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Revisiting the nonlinear elastic problem of internally constrained beams in a perturbation perspective

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ABSTRACT

Unshearable and inextensible planar beams, in a static regime of finite displacements, are studied in this paper. A nonlinear mixed model is derived via a direct approach, in which displacements and reactive internal forces are taken as unknowns. The elasto-static problem is then addressed, and the role of the boundary conditions is systematically discussed. The relevant solutions for selected classes of problems are pursued via a perturbation method. It is shown that each considered class calls for a specific algorithm, accounting for a proper scaling and expansion of the variables. Finally, the asymptotic solutions are compared with benchmark numerical computations, grounded on finite-element analyses. The paper is focused on the case of longitudinal force significantly smaller than the buckling load, leaving the case of large force to future developments, where a different perturbation scheme is required.

1. Introduction

Slender beams, the only ones for which geometrical nonlinearities entail a significant effect, have been investigated in the literature for many decades (Reissner, 1973; Antman, 1973; Holden, 1972; Simo and Vu-Quoc, 1988). Such class of beams is characterized by an essentially shear-undeformable behavior (Timoshenko and Goodier, 1951). Real beams, moreover, are also almost inextensible, since elongations are much smaller than (nondimensionalized) bending curvatures. Therefore, it is customary to neglect shear strains and elongations, and to resort to an internally constrained model (Antman, 1974; Takahashi, 1979; McHugh and Dowell, 2018; Luongo and Zulli, 2013).

In such a perspective, purely flexible beams (i.e., unshearable and inextensible, also named the Euler elastica), have often been considered in the literature, both in statics and dynamics, either as single beams (Mata et al., 2007; Luongo et al., 1986; Rincón-Casado et al., 2021), or as members of frames (Wood and Zienkiewicz, 1977; Contento and Luongo, 2013). Models have been obtained either via constrained variational principles (Di Carlo et al., 1981; Pignataro et al., 1990), in which the internal constraints are enforced via Lagrangian multiplier, or direct approaches (Di Egidio et al., 2007), in which the reactive internal stresses appear in the balance equations, together with kinematic descriptors. In all the cases, and limiting to consider planar beams, a different primary kinematic variable can be used (namely, either the slope θ of the tangent to the axis line, or the transverse displacement v), the remaining ones (always including the longitudinal displacement u) being related to the former ones by differential relationships. Moreover, different forms of the balance equations can be given, as a consequence of the application of the condensation of the reactive stresses, which, however, introduce arbitrary integration constants (Luongo and Zulli, 2013). The use of perturbation methods in such kinds of differential systems might be not straightforward; examples of their extensive use, specifically relevant to assemblies of purely flexible beams, is given in Pignataro et al. (1980) and Rizzi et al. (1980) where completely hand-worked applications dealing with simple frames are presented.

In this context, it is worth underlining that, while the unshearability can always be accepted, inextensibility is subordinated to the external constraints. If these allow a free shortening of the beam, i.e., if they allow the reduction of the chord connecting the two end-points with respect to the original length of the beam (as, e.g., it occurs for a hinged-supported beam), the inextensible model, although approximated, does not violate any kinematic prescription. In contrast, if the constraints prevent the shortening (as, e.g., it happens for a hinged-hinged beam), the inextensible model is kinematically not compatible, since in finite kinematics transverse displacements entail elongations. In such circumstances, the inextensible model must be abandoned, and elongations accounted for. An exception, however, is expected to occur

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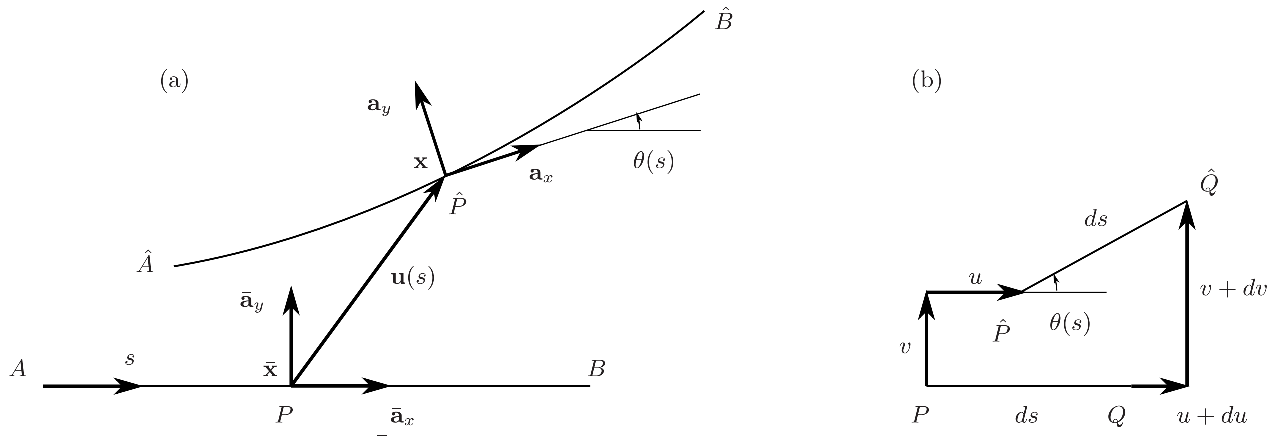

Fig. 1. Beam model: (a) triads $\bar{\mathcal{B}}, \mathcal{B}$, rotation $\theta(s)$ and displacement $\mathbf{u}(s)$; (b) displacement of an infinitesimal beam element.

when a non-zero shortening is assigned as imposed displacement of the boundary constraints, respectful of compatibility, and for which it is of interest to evaluate the associated state of stress.

In the literature, in spite of a considerable attention to internally constrained beams and to perturbation methods able to deal with them, it seems that a thorough and systematic analysis of the order of magnitude of the quantities involved is still missing. Namely, the following questions would desire to be answered. Are the internal stress components, dual of the internal constraints, of the same order of magnitude? Do they depend, and in which manner, on the boundary conditions? Can the problem, as usual in dealing with a weak nonlinear system, be brought to a sequence of linear problems, or do some cases exist in which the generating solution cannot be linearized? How to rescale data and unknowns so that a perturbation parameter appear in the equations? Overall: which is the proper asymptotic treatment of the equations governing the mechanics of inextensible and unshearable beams?

An attempt to answer the previous questions is made here, so far confining ourselves to static problems for a single beam. It will be shown that, depending on the boundary conditions, which make the beam non-redundant or redundant, in a sense to be specified later, as well as on the magnitude of the longitudinal loads, specific asymptotic treatments are required.

In particular, here, the position of the problem is provided, with the formulation of the relevant equilibrium equations. The description of the boundary conditions is also given, defining four different cases, which call for different perturbation approaches. In particular, two of the four mentioned cases are explicitly addressed here, namely those for which the longitudinal force can assume small values, i.e., far from the one which triggers buckling in the beam. However, the case of large longitudinal force is left for future developments, as it requires different treatment in a perturbation perspective.

Comparison with numerically exact solutions are also performed, to validate the asymptotic solutions, and to check their range of validity.

The paper is organized as follows: Section 2 deals with model formulation and strategy of solutions; Section 3 with the asymptotic treatment; Section 4 with numerical analyses; Section 5 with Conclusions. Finally, an Appendix supplies some details.

