qquad u_{n+1}(0) = u_0.$$

By invoking again Lemma (1.2), we obtain

$$|u_{n+1}(t) - u_n(t)| \leq \int_0^t \omega |u_n(s) - u_{n-1}(s)|\, ds,$$

for $0 \leq t \leq T$ and $n \geq 1$. Therefore,

$$|u_{n+1}(t) - u_n(t)| \leq \frac{(\omega t)^n}{n!} \|u_1 - u_0\|_{L^\infty}.$$

This implies that the sequence u_n converges uniformly on $[0,T]$ to a function u which is a weak solution to (1.18). □

1.8 A literature review on aggregation models

The aggregation models with nonlocal interactions are extensively studied by adopting many approaches. In this Section we cite some works, but this list is not exhaustive. Since in this thesis we deal with no pressure term, we brefly list here some references on the existence theory for systems of the form (1.7). We mention [33, 37, 46, 47] for the one-dimensional case and [38, 61], for the multi-dimensional case. In particular, in [33] authors focus on a model described a isentropic gas through porous media; in [37] a inhomogeneous system of isentropic gas dynamics is studied; in [47] compressible Euler equations with damping is presented; in [46] authors investigate a damped compressible isentropic flow. For the multi-dimensional case, in [61] the isentropic damped Euler equation is studied; in [38] the asymptotic behaviour of compressible Euler equation with damping is analysed.

1.8.1 First order models

Let us now switch to models with no pressure terms. There are many works concerning the study of equation (1.1) and its variants. In [50] authors study models for swarming phenomena with nonlocal interactions in the one-dimensional case, and in particular they focus on advection-diffusion equations with convolution terms describing long range attraction and repulsion. In [57] a continuum model for aggregation equations applied to biology is studied both in the one-dimensional case and in higher dimensions. In [56] authors investigate a two-dimensional continuum model describing the behaviour of biological individuals that interact through nonlocal terms. Here pattern formations are studied. In [5] the aggregation equation (1.1) is studied in several dimensions with initial data in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and a finite time blow-up result is proved with a Lipschitz assumption on the kernel. Aggregation equations (1.1) with singular kernels are approached in many ways. In [8] a well-posedness result is provided in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for sufficiently large p. In [7] authors study the well-posedness of non-negative solutions of equation (1.1) where the kernel is singular at the origin, and the finite time blow-up in multiple space dimensions is investigated. In [6] authors analyse the Osgood condition associated to equation (1.1) with singular kernels. In [15] the large time behaviour of one-dimensional nonlocal models for aggregation phenomena is studied via a gradient flow formulation considering the space of probability measure with the Wasserstein metric. In [40] authors study an aggregation phenomenon incuding the alignment interactions and the results are obtained by using mass trasportation theory. In [23] multi-dimensional continuum models for interacting particle systems through nonlocal terms are investigated. Here existence and uniqueness of the solutions are provided by adopting the theory of gradient flows in the space of probability measures equipped with the Wasserstein distance, and a finite time collapse is proved. In the last two decades, the attention of many reaserchers has turned to systems with many species. The first order approach is considered in many papers, see [22, 34, 35]. In particular, in [34] a one-dimensional system with cross-diffusion and nonlocal interactions is performed; in [35] weak solutions to a two-species model with nonlocal interaction are investigated; in [22] a first order system with potentials driven by Newtonian potentials is considered, and a notion of gradient flow solutions is provided, cf. [2]. See also [32] for an application to chemotaxis in biology.

Finite volume methods are adopted in order to study aggregation models with many species. We mention [4, 24, 25] for macroscopic one-dimensional systems. In details, in [25] a nonlocal two-species cross-interaction model with cross diffusion is studied, and the solutions are analysed by considering many regimes, both analytically and numerically. In [24] a semi-discrete finite volume scheme for a coupled system of two nonlocal equations with cross-diffusion is investigated and the convergence of the scheme is proved. Moreover, in [4] an implicit finite volume scheme for nonlinear and nonlocal aggregation equations is studied and the convergence of the scheme is provided under suitable assumptions on the potentials and diffusion functions.

1.8.2 Kinetic models

Kinetic theory provides a classical mesoscopic approach to fluid mechanics. For a general overview we refer to [26, 58]. The kinetic approach is largely used in the study of aggregation phenomena. Some of the techniques developed in this context are inspired by [48]. In [17] equation (1.5) is studied by tacking into account a self-propulsion term and a friction term. The first parameter provides an effect on individuals that are independent of the other individuals, whereas with the second parameter a velocity-averaging effect is considered and agents are forced to adapt their own velocity to that of other close agents. Here, well-posedness, existence, uniqueness and continuous dependence results are presented in the space of probability measures $\mathcal{P}_1(\mathbb{R}^d)$ equipped with the Monge-Kantorovich-Rubinstein distance. Moreover, the corresponding microscopic system is studied and a convergence result of the particle system to the kinetic equation is provided. Equation (1.5) is investigated also in [39] in a multi-dimensional space, considering in addition inertial effects. Here, by assuming smooth assumptions on the kernel and by applying the theory developed in [17], an existence and uniqueness result in the sense of measures is proved and the small inertia limit is performed, showing a convergence result to the corresponding macroscopic system. Concerning the existence of smooth solutions, it can be obtained using the classical framework for Vlasov-type equations, [43]. See also [27] for recent treatments of the theory of Vlasov-Poisson equation.

A lot of attention is devoted to aggregation systems with singular kernels also at mesoscopic level. In [30], the Vlasov-Manev-Fokker-Planck system in dimension 3 is considered, which has the gravitational potential of the form $-1/r - 1/r^2$. Existence of weak solutions are showed under suitable initial data. In [29], the fractional porous medium equation is investigated and a local-in-time existence and uniqueness result is proved. We also mention [20] for a recent contribution in the theory of Vlasov-Poisson-Fokker-Planck system.

From the mesoscopic point of view, in [42] a finite volume scheme to approximate the one-dimensional Vlasov-Poisson system is presented. Here the author proves that, under suitable assumptions, the numerical approximation converges to the weak solution of the system in L^∞. Moreover, if initial data are in BV, then the convergence is strong in $\mathcal{C}^0((0,T); L^1_{loc})$.

1.8.3 Second order models

The microscopic system (1.4) is studied in several ways and many continuum models are derived from it. In [10] system (1.4) is investigated by adding a friction term. Many properties of the model are presented and several macroscopic equations are derived.

The macroscopic system (1.3) and its variants have been intensively studied with the pressure term $p = 0$ and results of such "sticky particle" solutions have attracted the attention of many researchers. We mention [9] for results related with existence and uniqueness in the multi-dimensional case. In one space dimension there are many results in the literature, cf. [12, 13, 51]. In particular, in [51] the one-species case is investigated with $W = 0$ in the space of probability measures $\mathcal{P}_2(\mathbb{R})$ with the quadratic Wasserstein distance and many interesting properties of the solution are presented. In [12] system (1.3) is performed and we briefly describe it in Section 1.3. Here existence and uniqueness results are proved and many properties of the solution are provided in the metric space $(\mathcal{P}_2(\mathbb{R}), W_2)$. We also mention [53] for a very interesting application to the Euler-Poisson model, in which W is the solution operator to Poisson equation. For one-species case, a large friction limit is investigated in [21] where a multi-dimensional damped Euler system is studied under the influence of a external potential with respect to the 2-Wasserstein distance, and a rigorous proof of the overdamped limit is provided in one space dimension. Very little attention has been devoted to multi-species second order models, see [3] for an application to pedestrian movements.

1.9 Outline of the thesis

This thesis is organised as follows.

Chapter 2 and Chapter 3 are based on a joint work with Young-Pil Choi and Simone Fagioli. In particular, Chapter 2 is devoted to the study of a multi-dimension system made up by kinetic equations describing the evolution of many species coupled through nonlocal interaction. We consider inertial effects in the model and require smooth assumptions on all the kernels. We prove existence and uniqueness of a measure solution to the kinetic system. The main result in this Chapter concerns the *small inertia limit*: we prove that the solution to the kinetic system converges towards a solution to the associated first order macroscopic system as the inertia vanishes. More in details, in Section 2.1 we describe the kinetic model we deal with, and we provide a formal derivation of this model from a microscopic system when the number of individuals increases to infinity. In Section 2.2 we provide the measure space we consider for measure solutions. It is the product of spaces of probability measures on $\mathbb{R}^d$ with finite first moment equipped with the 1-Wasserstein distance. In Section 2.3 we study the well-posedness of the kinetic system. In particular, we associate to model its characteristic system and we fall into the classical ordinary differential equations theory. Taking the time dependent flow map associated to the characteristic system, we define the notion of measure solution. In this Section we also provide some a priori estimates on the characteristic system and we prove existence and uniqueness of measure solutions to the kinetic system in Theorem 2.1, and existence of smooth solutions to the kinetic system in Theorem 2.2. In Section 2.4 we gather all the uniform estimates in the inertia parameter, holding both for smooth solutions and for measure solutions. Finally, in Section 2.5 the main result of this Chapter is presented, that is Theorem 2.3. We prove that, under suitable smooth assumption on the kernels, the solution to the kinetic system converges towards a solution to the associated first order macroscopic system as the inertia parameter vanishes. The results of this Chapter are inspired by the paper [39] by Fetecau and Sun: we extend to many species their model without requiring further assumptions.

In Chapter 3 we study a mesoscopic multi-dimension system describing the behaviour of many interacting species and we consider inertial effects and singular self-potentials. We prove existence of solutions to the kinetic system, and we provide a small-inertia result. We cite the papers [30] by Choi and Jeong and [19] by Carrillo, Choi and Jung where the authors study the one-species case considering different effects on the model. Entering in details, in Section 3.1 we present the mesoscopic and macroscopic models with singular self-potentials. In Section 3.2 we consider a regularised version of the kinetic system obtained by perturbing the self-potentials and we prove some uniform estimates with respect to the perturbation. In Section 3.3 we prove existence of weak solutions to the kinetic system, see Theorem 3.1. In Section 3.4 we show rigorously the convergence of the solutions to the kinetic system towards solutions to the corresponding macroscopic model. This result is contained in Theorem 3.3.

In Chapter 4 we propose an upwind finite volume scheme for a system of two kinetic equations coupled through nonlocal interaction terms. In this model we do not consider inertial effects and we do not require any constrains on the mesh size. The finite volume method we construct conserves mass and preserves positivity. The main result of this Chapter concerns the convergence of the scheme. The notion of solution we adopt is the one of *weak solution*. The results of this Chapter are inspired by the paper [42] by Filbet, where the author constructs a finite volume scheme to discretize the one-dimensional Vlasov-Poisson system, showing that the approximated solution converges to the continuous one as square roots of the time, space and velocity steps go to zero. We improve the convergence rate in time, avoiding to take the square root of the time step. In details, in Section 4.1 we introduce the model and some of its properties at continuum level. In Section 4.2, we construct the numerical mesh and define the notion of weak solutions. In Section 4.3 we provide some a priori estimates, used for passing to the limit and obtain weak solutions of continuum system. In particular we show that the numerical solutions remain non-negative and bounded under a suitable timestep constraint. In Section 4.4 we prove the convergence of the numerical solution of the scheme to a weak solution to the continuum system, see Theorem 4.1. The work presented in this Chapter is based on a joint work with Markus Schmidtchen and Julia Hauser.

In Chapter 5 we investigate a second order system with two species interacting through nonlocal interactions and subject to linear damping from the macroscopic point of view. Our results only deal with the one space dimensional case. We first consider the case of smooth potentials and prove existence and uniqueness of the solution. Then, after a suitable rescaling of the time variable, we consider a large-time large-damping version of the system and show convergence to solution to the corresponding first order system. We also study the case of Newtonian potentials in the self-interaction terms, considering additional external coercive potentials. Once provided an existence result, we prove a collapse result showing that for large time a solution converges toward Dirac delta solutions. We mention the papers [12] by Brenier at al. where the author provide the existence of solution considering the notion of "sticky particles" dealing only with the one-species case and by not including the damping parameter, and [21] by Carrillo, Choi and Tse where the authors study the large damping limit for the one-species case. Entering more in details, in Section 5.1 we present the model from the microscopic and macroscopic points of view. In Section 5.2 we define the metric structure. The metric space we consider is the product of the probability spaces with finite second moment equipped with the 2-Wasserstein distance. We introduce formally the large-damping limit and we rephrase the model in a Lagrangian framework. In Section 5.3 we prove existence and uniqueness of solutions to the second order system considering the Lagrangian formulation and applying the Brezis theory, [14]. This result is stated in Proposition 5.1. In Section 5.4 we perform the large damping limit rigorously, obtaining in Theorem 5.2 that solutions to second order system converge to the solutions of the corresponding first order system. We observe that the technique used in this theorem was never used in the one-species case. In Section 5.5 we consider the case of Newtonian potentials and prove existence of sticky solutions and the large-time collapse to Dirac deltas, see Theorem 5.3. In Section 5.6 we provide some numerical simulations.

The results of this Chapter are contained in [36], joint work with Marco Di Francesco and Simone Fagioli.

Chapter 2

Small inertia limit to first order nonlocal system: smooth case

This Chapter is dedicated to investigating a mesoscopic multi-dimensional system with many species coupled through nonlocal smooth interaction, considering also an inertia parameter. Taking the associated characteristic system, we provide some a priori estimates. Then we prove existence and uniqueness of a measure solution to the kinetic system in the space of probability measures with finite first moment endowed with the 1-Wasserstein distance. The main result concerns the small inertia limit: we prove that the solution of the kinetic system converges towards a solution to the corresponding first order macroscopic system as the inertia vanishes.

2.1 The model

The system we deal with is the following first order macroscopic system

$$\begin{cases} \partial_t \rho_i + \nabla \cdot (\rho_i v_i) = 0, \\ v_i = -\sum_{j=1}^{N} \nabla K_{ij} * \rho_j, \end{cases} \tag{2.1}$$

for $i = 1, \dots, N$, where N is the number of species, $\rho_i(t, x)$ is the probability measure in $\mathbb{R}^d$ modelling the i-th species, $v_i(t, x)$ is its Eulerian velocities, K_{ii} are *self-interaction* kernels and K_{ij} are *cross-interaction* kernels. The self-interaction kernels model the interactions between agents of the same species, whereas cross-interaction kernels describe the interactions of individuals of different species.

System (2.1) admits a discrete counterpart constructed as follows: consider M particles for each species and let z_i^k, $k = 1, \dots, M$, be the locations of M particles of the i-th species, for $i = 1, \dots, N$. Denoting by u_i^k the velocities of z_i^k, the dynamics of z_i^k is determined by the first order ODE system

$$\begin{cases} \dfrac{dz_i^k}{dt} = u_i^k, \\ u_i^k = -\dfrac{1}{M} \sum_{j=1}^{N} \sum_{h=1}^{M} \nabla K_{ij}(z_i^k - z_i^h), \end{cases} \tag{2.2}$$

as $i = 1, \dots, N$, $k = 1, \dots, M$, where K_{ij} are the same kernels as in (2.1). System (2.2) was derived in [10] considering the second order model

$$\begin{cases} \varepsilon \dfrac{d^2 z_i^k}{dt^2} + \dfrac{dz_i^k}{dt} = F_i^k, \\ F_i^k = -\dfrac{1}{M} \displaystyle\sum_{j=1}^{N} \sum_{h=1}^{M} \nabla K_{ij}(z_i^k - z_i^h), \end{cases} \tag{2.3}$$

as $i = 1, \dots, N$, $k = 1 \dots, M$. In (2.3), $\varepsilon > 0$ represents a small *inertia* time of individuals. In system (2.2) it is assumed that the ε-terms in (2.3) are negligible, but this choice is quite restrictive in many cases since in this way a "reaction" time is not taking into account and velocities change instantaneously, [39]. Considering (2.2), we can write system (2.3) as

$$\begin{cases} \dfrac{d}{dt} z_i^k = u_i^k, \\ \varepsilon \dfrac{d}{dt} u_i^k = -u_i^k - \dfrac{1}{M} \displaystyle\sum_{j=1}^{N} \sum_{h=1}^{M} \nabla K_{ij}(z_i^k - z_i^h), \end{cases} \tag{2.4}$$

with $\varepsilon > 0$, $i = 1, \dots, N$, $k = 1, \dots, M$. Formally, considering the limit as $\varepsilon \to 0$ in (2.4), we obtain system (2.2).

Taking the formal limit as the number of particles increases to infinity, namely $M \to \infty$, we can associate to (2.4) the kinetic system

$$\partial_t f_i + v \cdot \nabla_x f_i = \frac{1}{\varepsilon} \nabla_v \cdot (v f_i) + \frac{1}{\varepsilon} \nabla_v \cdot \left(\left(\sum_{j=1}^{N} \nabla K_{ij} * \rho_j \right) f_i \right), \tag{2.5}$$

for $i = 1, \dots, N$, where $f_i(t, x, v)$ is the mesoscopic density of the i-th species at position $x \in \mathbb{R}^d$ with velocity $v \in \mathbb{R}^d$, and $\rho_i(t, x)$ is the associated macroscopic population density, i.e.,

$$\rho_i(t, x) = \int_{\mathbb{R}^d} f_i(t, x, v)\, dv. \tag{2.6}$$

Our aim is to investigate the small inertia limit at continuum level. In particular, we want to study the $\varepsilon \to 0$ limit in (2.5) and prove that it converges towards to the first order PDEs model (2.1).

2.1.1 Formal derivation of the kinetic model

We now formally derive the kinetic model (2.5) from the particle system (2.4) as the number of particles increases to infinity. Let $f_{i,M}$, for $i = 1, \dots, N$, be the empirical distribution density associated to the solution $(z_i^k(t), u_i^k(t))$ to (2.4), as $k = 1, \dots, M$, defined by

$$f_{i,M}(t, x, v) = \frac{1}{M} \sum_{k=1}^{M} \delta(x - z_i^k(t)) \delta(v - u_i^k(t)).$$

Observe that

$$\rho_{i,M}(t, x) := \int_{\mathbb{R}^d} f_{i,M}(t, x, v)\, dv = \frac{1}{M} \sum_{k=1}^{M} \delta(x - z_i^k(t)),$$

thus, if Q is a generic kernel, we get

$$\nabla Q * \rho_{i,M}(x) = \int_{\mathbb{R}^d} \nabla Q(x-y)\rho_{i,M}(y)\,dy = \frac{1}{M}\sum_{k=1}^{M} \nabla Q(x - z_i^k).$$

Now, let $\varphi := \varphi(x,v) \in \mathcal{C}_c^1(\mathbb{R}^{2d})$ be a test function. Then

$$\begin{aligned}
\frac{d}{dt}\langle f_{i,M}(t), \varphi\rangle =& \frac{1}{M}\sum_{k=1}^{M}\frac{d}{dt}\varphi(z_i^k(t), u_i^k(t)) \\
=& \frac{1}{M}\sum_{k=1}^{M}\nabla_x\varphi(z_i^k(t), u_i^k(t))\dot{z}_i^k(t) + \frac{1}{M}\sum_{k=1}^{M}\nabla_v\varphi(z_i^k(t), u_i^k(t))\dot{u}_i^k(t) \\
=& \frac{1}{M}\sum_{k=1}^{M}\nabla_x\varphi(z_i^k(t), u_i^k(t))u_i^k(t) \\
&+ \frac{1}{M}\sum_{k=1}^{M}\nabla_v\varphi(z_i^k(t), u_i^k(t))\frac{1}{\varepsilon}\Big(- u_i^k - \frac{1}{M}\sum_{j=1}^{N}\sum_{h=1}^{M}\nabla K_{ij}(z_i^k - z_i^h)\Big) \\
=& \langle f_{i,M}(t), \nabla_x\varphi \cdot v\rangle + \Big\langle f_{i,M}(t), \nabla_v\varphi \cdot \frac{1}{\varepsilon}\Big(- v - \sum_{j=1}^{N}\nabla K_{ij} * \rho_{j,M}(x)\Big)\Big\rangle.
\end{aligned}$$

Integrating by parts in x and v, we have

$$\Big\langle \partial_t f_{i,M} + v\cdot\nabla_x f_{i,M} - \frac{1}{\varepsilon}\nabla_v\cdot\Big(\Big(v + \sum_{j=1}^{N}\nabla K_{ij} * \rho_{j,M}\Big) f_{i,M}\Big), \varphi\Big\rangle = 0,$$

hence

$$\partial_t f_{i,M} + v\cdot\nabla_x f_{i,M} = \frac{1}{\varepsilon}\nabla_v\cdot(v f_{i,M}) + \frac{1}{\varepsilon}\nabla_v\cdot\Big(\Big(\sum_{j=1}^{N}\nabla K_{ij} * \rho_{j,M}\Big) f_{i,M}\Big).$$

Assuming that $f_{i,M}$ converges to f_i as $M\to\infty$ for all $i = 1,\ldots,N$, we formally obtain the mesoscopic system (2.5).

2.2 Main assumptions

Since we deal with N interacting species, the measure space we consider for measure solutions is $(\mathcal{P}_1(\mathbb{R}^d)^N, \mathcal{W}_1)$, where $\mathcal{P}_1(\mathbb{R}^d)$ denotes the space of probability measures on $\mathbb{R}^d$ having finite first moment, i.e.,

$$\mathcal{P}_1(\mathbb{R}^d) = \Big\{ f \in \mathcal{P}(\mathbb{R}^d) \,:\, \int_{\mathbb{R}^d} |x| f(x)\,dx < \infty \Big\},$$

and $\mathcal{W}_1$ is the 1-Wasserstein distance on $\mathcal{P}_1(\mathbb{R}^d)^N$ defined below. In order to fix the notation, we write

$$\mathbf{f} = (f_i)_{i=1}^N \in \mathcal{P}_1(\mathbb{R}^d)^N \tag{2.7}$$

to denote a N-tuple of probability measures in the product space $\mathcal{P}_1(\mathbb{R}^d)^N$.

Definition 2.1. (1-Wasserstein distance) Let $\mathbf{f} = (f_i)_{i=1}^N$, $\mathbf{g} = (g_i)_{i=1}^N \in \mathcal{P}_1(\mathbb{R}^d)^N$. The 1-*Wasserstein distance between* $\mathbf{f}$ *and* $\mathbf{g}$ is defined as

$$\mathcal{W}_1(\mathbf{f}, \mathbf{g}) := \sup_{t\in[0,T]} \big[W_1(f_1, g_1) + \cdots + W_1(f_N, g_N)\big],$$

where W_1 is the Monge-Kantorovich-Rubinstein distance introduced in Section 1.6.

We assume the following condition, labelled by (**Pot**), on the potentials involved in system (2.5). Denoting by Q a generic potential, we require

$Q \in W^{2,\infty}(\mathbb{R}^d)$ such that ∇Q is locally Lipschitz and sub-linear, i.e., (**Pot**)

1. for all $x \in \mathbb{R}^d$ there is a constant $C > 0$ such that

$$|\nabla Q(x)| \leq C(1 + |x|);$$

2. for any compact set $K \subset \mathbb{R}^d$ there exists a positive constant L such that

$$|\nabla Q(x) - \nabla Q(y)| \leq L|x - y|,$$

   for all $x, y \in K$.

Remark 2.1 (Lipschitz constant)**.** Denote by B_R a closed ball in $\mathbb{R}^d$ centered in 0 and with radius $R > 0$ and consider a function $Q : \mathbb{R}^d \to \mathbb{R}^n$. We denote by $Lip_R(Q)$ the Lipschitz constant of Q in the ball $B_R \subset \mathbb{R}^d$. If Q depends also on time, i.e., $Q : [0, T] \times \mathbb{R}^d \to \mathbb{R}^n$, $Q = Q(t, x)$, we write $Lip_R(Q)$ to denote the Lipschitz constant of Q with respect to x in the ball $B_R \subset \mathbb{R}^d$, that is the smallest constant such that

$$|Q(t, x) - Q(t, y)| \leq Lip_R(Q)|x - y|,$$

for all $x, y \in B_R$, and for all $t \in [0, T]$.

2.3 Well-posedness for the kinetic system for $\varepsilon > 0$ fixed

This Section is devoted to the study of the well-posedness for system (2.5) for $\varepsilon > 0$ fixed, in the spirt of [17]. We start observing that such existence theory can be studied for a more general class of force fields

$$\mathbf{E} := (E_i)_{i=1}^N(t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{Nd},$$

for $i = 1, \ldots, N$, fulfilling the following general set of assumptions:

($\mathbf{E_1}$) E_i are continuous on $[0, T] \times \mathbb{R}^d$, for all $i = 1, \ldots, N$.

($\mathbf{E_2}$) There exist some positive constants C_i such that

$$|E_i(t, x)| \leq C_i(1 + |x|),$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$, for all $i = 1, \ldots, N$.

($\mathbf{E_3}$) E_i are locally Lipschitz with respect to x uniformly in t, for $i = 1, \ldots, N$, that is for any compact set $K \subset \mathbb{R}^d$ there exist positive constants L_i such that

$$|E_i(t, x) - E_i(t, y)| \leq L_i|x - y|,$$

for all $x, y \in K$, and for all $t \in [0, T]$.

The kinetic system we are going to study takes then the following form

$$\partial_t f_i + v \cdot \nabla_x f_i - \frac{1}{\varepsilon} \nabla_v \cdot (v f_i) + \frac{1}{\varepsilon} E_i \cdot \nabla_v f_i = 0, \tag{2.8}$$

as $i = 1, \ldots, N$. Following the approach in [17], we will construct solutions to (2.8) by considering the following characteristic system associated to (2.8)

$$\begin{cases} \dfrac{dX}{dt} = V, \\ \dfrac{dV}{dt} = -\dfrac{1}{\varepsilon} V + \dfrac{1}{\varepsilon} E_i(t, X), \end{cases} \tag{2.9}$$

as $i = 1, \ldots, N$. Introducing $P \coloneqq (X, V) \in \mathbb{R}^d \times \mathbb{R}^d$, and denoting by

$$\Psi_{E_i} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$$

the right-hand side of system (2.9), we can rewrite system (2.9) as

$$\frac{d}{dt} P = \Psi_{E_i}(t, P), \tag{2.10}$$

for $i = 1, \ldots, N$, subject to the initial condition $(X_0, V_0) \in \mathbb{R}^d \times \mathbb{R}^d$. Throughout the Chapter, we will use the compact notation

$$\Psi_{\mathbf{E}} \coloneqq (\Psi_{E_i})_{i=1}^N.$$

Note that system (2.5) falls back to this formalism by setting

$$\mathbf{E} \coloneqq (E_i)_{i=1}^N = \Big(- \sum_{j=1}^N \nabla K_{ij} * \rho_j \Big)_{i=1}^N.$$

The following Lemma concerns existence and uniqueness of solutions to system (2.9), and falls into the classical ordinary differential equations theory, see [55].

Lemma 2.1. *Fix $T > 0$ and $\varepsilon > 0$. Consider a vector field $\mathbf{E} = (E_i)_{i=1}^N$ satisfying $(\mathbf{E_1})$-$(\mathbf{E_2})$-$(\mathbf{E_3})$. Let $P_0 \in \mathbb{R}^{2d}$ be a given vector. Then, for each $i = 1, \ldots, N$, there exists a unique solution P to system (2.10) with initial condition P_0 (or equivalently (X, V) solution to system (2.9) with initial condition (X_0, V_0), for each $i = 1, \ldots, N$) such that $P \in \mathcal{C}^1([0, T]; \mathbb{R}^{2d})$. Furthermore, there exists a constant C depending on T, $|X_0|$, $|V_0|$, and on the constants C_i in Assumption $(\mathbf{E_2})$ such that*

$$|P| \leq |P_0| e^{Ct},$$

for all $t \in [0, T]$.

Proof. The statement follows from the regularity of the fields E_i, as $i = 1, \ldots, N$, using the standard theory of ordinary differential equations. The bound follows from a direct estimate on the equations, using the Assumption $(\mathbf{E_2})$. □

Thanks to the existence result above, we can introduce the time dependent flow map associated to system (2.9) by

$$\mathcal{T}^t_{E_i} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d,$$

such that

$$\mathcal{T}^t_{E_i}((X_0, V_0)) = (X(t), V(t)),$$

where $(X(t), V(t))$ is the unique solution to (2.9) at time $t > 0$ under the initial condition (X_0, V_0).

We are now in the position to introduce the notion of measure solution to system (2.8). Consider $f_{i0} \in \mathcal{P}_1(\mathbb{R}^{2d})$ the initial datum of the i-th species and let $T > 0$. Then, a measure solution to (2.8) can be defined as

$$f_i(t) = \mathcal{T}^t_{E_i} \# f_{i0}, \tag{2.11}$$

as $i = 1, \dots, N$. With a slight abuse of notation, for using a compact formulation we set

$$\mathcal{T}^t_{\mathbf{E}} = (\mathcal{T}^t_{E_i})_{i=1}^N,$$

and given the initial datum $\mathbf{f}_0 \in \mathcal{P}(\mathbb{R}^{2d})^N$ and a time $T > 0$, we define the measure solution to (2.8) as

$$\mathbf{f}(t) = \mathcal{T}^t_{\mathbf{E}} \# \mathbf{f}_0. \tag{2.12}$$

Going back to system (2.5), we define the vector field $\mathbf{E}[\mathbf{f}]$ associated to a N-tuple of measures $\mathbf{f}$ as

$$\mathbf{E}[\mathbf{f}] = (E_i[\mathbf{f}])_{i=1}^N = \left(- \sum_{j=1}^N \nabla K_{ij} * \rho_j \right)_{i=1}^N. \tag{2.13}$$

We can now give the notion of measure solution to system (2.5) as in [17, 39].

Definition 2.2 (Measure solution to (2.5))**.** Fix $T > 0$ and $\varepsilon > 0$. Let $\mathbf{f}_0 \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ be a given initial condition and let $\mathbf{E}[\mathbf{f}]$ be defined as in (2.13). A N-tuple $\mathbf{f} : [0, T] \to \mathcal{P}_1(\mathbb{R}^{2d})^N$ is a measure solution to system (2.5) with initial condition $\mathbf{f}_0$ if:

1. the field $\mathbf{E}[\mathbf{f}]$ defined in (2.13) satisfies the conditions ($\mathbf{E_1}$)-($\mathbf{E_2}$)-($\mathbf{E_3}$);
2. it holds $\mathbf{f}(t) = \mathcal{T}^t_{\mathbf{E}[\mathbf{f}]} \# \mathbf{f}_0$.

2.3.1 A priori estimates on the characteristics system

In this Subsection we collect some results on the solution to the characteristic system (2.9). We start with two standard regularity results. The proofs of the two Lemmas below can be obtained directly from system (2.9) and by definition of $\Psi_{\mathbf{E}}$ in (2.10).

Lemma 2.2 (Regularity of the characteristic system)**.** *Let $\mathbf{E} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{Nd}$ be a field that satisfies ($\mathbf{E_1}$)-($\mathbf{E_2}$)-($\mathbf{E_3}$). Consider $R > 0$ and the closed ball $B_R \subset \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$. Then*

1. *$\Psi_{\mathbf{E}}$ is bounded in compact sets, i.e.,*

$$|\Psi_{\mathbf{E}}(t, P)| \le C,$$

for all $P \in B_R$, $t \in [0, T]$ and for some $C > 0$ which depends on R and $\|\mathbf{E}\|_{L^\infty([0,T]\times B^1_R)}$, where B^1_R is the ball in $\mathbb{R}^d$ with radius R.

2. *$\Psi_{\mathbf{E}}$ is locally Lipschitz with respect to X and V, i.e.,*

$$|\Psi_{\mathbf{E}}(t, P_1) - \Psi_{\mathbf{E}}(t, P_2)| \le C(1 + Lip_R(\mathbf{E}))|P_1 - P_2|,$$

for all $P_1, P_2 \in B_R$, $t \in [0, T]$ and $C > 0$.

Lemma 2.3 (Dependence of the characteristic equations on $\mathbf{E}$)**.** *Consider two fields* $\mathbf{E}$ *and* $\mathbf{D}$ *satisfying* ($\mathbf{E_1}$)-($\mathbf{E_2}$)-($\mathbf{E_3}$)*, and consider the functions* $\Psi_{\mathbf{E}}, \Psi_{\mathbf{D}}$ *as in* (2.10)*. Then, for any compact set* $B \subset \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$*,*

$$\|\Psi_{\mathbf{E}} - \Psi_{\mathbf{D}}\|_{L^\infty(B)} \leq \frac{1}{\varepsilon}\|\mathbf{E} - \mathbf{D}\|_{L^\infty(B^1)}.$$

Now we provide some results that concern the dependence of the characteristics on the field $\mathbf{E}$ and a quantitative bound on the regularity of the flow $\mathcal{T}^t_{\mathbf{E}}$.

Lemma 2.4 (Dependence of the characteristics on $\mathbf{E}$)**.** *Fix* $T > 0$ *and consider two vector fields* $\mathbf{E}$ *and* $\mathbf{D}$ *satisfying* ($\mathbf{E_1}$)-($\mathbf{E_2}$)-($\mathbf{E_3}$)*. Fix* $P_0 \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$ *and* $R > 0$*. Assume*

$$|\mathcal{T}^t_{\mathbf{E}}(P_0)| \leq R, \quad |\mathcal{T}^t_{\mathbf{D}}(P_0)| \leq R,$$

for $t \in [0, T]$*. Then, it holds that*

$$|\mathcal{T}^t_{\mathbf{E}}(P_0) - \mathcal{T}^t_{\mathbf{D}}(P_0)| \leq \frac{e^{Ct} - 1}{C\varepsilon} \sup_{s\in[0,T)} \|\mathbf{E}(s) - \mathbf{D}(s)\|_{L^\infty(B^1_R)},$$

for $t \in [0, T]$*, where the constant* C *depends on* R *and* $Lip_R(\mathbf{E})$*.*

Proof. Let $P_{\mathbf{E}}(t) = \mathcal{T}^t_{\mathbf{E}}(P_0)$ and $P_{\mathbf{D}}(t) = \mathcal{T}^t_{\mathbf{D}}(P_0)$ be the solutions to system (2.9) with vector fields $\mathbf{E}$ and $\mathbf{D}$ respectively, that is

$$\frac{d}{dt}P_{\mathbf{E}}(t) = \Psi_{\mathbf{E}}(t, P_{\mathbf{E}}(t)), \qquad \frac{d}{dt}P_{\mathbf{D}}(t) = \Psi_{\mathbf{D}}(t, P_{\mathbf{D}}(t)).$$

Using Lemma 2.2 and Lemma 2.3 we have

$$\begin{aligned}
|P_{\mathbf{E}}(t) - P_{\mathbf{D}}(t)| &\leq \int_0^t |\Psi_{\mathbf{E}}(s, P_{\mathbf{E}}(s)) - \Psi_{\mathbf{D}}(s, P_{\mathbf{D}}(s))|\, ds \\
&\leq \int_0^t |\Psi_{\mathbf{E}}(s, P_{\mathbf{E}}(s)) - \Psi_{\mathbf{E}}(s, P_{\mathbf{D}}(s))|\, ds \\
&\qquad + \int_0^t |\Psi_{\mathbf{E}}(s, P_{\mathbf{D}}(s)) - \Psi_{\mathbf{D}}(s, P_{\mathbf{D}}(s))|\, ds \\
&\leq C\int_0^t |P_{\mathbf{E}}(s) - P_{\mathbf{D}}(s)|\, ds + \int_0^t \frac{1}{\varepsilon}\|\mathbf{E}(s) - \mathbf{D}(s)\|_{L^\infty(B^1_R)}\, ds.
\end{aligned}$$

By Grönwall's lemma we obtain that

$$\begin{aligned}
|P_{\mathbf{E}}(t) - P_{\mathbf{D}}(t)| &\leq \int_0^t e^{C(t-s)}\frac{1}{\varepsilon}\|\mathbf{E}(s) - \mathbf{D}(s)\|_{L^\infty(B^1_R)}\, ds \\
&\leq \frac{e^{Ct} - 1}{C\varepsilon} \sup_{s\in[0,T)} \|\mathbf{E}(s) - \mathbf{D}(s)\|_{L^\infty(B^1_R)},
\end{aligned}$$

that concludes the proof. ∎

Remark 2.2. Note that the sub-linearity assumption on the vector field $\mathbf{E}$ ensures global existence for solution for $t \in \mathbb{R}$. The boundedness assumption in Lemma 2.4 on the initial flow $\mathcal{T}^t_{\mathbf{E}}(P_0)$ is only needed to prove a quantitative estimate on the flow map for every time $t \in [0, T]$.

Lemma 2.5 (Regularity of the characteristics with respect to the initial conditions)**.** *Fix $T > 0$ and a vector field $\mathbf{E}$ satisfying* $(\mathbf{E_1})$-$(\mathbf{E_2})$-$(\mathbf{E_3})$. *Consider $P_1, P_2 \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$ and $R > 0$ and assume*

$$|\mathcal{T}^t_{\mathbf{E}}(P_1)| \leq R, \quad |\mathcal{T}^t_{\mathbf{E}}(P_2)| \leq R,$$

for $t \in [0, T]$. Then

$$|\mathcal{T}^t_{\mathbf{E}}(P_1) - \mathcal{T}^t_{\mathbf{E}}(P_2)| \leq |P_1 - P_2| e^{C \int_0^t (Lip_R(\mathbf{E}(s))+1)\, ds},$$

for $t \in [0, T]$, where the constant C depends on R.

Proof. Set $P_i(t) = \mathcal{T}^t_{\mathbf{E}}(P_i)$, for $i = 1, 2$, and $t \in [0, T]$. These functions fulfil

$$\frac{d}{dt} P_i(t) = \Psi_{\mathbf{E}}(t, P_i(t)), \quad P_i(0) = P_i,$$

for $i = 1, 2$. Using Lemma 2.2, for $t \in [0, T]$ we have that

$$\begin{aligned} |P_1(t) - P_2(t)| &\leq |P_1 - P_2| + \int_0^t |\Psi_{\mathbf{E}}(s, P_1(s)) - \Psi_{\mathbf{E}}(s, P_2(s))|\, ds \\ &\leq |P_1 - P_2| + C \int_0^t (Lip_R(\mathbf{E}(s)) + 1)|P_1(s) - P_2(s)|\, ds. \end{aligned}$$

Applying Grönwall's lemma to inequality above, we get the statement. □

Remark 2.3. Lemma 2.5 ensures that the flow $\mathcal{T}^t_{\mathbf{E}}$ is Lipschitz on $B_R \subset \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$, with constant

$$Lip_R(\mathcal{T}^t_{\mathbf{E}}) \leq e^{C \int_0^t (Lip_R(\mathbf{E}(s))+1)\, ds},$$

for $t \in [0, T]$.

Lemma 2.6 (Regularity of the characteristics with respect to time)**.** *Let $T > 0$ and $\mathbf{E}$ be a vector field satisfying* $(\mathbf{E_1})$-$(\mathbf{E_2})$-$(\mathbf{E_3})$. *Let $P_0 \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$, $R > 0$ and assume*

$$|\mathcal{T}^t_{\mathbf{E}}(P_0)| \leq R,$$

for $t \in [0, T]$. Then, it holds that

$$|\mathcal{T}^t_{\mathbf{E}}(P_0) - \mathcal{T}^s_{\mathbf{E}}(P_0)| \leq C|t - s|,$$

for $s, t \in [0, T]$, where the constant C depends on R and $\|\mathbf{E}\|_{L^\infty([0,T] \times B^1_R)}$.

Proof. Since we are assuming that $\mathcal{T}^t_{\mathbf{E}}(P_0)$ is in a compact subset of $\mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$ for every time, then the statement holds by definition of $\mathcal{T}^t_{\mathbf{E}}(P_0)$ and by Lemma 2.2. □

In the following Lemmas we collect some contraction results in the Wasserstein distance $\mathcal{W}_1$ that are crucial in proving existence of measure solutions for (2.8). What we reproduce is the extension to multiple species of the results in [17, Lemma 3.11, Lemma 3.12, Lemma 3.13].

Lemma 2.7. *Let $\mathbf{E}$, $\mathbf{D} : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ be two Borel measurable maps and let $\mathbf{f} \in \mathcal{P}_1(\mathbb{R}^d)^N$. Then*

$$\mathcal{W}_1(\mathbf{E}\#\mathbf{f}, \mathbf{D}\#\mathbf{f}) \leq \|\mathbf{E} - \mathbf{D}\|_{L^\infty(\operatorname{supp} \mathbf{f})}.$$

Proof. Thanks to the Definition 2.1 of 1-Wasserstain distance in the product space, it holds that

$$\mathcal{W}_1(\mathbf{E}\#\mathbf{f}, \mathbf{D}\#\mathbf{f}) \leq \sum_{i=1}^N W_1(E_i\#f_i, D_i\#f_i)$$

$$\leq \sum_{i=1}^{N} \|E_i - D_i\|_{L^\infty(\operatorname{supp} \mathbf{f})}$$
$$\leq \|\mathbf{E} - \mathbf{D}\|_{L^\infty(\operatorname{supp} \mathbf{f})},$$

thus the statement is proved. □

Lemma 2.8. *Let $T > 0$. Let $\mathbf{E} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{Nd}$ be a field that satisfies ($\mathbf{E_1}$)-($\mathbf{E_2}$)-($\mathbf{E_3}$) and let $\mathbf{f}$ be a N-tuple of measures on $\mathbb{R}^d$ with compact support contained in a ball $B_R \subset \mathbb{R}^d$. Then, there exists a positive constant C depending on N, R and $\|\mathbf{E}\|_{L^\infty([0,T]\times B_R^1)}$ such that*

$$\mathcal{W}_1(\mathcal{T}^t_{\mathbf{E}} \# \mathbf{f}, \mathcal{T}^s_{\mathbf{E}} \# \mathbf{f}) \leq C|t-s|,$$

for any $s, t \in [0,T]$.

Proof. By the definition of flow in (2.12), we have that

$$\mathcal{W}_1\big(\mathcal{T}^t_{\mathbf{E}} \# \mathbf{f}, \mathcal{T}^s_{\mathbf{E}} \# \mathbf{f}\big) \leq \sum_{i=1}^{N} W_1(\mathcal{T}^t_{E_i} \# f_i, \mathcal{T}^s_{E_i} \# f_i)$$
$$\leq \sum_{i=1}^{N} C_i|t-s|$$
$$\leq C|t-s|,$$

therefore the statement holds. □

Lemma 2.9. *Let $\mathcal{T} : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ be a Lipschitz map and let $\mathbf{f}, \mathbf{g} \in \mathcal{P}_1(\mathbb{R}^d)^N$ both have compact support contained in a ball B_R. Then*

$$\mathcal{W}_1\big(\mathcal{T} \# \mathbf{f}, \mathcal{T} \# \mathbf{g}\big) \leq L \mathcal{W}_1(\mathbf{f}, \mathbf{g}),$$

where L is the Lipschitz constant of $\mathcal{T}$ on the ball B_R.

Proof. By a simple application of the triangular inequality we have

$$\mathcal{W}_1\big(\mathcal{T} \# \mathbf{f}, \mathcal{T} \# \mathbf{g}\big) = \mathcal{W}_1\big((\mathcal{T}_1 \# f_1, \cdots, \mathcal{T}_N \# f_N), (\mathcal{T}_1 \# g_1, \cdots, \mathcal{T}_N \# g_N)\big)$$
$$\leq \sum_{i=1}^{N} \big[W_1(\mathcal{T}_i \# f_i, \mathcal{T}_i \# g_i)\big]$$
$$\leq \sum_{i=1}^{N} L_i W_1(f_i, g_i)$$
$$\leq L\ \mathcal{W}_1(\mathbf{f}, \mathbf{g}),$$

that concludes the proof. □

2.3.2 Existence and uniqueness for smooth potentials

We turn now into the existence and uniqueness of measure solutions to system (2.5). We first provide the following preliminary Lemmas, whose proof is straightforward and we omit.

Lemma 2.10. *Assume the potentials K_{ij} under assumption* (**Pot**). *Let $\mathbf{f} \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ be with compact support contained in a ball $B_R \subset \mathbb{R}^{2d}$. Set $B_R^1 := \{x : (x,v) \in B_R\}$. Consider the vector field defined in* (2.13). *Then,*

$$\|\mathbf{E}[\mathbf{f}]\|_{L^\infty(B_R^1)} \leq \Xi, \quad \text{and} \quad Lip_R(\mathbf{E}[\mathbf{f}]) \leq \Upsilon,$$

where the constants Ξ and Υ are defined by

$$\Xi := \sum_{i,j=1}^{N} \|\nabla K_{ij}\|_{L^\infty(B_{2R})},$$

and

$$\Upsilon := \sum_{i,j=1}^{N} Lip_{2R}(\nabla K_{ij}).$$

Lemma 2.11. *Assume the potentials K_{ij} as in* (**Pot**). *Let $\mathbf{f}, \mathbf{g} \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ and $R > 0$. Then,*

$$\|\mathbf{E}[\mathbf{f}] - \mathbf{E}[\mathbf{g}]\|_{L^\infty(B_R^1)} \leq \Upsilon \mathcal{W}_1(\mathbf{f}, \mathbf{g}).$$

Existence and uniqueness of measure solutions to the kinetic system (2.5) is stated and proved in the following Theorem.

Theorem 2.1. *Assume the potentials K_{ij} under assumption* (**Pot**). *Let $\mathbf{f}_0 \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ be with compact support. Then there exists a unique measure solution $\mathbf{f} \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ to system* (2.5) *with initial condition $\mathbf{f}_0$ in the sense of Definition 2.2. In particular,*

$$\mathbf{f} \in \mathcal{C}([0, +\infty); \mathcal{P}_1(\mathbb{R}^{2d})^N), \tag{2.14}$$

and there exists an increasing function $R = R(T)$ such that for all $T > 0$,

$$\operatorname{supp}(\mathbf{f}) \subset B_{R(T)} \subset \mathbb{R}^d \times \mathbb{R}^d, \tag{2.15}$$

for all $t \in [0, T]$.