2. The beam model

In what follows, we derive the constrained problem for an inextensible and unshearable beam. We consider an initially straight beam, modeled as one-dimensional polar continuum embedded in a 2D space (Luongo and Zulli, 2013). The beam is referred to a material abscissa $s \in [0, l]$ along the axis, running from the left A to the right B points (Fig. 1-a), with l the length of the line. We consider rigid cross-sections which are normal to the axis, and attach to them an orthonormal triad of directors, $\bar{\mathcal{B}} := (\bar{\mathbf{a}}_x, \bar{\mathbf{a}}_y, \bar{\mathbf{a}}_z)$, in which $\bar{\mathbf{a}}_x$ is the tangent to the beam axis, $\bar{\mathbf{a}}_y$ is the (in-plane) normal, and $\bar{\mathbf{a}}_z$ is the binormal, all independent of s. Therefore, indicating with $\bar{\mathbf{x}}(s)$ the position of a generic point P of the axis line at abscissa s, then $\bar{\mathbf{x}}'(s) = \bar{\mathbf{a}}_x$, where prime stands for differentiation with respect to s.

2.1. Deriving the constrained field equations

2.1.1. Kinematics

When the beam moves to the current configuration (Fig. 1), the beam axis bends and occupies a smooth curve of ends $\hat{A}, \hat{B}$. Owing to unshearability, the cross-sections remain normal to the curved axis, so that the triad $\bar{\mathcal{B}}$ changes into $\mathcal{B} = \left(\mathbf{a}_x(s), \mathbf{a}_y(s), \mathbf{a}_z \equiv \bar{\mathbf{a}}_z\right)$, where $\mathbf{a}_x = \cos\theta(s)\bar{\mathbf{a}}_x + \sin\theta(s)\bar{\mathbf{a}}_y$ is the tangent vector to the current axis line, $\mathbf{a}_y(s)$ the normal and $\mathbf{a}_z$ the binormal; finally, $\theta(s)$ is the rotation, equal to the angle between $\mathbf{a}_x$ and $\bar{\mathbf{a}}_x$. The derivative of the rotation is defined as the bending curvature of the beam, i.e. $\kappa := \theta'(s)$.

Bending of the beam entails that the position $\bar{\mathbf{x}}$ moves to the new position $\mathbf{x} := \bar{\mathbf{x}}(s) + \mathbf{u}(s)$, where $\mathbf{u}(s) := u(s)\bar{\mathbf{a}}_x + v(s)\bar{\mathbf{a}}_y$ is the displacement of P. Owing to inextensibility, the tangent in the current configuration, $\mathbf{x}' = \lambda \mathbf{a}_x$ must have unitary modulus $\lambda = 1$, i.e., $\bar{\mathbf{a}}_x + \mathbf{u}' = \mathbf{a}_x$ must hold. When this vector relationship is projected onto the basis $\bar{\mathcal{B}}$, it follows that:

$$
\begin{aligned}
u' &= \cos\theta(s) - 1, \\
v' &= \sin\theta(s).
\end{aligned}
\tag{1}
$$

The scalar relationships (1) are *internal constraints*; their geometrical meaning clearly emerges from Fig. 1-b. The relationships in Eq. (1) can be integrated to express displacements in terms of rotations, namely:

$$
\begin{aligned}
u(s) &= u_A + \int_0^s (\cos\theta(s) - 1) ds, \\
v(s) &= v_A + \int_0^s \sin\theta(s) ds,
\end{aligned}
\tag{2}
$$

where $u_A := u(0)$, $v_A := v(0)$ are integration constants representing displacements at the left end. When $s = l$, it also follows:

$$
\begin{aligned}
&\int_0^l (\cos\theta(s) - 1) ds = u_B - u_A, \\
&\int_0^l \sin\theta(s) ds = v_B - v_A,
\end{aligned}
\tag{3}
$$

where $u_B := u(l)$, $v_B := v(l)$ are displacements at the right end. The conditions (1) express the fact that, because of inextensibility (and unshearability), the differences of the end displacements are not independent of the rotation field. In particular, they state that, when $|\theta| < \pi/2$ (i.e., in the field of technical interest), then $\Delta := u_B - u_A < 0$.

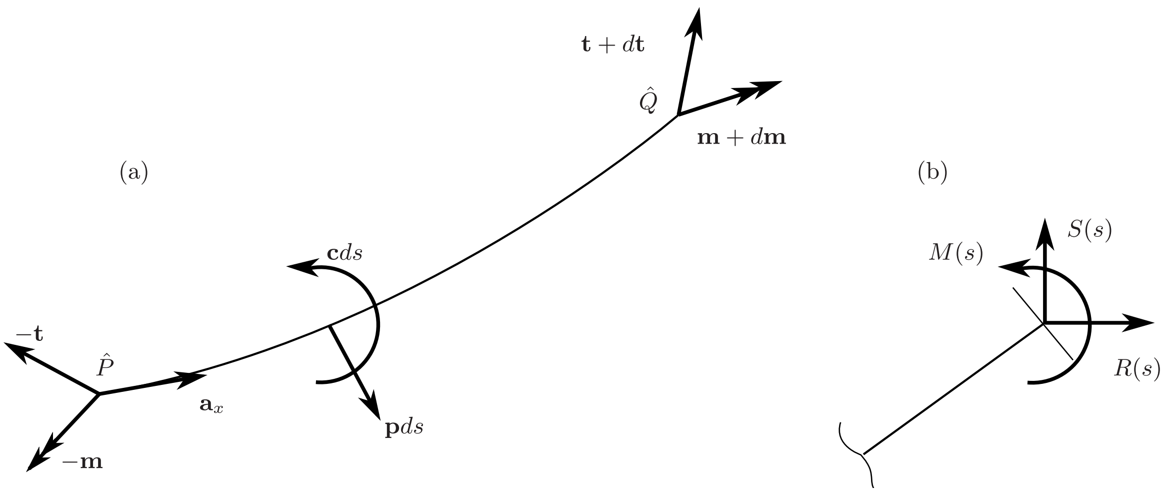

Fig. 2. Element of a planar beam: (a) forces and stresses; (b) scalar components of stresses.

The quantity Δ is usually referred to as the *shortening* of the beam. On the other hand, if u_A, v_A, u_B, v_B are freely assigned (respectfully of $\Delta < 0$), then $\theta(s)$ cannot be chosen arbitrarily, but it has to satisfy the two integral constraints (1). For all these reasons, we will call Eqs. (3) the (internal) *compatibility conditions*.

2.1.2. Equilibrium

The balance equations, expressing the equilibrium of forces and moments, in vector form are (see Fig. 2-a) (Luongo and Zulli, 2013):

$$\begin{aligned} &\mathbf{t}'(s) + \mathbf{p}(s) = \mathbf{0}, \\ &\mathbf{m}'(s) + \mathbf{a}_x(s) \times \mathbf{t}(s) = \mathbf{c}(s), \end{aligned} \tag{4}$$

where $\mathbf{t}(s)$ is the stress-force and $\mathbf{m}(s)$ the stress-couple at the material abscissa s; $\mathbf{p} := p_x(s)\bar{\mathbf{a}}_x + p_y(s)\bar{\mathbf{a}}_y, \mathbf{c} = c(s)\,\bar{\mathbf{a}}_z$ are the linear density of the external body forces and couples, respectively. For the sake of simplicity, $\mathbf{c}(s) = \mathbf{0}$ will be taken ahead. The following representation (in $\bar{\mathcal{B}}$) is introduced: $\mathbf{m}(s) := M(s)\bar{\mathbf{a}}_z$, $\mathbf{t}(s) := R(s)\bar{\mathbf{a}}_x + S(s)\bar{\mathbf{a}}_y$, in which $M(s)$ is the bending moment, $R(s)$ the longitudinal component and $S(s)$ the transverse component of the force-stress (see Fig. 2-b). Note that these latter are different from the more usual normal and shear forces, which are instead defined in $\mathcal{B}$. By projecting the balance Eqs. (4) onto the basis $\bar{\mathcal{B}}$, we obtain:

$$\begin{aligned} &R' + p_x = 0, \\ &S' + p_y = 0, \\ &M' + S\cos\theta - R\sin\theta = 0. \end{aligned} \tag{5}$$

After integrating the two equations ((5)-a,b), we get:

$$\begin{aligned} &R(s) = R_B + \int_s^l p_x(s)ds, \\ &S(s) = S_B + \int_s^l p_y(s)ds, \end{aligned} \tag{6}$$

where $R_B := R(l)$, $S_B := S(l)$; consequently, the moment equation ((5)-c) reads:

$$M'(s) + \left(S_B + \int_s^l p_y(s)ds\right)\cos\theta(s) - \left(R_B + \int_s^l p_x(s)ds\right)\sin\theta(s) = 0. \tag{7}$$

This is a differential equation for the unknown bending moment $M(s)$; however, it involves two additional quantities, i.e., R_B, S_B, which are generally unknown (and, therefore, referred to as redundant unknowns).

2.1.3. The constrained elastic problem

By assuming the linear elastic law $M = EJ\kappa(s)$, in which EJ is the bending stiffness, the elasto-static problem is therefore governed by the following integro-differential problem:

$$\begin{aligned} &EJ\theta'' + \left(S_B + \int_s^l p_y ds\right)\cos\theta - \left(R_B + \int_s^l p_x ds\right)\sin\theta = 0, \\ &\int_0^l (\cos\theta - 1)ds = u_B - u_A, \\ &\int_0^l \sin\theta ds = v_B - v_A, \end{aligned} \tag{8}$$

with relevant boundary conditions, to be discussed soon. Here, Eq. ((8)-a) is the equilibrium equation descending from Eq. (7) and the definition of the bending curvature, and Eqs. ((8)-b,c) are compatibility conditions.

From a mechanical point of view, the relationships in Eqs. (8) display the mixed (displacements and forces) nature of the approach; from a mathematical point of view, they constitute a mixed differential–algebraic problem in the unknown field $\theta(s)$ and constants. Once the problem is solved, then: (a) the bending moment follows from $M(s) = EJ\theta'(s)$; (b) the reactive stresses $R(s), S(s)$ from the equilibrium equations (6); (c) the displacements $u(s), v(s)$ from the kinematic compatibility (2).

2.2. A discussion on the boundary conditions

If the beam is isolated, i.e., it does not interact with adjacent beams, boundary conditions at the ends A and B alternatively prescribe either displacements and rotations u_H, v_H, θ_H or force- and couple-stresses R_H, S_H, M_H, with $H = A, B$. The six boundary conditions and the two integral equations ((8)-b,c) balance the six constants $u_A, v_A, u_B, v_B, R_B, S_B$ and the two constants arising from integration of the differential equation ((8)-a). To discuss the different classes of existing problems, we observe that, due to the mixed nature of formulation: (a) the conditions prescribing rotations θ_H or bending moments M_H at the ends do not entail any differences in the procedure, since M_H is active, and equilibrium is expressed in terms of rotations, according to the spirit of the displacement method; (b) in contrast, conditions on translations u_H, v_H, or, alternatively, on R_H, S_H, do affect the procedure, since R_H, S_H are reactive stresses, which are balanced by compatibility conditions, according to the spirit of the force method. To exclude kinematically undetermined cases, we assume that, e.g., u_A, v_A are always prescribed; then, according to conditions on the translation of B, the local stresses R_B, S_B, can be, either: (i) both known, (ii) one known and the other unknown, or (iii) both unknown.

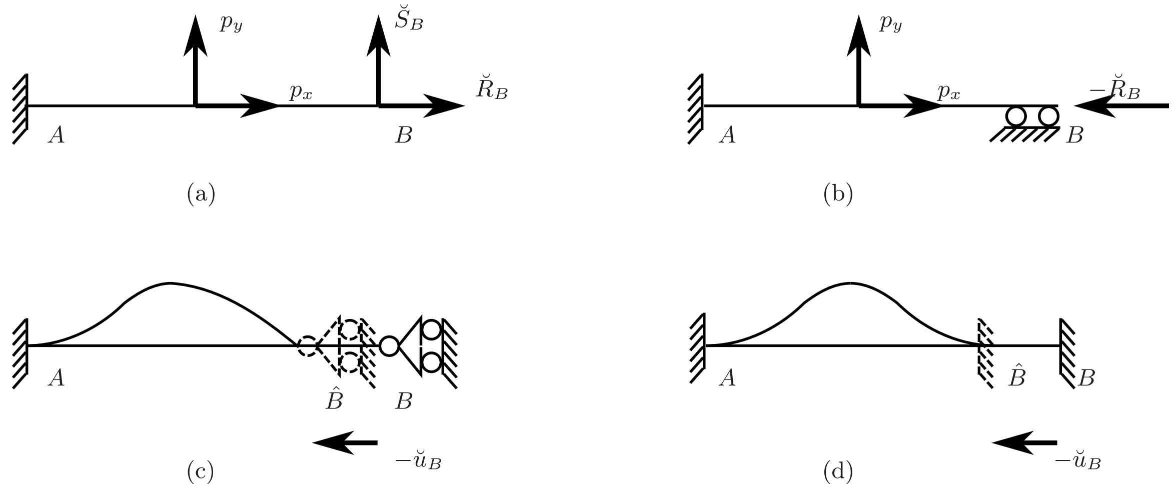

Fig. 3. Representative systems: (a) non-redundant; (b) S-redundant; (c) R-redundant, (d) RS-redundant.

We call the unknown R_B, S_B the *redundant stresses* (which should not be confused with the hyperstatic unknowns of the equilibrium equations in terms of stresses — for instance, a beam hinged at A and simply supported at B, which is isostatic, possesses a redundant stress S_B in the mixed formulation).

By summarizing, we distinguish four classes of problems, according to the nature of R_B, S_B: (1) *non-redundant*, (2) *S-redundant*, (3) *R-redundant*, (4) *RS-redundant*. Systems representative of these classes are, e.g. (see Fig. 3): (1) the clamped-free beam (Fig. 3-a), in which no unknown stresses appear, since R_B, S_B are both prescribed (as $\check{R}_B, \check{S}_B$, where the symbol ˇ indicates the prescribed value); (2) the clamped-sliding beam (Fig. 3-b), in which only S_B is unknown; (3) the clamped-horizontally supported beam (Fig. 3-c), in which only R_B is unknown; (3) the clamped-clamped beam (Fig. 3-d), in which both R_B, S_B are unknown. In cases (1), (2) the shortening $\Delta := u_B - u_A$ is free; in cases (3), (4) it is prescribed. The relevant boundary conditions, and the solution strategy for the four representative problems are discussed in what follows:

- *Non-redundant* clamped-free beam:

$$\begin{aligned} &u_A = \check{u}_A, \quad v_A = \check{v}_A, \quad \theta_A = \check{\theta}_A, \\ &R_B = \check{R}_B, \quad S_B = \check{S}_B, \quad EJ\theta'_B = \check{M}_B, \end{aligned} \tag{9}$$

  where use has been made of the constitutive law. Since the parameters R_B, S_B are known, as directly determined by the boundary conditions, the field equation ((8)-a) can be integrated (with the prescribed values at the boundaries of θ_A, θ'_B), to supply $\theta(s)$. Consequently, the conditions ((8)-b,c) furnish the free displacements u_B, v_B, once the known values of u_A, v_A have been accounted for. Overall, the problem (8) degenerates into a sequence of independent steps, as it also happens in the linear field.
- *S-redundant,* clamped-sliding beam:

$$\begin{aligned} &u_A = \check{u}_A, \quad v_A = \check{v}_A, \quad \theta_A = \check{\theta}_A, \\ &R_B = \check{R}_B, \quad v_B = \check{v}_B, \quad \theta_B = \check{\theta}_B. \end{aligned} \tag{10}$$