Proof. Let $\mathbf{f}_0$ be such that

$$\operatorname{supp}(\mathbf{f}_0) \subset B_{R_0} \subset \mathbb{R}^d \times \mathbb{R}^d,$$

for some $R_0 > 0$. In order to prove existence and uniqueness of the solution, we are going to use a contraction argument. In particular, we introduce the metric space

$$\mathcal{F} = \left\{ \mathbf{f} \in \mathcal{C}((0, T], \mathcal{P}_1(\mathbb{R}^{2d})^N) \,:\, \operatorname{supp}(\mathbf{f}) \subset B_R \text{ for all } t \in [0, T] \right\},$$

where $R := 2R_0$ and $T > 0$ is a fixed time we will choose later. This metric space is equipped with the distance $\mathcal{W}_1$, see Definition 2.1. On this space we define a map as follows. For $\mathbf{f} \in \mathcal{F}$, consider $\mathbf{E}[\mathbf{f}]$ defined as in (2.13). Then, by Lemmas 2.10 and 2.11 and by assumption (**Pot**), we obtain that $\mathbf{E}[\mathbf{f}]$ satisfies ($\mathbf{E_1}$)-($\mathbf{E_2}$)-($\mathbf{E_3}$) and thus we can define

$$\Gamma[\mathbf{f}](t) := \mathcal{T}^t_{\mathbf{E}[\mathbf{f}]} \# \mathbf{f}_0.$$

The aim is to prove that this map is a contraction and its unique fixed point in $\mathcal{F}$ is the solution to (2.5). We start proving that the operator $\Gamma[\mathbf{f}]$ is well-posed in the space $\mathcal{F}$. From Lemma 2.10 we have that

$$\|\mathbf{E}[\mathbf{f}]\|_{L^\infty([0,T]\times B_R^1)} \leq \Xi,$$

and from Lemma 2.2,

$$\left|\frac{d}{dt}\mathcal{T}^t_{\mathbf{E}[\mathbf{f}]}(P)\right| \leq C_1,$$

for all $P \in B_{R_0} \subset \mathbb{R}^d \times \mathbb{R}^d$, with C_1 depending on R_0 and Ξ. For $T < R_0/C_1$, we have that $\mathcal{T}^t_{\mathbf{E}[\mathbf{f}]}\#\mathbf{f}_0$ has support contained in B_R for all $t \in [0,T]$. Then, for each $t \in [0,T]$, $\Gamma[\mathbf{f}](t) \in \mathcal{P}_1(\mathbb{R}^{2d})^N$, the support of $\Gamma[\mathbf{f}](t)$ is contained in B_R and the map $t \mapsto \Gamma[\mathbf{f}](t)$ is continuous by Lemma 2.8. Thus the map $\Gamma : \mathcal{F} \to \mathcal{F}$ is well defined.
We show now that the map is a contraction, i.e., considering two functions $\mathbf{f}, \mathbf{g} \in \mathcal{F}$ and taking $\Gamma[\mathbf{f}]$ and $\Gamma[\mathbf{g}]$, we want to prove that

$$\mathcal{W}_1(\Gamma[\mathbf{f}], \Gamma[\mathbf{g}]) \leq C\mathcal{W}_1(\mathbf{f}, \mathbf{g})$$

for $0 < C < 1$ which does not depend on the functions $\mathbf{f}$ and $\mathbf{g}$. By definition of Γ we have that

$$\mathcal{W}_1(\Gamma[\mathbf{f}], \Gamma[\mathbf{g}]) = \mathcal{W}_1(\mathcal{T}^t_{\mathbf{E}[\mathbf{f}]}\#\mathbf{f}_0, \mathcal{T}^t_{\mathbf{E}[\mathbf{g}]}\#\mathbf{f}_0).$$

Using Lemmas 2.7, 2.4 and 2.11, the above distance can be estimated as follows

$$\begin{aligned}
\mathcal{W}_1(\mathcal{T}^t_{\mathbf{E}[\mathbf{f}]}\#\mathbf{f}_0, \mathcal{T}^t_{\mathbf{E}[\mathbf{g}]}\#\mathbf{f}_0) &\leq \|\mathcal{T}^t_{\mathbf{E}[\mathbf{f}]} - \mathcal{T}^t_{\mathbf{E}[\mathbf{g}]}\|_{L^\infty(\operatorname{supp}\mathbf{f}_0)} \\
&\leq C(t) \sup_{s\in[0,T]} \|\mathbf{E}[\mathbf{f}](s) - \mathbf{E}[\mathbf{g}](s)\|_{L^\infty(B_R^1)} \\
&\leq C(t)\Upsilon\mathcal{W}_1(\mathbf{f}, \mathbf{g}),
\end{aligned}$$

where $C(t) = (e^{C_2 t} - 1)/\varepsilon C_2$ is the function in the statement of Lemma 2.4, with C_2 a constant depending on R and Υ. Therefore, we obtain that

$$\mathcal{W}_1(\Gamma[\mathbf{f}], \Gamma[\mathbf{g}]) \leq C(t)\Upsilon\mathcal{W}_1(\mathbf{f}, \mathbf{g}).$$

Since it holds that

$$\lim_{t\to 0} C(t) = 0,$$

we get

$$\mathcal{W}_1(\Gamma[\mathbf{f}], \Gamma[\mathbf{g}]) \leq C(T)\Upsilon\mathcal{W}_1(\mathbf{f}, \mathbf{g}).$$

We can choose T small enough so that $C(T)\Upsilon < 1$. In this way, the functional Γ is contractive and then there is a unique fixed point of Γ in $\mathcal{F}$. By construction it is easy to see that this fixed point of Γ is a solution to (2.5) on $[0,T]$. Finally, since the growth of characteristic is bounded, as proved in Lemma 2.1, we can construct a unique global solution satisfying (2.14) and (2.15). □

Proposition 2.1 (Stability of the solutions)**.** *Assume that the potentials K_{ij} are under assumption* (**Pot**)*. Let $\mathbf{f}_0, \mathbf{g}_0 \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ be with compact support, and consider the solutions $\mathbf{f}, \mathbf{g}$ to* (2.5) *with initial conditions $\mathbf{f}_0$ and $\mathbf{g}_0$, respectively. Then, there exists an increasing function $r(t) : [0,\infty) \to \mathbb{R}^+$ with $r(0) = 1$ that depends only on the supports of $\mathbf{f}_0$ and $\mathbf{g}_0$ such that*

$$\mathcal{W}_1(\mathbf{f}(t), \mathbf{g}(t)) \leq r(t)\mathcal{W}_1(\mathbf{f}_0, \mathbf{g}_0), \tag{2.16}$$

for $t \geq 0$.

Proof. Take $T > 0$ and let $R > 0$ such that the supports of $\mathbf{f}(t)$ and $\mathbf{g}(t)$ are contained in $B_R \subset \mathbb{R}^{2d}$ for each $t \in [0,T]$. In order to prove the stability result, we follow the strategy introduced in [17, Theorem 3.16], where a triangulation argument mixes the estimates obtained in the previous Lemmas. We get

$$\begin{aligned}\mathcal{W}_1(\mathbf{f}(t),\mathbf{g}(t)) &= \mathcal{W}_1(\mathcal{T}^t_{\mathbf{E}[\mathbf{f}]}\#\mathbf{f}_0, \mathcal{T}^t_{\mathbf{E}[\mathbf{g}]}\#\mathbf{g}_0)\\ &\le \mathcal{W}_1(\mathcal{T}^t_{\mathbf{E}[\mathbf{f}]}\#\mathbf{f}_0, \mathcal{T}^t_{\mathbf{E}[\mathbf{g}]}\#\mathbf{f}_0) + \mathcal{W}_1(\mathcal{T}^t_{\mathbf{E}[\mathbf{g}]}\#\mathbf{f}_0, \mathcal{T}^t_{\mathbf{E}[\mathbf{g}]}\#\mathbf{g}_0)\\ &=: A_1 + A_2.\end{aligned}$$

Using Lemmas 2.7, 2.4 and 2.11, we get

$$\begin{aligned}A_1 &\le \|\mathcal{T}^t_{\mathbf{E}[\mathbf{f}]} - \mathcal{T}^t_{\mathbf{E}[\mathbf{g}]}\|_{L^\infty(\operatorname{supp}\mathbf{f}_0)}\\ &\le \int_0^t e^{C_1(t-s)}\frac{1}{\varepsilon}\|\mathbf{E}[\mathbf{f}](s) - E[\mathbf{g}](s)\|_{L^\infty(B^1_R)}\,ds\\ &\le C_1\frac{\Upsilon}{\varepsilon}\int_0^t e^{C_1(t-s)}\mathcal{W}_1(\mathbf{f}(s),\mathbf{g}(s))\,ds.\end{aligned}$$

Moreover, by Lemma 2.5, calling L the Lipschitz constant of $\mathcal{T}^t_{\mathbf{E}[\mathbf{g}]}$ on B_R, we have that

$$A_2 \le L\mathcal{W}_1(\mathbf{f}_0,\mathbf{g}_0) \le e^{C_2 t}\mathcal{W}_1(\mathbf{f}_0,\mathbf{g}_0),$$

for all $t \in [0,T]$. Thus, we obtain

$$\mathcal{W}_1(\mathbf{f}(t),\mathbf{g}(t)) \le \frac{\Upsilon}{\varepsilon}\int_0^t e^{C_1(t-s)}\mathcal{W}_1(\mathbf{f}(s),\mathbf{g}(s))\,ds + e^{C_2 t}\mathcal{W}_1(\mathbf{f}_0,\mathbf{g}_0).$$

Setting $C = \max\{\Upsilon/\varepsilon, C_1, C_2\}$, multiplying by e^{-Ct} we get

$$e^{-Ct}\mathcal{W}_1(\mathbf{f}(t),\mathbf{g}(t)) \le C\int_0^t e^{-Cs}\mathcal{W}_1(\mathbf{f}(s),\mathbf{g}(s))\,ds + \mathcal{W}_1(\mathbf{f}_0,\mathbf{g}_0).$$

By Grönwall's lemma we derive

$$e^{-Ct}\mathcal{W}_1(\mathbf{f}(t),\mathbf{g}(t)) \le \mathcal{W}_1(\mathbf{f}_0,\mathbf{g}_0)e^{Ct},$$

for $t \in [0,T]$, thus (2.16) holds. $\square$

Theorem 2.2 (Existence of smooth solutions)**.** *Let $T > 0$ be a positive time. Assume $\nabla K_{ij} \in W^{1,\infty}(\mathbb{R}^d)$. Let $\mathbf{f}_0 \in \mathcal{C}^2(\mathbb{R}^{2d})^N \cap L^1(\mathbb{R}^{2d})^N$. Then system (2.5) has a solution $\mathbf{f}^\varepsilon \in \mathcal{C}([0,T];\mathcal{C}^1(\mathbb{R}^{2d})^N)$ with initial datum $\mathbf{f}_0$.*

Sketch of Proof. We provide a sketch of proof. The details can be found in [43] for Vlasov-Poisson system and Vlasov-Maxwell system. The proof is divided in three steps. One first constructs an approximating sequence $\mathbf{f}^{\varepsilon,n} \in \mathcal{C}([0,T];\mathcal{C}^2(\mathbb{R}^d\times\mathbb{R}^d)^N)$ by iterations, defining $\mathbf{f}^{\varepsilon,n+1}$ to be the solution of

$$\begin{gathered}\partial_t f_i^{\varepsilon,n+1} + v\cdot\nabla_x f_i^{\varepsilon,n+1} - \frac{1}{\varepsilon}\nabla_v\cdot(v f_i^{\varepsilon,n+1}) + \frac{1}{\varepsilon}E_i^n\cdot\nabla_v f_i^{\varepsilon,n+1} = 0,\\ f_i^{\varepsilon,n+1}(0,x,v) = f_{i0}(x,v),\end{gathered}$$

as $i = 1,\dots,N$. Once we observe that the characteristics associated to the system above depend on n, but still satisfy the features in Lemmas above, then it holds that $\mathbf{f}^{\varepsilon,n} \in \mathcal{C}^1([0,T];\mathcal{C}^1(\mathbb{R}^d\times\mathbb{R}^d)^N)$ is uniformly bounded with respect to n, since all the constants in the estimates above depend on the support of the initial datum and on the Lipschitz constant of the kernels. Next, showing that $\mathbf{f}^{\varepsilon,n}$ is a Cauchy sequence in $\mathcal{C}([0,T];\mathcal{C}^1(\mathbb{R}^d\times\mathbb{R}^d)^N)$ converging to the solution to (2.5), we complete the proof. $\square$

2.4 Uniform estimates in ε

In this Section we gather some uniform in ε estimates we will use to prove the convergence of solutions to (2.5) towards the solution to (2.1) as $\varepsilon \to 0$. For this reason, we make the ε-dependence explicit, i.e., we deal with the system

$$\partial_t f_i^\varepsilon + v \cdot \nabla_x f_i^\varepsilon = \frac{1}{\varepsilon} \nabla_v \cdot \left(\left(v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right) f_i^\varepsilon \right), \tag{2.17}$$

for $i = 1, \ldots, N$, equipped with initial data

$$f_i^\varepsilon(t, x, v)|_{t=0} = f_{i0}^\varepsilon(x, v) \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d), \tag{2.18}$$

and where

$$\rho_i^\varepsilon(t, x) = \int_{\mathbb{R}^d} f_i^\varepsilon(t, x, v)\, dv.$$

Throughout this Section, we assume the initial data with compact support.

Proposition 2.2 (Uniform estimate for the support)**.** *Assume all the potentials under assumption* (**Pot**)*. Let* $\mathbf{f}^\varepsilon$ *be a solution to the system* (2.17)-(2.18) *as proved in Theorem 2.1. Then, there exists an increasing function* $R(T)$ *independent on* ε *such that for all* $T > 0$*,*

$$\mathrm{supp}(\mathbf{f}^\varepsilon)(t) \subset B_{R(T)}, \tag{2.19}$$

for all $t \in [0, T]$ *and* $\varepsilon > 0$*. The function* $R(T)$ *depends only on the support of* $\mathbf{f}_0^\varepsilon$ *and* Ξ*.*

Proof. Consider the initial point $(x_0, v_0) \in \mathrm{supp}(\mathbf{f}_0^\varepsilon)$. The support of $\mathbf{f}^\varepsilon$ evolves according to the flow associated to the following characteristic equations

$$\frac{dx_i^\varepsilon}{dt} = v_i^\varepsilon, \tag{2.20}$$

$$\varepsilon \frac{dv_i^\varepsilon}{dt} = -v_i^\varepsilon - \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon, \tag{2.21}$$

starting from (x_{i0}, v_{i0}), as $i = 1, \ldots, N$. Since

$$\left| \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right| \le \sum_{j=1}^N \|\nabla K_{ij}\|_{L^\infty} =: C_{i,1},$$

for $i = 1, \ldots, N$, we have that the Euclidean norms $|v_i^\varepsilon(t)|$ of the trajectories of v_i^ε satisfies

$$\frac{d|v_i^\varepsilon|}{dt} \le -\frac{1}{\varepsilon}|v_i^\varepsilon| + \frac{1}{\varepsilon} C_1, \qquad v_i^\varepsilon(0) = v_{i0},$$

with $C_1 = \max_i C_{i,1}$. Therefore, there exists a constant C_2 depending only on the support of $\mathbf{f}_0$ and Ξ such that all the characteristic trajectories starting within the supports of $\mathbf{f}_0$ satisfy

$$|v_i^\varepsilon(t)| \le C_2,$$

as $i = 1, \ldots, N$, for all $t > 0$ and $\varepsilon > 0$. The trajectories of x_i^ε grow at most linearly in time since

$$\frac{d|x_i^\varepsilon|}{dt} \le |v_i^\varepsilon|.$$

Then, there exists a function $R(T)$ that depends only on the supports of $\mathbf{f}_0$, T and Ξ such that (2.19) holds. □

2.4.1 Estimate for smooth solutions

We first produce uniform in ε estimates in case of smooth solutions, namely solutions given by Theorem 2.2. In the next Subsection, we will deal with uniform in ε estimates for measure solutions.

Proposition 2.3. *Assume all the potentials under assumption* (**Pot**). *Suppose that the initial datum* $\mathbf{f}_0$ *in* (2.18) *has a finite first moment in* v*, i.e.,* $|v|f_{i0} \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$ *for all* $i = 1, \dots, N$*. Let* $\mathbf{f}^\varepsilon$ *be the classical solution to* (2.17)*, as in Theorem 2.2. Then there exist some positive constants* C_i*, as* $i = 1, \dots, N$*, and a function* $M(\varepsilon)$ *depending on* ε*, such that*

$$\iint_{\mathbb{R}^{2d}} \left| v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right| f_i^\varepsilon \, dx \, dv \le C_i M(\varepsilon), \tag{2.22}$$

for all $t \in [0, T]$*, where* C_i *depends on* $\|(1 + |v|)f_{i0}\|_{L^1(\mathbb{R}^{2d})}$ *and* Ξ*. Moreover,*

$$\lim_{\varepsilon \downarrow 0} M(\varepsilon) = 0.$$

Proof. We set

$$I_i(t) = \iint_{\mathbb{R}^{2d}} \left| v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right| f_i^\varepsilon \, dx \, dv,$$

as $i = 1, \dots, N$. We want to prove that there exist C_i and $M(\varepsilon)$ as in the statement such that

$$\sup_{t \in [0,T]} I_i(t) \le C_i M(\varepsilon),$$

for a small ε. Straightforward computation shows that

$$\begin{aligned} \frac{d}{dt} I_i(t) = & \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\partial_t \left| v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right| \right) f_i^\varepsilon \, dx \, dv \\ & + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right| \partial_t f_i^\varepsilon \, dx \, dv. \end{aligned}$$

By using system (2.17) and integration by parts, we get

$$\begin{aligned} & \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right| \partial_t f_i^\varepsilon \, dx \, dv \\ & \quad = - \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v \cdot \nabla_x f_i^\varepsilon) \left| v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right| dx \, dv \\ & \qquad + \frac{1}{\varepsilon} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \cdot \left(\left(v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right) f_i^\varepsilon \right) \left| v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right| dx \, dv \\ & \quad = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(v \cdot \nabla_x \left| v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right| \right) f_i^\varepsilon \, dx \, dv \\ & \qquad - \frac{1}{\varepsilon} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right| f_i^\varepsilon \, dx \, dv. \end{aligned}$$

Therefore, we have that

$$\frac{d}{dt} I_i(t) = -\frac{1}{\varepsilon} I_i(t) + I_i^1(t) + I_i^2(t),$$

with

$$I_i^1(t) = \iint_{\mathbb{R}^{2d}} \left(\partial_t \left| v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right| \right) f_i^\varepsilon \, dx \, dv,$$

$$I_i^2(t) = \iint_{\mathbb{R}^{2d}} \left(v \cdot \nabla_x \left| v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right| \right) f_i^\varepsilon \, dx \, dv,$$

for $i = 1, \ldots, N$. In order to obtain our claim, we want to show that $I_i^1(t)$ and $I_i^2(t)$ are bounded linearly by $I_i(t)$, and then derive a differential inequality to bound $I_i(t)$. Setting

$$\langle f_i \rangle := \int_{\mathbb{R}^d} f_i \, dv,$$

integrating (2.17) in v, we have that

$$\partial_t \rho_i^\varepsilon + \nabla_x \cdot \langle v f_i^\varepsilon \rangle = 0, \tag{2.23}$$

and the conservation of masses

$$\|\rho_i^\varepsilon(t)\|_{L^1(\mathbb{R}^d)} = \|f_{i0}\|_{L^1(\mathbb{R}^{2d})},$$

for all $t > 0$ and for all $i = 1, \ldots, N$. Thus, using the equation for ρ_i^ε in (2.23), we can preliminary estimate

$$\left| \partial_t \left| v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right| \right| \le \left| \sum_{j=1}^N \nabla K_{ij} * \partial_t \rho_j^\varepsilon \right| \le \sum_{j=1}^N \left| \Delta K_{ij} * \langle v f_j^\varepsilon \rangle \right|.$$

By adding and subtracting $\sum_{h=1}^N \nabla K_{ih} * \rho_h^\varepsilon$ in the absolute value in the right hand side of the inequality above, and using assumption (**Pot**) we get

$$\begin{aligned} \sum_{j=1}^N \left| \nabla K_{ij} * \langle v f_j^\varepsilon \rangle \right| \le & \sum_{j=1}^N \left| \Delta K_{ij} * \left\langle \left(v + \sum_{h=1}^N \nabla K_{ih} * \rho_h^\varepsilon \right) f_j^\varepsilon \right\rangle \right| \\ & + \sum_{j=1}^N \|\Delta K_{ij} * \rho_j^\varepsilon\|_{L^\infty} \sum_{h=1}^N \|\nabla K_{ih} * \rho_h^\varepsilon\|_{L^\infty} \\ \le & \sum_{j=1}^N \|\Delta K_{ij}\|_{L^\infty} \int_{\mathbb{R}^d} \left| w + \sum_{h=1}^N \nabla K_{ih} * \rho_h^\varepsilon \right| f_j^\varepsilon(x, w, t) \, dw \\ & + \sum_{j=1}^N \|\Delta K_{ij}\|_{L^\infty} \|\nabla K_{ij}\|_{L^\infty} \|\rho_j^\varepsilon\|_{L^1}^2. \end{aligned}$$

Thus, integrating the above inequality in x and v we get

$$\begin{aligned}
|I_i^1(t)| \le & \sum_{j=1}^N \|\Delta K_{ij}\|_{L^\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left| v + \sum_{h=1}^N \nabla K_{ih} * \rho_h^\varepsilon \right| f_i^\varepsilon(x,v,t) f_j^\varepsilon(x,w,t)\, dx\, dv\, dw \\
& + \sum_{j=1}^N \|\Delta K_{ij}\|_{L^\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (|v|+|w|) f_j^\varepsilon(x,w,t) f_i^\varepsilon(x,v,t)\, dx\, dv\, dw \\
& + \sum_{j=1}^N \|\Delta K_{ij}\|_{L^\infty} \|\nabla K_{ij}\|_{L^\infty} \|\rho_j^\varepsilon\|_{L^1}^2 \|\rho_i^\varepsilon\|_{L^1} \\
\le & \sum_{j=1}^N \|\Delta K_{ij}\|_{L^\infty} I_i(t) \|\rho_j^\varepsilon\|_{L^1} \\
& + \sum_{j=1}^N \|\Delta K_{ij}\|_{L^\infty} \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |v| f_i^\varepsilon(x,v,t)\, dx\, dv + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |w| f_j^\varepsilon(x,w,t)\, dx, dw \right) \\
& + \sum_{j=1}^N \|\Delta K_{ij}\|_{L^\infty} \|\nabla K_{ij}\|_{L^\infty} \|\rho_j^\varepsilon\|_{L^1}^2 \|\rho_i^\varepsilon\|_{L^1}.
\end{aligned}$$

Since $f_i^\varepsilon \in \mathcal{P}_1(\mathbb{R}^{2d})$ for all $i = 1, \dots, N$, we obtain that for each i there exist two positive constants A_i^1 and A_i^2 depending on Ξ and all $\|(1+|v|)f_{i0}\|_{L^1}$ such that

$$I_i^1(t) \le A_i^1 I_i(t) + A_i^2.$$

Concerning the terms I_i^2, we can estimate

$$\begin{aligned}
|I_i^2(t)| &\le \sum_{j=1}^N \int_{\mathbb{R}^d \times \mathbb{R}^d} |v| |\Delta K_{ij} * \rho_j^\varepsilon| f_i^\varepsilon \, dx\, dv \\
&\le \sum_{j=1}^N \|\Delta K_{ij}\|_{L^\infty} \|\rho_j^\varepsilon\|_{L^1} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v| f_i^\varepsilon \, dx\, dv \\
&\le \sum_{j=1}^N \|\Delta K_{ij}\|_{L^\infty} \|\rho_j^\varepsilon\|_{L^1} \Bigg[\iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| v + \sum_{h=1}^N \nabla K_{ih} \rho_h^\varepsilon \right| f_i^\varepsilon \, dx\, dv \\
&\qquad\qquad + \sum_{h=1}^N \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| \nabla K_{ih} * \rho_h^\varepsilon \right| f_i^\varepsilon \, dx\, dv \Bigg] \\
&\le \sum_{j=1}^N \|\Delta K_{ij}\|_{L^\infty} \|\rho_j^\varepsilon\|_{L^1} \left[I_i(t) + \sum_{h=1}^N \|\nabla K_{ih}\|_{L^\infty} \|\rho_h^\varepsilon\|_{L^1} \|\rho_i^\varepsilon\|_{L^1} \right].
\end{aligned}$$

Thus, we derive that for each i there exist two positive constants B_i^1 and B_i^2 depending on Ξ and all $\|f_{i0}\|_{L^1}$ such that

$$|I_i^2(t)| \le B_i^1 I_i(t) + B_i^2.$$

Hence, considering the estimates above, we obtain that

$$\frac{d}{dt} I_i(t) \le -\frac{1}{\varepsilon} I_i(t) + C_i^1 I_i(t) + C_i^2, \tag{2.24}$$

where $C_i^k = A_i^k + B_i^k$ for $i = 1, \dots, N$ and $k = 1, 2$. Furthermore, at time $t = 0$ we get

$$I_i(0) \le \||v|f_{i0}\|_{L^1} + D_i, \tag{2.25}$$

as $i = 1, \dots, N$, where the positive constants D_i depend on Ξ and all $\|f_{i0}\|_{L^1}$. Combining (2.24) and (2.25) and using Grönwall's lemma, we obtain that

$$\sup_{t\in[0,T]} I_i(t) \le C_i M_i(\varepsilon),$$

where the constants C_i depend on Ξ and all $\|(1+|v|)f_{i0}\|_{L^1}$. Finally, is it enough to note that $M(\varepsilon) := \max_i\{M_i(\varepsilon)\}$ decays to 0 as $\varepsilon \to 0$. □

2.4.2 Estimate for measure solutions

In this Section our aim is to find an estimate as in (2.22) for a measure solution $\mathbf{f}$ to system (2.17). In order to proceed, we introduce the mollifier

$$\gamma^{(n)}(x, v) = n^{2d}\gamma^{(1)}(nx, nv) \in \mathcal{C}_c^\infty(\mathbb{R}^{2d}),$$

where

$$\operatorname{supp}(\gamma^{(1)}) \subset \overline{B(0,1)} \subset \mathbb{R}^{2d}, \qquad \gamma^{(1)} \ge 0, \qquad \iint_{\mathbb{R}^{2d}} \gamma^{(1)}(x,v)\,dx\,dv = 1,$$
$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|\gamma^{(1)}(x,v)\,dx\,dv \le 1.$$

Now, let $\mathbf{f}_0 \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ be with compact support and let $\varepsilon > 0$ fixed. Define

$$\mathbf{f}_0^{(n)} = \mathbf{f}_0 * \gamma^{(n)}, \tag{2.26}$$

i.e.,

$$f_{i0}^{(n)} = f_{i0} * \gamma^{(n)} \in \mathcal{C}^2(\mathbb{R}^{2d}),$$

for all $i = 1, \dots, N$. The following is a classical result concerning the mollifier γ, see [1].

Lemma 2.12. *Let $\mathbf{f} \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ be with $\operatorname{supp}(\mathbf{f}) \subset B(R_0) \subset \mathbb{R}^{2d}$. Then*

(i) $\operatorname{supp}(\mathbf{f}^{(n)}) \subset B(R_0 + 1)$ *for all* $n \ge 1$.

(ii) $\mathbf{f}^{(n)} \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ *and*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|f_i^{(n)}(x,v)\,dx\,dv$$

are uniformly bounded, for all $i = 1, \dots, N$.

(iii) $\{\mathbf{f}^{(n)}\}_{n\ge1}$ *is a Cauchy sequence in $\mathcal{P}_1(\mathbb{R}^{2d})^N$ equipped with the Wasserstein distance $\mathcal{W}_1$ and $\|\mathbf{f}^{(n)} - \mathbf{f}\|_{\mathcal{W}_1} \to 0$ as $n \to +\infty$.*

Consider that the approximating sequence $\mathbf{f}^{\varepsilon,(n)}$ satisfies the system

$$\partial_t f_i^{\varepsilon,(n)} + v\cdot\nabla_x f_i^{\varepsilon,(n)} = \frac{1}{\varepsilon}\nabla_v \cdot \left(\left(v + \sum_{j=1}^N \nabla K_{ij} * \rho_i^{\varepsilon,(n)}\right) f_i^{\varepsilon,(n)}\right), \tag{2.27}$$

for $i = 1, \dots, N$, equipped with initial data

$$f_i^{\varepsilon,(n)} \mid_{t=0} = f_{i0}^{(n)}(x, v),$$

with

$$\rho_i^{\varepsilon,(n)} = \int_{\mathbb{R}^d} f_i^{\varepsilon,(n)} \, dv.$$

Lemma 2.13. *Assume all the potentials under assumption (**Pot**). Let $\mathbf{f}_0 \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ be with compact support and $\mathbf{f}_0^{(n)}$ defined as in (2.26). Then for each $T > 0$ there exists a solution $\mathbf{f}^{\varepsilon,(n)} \in \mathcal{C}([0,T]; \mathcal{C}^1(\mathbb{R}^{2d})^N)$ to (2.27) whose support depends only on T and Ξ and is uniformly bounded both in ε and n. Furthermore, if $\mathbf{f}^\varepsilon \in \mathcal{C}([0,T), \mathcal{P}_1(\mathbb{R}^{2d})^N)$ is the unique measure solution to (2.17) as provided in Theorem 2.1, then*

$$\mathbf{f}^{\varepsilon,(n)}(t,\cdot,\cdot) \xrightarrow{\mathcal{W}_1} \mathbf{f}^\varepsilon(t,\cdot,\cdot) \quad \text{in } \mathcal{P}_1(\mathbb{R}^{2d})^N, \tag{2.28}$$

uniformly in t as $n \to \infty$.

Proof. Since $\mathbf{f}_0^{(n)} \in \mathcal{C}_c^2(\mathbb{R}^{2d})^N$, we can apply Theorem 2.2 and we find that there exists a smooth solution $\mathbf{f}^{\varepsilon,(n)} \in \mathcal{C}([0,t), \mathcal{C}^1(\mathbb{R}^{2d})^N)$ to (2.17) for every $\varepsilon > 0$ and every $n \geq 1$, with compact support. By Proposition 2.2, we have that $\operatorname{supp}(\mathbf{f}^{\varepsilon,(n)})$ is independent of ε and depends on T, Ξ and the support of $\mathbf{f}_0^{(n)}$. Since by Lemma 2.12 $\operatorname{supp}(\mathbf{f}_0^{(n)})$ is contained in a ball for all $n \geq 1$, we deduce that $\operatorname{supp}(\mathbf{f}^{\varepsilon,(n)})$ is uniformly bounded both in ε and in n for all $t \in [0,T]$. Now, let $\mathbf{f}^\varepsilon \in \mathcal{C}([0,t), \mathcal{P}_1(\mathbb{R}^{2d})^N)$ be the unique solution to (2.17) as in Theorem 2.1. By Proposition 2.1, for all $t \geq 0$,

$$\|\mathbf{f}^{\varepsilon,(n)} - \mathbf{f}^\varepsilon\|_{\mathcal{W}_1} \leq r(T) \|\mathbf{f}_0^{(n)} - \mathbf{f}_0\|_{\mathcal{W}_1}.$$

By Lemma 2.12, we have the assertion. □

From this result it follows that

$$\boldsymbol{\rho}^{\varepsilon,(n)}(t,\cdot) \to \boldsymbol{\rho}^\varepsilon(t,\cdot) \quad \text{weakly as measures} \tag{2.29}$$

for each $t \in [0,T)$ as $n \to \infty$, where $\boldsymbol{\rho} = (\rho_i)_{i=1}^N$.

Lemma 2.14. *Let $\mathbf{f}^\varepsilon$ be the solution to (2.17) obtained as limit of approximating sequences $\mathbf{f}^{\varepsilon,(n)}$ as in Lemma 2.13. Then, for all $t \geq 0$, $\nabla K_{ij} * \rho_j^\varepsilon$ are continuous functions in $\mathbb{R}^d$ for all $i, j = 1 \dots, N$ and*

$$\nabla K_{ij} * \rho_j^{\varepsilon,(n)}(t,\cdot) \to \nabla K_{ij} * \rho_j^\varepsilon(t,\cdot)$$

strongly in $L^\infty_{loc}(\mathbb{R}^d)$, as $n \to \infty$.

Proof. Given the regularity of $\mathbf{f}^{\varepsilon,(n)}$, i.e., $\mathbf{f}^{\varepsilon,(n)} \in \mathcal{C}([0,T); \mathcal{C}^1(\mathbb{R}^{2d})^2)$ and the assumption (**Pot**), we get the continuity of the convolutions. Moreover, we can easily estimate

$$|\nabla K_{ij} * \rho_j^{\varepsilon,(n)}| \leq \|\nabla K_{ij}\|_{L^\infty},$$

and for all $x_1, x_2 \in \mathbb{R}^d$, we have

$$|\nabla K_{ij} * \rho_j^{\varepsilon,(n)}(x_1) - \nabla K_{ij} * \rho_j^{\varepsilon,(n)}(x_2)| \leq \|\nabla K_{ij}\|_{L^\infty} |x_1 - x_2|,$$

for $i, j = 1, \ldots, N$. Thus the sequences

$$\{\nabla K_{ij} * \rho_j^{\varepsilon,(n)}\}_{n\geq 1}$$

are equicontinuous and uniformly bounded. Hence, by Ascoli-Arzelà theorem, they strongly converge on a subsequence on compact sets in $\mathbb{R}^d$. Furthermore, by (2.29) we have that the limit functions are

$$\nabla K_{ij} * \rho_j^{\varepsilon},$$

respectively. These limit functions are also continuous on $\mathbb{R}^d$ by inequalities above (using ρ_j^{ε} in place of $\rho_j^{\varepsilon,(n)}$). Then it follows the assertion. □

Since the approximating sequence $\mathbf{f}^{\varepsilon,(n)}$ is smooth, we can apply to it Proposition 2.3 with ε fixed. In particular, with $n \geq 1$ fixed, we can say that there exist N positive constants C_i depending on Ξ and all $\|(1 + |v|f_{i0}^{(n)})\|_{L^1}$ and a function $M(\varepsilon)$ depending on ε such that for $\varepsilon < \varepsilon_0$

$$\left|\iint_{\mathbb{R}^{2d}} \left(v + \sum_{j=1}^{N} \nabla K_{ij} * \rho_j^{\varepsilon,(n)}\right) f_i^{\varepsilon,(n)}\, dx\, dv\right| \leq C_i M(\varepsilon).$$

By part (ii) in Lemma 2.12, we have that $\|(1+|v|f_{i0}^{(n)})\|_{L^1}$ are uniform bound in n for all $i = 1, \ldots, N$, thus the function $M(\varepsilon)$ and the constants C_i can be chosen independent on n. Therefore the estimates

$$\left|\iint_{\mathbb{R}^{2d}} \left(v + \sum_{j=1}^{N} \nabla K_{ij} * \rho_j^{\varepsilon,(n)}\right) f_i^{\varepsilon,(n)}\, dx\, dv\right| \leq C_i M(\varepsilon) \tag{2.30}$$

hold for all $n \geq 1$ and $t \in [0, T]$, as $i = 1, \ldots, N$.

Proposition 2.4 (Main estimates for measure solutions). *Assume $\varepsilon > 0$ fixed such that (2.30) holds and assume that assumptions in Lemma 2.13 are satisfied. Then for any $(\phi_i)_{i=1}^N \in \mathcal{C}_b(\mathbb{R}^{2d})^N$ there exist N constants $\overline{C}_i$ such that*

$$\left|\iint_{\mathbb{R}^{2d}} \phi_i(x,v)\left(v + \sum_{j=1}^{N} \nabla K_{ij} * \rho_j^{\varepsilon}(x)\right) f_i^{\varepsilon}(x,v)\, dx\, dv\right| \leq \overline{C}_i M(\varepsilon)$$

hold for all $t \in [0, T]$, as $i = 1, \ldots, N$. In particular, the constants $\overline{C}_i$ are independent of ε and t, and $\overline{C}_i = \|\phi_i\|_{L^\infty} C_i$, where C_i are constants depending on all $\iint (1+|v|) f_{i0}\, dx\, dv$ and Ξ.

Proof. Multiplying (2.30) by ϕ_i we have

$$\left|\iint_{\mathbb{R}^{2d}} \phi(x,v)\left(v + \sum_{j=1}^{N} \nabla K_{ij} * \rho_j^{\varepsilon,(n)}\right) f_i^{\varepsilon,(n)}\, dx\, dv\right| \leq C_i \|\phi_i\|_{L^\infty} M(\varepsilon), \tag{2.31}$$

where C_i are constants depending on Ξ and the first moment of f_{i0} in v. Let $\Omega(T)$ be the common support of $\mathbf{f}^{\varepsilon,(n)}(t)$ for all $\varepsilon > 0$, $n \geq 1$ and $t \in [0, T]$. Then, by Lemma

2.14 and Proposition 1.2, we obtain that for each $t \in [0, T]$,

$$\iint_{\mathbb{R}^{2d}} \phi_i(x,v)\left(v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^{\varepsilon,(n)}\right) f_i^{\varepsilon,(n)}\, dxdv$$
$$= \iint_{\Omega(T)} \phi_i(x,v)\left(v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^{\varepsilon,(n)}\right) f_i^{\varepsilon,(n)}\, dxdv$$

converges to

$$\iint_{\mathbb{R}^{2d}} \phi_i(x,v)\left(v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^{\varepsilon}\right) f_i^{\varepsilon}\, dxdv$$

as $n \to \infty$, for all $i = 1, \dots, N$. Therefore, considering the limit as $n \to \infty$ in (2.31), we find the assertion. □

2.5 Small inertia limit

We now tackle the $\varepsilon \to 0$ limit. More precisely, we consider $\mathbf{f}^\varepsilon$ solution to (2.5), satisfying the uniform bounds as stated in Proposition 2.4 and we show that the marginals

$$\rho_i^\varepsilon(t,x) = \int_{\mathbb{R}^d} f_i^\varepsilon(t,x,v)\, dv, \tag{2.32}$$

as $i = 1, \dots, N$, converge to a solution $\boldsymbol{\rho} = (\rho_i)_{i=1}^N$ to the first order system

$$\partial_t \rho_i - \nabla \cdot \left(\left(\sum_{j=1}^N \nabla K_{ij} * \rho_j\right)\rho_i\right) = 0, \tag{2.33}$$

for $i = 1, \dots, N$, equipped with initial data

$$\rho_i(t,x) \mid_{t=0} = \rho_{i0}(x).$$

Next we define the weak solutions to (2.33).

Definition 2.3. A weak solution to (2.33) is a N-tuple $\boldsymbol{\rho} = (\rho_i)_{i=1}^N \in \mathcal{C}([0,T), \mathcal{P}(\mathbb{R}^d)^N)$ that satisfies

$$\int_0^T \int_{\mathbb{R}^d} \partial_t \phi_i \rho_i\, dx\, dt - \int_0^T \int_{\mathbb{R}^d} \nabla_x \phi_i \cdot \left(\sum_{j=1}^N K_{ij} * \rho_j\right)\rho_i\, dx\, dt + \int_{\mathbb{R}^d} \phi_i(0)\rho_{i0}\, dx = 0, \tag{2.34}$$

for each $\phi_i \in \mathcal{C}_c^1([0,T); \mathcal{C}_b^1(\mathbb{R}^d))$, as $i = 1, \dots, N$.

Theorem 2.3 (Small inertia limit)**.** *Let $T > 0$. Assume all the potentials as in* (**Pot**)*. Consider $\mathbf{f}_0 \in \mathcal{P}_1(\mathbb{R}^{2d})^N$ with compact support. Let $\mathbf{f}^\varepsilon \in \mathcal{C}([0,T); \mathcal{P}_1(\mathbb{R}^{2d})^N)$ be the solution to system (2.5) given by Theorem 2.1. Let ρ_i^ε be given by (2.32), for $i = 1, \dots, N$. Then there exists $\boldsymbol{\rho} \in \mathcal{C}([0,T); \mathcal{P}_1(\mathbb{R}^d)^N)$ such that for each $t \in [0,T)$,*

$$\boldsymbol{\rho}^\varepsilon(t,\cdot) \xrightarrow{\mathcal{W}_1} \boldsymbol{\rho}(t,\cdot) \qquad \text{in } \mathcal{P}_1(\mathbb{R}^d)^N$$

as $\varepsilon \to 0$. Moreover, $\boldsymbol{\rho}$ is a weak solution to system (2.33) in the sense of Definition 2.3.