  Now, R_B is known, while S_B is a redundant stress. The field equation ((8)-a) and the boundary conditions on θ_A, θ_B furnish ∞^1 solutions, denoted as $\theta(s; S_B)$. Then, substitution into the compatibility conditions ((8)-c), leads to a *nonlinear compatibility equation* for the redundant unknown. Once this has been solved, Eq. ((8)-b) furnishes u_B.
- *R-redundant,* clamped-horizontally supported beam:

$$\begin{aligned} &u_A = \check{u}_A, \quad v_A = \check{v}_A, \quad \theta_A = \check{\theta}_A, \\ &S_B = \check{S}_B, \quad u_B = \check{u}_B, \quad \theta_B = \check{\theta}_B. \end{aligned} \tag{11}$$

  In this case, S_B is known, while R_B is redundant. The field equation ((8)-a) and the boundary conditions on θ_A, θ_B furnish ∞^1 solutions, of type $\theta(s; R_B)$. Then, substitution into the compatibility conditions ((8)-b), leads to a *nonlinear compatibility equation* for R_B. Successively, Eq. ((8)-c) furnishes v_B.
- *RS-redundant,* clamped-clamped beam

$$u_H = \check{u}_H, \quad v_H = \check{v}_H, \quad \theta_H = \check{\theta}_H, \quad H = A, B. \tag{12}$$

  Since R_B, S_B are both redundant, ∞^2 solutions $\theta(s; R_B, S_B)$ are found from the balance equation. The two compatibility conditions ((8)-b,c), supply the *coupled nonlinear compatibility equations* for the two redundant unknowns.

3. Asymptotic analysis

We now address the nonlinear elasto-static problem (8) for a single beam. Since the field equation is nonlinear, and cannot be solved in closed-form, we apply a perturbation method.

It is worth noticing that, for the first two types of boundary problems represented in Fig. 3-a,b, namely the non-redundant and the S-redundant cases, respectively, the longitudinal force R_B is prescribed and therefore it can assume either small values, i.e. much smaller than the first buckling load, or large values, i.e. comparable or larger than the first buckling load. However, in the boundary problems represented in Fig. 3-c,d, namely the R-redundant and the RS-redundant cases, respectively, the imposition of a non-negligible shortening induces large values for R_B. The latter aspect is related to the inextensional feature of the beam model, where an imposed shortening is necessarily accompanied by transverse force or imposed displacement, and induces large longitudinal reaction.

Here, we focus on the perturbation solution of the first two types of boundary problems, namely the non-redundant and the S-redundant cases represented in Fig. 3-a,b, respectively, when R_B is significantly smaller that the buckling load. However, the proper asymptotic treatment of the remaining cases, namely the R-redundant and the RS-redundant cases, represented in Fig. 3-c,d, respectively, will be object of future developments. Indeed, these latter boundary conditions require a scaling for R different from the one adopted in what follows, i.e., $R_B = O(1)$, to address the shortening problem properly. Coherently, the study of the first two cases (Fig. 3-a,b) when $R_B = O(1)$, is also postponed to future developments.

3.1. The non-redundant case

We study a clamped-free beam, for which the boundary conditions (9) hold. The Fundamental Problem (8), consequently becomes:

$$
\begin{aligned}
&EJ\theta'' + \left(\check{S}_B + \int_s^l p_y ds\right)\cos\theta - \left(\check{R}_B + \int_s^l p_x ds\right)\sin\theta = 0,\\
&\theta_A = \check{\theta}_A, \quad \theta'_B = \check{M}_B/EJ,\\
&u_B = \check{u}_A + \int_0^l (\cos\theta - 1)ds, \qquad v_B = \check{v}_A + \int_0^l \sin\theta ds.
\end{aligned} \tag{13}
$$

First, we transform nonlinearities in polynomial form, by Taylor-expanding the harmonic functions for small rotations θ, i.e.:

$$
\begin{aligned}
&\cos\theta = 1 - \frac{1}{2}\theta^2 + \cdots\\
&\sin\theta = \theta - \frac{1}{6}\theta^3 + \cdots
\end{aligned} \tag{14}
$$

Then, we have to properly rescale the known terms, in order a perturbation parameter $0 < \epsilon \ll 1$ appears; moreover, we rescale and expand the unknowns in series of ϵ. Thus, by assuming that the known-terms are small quantities, we put:

$$
\begin{aligned}
&p_x = \epsilon p_x^\star, \quad p_y = \epsilon p_y^\star, \quad \check{\theta}_A = \epsilon\check{\theta}_A^\star, \quad \check{M}_B = \epsilon\check{M}_B^\star,\\
&\check{u}_A = \epsilon\check{u}_A^\star, \quad \check{v}_A = \epsilon\check{v}_A^\star, \quad \check{R}_B = \epsilon\check{R}_B^\star, \quad \check{S}_B = \epsilon\check{S}_B^\star.
\end{aligned} \tag{15}
$$

Then, we both order and expand the unknowns as (star omitted on all variables):

$$
\begin{aligned}
&\theta(s) = \epsilon\theta_1(s) + \epsilon^2\theta_2(s) + \epsilon^3\theta_3(s) + \cdots\\
&u_B = \epsilon u_{B1} + \epsilon^2 u_{B2} + \epsilon^3 u_{B3} + \cdots\\
&v_B = \epsilon v_{B1} + \epsilon^2 v_{B2} + \epsilon^3 v_{B3} + \cdots
\end{aligned} \tag{16}
$$

Finally, substituting Eqs. (14)–(16) in Eqs. (13), and separately equating to zero the coefficients with the same power of ϵ, we obtain the perturbation equations:

Order ϵ^1:

$$
\begin{aligned}
&EJ\theta_1'' = -\check{S}_B - \int_s^l p_y ds,\\
&\theta_{A1} = \check{\theta}_A, \quad \theta'_{B1} = \check{M}_B/EJ,\\
&u_{B1} = \check{u}_A, \qquad v_{B1} = \check{v}_A + \int_0^l \theta_1 ds.
\end{aligned} \tag{17}
$$

Order ϵ^2:

$$
\begin{aligned}
&EJ\theta_2'' = \left(\check{R}_B + \int_s^l p_x ds\right)\theta_1,\\
&\theta_{A2} = 0, \quad \theta'_{B2} = 0,\\
&u_{B2} = -\frac{1}{2}\int_0^l \theta_1^2 ds, \qquad v_{B2} = \int_0^l \theta_2 ds.
\end{aligned} \tag{18}
$$

Order ϵ^3:

$$
\begin{aligned}
&EJ\theta_3'' = \frac{1}{2}\left(\check{S}_B + \int_s^l p_y ds\right)\theta_1^2 + \left(\check{R}_B + \int_s^l p_x ds\right)\theta_2,\\
&\theta_{A3} = 0, \quad \theta'_{B3} = 0,\\
&u_{B3} = -\int_0^l \theta_1\theta_2 ds, \qquad v_{B3} = \int_0^l \left(\theta_3 - \frac{1}{6}\theta_1^3\right) ds.
\end{aligned} \tag{19}
$$

All the perturbation equations call for chain-solving the same *linear* differential equation:

$$
\left[EJ\frac{d^2}{ds^2}\right]\theta_k = q_k(s), \qquad k = 1, 2, \ldots \tag{20}
$$

where the differential operator has been formally expressed in the square brackets, and $q_k(s)$ are know terms. By using the boundary conditions, a unique $\theta_k(s)$ is evaluated; then, from the compatibility conditions, u_{Bk}, v_{Bk} are computed in turn. Thus, all the coefficients of the series (16) are determined.