Proof. We start noting that, for each $\phi_i \in \mathcal{C}_c^1([0,T);\mathcal{C}_b^1(\mathbb{R}^{2d}))$, the measure solution $\mathbf{f}^\varepsilon$ satisfies

$$\begin{aligned}\int_0^T \iint_{\mathbb{R}^{2d}} \partial_t \phi_i f_i^\varepsilon \, dx\, dv\, dt + \iint_{\mathbb{R}^{2d}} \phi_i(0) f_{i0} \, dx\, dv + \int_0^T \iint_{\mathbb{R}^{2d}} \nabla_x \phi_i \cdot v f_i^\varepsilon \, dx\, dv\, dt \\ - \frac{1}{\varepsilon} \int_0^T \iint_{\mathbb{R}^{2d}} \nabla_v \phi_i \cdot \left(v + \sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right) f_i^\varepsilon \, dx\, dv\, dt = 0,\end{aligned} \tag{2.35}$$

for all $i = 1, \dots, N$. Consider $\psi_i \in \mathcal{C}_c^1(0,T)$, and $\chi_i \in \mathcal{C}_b^1(\mathbb{R}^d)$, as $i = 1, \dots, N$, and define

$$\phi_i(t,x,v) = \psi_i(t)\chi_i(x). \tag{2.36}$$

Using the test functions defined in (2.36) in system (2.35) we have

$$\int_0^T \psi_i'(t) \int_{\mathbb{R}^d} \chi_i(x) \rho_i^\varepsilon(t,x) \, dx\, dt = - \int_0^T \psi_i(t) \iint_{\mathbb{R}^{2d}} \nabla_x \chi_i(x) \cdot v f_i^\varepsilon \, dx\, dv\, dt.$$

Set

$$\xi_i(t) := \int_{\mathbb{R}^d} \chi_i(x) \rho_i^\varepsilon(t,x) \, dx.$$

Thus, it follows

$$\int_0^T \psi_i'(t) \xi_i(t) \, dt = - \int_0^T \psi_i(t) \iint_{\mathbb{R}^{2d}} \nabla_x \chi_i(x) \cdot v f_i^\varepsilon \, dx\, dv\, dt,$$

for any $\psi_i \in \mathcal{C}_c^1(0,T)$. Therefore, we deduce that the weak derivative of ξ_i is

$$\xi_i'(t) = \iint_{\mathbb{R}^{2d}} \nabla_x \chi_i \cdot v f_i^\varepsilon \, dx\, dv \in L^\infty(0,T).$$

Let $\Omega(T)$ be the common support of $\mathbf{f}^\varepsilon$ for every $\varepsilon > 0$ and for all $t \in [0,T]$. By Theorem 2.1, $\mathbf{f}^\varepsilon$ is uniformly supported on $\Omega(T)$, thus

$$\|\xi_i\|_{W^{1,\infty}(0,T)} \le C_i(T) \|\chi_i\|_{\mathcal{C}_b^1(\mathbb{R}^d)}, \tag{2.37}$$

where C_i depend on T and are independent of ε. Since $\xi_i(t)$ are uniformly bounded in $W^{1,\infty}(0,T)$, as $i = 1, \dots, N$, by Ascoli-Arzelà theorem there exist a subsequence ε_k and a function $\mu_i(t) \in \mathcal{C}([0,T))$ such that

$$\int_{\mathbb{R}^d} \chi_i(x) \rho_i^{\varepsilon_k}(x,t) \, dx \to \mu_i(t) \tag{2.38}$$

uniformly on $[0,T)$ as $\varepsilon_k \to 0$. Furthermore, Proposition 2.2 ensures that the support of $\mathbf{f}^\varepsilon$ is uniformly bounded in ε, then the sequence $\boldsymbol{\rho}^\varepsilon(t,\cdot)$ is tight. By Prokhorov's theorem 1.1, for each $t \in [0,T)$, $\boldsymbol{\rho}^\varepsilon(t,\cdot)$ converges weakly-*, up to a subsequence, to $\boldsymbol{\rho}(t,\cdot) \in \mathcal{P}(\mathbb{R}^d)^N$. By Proposition 1.2, we have that this implies convergence in $\mathcal{P}_1(\mathbb{R}^d)^N$ with respect to $\mathcal{W}_1$-distance. Hence, for each $t > 0$, there exists a subsequence of $\boldsymbol{\rho}^{\varepsilon_k}$ denoted by $\boldsymbol{\rho}^{\varepsilon_{k_n}}$, where k_n may depend on time, such that

$$\boldsymbol{\rho}^{\varepsilon_{k_n}}(t,\cdot) \xrightarrow{\mathcal{W}_1} \boldsymbol{\rho}(t,\cdot) \quad \text{in } \mathcal{P}_1(\mathbb{R}^d)^N$$

as $\varepsilon_{k_n} \to 0$. It follows that for each $t \in [0,T)$ and all $\chi_i \in \mathcal{C}_b^1(\mathbb{R}^d)$ we get

$$\int_{\mathbb{R}^d} \rho_i^{\varepsilon_{k_n}}(t,x)\chi_i(x)\,dx \to \int_{\mathbb{R}^d} \rho_i(t,x)\chi_i(x)\,dx \tag{2.39}$$

as $\varepsilon_{k_n} \to 0$. The limit $\mu_i(t)$ in (2.38) is unique at each $t \in [0,T)$. Combining this with (2.39), we deduce that the sequence $\boldsymbol{\rho}_{\varepsilon_k}(t,\cdot)$, with ε_k independent of time, and $\boldsymbol{\rho}(t,\cdot) \in \mathcal{P}_1(\mathbb{R}^d)^N$ satisfy

$$\int_{\mathbb{R}^d} \chi_i(x)\rho_i^{\varepsilon_k}(t,x)\,dx \to \int_{\mathbb{R}^d} \chi_i(x)\rho_i(t,x)\,dx \tag{2.40}$$

uniformly on $[0,T)$ as $\varepsilon_k \to 0$, for any $\chi_i \in \mathcal{C}_b^1(\mathbb{R}^d)$. Moreover,

$$\boldsymbol{\rho}^{\varepsilon_k}(t,\cdot) \xrightarrow{\mathcal{W}_1} \boldsymbol{\rho}(t,\cdot) \quad \text{in } \mathcal{P}_1(\mathbb{R}^d)^N \tag{2.41}$$

as $\varepsilon_k \to 0$. Now we want to prove that in (2.40) we can consider test functions χ_i depending also on t. In particular, taking $\zeta_i(t,x) \in \mathcal{C}_c([0,T);\mathcal{C}_b^1(\mathbb{R}^d))$ we have that

$$\int_{\mathbb{R}^d} \zeta_i(t,x)\rho_i^{\varepsilon_k}(t,x)\,dx$$

are equicontinuous on $[0,T)$. Indeed, considering $s,t \in [0,T)$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} \zeta_i(t,x)\rho_i^{\varepsilon_k}(t,x)\,dx - \int_{\mathbb{R}^d} \zeta_i(s,x)\rho_i^{\varepsilon_k}(s,x)\,dx\right| \\
&\leq \int_{\mathbb{R}^d} |\zeta_i(t,x)-\zeta_i(s,x)|\rho_i^{\varepsilon_k}(t,x)\,dx + \left|\int_{\mathbb{R}^d} \zeta_i(s,x)[\rho_i^{\varepsilon_k}(t,x)-\rho_i^{\varepsilon_k}(s,x)]\,dx\right| \\
&\leq \sup_{x\in\mathbb{R}^d} |\zeta_i(t,x)-\zeta_i(s,x)| + C_i(T)\sup_{t\in(0,T)} \|\zeta_i\|_{\mathcal{C}_b^1(\mathbb{R}^d)}|t-s|.
\end{aligned}$$

Since ζ_i is uniformly continuous on $[0,T)\times\mathbb{R}^d$, then

$$\sup_{x\in\mathbb{R}^d} |\zeta_i(t,x)-\zeta_i(s,x)| \to 0 \quad \text{as } |t-s| \to 0,$$

and we get equicontinuity. Thus, up to a subsequence,

$$\int_{\mathbb{R}^d} \zeta_i(t,x)\rho_i^{\varepsilon_k}(t,x)\,dx \to \int_{\mathbb{R}^d} \zeta_i(t,x)\rho_i(t,x)\,dx \tag{2.42}$$

uniformly on $[0,T)$ as $\varepsilon_k \to 0$, for any test functions $\zeta_i \in \mathcal{C}_c([0,T);\mathcal{C}_b^1(\mathbb{R}^d))$.
Now, set

$$\Omega_1(T) \coloneqq \{x \,:\, (x,v) \in \Omega(T)\}.$$

We can deduce that Ω_1 is bounded and both $\text{supp}(\boldsymbol{\rho})$ and $\text{supp}(\boldsymbol{\rho}^{\varepsilon_k})$ are in $\Omega_1(T)$ for all $t \in [0,T]$. Consider $\Psi_i \in \mathcal{C}_c^1([0,T);\mathcal{C}_b^1(\mathbb{R}^d))$ and let $\phi_i(x,v,t) = \Psi_i(x,t)$ in (2.35), as $i = 1,\dots,N$. Hence

$$\int_0^T \int_{\mathbb{R}^d} \partial_t\Psi_i\rho_i^{\varepsilon_k}\,dx\,dt + \int_0^T \iint_{\mathbb{R}^{2d}} \nabla_x\Psi_i\cdot vf_i^{\varepsilon_k}\,dx\,dv\,dt + \int_{\mathbb{R}^d} \Psi_i(0)\rho_{i0}(x)\,dx = 0. \tag{2.43}$$

Regarding the first integral in (2.43), by (2.42) we have that

$$\int_0^T \int_{\mathbb{R}^d} \partial_t\Psi_i\rho_i^{\varepsilon_k}\,dx\,dt \to \int_0^T \int_{\mathbb{R}^d} \partial_t\Psi_i\rho_i\,dx\,dt,$$

as $\varepsilon_k \to 0$. Concerning the integrand of the second term in (2.43), it can be rewritten as

$$\iint_{\mathbb{R}^{2d}} \nabla_x \Psi_i \cdot v f_i^{\varepsilon_k}\, dx\, dv = \iint_{\mathbb{R}^{2d}} \nabla_x \Psi_i \cdot \left(v + \sum_{j=1}^{N} \nabla K_{ij} * \rho_j^{\varepsilon_k} \right) f_i^{\varepsilon_k}\, dx\, dv$$
$$- \iint_{\mathbb{R}^{2d}} \nabla_x \Psi_i \cdot \left(\sum_{j=1}^{N} \nabla K_{ij} * \rho_j^{\varepsilon_k} \right) f_i^{\varepsilon_k}\, dx\, dv.$$

By Proposition 2.4, we have that

$$\iint_{\mathbb{R}^{2d}} \nabla_x \Psi_i \cdot \left(v + \sum_{j=1}^{N} \nabla K_{ij} * \rho_j^{\varepsilon_k} \right) f_i^{\varepsilon_k}\, dx\, dv \to 0, \tag{2.44}$$

as $\varepsilon_k \to 0$, uniformly in t. The families $\{\nabla K_{ij} * \rho_j^{\varepsilon_k}(t,\cdot)\}$ are bounded in $W^{1,\infty}(\mathbb{R}^d)$ for all $t \in [0,T)$. In particular,

$$\|\nabla K_{ij} * \rho_j^{\varepsilon_k}(t,\cdot)\|_{W^{1,\infty}(\mathbb{R}^d)} \le \|\nabla K_{ij}\|_{W^{1,\infty}(\mathbb{R}^d)}.$$

Now, we want to prove that $\{\nabla K_{ij} * \rho_j^{\varepsilon_k}\}$ are equicontinuous in t. In order to use inequalities in (2.37) with the kernels in places of χ_i, we should mollify K_{ij}. Let

$$K_{ij}^{(n)} = K_{ij} * \gamma^{(n)},$$

where $\gamma^{(n)}$ is the mollifier defined in Subsection 2.4.2. It follows that

$$\nabla K_{ij}^{(n)} = \nabla K_{ij} * \gamma^{(n)},$$

thus, we have

$$\|\nabla K_{ij}^{(n)}\|_{\mathcal{C}_b^1} \le \|\nabla K_{ij}\|_{W^{1,\infty}},$$

for all $n \ge 1$. Now, considering the mollified interaction kernels acting on the i-th species in estimates (2.37) in places of χ_i, we get

$$\sup_x \|\sum_{j=1}^{N} \nabla K_{ij}^{(n)} * \rho_j^{\varepsilon_k}\|_{W^{1,\infty}(0,T)} \le C_i(T) \sum_{j=1}^{N} \|\nabla K_{ij}^{(n)}\|_{\mathcal{C}_b^1} \le C_i(T) \sum_{j=1}^{N} \|\nabla K_{ij}\|_{W^{1,\infty}}.$$

Furthermore,

$$\left\| \sum_{j=1}^{N} \nabla K_{ij}^{(n)} * \rho_j^{\varepsilon_k} \right\|_{W^{1,\infty}(\mathbb{R}^d)} \le \sum_{j=1}^{N} \|\nabla K_{ij}^{(n)}\|_{W^{1,\infty}} \le \sum_{j=1}^{N} \|\nabla K_{ij}\|_{W^{1,\infty}},$$

thus, we find that

$$\left\| \sum_{j=1}^{N} \nabla K_{ij}^{(n)} * \rho_j^{\varepsilon_k} \right\|_{W^{1,\infty}(\mathbb{R}^d \times (0,T))} \le (1 + C_i(T)) \sum_{j=1}^{N} \|\nabla K_{ij}\|_{W^{1,\infty}}.$$

Therefore, for all $x, y \in \mathbb{R}^d$ and $s, t \in [0, T)$, we get

$$\begin{aligned} &\left|\sum_{j=1}^{N} \left[\nabla K_{ij}^{(n)} * \rho_j^{\varepsilon_k}(t,x) - \nabla K_{ij}^{(n)} * \rho_j^{\varepsilon_k}(s,y)\right]\right| \\ &\qquad \leq (C_i(T)+1)\left(\sum_{j=1}^{N} \|\nabla K_{ij}\|_{W^{1,\infty}}\right)\big(|t-s| + |x-y|\big). \end{aligned} \tag{2.45}$$

Since ∇K_{ij} are continuous, by Lemma 2.14 we get

$$\nabla K_{ij}^{(n)} \to \nabla K_{ij},$$

uniformly on compact sets in $\mathbb{R}^d$. Since

$$\left|\sum_{j=1}^{N} \left[\nabla K_{ij}^{(n)} * \rho_j^{\varepsilon_k}(t,x) - \nabla K_{ij} * \rho_j^{\varepsilon_k}(t,x)\right|\right. \leq \sup_x \left[\left|\sum_{j=1}^{N} \nabla K_{ij}^{(n)}(x) - \nabla K_{ij}(x)\right|\right],$$

we have that for any compact set $A \subset \mathbb{R}^d$,

$$\sum_{j=1}^{N} \nabla K_{ij}^{(n)} * \rho_j^{\varepsilon_k}(t,x) \xrightarrow{n\to\infty} \sum_{j=1}^{N} \nabla K_{ij} * \rho_j^{\varepsilon_k}(t,x),$$

uniformly for $t \in [0, T)$, for $x \in A$, $k \in \mathbb{N}$. Therefore, considering the limit as $n \to \infty$ in (2.45) on a compact set $A \subset \mathbb{R}^d$, we get

$$\begin{aligned} &\left|\sum_{j=1}^{N} \left[\nabla K_{ij} * \rho_j^{\varepsilon_k}(t,x) - \nabla K_{ij} * \rho_j^{\varepsilon_k}(s,y)\right]\right| \\ &\qquad \leq (C(T)+1)\left(\sum_{j=1}^{N} \|\nabla K_{ij}\|_{W^{1,\infty}}\right)\big(|t-s| + |x-y|\big). \end{aligned}$$

Thus, by Ascoli-Arzelà theorem, there exist N subsequences still denoted by $\rho_i^{\varepsilon_k}$, as $i = 1, \ldots, N$, such that

$$\sum_{j=1}^{N} \nabla K_{ij} * \rho_j^{\varepsilon_k} \to \sum_{j=1}^{N} \nabla K_{ij} * \rho_j,$$

as $\varepsilon_k \to 0$, strongly in $L^\infty([0,T) \times A)$, with $A \subset \mathbb{R}^d$ compact set. Hence, for every $t \in [0, T)$,

$$\begin{aligned} &\left|\int_{\mathbb{R}^d} \nabla \Psi_i \cdot \left[\sum_{j=1}^{N} (\nabla K_{ij} * \rho_j^{\varepsilon_k})\rho_i^{\varepsilon_k} - \sum_{j=1}^{N} (\nabla K_{ij} * \rho_j)\rho_i\right] dx\right| \\ &\quad \leq \sum_{j=1}^{N} \int_{\Omega_1(T)} |\nabla \Psi_i| \cdot |\nabla K_{ij} * \rho_j^{\varepsilon_k} - \nabla K_{ij} * \rho_j| \rho_i^{\varepsilon_k}\, dx \\ &\qquad + \sum_{j=1}^{N} \left|\int_{\mathbb{R}^d} \nabla \Psi_i \cdot (\nabla K_{ij} * \rho_j)(\rho_i^{\varepsilon_k} - \rho_i)\, dx\right| \end{aligned}$$

$$\leq \sum_{j=1}^{N} \|\nabla K_{ij} * \rho_j^{\varepsilon_k} - \nabla K_{ij} * \rho_j\|_{L^\infty(\Omega_1(T))} \|\nabla \Psi_i\|_{L^\infty}$$
$$+ \sum_{j=1}^{N} \left| \int_{\Omega_1(T)} \nabla \Psi_i \cdot (\nabla K_{ij} * \rho_j)(\rho_i^{\varepsilon_k} - \rho_i)\, dx \right|,$$

and the first term goes to zero as $\varepsilon_k \to 0$ uniformly on $[0, T)$ and the second integral vanishes as $\varepsilon_k \to 0$ by (2.41). Combining this with (2.44) we obtain that, for each $t \in (0, T)$,

$$\iint_{\mathbb{R}^{2d}} \nabla \Psi_i \cdot v f_i^{\varepsilon_k}\, dx\, dv \to - \int_{\mathbb{R}^d} \nabla \Psi_i \cdot \left[\sum_{j=1}^{N} (\nabla K_{ij} * \rho_j) \rho_i \right] dx$$

as $\varepsilon_k \to 0$. Finally, define

$$\Omega_2(T) = \{v \in \mathbb{R}^d \,:\, (x, v) \in \Omega(T)\}.$$

We have that $\Omega_2(T)$ is bounded for all $t \in (0, T)$ and the following uniform estimate holds:

$$\left| \iint_{\mathbb{R}^{2d}} \nabla \Psi_i \cdot v f_i^{\varepsilon_k}\, dx\, dv \right| \leq D_i \|\nabla \Psi_i\|_{L^\infty(\mathbb{R}^d)},$$

where the constant D_i depends only on $\Omega_2(T)$. This implies, by Lebesgue's dominated convergence theorem, that

$$\int_0^T \iint_{\mathbb{R}^{2d}} \nabla \Psi_i \cdot v f_i^{\varepsilon_k}\, dx\, dv\, dt \to - \int_0^T \int_{\mathbb{R}^d} \nabla \Psi_i \cdot \left[\sum_{j=1}^{N} (\nabla K_{ij} * \rho_j) \rho_i \right] dx\, dt$$

as $\varepsilon_k \to 0$. Thus the limiting N-tuple of measures $\boldsymbol{\rho} \in \mathcal{C}([0, T); \mathcal{P}(\mathbb{R})^N)$ is a solution to system (2.33) in the weak sense. □

Corollary 2.1 (Uniqueness)**.** *Assume that the assumptions in Theorem 2.3 and Proposition 2.1 hold. Then, the N-tuple $\boldsymbol{\rho} \in \mathcal{C}([0, T); \mathcal{P}_1(\mathbb{R}^d)^N)$ obtained in Theorem 2.3 is the unique solution to system (2.1).*

Proof. The proof follows by Proposition 2.1. Indeed, if we assume that there are two solutions starting from the same initial datum, by (2.16) we have the statement. □

Corollary 2.2 (Uniqueness with gradient flow structure)**.** *Assume that assumptions in Theorem 2.3 hold. Moreover, assume that the cross-interaction kernels are equal, i.e., $H := K_{ij}$, for $i \neq j$. Then the solution to system (2.1) obtained in Theorem 2.3 is unique.*

Proof. Since $\boldsymbol{\rho} \in \mathcal{C}([0, T), \mathcal{P}_1(\mathbb{R}^d)^N)$ is a weak solution to (2.33), by [39, Theorem 5.1] and the references therein, we can say that $\boldsymbol{\rho}$ is the push-forward of $\boldsymbol{\rho}_0$ via the flow $\mathcal{T}^t_{\mathbf{E}[\mathbf{f}]}$ where $\mathbf{E}[\mathbf{f}] = (E_i[\mathbf{f}])_{i=1}^N$ with

$$E_i[\mathbf{f}] = - \sum_{j=1}^{N} \nabla K_{ij} * \rho_j \in L^\infty([0, T) \times \mathbb{R}^d),$$

that is

$$\boldsymbol{\rho} = \mathcal{T}^t_{\mathbf{E}[\mathbf{f}]} \# \boldsymbol{\rho}_0.$$

Furthermore, $\boldsymbol{\rho}(t,\cdot)$ has compact support and it is narrowly continuous in time, since we get that $\boldsymbol{\rho}(t,\cdot) \in \mathcal{C}([0,T);\mathcal{P}_1(\mathbb{R}^d)^N)$ where the continuity is in the $\mathcal{W}_1$ metric, (see Proposition 1.2). Then $\boldsymbol{\rho}$ is the unique solution to (2.33) in the mass transportation sense. □

Chapter 3

Small inertia limit to first order nonlocal system: singular case

In this Chapter we deal with a multi-dimensional system with many species subject to smooth cross-potentials and singular self-potentials. We consider in addition an inertial effect. Once we introduce the mesoscopic and macroscopic models we want to study, we perturb the self-potentials in order to switch to a regularised system. After providing some uniform estimates with respect to the perturbation, we prove existence of weak solutions to the kinetic system. Then we show rigorously that a solution to the kinetic system converges towards a solution to the corresponding macroscopic system as the inertia goes to zero.

3.1 The model

The kinetic system we investigate in this Chapter is

$$\partial_t f_i + v \cdot \nabla_x f_i = \frac{1}{\varepsilon}\nabla_v \cdot (v f_i) + \frac{1}{\varepsilon}\left(\sum_{j=1}^{N} \nabla K_{ij} * \rho_j\right) \cdot \nabla_v f_i, \tag{3.1}$$

for $i = 1, \dots, N$, with smooth cross-potentials K_{ij}, $i \neq j$, as in assumption (**Pot**) and singular self-potentials K_{ii} of the form

$$K_{ii}(x) := \frac{C_i}{|x|^{\alpha_i}}, \tag{3.2}$$

with $\alpha_i \in (0, d-1]$, and some positive constants C_i. As said, $\rho_i(t,x)$ is the macroscopic population density of the i-th species, namely

$$\rho_i(t,x) = \int_{\mathbb{R}^d} f_i(t,x,v)\, dv.$$

We consider system (3.1) equipped with initial data $\mathbf{f}_0 = (f_{i0})_{i=1}^N$ such that

$$f_{i0} \in L^1_+ \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d), \qquad \text{and} \qquad (|x|^2 + |v|^2) f_{i0} \in L^1(\mathbb{R}^d \times \mathbb{R}^d).$$

We want to study the $\varepsilon \to 0$ limit in (3.1) to derive the first order macroscopic system

$$\begin{cases} \partial_t \rho_i = \nabla \cdot (\rho_i u_i), \\ u_i = \displaystyle\sum_{j=1}^{N} \nabla K_{ij} * \rho_j, \end{cases} \tag{3.3}$$

for $i = 1, \ldots, N$, with smooth cross-potentials as in (**Pot**) and singular self-potentials as in (3.2). Note that if $\alpha_i \in ((d-2) \vee 0, d)$, then $K_{ii} * \rho_i = \Lambda^{\alpha_i - d}\rho_i$ with $\Lambda = (-\Delta)^{\frac{1}{2}}$ up to constant. Thus in this case the system (3.3) becomes the following coupled *fractional porous medium flows*, [16]:

$$\partial_t \rho_i = \nabla \cdot \left(\rho_i \left(\nabla \Lambda^{\alpha_i - d} \rho_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \nabla K_{ij} * \rho_j \right) \right),$$

for $i = 1, \ldots, N$. The notion of solution we adopt for the kinetic system (3.1) is that of weak solution contained in the next Definition.

Definition 3.1 (Weak solution to (3.1))**.** Let $\mathbf{f}_0 = (f_{i0})_{i=1}^N \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)^N$ be the initial datum. A weak solution to (3.1) is a N-tuple $\mathbf{f} = (f_i)_{i=1}^N \in \mathcal{C}([0,T), \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)^N)$ that fulfils

$$\begin{aligned} \int_0^T \iint_{\mathbb{R}^d \times \mathbb{R}^d} \partial_t \phi_i f_i \, dx\, dv\, dt &+ \int_0^T \iint_{\mathbb{R}^d \times \mathbb{R}^d} v \cdot \nabla_x \phi_i f_i \, dx\, dv\, dt \\ &+ \iint_{\mathbb{R}^d \times \mathbb{R}^d} \phi_i(0) f_{i0} \, dx\, dv - \frac{1}{\varepsilon} \int_0^T \iint_{\mathbb{R}^d \times \mathbb{R}^d} v \cdot \nabla_v \phi_i f_i \, dx\, dv\, dt \\ &\qquad + \frac{1}{\varepsilon} \int_0^T \iint_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi_i \cdot \left(\sum_{j=1}^N \nabla K_{ij} * \rho_i \right) f_i \, dx\, dv\, dt = 0, \end{aligned}$$

for each $\phi_i \in \mathcal{C}_c^1([0,T); \mathcal{C}_b^1(\mathbb{R}^d \times \mathbb{R}^d))$, as $i = 1, \ldots, N$.

3.2 Regularised system

We start by considering a regularised version of the system (3.1). For this purpose, we perturb the self-potentials and consider the following system

$$\partial_t f_i^\delta + v \cdot \nabla_x f_i^\delta = \frac{1}{\varepsilon} \nabla_v \cdot (v f_i^\delta) + \frac{1}{\varepsilon} \left(\sum_{j=1}^N \nabla K_{ij}^\delta * \rho_j^\delta \right) \cdot \nabla_v f_i^\delta, \tag{3.4}$$

for $i = 1, \ldots, N$, with

$$K_{ii}^\delta(x) := \frac{C_i}{|x|^{\alpha_i} + \delta},$$

and

$$\rho_i^\delta(t,x) := \int_{\mathbb{R}^d} f_i^\delta(t,x,v)\, dv.$$

In system (3.4) we setted $K_{ij}^\delta := K_{ij}$, for $i \neq j$, in order to keep the notation to a minimum. Notice that the global-in-time existence and uniqueness of a weak solution to the regularised system (3.4) follows by the results developed in Chapter 2, since the force fields $\nabla K_{ij}^\delta * \rho_j^\delta$ are bounded and Lipschitz continuous. See in particular Definition 2.2 and Theorem 2.1.

3.2.1 Uniform in δ estimates

In this Subsection we gather some uniform in δ estimates that we will apply for proving existence of solutions to system (3.1). Let us begin with L^∞ bound estimates.

Lemma 3.1. *Let $T > 0$ and $\mathbf{f}^\delta := (f_1^\delta, \dots, f_N^\delta)$ be the weak solution to (3.4) on the interval $[0, T]$ in the sense of Definition 2.2. Then we have*

$$\sup_{0\le t\le T} \|f_i^\delta(\cdot,\cdot,t)\|_{L^p} \le \|f_{i0}^\delta\|_{L^p} e^{d\frac{1}{\varepsilon}(1-\frac{1}{p})T},$$

for $p \in [1, +\infty)$, and

$$\sup_{0\le t\le T} \|f_i^\delta(\cdot,\cdot,t)\|_{L^\infty} \le \|f_{i0}^\delta\|_{L^\infty} e^{d\frac{1}{\varepsilon}T}.$$

Proof. By integrating by parts with respect to x and v we get

$$\begin{aligned} \frac{d}{dt}\iint_{\mathbb{R}^d\times\mathbb{R}^d} (f_i^\delta)^p \, dx\, dv = &- \frac{1}{\varepsilon} p(p-1) \iint_{\mathbb{R}^d\times\mathbb{R}^d} (f_i^\delta)^{p-2} \nabla_x f_i^\delta \cdot v f_i^\delta \\ &- \frac{1}{\varepsilon} p(p-1) \iint_{\mathbb{R}^d\times\mathbb{R}^d} (f_i^\delta)^{p-2} \nabla_v f_i^\delta \cdot \left(\sum_{j=1}^N K_{ij}^\delta * \rho_j^\delta \right) f_i^\delta \\ &+ p(p-1) \iint_{\mathbb{R}^d\times\mathbb{R}^d} (f_i^\delta)^{p-2} \nabla_x f_i^\delta \cdot v f_i^\delta. \end{aligned}$$

Thus,

$$\frac{d}{dt}\iint_{\mathbb{R}^d\times\mathbb{R}^d} (f_i^\delta)^p \, dx\, dv = d\frac{1}{\varepsilon}(p-1) \iint_{\mathbb{R}^d\times\mathbb{R}^d} (f_i^\delta)^p \, dx\, dv,$$

for $p \in [1, +\infty)$. Therefore, by Grönwall's lemma we have

$$\|f_i^\delta(\cdot,\cdot,t)\|_{L^p}^p = \|f_{i0}^\delta\|_{L^p}^p e^{d\frac{1}{\varepsilon}(p-1)t}.$$

Then, it follows that

$$\sup_{0\le t\le T} \|f_i^\delta(\cdot,\cdot,t)\|_{L^p} \le \|f_{i0}^\delta\|_{L^p} e^{d\frac{1}{\varepsilon}(1-\frac{1}{p})T},$$

for $p \in [1, +\infty)$. Sending $p \to +\infty$ in the previous line, we obtain that

$$\sup_{0\le t\le T} \|f_i^\delta(\cdot,\cdot,t)\|_{L^\infty} \le \|f_{i0}^\delta\|_{L^\infty} e^{d\frac{1}{\varepsilon}T},$$

that concludes the proof. □

Now we prove a Lemma that points out the relationship between the local density and the kinetic energy (cf. [44, Lemma 3.1]), that we will use to estimate the interaction energy. Notice that in the next result we consider generic functions and we do not work along the solutions of system (3.4).

Lemma 3.2. *Assume that $f_i \in L^1_+ \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $|v|^2 f_i \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$, as $i = 1, \dots, N$. Then, there exists a positive constant C such that*

$$\|\rho_i\|_{L^{\frac{d+2}{d}}} \le C \|f_i\|_{L^\infty}^{\frac{2}{d+2}} \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 f_i \, dx\, dv \right)^{\frac{d}{d+2}}.$$

In particular, we find that

$$\|\rho_i\|_{L^p} \le C\|f_i\|_{L^\infty}^{\frac{2}{d+2}\beta}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2 f_i\,dx\,dv\right)^{\frac{d}{d+2}\beta}\|\rho_i\|_{L^1}^{1-\beta},$$

for all $p\in[1,\frac{d+2}{d}]$, *with* $\rho_i=\int_{\mathbb{R}^d} f_i\,dv$ *and* $\beta=\frac{d+2}{2}(1-\frac{1}{p})$.

Proof. Let $R>0$. Then

$$\rho_i=\int_{\mathbb{R}^d} f_i\,dv=\left(\int_{|v|\ge R}+\int_{|v|\le R}\right)f_i\;\;dv\le\frac{1}{R^2}\int_{\mathbb{R}^d}|v|^2 f_i\,dv+C\|f_i\|_{L^\infty}R^d.$$

For $R=(\int_{\mathbb{R}^d}|v|^2 f_i\,dv/\|f_i\|_{L^\infty})^{\frac{1}{d+2}}$ we have that

$$\rho_i\le C\|f_i\|_{L^\infty}^{\frac{2}{d+2}}\left(\int_{\mathbb{R}^d}|v|^2 f_i\,dv\right)^{\frac{d}{d+2}}.$$

Taking the power to $\frac{d+2}{d}$ and integrating with respect to x, we obtain that

$$\|\rho_i\|_{L^{\frac{d+2}{d}}}\le C\|f_i\|_{L^\infty}^{\frac{2}{d+2}}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2 f_i\,dx\,dv\right)^{\frac{d}{d+2}}.$$

By using the L^p interpolation inequality, we obtain the result. □

Let us now provide a bound estimate on the interaction energy.

Lemma 3.3. *Let* $T>0$ *and* $\mathbf{f}^\delta$ *be the weak solution to* (3.4) *on the interval* $[0,T]$. *Then*

$$\left|\iint_{\mathbb{R}^d\times\mathbb{R}^d}K_{ii}^\delta(x-y)\rho_i^\delta(x)\rho_i^\delta(y)\,dx\,dy\right|\le C_i\|\rho_{i0}\|_{L^1}^{2-\frac{5}{2d}\alpha_i}\|\rho_i^\delta\|_{L^{\frac{d+2}{d}}}^{\frac{5}{2d}\alpha_i},$$

where $C_i>0$ *is independent of* δ.

Proof. We recall the classical Hardy-Littlewood-Sobolev inequality, that is

$$\left|\iint_{\mathbb{R}^d\times\mathbb{R}^d}\mu(x)|x-y|^{-\lambda}\nu(y)\,dx\,dy\right|\le C_{p,\lambda,d}\|\mu\|_{L^p}\|\nu\|_{L^q},$$

for $\mu\in L^p(\mathbb{R}^d)$, $\nu\in L^q(\mathbb{R}^d)$, $1<p,q<\infty$, $1/p+1/q+\lambda/d=2$, and $0<\lambda<d$. By L^p-interpolation we know that for $1\le p,q\le\gamma$,

$$\|\rho_i\|_{L^p}\le\|\rho_i\|_{L^1}^{1-a}\|\rho_i\|_{L^\gamma}^{a},\qquad\frac{1}{p}=1-a+\frac{a}{\gamma},$$

and

$$\|\rho_i\|_{L^q}\le\|\rho_i\|_{L^1}^{1-b}\|\rho_i\|_{L^\gamma}^{b},\qquad\frac{1}{q}=1-b+\frac{b}{\gamma}.$$

Thus

$$\|\rho_i\|_{L^p}\|\rho_i\|_{L^q}\le\|\rho_i\|_{L^1}^{2-(a+b)}\|\rho_i\|_{L^\gamma}^{a+b}.$$

If $1/p+1/q+\lambda/d=2$, then

$$a+b=\frac{\gamma}{\gamma-1}\frac{\lambda}{d}.$$

If we take $\gamma = \frac{d+2}{d}$ and $\lambda = \alpha_i$, we obtain

$$
\begin{aligned}
\iint_{\mathbb{R}^d\times\mathbb{R}^d} K_{ii}^\delta(x-y)\rho_i^\delta(x)\rho_i^\delta(y)\,dx\,dy &\le \iint_{\mathbb{R}^d\times\mathbb{R}^d} K_{ii}(x-y)\rho_i^\delta(x)\rho_i^\delta(y)\,dx\,dy \\
&\le C_i\|\rho_i^\delta\|_{L^p}\|\rho_i^\delta\|_{L^q} \\
&\le C_i\|\rho_i^\delta\|_{L^1}^{2-\frac{5}{2d}\alpha_i}\|\rho_i^\delta\|_{L^{\frac{d+2}{d}}}^{\frac{5}{2d}\alpha_i} \\
&\le C_i\|\rho_{i0}\|_{L^1}^{2-\frac{5}{2d}\alpha_i}\|\rho_i^\delta\|_{L^{\frac{d+2}{d}}}^{\frac{5}{2d}\alpha_i},
\end{aligned}
$$

with $C_i > 0$ independent of δ. □

Next we provide a uniform in δ estimate on the second moments of the weak solution $\mathbf{f}^\delta$ to system (3.4).

Proposition 3.1. *Let $T > 0$ and $\mathbf{f}^\delta$ be the weak solution to system (3.4) on the interval $[0, T]$. Assume that*

$$
\int_{\mathbb{R}^d} \rho_{i0}K_{ii} * \rho_{i0}\,dx < \infty.
$$

Then the following estimate on the second moment holds:

$$
\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{|x|^2}{2}+\frac{|v|^2}{2}\right)f_i^\delta\,dx\,dv + \frac{1}{\varepsilon}\int_0^t\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{f_i^\delta}|vf_i^\delta|^2\,dx\,dv\,ds \le C,
$$

for all $t \in [0, T]$ and for some $C > 0$ independent of δ.

Proof. A direct computation gives that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2 f_i^\delta\,dx\,dv\right) &= \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2\partial_t f_i^\delta\,dx\,dv \\
&= \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2\left[\frac{1}{\varepsilon}\nabla_v\cdot(vf_i^\delta)+\frac{1}{\varepsilon}(\sum_{j=1}^N\nabla K_{ij}^\delta*\rho_j^\delta)\cdot\nabla_v f_i^\delta - v\cdot\nabla_x f_i^\delta\right]dx\,dv \\
&= -\frac{1}{\varepsilon}\iint_{\mathbb{R}^d\times\mathbb{R}^d}(\sum_{j=1}^N\nabla K_{ij}^\delta*\rho_j^\delta)\cdot vf_i^\delta\,dx\,dv - \frac{1}{\varepsilon}\iint_{\mathbb{R}^d\times\mathbb{R}^d}v\cdot vf_i^\delta\,dx\,dv \\
&= -\frac{1}{\varepsilon}\sum_{\substack{j=1\\ j\neq i}}^N\iint_{\mathbb{R}^d\times\mathbb{R}^d}(\nabla K_{ij}^\delta*\rho_j^\delta)\cdot vf_i^\delta\,dx\,dv \\
&\quad -\frac{1}{\varepsilon}\iint_{\mathbb{R}^d\times\mathbb{R}^d}(\nabla K_{ii}^\delta*\rho_i^\delta)\cdot vf_i^\delta\,dx\,dv \\
&\quad -\frac{1}{\varepsilon}\iint_{\mathbb{R}^d\times\mathbb{R}^d}v\cdot vf_i^\delta\,dx\,dv.
\end{aligned}
$$

For $i \neq j$, we have that $|\nabla K_{ij}*\rho_j| \le \|\nabla K_{ij}\|_{L^\infty}$, thus

$$
\left|\iint_{\mathbb{R}^d\times\mathbb{R}^d}\nabla K_{ij}^\delta*\rho_j^\delta\,dx\,dv\right| \le \|\nabla K_{ij}^\delta\|_{L^\infty}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|f_i^\delta\,dx\,dv.
$$

If, instead, $i = j$, by using (2.23) in Proposition 2.3 we get

$$\begin{aligned}
\frac{d}{dt}&\left(\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} K_{ii}^\delta(x-y)\rho_i^\delta(x)\rho_i^\delta(y)\,dx\,dy\right)\\
&=\iint_{\mathbb{R}^d\times\mathbb{R}^d} K_{ii}^\delta(x-y)\partial_t\rho_i^\delta(x)\rho_i^\delta(y)\,dx\,dy\\
&=-\iint_{\mathbb{R}^d\times\mathbb{R}^d} K_{ii}^\delta(x-y)\nabla_x\cdot\langle vf_i^\delta\rangle(x)\rho_i^\delta(y)\,dx\,dy\\
&=\iint_{\mathbb{R}^d\times\mathbb{R}^d} \nabla K_{ii}^\delta(x-y)\rho_i^\delta(y)\cdot\langle vf_i^\delta\rangle(x)\,dx\,dy\\
&=\iint_{\mathbb{R}^d\times\mathbb{R}^d} (\nabla K_{ii}^\delta*\rho_i^\delta)\cdot vf_i^\delta\,dx\,dv.
\end{aligned}$$

Thus, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2f_i^\delta\,dx\,dv\right)+\frac{1}{2}\frac{1}{\varepsilon}\frac{d}{dt}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}K_{ii}^\delta(x-y)\rho_i^\delta(x)\rho_i^\delta(y)\,dx\,dy\right)\\
&\quad=-\frac{1}{\varepsilon}\sum_{\substack{j=1\\j\neq i}}^N\iint_{\mathbb{R}^d\times\mathbb{R}^d}(\nabla K_{ij}*\rho_j^\delta)\cdot vf_i^\delta\,dx\,dv-\frac{1}{\varepsilon}\iint_{\mathbb{R}^d\times\mathbb{R}^d}v\cdot vf_i^\delta\,dx\,dv.
\end{aligned}$$

In the spatial variable, we have the following estimate for the second order moment

$$\begin{aligned}
\frac{d}{dt}&\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|x|^2}{2}f_i^\delta\,dx\,dv\right)=\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|x|^2}{2}\partial_tf_i^\delta\,dx\,dv\\
&=\iint_{\mathbb{R}^d\times\mathbb{R}^d}x\cdot vf_i^\delta\,dx\,dv\le\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{|x|^2}{2}+\frac{|v|^2}{2}\right)f_i^\delta\,dx\,dv.
\end{aligned}$$

Now, considering the estimates above, we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{|x|^2}{2}+\frac{|v|^2}{2}\right)f_i^\delta\,dx\,dv+\frac{1}{\varepsilon}\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}K_{ii}^\delta(x-y)\rho_i^\delta(x)\rho_i^\delta(y)\,dx\,dy\\
&\qquad+\frac{1}{\varepsilon}\int_0^t\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{f_i^\delta}|vf_i^\delta|^2\,dx\,dv\,ds\\
&\quad\le\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{|x|^2}{2}+\frac{|v|^2}{2}\right)f_{i0}^\delta\,dx\,dv+\frac{1}{\varepsilon}\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}K_{ii}^\delta(x-y)\rho_{i0}^\delta(x)\rho_{i0}^\delta(y)\,dx\,dy\\
&\qquad+\frac{1}{\varepsilon}\sum_{\substack{j=1\\j\neq i}}^N\int_0^t\iint_{\mathbb{R}^d\times\mathbb{R}^d}(\nabla K_{ij}^\delta*\rho_j^\delta)\cdot vf_i^\delta\,dx\,dv\\
&\qquad+\int_0^t\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{|x|^2}{2}+\frac{|v|^2}{2}\right)f_i^\delta\,dx\,dv\,ds.
\end{aligned}$$

By previous Lemmas we know that

$$\left|\iint_{\mathbb{R}^d\times\mathbb{R}^d}K_{ii}^\delta(x-y)\rho_i^\delta(x)\rho_i^\delta(y)\,dx\,dy\right|\le C,$$

where C is independent of δ. We derive

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{|v|^2}{2}+\frac{|x|^2}{2}\right)f_i^\delta\,dx\,dv+\frac{1}{\varepsilon}\int_0^t\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{f_i^\delta}|vf_i^\delta|^2\,dx\,dv\,ds$$
$$\leq\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{|x|^2}{2}+\frac{|v|^2}{2}\right)f_{i0}^\delta\,dx\,dv+\ C\int_0^t\iint_{\mathbb{R}^d\times\mathbb{R}^d}(|v|^2+|x|^2)f_i^\delta\,dx\,dv\,ds+C$$

for some $C>0$ independent of δ. Then, by Grönwall's lemma we obtain the result. □

Remark 3.1. From Proposition 3.1, we derive the following total energy estimates for f_i^δ, $i=1,\dots,N$, and for all $t\in[0,T]$:

$$\sum_{i=1}^N\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|v|^2}{2}f_i^\delta\,dx\,dv+\frac{1}{2\varepsilon}\sum_{i=1}^N\int_{\mathbb{R}^d}\rho_i^\delta K_{ii}*\rho_i^\delta\,dx+\frac{1}{\varepsilon}\int_0^t\sum_{i=1}^N\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2f_i^\delta\,dx\,dv\,ds$$
$$\leq\sum_{i=1}^N\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|v|^2}{2}f_{i0}^\delta\,dx\,dv+\frac{1}{2\varepsilon}\sum_{i=1}^N\int_{\mathbb{R}^d}\rho_{i0}^\delta K_{ii}*\rho_{i0}^\delta\,dx+\frac{1}{\varepsilon}\sum_{i\neq j}\|\nabla K_{ij}\|_{L^\infty}t.$$

Remark 3.2. If $K_{ij}=K_{ji}$ for all $i,j=1,\dots,N$, we find

$$\frac{d}{dt}\left(\sum_{i=1}^N\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|v|^2}{2}f_i^\delta\,dx\,dv+\frac{1}{2\varepsilon}\sum_{i=1}^N\int_{\mathbb{R}^d}\rho_i^\delta K_{ii}*\rho_i^\delta\,dx+\frac{1}{\varepsilon}\sum_{i>j}\int_{\mathbb{R}^d}\rho_i^\delta K_{ij}*\rho_j^\delta\,dx\right)$$
$$=-\frac{1}{\varepsilon}\sum_{i=1}^N\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2f_i^\delta\,dx\,dv.$$

Moreover, if $K_{ij}\in L^\infty$ for $i\neq j$, then

$$\sum_{i=1}^N\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|v|^2}{2}f_i^\delta\,dx\,dv+\frac{1}{2\varepsilon}\sum_{i=1}^N\int_{\mathbb{R}^d}\rho_i^\delta K_{ii}*\rho_i^\delta\,dx+\frac{1}{\varepsilon}\int_0^t\sum_{i=1}^N\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2f_i^\delta\,dx\,dv\,ds$$
$$\leq\sum_{i=1}^N\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|v|^2}{2}f_{i0}^\delta\,dx\,dv+\frac{1}{2\varepsilon}\sum_{i=1}^N\int_{\mathbb{R}^d}\rho_{i0}^\delta K_{ii}*\rho_{i0}^\delta\,dx+\frac{2}{\varepsilon}\sum_{i>j}\|K_{ij}\|_{L^\infty}t.$$

In this case, if we define a free energy $\mathcal{F}(\rho)$ as

$$\mathcal{F}(\rho)=\sum_{i,j=1}^N\int_{\mathbb{R}^d}\rho_iK_{ij}*\rho_j\,dx,$$

then the limiting system (3.3) has a gradient flow structure:

$$\partial_t\rho_i=\nabla\cdot\left(\rho_i\nabla\frac{\delta\mathcal{F}(\rho)}{\delta\rho_i}\right),$$

as $i=1,\dots,N$.

3.3 Existence of weak solution to the kinetic system

Now, we prove the existence of weak solutions to system (3.1). For this porpouse, we need the following lemma, cf. [43, 49].

Lemma 3.4. *Let $\{f^n\}_n$ be bounded in $L^p_{loc}(\mathbb{R}^d \times \mathbb{R}^d \times [0,T])$ with $1 < p < \infty$, and $\{G^n\}_n$ be bounded in $L^p_{loc}(\mathbb{R}^d \times \mathbb{R}^d \times [0,T])$. Assume that f^n and G^n satisfy*

$$\partial_t f^n + v \cdot \nabla_x f^n = \nabla_v G^n, \qquad f^n \,|_{t=0} = f_0 \in L^p(\mathbb{R}^d \times \mathbb{R}^d),$$

and

$$f^n \text{ is bounded in } L^\infty(\mathbb{R}^d \times \mathbb{R}^d),$$
$$(|v|^2 + |x|^2) f^n \text{ is bounded in } L^\infty((0,T); L^1(\mathbb{R}^d \times \mathbb{R}^d)).$$

Then, for any $q < \frac{d+2}{d+1}$, the sequence

$$\left\{ \int_{\mathbb{R}^d} f^n \, dv \right\}_n$$

is relatively compact in $L^q((0,T) \times \mathbb{R}^d)$.

The existence result of weak solutions to system (3.1) is contained in the following Theorem.

Theorem 3.1 (Existence of weak solutions)**.** *Assume that the initial datum $\mathbf{f}_0$ satisfies*

$$f_{i0} \in L^1_+ \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d), \qquad (|x|^2 + |v|^2) f_{i0} \in L^1(\mathbb{R}^d \times \mathbb{R}^d),$$

and

$$(K_{ii} * \rho_{i0}) f_{i0} \in L^1(\mathbb{R}^d \times \mathbb{R}^d).$$

Then there exists a weak solution $\mathbf{f}$ to (3.1) such that

$$\mathbf{f} \in \mathcal{C}([0,T]; \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)^N).$$

Proof. By the uniform in δ bound estimates obtained above we know

$$\|f_i^\delta\|_{L^\infty((0,T);L^p(\mathbb{R}^d \times \mathbb{R}^d))} + \|\rho_i^\delta\|_{L^\infty((0,T);L^q(\mathbb{R}^d))} \leq C,$$

with $p \in [1 + \infty]$, $q \in [1, \frac{d+2}{d}]$, $C > 0$ independent of δ. Therefore, by compactness theory, we have that as $\delta \to 0$, up to a subsequence,

$$f_i^\delta \overset{*}{\rightharpoonup} f_i \quad \text{in} \quad L^\infty((0,T); L^p(\mathbb{R}^d \times \mathbb{R}^d)), \quad p \in [1, +\infty],$$
$$\rho_i^\delta \overset{*}{\rightharpoonup} \rho_i \quad \text{in} \quad L^\infty((0,T); L^p(\mathbb{R}^d)), \quad p \in [1, \tfrac{d+2}{d}].$$

Set

$$G_i^\delta := \frac{1}{\varepsilon} v f_i^\delta + \frac{1}{\varepsilon} \sum_{j=1}^N (\nabla K_{ij}^\delta * \rho_j^\delta) f_i^\delta.$$

We want to prove that $G_i^\delta \in L^p_{loc}(\mathbb{R}^d \times \mathbb{R}^d \times [0,T])$ for some $p \in (1,\infty)$, in order to apply Lemma 3.4. We need to check the self terms. Let $q < 2$. Then

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v f_i^\delta|^q \,dx\,dv &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{f_i^\delta}|v f_i^\delta|^2\right)^{\frac{q}{2}} (f_i^\delta)^{\frac{q}{2}} \,dx\,dv \\ &\leq \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{f_i^\delta}|v f_i^\delta|^2 \,dx\,dv\right)^{\frac{q}{2}} \|f_i^\delta\|^{\frac{q}{2}}_{L^{\frac{q}{2-q}}}.\end{aligned}$$

For the second term, by using Young's inequality for convolution, we obtain that

$$\|(\nabla K_{ii}^\delta * \rho_i^\delta) f_i^\delta\|_{L^p} \leq C\|f_i^\delta\|_{L^\infty}\|\nabla K_{ii} * \rho_i^\delta\|_{L^p} \leq C\|f_i^\delta\|_{L^\infty}\|\rho_i^\delta\|_{L^p},$$

for $p < \frac{d+2}{d}$. Thus, by Lemma 3.4, we derive that

$$\rho_i^\delta \to \rho_i \text{ in } L^q(\mathbb{R}^d \times (0,T)) \text{ and a.e.},$$

up to a subsequence, as $\delta \to 0$, for $q < \frac{d+2}{d+1}$. Now we want to prove that

$$(\nabla K_{ii}^\delta * \rho_i^\delta) f_i^\delta \to (\nabla K_{ii} * \rho_i) f_i,$$

in the sense of distributions. Let $\Psi_i \in \mathcal{C}_c^\infty([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$.

$$\begin{aligned}&\int_0^T \iint_{\mathbb{R}^d\times\mathbb{R}^d} [(\nabla K_{ii}^\delta * \rho_i^\delta) f_i^\delta - (\nabla K_{ii} * \rho_i) f_i] \Psi_i \,dx\,dv\,ds \\ &= \int_0^T \int_{\mathbb{R}^d} (\nabla(K_{ii}^\delta - K_{ii}) * \rho_i)\rho_{i,\Psi} \,dx\,ds \\ &\quad + \int_0^T \int_{\mathbb{R}^d} \nabla K_{ii}^\delta * (\rho_i^\delta - \rho_i)\rho_{i,\Psi}^\delta \,dx\,ds \\ &\quad + \int_0^T \int_{\mathbb{R}^d} (\nabla K_{ii}^\delta * \rho_i)(\rho_{i,\Psi}^\delta - \rho_{i,\Psi}) \,dx\,ds \\ &=: A + B + C,\end{aligned}$$

with $\rho_{i,\Psi} := \int_{\mathbb{R}^d} f_i \Psi \,dv$ and $\rho_{i,\Psi}^\delta := \int_{\mathbb{R}^d} f_i^\delta \Psi \,dv$. Thanks to the uniform in δ estimate for f_i^δ in $L^\infty((0,T) \times \mathbb{R}^d \times \mathbb{R}^d)$ and the compact support of Ψ_i, we find

$$\rho_{i,\Psi},\ \rho_{i,\Psi}^\delta \in L^p((0,T); L^q(\mathbb{R}^d)),$$

for any $p, q \in [1,\infty]$, uniformly in δ.