The perturbation equations possess the following mechanical meaning. Order-ϵ^1 equations are the equilibrium and compatibility equations of the *linear* beam, for which equilibrium is enforced in the straight undeformed configuration. Accordingly, while $\check{S}_B$ enters in the moment equation, $\check{R}_B$ does not contribute to it. Moreover, linear kinematics states that, as a consequence on inextensibility, the *longitudinal displacements at the ends are equal* at this order, i.e., $u_B - u_A = 0$. Higher-order equations are still the equations for the linear beam, but carrying 'fictitious loads' and 'fictitious distortions' generated by nonlinearities. Thus, the order-ϵ^2 equilibrium equation accounts for bending moments caused by the known longitudinal forces, $\check{R}_B, p_x$, acting on the deformed configuration, evaluated at the previous order. Similarly, the order-ϵ^3 equation considers: (a) the same forces acting on an updated configuration and, (b) the transverse forces $\check{S}_B, p_y$, again, whose arm, however, is modified to account for deformation. On the other hand, higher-order compatibility equations state that the free end B undergoes a longitudinal displacement, such that $\Delta := u_B - u_A < 0$.

It should be noticed that, in an inextensible beam, the shortening is a second-order effect. Therefore, while u_A, u_B are individually first-order quantities, their difference $u_B - u_A$ is a second-order quantity. Said in other words, u_A, u_B are not only small, but *nearly-identical*.

3.2. The S -redundant case

We investigate the effects of the redundant constraints, when the shortening is free. Considering a beam clamped at A and constrained by a slider at B, for which the boundary conditions (10) hold (Fig. 3-b). For this case, we still assume here that the longitudinal force $R_B = \check{R}_B$ is small, of the same order of the integral of the transverse forces.

The Fundamental Problem (8) becomes:

$$
\begin{aligned}
&EJ\theta'' + \left(S_B + \int_s^l p_y ds\right)\cos\theta - \left(\check{R}_B + \int_s^l p_x ds\right)\sin\theta = 0,\\
&\theta_A = \check{\theta}_A, \quad \theta_B = \check{\theta}_B,\\
&\int_0^l (\cos\theta(s) - 1)\, ds = u_B - \check{u}_A, \qquad \int_0^l \sin\theta(s) ds = \check{v}_B - \check{v}_A.
\end{aligned} \tag{21}
$$

As before, we assume that all known-terms are of order ϵ, according to:

$$
\begin{aligned}
&p_x = \epsilon p_x^\star, \quad p_y = \epsilon p_y^\star, \quad \check{\theta}_A = \epsilon\check{\theta}_A^\star, \quad \check{\theta}_B = \epsilon\check{\theta}_B^\star,\\
&\check{u}_A = \epsilon\check{u}_A^\star, \quad \check{v}_A = \epsilon\check{v}_A^\star, \quad \check{R}_B = \epsilon\check{R}_B^\star, \quad \check{v}_B = \epsilon\check{v}_B^\star.
\end{aligned} \tag{22}
$$

Moreover, we order and expand the unknowns as (star omitted on all variables):

$$
\begin{aligned}
&\theta(s) = \epsilon\theta_1(s) + \epsilon^2\theta_2(s) + \epsilon^3\theta_3(s) + \cdots\\
&u_B = \epsilon u_{B1} + \epsilon^2 u_{B2} + \epsilon^3 u_{B3} + \cdots\\
&S_B = \epsilon S_{B1} + \epsilon^2 S_{B2} + \epsilon^3 S_{B3} + \cdots
\end{aligned} \tag{23}
$$

from which the following perturbation equations are obtained:

Order ϵ^1:

$$
\begin{aligned}
&EJ\theta_1'' + S_{B1} = -\int_s^l p_y ds,\\
&\theta_{A1} = \check{\theta}_A, \quad \theta_{B1} = \check{\theta}_B,\\
&u_{B1} = \check{u}_A, \qquad \int_0^l \theta_1 ds = \check{v}_B - \check{v}_A.
\end{aligned} \tag{24}
$$

Order ϵ^2:

$$
\begin{aligned}
&EJ\theta_2'' + S_{B2} = \left(\check{R}_B + \int_s^l p_x ds\right)\theta_1,\\
&\theta_{A2} = 0, \quad \theta_{B2} = 0,\\
&u_{B2} = -\frac{1}{2}\int_0^l \theta_1^2 ds, \qquad \int_0^l \theta_2 ds = 0.
\end{aligned} \tag{25}
$$

Order ϵ^3:

$$
\begin{aligned}
&EJ\theta_3'' + S_{B3} = \frac{1}{2}\left(S_{B1} + \int_s^l p_y ds\right)\theta_1^2 + \left(\check{R}_B + \int_s^l p_x ds\right)\theta_2,\\
&\theta_{A3} = 0, \quad \theta_{B3} = 0,\\
&u_{B3} = -\int_0^l \theta_1\theta_2 ds, \qquad \int_0^l \theta_3 ds = \int_0^l \frac{1}{6}\theta_1^3 ds.
\end{aligned} \tag{26}
$$

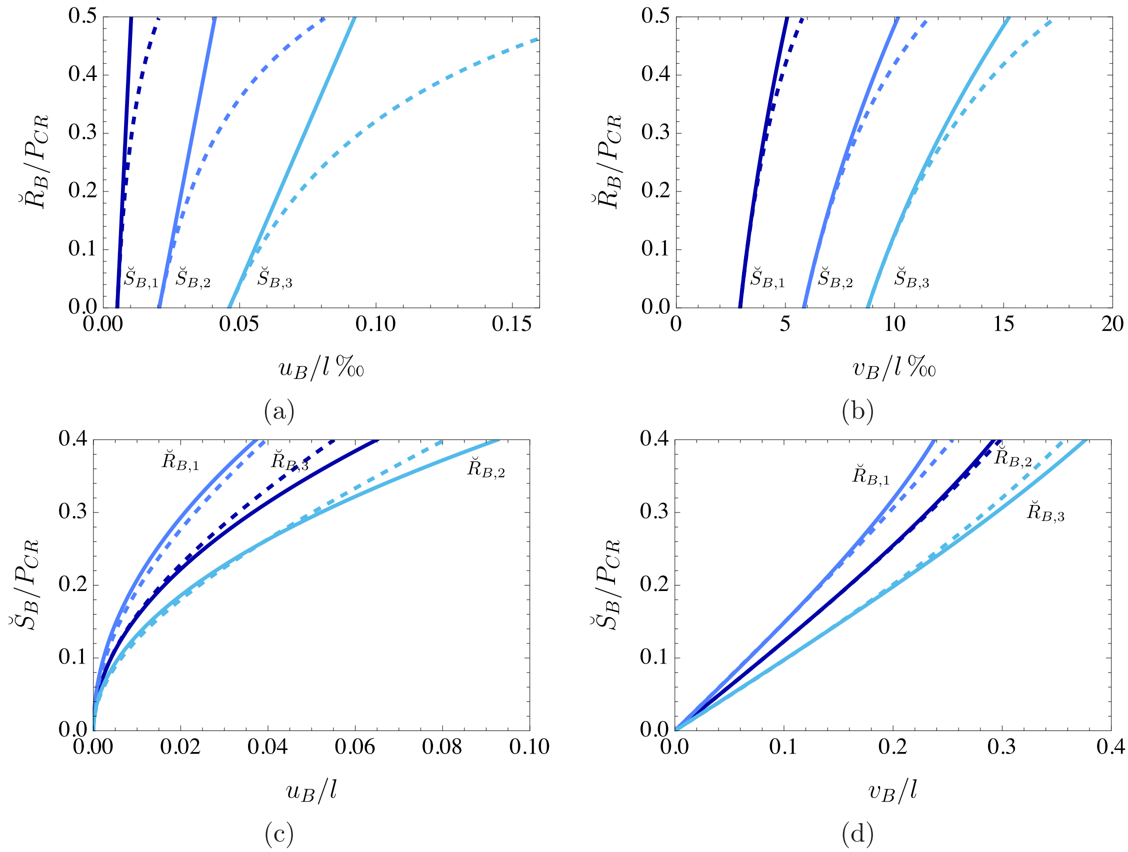

Fig. 4. Static response of the non-redundant beam at increasing $\breve{R}_B \in (0, 0.5)P_{CR}$ and fixed values of $\breve{S}_B = 5, 10, 15$ N: (a) u_B/l ‰ vs. $\breve{R}_B/P_{CR}$, (b) v_B/l ‰ vs. $\breve{R}_B/P_{CR}$; at increasing $S_B \in (0, 0.4)P_{CR}$ and fixed values of $\breve{R}_B = (-0.2, 0, 0.2)P_{CR}$: (c) u_B/l vs. S_B/P_{CR}, (d) v_B/l vs. S_B/P_{CR}. The solid lines denote the perturbation solution, while the dashed lines represent the numerical one. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

All the perturbation equations call for chain-solving *linear differential–algebraic systems* in the unknowns $\theta_k(s), S_{Bk}$. They are constituted by the balance equation and the transverse compatibility condition, having the following common form:

$$\begin{bmatrix} EJ\frac{d^2}{ds^2} & 1 \\ \int_0^l(\cdot)\,ds & 0 \end{bmatrix} \begin{pmatrix} \theta_k(s) \\ S_{Bk} \end{pmatrix} = \begin{pmatrix} q_{1k}(s) \\ q_{2k}(s) \end{pmatrix}, \qquad k = 1, 2, 3, \dots \tag{27}$$

where the linear operator has been formally represented in the square brackets, and where $q_{1k}(s), q_{2k}(s)$ are known terms. Eqs. (27) are combined with boundary conditions on $\theta_k(s)$, which allow the evaluation of the integration constants. Once $\theta_k(s), S_{Bk}$ are computed, the longitudinal compatibility condition furnishes u_{Bk}. As a solution strategy, (a) the field equation ((27)-a) (with the boundary conditions) is solved for $\theta_k(s)$ as a function of S_{Bk}; (b) this latter is evaluated by the integral equation ((27)-b); (c) finally, S_{Bk} is substituted back in $\theta_k(s)$. Appendix furnishes some details on the computational aspects.

It is useful to remark that the shortening $\Delta = u_B - \breve{u}_A$ is a second-order quantity and it depends on $\breve{R}_B$ only at the third-order, through the product $\theta_1\theta_2$ (weak effect). In contrast, first-order transverse displacements and rotations prescribed at the ends, trigger a shortening of second order, via θ_1^2 (strong effect). Note that the shortening can be found to depend on the longitudinal force at second-order, when $\breve{R}_B$ becomes large (left for future developments).

4. Numerical results

Numerical simulations are carried out here to evaluate the response of the inextensible beam by considering the different types of boundary conditions discussed above. The values adopted for the mechanical parameters are: $l = 3$ m, $E = 3$ GPa, $J = 1.71 \cdot 10^{-6}$ mm^4.

The behavior of the beam is studied in the pre-critical condition to analyze the nonlinear equilibrium paths when R_B is much lower than the values corresponding to the onset of buckling, i.e., the critical load indicated as P_{CR}. The asymptotic solution is systematically compared to that obtained via a numerical procedure. The latter is derived by the direct integration of the ensuing nonlinear ordinary differential equations descending from an index-reduction procedure performed on the original differential–algebraic equations, through the built-in function of Wolfram Mathematica (Wolfram Research Inc, 2021).

4.1. Non-redundant case

The beam with the boundary conditions of Fig. 3-a, namely the non-redundant case, is first analyzed. The considered longitudinal load at the free end $\breve{R}_B$ is varied up to $0.5P_{CR}$ that is a relatively large value, but it is considered to explore the validity range of the perturbation approach. The results are illustrated in Fig. 4; there (and in what follows), positive values $\breve{R}_B$ are associated to a compressive load and u_B is taken positive if directed opposite to the $\bar{\mathbf{a}}_x$, as indicated in Fig. 1-a. In particular, the longitudinal load is varied in the range $\breve{R}_B \in (0, 0.5)P_{CR}$ at selected values of the transverse force applied at the free end, namely $\breve{S}_B = 5, 10, 15$ N and the solution is illustrated in terms of: u_B/l ‰ vs. $\breve{R}_B/P_{CR}$ in Fig. 4-a; v_B/l ‰ vs. $\breve{R}_B/P_{CR}$ in Fig. 4-b.

For the same problem, it is also considered the case where the transverse load is varied in the range $S_B \in (0, 0.4)P_{CR}$, and the solution is illustrated in terms of: u_B/l vs. S_B/P_{CR} in Fig. 4-c; v_B/l vs. S_B/P_{CR} in Fig. 4-d. The perturbation solution is represented by the solid blue-scaled lines, while the numerically one is denoted by the dashed lines. It can be noted that the beam behavior is remarkably nonlinear even when the longitudinal load is small. However, the proposed asymptotic solution is in a general good agreement with the numerical one up to