Estimate of A

We have that $|(\nabla K_{ii}^\delta * \rho_i)\rho_{i,\Psi}| \leq |\nabla K_{ii} * \rho_i||\rho_{i,\Psi}|$ and $(\nabla K_{ii}^\delta * \rho_i)\rho_{i,\Psi}$ converges pointwise to $(\nabla K_{ii} * \rho_i)\rho_{i,\Psi}$ as $\delta \to 0$. Moreover, by Hardy-Littlewood-Sobolev inequality, we get

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} (|\nabla K_{ii}| * \rho_i)|\rho_{i,\Psi}| \,dx\,ds &\leq \int_0^T \iint_{\mathbb{R}^d\times\mathbb{R}^d} \rho_i(x)|x-y|^{-(\alpha_i+1)}|\rho_{i,\Psi}|(y) \,dx\,dy\,ds \\ &\leq C\|\rho_i\|_{L^p(\mathbb{R}^d\times(0,T))}\|\rho_{i,\Psi}\|_{L^{p'}((0,T);L^q(\mathbb{R}^d))},\end{aligned}$$

where $p \in (1, \frac{d+2}{d})$, $\frac{\alpha_i+1}{d} = 1 - \frac{1}{q} + \frac{1}{p}$, and p' is the Holder conjugate of p. Therefore, by Lebesgue's dominated convergence theorem, we obtain that A vanishes as $\delta \to 0$.

Estimate of B

As in the previous estimate, we have that

$$\left| \int_0^T \int_{\mathbb{R}^d} \nabla K_{ii} * (\rho_i^\delta - \rho_i) \rho_{i,\Psi}^\delta \, dx \, ds \right|$$
$$\leq \int_0^T \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\rho_i^\delta - \rho_i|(x) |x-y|^{-(\alpha_i+1)} |\rho_{i,\Psi}^\delta|(y) \, dx \, dy \, ds$$
$$\leq C \|\rho_i^\delta - \rho_i\|_{L^p(\mathbb{R}^d \times (0,T))} \|\rho_i \Psi\|_{L^{p'}((0,T);L^q(\mathbb{R}^d))},$$

with $p \in (1, \frac{d+2}{d+1})$ and $\frac{\alpha_i+1}{d} = 1 - \frac{1}{q} + \frac{1}{p}$. Thus, $B \to 0$ as $\delta \to 0$.

Estimate of C

As said, we know that

$$(\nabla K_{ii}^\delta * \rho_i)\Psi \in L^1((0,T); L^q(\mathbb{R}^d)),$$

with $q < 2$ uniformly in δ. Then, since $f_i^\delta \overset{*}{\rightharpoonup} f_i$, we obtain that $C \to 0$ as $\delta \to 0$. We conclude that $\mathbf{f}$ is a weak solution to system (3.1). □

3.4 Small inertia limit

In this Section we prove rigorously the small inertia limit. Since we want to study the behaviour of solutions to kinetic system (3.1) with respect to the inertia parameter $\varepsilon > 0$, we explicit the ε-dependence, namely we define $\mathbf{f}^\varepsilon = (f_i^\varepsilon)_{i=1}^N$ to be a weak solution to system

$$\partial_t f_i^\varepsilon + v \cdot \nabla_x f_i^\varepsilon = \frac{1}{\varepsilon} \nabla_v \cdot (v f_i^\varepsilon) + \frac{1}{\varepsilon} \left(\sum_{j=1}^N \nabla K_{ij} * \rho_j^\varepsilon \right) \cdot \nabla_v f_i^\varepsilon, \tag{3.5}$$

for $i = 1, \ldots, N$, with smooth cross-potentials as in assumption (**Pot**) and singular self-potentials of the form

$$K_{ii}(x) := \frac{C_i}{|x|^{\alpha_i}},$$

with $\alpha_i \in (0, d-1]$, and some positive constants C_i. As above, $\rho_i^\varepsilon(t,x)$ is the macroscopic population density of the i-th species, namely

$$\rho_i^\varepsilon(t,x) = \int_{\mathbb{R}^d} f_i^\varepsilon(t,x,v) \, dv.$$

The main purpose in this Section is to consider the limit $\varepsilon \to 0$ in (3.5) to derive

$$\begin{cases} \partial_t \rho_i = \nabla \cdot (\rho_i u_i), \\ u_i = \displaystyle\sum_{j=1}^N \nabla K_{ij} * \rho_j, \end{cases} \tag{3.6}$$

as $i = 1, \ldots, N$.

Remark 3.3. In literature, the macroscopic velocity is defined by, see [19],

$$u(t,x) = \frac{\int_{\mathbb{R}^d} v f(t,x,v)\, dv}{\int_{\mathbb{R}^d} f(t,x,v)\, dv}.$$

Since

$$\rho_i^\varepsilon |u_i^\varepsilon|^2 \le \int_{\mathbb{R}^d} |v|^2 f_i^\varepsilon \, dv,$$

Proposition 3.1 shows that for $\varepsilon > 0$

$$\sum_{i=1}^N \int_{\mathbb{R}^d} \rho_i^\varepsilon |u_i^\varepsilon|^2 \, dx < \infty$$

on some time interval $[0,T]$.

Next we recall from [31] (see also [52]) the following modulated interaction energy estimates.

Theorem 3.2. *Let $T > 0$ and K be given by*

$$K(x) = \frac{1}{|x|^\alpha} \quad \text{with } \alpha \in (0,d).$$

Suppose that the pairs $(\bar\rho, \bar u)$ and (ρ, u) satisfy the followings:

(i) $(\bar\rho, \bar u)$ and (ρ, u) satisfy the continuity equations in the sense of distribution:

$$\partial_t \bar\rho + \nabla \cdot (\bar\rho \bar u) = 0 \quad \text{and} \quad \partial_t \rho + \nabla \cdot (\rho u) = 0,$$

(ii) $(\bar\rho, \bar u)$ and (ρ, u) satisfy the energy inequality:

$$\sup_{0 \le t \le T} \left(\int_{\mathbb{R}^d} \bar\rho |\bar u|^2 \, dx + \int_{\mathbb{R}^d} \bar\rho K * \bar\rho \, dx \right) < \infty,$$

and

$$\sup_{0 \le t \le T} \left(\int_{\mathbb{R}^d} \rho |u|^2 \, dx + \int_{\mathbb{R}^d} \rho K * \rho \, dx \right) < \infty,$$

(iii) $\bar\rho, \rho \in \mathcal{C}((0,T); L^1(\mathbb{R}^d))$, $\nabla u \in L^\infty(\mathbb{R}^d \times (0,T))$ and if $\alpha < d-2$,

$$\begin{cases} \nabla^{[(d-\alpha)/2]+1} u \in L^\infty((0,T); L^{\frac{d}{[(d-\alpha)/2]}}(\mathbb{R}^d)), & \text{if} \quad \alpha \in (0, d-2) \setminus (d - 2\mathbb{N}), \\ \nabla^{\frac{d-\alpha}{2}} u \in L^\infty((0,T); L^{\frac{2d}{d-\alpha-2}}(\mathbb{R}^d)), & \text{if} \quad \alpha \equiv d \mod 2, \end{cases}$$

where $d - 2\mathbb{N} := \{d - 2n \ : \ n \in \mathbb{N}\}$ and $[\,\cdot\,]$ denotes the floor function.

Then we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} (\rho - \bar\rho) K * (\rho - \bar\rho)\, dx \le \int_{\mathbb{R}^d} \bar\rho (u - \bar u) \cdot \nabla K * (\rho - \bar\rho)\, dx + C \int_{\mathbb{R}^d} (\rho - \bar\rho) K * (\rho - \bar\rho)\, dx$$

for $t \in [0,T)$ and some $C > 0$ which depends only on α, d and $\|\nabla u\|_{L^\infty(\mathbb{R}^d \times (0,T))}$, and if $d < \alpha - 2$, additionally

$$\begin{cases} \|\nabla^{[(d-\alpha)/2]+1} u\|_{L^\infty((0,T); L^{\frac{d}{[(d-\alpha)/2]-1}})}, & \text{if} \quad \alpha \in (0, d-2) \setminus (d - 2\mathbb{N}), \\ \|\nabla^{(d-\alpha)/2} u\|_{L^\infty((0,T); L^{\frac{2d}{d-\alpha-2}})}, & \text{if} \quad \alpha \equiv d \bmod 2. \end{cases}$$

We also recall from [19, Lemma 4.1] (see also [28, Proposition 3.1], [60, Theorem 23.9], [2, 18, 41]) the following lemma which gives a relation between the bounded Lipschitz distance and modulated kinetic energy.

Lemma 3.5. *Let $T > 0$ and $\bar{\rho} : [0, T] \to \mathcal{P}(\mathbb{R}^d)$ be a narrowly continuous solution of*

$$\partial_t \bar{\rho} + \nabla \cdot (\bar{\rho}\bar{u}) = 0,$$

that is, $\bar{\rho}$ is continuous in the duality with continuous bounded functions, for a Borel vector field $\bar{u}$ satisfying

$$\int_0^T \int_{\mathbb{R}^d} |\bar{u}(x,t)|^p \bar{\rho}(x,t)\, dx\, dt < \infty$$

for some $p > 1$. Let $\rho \in \mathcal{C}([0, T]; \mathcal{P}_p(\mathbb{R}^d))$ be a solution of the following continuity equation

$$\partial_t \rho + \nabla \cdot (\rho u) = 0$$

with the velocity fields $u \in L^\infty((0, T); \dot{W}^{1,\infty}(\mathbb{R}^d))$. Then there exists a $C_{u,T} > 0$ depending only on T and $\|\nabla u\|_{L^\infty}$ such that for all $t \in [0, T]$

$$d_{BL}^2(\rho, \bar{\rho}) \le C_{u,T} \left(d_{BL}^2(\rho_0, \bar{\rho}_0) + \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon - u|^2\, dx\, ds \right),$$

where ρ^ε and u^ε are defined in Remark 3.3.

Remark 3.4. Since

$$\rho_i^\varepsilon |u_i^\varepsilon - u_i|^2 \le \int_{\mathbb{R}^d} f_i^\varepsilon |v - u_i|^2\, dv,$$

Lemma 3.5 particularly implies

$$d_{\rm BL}^2(\rho_i, \bar{\rho}_i) \le C_{u,T} \left(d_{\rm BL}^2(\rho_{i0}, \bar{\rho}_{i0}) + \int_0^t \iint_{\mathbb{R}^d \times \mathbb{R}^d} f_i^\varepsilon |v - u_i|^2\, dx\, dv\, ds \right),$$

as $i = 1, \dots, N$.

Theorem 3.3. *Let $T > 0$ and $d \ge 1$. Let $\mathbf{f}^\varepsilon = (f_i^\varepsilon)_{i=1}^N \in \mathcal{C}([0, T); \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)^N)$ be a solution to system (3.5) in the sense of distributions, and let $(\boldsymbol{\rho}, \mathbf{u}) = (\rho_i, u_i)_{i=1}^N$ be the unique classical solution of the coupled fractional porous medium flows (3.6) with $\rho_i > 0$ on $\mathbb{R}^d \times [0, T)$, $\partial_t u_i + u_i \cdot \nabla u_i \in L^\infty(\mathbb{R}^d \times (0, T))$, and if $\alpha < d - 2$, $\nabla^{[(d-\alpha)/2]+1} u_i \in L^\infty((0, T); L^{\frac{d}{[(d-\alpha)/2]}}(\mathbb{R}^d))$ up to time $T > 0$ with the initial data ρ_{i0}. If*

$$\sup_{\varepsilon > 0} \sum_{i=1}^N \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v - u_{i0}(x)|^2 f_{i0}^\varepsilon(x, v)\, dx\, dv < \infty$$

and

$$\sum_{i=1}^N \int_{\mathbb{R}^d} (\rho_{i0} - \rho_{i0}^\varepsilon) K_{ii} * (\rho_{i0} - \rho_{i0}^\varepsilon)\, dx + \sum_{i=1}^N d_{BL}(\rho_{i0}, \rho_{i0}^\varepsilon) \to 0,$$

as $\varepsilon \to 0$, then for each $i = 1, \dots, N$, we have

$$\int_{\mathbb{R}^d} f_i^\varepsilon\, dv \rightharpoonup \rho_i \quad \text{weakly in } L^\infty((0, T); \mathcal{M}(\mathbb{R}^d)),$$
$$\int_{\mathbb{R}^d} v\, f_i^\varepsilon\, dv \rightharpoonup \rho_i u_i \quad \text{weakly in } L^2((0, T); \mathcal{M}(\mathbb{R}^d)),$$

and

$$f_i^\varepsilon \rightharpoonup \rho_i \delta_{u_i} \quad \text{weakly in } L^2((0,T);\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)),$$

where d_{BL} stands for the bounded-Lipschitz distance introduced in Section 1.6, and we denoted by $\mathcal{M}(\mathbb{R}^n)$ the space of signed Radon measures on $\mathbb{R}^n$ with $n \in \mathbb{N}$.

Proof. We first rewrite the system (3.6) as

$$\partial_t \rho_i + \nabla \cdot (\rho_i u_i) = 0,$$

$$\varepsilon \partial_t u_i + \varepsilon u_i \cdot \nabla u_i = -u_i - \sum_{j=1}^N \nabla K_{ij} * \rho_j + \varepsilon \mathrm{e}_i,$$

where $\mathrm{e}_i \coloneqq \partial_t u_i + u_i \cdot \nabla u_i$, for $i = 1, \dots, N$. For the error estimates, we consider the modulated kinetic and interaction energies:

$$\mathcal{E}_K(f_i^\varepsilon | \rho_i, u_i) \coloneqq \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_i - v|^2 f_i^\varepsilon \, dx \, dv + \frac{1}{2\varepsilon} \int_{\mathbb{R}^d} (\rho_i - \rho_i^\varepsilon) K_{ii} * (\rho_i - \rho_i^\varepsilon) \, dx.$$

Straightforward computation yields that for each $i = 1, \dots, N$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_i - v|^2 f_i^\varepsilon \, dx \, dv + \frac{1}{\varepsilon} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_i - v|^2 f_i^\varepsilon \, dx \, dv \\
&\quad = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u_i - v) \otimes (v - u_i) : \nabla_x u_i f_i^\varepsilon \, dx \, dv - \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v - u_i) \cdot \mathrm{e}_i f_i^\varepsilon \, dx \, dv \\
&\qquad + \frac{1}{\varepsilon} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v - u_i) \cdot \left(\sum_{j=1}^N \nabla K_{ij} * (\rho_j - \rho_j^\varepsilon) \right) f_i^\varepsilon \, dx \, dv \\
&\quad =: I + II + III,
\end{aligned}$$

where

$$I \le \|\nabla u_i\|_{L^\infty} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_i - v|^2 f_i^\varepsilon \, dx \, dv,$$

and

$$II \le 4\varepsilon \|\mathrm{e}_i\|_{L^\infty} + \frac{1}{\varepsilon} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_i - v|^2 f_i^\varepsilon \, dx \, dv.$$

For III, we use $\nabla K_{ij} \in W^{1,\infty}$ for $i, j = 1, \dots, N$ with $i \neq j$ to obtain

$$\begin{aligned}
III \le &\frac{1}{\varepsilon} \int_{\mathbb{R}^d} \rho_i^\varepsilon (u_i^\varepsilon - u_i) \cdot \nabla K_{ii} * (\rho_i - \rho_i^\varepsilon) \, dx \\
&+ \frac{1}{\varepsilon} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_i - v|^2 f_i^\varepsilon \, dx \, dv \right)^{1/2} \sum_{j \neq i} \|\nabla K_{ij}\|_{W^{1,\infty}} d_{\mathrm{BL}}(\rho_j, \rho_j^\varepsilon) \\
\le &\frac{1}{\varepsilon} \int_{\mathbb{R}^d} \rho_i^\varepsilon (u_i^\varepsilon - u_i) \cdot \nabla K_{ii} * (\rho_i - \rho_i^\varepsilon) \, dx \\
&+ \frac{c_K}{2\varepsilon} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_i - v|^2 f_i^\varepsilon \, dx \, dv + \frac{c_K}{2\varepsilon} \sum_{j \neq i} d_{\mathrm{BL}}^2(\rho_j, \rho_j^\varepsilon),
\end{aligned}$$

where

$$c_K \coloneqq \max_{i=1,\dots,N} \sum_{j \neq i} \|\nabla K_{ij}\|_{W^{1,\infty}}.$$

We then apply Theorem 3.2 and Lemma 3.5 to deduce

$$
\begin{aligned}
&\mathcal{E}_K(f_i^\varepsilon|\rho_i,u_i) + \frac{1}{\varepsilon}\left(1 - \max_{i=1,\dots,N}\|\nabla u_i\|_{L^\infty}\varepsilon - 1 - \frac{c_K}{2}\right)\int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} |u_i - v|^2 f_i^\varepsilon\,dx\,dv\,ds \\
&\quad\le \mathcal{E}_K(f_{i0}^\varepsilon|\rho_{i0},u_{i0}) + \frac{Cc_K}{2\varepsilon}\sum_{j\neq i} d_{\rm BL}^2(\rho_{j0},\rho_{j0}^\varepsilon) + C\varepsilon \\
&\qquad + \frac{C}{\varepsilon}\sum_{i=1}^N \int_0^t \int_{\mathbb{R}^d} (\rho_i - \rho_i^\varepsilon)K_{ii} * (\rho_i - \rho_i^\varepsilon)\,dx\,ds \\
&\qquad + \frac{Cc_K}{2\varepsilon}\sum_{j\neq i}\int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} |u_j - v|^2 f_j^\varepsilon\,dx\,dv\,ds,
\end{aligned}
$$

where $C > 0$ depends only on $u_1, \dots, u_N$, and T, but independent of $\varepsilon > 0$. We now sum over $i = 1, \dots, N$, apply Grönwall's inequality to have

$$
\begin{aligned}
&\sum_{i=1}^N \mathcal{E}_K(f_i^\varepsilon|\rho_i,u_i) + \frac{1}{\varepsilon}\sum_{i=1}^N \int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} |u_i - v|^2 f_i^\varepsilon\,dx\,dv\,ds \\
&\quad \le c_0 \sum_{i=1}^N \mathcal{E}_K(f_{i0}^\varepsilon|\rho_{i0},u_{i0}) + \frac{c_0}{\varepsilon}\sum_{i=1}^N d_{\rm BL}^2(\rho_{i0},\rho_{i0}^\varepsilon),
\end{aligned}
$$

where $c_0 > 0$ is independent of $\varepsilon > 0$. □

Remark 3.5. If $\nabla K_{ij} \in W^{1,\infty}$ for all $i, j = 1, \dots, N$, then we only need to assume that

$$
\sup_{\varepsilon>0}\sum_{i=1}^N \iint_{\mathbb{R}^d\times\mathbb{R}^d} |v - u_{i0}(x)|^2 f_{i0}^\varepsilon(x,v)\,dx\,dv < \infty
$$

and

$$
\sum_{i=1}^N d_{\rm BL}(\rho_{i0},\rho_{i0}^\varepsilon) \to 0
$$

as $\varepsilon \to 0$ for the desired convergences. That is, the modulated interaction energies are not required when the interaction potentials are smooth enough. In this case, we also readily find $\partial_t u_i + u_i \cdot \nabla u_i \in L^\infty(\mathbb{R}^d \times (0,T))$ for all $i = 1, \dots, N$. Indeed,

$$
\|u_i\|_{W^{1,\infty}} \le \sum_{j=1}^N \|\nabla K_{ij}\|_{W^{1,\infty}} < \infty
$$

and

$$
|\partial_t u_i| = \left|\sum_{j=1}^N \nabla K_{ij} * \partial_t \rho_j\right| \le \|u_i\|_{L^\infty}\sum_{j=1}^N \|\Delta K_{ij}\|_{L^\infty} < \infty.
$$

Remark 3.6. In Theorem 3.3, if we assume

$$
\sum_{i=1}^N \iint_{\mathbb{R}^d\times\mathbb{R}^d} |v - u_{i0}(x)|^2 f_{i0}^\varepsilon(x,v)\,dx\,dv \to 0
$$

and

$$
\frac{1}{\varepsilon}\sum_{i=1}^N \int_{\mathbb{R}^d} (\rho_{i0} - \rho_{i0}^\varepsilon)K_{ii} * (\rho_{i0} - \rho_{i0}^\varepsilon)\,dx + \frac{1}{\varepsilon}\sum_{i=1}^N d_{\rm BL}^2(\rho_{i0},\rho_{i0}^\varepsilon) \to 0
$$

as $\varepsilon \to 0$, then for each $i = 1, \dots, N$, we have

$$\int_{\mathbb{R}^d} f_i^\varepsilon \, dv \rightharpoonup \rho_i, \qquad \int_{\mathbb{R}^d} v f_i^\varepsilon \, dv \rightharpoonup \rho_i u_i \quad \text{weakly in } L^\infty((0,T);\mathcal{M}(\mathbb{R}^d)),$$

and

$$f_i^\varepsilon \rightharpoonup \rho_i \delta_{u_i} \quad \text{weakly in } L^\infty((0,T);\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d))$$

as $\varepsilon \to 0$.

Remark 3.7 (Existence of unique classical solution to system (3.6))**.** In the statement of Theorem 3.3 we assume that there exists a unique classical solution to system (3.6). The strategy for proving existence of unique smooth solution to system (3.6) could be to adapt the argument in [29] by exploiting a proper splitting argument between singular self-potentials and smooth cross-potentials. This will be object of future investigations.

Chapter 4

A finite volume method for a kinetic model for interacting species

In this Chapter we propose an upwind finite volume scheme for a system of two kinetic equations coupled through nonlocal interaction terms. Once we construct the numerical mesh, we define the numerical approximation of the solution. After showing some a priori estimates, we prove the convergence of the numerical solution of the scheme to the solution to the continuum system.

4.1 The model

The system we deal with is the following two-species kinetic system

$$\begin{cases} \dfrac{\partial f}{\partial t} + v\dfrac{\partial f}{\partial x} &= (K'_{11} * \rho + K'_{12} * \eta)\dfrac{\partial f}{\partial v}, \\ \dfrac{\partial g}{\partial t} + v\dfrac{\partial g}{\partial x} &= (K'_{22} * \eta + K'_{21} * \rho)\dfrac{\partial g}{\partial v}, \end{cases} \tag{4.1}$$

equipped with some non-negative initial data $f_0, g_0 \in L^1(\mathbb{R} \times \mathbb{R})$, i.e.,

$$f(0, x, v) = f_0(x, v), \quad \text{and} \quad g(0, x, v) = g_0(x, v).$$

Here, $(f, g)(t, x, v)$ is a pair of densities describing the distribution of the two species on the domain $[0, T] \times \mathbb{R} \times \mathbb{R}$, and K_{ij} are the interaction potentials. Moreover, $\rho(t, x)$ and $\eta(t, x)$ denote the macroscopic population densities, i.e.,

$$\rho(t, x) = \int_{\mathbb{R}} f(t, x, v)\, dv, \quad \text{and} \quad \eta(t, x) = \int_{\mathbb{R}} g(t, x, v)\, dv.$$

The existence theory for system (4.1) has been studied in arbitrary dimension and considering many species in Chapter 2.

In particular, here we consider a two-species version in order to construct an upwind finite volume scheme, and we do not include the inertia term. Before constructing the numerical scheme and studying its properties, let us present some formal properties of the solutions at the continuous level.

For convenience, we shall, henceforth, use the notation

$$\Upsilon_f(t, x) := K'_{11} * \rho + K'_{12} * \eta, \quad \text{and} \quad \Upsilon_g(t, x) := K'_{22} * \eta + K'_{21} * \rho. \tag{4.2}$$

Now, we show that the solutions on the continuous level are bounded and positive. Indeed, for any smooth function $\phi : \mathbb{R} \to \mathbb{R}$ and $p \in \{f, g\}$, a straightforward computation shows

$$\begin{aligned}
\frac{d}{dt}\iint_{\mathbb{R}\times\mathbb{R}} \phi \circ p \, dx\, dv &= \iint_{\mathbb{R}\times\mathbb{R}} \phi'(p)\partial_t p \, dx\, dv \\
&= \iint_{\mathbb{R}\times\mathbb{R}} \phi'(p)\left(-v\frac{\partial p}{\partial x} + \Upsilon_p \frac{\partial p}{\partial v}\right) dx\, dv \\
&= \iint_{\mathbb{R}\times\mathbb{R}} \begin{pmatrix} -v \\ \Upsilon_p \end{pmatrix} \cdot \nabla_{(x,v)}\phi(p)\, dx\, dv \\
&= 0.
\end{aligned}$$

We introduce the notation

$$[x]^+ := \max\{x, 0\}, \quad \text{and} \quad [x]^- := -\min\{x, 0\}$$

for the positive part and the negative part, respectively, of a real number x. Using $\phi(s) = [s]^-$, we observe that the solutions of system (4.1) remain non-negative if they were non-negative initially. Furthermore, if $\phi(s) = [s - \|p_0\|_{L^\infty}]^+$, with $p_0 \in \{f_0, g_0\}$, we obtain that the solution is bounded at the continuous level. Moreover, considering $\phi(s) = |s|^q$, we see that the L^q-norms of the solution are preserved.

4.2 Derivation of the numerical method

In this Section, we shall derive a finite volume scheme to approximate the solutions of system (4.1) on the domain $Q_T := (0, T) \times (-L, L) \times \mathbb{R}$, equipped with some periodic boundary conditions in the physical space. Throughout this Chapter, we will use the following domains $Q := (-L, L) \times \mathbb{R}$ and $\Omega_T := (0, T) \times (-L, L)$.

4.2.1 Numerical mesh

We discretize the phase space by introducing cells

$$C_{i,j} = (x_{i-1/2}, x_{i+1/2}) \times (v_{j-1/2}, v_{j+1/2}),$$

for $(i, j) \in \mathcal{I} \times \mathbb{Z}$, where $\mathcal{I} = \{0, \ldots, N_x - 1\}$. Here, $(x_{i-1/2})_{i\in\{0,\ldots,N_x\}}$ is a strictly increasing family of interfaces with $x_{-1/2} = -L$ and $x_{N_x-1/2} = L$. Similarly, $(v_{j-1/2})_{j\in\mathbb{Z}}$ denotes a strictly increasing sequence in $\mathbb{R}$, with $v_{j+1/2} \to \pm\infty$, as $j \to \pm\infty$.

We denote by $\Delta x_i = x_{i+1/2} - x_{i-1/2}$, for $i \in \mathcal{I}$, the width of the spatial interval $(x_{i-1/2}, x_{i+1/2})$. Additionally, we set $\Delta v_j = v_{j+1/2} - v_{j-1/2}$, for $j \in \mathbb{Z}$, to denote the width of the velocity interval $(v_{j-1/2}, v_{j+1/2})$.

We associate with the mesh the parameter h as the maximum of all space and velocity steps, i.e.,

$$h := \max_{i\in\mathcal{I}, j\in\mathbb{Z}} \{\Delta x_i, \Delta v_j\} > 0.$$

We denote by x_i the centre of the interval $(x_{i-1/2}, x_{i+1/2})$ and by v_j the center of the interval $(v_{j-1/2}, v_{j+1/2})$.

Additionally we call the mesh admissible if there exists $\alpha \in (0, 1)$ such that

$$\alpha h \le \Delta x_i, \Delta v_j \le h,$$

for all $(i,j) \in \mathcal{I} \times \mathbb{Z}$. Henceforth, we assume that our mesh admits the existence of such an $\alpha > 0$.

Finally, for some $N_T \in \mathbb{N}$, we set $\Delta t := T/N_T$ for the time step and $t^n := n\Delta t$, $n = 0, \dots, N_T$, to denote the discrete time instances.

4.2.2 Discretization of the data

We discretize the initial data by a piecewise constant function. We set

$$f^0_{i,j} := \fint_{C_{i,j}} f_0(x,v)\,dx\,dv, \quad \text{and} \quad g^0_{i,j} := \fint_{C_{i,j}} g_0(x,v)\,dx\,dv,$$

for $(i,j) \in \mathcal{I} \times \mathbb{Z}$ as the averaged integral $\fint$ of the initial datum (f_0, g_0) over the cell $C_{i,j}$.

To approximate the functions f and g we use piecewise constant functions on each cell $(t^n, t^{n+1}) \times C_{i,j}$, $n = 0, \dots, N_T - 1$, $(i,j) \in \mathcal{I} \times \mathbb{Z}$. For that purpose, we write these approximations as

$$f^n_{i,j} \approx \fint_{C_{i,j}} f(t^n,x,v)\,dx\,dv, \quad \text{and} \quad g^n_{i,j} \approx \fint_{C_{i,j}} g(t^n,x,v)\,dx\,dv.$$

Besides, we define the piecewise constant approximations, ρ_h and η_h, of the macroscopic densities ρ and η as

$$\rho_h(t,x) = \rho^n_i, \quad \text{and} \quad \eta_h(t,x) = \eta^n_i,$$

for $(t,x) \in [t^n, t^{n+1}) \times [x_{i-1/2}, x_{i+1/2})$, with $i \in \mathcal{I}$, and

$$\rho^n_i := \sum_{j\in\mathbb{Z}} \Delta v_j f^n_{i,j}, \quad \text{and} \quad \eta^n_i := \sum_{j\in\mathbb{Z}} \Delta v_j g^n_{i,j}.$$

However, these sums are over infinitely many entries $j \in \mathbb{Z}$. To implement the scheme, we have to work in a bounded domain. Therefore we need to truncate the velocity domain. Hence, we choose an arbitrary $v_h > 0$ sufficiently large, such that $v_h \to \infty$ as $h \to 0$ and restrict the velocity domain to $(-v_h, v_h)$. We introduce the index set $\mathcal{J} := \{j \in \mathbb{Z} : |v_{j+1/2}| \le v_h\}$ which consists of all indices j of the interfaces $(v_{j-1/2})_j$ that are inside the truncated velocity domain. Note that the choice of $v_h > 0$ is made precise in Remark 4.2.

Therefore, we define the *piecewise constant approximation* associated with the iterates obtained from the scheme, (f_h, g_h) on $[0,T) \times [-L,L] \times (-v_h, v_h)$. They are extended by zero to the whole domain $[0,T) \times [-L,L] \times \mathbb{R}$, such that

$$(f_h, g_h)(t,x,v) := \begin{cases} (f^n_{i,j}, g^n_{i,j}), & \text{if } (t,x,v) \in [t^n, t^{n+1}) \times C_{i,j} \text{ and } (i,j) \in \mathcal{I} \times \mathcal{J}, \\ (0,0), & \text{else.} \end{cases}$$

4.2.3 Construction of the method

We obtain the finite volume approximation by integrating system (4.1) over a test cell, $(t^n, t^{n+1}) \times C_{i,j}$ for a fixed $n \in \{0, \dots N_T - 1\}$, $i \in \mathcal{I}$ and $j \in \mathbb{Z}$. A formal computation yields

$$\begin{cases} \displaystyle\fint_{C_{i,j}} f(t^{n+1},x,v) - f(t^n,x,v)\,dx\,dv = -\frac{{}^xF^n_{i+1/2,j} - {}^xF^n_{i-1/2,j} + {}^vF^n_{i,j+1/2} - {}^vF^n_{i,j-1/2}}{|C_{i,j}|}, \\ \displaystyle\fint_{C_{i,j}} g(t^{n+1},x,v) - g(t^n,x,v)\,dx\,dv = -\frac{{}^xG^n_{i+1/2,j} - {}^xG^n_{i-1/2,j} + {}^vG^n_{i,j+1/2} - {}^vG^n_{i,j-1/2}}{|C_{i,j}|}, \end{cases}$$

where ${}^xF^n_{i+1/2,j}$, ${}^vF^n_{i,j+1/2}$ are the fluxes of f on the respective parts of the boundary of the cell $C_{i,j}$ given by

$$\begin{cases} \displaystyle {}^xF^n_{i+1/2,j} = \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} v f(t, x_{i+1/2}, v)\,dv\,dt, \\ \displaystyle {}^vF^n_{i,j+1/2} = \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} -\Upsilon_f(t,x) f(t,x,v_{j+1/2})\,dx\,dt, \end{cases}$$

for $(i,j) \in \mathcal{I} \times \mathbb{Z}$. Similarly, ${}^xG^n_{i+1/2,j}$, ${}^vG^n_{i,j+1/2}$ are the fluxes of g on the boundary of the cell $C_{i,j}$, i.e.,

$$\begin{cases} \displaystyle {}^xG^n_{i+1/2,j} = \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} v g(t, x_{i+1/2}, v)\,dv\,dt, \\ \displaystyle {}^vG^n_{i,j+1/2} = \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} -\Upsilon_g(t,x) g(t,x,v_{j+1/2})\,dx\,dt, \end{cases}$$

with $(i,j) \in \mathcal{I} \times \mathbb{Z}$.

If we apply the piecewise constant approximation for f, g, ρ and η as in Section 4.2.2, we arrive at the discrete version of (4.1):

$$\begin{cases} \displaystyle f^{n+1}_{i,j} = f^n_{i,j} - \frac{1}{|C_{i,j}|}\big({}^x\bar{F}^n_{i+1/2,j} - {}^x\bar{F}^n_{i-1/2,j} + {}^v\bar{F}^n_{i,j+1/2} - {}^v\bar{F}^n_{i,j-1/2}\big), \\ \displaystyle g^{n+1}_{i,j} = g^n_{i,j} - \frac{1}{|C_{i,j}|}\big({}^x\bar{G}^n_{i+1/2,j} - {}^x\bar{G}^n_{i-1/2,j} + {}^v\bar{G}^n_{i,j+1/2} - {}^v\bar{G}^n_{i,j-1/2}\big), \end{cases} \tag{4.3a}$$

for $(i,j) \in \mathcal{I} \times \mathbb{Z}$ and $n \in \{0, \dots N_T - 1\}$. Note that we have replaced the continuous fluxes above by the discrete upwind fluxes ${}^x\bar{F}^n_{i+1/2,j}$, ${}^v\bar{F}^n_{i,j+1/2}$, ${}^x\bar{G}^n_{i+1/2,j}$, ${}^v\bar{G}^n_{i,j+1/2}$, defined as

$$\begin{cases} {}^x\bar{F}^n_{i+1/2,j} = \Delta t \Delta v_j \left(f^n_{i,j}[v_j]^+ - f^n_{i+1,j}[v_j]^-\right), \\ {}^v\bar{F}^n_{i,j+1/2} = \Delta t \Delta x_i \left(f^n_{i,j}[(\Upsilon_f)^n_i]^- - f^n_{i,j+1}[(\Upsilon_f)^n_i]^+\right), \end{cases} \tag{4.3b}$$

and, similarly,

$$\begin{cases} {}^x\bar{G}^n_{i+1/2,j} = \Delta t \Delta v_j \big(g^n_{i,j}[v_j]^+ - g^n_{i+1,j}[v_j]^-\big), \\ {}^v\bar{G}^n_{i,j+1/2} = \Delta t \Delta x_i \big(g^n_{i,j}[(\Upsilon_g)^n_i]^- - g^n_{i,j+1}[(\Upsilon_g)^n_i]^+\big), \end{cases} \tag{4.3c}$$

for $(i,j) \in \mathcal{I} \times \mathbb{Z}$.

The terms $(\Upsilon_f)^n_i$ and $(\Upsilon_g)^n_i$ are the approximations of the interaction terms Υ_f and Υ_g

at the point (t^n, x_i), and are defined by

$$\begin{cases} (\Upsilon_f)_i^n := \sum_{k\in\mathcal{I}} \left(\rho_k^n \int_{x_{k-1/2}}^{x_{k+1/2}} K_{11}'(x_i - y)\,dy + \eta_k^n \int_{x_{k-1/2}}^{x_{k+1/2}} K_{12}'(x_i - y)\,dy \right), \\ (\Upsilon_g)_i^n := \sum_{k\in\mathcal{I}} \left(\eta_k^n \int_{x_{k-1/2}}^{x_{k+1/2}} K_{22}'(x_i - y)\,dy + \rho_k^n \int_{x_{k-1/2}}^{x_{k+1/2}} K_{21}'(x_i - y)\,dy \right). \end{cases} \tag{4.3d}$$

The scheme is complemented with periodic boundary conditions in space, i.e.,

$$f_{N_x,j}^n = f_{0,j}^n, \quad g_{N_x,j}^n = g_{0,j}^n, \tag{4.4a}$$

$$f_{-1,j}^n = f_{N_x-1,j}^n, \quad g_{-1,j}^n = g_{N_x-1,j}^n, \tag{4.4b}$$

where the values $f_{-1,j}^n$, $g_{-1,j}^n$, $f_{N_x,j}^n$, $g_{N_x,j}^n$ represent an approximation on a "virtual cell". In velocity we have no-flux boundaries, i.e.,

$${}^v\bar{F}_{i,j+1/2}^n = 0 = {}^v\bar{G}_{i,j+1/2}^n \tag{4.4c}$$

for all $(i,j) \in \mathcal{I} \times \mathbb{Z} \setminus \mathcal{J}$.

4.2.4 The finite volume scheme

Throughout the Chapter we will use the following two representations of our scheme. First, we consider

$$\begin{cases} f_{i,j}^{n+1} = f_{i,j}^n - \dfrac{1}{|C_{i,j}|}\left({}^x\bar{F}_{i+1/2,j}^n - {}^x\bar{F}_{i-1/2,j}^n + {}^v\bar{F}_{i,j+1/2}^n - {}^v\bar{F}_{i,j-1/2}^n\right), \\ g_{i,j}^{n+1} = g_{i,j}^n - \dfrac{1}{|C_{i,j}|}\left({}^x\bar{G}_{i+1/2,j}^n - {}^x\bar{G}_{i-1/2,j}^n + {}^v\bar{G}_{i,j+1/2}^n - {}^v\bar{G}_{i,j-1/2}^n\right), \end{cases} \tag{4.3a}$$

for $n = 0, \ldots, N_T - 1$ and $(i,j) \in \mathcal{I} \times \mathcal{J}$, where ${}^x\bar{F}$, ${}^v\bar{F}$, ${}^x\bar{G}$ and ${}^v\bar{G}$ are defined in (4.3b) and (4.3c). Second, we can rewrite the scheme (4.3) and get by a short computation

$$\begin{aligned} p_{i,j}^{n+1} = &\left(1 - \Delta t\left[\frac{|v_j|}{\Delta x_i} + \frac{|(\Upsilon_p)_i^n|}{\Delta v_j}\right]\right) p_{i,j}^n + \Delta t \frac{[v_j]^-}{\Delta x_i} p_{i+1,j}^n + \Delta t \frac{[v_j]^+}{\Delta x_i} p_{i-1,j}^n \\ &+ \Delta t \frac{[(\Upsilon_p)_i^n]^+}{\Delta v_j} p_{i,j+1}^n + \Delta t \frac{[(\Upsilon_p)_i^n]^-}{\Delta v_j} p_{i,j-1}^n, \end{aligned} \tag{4.5}$$

for $p \in \{f, g\}$ and $n = 0, \ldots, N_T - 1$ and $(i,j) \in \mathcal{I} \times \mathcal{J}$. For both representations we use the boundary conditions (4.4).

Before proving some a priori estimates, let us introduce our notion of solutions.

Definition 4.1 (Weak solution)**.** We call the pair (f, g) a *weak solution* to system (4.1) if it satisfies

$$\begin{cases} \displaystyle\int_{Q_T} f\left(\frac{\partial\varphi}{\partial t} + v\frac{\partial\varphi}{\partial x} - \Upsilon_f \frac{\partial\varphi}{\partial v}\right) dt\,dx\,dv + \int_Q f_0(x,v)\varphi(0,x,v)\,dx\,dv = 0, \\ \displaystyle\int_{Q_T} g\left(\frac{\partial\varphi}{\partial t} + v\frac{\partial\varphi}{\partial x} - \Upsilon_g \frac{\partial\varphi}{\partial v}\right) dt\,dx\,dv + \int_Q g_0(x,v)\varphi(0,x,v)\,dx\,dv = 0, \end{cases}$$

for every $\varphi \in \mathcal{C}_c^\infty([0,T) \times Q)$.

4.3 Properties of the numerical method and a priori estimates

4.3.1 A priori estimates

This Section is dedicated to establishing the positivity and boundedness of the discrete approximation obtained in Section 4.2.4.

4.3.2 Positivity of the solution and the CFL condition

To mimic the structure-preserving properties of system (4.1) on the level of the approximations, a stepsize restriction is required. Indeed, we assume that there exists $\xi \in (0,1)$ such that, for both species, $p \in \{f, g\}$,

$$\frac{\Delta t}{|C_{i,j}|}(\Delta v_j |v_j| + \Delta x_i |(\Upsilon_p)_i^n|) \leq 1 - \xi, \tag{4.6}$$

for all $(i,j) \in \mathcal{I} \times \mathcal{J}$, and all $n \in \mathbb{N}$.

It is absolutely crucial to stress that, albeit apparently dependent on n, the stepsize restriction, (4.6), can be shown to be satisfied uniformly in n. Indeed, we shall see in the subsequent Proposition that it is independent of n using a short induction argument.

Proposition 4.1 (Positivity preservation of the scheme)**.** *Let $K_{ij} \in W^{1,\infty}(-L, L)$, $i, j \in \{1, 2\}$, $p \in \{f, g\}$ be with non-negative initial condition $p_0 \in \{f_0, g_0\}$ with $\|p_0\|_{L^1(Q)} = 1$. Assume that there exists $\xi \in (0,1)$ such that the stepsize restriction*

$$\frac{\Delta t}{\Delta x_i \Delta v_j}(\Delta v_j |v_j| + \Delta x_i |(\Upsilon_p)_i^0|) \leq 1 - \xi, \tag{4.7}$$

is satisfied. Then, the following holds true:

(i) $p_{i,j}^n \geq 0$, for all $(i,j) \in \mathcal{I} \times \mathcal{J}$, and $\|p_h(t^n)\|_{L^1(Q)} = \|p_h(t = 0)\|_{L^1(Q)}$, for all $n \in \mathbb{N}$.

(ii) If Δt is chosen such that

$$\Delta t \leq \frac{(1-\xi)\alpha}{v_h + \mathcal{C}_{\mathcal{W}}} h, \tag{4.8}$$

where ξ is as in (4.7) and $\mathcal{C}_{\mathcal{W}}$ is defined by

$$\mathcal{C}_{\mathcal{W}} := \max_{i \in \{1,2\}} \sum_{j=1}^{2} \|K'_{ij}\|_{L^\infty(-L,L)}, \tag{4.9}$$

then the CFL condition (4.6) is satisfied for the two species uniformly in $n \in \mathbb{N}$.

Remark 4.1. Note that, by Proposition 4.1, the positivity of f_h and g_h is guaranteed, and the scheme conserves the mass.

Proof. We proceed by induction. First, let us consider $n = 0$. Since p_0 is non-negative, we know that $p_{ij}^0 \geq 0$ which implies (i). On the other hand, for $n = 0$ the CFL condition (4.6) is satisfied by assumption. Next, let us assume for n fixed that the statement (i) and condition (4.6) hold true. Let us prove *(i)* for $n+1$. We consider the representation

(4.5) of our scheme. Since, by induction assumption, $p_{ij}^n \geq 0$ for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$, and condition (4.6) is met for n, we derive from the representation (4.5) that

$$p_{i,j}^{n+1} \geq 0.$$

Next, we prove the conservation of mass. Using the non-negativity in conjunction with the scheme, we compute

$$\begin{aligned}
\|p_h(t^{n+1})\|_{L^1(Q)} &= \sum_{i\in\mathcal{I},j\in\mathcal{J}} |C_{i,j}|\, p_{i,j}^{n+1} \\
&= \sum_{i\in\mathcal{I},j\in\mathcal{J}} |C_{i,j}|\, p_{i,j}^{n} - \sum_{i\in\mathcal{I},j\in\mathcal{J}} \left({}^x\bar{P}^n_{i+1/2,j} - {}^x\bar{P}^n_{i-1/2,j}\right) \\
&\qquad - \sum_{i\in\mathcal{I},j\in\mathcal{J}} \left({}^v\bar{P}^n_{i+1/2,j} - {}^v\bar{P}^n_{i-1/2,j}\right) \\
&= \sum_{i\in\mathcal{I},j\in\mathcal{J}} |C_{i,j}|\, p_{i,j}^{n},
\end{aligned}$$

since both sums over the fluxes are telescopic sums and having exploited the periodic and no-flux boundary conditions. Therefore, we obtain

$$\|p_h(t^{n+1})\|_{L^1(Q)} = \sum_{i\in\mathcal{I},j\in\mathcal{J}} |C_{i,j}|\, p_{i,j}^{n} = \|p_h(t^n)\|_{L^1(Q)} = \|p_h(0)\|_{L^1(Q)},$$

where the last equality holds by assumption. Thus, the conservation of mass, (i), is guaranteed on the numerical level.

Next, we prove statement *(ii)*. Let $\zeta_h \in \{\rho_h, \eta_h\}$ be the respective macroscopic density of $p_h \in \{f_h, g_h\}$. We know that

$$\begin{aligned}
\int_{-L}^{L} \zeta_h(t^{n+1}, x)\, dx &= \sum_{i\in\mathcal{I},j\in\mathcal{J}} |C_{i,j}|\, p_{i,j}^{n+1} \\
&= \sum_{i\in\mathcal{I},j\in\mathcal{J}} |C_{i,j}|\, p_{i,j}^{0} \\
&\leq \sum_{i\in\mathcal{I},j\in\mathbb{Z}} |C_{i,j}|\, p_{i,j}^{0} \\
&= 1.
\end{aligned}$$

Then we compute for $p_h = f_h$

$$\begin{aligned}
|(\Upsilon_f)_i^{n+1}| &= \left| \int_{-L}^{L} K'_{11}(x_i - y)\rho_h(t^{n+1}, y)\, dy + K'_{12}(x_i - y)\eta_h(t^{n+1}, y)\, dy \right| \\
&\leq \|K'_{11}\|_{L^\infty} \int_{-L}^{L} \rho_h(t^{n+1}, y)\, dy + \|K'_{12}\|_{L^\infty} \int_{-L}^{L} \eta_h(t^{n+1}, y)\, dy \\
&\leq \mathcal{C}_{\mathcal{W}}.
\end{aligned} \tag{4.10}$$

The same estimate can be established for the other species, $p_h = g_h$. Overall, this shows that

$$\Delta t\left(\frac{|v_j|}{\Delta x_i} + \frac{|(\Upsilon_p)_i^{n+1}|}{\Delta v_j}\right) \leq \Delta t\left(\frac{v_h}{\alpha h} + \frac{\mathcal{C}_{\mathcal{W}}}{\alpha h}\right)$$

for $p \in \{f, g\}$. So, if we choose Δt such that

$$\frac{(1-\xi)}{v_h + \mathcal{C}_{\mathcal{W}}} h > \Delta t,$$

we can guarantee

$$\Delta t\left(\frac{|v_j|}{\Delta x_i} + \frac{|(\Upsilon_p)_i^{n+1}|}{\Delta v_j}\right) \leq 1 - \xi.$$

Therefore, the stepsize condition (4.6) is satisfied for both species and all $n \in \mathbb{N}$. □

4.3.3 Boundedness of the solution and an a priori estimate

We will begin by proving that the solutions of the scheme described in Section 4.2.4 are bounded in $L^p(Q)$ for each time $t \in (0, T)$. This we will prove using the next Proposition.

Proposition 4.2. *Consider a non-negative, convex function $\phi : \mathbb{R} \to \mathbb{R}$ such that*

$$\int_{-L}^{L} \int_{\mathbb{R}} \phi(p_0(x, v))\, dx\, dv < +\infty,$$

for $p_0 \in \{f_0, g_0\}$. Let the assumptions of Proposition 4.1 hold true. Then, under the CFL condition (4.6), the numerical solution satisfies

$$\int_{-L}^{L} \int_{\mathbb{R}} \phi(p_h(t+\tau, x, v))\, dx\, dv \leq \int_{-L}^{L} \int_{\mathbb{R}} \phi(p_h(t, x, v))\, dx\, dv,$$

for $p \in \{f, g\}$ and every $t, \tau \geq 0$.

Proof. Consider the representation (4.5) of the discrete scheme. Under the CFL condition (4.6), we can observe that $p_{i,j}^{n+1}$ is a convex combination of $p_{i,j}^n$, $p_{i+1,j}^n$, $p_{i-1,j}^n$, $p_{i,j+1}^n$, $p_{i,j-1}^n$. By convexity of ϕ, we obtain

$$\begin{aligned}\phi(p_{i,j}^{n+1}) \leq & \left(1 - \Delta t\left[\frac{|v_j|}{\Delta x_i} + \frac{|(\Upsilon_p)_i^n|}{\Delta v_j}\right]\right)\phi(p_{i,j}^n) + \Delta t\frac{[v_j]^-}{\Delta x_i}\phi(p_{i+1,j}^n) + \Delta t\frac{[v_j]^+}{\Delta x_i}\phi(p_{i-1,j}^n) \\ & + \Delta t\frac{[(\Upsilon_p)_i^n]^+}{\Delta v_j}\phi(p_{i,j+1}^n) + \Delta t\frac{[(\Upsilon_p)_i^n]^-}{\Delta v_j}\phi(p_{i,j-1}^n).\end{aligned}$$

Integrating in space and velocity, we have

$$\begin{aligned}&\int_{-L}^{L} \int_{\mathbb{R}} \phi(p_h(t^{n+1}, x, v))\, dx\, dv \\ &\qquad = \sum_{i \in \mathcal{I}, j \in J} |C_{i,j}|\, \phi(p_{i,j}^{n+1}) \\ &\qquad \leq \sum_{i=0}^{N_x-1} \sum_{j \in \mathcal{J}} \Big[\big(\Delta x_i \Delta v_j - \Delta t(|v_j|\Delta v_j + |(\Upsilon_p)_i^n|\Delta x_i)\big)\phi(p_{i,j}^n) \\ &\qquad\quad + \Delta t \Delta v_j v_j^- \phi(p_{i+1,j}^n) + \Delta t \Delta v_j v_j^+ \phi(p_{i-1,j}^n) \\ &\qquad\quad + \Delta t \Delta x_i [(\Upsilon_p)_i^n]^+ \phi(p_{i,j+1}^n) + \Delta t \Delta x_i [(\Upsilon_p)_i^n]^- \phi(p_{i,j-1}^n)\Big].\end{aligned}$$

By shifting the indices and applying the boundary conditions (4.4), we get

$$\begin{aligned}\int_{-L}^{L}\int_{\mathbb{R}} \phi(p_h(t^{n+1},x,v))\,dx\,dv &\le \sum_{i\in\mathcal{I},j\in\mathcal{J}} \Delta x_i \Delta v_j \phi(p^n_{i,j}) \\ &= \int_{-L}^{L}\int_{\mathbb{R}} \phi(p_h(t^{n},x,v))\,dx\,dv\,.\end{aligned} \tag{4.11}$$

Finally, let $t,\tau \ge 0$ be given. The statement follows from fixing integers, $n_0, n_1 \in \mathbb{N}$ such that $t \in [t^{n_0}, t^{n_0+1})$ and $t+\tau \in [t^{n_1}, t^{n_1+1})$ and applying estimate (4.11) iteratively. □

In the subsequent analysis, more refined bounds are required. To this end, we estimate the tails of (f_h, g_h).

Proposition 4.3. *Let the initial datum of both species be non-negative and bounded from above by a function R of the following type*

$$R(x,v) = \frac{C}{1+|v|^{\lambda_1}+|x|^{\lambda_2}},$$

for some $\lambda_1 > 1$, $\lambda_2 \ge 1$, with $\lambda_2 \le \lambda_1$, i.e., $0 \le p_0(x,v) \le R(x,v)$ with $p \in \{f,g\}$. Then, there exists a constant $C_T > 0$ depending on α, λ_1, λ_2, $\mathcal{C}_{\mathcal{W}}$ and the final time $T > 0$ such that

$$0 \le p_h(t,x,v) \le C_T R_h(x,v), \tag{4.12}$$

for $(t,x,v) \in Q_T$, $p_h \in \{f_h, g_h\}$, where

$$R_h(x,v) := \frac{C}{1+|v_j|^{\lambda_1}+|x_i|^{\lambda_2}},$$

for $(x,v) \in C_{i,j}$. As a consequence, for h small enough

$$0 \le \zeta_h(t,x) \le C_T,$$

for $(t,x) \in \Omega_T$, and where $\zeta_h \in \{\rho_h, \eta_h\}$ is the respective macroscopic density of $p_h \in \{f_h, g_h\}$.

Proof. Let $p_h \in \{f_h, g_h\}$. By Proposition 4.1, we know that p_h is non-negative. Next, since $x_i = x_{i+1} - \frac{1}{2}(\Delta x_i + \Delta x_{i+1})$, setting $\Delta x_{i+1/2} = \frac{1}{2}(\Delta x_i + \Delta x_{i+1})$, by definition of R_h we have

$$\begin{aligned}\frac{R_h(x_{i+1},v_j)}{R_h(x_i,v_j)} &\le \frac{1+|v_j|^{\lambda_1}+(|x_{i+1}|+\Delta x_{i+1/2})^{\lambda_2}}{1+|v_j|^{\lambda_1}+|x_{i+1}|^{\lambda_2}} \\ &\le \frac{1+|v_j|^{\lambda_1}+|x_{i+1}|^{\lambda_2}+C|x_{i+1}|^{\lambda_2-1}\Delta x_{i+1/2}+\mathcal{O}((\Delta x_{i+1/2})^2)}{1+|v_j|^{\lambda_1}+|x_{i+1}|^{\lambda_2}} \\ &\le 1 + C\frac{|x_{i+1}|^{\lambda_2-1}}{1+|v_j|^{\lambda_1}+|x_{i+1}|^{\lambda_2}}\Delta x_{i+1/2} + \mathcal{O}((\Delta x_{i+1/2})^2).\end{aligned}$$

In the same way, we obtain

$$\frac{R_h(x_{i-1},v_j)}{R_h(x_i,v_j)} \le 1 + C\frac{|x_i|^{\lambda_2-1}}{1+|v_j|^{\lambda_1}+|x_i|^{\lambda_2}}\Delta x_{i-1/2} + \mathcal{O}((\Delta x_{i-1/2})^2).$$

Since by assumption $\lambda_2 \le \lambda_1$, we derive, for $i \in \mathcal{I}$,

$$\frac{|x_i|^{\lambda_2-1}|v_j|}{1+|v_j|^{\lambda_1}+|x_i|^{\lambda_2}} \le 1.$$

Indeed, if $|v_j| \le |x_i|$, we get

$$|x_i|^{\lambda_2-1}|v_j| \le |x_i|^{\lambda_2} \le 1+|v_j|^{\lambda_1}+|x_i|^{\lambda_2}.$$

If, instead, $|x_i| < |v_j|$ and $|v_j| \ge 1$, we obtain

$$|x_i|^{\lambda_2-1}|v_j| \le |v_j|^{\lambda_2} \le |v_j|^{\lambda_1} \le 1+|v_j|^{\lambda_1}+|x_i|^{\lambda_2}.$$

If, finally, $|x_i| < |v_j|$ and $|v_j| < 1$, then

$$|x_i|^{\lambda_2-1}|v_j| \le |x_i|^{\lambda_2-1} \le 1+|x_i|^{\lambda_2} \le 1+|x_i|^{\lambda_2}+|v_j|^{\lambda_1}.$$

Therefore, we derive that

$$\begin{aligned}[v_j]^-\frac{R_h(x_{i+1},v_j)}{R_h(x_i,v_j)}+[v_j]^+\frac{R_h(x_{i-1},v_j)}{R_h(x_i,v_j)} &\le |v_j|+c_1(\lambda_2)\Delta x_i\left(2+\frac{\Delta x_{i+1}+\Delta x_{i-1}}{\Delta x_i}\right)\\ &\le |v_j|+c_1(\alpha,\lambda_2)\Delta x_i.\end{aligned}$$

Now, setting $\Delta v_{j+1/2} = \frac{1}{2}(\Delta v_j + \Delta v_{j+1})$, we have

$$\begin{aligned}\frac{R_h(x_i,v_{j+1})}{R_h(x_i,v_j)} &= \frac{1+|v_j|^{\lambda_1}+|x_i|^{\lambda_2}}{1+|v_{j+1}|^{\lambda_1}+|x_i|^{\lambda_2}} \le \frac{1+(|v_{j+1}|+\Delta v_{j+1/2})^{\lambda_1}+|x_i|^{\lambda_2}}{1+|v_{j+1}|^{\lambda_1}+|x_i|^{\lambda_2}}\\ &\le 1+C\frac{|v_{j+1}|^{\lambda_1-1}\Delta v_{j+1/2}+\mathcal{O}((\Delta v_{j+1/2})^2)}{1+|v_{j+1}|^{\lambda_1}+|x_i|^{\lambda_2}}\\ &\le 1+C\frac{|v_{j+1}|^{\lambda_1-1}}{1+|v_{j+1}|^{\lambda_1}}\Delta v_{j+1/2}+\mathcal{O}((\Delta v_{j+1/2})^2)\\ &\le 1+c_2(\alpha,\lambda_1)\Delta v_j.\end{aligned}$$

In the same way, we obtain

$$\frac{R_h(x_i,v_{j-1})}{R_h(x_i,v_j)} \le 1+c_3(\alpha,\lambda_1)\Delta v_j.$$

Set $c_0(\alpha,\lambda_1,\lambda_2) = \max\{c_1,c_2,c_3\}$. Set $A := (1+\Delta t c_0(1+\mathcal{C}_{\mathcal{W}}))$. We know that $p_0(x,v) \le A^0 R_h(x,v)$.

Let us proceed by induction. Assume $p_h(t^n,x,v) \le A^n R_h(x,v)$. Using the numerical scheme (4.5) we have

$$\begin{aligned}\frac{p^{n+1}_{i,j}}{R_h(x_i,v_j)} =&\left(1-\Delta t\left[\frac{|v_j|}{\Delta x_i}+\frac{|(\Upsilon_p)^n_i|}{\Delta v_j}\right]\right)\frac{p^n_{i,j}}{R_h(x_i,v_j)}\\ &+\Delta t\frac{[v_j]^-}{\Delta x_i}\frac{p^n_{i+1,j}}{R_h(x_{i+1},v_j)}\frac{R_h(x_{i+1},v_j)}{R_h(x_i,v_j)}\\ &+\Delta t\frac{[v_j]^+}{\Delta x_i}\frac{p^n_{i-1,j}}{R_h(x_{i-1},v_j)}\frac{R_h(x_{i-1},v_j)}{R_h(x_i,v_j)}\\ &+\Delta t\frac{[(\Upsilon_p)^n_i]^+}{\Delta v_j}\frac{p^n_{i,j+1}}{R_h(x_i,v_{j+1})}\frac{R_h(x_i,v_{j+1})}{R_h(x_i,v_j)}\\ &+\Delta t\frac{[(\Upsilon_p)^n_i]^-}{\Delta v_j}\frac{p^n_{i,j-1}}{R_h(x_i,v_{j-1})}\frac{R_h(x_i,v_{j-1})}{R_h(x_i,v_j)}.\end{aligned}$$

Hence, using the estimates above and (4.10) we arrive at

$$\begin{aligned}\frac{p_{i,j}^{n+1}}{R_h(x_i,v_j)} \leq & \left(1-\Delta t\left[\frac{|v_j|}{\Delta x_i}+\frac{|(\Upsilon_p)_i^n|}{\Delta v_j}\right]\right)A^n + \Delta t\frac{|v_j|}{\Delta x_i}A^n\left(1+c_0\frac{\Delta x_i}{|v_j|}\right)\\ & + \Delta t\frac{|(\Upsilon_p)_i^n|}{\Delta v_j}A^n(1+c_0\Delta v_j)\\ \leq & A^n\Big(1+\Delta t c_0(1+\mathcal{C}_{\mathcal{W}})\Big) = A^{n+1}.\end{aligned}$$

Thus, we obtain that for all $(i,j)\in\mathcal{I}\times\mathbb{Z}$,

$$\frac{p_{i,j}^{n+1}}{R_h(x_i,v_j)} \leq A^{n+1}.$$

By definition of A^n, we have for all $n\in\{0,\dots\lceil T/\Delta t\rceil\}$, $A^{n+1} < e^{c_0(1+\mathcal{C}_{\mathcal{W}})T}$. Therefore, as in the continuous case, at discrete level there exists $C_T>0$ depending on $\alpha,\lambda_1,\lambda_2,\mathcal{C}_{\mathcal{W}},T$ such that

$$p_h(t,x,v)\leq C_T R_h(x,v),$$

for $(t,x,v)\in Q_T$. Moreover, we have that

$$\begin{aligned}\int_{\mathbb{R}} R_h(x,v)\,dv &= C\sum_{j\in\mathbb{Z}}\frac{\Delta v_j}{1+|v_j|^{\lambda_1}+|x_i|^{\lambda_2}} \leq 2C\sum_{j\in\mathbb{N}}\frac{h}{1+(\alpha[j-1]h)^{\lambda_1}+|x_i|^{\lambda_2}}\\ &\leq \frac{2C}{\alpha}\sum_{j\in\mathbb{N}}\frac{\Delta v_{j-1}}{1+(\alpha[j-1]h)^{\lambda_1}} \leq \frac{2C}{\alpha^{1+\lambda_1}}\int_0^\infty\frac{1}{1+v^{\lambda_1}}\,dv.\end{aligned}$$

Now, we have that

$$\begin{aligned}\int_0^\infty\frac{1}{1+v^{\lambda_1}}\,dv &= \int_0^1\frac{1}{1+v^{\lambda_1}}\,dv + \int_1^\infty\frac{1}{1+v^{\lambda_1}}\,dv\\ &\leq 1+\int_1^\infty\frac{1}{v^{\lambda_1}}\,dv = 1+\frac{1}{\lambda_1-1}.\end{aligned}$$

Thus, for h small enough, we obtain that

$$\zeta_h(t,x) = \int_{\mathbb{R}} p_h(t,x,v)\,dv \leq C_T\left(\frac{1}{\alpha^{1+\lambda_1}}\int_{\mathbb{R}}\frac{dv}{1+|v|^{\lambda_1}}\right) < +\infty,$$

for $\zeta_h\in\{\rho_h,\eta_h\}$. □

Remark 4.2. With this proposition we now choose an appropriate v_h which is applied for the cut off in the velocity domain in Section 4.2.2. First, let $x\in(-L,L)$ be fixed and $\varepsilon>0$. Now we want to choose v_h such that

$$\int_{\mathbb{R}\setminus(-v_h,v_h)} p_h(t,x,v)\,dv < \varepsilon,$$

with $p_h\in\{f_h,g_h\}$ and $t\in(0,T)$. Indeed we derive as in the proof of Proposition 4.3

$$\begin{aligned}\int_{\mathbb{R}\setminus(-v_h,v_h)} p_h(t,x,v)\,dv &\leq \int_{\mathbb{R}\setminus(-v_h,v_h)} C_T R_h(x,v)\,dv \leq C_T\frac{2}{\alpha^{1+\lambda_1}}\int_{v_h}^\infty\frac{1}{1+v^{\lambda_1}}dv\\ &\leq \frac{2\,C_T}{\alpha^{1+\lambda_1}}\frac{1}{\lambda_1-1}v_h^{-\lambda_1+1},\end{aligned}$$

for $t \in (0,T)$. Then we choose

$$v_h = \left(\frac{2C_T}{\alpha^{1+\lambda_1}} \frac{1}{\lambda_1 - 1} \varepsilon^{-1} \right)^{\frac{1}{\lambda_1+1}}.$$

Such a choice of v_h guarantees that the mass outside of $(-v_h, v_h)$ is less then ε for all time $t \in (0,T)$. If ε is much smaller than the machine epsilon then the error that we are making by cutting off the functions f and g in the velocity domain is minimal with respect to the computational error.

4.4 Convergence of the scheme

Before we will prove the convergence of the scheme, we will introduce the piecewise constant approximation of the interaction terms that we will use in the following proof.

Definition 4.2 (Piecewise constant interpolation)**.** We define the piecewise constant approximation of the interaction terms as

$$(\Upsilon_f)_h(t,x) := \int_{-L}^{L} K'_{11}(x-y)\rho_h(t,y)\,dy + \int_{-L}^{L} K'_{12}(x-y)\eta_h(t,y)\,dy, \tag{4.13a}$$

$$(\Upsilon_g)_h(t,x) := \int_{-L}^{L} K'_{22}(x-y)\eta_h(t,y)\,dy + \int_{-L}^{L} K'_{21}(x-y)\rho_h(t,y)\,dy. \tag{4.13b}$$

We are now in the position to prove the main result of this Chapter concerning the convergence of the scheme.

Theorem 4.1 (Convergence of the scheme)**.** *Assume $p_0 \in \{f_0, g_0\}$ non-negative and bounded from above by a function R, where*

$$R(x,v) := \frac{C}{1 + |v_j|^{\lambda_1} + |x_i|^{\lambda_2}},$$

for $(x,v) \in (x_{i-1/2}, x_{i+1/2}) \times (v_{j-1/2}, v_{j+1/2})$, with $\lambda_1 > 1$, $\lambda_2 \geq 1$ and $\lambda_2 \leq \lambda_1$, for some $C > 0$. Assume that the CFL condition (4.6) is satisfied and that Δt satisfies (4.8). Let $K_{ij} \in W^{2,\infty}(-L,L)$, for $i,j \in \{1,2\}$. Denoting by $(f_h, g_h)(t,x,v)$ the numerical solution to the scheme (4.3), then we have

$$f_h(t,x,v) \rightharpoonup f(t,x,v), \qquad g_h(t,x,v) \rightharpoonup g(t,x,v),$$

weakly-$*$ in $L^\infty(Q_T)$ as $h \to 0$, where (f,g) is a solution to system (4.1), in the sense of Definition 4.1.

Proof. Let $p_h \in \{f_h, g_h\}$ and $\zeta_h \in \{\rho_h, \eta_h\}$ the respective density. By Proposition 4.3 we know that p_h is bounded in $L^\infty(Q_T)$, thus, by Banach-Alaoglu Theorem we have that, up to a subsequence, there exists a function $p \in L^\infty(Q_T)$ such that

$$p_h(t,x,v) \rightharpoonup p(t,x,v),$$

weakly-$*$ in $L^\infty(Q_T)$, as $h \to 0$. We also know that ζ_h is bounded in $L^\infty(\Omega_T)$, thus, up to a subsequence,

$$\zeta_h(t,x) \rightharpoonup \zeta(t,x),$$

weakly-$*$ in $L^\infty(\Omega_T)$, as $h \to 0$. Moreover, we have that $\zeta(t,x)$ is equal to $\int_{\mathbb{R}} p(t,x,v)\,dv$ a.e.. Indeed, considering $\psi \in L^1(\Omega_T)$ we have

$$\int_0^T \int_{-L}^L \left(\zeta_h - \int_{\mathbb{R}} p\,dv\right)\psi(t,x)\,dx\,dt = \int_0^T \int_{-L}^L \int_{\mathbb{R}} (p_h - p)\psi(t,x)\,dv\,dx\,dt \to 0,$$

since $p_h \rightharpoonup p$ weakly-$*$ in $L^\infty(Q_T)$. Furthermore, if $p = f$, it holds that

$$\int_{Q_T} \frac{\partial \varphi}{\partial v}(\Upsilon_f)_h f_h \,dx\,dv\,dt \to \int_{Q_T} \frac{\partial \varphi}{\partial v}\Upsilon_f f\,dx\,dv\,dt,$$

since $f_h \rightharpoonup f$ weakly-$*$ in $L^\infty(Q_T)$, and $(\Upsilon_f)_h \to \Upsilon_f$ strongly in $L^1(-L,L)$. Indeed, for any $x \in (-L,L)$ fixed, we get

$$\begin{aligned}
(\Upsilon_f)_h(t,x) - \Upsilon_f(t,x) =& \int_{-L}^L K_{11}'(x-y)\rho_h(t,y)\,dy + \int_{-L}^L K_{12}'(x-y)\eta_h(t,y)\,dy \\
&- \int_{-L}^L K_{11}'(x-y)\rho(t,y)\,dy - \int_{-L}^L K_{12}'(x-y)\eta(t,y)\,dy \\
\longrightarrow& \int_{-L}^L K_{11}'(x-y)\rho(t,y)\,dy + \int_{-L}^L K_{12}'(x-y)\eta(t,y)\,dy \\
&- \int_{-L}^L K_{11}'(x-y)\rho(t,y)\,dy - \int_{-L}^L K_{12}'(x-y)\eta(t,y)\,dy \\
=&0,
\end{aligned}$$

since $\rho_h \rightharpoonup \rho$ and $\eta_h \rightharpoonup \eta$ weakly-$*$ in $L^\infty(\Omega_T)$. Thus, we have pointwise convergence. Since

$$\|K_{11}' * \rho_h\|_{L^\infty} \le \|K_{11}'\|_{L^\infty}\|\rho_h\|_{L^1} \le C \quad \text{and} \quad \|K_{12}' * \eta_h\|_{L^\infty} \le \|K_{12}'\|_{L^\infty}\|\eta_h\|_{L^1} \le C$$

with C independent of h and $\int_\Omega C\,dx < \infty$, then by Lebesgue's dominated convergence theorem we get $(\Upsilon_f)_h \to \Upsilon_f$ strongly in $L^1(\Omega_T)$. The same argumentation can be done for $p = g$.

Having garnered all information necessary, we are now ready to identify the limit. The following notation will be convenient:

$$\begin{aligned}
\mathcal{I}_t^h &:= \int_{Q_T} p_h(t,x,v)\frac{\partial \varphi}{\partial t}(t,x,v)\,dt\,dx\,dv + \int_Q p_0(x,v)\varphi(0,x,v)\,dx\,dv, \\
\mathcal{I}_x^h &:= \int_{Q_T} p_h(t,x,v)v\frac{\partial \varphi}{\partial x}(t,x,v)\,dt\,dx\,dv, \\
\mathcal{I}_v^h &:= -\int_{Q_T} p_h(t,x,v)(\Upsilon_p)_h(t,x)\frac{\partial \varphi}{\partial v}(t,x,v)\,dt\,dx\,dv,
\end{aligned}$$

where $\varphi \in C_c^\infty([0,T) \times Q)$ is arbitrary but fixed throughout. With the compactness from above, it is immediate to see that

$$\lim_{\Delta t,\, h \to 0} I_t^h = \int_{Q_T} p(t,x,v)\frac{\partial \varphi}{\partial t}(t,x,v)\,dt\,dx\,dv + \int_Q p_0(x,v)\varphi(0,x,v)\,dt\,dx\,dv,$$

as well as

$$\lim_{\Delta t,\, h \to 0} I_x^h = \int_{Q_T} p(t,x,v)\,v\frac{\partial \varphi}{\partial x}(t,x,v)\,dt\,dx\,dv,$$

and

$$\lim_{\Delta t,\, h \to 0} \mathcal{I}_v^h = -\int_{Q_T} p(t,x,v)\, \Upsilon_p(t,x) \frac{\partial \varphi}{\partial v}(t,x,v)\, dt\, dx\, dv.$$

It remains to show that

$$\lim_{\Delta t,\, h \to 0} \mathcal{I}_t^h + \mathcal{I}_x^h + \mathcal{I}_v^h = 0. \tag{4.14}$$

In order to establish this limit, we exploit the discrete scheme, (4.5). Indeed, let us observe that (4.5) can be rewritten as

$$\begin{aligned} \frac{p_{i,j}^{n+1} - p_{i,j}^n}{\Delta t} &= \frac{[v_j]^-}{\Delta x_i}(p_{i+1,j}^n - p_{i,j}^n) + \frac{[v_j]^+}{\Delta x_i}(p_{i-1,j}^n - p_{i,j}^n) \\ &\quad + \frac{[(\Upsilon_p)_i^n]^-}{\Delta v_j}(p_{i,j-1}^n - p_{i,j}^n) + \frac{[(\Upsilon_p)_i^n]^+}{\Delta v_j}(p_{i,j+1}^n - p_{i,j}^n). \end{aligned} \tag{4.15}$$

Multiplying (4.15) by

$$\varphi_{i,j}^n \coloneqq \int_{C_{i,j}^n} \varphi(t,x,v)\, dt\, dx\, dv,$$

where $C_{i,j}^n \coloneqq [t^n, t^{n+1}) \times C_{i,j}$, and summing over $i \in \mathcal{I}$, $j \in \mathcal{J}$ and $n \in \{0, \ldots, N_T - 1\}$, we obtain

$$\mathcal{J}_t^h + \mathcal{J}_x^h + \mathcal{J}_v^h = 0,$$

with

$$\begin{aligned} \mathcal{J}_t^h &\coloneqq \sum_{n,i,j} \frac{p_{i,j}^{n+1} - p_{i,j}^n}{\Delta t} \varphi_{i,j}^n, \\ \mathcal{J}_x^h &\coloneqq -\sum_{n,i,j} \left[\frac{[v_j]^-}{\Delta x_i}(p_{i+1,j}^n - p_{i,j}^n)\varphi_{i,j}^n + \frac{[v_j]^+}{\Delta x_i}(p_{i-1,j}^n - p_{i,j}^n)\varphi_{i,j}^n \right], \\ \mathcal{J}_v^h &\coloneqq -\sum_{n,i,j} \left[\frac{[(\Upsilon_p)_i^n]^-}{\Delta v_j}(p_{i,j-1}^n - p_{i,j}^n)\varphi_{i,j}^n + \frac{[(\Upsilon_p)_i^n]^+}{\Delta v_j}(p_{i,j+1}^n - p_{i,j}^n)\varphi_{i,j}^n \right]. \end{aligned}$$

The strategy is to show that

$$|\mathcal{I}_t^h + \mathcal{J}_t^h|,\ |\mathcal{I}_x^h + \mathcal{J}_x^h|,\ |\mathcal{I}_v^h + \mathcal{J}_v^h|, \to 0, \tag{4.16}$$

as Δt, $h \to 0$, which shows the convergence of each of the terms on the one hand and establishes the limit in (4.14) on the other hand. We proceed term by term.

Estimating $\mathcal{J}_t^h$

We consider

$$\mathcal{J}_t^h = \frac{1}{\Delta t} \sum_{i,j} \mathfrak{j}_{i,j}, \tag{4.17}$$

where

$$
\begin{aligned}
\mathfrak{j}_{i,j} &:= \sum_{n=0}^{N_T-1} (p_{i,j}^{n+1} - p_{i,j}^{n})\varphi_{i,j}^{n} \\
&= -\sum_{n=1}^{N_T} p_{i,j}^{n}(\varphi_{i,j}^{n} - \varphi_{i,j}^{n-1}) - p_{i,j}^{0}\varphi_{i,j}^{0} \\
&= -\sum_{n=0}^{N_T-1} p_{i,j}^{n+1}(\varphi_{i,j}^{n+1} - \varphi_{i,j}^{n}) - p_{i,j}^{0}\varphi_{i,j}^{0},
\end{aligned} \tag{4.18}
$$

having used integration by parts and the fact that $\varphi_{i,j}^{N_T} = 0$ due to the compact support of φ. By using Taylor expansion we derive

$$
\varphi(t-\Delta t, x, v) = \varphi(t,x,v) - \Delta t \frac{\partial \varphi}{\partial t}(t,x,v) + \frac{\Delta t^2}{2}\frac{\partial^2 \varphi}{\partial t^2}(t,x,v) + \mathcal{O}(\Delta t^3).
$$

Then, using the definition of $\varphi_{i,j}^{n}$ we get

$$
\begin{aligned}
\varphi_{i,j}^{n+1} - \varphi_{i,j}^{n} &= \int_{C_{i,j}} \left[\int_{t^{n+1}}^{t^{n+2}} \varphi(t,x,v)\, dt - \int_{t^n}^{t^{n+1}} \varphi(t,x,v)\, dt \right] dx\, dv \\
&= \int_{C_{i,j}^{n+1}} \big[\varphi(t,x,v) - \varphi(t-\Delta t, x, v)\big]\, dt\, dx\, dv \\
&= \int_{C_{i,j}^{n+1}} \left[\Delta t \frac{\partial \varphi}{\partial t}(t,x,v) - \frac{\Delta t^2}{2}\frac{\partial^2 \varphi}{\partial t^2}(t,x,v) + \mathcal{O}(\Delta t^3) \right] dt\, dx\, dv.
\end{aligned}
$$

Substituting this expression in (4.18), we obtain

$$
\mathfrak{j}_{i,j} = -\sum_{n=0}^{N_T-1} p_{i,j}^{n+1} \int_{C_{i,j}^{n+1}} \left[\Delta t \frac{\partial \varphi}{\partial t} - \frac{\Delta t^2}{2}\frac{\partial^2 \varphi}{\partial t^2} + \mathcal{O}(\Delta t^3) \right] dt\, dx\, dv - p_{i,j}^{0}\varphi_{i,j}^{0}. \tag{4.19}
$$

Substituting (4.19) into (4.17) yields

$$
\begin{aligned}
\mathcal{J}_t^h &= \frac{1}{\Delta t}\sum_{i,j}\Bigg[-\sum_{n=0}^{N_T-1}\left[\int_{C_{i,j}^{n+1}} p_{i,j}^{n+1}\Delta t \frac{\partial \varphi}{\partial t}\, dt\, dx\, dv + \int_{C_{i,j}^{n+1}} p_{i,j}^{n+1}\frac{\Delta t^2}{2}\frac{\partial^2 \varphi}{\partial t^2}\, dt\, dx\, dv \right] \\
&\qquad - p_{i,j}^{0}\varphi_{i,j}^{0} - \sum_{n=1}^{N_T} p_{i,j}^{n}|C_{i,j}|\Delta t^3 \mathcal{O}(1) \Bigg] \\
&= \sum_{i,j}\Bigg[-\sum_{n=1}^{N_T-1}\int_{C_{i,j}^{n}} p_{i,j}^{n}\frac{\partial \varphi}{\partial t}\, dt\, dx\, dv + \sum_{n=0}^{N_T-1}\int_{C_{i,j}^{n+1}} p_{i,j}^{n+1}\frac{\Delta t}{2}\frac{\partial^2 \varphi}{\partial t^2}\, dt\, dx\, dv \\
&\qquad - \frac{1}{\Delta t}p_{i,j}^{0}\varphi_{i,j}^{0} - \sum_{n=1}^{N_T} p_{i,j}^{n}|C_{i,j}|\Delta t^2 \mathcal{O}(1) \Bigg],
\end{aligned}
$$

since the test function has compact support. Rearranging the expression in $\mathcal{I}_t^h$, we find

$$
\begin{aligned}
\mathcal{J}_t^h + \mathcal{I}_t^h = &\sum_{i,j}\Bigg[-\frac{1}{\Delta t}p_{i,j}^{0}\varphi_{i,j}^{0} + p_{i,j}^{0}\int_{C_{ij}}\int_{0}^{t_1}\frac{\partial \varphi}{\partial t}\, dt\, dx\, dv + \int_{C_{ij}} p_0(x,v)\varphi(0,x,v)\, dx\, dv \\
&+ \sum_{n=0}^{N_T-1}\int_{C_{i,j}^{n+1}} p_{i,j}^{n+1}\frac{\Delta t}{2}\frac{\partial^2 \varphi}{\partial t^2}\, dt\, dx\, dv - \sum_{n=1}^{N_T} p_{i,j}^{n}|C_{i,j}|\Delta t^2 \mathcal{O}(1) \Bigg].
\end{aligned}
$$

Now, noting that

$$\frac{1}{\Delta t}\varphi^0_{i,j} = \frac{1}{\Delta t}\int_{C_{i,j}}\int_0^{t^1}\varphi(t,x,v)\,dt\,dx\,dv,$$

we use the Taylor expansion for both $\varphi(t,x,v)$ and $\frac{\partial\varphi}{\partial t}(t,x,v)$ around the point $(0,x,v)$ to get

$$\begin{aligned}
-\frac{1}{\Delta t}&\int_{C_{i,j}}\int_0^{t^1}\varphi(t,x,v)\,dt\,dx\,dv + \int_{C_{i,j}}\int_0^{t^1}\frac{\partial\varphi}{\partial t}(t,x,v)\,dt\,dx\,dv \\
&= -\int_{C_{i,j}}\varphi(0,x,v)\,dx\,dv + \int_{C_{i,j}}\frac{\partial\varphi}{\partial t}(0,x,v)\int_0^{t^1}\left(1-\frac{t}{\Delta t}\right)dt\,dx\,dv \\
&\quad + \int_{C_{i,j}}\frac{\partial^2\varphi}{\partial t^2}(0,x,v)\int_0^{t^1}\left[t\left(1-\frac{t}{2}\frac{1}{\Delta t}\right)\right]dt\,dx\,dv + \mathcal{O}(\Delta t^3\Delta x_i\Delta v_j),
\end{aligned}$$

for $i\in\mathcal{I}$ and $j\in\mathcal{J}$. Therefore, we derive

$$\begin{aligned}
|\mathcal{J}^h_t + \mathcal{I}^h_t| \le \sum_{i,j}\Bigg[&\int_{C_{i,j}}|p_0(x,v)-p^0_{i,j}||\varphi(0,x,v)|\,dx\,dv + p^0_{i,j}\|\frac{\partial\varphi}{\partial t}(0)\|_{L^\infty(Q)}|C_{i,j}|\frac{\Delta t}{2} \\
&+ p^0_{i,j}\|\frac{\partial^2\varphi}{\partial t^2}(0)\|_{L^\infty(Q)}|C_{i,j}|\frac{5}{12}\Delta t^2 + p^0_{i,j}|C_{i,j}|\Delta t^3\mathcal{O}(1) \\
&+ \Delta t\sum_{n=0}^{N_T-1}\frac{p^{n+1}_{i,j}}{2}\|\frac{\partial^2\varphi}{\partial t^2}\|_{L^\infty(Q_T)}|C_{i,j}|\Delta t - \sum_{n=1}^{N_T}p^n_{i,j}|C_{i,j}|\Delta t^2\mathcal{O}(1)\Bigg].
\end{aligned}$$

Thus, we obtain that

$$|\mathcal{J}^h_t + \mathcal{I}^h_t| \le C\|p_0 - p_h(0)\|_{L^1(Q)} + C\Delta t + \mathcal{O}(\Delta t^2) \to 0,$$

as Δt, $h\to 0$.