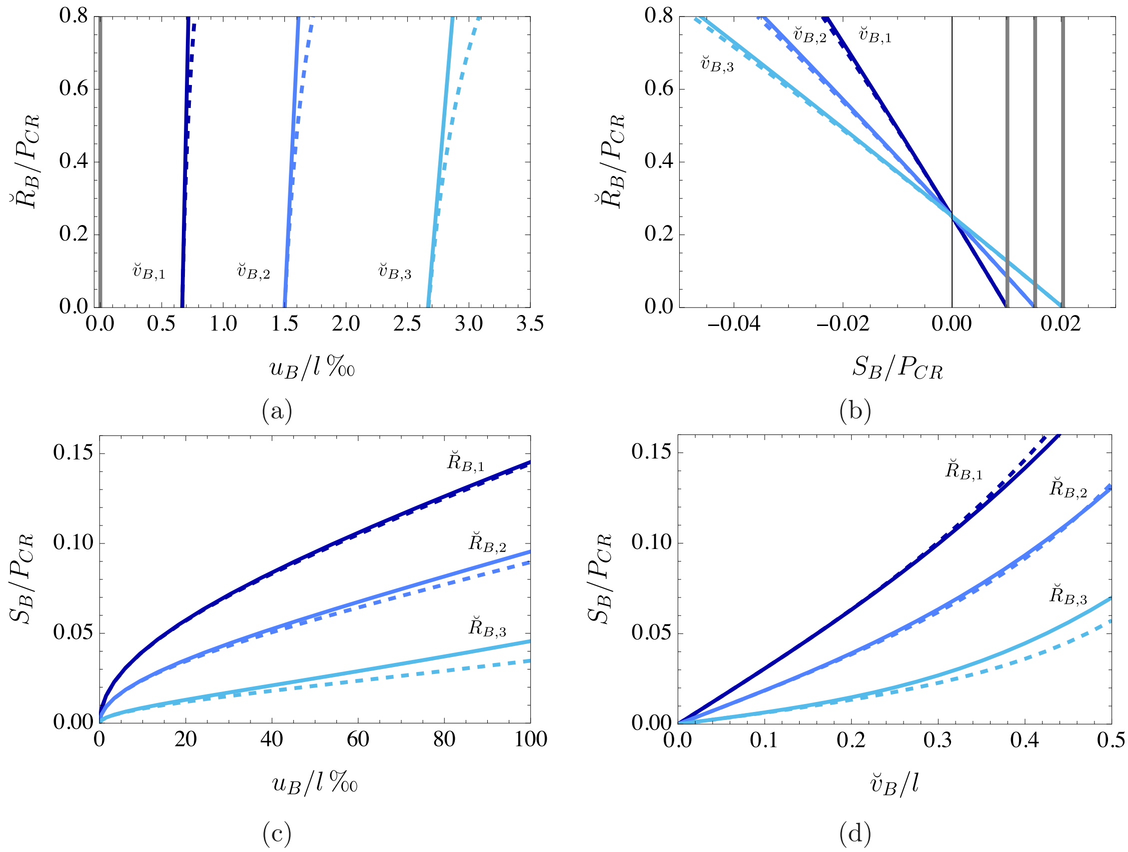

Fig. 5. Static response of the S-redundant beam at increasing $R_B \in (0, 0.5)P_{CR}$ and fixed values of $\breve{v}_B = (1/30, 1/50, 1/15)l$: (a) u_B/l ‰ vs. R_B/P_{CR}, (b) R_B/P_{CR} vs. S_B/P_{CR} (the gray lines denote the corresponding linear solution); at increasing $v_B \in (0, 0.5)l$ and fixed values of $\breve{R}_B = (0, 0.1, 0.2)P_{CR}$: (c) u_B/l ‰ vs. S_B/P_{CR}, (d) v_B/l vs. S_B/P_{CR}. The solid lines denote the perturbation solution, while the dashed lines represent the numerical one. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

relatively large values of R_B (remember that the perturbation scheme assumes $\breve{R}_B$ small). In other words, a better agreement especially in terms of u_B for larger values of $\breve{R}_B$ would require a different perturbation scheme, i.e., with $\breve{R}_B = O(1)$ which, by the way, could involve the occurrence of the buckling phenomenon and induce the singularity of the linear operator. This goes beyond the scope of the present work and is left for future developments.

4.2. S-redundant case

Here, the beam is restrained as displayed in Fig. 3-b, namely we are considering the S-redundant case. To emphasize the nonlinear behavior and explore the perturbation solution validity, the range of R_B is increased up to $0.8P_{CR}$. The results are illustrated in Fig. 5 with a logic analogous to that adopted before. In particular, the longitudinal load is varied in the range $\breve{R}_B \in (0, 0.8)P_{CR}$ at selected values of the transverse displacement prescribed at the free end, namely $v_B = \breve{v}_{B,k}$ with $\breve{v}_B = (1/30, 1/50, 1/15)l$ m and the solution is illustrated in terms of: u_B/l ‰ vs. $\breve{R}_B/P_{CR}$ in Fig. 5-a; $\breve{R}_B/P_{CR}$ vs. S_B/P_{CR} in Fig. 5-b. It is also considered the case where the transverse displacement is varied in the range $v_B \in (0, 0.5)l$ by taking $\breve{R}_B = (0, 0.1, 0.2)P_{CR}$ and here the solution is illustrated in terms of: u_B/l ‰ vs. S_B/P_{CR} in Fig. 5-c; v_B/l vs. S_B/P_{CR} in Fig. 5-d. The same color legend of the previous case is adopted, while the gray lines indicate the solution corresponding to the linear problem. In this case, the agreement between the perturbation and the numerical solutions is very good also for larger values of $\breve{R}_B$. The great difference of the solution as compared to the gray lines indicates that the analyzed behavior is significantly nonlinear.

5. Conclusions

The nonlinear elastic problem for inextensible and unshearable planar beams has been discussed. A mixed displacement–force formulation, well-known in literature, has been revisited. Here, however, a systematic discussion has been carried out on the role played by the boundary conditions on the solution. Moreover, the order of magnitude of all the involved quantities has been thoroughly investigated. Perturbations methods have been implemented for different classes of problems, and results compared with exact numerical computations. The following conclusions are drawn.

1. Depending on the boundary conditions, a single beam can exhibit 0, 1 or 2 redundant stresses R_B, S_B, consisting of reactive internal forces at end B, dual of the internal constraints. Four classes of problems have been determined: non-redundant, S-redundant, R-redundant, RS-redundant. In the former two classes, the shortening (i.e. the reduction of the chord of the beam) is free, while R_B is assigned; the opposite occurs in the latter two classes.
2. In the discussed perturbation schemes, all known and unknown terms are taken small of the same order. The chain of the perturbation equations is linear in the rotation field. The shortening, however, is found to be as a second-order variable, difference of two nearly-identical longitudinal displacements.
3. When a S-redundant system is dealt with, the chain of the perturbation equations is linear in the rotation field and the transverse force S_B. The shortening is found to depend on the longitudinal force only at third-order.

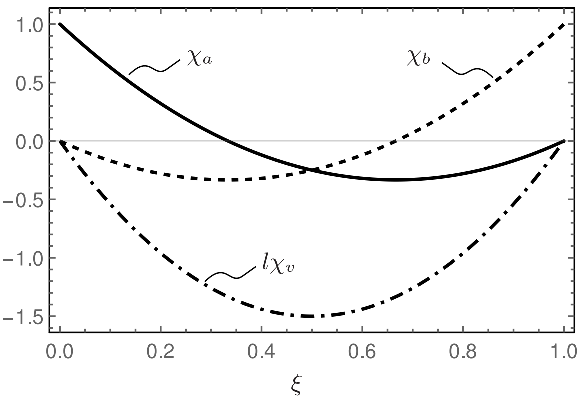

Fig. 6. Shape-functions from Eq. (29).

4. Numerical results have shown a good accordance with the perturbation solutions, in all the analyzed cases. However, for moderately large u_B values, it is shown that some quantitative differences between the two solutions arise, as R_B rapidly increases and cannot be considered as a small quantity. In this case, a proper perturbation scheme with $R_B = O(1)$ should be adopted, leading to a better match for the two solutions, but this is object of future developments.

CRediT authorship contribution statement

A. Luongo: Writing – review & editing, Writing – original draft, Supervision, Methodology, Formal analysis, Conceptualization. **D. Zulli:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis. **F. D'Annibale:** Validation, Methodology, Investigation. **A. Casalotti:** Writing – review & editing, Visualization, Validation, Software, Investigation, Formal analysis, Data curation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Appendix. Solution to Eqs. (24)–(26)

The perturbation Eqs. (24)–(26) are solved. To limit algebra, the simpler case $p_x = p_y = 0$ is considered. Eqs. (24) admit a (unique) solution of the type:

$$\begin{aligned}
\theta_1 &= \breve{\theta}_A \chi_a(s) + \breve{\theta}_B \chi_b(s) + \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right) \chi_v(s), \\
S_{B1} &= \breve{\theta}_A k_a + \breve{\theta}_B k_b + \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right) k_v, \\
u_{B1} &= \breve{u}_A,
\end{aligned} \tag{28}$$

where:

$$\begin{aligned}
&\chi_a(\xi) := 1 - 4\xi + 3\xi^2, \quad \chi_b(\xi) := 3\xi^2 - 2\xi, \quad \chi_v(\xi) := \frac{6}{l}\left(\xi - \xi^2\right), \\
&k_a := -\frac{6EJ}{l^2}, \quad k_b := -\frac{6EJ}{l^2}, \quad k_v := \frac{12EJ}{l^3}, \qquad \xi := \frac{s}{l},
\end{aligned} \tag{29}$$

are the well-known (exact) *shape-functions* (or Green functions) and stiffness coefficients of the linear Euler–Bernoulli beam (see Fig. 6).