Estimating $\mathcal{J}^h_x$

Next, let us consider

$$\begin{aligned}
\mathcal{J}^h_x &= \sum_{n,i,j}\frac{[v_j]^-}{\Delta x_i}(p^n_{i,j}-p^n_{i+1,j})\varphi^n_{i,j} + \frac{[v_j]^+}{\Delta x_i}(p^n_{i,j}-p^n_{i-1,j})\varphi^n_{i,j} \\
&= \mathcal{J}^{h,-}_x + \mathcal{J}^{h,+}_x,
\end{aligned}$$

with

$$\mathcal{J}^{h,+}_x := \sum_{n,i,j}\frac{[v_j]^+}{\Delta x_i}(p^n_{i,j}-p^n_{i-1,j})\varphi^n_{i,j}, \quad \text{and} \quad \mathcal{J}^{h,-}_x := \sum_{n,i,j}\frac{[v_j]^-}{\Delta x_i}(p^n_{i,j}-p^n_{i+1,j})\varphi^n_{i,j}.$$

By using Taylor expansion on the test function, we have

$$\int_{x_{i-1/2}}^{x_{i+1/2}}\varphi(t,x,v)\,dx = \varphi(t,x_{i\pm1/2},v)\Delta x_i + \mathcal{O}(\Delta x_i^2).$$

Thus, using integration by parts and the spatial boundary conditions, we obtain

$$\mathcal{J}^{h,+}_x = \sum_{n,j}\sum_{i=0}^{N_x-1}[v_j]^+(p^n_{i,j}-p^n_{i-1,j})\int_{t^n}^{t^{n+1}}\int_{v_{j-1/2}}^{v_{j+1/2}}\varphi(t,x_{i-1/2},v)\,dv\,dt + \mathcal{E}^{h,+}_x, \quad (4.20)$$

where

$$|\mathcal{E}_x^{h,+}| \le C \sum_{n,i,j} |C_{i,j}^n| [v_j]^+ |p_{i,j}^n - p_{i-1,j}^n|. \tag{4.21}$$

Manipulating the first term, we get

$$\begin{aligned}
&\sum_{n,j} [v_j]^+ \Bigg[p_{N_x-1,j}^n \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} \varphi(t, x_{N_x-3/2}, v)\, dv\, dt \\
&\qquad - p_{-1,j}^n \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} \varphi(t, x_{-3/2}, v)\, dv\, dt \\
&\qquad - \sum_{i=-1}^{N_x-2} p_{i,j}^n \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} (\varphi(t, x_{i+1/2}, v) - \varphi(t, x_{i-1/2}, v))\, dv\, dt \Bigg] \\
&= - \sum_{n,j} \sum_{i=0}^{N_x-1} [v_j]^+ p_{i,j}^n \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} (\varphi(t, x_{i+1/2}, v) - \varphi(t, x_{i-1/2}, v))\, dv\, dt \\
&= - \sum_{n,j} \sum_{i=0}^{N_x-1} [v_j]^+ \int_{C_{i,j}^n} p_h(t,x,v) \frac{\partial \varphi}{\partial x}(t,x,v)\, dt\, dx\, dv.
\end{aligned}$$

Substituting this into (4.20), we obtain

$$\mathcal{J}_x^{h,+} = - \sum_{n,i,j} [v_j]^+ \int_{C_{i,j}^n} p_h(t,x,v) \frac{\partial \varphi}{\partial x}(t,x,v)\, dt\, dx\, dv + \mathcal{E}_x^{h,+}. \tag{4.22}$$

Next, let us address $\mathcal{J}_x^{h,-}$. We have

$$\mathcal{J}_x^{h,-} = - \sum_{n,j} \sum_{i=0}^{N_x-1} [v_j]^- (p_{i+1,j}^n - p_{i,j}^n) \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} \varphi(t, x_{i+1/2}, v)\, dv\, dt + \mathcal{E}_x^{h,-}, \tag{4.23}$$

where

$$|\mathcal{E}_x^{h,-}| \le C \sum_{n,i,j} |C_{i,j}^n| [v]^- |p_{i+1,j}^n - p_{i,j}^n|. \tag{4.24}$$

Estimating the first term of $\mathcal{J}_x^{h,-}$, we find

$$\begin{aligned}
&- \sum_{n,j} [v_j]^- \Bigg[p_{N_x,j}^n \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} \varphi(t, x_{N_x-1/2}, v)\, dv\, dt \\
&\qquad - p_{0,j}^n \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} \varphi(t, x_{1/2}, v)\, dv\, dt \\
&\qquad - \sum_{i=1}^{N_x} p_{i,j}^n \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} (\varphi(t, x_{i+1/2}, v) - \varphi(t, x_{i-1/2}, v))\, dv\, dt \Bigg] \\
&= \sum_{n,j} \sum_{i=0}^{N_x-1} [v_j]^- p_{i,j}^n \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} (\varphi(t, x_{i+1/2}, v) - \varphi(t, x_{i-1/2}, v))\, dv\, dt \\
&= \sum_{n,i,j} [v_j]^- \int_{C_{i,j}^n} p_h(t,x,v) \frac{\partial \varphi}{\partial x}(t,x,v)\, dt\, dx\, dv.
\end{aligned}$$

Substituting this into (4.23), we obtain

$$\mathcal{J}_x^{h,-} = \sum_{n,i,j} [v_j]^- \int_{C_{i,j}^n} p_h(t,x,v) \frac{\partial \varphi}{\partial x}(t,x,v)\, dt\, dx\, dv + \mathcal{E}_x^{h,-}. \tag{4.25}$$

Adding up (4.22) and (4.25), we get

$$\begin{aligned}\mathcal{J}_x^h &= -\sum_{n,i,j} \int_{C_{i,j}^n} p_h(t,x,v) v_j \frac{\partial \varphi}{\partial x}(t,x,v)\, dt\, dx\, dv + \mathcal{E}_x^{h,+} + \mathcal{E}_x^{h,-} \\ &= -\sum_{n,i,j} \int_{C_{i,j}^n} p_h(t,x,v) v \frac{\partial \varphi}{\partial x}(t,x,v)\, dt\, dx\, dv + \mathcal{E}_x^{h,+} + \mathcal{E}_x^{h,-} + \mathcal{O}(h).\end{aligned}$$

Thus, we may conclude our estimate by summarising

$$\mathcal{J}_x^h + \mathcal{I}_x^h = \mathcal{E}_x^{h,+} + \mathcal{E}_x^{h,-} + \mathcal{O}(h). \tag{4.26}$$

Estimating $\mathcal{J}_v^h$

We consider

$$\begin{aligned}\mathcal{J}_v^h &= \sum_{n,i,j} \frac{[(\Upsilon_p)_i^n]^-}{\Delta v_j}(p_{i,j}^n - p_{i,j-1}^n)\varphi_{i,j}^n + \frac{[(\Upsilon_p)_i^n]^+}{\Delta v_j}(p_{i,j}^n - p_{i,j+1}^n)\varphi_{i,j}^n \\ &= \mathcal{J}_v^{h,-} + \mathcal{J}_v^{h,+},\end{aligned}$$

where

$$\mathcal{J}_v^{h,+} := \sum_{n,i,j} \frac{[(\Upsilon_p)_i^n]^+}{\Delta v_j}(p_{i,j}^n - p_{i,j+1}^n)\varphi_{i,j}^n, \quad \text{and} \quad \mathcal{J}_v^{h,-} := \sum_{n,i,j} \frac{[(\Upsilon_p)_i^n]^-}{\Delta v_j}(p_{i,j}^n - p_{i,j-1}^n)\varphi_{i,j}^n.$$

Again, we proceed by Taylor expanding the test function, i.e.,

$$\int_{v_{j-1/2}}^{v_{j+1/2}} \varphi(t,x,v)\, dv = \varphi(t,x,v_{j\pm 1/2})\Delta v_j + \mathcal{O}(\Delta v_j^2).$$

Now, let $J \in \mathbb{N}$ such that $\operatorname{supp}(\varphi(t,x,\cdot)) \subset (v_{-J-1/2}, v_{J+1/2})$. Then we have

$$\begin{aligned}\mathcal{J}_v^{h,-} &= \sum_{n,i} \sum_{j=-J}^{J} \frac{[(\Upsilon_p)_i^n]^-}{\Delta v_j}(p_{i,j}^n - p_{i,j-1}^n)\varphi_{i,j}^n \\ &= \sum_{n,i} \sum_{j=-J}^{J} [(\Upsilon_p)_i^n]^-(p_{i,j}^n - p_{i,j-1}^n) \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(t,x,v_{j-1/2})\, dx\, dt + \mathcal{E}_v^{h,-},\end{aligned}$$

where

$$|\mathcal{E}_v^{h,-}| \leq C \sum_{n,i,j} |C_{i,j}^n| [(\Upsilon_p)_i^n]^- |p_{i,j}^n - p_{i,j-1}^n|. \tag{4.27}$$

By manipulating the first term in $\mathcal{J}_v^{h,-}$ we get

$$
\begin{aligned}
\sum_{n,i} [(\Upsilon_p)_i^n]^- \Big[& p_{i,J}^n \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(t,x,v_{J-1/2})\,dx\,dt \\
& - p_{i,-J-1}^n \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(t,x,v_{-J-3/2})\,dx\,dt \\
& - \sum_{j=-J-1}^{J-1} p_{i,j}^n \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} (\varphi(t,x,v_{j+1/2}) - \varphi(t,x,v_{j-1/2}))\,dx\,dt \Big] \\
= -\sum_{n,i} [(\Upsilon_p)_i^n]^- & \sum_{j=-J}^{J} p_{i,j}^n \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} (\varphi(t,x,v_{j+1/2}) - \varphi(t,x,v_{j-1/2}))\,dx\,dt \\
= -\sum_{n,i,j} \int_{C_{i,j}^n} & p_h(t,x,v)[(\Upsilon_p)_i^n]^- \frac{\partial \varphi}{\partial v}(t,x,v)\,dv\,dx\,dt,
\end{aligned}
$$

having used the compact support of the test function. Next, let us consider

$$
\begin{aligned}
\mathcal{J}_v^{h,+} &= -\sum_{n,i,j} \frac{[(\Upsilon_p)_i^n]^+}{\Delta v_j} (p_{i,j+1}^n - p_{i,j}^n)\varphi_{i,j}^n \\
&= -\sum_{n,i} \sum_{j=-J}^{J} [(\Upsilon_p)_i^n]^+ (p_{i,j+1}^n - p_{i,j}^n) \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(t,x,v_{j+1/2})\,dx\,dt + \mathcal{E}_v^{h,+},
\end{aligned}
$$

where

$$
|\mathcal{E}_v^{h,+}| \le C \sum_{n,i,j} |C_{i,j}^n| [(\Upsilon_p)_i^n]^+ \big| p_{i,j+1}^n - p_{i,j}^n \big|. \tag{4.28}
$$

We continue treating the first term of $\mathcal{J}_v^{h,+}$, obtaining

$$
\begin{aligned}
-\sum_{n,i} [(\Upsilon_p)_i^n]^+ \Big[& p_{i,J+1}^n \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(t,x,v_{J+3/2})\,dx\,dt \\
& - p_{i,-J}^n \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(t,x,v_{-J+1/2})\,dx\,dt \\
& - \sum_{j=-J+1}^{J+1} p_{i,j}^n \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} (\varphi(t,x,v_{j+1/2}) - \varphi(t,x,v_{j-1/2}))\,dx\,dt \Big] \\
= \sum_{n,i} [(\Upsilon_p)_i^n]^+ & \sum_{j=-J}^{J} p_{i,j}^n \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{v_{j-1/2}}^{v_{j+1/2}} \frac{\partial \varphi}{\partial v}(t,x,v)\,dv\,dx\,dt \\
= \sum_{n,i,j} \int_{C_{i,j}^n} & p_h(t,x,v)[(\Upsilon_p)_i^n]^+ \frac{\partial \varphi}{\partial v}(t,x,v)\,dv\,dx\,dt.
\end{aligned}
$$

Thus

$$
\mathcal{J}_v^{h,+} = \sum_{n,i,j} \int_{C_{i,j}^n} p_h(t,x,v)[(\Upsilon_p)_i^n]^+ \frac{\partial \varphi}{\partial v}(t,x,v)\,dv\,dx\,dt + \mathcal{E}_v^{h,+}.
$$

In conclusion, we have

$$
\begin{aligned}
\mathcal{J}_v^h &= \mathcal{J}_v^{h,-} + \mathcal{J}_v^{h,-} \\
&= \sum_{n,i,j} \int_{C_{i,j}^n} p_h(t,x,v)[(\Upsilon_p)_i^n] \frac{\partial \varphi}{\partial v}(t,x,v)\, dv\, dx\, dt + \mathcal{E}_v^{h,-} + \mathcal{E}_v^{h,+} \\
&= \sum_{n,i,j} \int_{C_{i,j}^n} p_h(t,x,v)\ (\Upsilon_p)_h \frac{\partial \varphi}{\partial v}(t,x,v)\, dv\, dx\, dt + \mathcal{E}_v^{h,-} + \mathcal{E}_v^{h,+} + \mathcal{O}(h),
\end{aligned}
$$

since

$$
\begin{aligned}
&\left| \sum_{n,i,j} \int_{C_{i,j}^n} p_h(t,x,v)[[(\Upsilon_p)_i^n] - (\Upsilon_p)_h] \frac{\partial \varphi}{\partial v}(t,x,v)\, dv\, dx\, dt \right| \\
&\leq C \sum_{n,i,j} |C_{i,j}^n| p_{i,j}^n | [(\Upsilon_p)_i^n] - (\Upsilon_p)_h| + \mathcal{O}(h).
\end{aligned}
$$

Thus

$$
\mathcal{I}_v^h + \mathcal{J}_v^h = \mathcal{E}_v^{h,+} + \mathcal{E}_v^{h,-} + \mathcal{O}(h).
$$

Combination of all Estimates

We have

$$
\begin{aligned}
e_h &:= |\mathcal{J}_t^h + \mathcal{J}_x^h + \mathcal{J}_v^h + \mathcal{I}_t^h + \mathcal{I}_x^h + \mathcal{I}_v^h| \\
&\leq |\mathcal{J}_t^h + \mathcal{I}_t^h| + |\mathcal{J}_x^h + \mathcal{I}_x^h| + |\mathcal{J}_v^h + \mathcal{I}_v^h| \\
&\leq |\mathcal{E}_x^{h,+}| + |\mathcal{E}_x^{h,-}| + |\mathcal{E}_v^{h,+}| + |\mathcal{E}_v^{h,-}| + \mathcal{O}(h).
\end{aligned}
$$

Using equations (4.21), (4.24), (4.27), (4.28), we have

$$
\begin{aligned}
e_h &\leq \sum_{n,i,j} |C_{i,j}^n| \Big[[v_j]^+ |p_{i,j}^n - p_{i-1,j}^n| + [v]^- |p_{i+1,j}^n - p_{i,j}^n| \\
&\qquad\qquad + [(\Upsilon_p)_i^n]^- |p_{i,j}^n - p_{i,j-1}^n| + [(\Upsilon_p)_i^n]^+ |p_{i,j+1}^n - p_{i,j}^n| \Big] + \mathcal{O}(h) \\
&\leq h \Delta t \sum_{n,i,j} \Big[\Delta v_j [v_j]^+ |p_{i,j}^n - p_{i-1,j}^n| + \Delta v_j [v]^- |p_{i+1,j}^n - p_{i,j}^n| \\
&\qquad\qquad + \Delta x_i [(\Upsilon_p)_i^n]^- |p_{i,j}^n - p_{i,j-1}^n| + \Delta x_i [(\Upsilon_p)_i^n]^+ |p_{i,j+1}^n - p_{i,j}^n| \Big] + \mathcal{O}(h) \\
&\leq h \Delta t \Big[\sum_{n,i,j} \Delta v_j |v_j| + \Delta x_i | [(\Upsilon_p)_i^n] | \Big]^{1/2} \\
&\qquad \times \Big[\sum_{n,i,j} \Delta v_j [v_j]^+ [p_{i,j}^n - p_{i-1,j}^n]^2 + \Delta v_j [v_j]^- [p_{i,j}^n - p_{i+1,j}^n]^2 \\
&\qquad\qquad + \Delta x_i [(\Upsilon_p)_i^n]^+ [p_{i,j}^n - p_{i,j+1}^n]^2 + \Delta x_i [(\Upsilon_p)_i^n]^- [p_{i,j}^n - p_{i,j-1}^n]^2 \Big]^{1/2} + \mathcal{O}(h),
\end{aligned}
$$

having used Cauchy-Schwarz inequality. Since

$$\Big[\sum_{n,i,j} \Delta v_j |v_j| + \Delta x_i |[(\Upsilon_p)_i^n]|\Big]^{1/2} \le C(v_h, \mathcal{C}_{\mathcal{W}}) \Big[\sum_{n,i,j} (\Delta v_j + \Delta x_i)\Big]^{1/2}$$
$$\le C(v_h, L, \mathcal{C}_{\mathcal{W}}, T) \frac{1}{h^{1/2}} \frac{1}{\Delta t^{1/2}},$$

we have

$$e_h \le C h^{1/2} \Delta t^{1/2} \mathcal{R}^{1/2}, \tag{4.29}$$

where

$$\mathcal{R} := \sum_{n,i,j} \Delta v_j [v_j]^+ |p_{i-1,j}^n - p_{i,j}^n|^2 + \Delta v_j [v_j]^- |p_{i+1,j}^n - p_{i,j}^n|^2$$
$$+ \Delta x_i [(\Upsilon_p)_i^n]^+ |p_{i,j+1}^n - p_{i,j}^n|^2 + \Delta x_i [(\Upsilon_p)_i^n]^- |p_{i,j-1}^n - p_{i,j}^n|^2.$$

Using the fact that

$$|\hat{p} - p_{i,j}^n|^2 = 2\big(p_{i,j}^n - \hat{p}\big) p_{i,j}^n + \hat{p}^2 - \big|p_{i,j}^n\big|^2,$$

in particular for $\hat{p} \in \{p_{i\pm1,j}^n, p_{i,j\pm1}\}$, we may rewrite $\mathcal{R}$ such that

$$\mathcal{R} = 2\sum_{n,i,j} \Big[\Delta v_j [v_j]^+ [p_{i,j}^n - p_{i-1,j}^n] p_{i,j}^n + \Delta v_j [v_j]^- [p_{i,j}^n - p_{i+1,j}^n] p_{i,j}^n$$
$$+ \Delta x_i [(\Upsilon_p)_i^n]^- [p_{i,j}^n - p_{i,j-1}^n] p_{i,j}^n + \Delta x_i [(\Upsilon_p)_i^n]^+ [p_{i,j}^n - p_{i,j+1}^n] p_{i,j}^n\Big]$$
$$+ \sum_{n,i,j} \Big[\Delta v_j [v_j]^+ (|p_{i-1,j}^n|^2 - |p_{i,j}^n|^2) + \Delta v_j [v_j]^- (|p_{i+1,j}^n|^2 - |p_{i,j}^n|^2)$$
$$+ \Delta x_i [(\Upsilon_p)_i^n]^- (|p_{i,j-1}^n|^2 - |p_{i,j}^n|^2) + \Delta x_i [(\Upsilon_p)_i^n]^+ (|p_{i,j+1}^n|^2 - |p_{i,j}^n|^2)\Big].$$

We observe that the last summation contains telescopic sums such that, indeed,

$$\mathcal{R} \le 2\sum_{n,i,j} p_{i,j}^n \Big[\Delta v_j [v_j]^+ [p_{i,j}^n - p_{i-1,j}^n] + \Delta v_j [v_j]^- [p_{i,j}^n - p_{i+1,j}^n]$$
$$+ \Delta x_i [(\Upsilon_p)_i^n]^- [p_{i,j}^n - p_{i,j-1}^n] + \Delta x_i [(\Upsilon_p)_i^n]^+ [p_{i,j}^n - p_{i,j+1}^n]\Big]$$
$$+ \sum_{n,i} \Big[\Delta x_i [(\Upsilon_p)_i^n]^+ (|p_{i,-J}^n|^2 - |p_{i,J+1}^n|^2) + \Delta x_i [(\Upsilon_p)_i^n]^- (|p_{i,-J-1}^n|^2 - |p_{i,J}^n|^2)\Big],$$

where we factored out a $p_{i,j}^n$ in the first term. Using the scheme (4.15), we see that

$$\mathcal{R} \le 2\sum_{n,i,j} |C_{i,j}| p_{i,j}^n \frac{p_{i,j}^{n+1} - p_{i,j}^n}{\Delta t} + \frac{C}{\Delta t},$$

where the last term comes from bounding the boundary terms, i.e., the second sum in the previous equation. By convexity of $x \mapsto x^2$, we can estimate further to get

$$\mathcal{R} \leq \sum_{n,i,j} \frac{|C_{i,j}|}{\Delta t} \left(\left(p_{i,j}^{n+1}\right)^2 - \left(p_{i,j}^{n}\right)^2 \right) + \frac{C}{\Delta t} \leq \frac{C}{\Delta t}. \tag{4.30}$$

Substituting (4.30) into (4.29), we finally obtain

$$e_h \leq C h^{1/2},$$

which goes to zero, as $h \to 0$. Therefore, we have established (4.16), and thus

$$\int_{Q_T} p_h \left(\frac{\partial \varphi}{\partial t} + v \frac{\partial \varphi}{\partial x} - (\Upsilon_p)_h \frac{\partial \varphi}{\partial v} \right) dt\, dx\, dv + \int_{Q} p_0(x,v) \varphi(0,x,v)\, dx\, dv \to 0$$

as $\Delta t, h \to 0$. Then the limit (f, g) of (f_h, g_h) is a solution to the weak formulation of system (4.1). □

Chapter 5

One-dimensional second order system with two species

In this Chapter we study a second order system with two species subject to nonlocal interactions and linear damping. We prove that, under smoothness assumptions on the potentials, a unique measure solution exists. We then consider a large-time large-damping scaled version of the system and prove convergence towards the solution to the corresponding first order system. Finally, we consider the case of self-interaction potentials driven by Newtonian potentials and external coercive potentials. After providing an existence result, we prove a collapse result, showing that for large times the solutions converge to Dirac delta measures. We complement the results with numerical simulations.

5.1 The model

The system we deal with is

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho v) = 0, \\ \dfrac{\partial \eta}{\partial t} + \dfrac{\partial}{\partial x}(\eta w) = 0, \\ \dfrac{\partial}{\partial t}(\rho v) + \dfrac{\partial}{\partial x}(\rho v^2) = -\sigma \rho v - \rho[K_{11}' * \rho + K_{12}' * \eta], \\ \dfrac{\partial}{\partial t}(\eta w) + \dfrac{\partial}{\partial x}(\eta w^2) = -\sigma \eta w - \eta[K_{22}' * \eta + K_{21}' * \rho], \end{cases} \tag{5.1}$$

equipped with initial data

$$\begin{cases} (\rho, v)(t=0) = (\overline{\rho}, \overline{v}), \\ (\eta, w)(t=0) = (\overline{\eta}, \overline{w}). \end{cases} \tag{5.2}$$

In system (5.1), $\rho(t,x)$ and $\eta(t,x)$ are probability measures modelling two species of agents, or individuals, $v(t,x)$ and $w(t,x)$ are the corresponding Eulerian velocities of the two species, $\sigma > 0$ is the damping parameter, K_{ij} are smooth (to an extent to be specified later) given space-depending potentials. The convolutions in (5.1) are meant with respect to the space variable.

System (5.1) has a natural discrete *particle* counterpart. Let us consider $x_1, \dots, x_N$ as N particles of the first species with masses $m_1, \dots, m_N$, and $y_1, \dots, y_M$ as M particles of the second species with masses $n_1, \dots, n_M$. The dynamics of x_i and y_j is determined by the following equations

$$\begin{cases} \ddot{x}_i = -\sigma \dot{x}_i - \sum_{k \neq i} m_k K'_{11}(x_i - x_k) - \sum_k n_k K'_{12}(x_i - y_k), \\ \ddot{y}_j = -\sigma \dot{y}_j - \sum_{k \neq j} n_k K'_{21}(y_j - y_k) - \sum_k m_k K'_{22}(y_j - x_k), \end{cases} \tag{5.3}$$

with $i = 1, \dots, N$ and $j = 1, \dots, M$ and the following initial data

$$\begin{cases} x_i(0) = \overline{x}_i, \\ \dot{x}_i(0) = \overline{v}_i, \end{cases} \qquad \begin{cases} y_j(0) = \overline{y}_j, \\ \dot{y}_j(0) = \overline{w}_j. \end{cases}$$

5.2 Main assumptions and particle system

In what follows we will set the assumptions and introduce definitions. Then, we precise description of system (5.1) in terms of particles and Lagrangian coordinates respectively. We also provide a formal argument for the large damping limit of system (5.1) towards the corresponding first order system.

We point out here that, since we are dealing with a two-species system, we will work on the product space $\mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R})$, where $\mathcal{P}_2(\mathbb{R})$ is the set of probability measures with finite second moment. For all $\boldsymbol{\mu} = (\mu_1, \mu_2)$, $\boldsymbol{\nu} = (\nu_1, \nu_2) \in \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R})$, we define the *product Wasserstein distance* as

$$\mathcal{W}_2^2(\boldsymbol{\mu}, \boldsymbol{\nu}) = W_2^2(\mu_1, \nu_1) + W_2^2(\mu_2, \nu_2),$$

where W_2 is the 2-Wasserstein distance introduced in Section 1.6. See Subsection 1.6.1 for the description of the 2-Wasserstein distance in the one-dimensional case and the bijection between the space of probability measures on $\mathbb{R}$ with finite second moment and the convex cone $\mathcal{K}$ of the non-decreasing $L^2(\Omega)$-functions, with $\Omega := (0, 1)$.

5.2.1 Main assumptions

Let us start by specifying the class of interaction potentials we are going to use.

Definition 5.1. A function $K : \mathbb{R} \to \mathbb{R}$ is called an *admissible potential* if

$$K \in W^{2,\infty}(\mathbb{R}),\ K(0) = 0 \text{ and } K(-x) = K(x). \tag{\textbf{A}}$$

An admissible potential K is said to be *sub-quadratic at infinity* if there exists a constant $C > 0$ such that

$$K(x) \leq C(1 + |x|^2) \quad \text{for all } x \in \mathbb{R}. \tag{\textbf{SQ}}$$

An admissible potential K has a *sub-linear gradient* if there exists $C > 0$ such that

$$K'(x) \leq C(1 + |x|) \quad \text{for all } x \in \mathbb{R}. \tag{\textbf{SL}}$$

We call an admissible potential *attractive* if

$$K(x) = k(|x|) \geq 0,\ \text{for all } x \in \mathbb{R} \text{ and } K'(r)r \geq 0 \text{ for all } r \in \mathbb{R}. \tag{\textbf{AT}}$$

In Section 5.5 we will also take into account the action of *external* potentials in the dynamics. More precisely, we consider $A \in \mathcal{C}^2(\mathbb{R})$ and assume that there exist the positive constants λ and α such that

$$A(x) \geq \lambda |x|^2 \tag{\textbf{H1}}$$

and

$$xA'(x) \geq \alpha |x|^2 \tag{\textbf{H2}}$$

for all $x \in \mathbb{R}$.

Denoting with $\langle \cdot, \cdot \rangle_{L^2(\Omega)^2}$ the inner product on the space $L^2(\Omega)^2$, that is

$$\langle Z_1, Z_2 \rangle_{L^2(\Omega)^2} = \int_\Omega \big[X_1(s) X_2(s) + Y_1(s) Y_2(s) \big] \, ds,$$

for $Z_1 = (X_1, Y_1)$ and $Z_2 = (X_2, Y_2)$ in $L^2(\Omega)^2$, we recall below the notion of Fréchet sub-differential for a generic operator $\mathfrak{F}$ on a general Hilbert space.

Definition 5.2. Let H be a Hilbert space. For a given, proper and lower semi-continuous functional $\mathfrak{F} : H \to (-\infty, +\infty]$, we say that $Z \in H$ belongs to the sub-differential of $\mathfrak{F}$ at $\widetilde{Z} \in H$ if and only if

$$\mathfrak{F}(R) - \mathfrak{F}(\widetilde{Z}) \geq \langle Z, R - \widetilde{Z} \rangle_H + o(\|R - \widetilde{Z}\|),$$

as $\|R - \widetilde{Z}\| \to 0$, with $R \in H$. The sub-differential of $\mathfrak{F}$ at $\widetilde{Z}$ is denoted by $\partial \mathfrak{F}(\widetilde{Z})$.

In particular, we will usually consider as Hilbert spaces $H = L^2(\Omega)$ or $H = L^2(\Omega)^2$.

Let $I_\mathcal{K} : L^2(\Omega) \to [0, +\infty)$ be the indicator function of the L^2-convex cone $\mathcal{K}$ introduced in (1.15), that is

$$I_\mathcal{K}(X) = \begin{cases} 0 & \text{if } X \in \mathcal{K}, \\ +\infty & \text{otherwise.} \end{cases}$$

For a given $X \in L^2(\Omega)$, the sub-differential of $I_\mathcal{K}$ in X is given by

$$\partial I_\mathcal{K}(X) = \left\{ Z \in L^2(\Omega) \, : \, I_\mathcal{K}(\widetilde{X}) \geq I_\mathcal{K}(X) + \int_\Omega Z(\widetilde{X} - X) dm, \text{ for all } \widetilde{X} \in \mathcal{K} \right\},$$

or in its alternative form

$$\partial I_\mathcal{K}(X) = \begin{cases} \{ Z \in L^2(\Omega) \, : \, 0 \geq \int_\Omega Z(\widetilde{X} - X) dm, \text{ for all } \widetilde{X} \in \mathcal{K} \}, & \text{if } X \in \mathcal{K}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The definitions above can be easily extended to any Hilbert space H different form L^2, as sometimes required.

We conclude this Subsection with the following definition, which we borrow from [12].

Definition 5.3. An operator $F : \mathcal{K} \to L^2(\Omega)$ is *bounded* if there exists a constant $C \geq 0$ such that

$$\|F[X]\|_{L^2(\Omega)} \leq C(1 + \|X\|_{L^2(\Omega)}) \qquad \text{for all } X \in \mathcal{K}.$$

An operator $F : \mathcal{K} \to L^2(\Omega)$ is *pointwise linearly bounded* if there exists a constant $C_p \geq 0$ such that

$$|F[X](m)| \leq C_p\big(1 + |X(m)| + \|X\|_{L^1(\Omega)}\big) \qquad \text{for a.e. } m \in \Omega \text{ and all } X \in \mathcal{K}.$$

An operator $F : \mathcal{K} \to L^2(\Omega)$ is *uniformly continuous* if there exists a modulus of continuity ω such that

$$\|F[X_1] - F[X_2]\|_{L^2(\Omega)} \leq \omega\big(\|X_1 - X_2\|_{L^2(\Omega)}\big) \qquad \text{for all } X_1, X_2 \in \mathcal{K}.$$

5.2.2 Particles system

We dedicate this Subsection to the study of sticky solutions in the finite dimensional case. Let $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and $y = (y_1, \ldots, y_M) \in \mathbb{R}^M$ be the positions of particles of the first and second species respectively. Since the particles do not overtake each other, the "sticky" condition preserves the ordering of the particles. Therefore their evolution is confined in the closed convex set

$$\mathbb{K}^N \times \mathbb{K}^M = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^M \,:\, x_1 \leq \ldots \leq x_N,\, y_1 \leq \ldots \leq y_M\}.$$

Setting $v = (v_1, \ldots, v_N) \in \mathbb{R}^N$ and $w = (w_1, \ldots, w_M) \in \mathbb{R}^M$ as the velocity vectors of particles of the first species and second species respectively, we consider the following system

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{y}_j(t) = w_j(t), \\ \dot{v}_i(t) = a_i(x(t)) + b_i(x(t), y(t)) - \sigma v_i(t), \\ \dot{w}_j(t) = c_j(y(t)) + d_j((x(t), y(t)) - \sigma w_j(t), \end{cases} \tag{5.4}$$

for $i = 1, \ldots, N$ and $j = 1, \ldots, M$. In system (5.4),

$$a_i(x) = -\sum_{k=1}^{N} m_k K'_\rho(x_i - x_k)\,, \quad b_i(x, y) = -\sum_{k=1}^{M} n_k H'_\rho(x_i - y_k)\,,$$
$$c_j(y) = -\sum_{k=1}^{M} n_k K'_\eta(y_i - y_k)\,, \quad d_j(x, y) = -\sum_{k=1}^{N} m_k H'_\eta(y_j - x_k)\,.$$

for $i = 1, \ldots, N$ and $j = 1, \ldots, M$. The i-th component of the vector field

$$a(x) : x \in \mathbb{K}^N \to \big(a_1(x), \ldots, a_N(x)\big) \in \mathbb{R}^N$$

models the interactions between particles of the first species and the i-th particle of the first species, while the i-th component of the vector field

$$b(x, y) : (x, y) \in \mathbb{K}^N \times \mathbb{K}^M \to (b_1(x, y), \ldots, b_N(x, y)) \in \mathbb{R}^N$$

describes the interactions between the i-th particle of the first species and particles of the second species. Similarly one can describe the j-th component of the terms

$$c(y) : y \in \mathbb{K}^M \to (c_1(y), \ldots, c_M(y)) \in \mathbb{R}^M,$$

and

$$d(x, y) : (x, y) \in \mathbb{K}^N \times \mathbb{K}^M \to (d_1(x, y), \ldots, d_M(x, y)) \in \mathbb{R}^M,$$

respectively.

Assuming that all the potentials in (5.4) are smooth enough (for example with $\mathcal{C}^2$ regularity), a unique solution to (5.4) exists as long as particles occupy distinct positions. When two or more particles collide, we apply the concept of sticky particle solution sketched in the introduction, Section 1.3. Following [12, 51], the precise formalisation of sticky collisions requires the definition of the following normal cones

$$\begin{aligned} N_x\mathbb{K}^N &:= \{l \in \mathbb{R}^N \,:\, l\cdot(\widetilde{x}-x) \le 0 \quad \text{for all } \widetilde{x} \in \mathbb{K}^N\}, \\ N_y\mathbb{K}^M &:= \{n \in \mathbb{R}^M \,:\, n\cdot(\widetilde{y}-y) \le 0 \quad \text{for all } \widetilde{y} \in \mathbb{K}^M\}. \end{aligned}$$

Note that the normal cone $N_x\mathbb{K}^N$ is equal to the sub-differential $\partial I_{\mathbb{K}^N}(x)$ of the indicator function of $\mathbb{K}^N$ at the point x. When two particles of the same species collide, an instantaneous force is released and the respective particles velocities evolve as elements of the normal cones $N_x\mathbb{K}^N$ and $N_y\mathbb{K}^M$ respectively. Given these premises, we can consider the second order system of differential inclusions

$$\begin{cases} \dot{x} = v, \\ \dot{y} = w, \\ \dot{v} + N_x\mathbb{K}^N \ni a(x) + b(x,y) - \sigma v, \\ \dot{w} + N_y\mathbb{K}^M \ni c(y) + d(x,y) - \sigma w. \end{cases} \tag{5.5}$$

System (5.5) is justified as follows. Introducing the vector $\mathcal{W}(t) = (V(t), W(t)) = e^{\sigma t}(v(t), w(t))$, from (5.4) we get

$$\dot{\mathcal{W}}(t) = e^{\sigma t}\mathcal{A}(x(t), y(t))\,,$$

where $\mathcal{A}(x,y)$ is the vector in $\mathbb{R}^{N+M}$ with components $a(x)+b(x,y)$ and $c(y)+d(x,y)$ respectively. Now, due to the smoothness of the interaction potentials, the vector field $\mathcal{A}(x,y)$ can be extended by continuity to the boundary of the cone $\mathbb{K}^N \times \mathbb{K}^M$. Therefore, as $\mathcal{W}$ and (v,w) only differ by a scalar factor, a suitable modified version of the differential equation for $\mathcal{W}$ that keeps the dynamics in $\mathbb{K}^N \times \mathbb{K}^M$ is the differential inclusion

$$\dot{\mathcal{W}}(t) \in e^{\sigma t}\mathcal{A}(x(t), y(t)) + N_{x(t)}\mathbb{K}^N \times N_{y(t)}\mathbb{K}^M\,,$$

which easily yields the last two differential inclusions in (5.5).

According to [12], if $x : [0,\infty) \to \mathbb{K}^N$ satisfies the *global sticky condition*, i.e., particles are not allowed to split after colliding, then the following monotonicity property on the family of normal cones $N_{x(t)}\mathbb{K}^N$ holds:

$$N_{x(s)}\mathbb{K}^N \subset N_{x(t)}\mathbb{K}^N \qquad \text{for all } s < t.$$

Hence, for any function $\zeta : [0,\infty) \to \mathbb{R}^N$ such that $\zeta(t) \in N_{x(t)}\mathbb{K}^N$, we have

$$\int_s^t \zeta(r)\,dr \in N_{x(t)}\mathbb{K}^N \qquad \text{for all } s < t.$$

Consequently, integrating the last two equations in (5.5) on a time interval $[s,t]$, one obtains

$$v(t) + \sigma x(t) + N_{x(t)}\mathbb{K}^N \ni v(s) + \sigma x(s) + \int_s^t a(x(r))\,dr + \int_s^t b(x(r), y(r))\,dr, \tag{5.6}$$

and

$$w(t) + \sigma y(t) + N_{y(t)}\mathbb{K}^M \ni w(s) + \sigma y(s) + \int_s^t c(y(r))\,dr + \int_s^t d(x(r), y(r))\,dr. \quad (5.7)$$

System (5.5), together with (5.6) and (5.7), can be rewritten in a more compact form in the new variables (x, y, p, q) where p and q are defined by

$$\begin{aligned} p(t) &= \int_s^t a(x(r))\,dr + \int_s^t b(x(r), y(r))\,dr + v(s) + \sigma x(s), \\ q(t) &= \int_s^t c(y(r))\,dr + \int_s^t d(x(r), y(r))\,dr + w(s) + \sigma y(s), \end{aligned}$$

yielding the following first order system of differential inclusions

$$\begin{cases} \dot{x} + \sigma x + N_x\mathbb{K}^N \ni p, \\ \dot{y} + \sigma y + N_y\mathbb{K}^M \ni q, \\ \dot{p} = a(x) + b(x, y), \\ \dot{q} = c(y) + d(x, y), \end{cases}$$

with the additional characterisation of v and w in terms of p and q given by

$$\begin{aligned} v(t) + \sigma \int_s^t v(r)\,dr + N_x\mathbb{K}^N \ni p(t), \\ w(t) + \sigma \int_s^t w(r)\,dr + N_y\mathbb{K}^M \ni q(t). \end{aligned}$$

5.2.3 Time scaling and formal large damping limit

One of the purposes is to study system (5.1) in the *large time / large damping regime*, namely we aim to send $\sigma \to +\infty$ in (5.1) after having suitably rescaled the time variable. We start performing the scaling at the level of particles, namely for system (5.3). Consider the new time variable τ defined by

$$\tau = \frac{t}{\sigma}, \quad (5.8)$$

and introduce the scaled particle trajectories as follows:

$$\begin{aligned} x_i(t) = \chi_i(\tau) = \chi_i(t/\sigma), \\ y_j(t) = \xi_j(\tau) = \xi_j(t/\sigma), \end{aligned}$$

as $i = 1, \ldots, N$ and $j = 1, \ldots, M$. Notice that we can scale the initial velocities accordingly as

$$\dot{\chi}_i(0) := \overline{\nu}_i = \sigma\overline{v}_i, \qquad \dot{\xi}_j(0) := \overline{\omega}_j = \sigma\overline{w}_j.$$

Hence, system (5.3) becomes

$$\begin{aligned} \sigma^{-2}\ddot{\chi}_i(\tau) &= -\dot{\chi}_i(\tau) - \sum_{k \neq i} m_k K'_\rho\big(\chi_i(\tau) - \chi_k(\tau)\big) - \sum_k n_k H'_\rho\big(\chi_i(\tau) - \xi_k(\tau)\big), \\ \sigma^{-2}\ddot{\xi}_j(\tau) &= -\dot{\xi}_j(\tau) - \sum_{k \neq j} n_k K'_\eta\big(\xi_j(\tau) - \xi_k(\tau)\big) - \sum_k m_k H'_\eta\big(\xi_j(\tau) - \chi_k(\tau)\big). \end{aligned}$$

A formal limit $\sigma \to +\infty$ leads to the following first order system of differential equations for particle positions

$$\dot{\chi}_i(\tau) = -\sum_{k\neq i} m_k K'_\rho\big(\chi_i(\tau) - \chi_k(\tau)\big) - \sum_k n_k H'_\rho\big(\chi_i(\tau) - \xi_k(\tau)\big),$$
$$\dot{\xi}_j(\tau) = -\sum_{k\neq j} n_k K'_\eta\big(\xi_j(\tau) - \xi_k(\tau)\big) - \sum_k m_k H'_\eta\big(\xi_j(\tau) - \chi_k(\tau)\big).$$

A similar time scaling can be performed at the level of (5.1). Using the definition of τ in (5.8) and considering $(\widetilde{\rho}, \widetilde{v}, \widetilde{\eta}, \widetilde{w})$ solution to

$$\begin{cases} \dfrac{\partial \widetilde{\rho}}{\partial t} + \dfrac{\partial}{\partial x}(\widetilde{\rho}\widetilde{v}) = 0, \\ \dfrac{\partial \widetilde{\eta}}{\partial t} + \dfrac{\partial}{\partial x}(\widetilde{\eta}\widetilde{w}) = 0, \\ \dfrac{\partial}{\partial t}(\widetilde{\rho}\widetilde{v}) + \dfrac{\partial}{\partial x}(\widetilde{\rho}\widetilde{v}^2) = -\sigma\widetilde{\rho}\widetilde{v} - \widetilde{\rho}[K'_{11} * \widetilde{\rho} + K'_{12} * \widetilde{\eta}], \\ \dfrac{\partial}{\partial t}(\widetilde{\eta}\widetilde{w}) + \dfrac{\partial}{\partial x}(\widetilde{\eta}\widetilde{w}^2) = -\sigma\widetilde{\eta}\widetilde{w} - \widetilde{\eta}[K'_{22} * \widetilde{\eta} + K'_{21} * \widetilde{\rho}], \end{cases}$$

we can introduce the rescaled densities and velocities as

$$\rho(\tau, x) = \widetilde{\rho}(t, x), \qquad v(\tau, x) = \sigma\widetilde{v}(t, x),$$
$$\eta(\tau, x) = \widetilde{\eta}(t, x), \qquad w(\tau, x) = \sigma\widetilde{w}(t, x).$$

Then the quadruple (ρ, v, η, w) solves

$$\begin{cases} \dfrac{\partial \rho}{\partial \tau} + \dfrac{\partial}{\partial x}(\rho v) = 0, \\ \dfrac{\partial \eta}{\partial \tau} + \dfrac{\partial}{\partial x}(\eta w) = 0, \\ \sigma^{-2}\left[\dfrac{\partial}{\partial \tau}(\rho v) + \dfrac{\partial}{\partial x}(\rho v^2)\right] = -\rho v - \rho[K'_{11} * \rho + K'_{12} * \eta], \\ \sigma^{-2}\left[\dfrac{\partial}{\partial \tau}(\eta w) + \dfrac{\partial}{\partial x}(\eta w^2)\right] = -\eta w - \eta[K'_{22} * \eta + K'_{21} * \rho], \end{cases} \tag{5.9}$$

and formally, as $\sigma \to \infty$, we get the first order system

$$\begin{cases} \dfrac{\partial \rho}{\partial \tau} = \dfrac{\partial}{\partial x}\big[\rho K'_{11} * \rho + \rho K'_{12} * \eta\big], \\ \dfrac{\partial \eta}{\partial \tau} = \dfrac{\partial}{\partial x}\big[\eta K'_{22} * \eta + \eta K'_{21} * \rho\big]. \end{cases} \tag{5.10}$$

5.2.4 Lagrangian description of the continuum model

We now transpose the considerations above in terms of a Lagrangian description for system (5.1). For any $X \in \mathcal{K}$, where $\mathcal{K}$ denotes the convex cone introduced in (1.15), we define the set

$$\Omega_X := \{m \in \Omega \,:\, X \text{ is constant in an open neighborhood of } m\}, \tag{5.11}$$

and the closed subspace

$$\mathcal{H}_X = \{Z \in L^2(0,1) \,:\, Z \text{ is constant on each interval } (a,b) \in \Omega_X\}. \tag{5.12}$$

A crucial quantity in the following analysis is the projection $\mathsf{P}_{\mathcal{H}_X} : L^2 \to \mathcal{H}_X$ given by

$$\mathsf{P}_{\mathcal{H}_X}(U) = \begin{cases} \fint_a^b U(m)\,dm & \text{in any maximal interval } (a,b) \subset \Omega_X, \\ U & \text{a.e. in } \Omega \setminus \Omega_X, \end{cases} \tag{5.13}$$

for all $U \in L^2(\Omega)$. The proof of the following Lemma is an easy consequence of Jensen's inequality, see [12, Lemma 2.2].

Lemma 5.1 ($\mathcal{H}_X$-contraction). *Let $\psi : \mathbb{R} \to [0,\infty)$ be a convex lower semi-continuous function. Then $\mathsf{P}_{\mathcal{H}_X}$ is dominated by X, namely*

$$\int_\Omega \psi\big(\mathsf{P}_{\mathcal{H}_X}(Y)\big)\,dm \le \int_\Omega \psi(Y)\,dm \qquad \text{for all } X \in \mathcal{K} \text{ and all } Y \in L^2(\Omega),$$

and we write $\mathsf{P}_{\mathcal{H}_X} \prec X$.

Consider a quadruple (ρ,η,v,w) solution to (5.1) and define the maps $X, Y : [0,\infty)\times\Omega \to \mathbb{R}$ and the velocities $V, W : [0,\infty)\times\Omega \to \mathbb{R}$ as follows

$$\begin{aligned} X(t,\cdot) &= \Psi(\rho(t,\cdot)), & V(t,\cdot) &= v(t,X(t,\cdot)) = \partial_t X(t,\cdot), \\ Y(t,\cdot) &= \Psi(\eta(t,\cdot)), & W(t,\cdot) &= w(t,Y(t,\cdot)) = \partial_t Y(t,\cdot), \end{aligned}$$

where Ψ is the isometry defined in (1.16) that associates to a probability measure its monotone rearrangement. In the new unknowns (X,Y,V,W), system (5.1) can be (formally) rephrased as

$$\begin{cases} \partial_t X(t) = V(t), \\ \partial_t Y(t) = W(t), \\ \partial_t V(t) = -\displaystyle\int_\Omega K_{11}'\big(X(m)-X(m')\big)\,dm' \\ \qquad\qquad - \displaystyle\int_\Omega K_{12}'\big(X(m)-X(m')\big)\,dm' - \sigma V(t), \\ \partial_t Y(t) = -\displaystyle\int_\Omega K_{21}'\big(Y(m)-Y(m')\big)\,dm' \\ \qquad\qquad - \displaystyle\int_\Omega K_{22}'\big(Y(m)-X(m')\big)\,dm' - \sigma W(t). \end{cases}$$

Similarly to Subsection 5.2.2, one can show that the previous system can be reformulated in terms of differential inclusions to incorporate particles collisions. Moreover, since we will investigate on the large-damping limit, through the chapter we consider the Lagrangian counterpart of the rescaled system (5.9). Then, according to the previous calculations, we get the system

$$\begin{cases} \varepsilon\dot{X}(t,m) + X(t,m) + \partial I_{\mathcal{K}}(X(t,m)) \ni \varepsilon\overline{V}(m) + \overline{X}(m) \\ \qquad\qquad\qquad\qquad + \displaystyle\int_0^t F[X(\cdot,r),Y(\cdot,r)](m)\,dr, \\ \varepsilon\dot{Y}(t,m) + Y(t,m) + \partial I_{\mathcal{K}}(Y(t,m)) \ni \varepsilon\overline{W}(m) + \overline{Y}(m) \\ \qquad\qquad\qquad\qquad + \displaystyle\int_0^t G[X(\cdot,r),Y(\cdot,r)](m)\,dr, \end{cases} \tag{5.14}$$

with $\varepsilon := \sigma^{-2}$ and where we have denoted by

$$F : \mathcal{K} \times \mathcal{K} \to L^2(\Omega) \ \text{ and } \ G : \mathcal{K} \times \mathcal{K} \to L^2(\Omega)$$

the operators

$$\begin{aligned} F[X,Y](m) = &- \int_\Omega K'_{11}\big(X(r,m) - X(r,m')\big)\, dm' \\ &- \int_\Omega K'_{12}\big(X(r,m) - Y(r,m')\big)\, dm', \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} G[X,Y](m) = &- \int_\Omega K'_{22}\big(Y(r,m) - Y(r,m')\big)\, dm' \\ &- \int_\Omega K'_{21}\big(Y(r,m) - X(r,m')\big)\, dm' . \end{aligned} \tag{5.16}$$

We observe that if K_{ij} are $\mathcal{C}^1$ functions that satisfy (**A**) and (**SL**) then the two operator F and G defined in (5.15) and (5.16) are uniformly continuous and bounded according to Definition 5.3.

Notice also that the parameter $\varepsilon = \sigma^{-2}$ is an *inertia* parameter; thus a *large damping limit* corresponds to a *small inertia limit*, i.e., send $\sigma \to \infty$ means to consider $\varepsilon \to 0$.

Definition 5.4 (Lagrangian solutions)**.** Let K_{11}, K_{12}, K_{21}, $K_{22} \in \mathcal{C}^1(\mathbb{R})$ potentials satisfying (**A**) and (**SL**). Let $\overline{X}, \overline{Y} \in \mathcal{K}$ and $\overline{V}, \overline{W} \in L^2(\Omega)$ be given. A *Lagrangian solution* to (5.14) with initial data $(\overline{X}, \overline{Y}, \overline{V}, \overline{W})$ is a pair $(X,Y) \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty);\mathcal{K}) \times \mathrm{Lip}_{\mathrm{loc}}([0,\infty);\mathcal{K})$ satisfying $X(0) = \overline{X}$, $Y(0) = \overline{Y}$ and (5.14) for a.e. $t \in [0,\infty)$.

In order to consider the case of Newtonian potentials, we introduce the following notion of *generalised Lagrangian solutions* for system (5.14) under globally sticky dynamics, see [12].

Definition 5.5. A *generalised solution* to the system (5.14) is a pair

$$(X,Y) \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty);\mathcal{K}) \times \mathrm{Lip}_{\mathrm{loc}}([0,\infty);\mathcal{K})$$

such that

1. *Differential inclusion:*

$$\begin{cases} \varepsilon\dot{X}(t) + X(t) + \partial I_{\mathcal{K}}(X(t)) \ni \varepsilon\overline{V} + \overline{X} + \displaystyle\int_0^t \Theta(s)\, ds, \\ \varepsilon\dot{Y}(t) + Y(t) + \partial I_{\mathcal{K}}(Y(t)) \ni \varepsilon\overline{W} + \overline{Y} + \displaystyle\int_0^t \Xi(s)\, ds, \end{cases}$$

holds for a.e. $t \in (0,\infty)$, for some maps

$$\Theta, \Xi \in L^\infty_{\mathrm{loc}}([0,\infty); L^2(\Omega)) \times L^\infty_{\mathrm{loc}}([0,\infty); L^2(\Omega))$$

with

$$\Theta - F[X(t),Y(t)] \in H^\perp_{X(t)} \quad \text{and} \quad \Theta \prec F[X(t),Y(t)] \quad \text{for a.e. } t \in (0,\infty), \tag{5.17}$$

and, similarly,

$$\Xi - G[X(t),Y(t)] \in H^\perp_{Y(t)} \quad \text{and} \quad \Xi \prec G[X(t),Y(t)] \quad \text{for a.e. } t \in (0,\infty), \tag{5.18}$$

where $F[X(t),Y(t)]$ and $G[X(t),Y(t)]$ are the operators defined in (5.15) and (5.16).

2. *Semigroup property:* for all $t \geq t_1 \geq 0$, the right derivatives $V = \frac{d^+}{dt}X$ and $W = \frac{d^+}{dt}Y$ satisfy

$$
\begin{cases} \varepsilon V(t) + X(t) + \partial I_{\mathcal{K}}(X(t)) \ni \varepsilon V(t_1) + X(t_1) + \int_{t_1}^{t} \Theta(s)\, ds, & (5.19) \\ \varepsilon Y(t) + W(t) + \partial I_{\mathcal{K}}(Y(t)) \ni \varepsilon W(t_1) + Y(t_1) + \int_{t_1}^{t} \Xi(s)\, ds. & (5.20) \end{cases}
$$

3. *Projection formula:* for all $t \geq t_1 \geq 0$

$$
\begin{cases} X(t) = \mathsf{P}_{\mathcal{K}}\Big(X(t_1) + \frac{1}{\varepsilon}(t - t_1)\big(X(t_1) + \varepsilon V(t_1)\big) \\ \qquad\qquad - \frac{1}{\varepsilon}\int_{t_1}^{t} X(s)\, ds + \frac{1}{\varepsilon}\int_{t_1}^{t} (t-s)\Theta(s)\, ds \Big), & (5.21) \\ Y(t) = \mathsf{P}_{\mathcal{K}}\Big(Y(t_1) + \frac{1}{\varepsilon}(t - t_1)\big(Y(t_1) + \varepsilon W(t_1)\big) \\ \qquad\qquad - \frac{1}{\varepsilon}\int_{t_1}^{t} Y(s)\, ds + \frac{1}{\varepsilon}\int_{t_1}^{t} (t-s)\Xi(s)\, ds \Big). & (5.22) \end{cases}
$$

Note that if we choose $\Theta(t) := F[X(t), Y(t)]$ and $\Xi(t) := G[X(t), Y(t)]$ with F and G as in (5.15) and (5.16) and the interaction potentials K_{11}, K_{12}, K_{21} and K_{22} satisfying (**A**) and (**SL**), then any Lagrangian solution is a generalised Lagrangian solution.

In the following we will make use of the auxiliary variables

$$
P(t, m) = \varepsilon \overline{V}(m) + \overline{X}(m) + \int_0^t F[X(\cdot, r), Y(\cdot, r)](m)\, dr, \tag{5.23}
$$

and

$$
Q(t, m) = \varepsilon \overline{W}(m) + \overline{Y}(m) + \int_0^t G[X(\cdot, r), Y(\cdot, r)](m)\, dr, \tag{5.24}
$$

that allow to rephrase system (5.14) in the equivalent form

$$
\begin{cases} \varepsilon \dot{X} + X + \partial I_{\mathcal{K}}(X) \ni P, \\ \varepsilon \dot{Y} + Y + \partial I_{\mathcal{K}}(Y) \ni Q, \\ \dot{P} = F[X, Y], \\ \dot{Q} = G[X, Y]. \end{cases} \tag{5.25}
$$

5.3 Existence and uniqueness for smooth potentials

In this Section we prove existence and uniqueness of solution to system (5.1). To perform this task, we pass through existence of solutions to the Lagrangian system (5.14), where we apply the theory of Maximal Monotone Operators subject to Lipschitz perturbations in the spirit of [14, Theorem 3.17], see Section 1.7.

We start proving the following Lemma.

Lemma 5.2. *Let* $(X,Y),(\widetilde{X},\widetilde{Y}) \in \mathcal{K}\times\mathcal{K}$ *be given. Consider the interaction kernels* K_{11}, K_{12}, K_{21}, K_{22} *under assumptions* (**A**) *and* (**SL**) *and let* F *and* G *be the operators defined in* (5.15) *and* (5.16) *respectively. Then there exist two positive constants* C_1 *and* C_2 *depending on the Lipschitz constants of the kernels, such that*

(i) $\|F[X,Y]-F[\widetilde{X},\widetilde{Y}]\|^2_{L^2(\Omega)} \le C_1\big(\|X-\widetilde{X}\|^2_{L^2(\Omega)}+\|Y-\widetilde{Y}\|^2_{L^2(\Omega)}\big)$,

(ii) $\|G[X,Y]-G[\widetilde{X},\widetilde{Y}]\|^2_{L^2(\Omega)} \le C_2\big(\|X-\widetilde{X}\|^2_{L^2(\Omega)}+\|Y-\widetilde{Y}\|^2_{L^2(\Omega)}\big)$.

Proof. We only prove (i) since (ii) follows from a similar argument. By the definition of F in (5.15) we have

$$\begin{aligned}
&\|F[X,Y]-F[\widetilde{X},\widetilde{Y}]\|^2_{L^2(\Omega)}\\
&=\int_\Omega\Big|-\int_\Omega K'_{11}\big(X(r,m)-X(r,m')\big)\,dm'-\int_\Omega K'_{12}\big(X(r,m)-Y(r,m')\big)\,dm'\\
&\quad+\int_\Omega K'_{11}\big(\widetilde{X}(r,m)-\widetilde{X}(r,m')\big)\,dm'+\int_\Omega K'_{12}\big(\widetilde{X}(r,m)-\widetilde{Y}(r,m')\big)\,dm'\Big|^2\,dm.
\end{aligned}\tag{5.26}$$

Using the fact that $|x+y|^2\le 2(|x|^2+|y|^2)$, the right hand side of (5.26) can be controlled by

$$\begin{aligned}
&2\int_\Omega\bigg(\Big|\int_\Omega\big[K'_{11}(X(r,m)-X(r,m'))-K'_{11}(\widetilde{X}(r,m)-\widetilde{X}(r,m'))\big]\,dm'\Big|^2\\
&\qquad+\Big|\int_\Omega\big[K'_{12}\big(X(r,m)-Y(r,m')\big)-K'_{12}\big(\widetilde{X}(r,m)-\widetilde{Y}(r,m')\big)\big]\,dm'\Big|^2\bigg)\,dm\\
&\le 2\int_\Omega\bigg(\int_\Omega|K'_{11}(X(r,m)-X(r,m'))-K'_{11}(\widetilde{X}(r,m)-\widetilde{X}(r,m'))|\,dm'\bigg)^2\\
&\qquad+\bigg(\int_\Omega|K'_{12}\big(X(r,m)-Y(r,m')\big)-K'_{12}(\widetilde{X}(r,m)-\widetilde{Y}(r,m'))|\,dm'\bigg)^2\,dm.
\end{aligned}\tag{5.27}$$

Let $L(K'_{11})$ and $L(K'_{12})$ be the Lipschitz constants of K'_{11} and K'_{12} respectively, then, using Jensen's inequality, the right hand side of (5.27) is bounded by

$$\begin{aligned}
&2\int_\Omega\bigg(\int_\Omega L(K'_{11})\big(|X(r,m)-\widetilde{X}(r,m)|+|X(r,m')-\widetilde{X}(r,m')|\big)\,dm'\bigg)^2\\
&\qquad+\bigg(\int_\Omega L(K'_{12})\big(|X(r,m)-\widetilde{X}(r,m)|+|Y(r,m')-\widetilde{Y}(r,m')|\big)\,dm'\bigg)^2\,dm\\
&\le 4\int_\Omega\bigg(\int_\Omega\big[L(K'_{11})^2|X(r,m)-\widetilde{X}(r,m)|^2+L(K'_{11})^2|X(r,m')-\widetilde{X}(r,m')|^2\big]\,dm'\bigg)\\
&\qquad+\bigg(\int_\Omega\big[L(K'_{12})^2|X(r,m)-\widetilde{X}(r,m)|^2+L(K'_{12})^2|Y(r,m')-\widetilde{Y}(r,m')|^2\big]\,dm'\bigg)\,dm.
\end{aligned}$$

Thus, there exists a positive constant $C_1=C_1\big(L(K'_{11}),L(K'_{12})\big)$ such that

$$\|F[X,Y]-F[\widetilde{X},\widetilde{Y}]\|^2_{L^2(\Omega)}\le C_1\big(\|X-\widetilde{X}\|^2_{L^2(\Omega)}+\|Y-\widetilde{Y}\|^2_{L^2(\Omega)}\big).$$

Analogously, one can prove the inequality (ii), we omit the details. □

We are now ready to state existence result for Lagrangian solution to system (5.14).

Proposition 5.1. *Let $T > 0$ and suppose that the kernels K_{11}, K_{12}, K_{21}, $K_{22} \in \mathcal{C}^1(\mathbb{R})$ satisfy* (**A**) *and* (**SL**)*. Then, for every $(\overline{X}, \overline{Y}, \overline{V}, \overline{W}) \in \mathcal{K}^2 \times L^2(\Omega)^2$ there exists a unique Lagrangian solution (X, Y) to* (5.14) *in* $[0, T]$.

Proof. According to the discussion in Subsection 5.2.4, system (5.14) can be rewritten in the following equivalent form

$$\begin{cases} \dot{X} + \partial\left(I_{\mathcal{K}}(X) + \dfrac{|X|^2}{2\varepsilon}\right) \ni \dfrac{P}{\varepsilon}, \\ \dot{Y} + \partial\left(I_{\mathcal{K}}(Y) + \dfrac{|Y|^2}{2\varepsilon}\right) \ni \dfrac{Q}{\varepsilon}, \\ \dot{P} = F[X, Y], \\ \dot{Q} = G[X, Y], \end{cases} \tag{5.28}$$

where P and Q are defined in (5.23) and (5.24) respectively. In order to prove the result we will follow the strategy in [14, Theorem 3.17]. Consider the operator

$$\mathcal{A}(X, Y, P, Q) := I_{\mathcal{K}}(X) + I_{\mathcal{K}}(Y) + \frac{|X|^2}{2\varepsilon} + \frac{|Y|^2}{2\varepsilon}$$

defined on the Hilbert space $H := L^2(\Omega)^2 \times L^2(\Omega)^2$. Note that $\mathcal{A}$ is convex and bounded from below. Consider the iterative sequence defined as follows: fix $U_0 := (\overline{X}, \overline{Y}, \overline{P}, \overline{Q}) \equiv (\overline{X}, \overline{Y}, \varepsilon\overline{V} + \overline{X}, \varepsilon\overline{W} + \overline{Y})$ and, for $n \geq 1$ construct

$$U_{n+1}(t) := (X_{n+1}(t), Y_{n+1}(t), P_{n+1}(t), Q_{n+1}(t))$$

recursively as the weak solution to the implicit-explicit system

$$\begin{cases} \dot{X}_{n+1} + \partial\left(I_{\mathcal{K}}(X_{n+1}) + \dfrac{|X_{n+1}|^2}{2\varepsilon}\right) \ni \dfrac{P_n}{\varepsilon}, & X_{n+1}(0) = \overline{X}, \\ \dot{Y}_{n+1} + \partial\left(I_{\mathcal{K}}(Y_{n+1}) + \dfrac{|Y_{n+1}|^2}{2\varepsilon}\right) \ni \dfrac{Q_n}{\varepsilon}, & Y_{n+1}(0) = \overline{Y}, \\ \dot{P}_{n+1} = F[X_n, Y_n], & P_{n+1}(0) = \overline{P}, \\ \dot{Q}_{n+1} = G[X_n, Y_n], & Q_{n+1}(0) = \overline{Q}. \end{cases} \tag{5.29}$$

Setting $R(U_n) = \big(P_n/\varepsilon, Q_n/\varepsilon, F[X_n, Y_n], G[X_n, Y_n]\big)$, the previous system (5.29) can be rewritten in the following compact form

$$\dot{U}_{n+1} + \partial\mathcal{A}(U_{n+1}) \ni R(U_n). \tag{5.30}$$

Since the functional $\mathcal{A}$ is convex, its sub-differential is a maximal monotone operator in the sense of [14] and R can be seen as a Lipschitz perturbation of it, see [14, Lemma 3.1]. A direct computation shows that

$$\frac{1}{2}\frac{d}{dt}\|U_{n+1} - U_n\|^2_{L^2(\Omega)} \leq \big(U_{n+1} - U_n, R(U_n) - R(U_{n-1})\big).$$

Proceeding as in [14, Lemma A.5], we may introduce the function

$$\begin{aligned} \psi_\delta(t) = \frac{1}{2}\left(\|U_{n+1}(0) - U_n(0)\|^2_{L^2(\Omega)} + \delta\right)^2 \\ + \int_0^t \big(U_{n+1}(r) - U_n(r), R(U_n)(r) - R(U_{n-1}(r))\big)\, dr \end{aligned}$$

and prove that it is absolutely continuous for all $t \in [0,T]$ with

$$\sqrt{\psi_\delta(t)} \leq \sqrt{\psi_\delta(0)} + \frac{1}{\sqrt{2}} \int_0^t \|R(U_n)(r) - R(U_{n-1})(r)\|_{L^2(\Omega)}\, dr,$$

uniformly in δ. Since $\|U_{n+1}(t) - U_n(t)\|_{L^2(\Omega)} \leq \sqrt{2}\sqrt{\psi_\delta(t)}$ for all $\delta > 0$, we have that

$$\|U_{n+1}(t) - U_n(t)\|_{L^2(\Omega)} \leq \int_0^t \|R(U_n)(r) - R(U_{n-1})(r)\|_{L^2(\Omega)}\, dr.$$

Invoking Lemma 5.2 and the definitions for P and Q in (5.23) and (5.24) respectively, we can say that there exists a positive constant C depending on T, ε and on the Lipschitz constants of the kernels $L(K'_{11})$, $L(K'_{22})$, $L(K'_{21})$, $L(K'_{12})$ such that

$$\|U_{n+1}(t) - U_n(t)\|_{L^2(\Omega)} \leq C \int_0^t \|U_n(r) - U_{n-1}(r)\|_{L^2(\Omega)}\, dr$$

for $0 \leq t \leq T$. An easy iterative procedure implies that

$$\|U_{n+1} - U_n\|_{L^2(\Omega)} \leq \frac{(Ct)^n}{n!} \|U_1 - U_0\|_{L^2(\Omega)},$$

thus, U_n uniformly converges on $[0,T]$ to some U. Due to the Lemma 5.2, R is continuous in L^2 in each component. Moreover, since the sub-differential of $\mathcal{A}$ is closed, we can pass to the limit in (5.30) and obtain that U is a weak solution to the system (5.28).

Concerning uniqueness, let $U_1 = (X_1, Y_1, P_1, Q_1)$ and $U_2 = (X_2, Y_2, P_2, Q_2)$ be two solutions to system (5.28) with the same initial condition $\overline{U}_1 = \overline{U}_2 = \overline{U}$. Proceeding in an analogous way as before, we can argue that

$$\|U_1(t) - U_2(t)\|_{L^2(\Omega)} \leq C \int_0^t \|U_1(r) - U_2(r)\|_{L^2(\Omega)}\, dr$$

for $0 \leq t \leq T$, where the positive constant C depends on T, ε, $L(K'_{11})$, $L(K'_{12})$, $L(K'_{21})$, $L(K'_{22})$. This implies that

$$\|U_1(t) - U_2(t)\|_{L^2(\Omega)} \leq e^{Ct} \|\overline{U}_1 - \overline{U}_2\|_{L^2(\Omega)} = 0,$$

that proves the uniqueness. □

The following Proposition collects some properties of Lagrangian solution.

Proposition 5.2. *Let $F, G : \mathcal{K} \times \mathcal{K} \to L^2(\Omega)$ be uniformly continuous operators in (5.15) and (5.16) and let (X,Y) be the Lagrangian solution to (5.14). Then, the following properties hold:*

(i) The right-derivatives

$$V = \frac{d^+}{dt} X, \qquad W = \frac{d^+}{dt} Y \tag{5.31}$$

exist for all $t \geq 0$.

(ii) V *and* W *are the unique elements of minimal norm in the closed convex sets*

$$\frac{1}{\varepsilon}\big(P(t) - \partial I_{\mathcal{K}}(X(t)) - X(t)\big) \quad \text{and} \quad \frac{1}{\varepsilon}\big(Q(t) - \partial I_{\mathcal{K}}(Y(t)) - Y(t)\big)$$

respectively, i.e.,

$$V(t) = \left(\frac{1}{\varepsilon}\big(P(t) - \partial I_{\mathcal{K}}(X(t)) - X(t)\big)\right)^{\circ} \tag{5.32}$$

and

$$W(t) = \left(\frac{1}{\varepsilon}\big(Q(t) - \partial I_{\mathcal{K}}(Y(t)) - Y(t)\big)\right)^{\circ} \tag{5.33}$$

respectively. In particular, by replacing $\dot{X}$ *by* V *and* $\dot{Y}$ *by* W*, (5.14) and (5.25) hold for all* $t \geq 0$*.*

(iii) *The functions* $t \mapsto V(t)$ *and* $t \mapsto W(t)$ *are right-continuous for all* $t \geq 0$*.*

(iv) *If* $\mathcal{T}_X^0 \subset (0,\infty)$ *and* $\mathcal{T}_Y^0 \subset (0,\infty)$ *denote the subsets of all times at which the maps* $s \to \|V(s)\|_{L^2(\Omega)}$ *and* $s \to \|W(s)\|_{L^2(\Omega)}$ *respectively are continuous, then* $(0,\infty)\setminus\mathcal{T}_X^0$ *and* $(0,\infty)\setminus\mathcal{T}_Y^0$ *are negligible,* V *and* W *are continuous,* X *and* Y *are differentiable in* $L^2(\Omega)$ *at every point of* $\mathcal{T}_X^0$ *and* $\mathcal{T}_Y^0$ *respectively.*

(v) *Setting* $\rho(t,\cdot) := \Psi^{-1}(X(t,\cdot))$ *and* $\eta(t,\cdot) := \Psi^{-1}(Y(t,\cdot))$ *where* Ψ *is the isometry introduced in (1.16), there exist a unique map* $v(t,\cdot) \in L^2(\mathbb{R},\rho)$ *and a unique map* $w(t,\cdot) \in L^2(\mathbb{R},\eta)$ *such that*

$$\dot{X}(t) = V(t) = \mathsf{P}_{\mathcal{H}_{X(t)}}\left(\frac{1}{\varepsilon}\big(P(t) - X(t)\big)\right) = v(t, X(t)) \in \mathcal{H}_{X(t)}, \tag{5.34}$$

for every $t \in \mathcal{T}_X^0$*, and*

$$\dot{Y}(t) = W(t) = \mathsf{P}_{\mathcal{H}_{Y(t)}}\left(\frac{1}{\varepsilon}\big(Q(t) - Y(t)\big)\right) = w(t, Y(t)) \in \mathcal{H}_{Y(t)}, \tag{5.35}$$

for every $t \in \mathcal{T}_Y^0$*.*

Proof. The results in $(i), (ii), (iii)$ are consequences of the general theory of [14, Theorem 3.5]. Concerning (iv) and (v), we follow [12, Theorem 3.5]. We prove only (5.34), since the proof of (5.35) is similar. By applying [14, Remark 3.9], one can see that if t is a point of differentiability of X, the derivative with respect to time of X in t is the projection of 0 onto the affine space generated by $P(t) - \partial I_{\mathcal{K}}(X(t)) - X(t)$, i.e., the orthogonal projection of $P(t) - X(t)$ onto the orthogonal complement of the space generated by $\partial I_{\mathcal{K}}(X(t))$. By using [12, Lemma 2.5], we obtain (5.34). Since any element of $\mathcal{H}_{X(t)}$ can be written as $v \circ X$, where $v \in L^2(\Omega)$ is a suitable Borel map, we have that there exists a Borel map $v : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ such that $v(t,\cdot) \in L^2(\mathbb{R}, \rho(t,\cdot))$ and $V(t,\cdot) = v(t, X(t))$ for $t \in \mathcal{T}_X^0$. □

We are now in the position of proving the main result, that concerns existence and uniqueness of the solution to system (5.1).

Theorem 5.1. *Let $T > 0$ and suppose that the kernels K_{11}, K_{12}, K_{21}, $K_{22} \in \mathcal{C}^1(\mathbb{R})$ satisfy (**A**) and (**SL**). Let $\overline{\rho}, \overline{\eta} \in \mathcal{P}_2(\mathbb{R})$ and $\overline{v} \in L^2(d\overline{\rho})$ and $\overline{w} \in L^2(d\overline{\eta})$. Then, there exists a unique quadruple*

$$(\rho, \eta, v, w) \in \mathrm{Lip}\big([0,T]; \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) \times L^2(d\rho(t)) \times L^2(d\eta(t))\big)$$

that is a distributional solution to system (5.1) such that

$$\lim_{t\downarrow 0} \rho(t,\cdot) = \overline{\rho} \quad in \quad \mathcal{P}_2(\mathbb{R}), \qquad \lim_{t\downarrow 0} \rho(t,\cdot)v(t,\cdot) = \overline{\rho}\,\overline{v} \quad in \quad \mathcal{M}(\mathbb{R}),$$

$$\lim_{t\downarrow 0} \eta(t,\cdot) = \overline{\eta} \quad in \quad \mathcal{P}_2(\mathbb{R}), \qquad \lim_{t\downarrow 0} \eta(t,\cdot)w(t,\cdot) = \overline{\eta}\,\overline{w} \quad in \quad \mathcal{M}(\mathbb{R}).$$

Proof. Let $\overline{\rho}, \overline{\eta} \in \mathcal{P}_2(\mathbb{R})$ and $\overline{v} \in L^2(d\overline{\rho})$, $\overline{w} \in L^2(d\overline{\eta})$ be given initial conditions. Define the $L^2(\Omega)$-functions $\overline{X} = \Psi(\overline{\rho})$ and $\overline{Y} = \Psi(\overline{\eta})$ and the compositions $\overline{V} = \overline{v} \circ \overline{X}$ and $\overline{W} = \overline{w} \circ \overline{Y}$. Then $(\overline{X}, \overline{Y}, \overline{V}, \overline{W})$ is an admissible initial condition for system (5.14), thus Proposition 5.1 ensures existence and uniqueness of a couple (X, Y) that is the Lagrangian solution to (5.14). According to Proposition 5.2 we can define the right-continuous functions V and W such that (5.31) holds for all $t \geq 0$ and introduce $\rho(t,\cdot) := \Psi^{-1}(X(t,\cdot))$ and $\eta(t,\cdot) := \Psi^{-1}(Y(t,\cdot))$. Let $v(t,\cdot)$ be the map given by Proposition 5.2 and φ be a test function on $(0,T) \times \mathbb{R}$, then

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \varepsilon\big(\partial_t \varphi(t,x) + \partial_x \varphi(t,x) v(t,x)\big) v(t,x) \rho(t,dx)\, dt \\
&\quad = \int_0^\infty \int_\Omega \varepsilon\big(\partial_t \varphi(t, X(t,m))\big) v(t, X(t,m))\, dm\, dt \\
&\qquad + \int_0^\infty \int_\Omega \varepsilon\big(\partial_x \varphi(t, X(t,m)) v(t, X(t,m))\big) v(t, X(t,m))\, dm\, dt.
\end{aligned} \tag{5.36}$$

Using (5.34) and integrating by parts, the right hand side of (5.36) is equal to

$$\begin{aligned}
&\int_0^\infty \int_\Omega \left(\frac{d}{dt}\varphi(t, X(t,m))\right) \big(P(t,m) - X(t,m)\big)\, dm\, dt \\
&= \int_0^\infty \int_\Omega \varphi(t, X(t,m)) \big(\dot{X}(t,m) - \dot{P}(t,m)\big)\, dm\, dt.
\end{aligned} \tag{5.37}$$

As proved in Proposition 5.2 we have that $\dot{X}(t,m) = V(t,m)$ and from the definition of the operator $P(t,m)$ in (5.23), one obtains that (5.37) equals

$$\begin{aligned}
&\int_0^\infty \int_\Omega \varphi(t, X(t,m)) \bigg(V(t,m) + \int_\Omega K_{11}'\big(X(s,m) - X(s,m')\big)\, dm' \\
&\qquad + \int_\Omega K_{12}'\big(X(s,m) - Y(s,m')\big)\, dm' \bigg)\, dm\, dt \\
&= \int_o^\infty \int_{\mathbb{R}} \varphi(t,x) \big(v(t,x) + K_{11}' * \rho(t,x) + K_{12}' * \eta(t,x)\big)\, \rho(t,dx)\, dt,
\end{aligned}$$

that is the distributional formulation of the momentum equation in (5.1). Similarly, for the continuity equation we have

$$\begin{aligned}
&\int_0^\infty \int_0^1 \left(\frac{d}{dt}\varphi(t, X(t,m))\right) dm\, dt \\
&= \int_0^\infty \int_0^1 \big(\partial_t \varphi(t, X(t,m)) + \partial_x \varphi(t, X(t,m)) V(t,m)\big)\, dm\, dt \\
&= \int_0^\infty \int_{\mathbb{R}} \big(\partial_t \varphi(t,x) + \varphi_x(t,x) v(t,x)\big)\, \rho(t,dx)\, dt = 0.
\end{aligned}$$

Concerning the initial conditions, since $\lim_{t\downarrow 0} X(t) = \overline{X}$ in $L^2(\Omega)$ for Proposition 5.1 and $\overline{X} = \Psi(\overline{\rho})$, we have that $\rho \to \overline{\rho}$ in $\mathcal{P}_2(\mathbb{R})$ as $t \to 0$. Moreover, $\overline{V} = \overline{v} \circ \overline{X}$, so that $\lim_{t\downarrow 0} V(t) = \overline{V}$ in $L^2(\Omega)$, therefore for every $\varphi \in \mathcal{C}_b(\mathbb{R})$ we have

$$\begin{aligned}\int_{\mathbb{R}} \varphi(x)\overline{v}(x)\overline{\rho}(dx) &= \int_0^1 \varphi(\overline{X}(m))\overline{V}(m)\,dm \\ &= \lim_{t\downarrow 0}\int_0^1 \varphi(X(t,m))V(t,m)\,dm = \lim_{t\downarrow 0}\int_{\mathbb{R}} \varphi(t,x)v(t,x)\rho(t,dx).\end{aligned}$$

A similar argument can be used for the pair (η, w). □

5.4 Large-damping limit

In this Section we study the large-damping limit of system (5.1) for the damping parameter $\sigma \to \infty$. In particular, we aim at making the formal argument introduced in Subsection 5.2.3 rigorous, and showing that solutions to system (5.9) converge to the ones of the first order system

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \dfrac{\partial}{\partial x}[\rho K_\rho' * \rho + \rho H_\rho' * \eta], \\ \dfrac{\partial \eta}{\partial t} = \dfrac{\partial}{\partial x}[\eta K_\eta' * \eta + \eta H_\eta' * \rho]. \end{cases} \tag{5.38}$$

In what follows we will assume that the potentials K_{11}, K_{12}, K_{21}, K_{22} are under assumptions (**A**) and (**SL**).

Recalling the definition of $F[X,Y](m)$ and $G[X,Y](m)$ in (5.15) and (5.16), we introduce the operator

$$L\big((X,Y)\big)(m) := \begin{pmatrix} F[X,Y](m) \\ G[X,Y](m) \end{pmatrix}.$$

By setting $Z_\varepsilon = (X_\varepsilon, Y_\varepsilon)$, $\overline{Z}_\varepsilon = (\overline{X}_\varepsilon, \overline{Y}_\varepsilon)$, $U_\varepsilon = (V_\varepsilon, W_\varepsilon)$ and $\overline{U}_\varepsilon = (\overline{V}_\varepsilon, \overline{W}_\varepsilon)$, system (5.14) can be rewritten in the following compact form

$$\varepsilon \dot{Z}_\varepsilon(t) + Z_\varepsilon(t) + \partial I_{\mathcal{K}^2}(Z_\varepsilon(t)) \ni \varepsilon \overline{U}_\varepsilon + \overline{Z}_\varepsilon + \int_0^t L(Z_\varepsilon(r))\,dr. \tag{5.39}$$

We are now in the position of proving

Theorem 5.2. *Let $T > 0$ and suppose that the kernels K_{11}, K_{12}, K_{21}, $K_{22} \in \mathcal{C}^1(\mathbb{R})$ satisfy (**A**) and (**SL**). Let $(\rho_\varepsilon, \eta_\varepsilon, v_\varepsilon, w_\varepsilon)$ be solution to system (5.9) with $\varepsilon = \sigma^{-2}$ under the initial condition $(\overline{\rho}_\varepsilon, \overline{\eta}_\varepsilon, \overline{v}_\varepsilon, \overline{w}_\varepsilon)$ and let (ρ, η) be solution to system (5.10) with initial data $(\overline{\rho}, \overline{\eta})$. Furthermore, assume that*

(i) $\overline{\rho}_\varepsilon \to \overline{\rho}$ and $\overline{\eta}_\varepsilon \to \overline{\eta}$ as $\varepsilon \to 0$ in $\mathcal{P}_2(\mathbb{R})$;

(ii) $\overline{v}_\varepsilon = o(1/\varepsilon)$ in $L^2(d\overline{\rho}_\varepsilon)$ and $\overline{w}_\varepsilon = o(1/\varepsilon)$ in $L^2(d\overline{\eta}_\varepsilon)$ as $\varepsilon \to 0$.

Then,

$$\lim_{\varepsilon\to 0}\int_0^T \mathcal{W}_2^2\big((\rho_\varepsilon, \eta_\varepsilon), (\rho, \eta)\big)\,dt = 0.$$

Remark 5.1 (Initial data are not *well-prepared* in the velocity variable)**.** In Theorem 5.2, recalling that $\overline{v}^\varepsilon = \frac{1}{\sqrt{\varepsilon}}\overline{v}$, assumption (ii) is satisfied in case $\overline{v} \in L^2(d\overline{\rho})$ and $\overline{w} \in L^2(d\overline{\eta})$ are given and independent of ε. Therefore, assumption (ii) is quite general in the context of singular limits. Assumption (i) instead imposes that the initial density should converge to the one of the limiting first order system.

Proof of Theorem 5.2. Let (ρ, η) be a solution to system (5.38) subject to the initial condition $(\overline{\rho}, \overline{\eta})$, and $(\rho_\varepsilon, \eta_\varepsilon, v_\varepsilon, w_\varepsilon)$ be a solution to system (5.9) subject to the initial condition $(\overline{\rho}_\varepsilon, \overline{\eta}_\varepsilon, \overline{v}_\varepsilon, \overline{w}_\varepsilon)$, Define $X_0 = \Psi(\rho)$ and $Y_0 = \Psi(\eta)$, then $Z_0 = (X_0, Y_0)$ is a solution to

$$Z_0(t) + \partial I_{\mathcal{K}^2}(Z_0(t)) \ni \overline{Z}_0 + \int_0^t L(Z_0(r))\, dr, \tag{5.40}$$

with $\overline{Z}_0 = (\overline{X}_0, \overline{Y}_0) = (\Psi(\overline{\rho}), \Psi(\overline{\eta}))$. Similarly, consider $Z_\varepsilon = (X_\varepsilon, Y_\varepsilon)$ that solves (5.39), with $X_\varepsilon = \Psi(\rho_\varepsilon)$ and $Y_\varepsilon = \Psi(\eta_\varepsilon)$. Adding $\varepsilon \dot{Z}_0(t)$ to both sides of (5.40) and taking the difference between (5.39) and (5.40), we get

$$\begin{aligned} &\varepsilon\big(\dot{Z}_\varepsilon(t) - \dot{Z}_0(t)\big) + Z_\varepsilon(t) - Z_0(t) + \partial I_{\mathcal{K}^2}(Z_\varepsilon(t)) - \partial I_{\mathcal{K}^2}(Z_0(t)) \\ &\ni \varepsilon\overline{U}_\varepsilon + \overline{Z}_\varepsilon - \overline{Z}_0 - \varepsilon\dot{Z}_0(t) + \int_0^t \big[L(Z_\varepsilon(r)) - L(Z_0(r))\big]\, dr. \end{aligned} \tag{5.41}$$

We now estimate the evolution of the L^2-norm of the quantity $Z_\varepsilon(t) - Z_0(t)$. In doing that, we use the monotonicity of the set valued operator $\partial I_{\mathcal{K}^2}$, which is a consequence of the convexity of the indicator function and of the definition of sub-differential.

$$\begin{aligned} &\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega \big(Z_\varepsilon(t,m) - Z_0(t,m)\big)^2\, dm + \int_\Omega \big(Z_\varepsilon(t,m) - Z_0(t,m)\big)^2\, dm \\ &\leq \int_\Omega \big[\varepsilon\overline{U}_\varepsilon(m) + \overline{Z}_\varepsilon(m) - \overline{Z}_0(m)\big]\big(Z_\varepsilon(t,m) - Z_0(t,m)\big)\, dm \\ &\quad - \varepsilon\int_\Omega \dot{Z}_0(t,m)\big(Z_\varepsilon(t,m) - Z_0(t,m)\big)\, dm \\ &\quad + \int_0^t\int_\Omega [L(Z_\varepsilon(r,m)) - L(Z_0(r,m))]\big(Z_\varepsilon(t,m) - Z_0(t,m)\big)\, dm\, dr. \end{aligned} \tag{5.42}$$

Using Young's inequality and the bounds in Lemma 5.2, (5.42) becomes

$$\begin{aligned} &\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega \big(Z_\varepsilon(t,m) - Z_0(t,m)\big)^2\, dm + \int_\Omega \big(Z_\varepsilon(t,m) - Z_0(t,m)\big)^2\, dm \\ &\leq \frac{1}{2}\int_\Omega \big[\varepsilon\overline{U}_\varepsilon(m) + \overline{Z}_\varepsilon(m) - \overline{Z}_0(m)\big]^2\, dm + \frac{1}{2}\int_\Omega \big(Z_\varepsilon(t,m) - Z_0(t,m)\big)^2\, dm \\ &\quad + \frac{\varepsilon}{2}\int_\Omega \dot{Z}_0^2(t,m)\, dm + \frac{\varepsilon}{2}\int_\Omega \big(Z_\varepsilon(t,m) - Z_0(t,m)\big)^2\, dm \\ &\quad + \frac{1}{2}\int_0^t\int_\Omega [L(Z_\varepsilon(r,m)) - L(Z_0(r,m))]^2\, dm\, dr \\ &\quad + \frac{1}{2}\int_0^t\int_\Omega \big(Z_\varepsilon(r,m) - Z_0(r,m)\big)^2\, dm\, dr\,, \end{aligned}$$

which implies

$$
\begin{aligned}
&\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega \big(Z_\varepsilon(t,m)-Z_0(t,m)\big)^2\,dm+\frac{1-\varepsilon}{2}\int_\Omega \big(Z_\varepsilon(t,m)-Z_0(t,m)\big)^2\,dm\\
&\le \frac{1}{2}\int_\Omega \big[\varepsilon\overline{U}_\varepsilon(m)+\overline{Z}_\varepsilon(m)-\overline{Z}_0(m)\big]^2\,dm+\frac{\varepsilon}{2}\int_\Omega \dot Z_0^2(t,m)\,dm\\
&\qquad + C\frac{1}{2}\int_0^t\int_\Omega \big(Z_\varepsilon(r,m)-Z_0(r,m)\big)^2\,dm\,dr\,,
\end{aligned}
$$

where C is a fixed constant depending on the operator L and coming from Lemma 5.2. Integrating over $[0,T]$ and denoting

$$
\begin{aligned}
A(\varepsilon,T) :=&(2\varepsilon+4T)\int_\Omega \big(\overline{Z}_\varepsilon(m)-\overline{Z}_0(m)\big)^2\,dm+4T\int_\Omega \big[\varepsilon\overline{U}_\varepsilon(m)\big]^2\,dm\\
&+2\varepsilon\int_0^T\int_\Omega \dot Z_0^2(t,m)\,dm\,dt,
\end{aligned}
$$

assuming $\varepsilon<1/2$, by using Cauchy-Schwarz inequality we have that

$$
\begin{aligned}
&\int_0^T\int_\Omega \big(Z_\varepsilon(t,m)-Z_0(t,m)\big)^2\,dm\,dt\\
&\le C\int_0^T\int_0^t\int_\Omega \big(Z_\varepsilon(r,m)-Z_0(r,m)\big)^2\,dm\,dr\,dt+A(\varepsilon,T)\,,
\end{aligned}
$$

by suitably renaming the constant C. By applying Grönwall's lemma we get

$$
\int_0^T\int_\Omega \big(Z_\varepsilon(t,m)-Z_0(t,m)\big)^2\,dm\,dt\le A(\varepsilon,T)e^{CT}.
$$

In order to conclude it is enough to see that $A(\varepsilon,T)\to 0$ as $\varepsilon\to 0$. We recall assumption (i) reads $\overline{\rho}_\varepsilon\to\overline{\rho}$ and $\overline{\eta}_\varepsilon\to\overline{\eta}$ in $\mathcal{P}_2(\mathbb{R})$, thus $\overline{Z}_\varepsilon\to\overline{Z}_0$ as $\varepsilon\to 0$ in $L^2(\Omega)^2$. Assumption (ii) implies initial velocities under the following conditions

$$
\overline{v}_\varepsilon=o(1/\varepsilon)\ \text{ in }\ L^2(d\overline{\rho}_\varepsilon)\ \text{ and }\ \overline{w}_\varepsilon=o(1/\varepsilon)\ \text{ in }\ L^2(d\overline{\eta}_\varepsilon)
$$

as $\varepsilon\to 0$, thus $\varepsilon\overline{U}_\varepsilon\to 0$ as $\varepsilon\to 0$. Finally, the last term in $A(\varepsilon,T)$ converges to zero since $\dot Z_0$ does not depend on ε. □

5.5 Newtonian potentials

This Section is devoted to study existence of solutions and asymptotic property of system (5.1) when self-attractive forces are driven by Newtonian potentials, i.e., $K_{11}(x)=K_{22}(x)=: N(x):=|x|$. We restrict the analysis to the case of equal cross potentials, namely $K_{12}=K_{21}=: H$. We also consider two uniformly convex external potentials A_ρ and A_η acting on the system. More precisely, we assume A_ρ, $A_\eta\in\mathcal{C}^2(\mathbb{R})$ under assumptions (**H1**) and (**H2**). These additional terms do not affect the study of existence of solutions, in the *generalised* sense specified in Definition 5.5, but are only required in the study of asymptotic behaviour in Theorem 5.3. The system we are dealing with is

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho v) = 0, \\ \dfrac{\partial \eta}{\partial t} + \dfrac{\partial}{\partial x}(\eta w) = 0, \\ \dfrac{\partial}{\partial t}(\rho v) + \dfrac{\partial}{\partial x}(\rho v^2) = -\sigma\rho v - \rho[N' * \rho + H' * \eta + A_\rho], \\ \dfrac{\partial}{\partial t}(\eta w) + \dfrac{\partial}{\partial x}(\eta w^2) = -\sigma\eta w - \eta[N' * \eta + H' * \rho + A_\eta], \end{cases} \tag{5.43}$$

and its Lagrangian counterpart is

$$\begin{cases} \partial_t X(t,m) = V(t,m), \\ \partial_t Y(t,m) = W(t,m), \\ \partial_t V(t,m) = -\displaystyle\int_\Omega \operatorname{sign}\big(X(t,m) - X(t,m')\big)\,dm' \\ \qquad\qquad - \displaystyle\int_\Omega H'\big(X(t,m) - Y(t,m')\big)\,dm' - \sigma V(t,m) - A'_\rho(X), \\ \partial_t W(t,m) = -\displaystyle\int_\Omega \operatorname{sign}\big(Y(t,m) - Y(t,m')\big)\,dm' \\ \qquad\qquad - \displaystyle\int_\Omega H'\big(Y(t,m) - X(t,m')\big)\,dm' - \sigma W(t,m) - A'_\eta(Y). \end{cases} \tag{5.44}$$

Stationary solutions in this case are $(\rho_s, \eta_s) = (\delta_0, \delta_0)$ where δ is the Dirac measure, which corresponds to $(X_s, Y_s) = (0,0)$ in terms of the Lagrangian description.

We can associate to the system (5.44) the following functional

$$\begin{aligned} \mathfrak{F}(X,Y) =& \frac{1}{2}\int_\Omega\int_\Omega |X(m) - X(m')|\,dm'\,dm + \frac{1}{2}\int_\Omega\int_\Omega |Y(m) - Y(m')|\,dm'\,dm \\ &+ \int_\Omega\int_\Omega H\big(Y(m) - X(m')\big)\,dm'\,dm \\ &+ \int_\Omega A_\rho(X(m))\,dm + \int_\Omega A_\eta(Y(m))\,dm. \end{aligned} \tag{5.45}$$

In particular, we write

$$\mathfrak{F}(X,Y) := S(X) + S(Y) + K(X,Y),$$

where

$$\begin{aligned} S(X) &:= \frac{1}{2}\int_\Omega\int_\Omega |X(m) - X(m')|\,dm'\,dm, \\ S(Y) &:= \frac{1}{2}\int_\Omega\int_\Omega |Y(m) - Y(m')|\,dm'\,dm, \\ K(X,Y) &:= \int_\Omega\int_\Omega H\big(Y(m) - X(m')\big)\,dm'\,dm \\ &\quad + \int_\Omega A_\rho(X(m))\,dm + \int_\Omega A_\eta(Y(m))\,dm. \end{aligned}$$

As shown in [11, 22], it is easy to prove that the self-interaction contributions in $\mathfrak{F}$ are linear when restricted to $\mathcal{K}$.

Lemma 5.3. *If $X \in \mathcal{K}$, then*

$$S(X) = \int_\Omega (2m-1)X(m)\,dm.$$

Proof. A direct computation shows that

$$S(X) = \frac{1}{2}\int_\Omega\int_\Omega |X(m)-X(s)|\,ds\,dm = \int\int_{\{X(m)\geq X(s)\}} \big(X(m)-X(s)\big)\,dm\,ds.$$

Since $X \in \mathcal{K}$, X is non-decreasing, then the set $\{X(m) \geq X(s)\}$ can be characterised as follows

$$\{X(m) \geq X(s)\} = \{m \geq s\} \cup \{m \leq s \leq \Sigma(m)\},$$

with

$$\Sigma(m) = \sup\{s \in [0,1] \ : \ X(s) = X(m)\}.$$

Moreover, $X(s) = X(m)$ on $\{m \leq s \leq \Sigma(m)\}$, then

$$\begin{aligned} S(X) &= \int\int_{m\geq s} \big(X(m)-X(s)\big)\,dm\,ds \\ &= \left(\int_\Omega\int_0^m X(m)\,ds\,dm - \int_\Omega\int_s^1 X(s)\,dm\,ds\right) \\ &= \int_\Omega mX(m)\,dm - \int_\Omega (1-s)X(s)\,ds \\ &= \int_\Omega (2m-1)X(m)\,dm, \end{aligned}$$

that proves the statement. □

The first result in this Section consists in proving the existence of a map $t \mapsto (X(t), Y(t))$ that is a generalised Lagrangian solution to (5.14) with respect to the choice $\Theta = \mathsf{P}_{\mathcal{H}_X}(F_1)(t,m)$ and $\Xi = \mathsf{P}_{\mathcal{H}_Y}(F_2)(t,m)$, i.e., the system (5.44) can be written as follows

$$\begin{cases} \partial_t X(t,m) = \mathsf{P}_{\mathcal{H}_X}(V)(t,m), \\ \partial_t Y(t,m) = \mathsf{P}_{\mathcal{H}_Y}(W)(t,m), \\ \partial_t V(t,m) = -\mathsf{P}_{\mathcal{H}_X}(F_1[X,Y])(m) - \sigma V(t,m), \\ \partial_t W(t,m) = -\mathsf{P}_{\mathcal{H}_Y}(F_2[X,Y])(m) - \sigma W(t,m), \end{cases} \tag{5.46}$$

where

$$F_1[X,Y](m) = 2m - 1 + \int_\Omega H'\big(X(m)-Y(m')\big)\,dm' + A_\rho'(X) \tag{5.47}$$

and

$$F_2[X,Y](m) = 2m - 1 + \int_\Omega H'\big(Y(m)-X(m')\big)\,dm' + A_\eta'(Y) \tag{5.48}$$

are the force operators and describe the external and interaction forces that act on the system.

The following Proposition ensures that a generalised Lagrangian solution exists.

Proposition 5.3. *Assume the cross-potential H under assumptions (**A**) and (**SL**). Assume the external potentials A_ρ, $A_\eta \in \mathcal{C}^2(\mathbb{R})$. Then for every $(\overline{X}, \overline{Y}, \overline{V}, \overline{W}) \in \mathcal{K}^2 \times \mathcal{H}_{\overline{X}} \times \mathcal{H}_{\overline{Y}}$ there exists a generalised Lagrangian solution to system (5.44) with initial data $(\overline{X}, \overline{Y}, \overline{V}, \overline{W})$ in the sense of Definition 5.5.*

Proof. The proof is based on a discretization argument, inspired by the result in [12, Theorem 4.5]. Consider the following two partitions of Ω:

$$0 =: l_0 < l_1 < \cdots < l_N := 1, \quad \text{and} \quad 0 =: z_0 < z_1 < \cdots < z_M := 1,$$

with

$$l_i := \sum_{j=1}^{i} m_j, \quad \text{and} \quad z_j := \sum_{i=1}^{j} n_i,$$

for $i = 1, \ldots, N-1$ and $j = 1, \ldots, M-1$, and introduce the piecewise constant functions

$$X(t,\cdot) = \sum_{i=1}^{N} x_i(t)\mathbb{I}_{L_i}, \qquad V(t,\cdot) = \sum_{i=1}^{N} v_i(t)\mathbb{I}_{L_i}, \tag{5.49}$$

$$Y(t,\cdot) = \sum_{j=1}^{M} y_j(t)\mathbb{I}_{Z_j}, \qquad W(t,\cdot) = \sum_{j=1}^{M} w_j(t)\mathbb{I}_{Z_j}, \tag{5.50}$$

defined on the intervals $L_i := [l_{i-1}, l_i)$ and $Z_j := [z_{j-1}, z_j)$, for $i = 1, \ldots, N-1$ and $j = 1, \ldots, M-1$. Consider the finite dimensional Hilbert space

$$\mathcal{H}_m \times \mathcal{H}_n := \left\{ (X,Y) = \left(\sum_{i=1}^{N} x_i \mathbb{I}_{L_i}, \sum_{j=1}^{M} y_j \mathbb{I}_{Z_j} \right) \, : \, (x,y) \in \mathbb{R}^N \times \mathbb{R}^M \right\} \subset L^2(\Omega) \times L^2(\Omega)$$

and its closed convex cone

$$\mathcal{K}_m \times \mathcal{K}_n := \left\{ (X,Y) = \left(\sum_{i=1}^{N} x_i \mathbb{I}_{L_i}, \sum_{j=1}^{M} y_j \mathbb{I}_{Z_j} \right) \, : \, (x,y) \in \mathbb{K}^N \times \mathbb{K}^M \right\} \subset \mathcal{K} \times \mathcal{K}.$$

Note that the projected forces

$$F_m[X,Y] := \mathsf{P}_{\mathcal{H}_m}(F_1[X,Y]) \quad \text{and} \quad F_n[X,Y] := \mathsf{P}_{\mathcal{H}_n}(F_2[X,Y])$$

are well defined and Lipschitz continuous according to the definitions in (5.47)-(5.48) and assumptions (**A**) and (**SL**).