When Eqs. (28) are substituted into the ϵ^2-order problem, and this is solved in a similar way, we find expressions of the type:

$$\begin{aligned}
\theta_2 &= \breve{R}_B \left(\breve{\theta}_A \chi_{ar}(s) + \breve{\theta}_B \chi_{br}(s) + \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right) \chi_{vr}(s)\right), \\
S_{B2} &= \breve{R}_B \left(\breve{\theta}_A k_{ar} + \breve{\theta}_B k_{br} + \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right) k_{vr}\right), \\
u_{B2} &= c_{aa}\breve{\theta}_A^2 + c_{ab}\breve{\theta}_A\breve{\theta}_B + c_{bb}\breve{\theta}_B^2 + c_{vv}\left(\breve{\upsilon}_B - \breve{\upsilon}_A\right)^2 \\
&\quad + c_{va}\left(\breve{\upsilon}_B - \breve{\upsilon}_A\right)\breve{\theta}_A + c_{vb}\left(\breve{\upsilon}_B - \breve{\upsilon}_A\right)\breve{\theta}_B.
\end{aligned} \tag{30}$$

At the next, ϵ^3-order, we obtain:

$$\begin{aligned}
\theta_3 &= \breve{R}_B^2 \left(\breve{\theta}_A \chi_{arr}(s) + \breve{\theta}_B \chi_{brr}(s) + \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right) \chi_{vrr}(s)\right) \\
&\quad + \breve{\theta}_A^3 \chi_{aaa}(s) + \breve{\theta}_B^3 \chi_{bbb}(s) + \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right)^3 \chi_{vvv}(s) \\
&\quad + 3\breve{\theta}_A^2\breve{\theta}_B \chi_{aab}(s) + 3\breve{\theta}_A\breve{\theta}_B^2 \chi_{abb}(s) + 3\breve{\theta}_A^2 \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right) \chi_{aav}(s) \\
&\quad + 3\breve{\theta}_A \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right)^2 \chi_{avv}(s) + 3\breve{\theta}_B^2 \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right) \chi_{bbv}(s) \\
&\quad + 3\breve{\theta}_B \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right)^2 \chi_{bvv}(s) + 6\breve{\theta}_A\breve{\theta}_B \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right) \chi_{abv}(s), \\
S_{B3} &= \breve{R}_B^2 \left(\breve{\theta}_A k_{arr} + \breve{\theta}_B k_{brr} + \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right) k_{vrr}\right) \\
&\quad + \breve{\theta}_A^3 k_{aaa} + \breve{\theta}_B^3 k_{bbb} + \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right)^3 k_{vvv} + 3\breve{\theta}_A^2\breve{\theta}_B k_{aab} \\
&\quad + 3\breve{\theta}_A\breve{\theta}_B^2 k_{abb} + 3\breve{\theta}_A^2 \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right) k_{aav} + 3\breve{\theta}_A \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right)^2 k_{avv} \\
&\quad + 3\breve{\theta}_B^2 \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right) k_{bbv} + 3\breve{\theta}_B \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right)^2 k_{bvv} + 6\breve{\theta}_A\breve{\theta}_B \left(\breve{\upsilon}_B - \breve{\upsilon}_A\right) k_{abv}, \\
u_{B3} &= \breve{R}_B (c_{aa}\breve{\theta}_A^2 + c_{ab}\breve{\theta}_A\breve{\theta}_B + c_{bb}\breve{\theta}_B^2 + c_{vv}\left(\breve{\upsilon}_B - \breve{\upsilon}_A\right)^2 \\
&\quad + c_{va}\left(\breve{\upsilon}_B - \breve{\upsilon}_A\right)\breve{\theta}_A + c_{vb}\left(\breve{\upsilon}_B - \breve{\upsilon}_A\right)\breve{\theta}_B).
\end{aligned} \tag{31}$$

Eqs. (28), (30) and (31) supply the coefficients of the series (23) solving the problem. The χ-functions and k-constants, appearing in Eqs. (30) and (31), assume cumbersome expressions not shown here.

References

Antman, S.S., 1973. The theory of rods. In: Linear Theories of Elasticity and Thermoelasticity. Springer, pp. 641–703.
Antman, S.S., 1974. Kirchhoff's problem for nonlinearly elastic rods. Quart. Appl. Math. 32 (3), 221–240.
Contento, A., Luongo, A., 2013. Static and dynamic consistent perturbation analysis for nonlinear inextensible planar frames. Comput. Struct. 123, 79–92.
Di Carlo, A., Pignataro, M., Rizzi, N., 1981. On the proper treatment of axial and shear undeformability constraints in post-buckling analysis of beams. Int. J. Non-Linear Mech. 16 (2), 221–229.
Di Egidio, A., Luongo, A., Paolone, A., 2007. Linear and non-linear interactions between static and dynamic bifurcations of damped planar beams. Int. J. Non-Linear Mech. 42 (1), 88–98.
Holden, J.T., 1972. On the finite deflections of thin beams. Int. J. Solids Struct. 8 (8), 1051–1055.
Luongo, A., Rega, G., Vestroni, F., 1986. On nonlinear dynamics of planar shear indeformable beams. J. Appl. Mech. 53 (3), 619–624.
Luongo, A., Zulli, D., 2013. Mathematical Models of Beams and Cables. John Wiley & Sons.
Mata, P., Oller, S., Barbat, A.H., 2007. Static analysis of beam structures under nonlinear geometric and constitutive behavior. Comput. Methods Appl. Mech. Engrg. 196 (45), 4458–4478.
McHugh, K., Dowell, E., 2018. Nonlinear responses of inextensible cantilever and free–free beams undergoing large deflections. J. Appl. Mech. 85 (5), 051008.
Pignataro, M., Rizzi, N., Di Carlo, A., 1980. Symmetric bifurcation of plane frames through a modified potential energy approach. J. Struct. Mech. 8 (3), 237–255.
Pignataro, M., Rizzi, N., Luongo, A., 1990. Bifurcation Stability and Postcritical Behaviour of Elastic Structures. Elsevier Science Publisher.
Reissner, E., 1973. On one-dimensional large-displacement finite-strain beam theory. Stud. Appl. Math. 52 (2), 87–95.
Rincón-Casado, A., González-Carbajal, J., García-Vallejo, D., Domínguez, J., 2021. Analytical and numerical study of the influence of different support types in the nonlinear vibrations of beams. Eur. J. Mech. A Solids 85, 104113.
Rizzi, N., Di Carlo, A., Pignataro, M., 1980. A parametric postbuckling analysis of an asymmetric two-bar frame. J. Struct. Mech. 8 (4), 435–448.
Simo, J.C., Vu-Quoc, L., 1988. On the dynamics in space of rods undergoing large motions - a geometrically exact approach. Comput. Methods Appl. Mech. Engrg. 66 (2), 125–161.
Takahashi, K., 1979. Non-linear free vibrations of inextensible beams. J. Sound Vib. 64 (1), 31–34.
Timoshenko, S., Goodier, J.N., 1951. Theory of Elasticity. Mc Graw-Hill Inc., New York.
Wolfram Research Inc, 2021. Mathematica, Version 13.1, 2021. Champaign, IL.
Wood, R.D., Zienkiewicz, O.C., 1977. Geometrically nonlinear finite element analysis of beams, frames, arches and axisymmetric shells. Comput. Struct. 7 (6), 725–735.