Now, assume that the initial condition $(\overline{X}, \overline{Y}, \overline{V}, \overline{W}) \in \mathcal{K}_m \times \mathcal{K}_n \times \mathcal{H}_{\overline{X}} \times \mathcal{H}_{\overline{Y}}$ does not hit the boundary of $\mathcal{K}_m \times \mathcal{K}_n$. Consider the time interval $[0, t_1)$ with $t_1 = \min\{t_1^X, t_1^Y\}$ where

$$t_1 = \inf\{t > 0 \, : \, X(t) \in \partial\mathcal{K}_m\}, \qquad t_1^Y = \inf\{t > 0 \, : \, Y(t) \in \partial\mathcal{K}_n\}.$$

Then, we obtain (5.49)-(5.50) by solving

$$\begin{aligned} \dot{X}(t) &= V(t), \quad \dot{V}(t) = \mathsf{P}_{\mathcal{H}_m}\left(\frac{1}{\varepsilon}(F_1[X(t),Y(t)] - V(t)) \right), \\ \dot{Y}(t) &= W(t), \quad \dot{W}(t) = \mathsf{P}_{\mathcal{H}_n}\left(\frac{1}{\varepsilon}(F_2[X(t),Y(t)] - W(t)) \right). \end{aligned} \tag{5.51}$$

We have that $\mathcal{H}_m = \mathcal{H}_{X(t)}$ and $\mathcal{H}_n = \mathcal{H}_{Y(t)}$ in $[0, t_1)$, thus the projection onto the set $\mathcal{H}_m$ yields functions defined on Ω that are constant on the same intervals where (X, V) is constant, and similarly the projection onto $\mathcal{H}_n$. Taking t_1 as the new initial time, we can consider a new initial condition $(\overline{X}', \overline{Y}', \overline{V}', \overline{W}') \in \mathcal{K}_{m'} \times \mathcal{K}_{n'} \times \mathcal{H}_{\overline{X}'} \times \mathcal{H}_{\overline{Y}'}$ of dimensions $N' \leq N$ and $M' \leq M$ and, proceeding in the same fashion, we can define $t_2 > t_1$ and consider the evolution in the time interval $[t_1, t_2)$. Iterating the procedure, we obtain a sequence of collision times $0 =: t_0 < t_1 < \cdots < t_K := \infty$ and the quadruple (X, Y, V, W) such that

$$\begin{aligned} \dot{X}(t) &= V(t), \quad \dot{V}(t) = \mathsf{P}_{\mathcal{H}_{X(t)}}\Big(\frac{1}{\varepsilon}\big(F_1[X(t), Y(t)] - V(t)\big)\Big), \\ \dot{Y}(t) &= W(t), \quad \dot{W}(t) = \mathsf{P}_{\mathcal{H}_{Y(t)}}\Big(\frac{1}{\varepsilon}\big(F_2[X(t), Y(t)] - W(t)\big)\Big), \end{aligned} \tag{5.52}$$

for all $t \in [t_{k-1}, t_k)$, $k = 1, \ldots, K$ with

$$\mathcal{H}_{X(t)} = \mathcal{H}_{X(t_{k-1})}, \quad \mathcal{H}_{Y(t)} = \mathcal{H}_{Y(t_{k-1})}. \tag{5.53}$$

When an inelastic collision occurs, we have that

$$\begin{aligned} X(t_k+) &= X(t_k-), \qquad V(t_k+) = \mathsf{P}_{\mathcal{H}_{X(t_k)}}(V(t_k-)), \\ Y(t_k+) &= Y(t_k-), \qquad W(t_k+) = \mathsf{P}_{\mathcal{H}_{Y(t_k)}}(W(t_k-)). \end{aligned} \tag{5.54}$$

In order to prove inclusion (5.19), it is not restrictive to assume $t_1 = 0$. We proceed by induction on the collision times. In the first time interval $[0, t_1)$, inclusion (5.19) holds by considering the empty set for the sub-differential $\partial I_{\mathcal{K}}(X(t))$. Now, suppose that (5.19) is satisfied in $[t_{k-1}, t_k)$. Hence, by induction assumption,

$$\varepsilon V(t_k-) + X(t_k-) + \xi = \varepsilon\overline{V} + \overline{X} + \int_0^{t_k} \mathsf{P}_{\mathcal{H}_{X(s)}}(F_1[X(s), Y(s)])\, ds \tag{5.55}$$

with $\xi \in \partial I_{\mathcal{K}}(X(t_k))$. By (5.52),

$$\begin{aligned} \varepsilon\dot{X}(t) + X(t) =& X(t_k+) + \varepsilon V(t_k+) + \int_{t_k}^{t} \mathsf{P}_{\mathcal{H}_{X(s)}}(F_1[X(s), Y(s)])\, ds \\ =& X(t_k+) + \varepsilon\big(V(t_k+) - V(t_k-)\big) + \varepsilon V(t_k-) \\ &+ \int_{t_k}^{t} \mathsf{P}_{\mathcal{H}_{X(s)}}(F_1[X(s), Y(s)])\, ds \end{aligned} \tag{5.56}$$

for any $t \in [t_k, t_{k+1})$. Combining equations (5.55) and (5.56) we get

$$\varepsilon\dot{X}(t) + X(t) + \varepsilon\big(V(t_k-) - V(t_k+)\big) + \xi = \varepsilon\overline{V} + \overline{X} + \int_0^{t} \mathsf{P}_{\mathcal{H}_{X(s)}}(F_1[X(s), Y(s)])\, ds.$$

Invoking again (5.52), we have

$$V(t_k-) = \lim_{h \to 0^+} \frac{X(t_k) - X(t_k - h)}{h},$$

hence using (5.54), we derive

$$\begin{aligned} V(t_k-) - V(t_k+) =& V(t_k-) - \mathsf{P}_{\mathcal{H}_{X(t_k)}}(V(t_k-)) \\ =& \lim_{h \to 0^+} \frac{X(t_k) - X(t_k - h) - \mathsf{P}_{\mathcal{H}_{X(t_k)}}\big(X(t_k) - X(t_k - h)\big)}{h} \\ =& \lim_{h \to 0^+} \frac{\mathsf{P}_{\mathcal{H}_{X(t_k)}}(X(t_k - h)) - X(t_k - h)}{h}. \end{aligned}$$

Applying [12, Lemma 2.6], we find that $V(t_k-) - V(t_k+) \in \partial I_{\mathcal{K}}(X(t_k))$, and using the monotonicity property of the sub-differential, one obtains that

$$\xi + V(t_k-) - V(t_k+) \in \partial I_{\mathcal{K}}(X(t))$$

for all $t \in [t_k, t_{k+1})$. Therefore inclusion (5.19) is satisfied. Now, let us prove that (5.21) holds. Consider system (5.25) with P replaced by

$$P_1(t,m) = \varepsilon\overline{V}(m) + \overline{X}(m) + \int_0^t F_1[X(r,\cdot), Y(r,\cdot)](m)\, dr.$$

Thus, we have that for any $t \geq s \geq 0$,

$$\frac{1}{\varepsilon}[P_1(s) - X(s)] - V(s) \in \partial I_{\mathcal{K}}(X(s)) \subset \partial I_{\mathcal{K}}(X(t)),$$

where we used the monotonicity of the sub-differential. Integrating on $s \in [0,t]$ we obtain

$$\int_0^t \frac{1}{\varepsilon}[P_1(s) - X(s)]\, ds + \overline{X} - X(t) \in \partial I_{\mathcal{K}}(X(t))$$

for a.e. $t \geq 0$. Since the following property holds (cf. [12])

$$Y = \mathsf{P}_{\mathcal{K}}(X) \iff X - Y \in \partial I_{\mathcal{K}}(Y),$$

we derive

$$X(t) = \mathsf{P}_{\mathcal{K}}\Big(\overline{X} - \frac{1}{\varepsilon}\int_0^t X(s)\, ds + \frac{1}{\varepsilon}t\big(\varepsilon\overline{V} + \overline{X}\big) + \frac{1}{\varepsilon}\int_0^t (t-s)F_1[X(s), Y(s)]\, ds\Big).$$

A similar proof holds for the equations (5.20) and (5.22). Finally, since the construction above starts form descrete initial data in the form of the piecewise constant functions as in (5.49)-(5.50), and since these functions are dense in $L^2(\Omega)$, we can approximate any given initial data and then combine the procedure into the proof with the stability Theorem 4.4 in [12]. $\square$

Now, we provide an estimate on the total energy of the system (5.46), used in the proof of next Theorem.

Lemma 5.4. *Let $(X, Y, V, W) \in \mathcal{K}^2 \times L^2(0,1)^2$ be the solution to the system (5.46) with initial data $(\overline{X}, \overline{Y}, \overline{V}, \overline{W})$. Then, the following uniform estimate holds:*

$$\begin{aligned} \sup_{t\geq 0} &\Big(\mathfrak{F}(X,Y) + \frac{1}{2}\|V\|^2_{L^2(\Omega)} + \frac{1}{2}\|W\|^2_{L^2(\Omega)}\Big) \\ &\leq \mathfrak{F}(\overline{X}, \overline{Y}) + \frac{1}{2}\|\overline{V}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\overline{W}\|^2_{L^2(\Omega)}. \end{aligned} \tag{5.57}$$

Proof. The proof is based on an estimate of the following total energy

$$\mathfrak{E}(X,Y,V,W) = \frac{1}{2}\int_\Omega |V|^2\, dm + \frac{1}{2}\int_\Omega |W|^2\, dm + \mathfrak{F}(X,Y).$$

Considering (X, Y, V, W) generalised solution to (5.46), we have

$$\begin{aligned}\frac{d^+}{dt}\mathfrak{E}(X,Y,V,W) = &-\sigma\int_\Omega \left(|V|^2+|W|^2\right)dm\\ &-\int_\Omega V\mathsf{P}_{\mathcal{H}_X}(F_1)\,dm - \int_\Omega W\mathsf{P}_{\mathcal{H}_Y}(F_2)\,dm\\ &+\int_\Omega \mathsf{P}_{\mathcal{H}_X}(V)\big[2m-1+\int_\Omega H'\big(X(m)-Y(m')\big)\,dm' + A_\rho'(X)\big]\,dm\\ &+\int_\Omega \mathsf{P}_{\mathcal{H}_Y}(W)\big[2m-1+\int_\Omega H'\big(Y(m)-X(m')\big)\,dm' + A_\eta'(Y)\big]\,dm.\end{aligned} \tag{5.58}$$

Thanks to the definitions of $F_1[X,Y]$ and $F_2[X,Y]$ in (5.47)-(5.48), we obtain that

$$\begin{aligned}\frac{d^+}{dt}\mathfrak{E}(X,Y,V,W) = &-\sigma\int_\Omega \left(|V|^2+|W|^2\right)dm - \int_\Omega V\mathsf{P}_{\mathcal{H}_X}(F_1)\,dm\\ &-\int_\Omega W\mathsf{P}_{\mathcal{H}_Y}(F_2)\,dm + \int_\Omega \mathsf{P}_{\mathcal{H}_X}(V)F_1\,dm + \int_\Omega \mathsf{P}_{\mathcal{H}_Y}(W)F_2\,dm.\end{aligned} \tag{5.59}$$

By definition of the projection operator in (5.13),

$$\int_\Omega \mathsf{P}_{\mathcal{H}_X}(V)\big(\mathsf{P}_{\mathcal{H}_X}(F_1[X,Y]) - F_1[X,Y]\big)\,dm = 0,$$

and

$$\int_\Omega \mathsf{P}_{\mathcal{H}_X}(F_1[X,Y])\big(\mathsf{P}_{\mathcal{H}_X}(V) - V\big)\,dm = 0,$$

then

$$\int_\Omega \big(F_1[X,Y]\mathsf{P}_{\mathcal{H}_X}(V) - V\mathsf{P}_{\mathcal{H}_X}(F_1[X,Y])\big)\,dm = 0,$$

and similarly

$$\int_\Omega \big(F_2[X,Y]\mathsf{P}_{\mathcal{H}_Y}(W) - W\mathsf{P}_{\mathcal{H}_Y}(F_2[X,Y])\big)\,dm = 0,$$

therefore (5.59) reduces to

$$\frac{d^+}{dt}\mathfrak{E}(X,Y,V,W) = -\sigma\int_\Omega |V|^2\,dm - \sigma\int_\Omega |W|^2\,dm \le 0, \tag{5.60}$$

from which we can easily deduce the uniform estimate (5.57). ☐

We can now provide the collapse result.

Theorem 5.3. *Let H be an interaction potential under assumptions* (**A**), (**SL**) *and* (**AT**). *Consider A_ρ, $A_\eta \in \mathcal{C}^2(\mathbb{R})$ as in* (**H1**) *and* (**H2**). *Let $(X,Y) \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty);\mathcal{K})^2$ be a generalised Lagrangian solution to* (5.44) *in the sense of Definition* 5.5. *Assume that the initial positions $(\overline{X},\overline{Y}) \in \mathcal{K}^2$ and velocities $(\overline{V},\overline{W}) \in \big(L^2(\Omega)\big)^2$ satisfy*

$$\|\overline{X}\|_{L^2} + \|\overline{Y}\|_{L^2} + \|\overline{V}\|_{L^2} + \|\overline{W}\|_{L^2} < \infty,$$

then

$$\lim_{t\to\infty}\left(\|X\|_{L^2} + \|Y\|_{L^2} + \|V\|_{L^2} + \|W\|_{L^2}\right) = 0.$$

Furthermore calling $\rho(t,\cdot) := \Psi^{-1}(X(t,\cdot))$ and $\eta(t,\cdot) := \Psi^{-1}(Y(t,\cdot))$, where Ψ is the isometry defined in (1.16), *we have*

$$\lim_{t\to\infty}\mathcal{W}_2^2\big((\rho,\eta),(\rho_s,\eta_s)\big) = 0.$$

Proof. Integrating in time the equation (5.60), we find that for all $T > 0$

$$\mathfrak{E}(X,Y,V,W)\mid_{t=T} + \sigma \int_0^T \int_\Omega \big(|V|^2 + |W|^2\big)\, dm\, dt = \mathfrak{E}(X,Y,V,W)\mid_{t=0}.$$

Thanks to the non-negativity of the cross-potential H, assumption (**H1**) and the fact that

$$\int_\Omega (2m-1)(X+Y)\, dm = -\int_\Omega (m^2-m)(\partial_m X + \partial_m Y)\, dm \geq 0, \tag{5.61}$$

which holds since $m^2 - m \leq 0$ for $m \in (0,1)$ and $\partial_m X + \partial_m Y \geq 0$ for $X, Y \in \mathcal{K}$, we obtain that

$$\sigma \int_0^T \int_\Omega \big(|V|^2 + |W|^2\big)\, dm\, dt \leq -\lambda \int_\Omega |X|^2 \mid_{t=T}\, dm - \mu \int_\Omega |Y|^2 \mid_{t=T}\, dm + C_1,$$

where C_1 is a constant depending on initial data, and λ and μ are the constants in assumption (**H1**) for both potentials. Thus

$$\int_0^\infty \int_\Omega \big(|V|^2 + |W|^2\big)\, dm\, dt < +\infty. \tag{5.62}$$

Computing the temporal derivative of the L^2-distance between (X,Y) and (X_s, Y_s), we derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \big(|X|^2 + |Y|^2\big)\, dm &= \int_\Omega X \mathsf{P}_{\mathcal{H}_X}(V)\, dm + \int_\Omega Y \mathsf{P}_{\mathcal{H}_Y}(W)\, dm \\ &= \int_\Omega X\big(\mathsf{P}_{\mathcal{H}_X}(V) - V\big)\, dm + \int_\Omega Y\big(\mathsf{P}_{\mathcal{H}_Y}(W) - W\big)\, dm \\ &\quad + \int_\Omega (XV + YW)\, dm \\ &= \int_\Omega (XV + YW)\, dm.\end{aligned} \tag{5.63}$$

In order to control the last term in the chain of equality above we compute

$$\begin{aligned}\frac{d}{dt}\int_\Omega (XV+YW)\, dm = &\int_\Omega X\big[-\sigma V - \mathsf{P}_{\mathcal{H}_X}(F_1)\big]\, dm + \int_\Omega V \mathsf{P}_{\mathcal{H}_X}(V)\, dm \\ &+ \int_\Omega Y\big[-\sigma W - \mathsf{P}_{\mathcal{H}_Y}(F_2)\big]\, dm + \int_\Omega W \mathsf{P}_{\mathcal{H}_Y}(W)\, dm.\end{aligned} \tag{5.64}$$

Using the definitions of F_1 and F_2 in (5.47) and (5.48) and the property for the projection operator we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega (XV+YW)\, dm = &\int_\Omega \big(-\sigma XV - \sigma YW + |V|^2 + |W|^2\big)\, dm \\ &- \int_\Omega (2m-1)(X+Y)\, dm \\ &- \int_\Omega\int_\Omega X(m) H'\big(X(m) - Y(m')\big)\, dm'\, dm \\ &- \int_\Omega\int_\Omega Y(m) H'\big(Y(m) - X(m')\big)\, dm'\, dm \\ &- \int_\Omega X A'_\rho(X)\, dm - \int_\Omega Y A'_\eta(Y)\, dm.\end{aligned} \tag{5.65}$$

Using assumption (**AT**) we can bound the terms involving the cross-interaction potential H as follows

$$
\begin{aligned}
&-\int_\Omega\int_\Omega X(m)H'\big(X(m)-Y(m')\big)-Y(m)H'\big(Y(m)-X(m')\big)\,dm'\,dm \\
&=-\int_\Omega\int_\Omega H'\big(X(m)-Y(m')\big)\big(X(m)-Y(m')\big)\,dm'\,dm\leq 0,
\end{aligned}
$$

thus, using assumption (**H2**) and (5.61), (5.65) can be bounded from above by

$$
\frac{d}{dt}\int_\Omega (XV+YW)\,dm\leq\int_\Omega\big(-\sigma XV-\sigma YW+|V|^2+|W|^2-\alpha|X|^2-\beta|Y|^2\big)\,dm. \tag{5.66}
$$

Note that for any $A>0$ we have $-XV\leq X^2A^2+\frac{V^2}{4A^2}$. Then, applying this inequality to $-\sigma XV$ and $-\sigma YW$, we obtain the following inequality holding for any $A_1,A_2>0$:

$$
\begin{aligned}
&\int_\Omega\big(-\sigma XV-\sigma YW+|V|^2+|W|^2-\alpha|X|^2-\beta|Y|^2\big)\,dm \\
&\leq-\int_\Omega|X|^2(\alpha-\sigma A_1^2)\,dm-\int_\Omega|Y|^2(\beta-\sigma A_2^2)\,dm \\
&\quad+\int_\Omega|V|^2\big(1+\frac{\sigma}{4A_1^2}\big)\,dm+\int_\Omega|W|^2\big(1+\frac{\sigma}{4A_2^2}\big)\,dm.
\end{aligned}
\tag{5.67}
$$

By taking sufficiently small A_1 and A_2, we have that (5.66) is bounded from above by

$$
\frac{d}{dt}\int_\Omega(XV+YW)\,dm\leq-\overline{C}_1\int_\Omega\big(|X|^2+|Y|^2\big)\,dm+\overline{C}_2\int_\Omega\big(|V|^2+|W|^2\big)\,dm \tag{5.68}
$$

for some constants $\overline{C}_1,\overline{C}_2>0$. Putting together estimates (5.63) and (5.68), we have that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\big(|X|^2+|Y|^2+XV+YW\big)\,dm \\
&\leq 2\int_\Omega\big(XV+YW\big)\,dm-\overline{C}_1\int_\Omega\big(|X|^2+|Y|^2\big)\,dm+\overline{C}_2\int_\Omega\big(|V|^2+|W|^2\big)\,dm.
\end{aligned}
\tag{5.69}
$$

Integrating in time inequality (5.69), for all $T>0$ we obtain

$$
\begin{aligned}
&\int_\Omega\big(|X|^2+|Y|^2+XV+YW\big)\,dm\,|_{t=T}-\int_\Omega\big(|X|^2+|Y|^2+XV+YW\big)\,dm\,|_{t=0} \\
&\leq 2\int_0^T\int_\Omega\big(XV+YW\big)\,dm\,dt-\overline{C}_1\int_0^T\int_\Omega\big(|X|^2+|Y|^2\big)\,dm\,dt \\
&\quad+\overline{C}_2\int_0^T\int_\Omega\big(|V|^2+|W|^2\big)\,dm\,dt,
\end{aligned}
$$

thus

$$
\begin{aligned}
\overline{C}_1\int_0^T\int_\Omega\big(|X|^2+|Y|^2\big)\,dm\,dt\leq{}&\overline{C}_2\int_0^T\int_\Omega\big(|V|^2+|W|^2\big)\,dm\,dt \\
&+2\int_0^T\int_\Omega\big(XV+YW\big)\,dm\,dt \\
&-\int_\Omega\big(|X|^2+|Y|^2+XV+YW\big)\,dm\,|_{t=T}+C_2,
\end{aligned}
$$

where C_2 is a constant which depends on initial data. Proceeding as in (5.67) and using the bound in (5.62), we have that

$$\int_0^\infty \int_\Omega \left(|X|^2 + |Y|^2\right) dm < +\infty. \tag{5.70}$$

Combining estimates (5.62) and (5.70) we get

$$\int_0^\infty \int_\Omega \left(|X|^2 + |Y|^2 + |V|^2 + |W|^2\right) dm\, dt < +\infty,$$

hence, there exists a subsequence $\{t_k\}_k$ such that

$$\int_\Omega \left(|X(t_k)|^2 + |Y(t_k)|^2 + |V(t_k)|^2 + |W(t_k)|^2\right) dm \to 0 \tag{5.71}$$

as $t_k \to +\infty$. Since the operator $\mathfrak{F}$ defined in (5.45) is a monotone operator, then

$$\mathfrak{F}(X,Y) + \frac{1}{2}\int_\Omega |V|^2\, dm + \frac{1}{2}\int_\Omega |W|^2\, dm \to \ell > 0$$

as $t \to +\infty$, and ℓ is unique. Moreover, Lemma 5.2 ensures that the operator $\mathfrak{F}$ is continuous, thus

$$\frac{1}{2}\int_\Omega |V|^2\, dm + \frac{1}{2}\int_\Omega |W|^2\, dm + \mathfrak{F}(X,Y)\mid_{t=t_k} \to \ell$$

as $t_k \to +\infty$. Using the coercivity of the external potentials A_ρ and A_η and (5.71), we have that ℓ is necessarily zero, hence the statement holds. □

5.6 Simulations

This last Section is devoted to provide some numerical examples on the behaviour of solutions to system (5.1). Numerical simulations will be performed by using the discrete particle counterpart of (5.1), namely solving numerically (5.3). We recall that the system of ODEs we are dealing with is the following

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{y}_j(t) = w_j(t), \\ \dot{v}_i(t) = -\sigma v_i(t) - \sum\limits_{k\neq i} m_k K_{11}'\big(x_i(t) - x_k(t)\big) \\ \qquad\qquad - \sum\limits_{k} n_k K_{12}'\big(x_i(t) - y_k(t)\big), \\ \dot{w}_j(t) = -\sigma w_j(t) - \sum\limits_{k\neq j} n_k K_{22}'\big(y_j(t) - y_k(t)\big) \\ \qquad\qquad - \sum\limits_{k} m_k K_{21}'\big(y_j(t) - x_k(t)\big), \end{cases} \tag{5.72}$$

where x_i and y_j denote the particles positions of first and second species respectively, v_i and w_j their velocities and m_i and n_j their masses, for $i = 1, \ldots, N$ and $j = 1, \ldots, M$. For simplicity we assume all the particles having the same mass. By a normalisation in

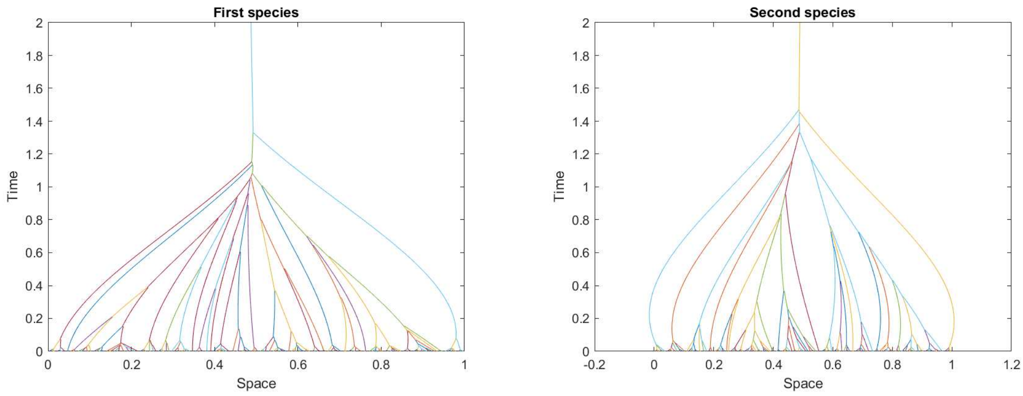

FIGURE 5.1: In this first example, we fix $N = 160$, and $M = 150$. All the potentials are attractive. In particular we set $K_\rho(x) = -e^{-|x|^3}$, $K_\eta(x) = -e^{-|x|^4}$, $H_\rho(x) = H_\eta(x) = -e^{-|x|^2}$.

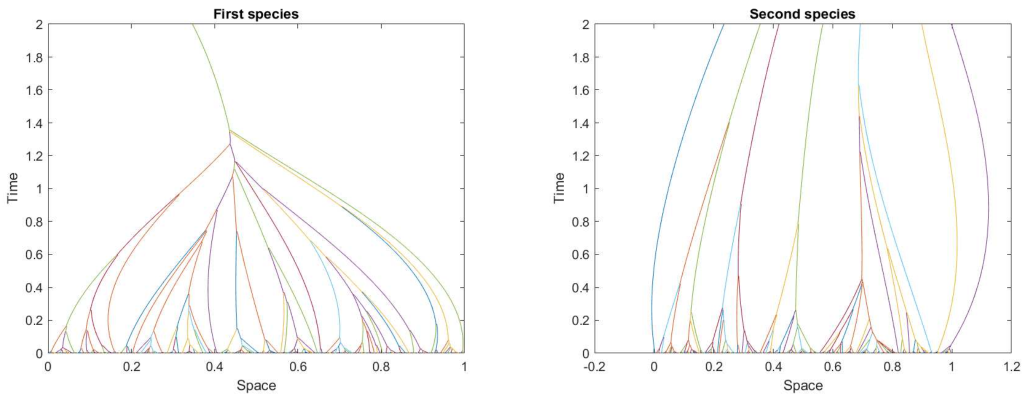

FIGURE 5.2: Evolution under the action of attractive self potentials that are given by $K_{11}(x) = -3e^{-|x|^2}$, and $K_{22}(x) = -2e^{-2|x|^3}$, and repulsive cross-potentials $K_{12}(x) = -|x|^2$, $K_{21}(x) = e^{-|x|^2}$. In this example, $N = 180$, $M = 200$.

the masses the total number of particles for each species, N and M respectively, will be modified in each of the examples below in order to highlights possible different changes in the solutions.

System (5.72) will be coupled with an uniform distributed set of particles in the space interval $[0, 1]$ and a random distrubution for the velocities. We then let the particles evolve by using an explicit second order three steps Runge-Kutta method, (cf. [45]) up to the first *collision*. In order to detect *collisions* between particles we fix a tolerance parameter *toll* and we assume that it occurs when the distance between two consecutive particles of the same species, for instance x_i and x_{i+1}, is smaller than *toll*. Once two consecutive particles *collide* they are replaced by a single particle with new position and velocity given by

$$x_{i+\frac{1}{2}}(t) = \frac{x_i(t) + x_{i+1}(t)}{2},$$
$$v_{i+\frac{1}{2}}(t) = \frac{v_i(t) + v_{i+1}(t)}{2},$$

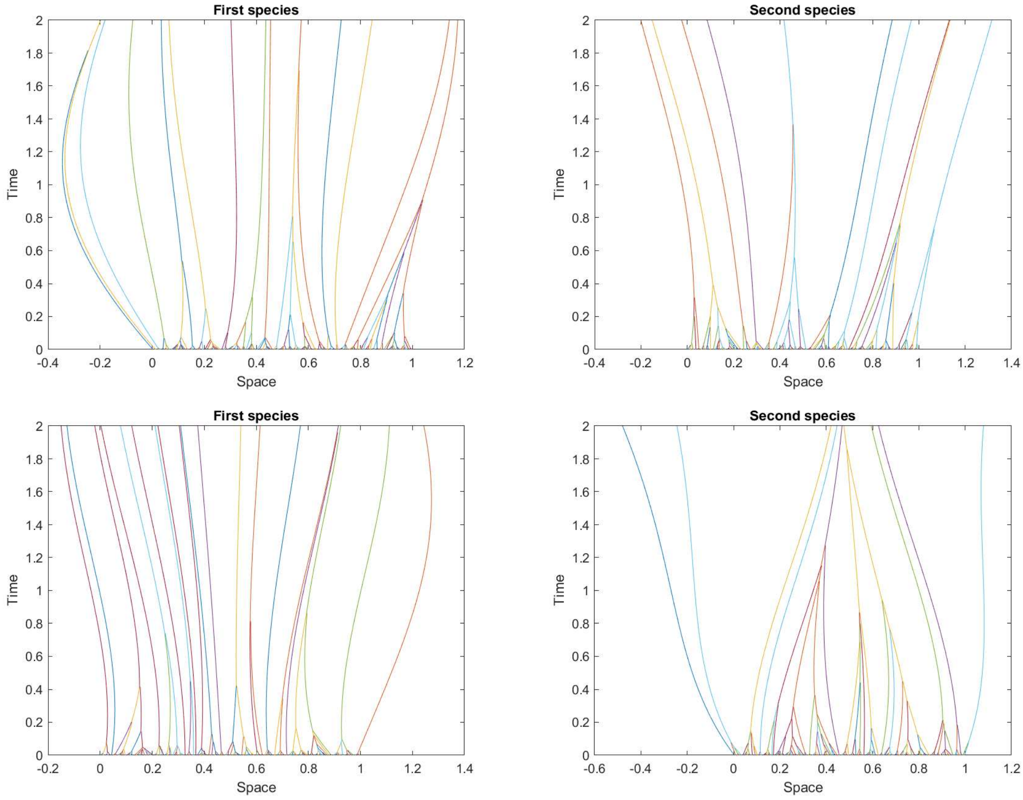

FIGURE 5.3: Two possible outcomes (top and bottom) for the evolution of the system under the action of self-repulsive potentials $K_{11}(x) = 2e^{-|x|^2}$ and $K_{22}(x) = e^{-|x|^3}$ and attractive cross-potentials $K_{12}(x) = |x|^2$ and $K_{21}(x) = -e^{-3|x|^2}$. In both the simulations the numbers of particles are fixed as $N = 170$ and $M = 160$, but initial velocities change (randomly).

and doubled mass, and we let the system evolve again with this new set of particles. In all the simulations below we fix $toll = 0.002$.

We study numerical solutions to the system (5.72) both in case of smooth potentials and in case of Newtonian self-potentials. Several examples are presented in the smooth case, where we highlight the possibility of a sticky dynamics, both in attractive and repulsive regime. Furthermore, we will compare solutions to second order system with solutions to first order one as the increasing values of the damping parameter σ, also comparing the Wasserstein distance between the solution to the second order system and the solution to the first order system as σ varies. Wasserstein distance is computed using its one-dimensional equivalence with the L^2-norm at the level of monotone rearrangements.

The first examples we provide concern the evolution of particles subject to the action of radial smooth potentials. Figure 5.1 displays the sticky particle dynamics when all the potentials are smooth and attractive. Instead, in Figure 5.2 the self-potentials are attractive and the cross-potentials are repulsive, while in Figure 5.3 the self-potentials are repulsive and the cross-potentials are attractive. In particular, we highlight how the behaviour is strongly different by comparing two simulations performed with the same potentials, number of particles and initial position, but different set of initial (random)

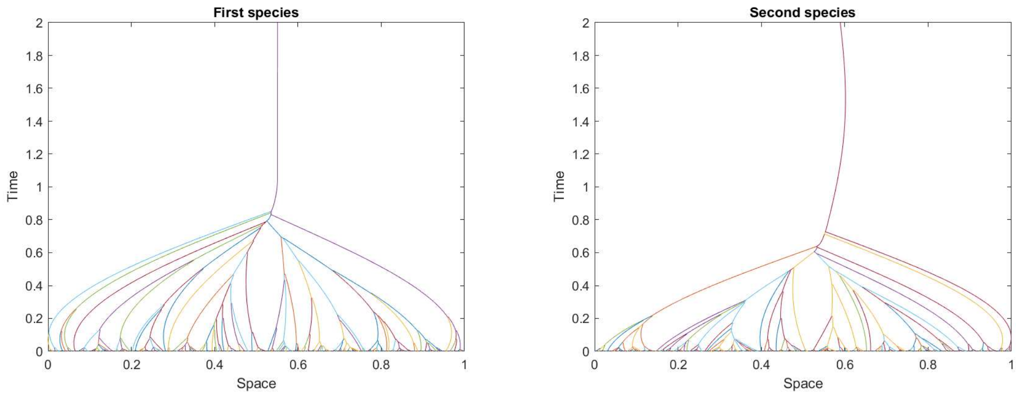

FIGURE 5.4: Evolution under the action of attractive Newtonian self-potentials and attractive Gaussian cross-potentials given by $K_{12}(x) = K_{21}(x) = -e^{-|x|^2}$. The external potentials are $A_\rho(x) = |x - \frac{1}{2}|^2$ and $A_\eta(x) = 2|x - \frac{1}{2}|^2$. In this example, $N = 200$ and $M = 210$.

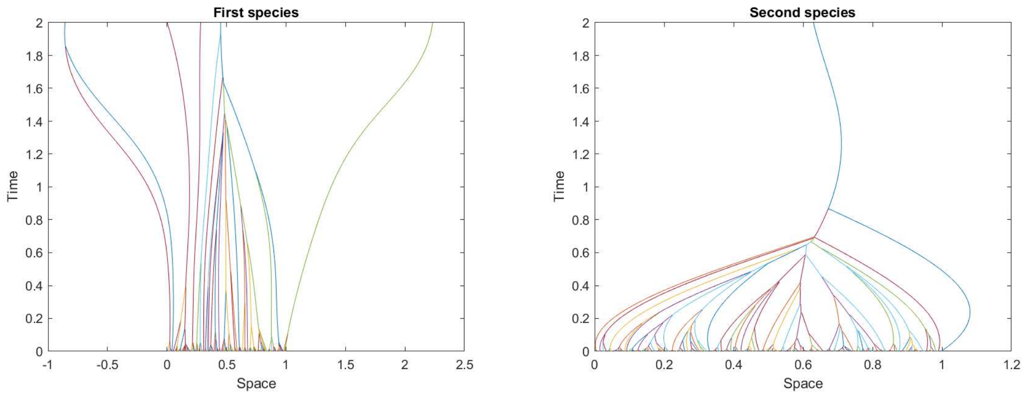

FIGURE 5.5: In this example, $N = 180$, $M = 190$, the self-potentials are Newtonian attractive and the cross-potentials are equal and repulsive. In particular they are $K_{12}(x) = K_{21}(x) = 3e^{-|x|^4}$. The external potentials are $A_\rho(x) = \frac{1}{2}|x - \frac{1}{2}|^2$ and $A_\eta(x) = 5|x - \frac{1}{2}|^2$.

velocities.

We then show a couple of simulations in which the self-potentials are attractive Newtonian, while the cross-potentials are symmetric, radial and smooth. In particular, in Figure 5.4, the cross-potentials are attractive, indeed the particles collide, while in Figure 5.5, they are repulsive and not all the particles collide. According to results in Section 5.5 also the effect of external potentials is taken into account.

We then focus on the numerical investigation of the large damping regime. Figures 5.6 and 5.7 show a comparison between the particle evolution associated to the second order system and the ones associated to the first order system (5.38), for various choices of potentials. We highlight numerically the relevance of the damping parameter σ in the evolution: increasing the value of σ solutions of the two different problems become indistinguishable.

Finally in Figure 5.8, considering the same potentials in Figure 5.4, we display the Wasserstein distance between the solution to the second order system and the ones to

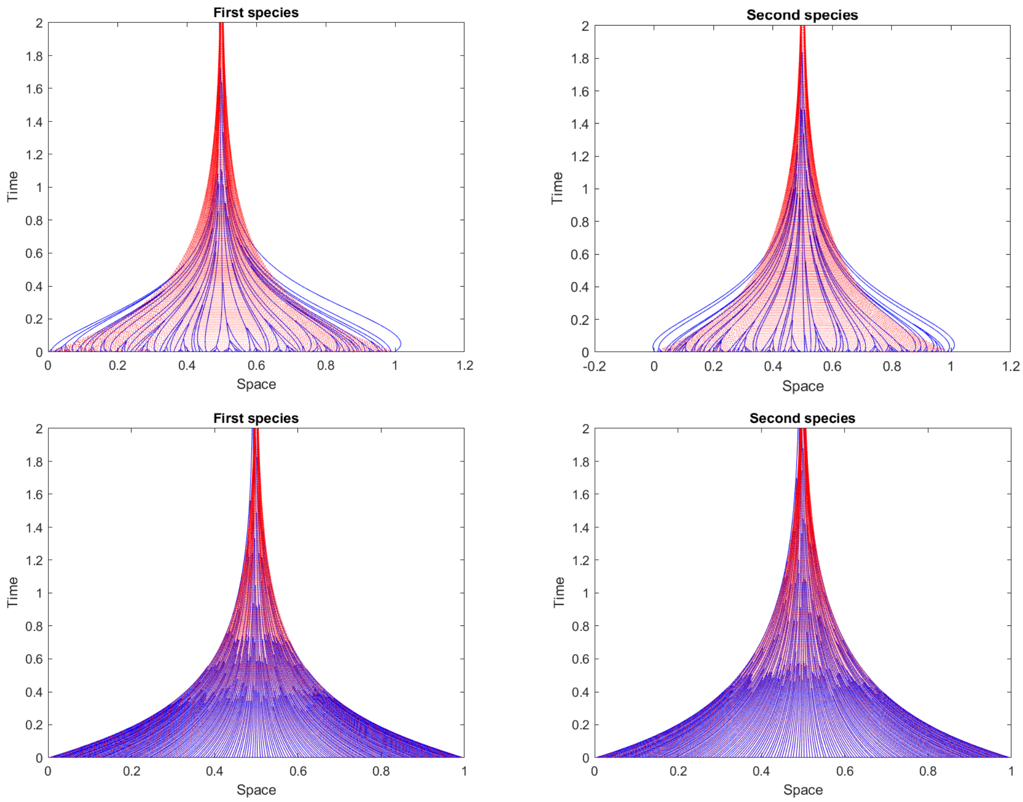

FIGURE 5.6: Solutions to the second order system (blue in the online version) and solutions to first order system (5.38) (in red) under the action of the following potentials are $K_{11}(x) = -e^{-|x|^3}$, $K_{22}(x) = -e^{-|x|^4}$, $K_{12}(x) = K_{21}(x) = -e^{-|x|^2}$. In this simulation we set $N = 160$, $M = 150$ and $\sigma = 10$ (top) and $\sigma = 1000$ (bottom).

the first order system for different values of σ. For small values of σ, the Wasserstein distance grows initially, and then decays in time. When σ is bigger, the distance remains controlled for all times.

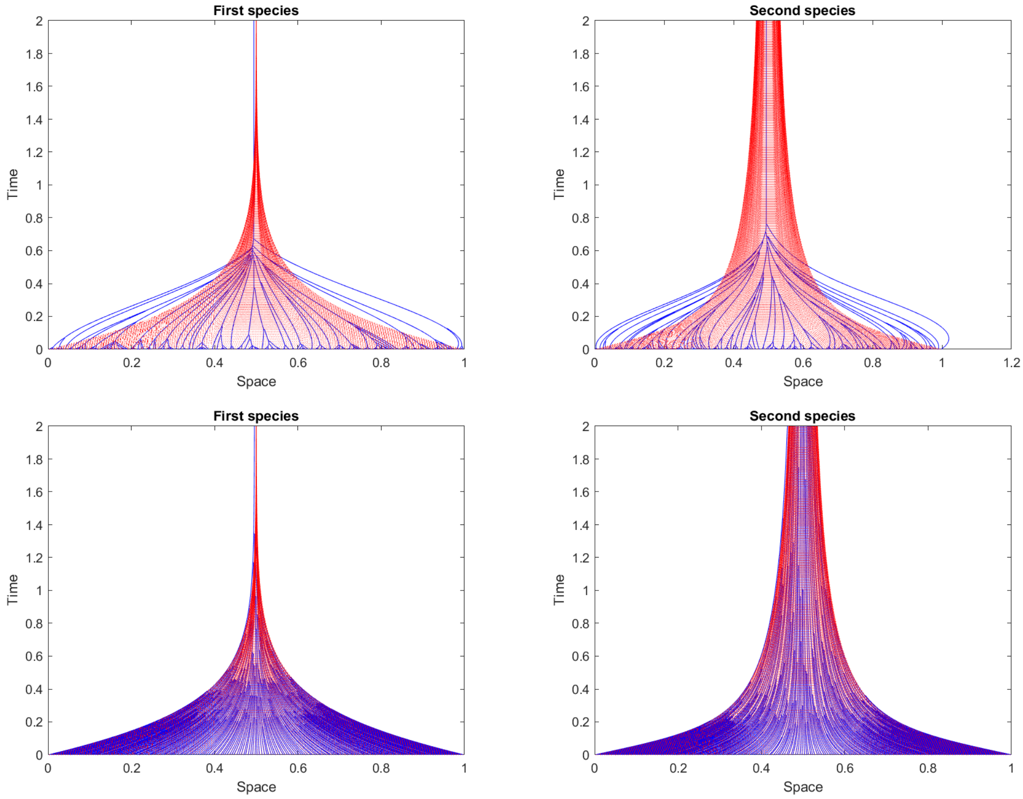

FIGURE 5.7: Solutions to the second order system (blue in the online version) and solutions to first order system (5.38) (in red) under the action of the following potentials are $K_{11}(x) = -e^{-|x|^2}$, $K_{22}(x) = -e^{-3|x|^3}$, $K_{12}(x) = |x|^2$, $K_{21}(x) = -e^{-2|x|^4}$. In this simulation we set $N = 180$, $M = 190$ and $\sigma = 5$ (top) and $\sigma = 900$ (bottom).

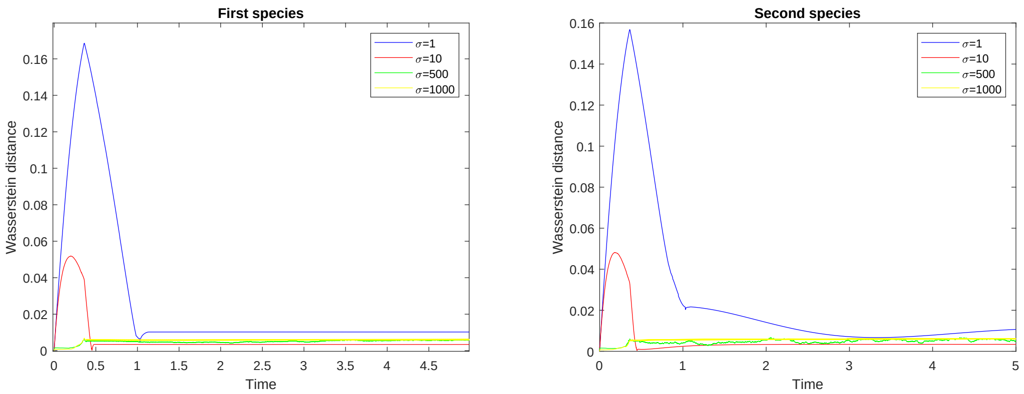

FIGURE 5.8: Behaviour of the Wasserstein distance between solutions of the first order system and solutions of the second order system. The self-potentials are Newtonian attractive potentials, while the cross-potentials are given by $K_{12}(x) = K_{21}(x) = -e^{-|x|^2}$. Increasing the damping parameter Wasserstein distance remain controlled.

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