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Superalgebras with graded involution: Classifying minimal varieties of quadratic growth

A. Ioppolo[a,1], R.B. dos Santos[b,*], M.L.O. Santos[b,2], A.C. Vieira[b,3]

[a] Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, Via Roberto Cozzi, 55, 20126 Milano, Italy
[b] Departamento de Matemática, Instituto de Ciências Exatas, UFMG, Av. Antônio Carlos, 6627, Belo Horizonte, Brazil

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ABSTRACT

Let $\mathcal{V}$ be a variety of superalgebras with graded involution and let $c_n^{\mathrm{gri}}(\mathcal{V})$ be its sequence of $*$-graded codimensions. We say that $\mathcal{V}$ has polynomial growth n^k if asymptotically $c_n^{\mathrm{gri}}(\mathcal{V}) \approx an^k$, for some $a \neq 0$. Furthermore, $\mathcal{V}$ is minimal of polynomial growth n^k if $c_n^{\mathrm{gri}}(\mathcal{V})$ grows as n^k and any proper subvariety of $\mathcal{V}$ has polynomial growth n^t, with $t < k$. In this paper, we classify superalgebras with graded involution generating minimal varieties of quadratic growth.

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[*] Corresponding author.

E-mail addresses: antonio.ioppolo@unimib.it (A. Ioppolo), rafaelsantos23@ufmg.br (R.B. dos Santos), mallumat@ufmg.br (M.L.O. Santos), anacris@ufmg.br (A.C. Vieira).

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1. Introduction

Let A be an associative algebra over a field F of characteristic zero. A polynomial identity of A is a polynomial in non-commuting variables vanishing under all evaluations by elements in A. The algebras having at least one such non-trivial relation are said to be PI-algebras.

In 1950, the celebrated theorem of Amitsur and Levitzki represented the beginning of a new approach in the theory of polynomial identities. They proved that the standard polynomial St_{2k} of degree $2k$ is an identity of minimal degree for the algebra $M_k(F)$ of $k \times k$ matrices ([1]). From that moment, one of the main goals in PI-theory is the description of the polynomial identities satisfied by a given algebra.

Nevertheless, knowing the polynomial identities satisfied by an algebra A is a very difficult problem and it was solved completely in very few cases. To overcome some of these difficulties, in 1972, Regev introduced the so-called sequence of codimensions of an algebra A, denoted $c_n(A)$ ([20]). Such a function measures, in some sense, the growth of the polynomial identities satisfied by A. In the same paper he established a fundamental result showing that, for a PI-algebra, the sequence of codimensions is exponentially bounded. It is exactly this exponential growth that Giambruno and Zaicev captured in their celebrated results ([8,9]), answering positively to a famous conjecture posed by Amitsur in the early 80's.

In light of some results of Kemer (see [14]) it turns out that the sequence of codimensions $c_n(A)$, $n \geq 1$, either grows exponentially or it is polynomially bounded. In the last case, we say that the algebra A has polynomial codimension growth, or simply we say that A has polynomial growth.

We recall that a variety of algebras $\mathcal{V}$ is minimal of polynomial growth n^k if asymptotically $c_n(\mathcal{V}) \approx an^k$, for some $a \neq 0$, and $c_n(\mathcal{U}) \approx bn^t$, with $t < k$, for any proper subvariety $\mathcal{U}$ of $\mathcal{V}$. In [6], Giambruno, La Mattina and Zaicev classified the minimal varieties generated by unitary algebras in general.

The concepts of codimensions, polynomial growth and minimal varieties have been extended to classes of algebras endowed with additional structures such as superalgebras and algebras with involution. In order to establish a common nomenclature for algebras in one of the previous situations we shall use the notion of φ-algebras, that is, any algebra A with an automorphism or an antiautomorphism φ of order at most 2. This means that any superalgebra or any algebra with involution is a φ-algebra and, in this case, we say that A generates a φ-variety $\mathcal{V}$ and we write $\mathcal{V} = \text{var}^{\varphi}(A)$.

In [11], Gouveia, dos Santos and Vieira proved that for $k \leq 2$, there is only a finite number of minimal φ-varieties generated by unitary algebras and also they provided a list of finite dimensional algebras generating each one of such minimal φ-varieties. For $k = 3$, they showed that the number of minimal φ-varieties generated by unitary algebras is infinity and they classified all minimal φ-varieties of cubic growth from their φ-ideals. Furthermore they extended the results for $k \geq 4$ by giving a receipt for the construction of the φ-ideals of minimal φ-varieties of polynomial growth n^k.

In this article we deal with the so called $*$-superalgebras. A $*$-superalgebra is a superalgebra $A = A_0 + A_1$ with graded involution $*$, that is, an involution preserving the grading: $A_i^* = A_i$, $i = 0, 1$. In [7], Giambruno, dos Santos and Vieira characterized varieties of $*$-superalgebras with polynomially bounded $*$-graded codimension sequence by excluding five $*$-superalgebras from the variety. As a consequence, they proved that there are only five varieties of $*$-superalgebras with almost polynomial growth (i.e. a variety with exponential growth such that any proper subvariety has polynomial growth).

The minimal subvarieties of the varieties of $*$-superalgebras with almost polynomial growth were completely determined in [13,17]. Moreover, in [13], Ioppolo and La Mattina classified the $*$-superalgebras with at most linear growth and, in addition, they determined the minimal varieties of $*$-superalgebras whose $*$-graded codimension sequence grows as a linear function.

The main goal of this paper is to give a complete list of minimal varieties of $*$-superalgebras with quadratic growth of the $*$-graded codimension sequence.

2. Preliminaries

Throughout this paper F will be a field of characteristic zero and A an associative F-algebra. We say that A is graded by $\mathbb{Z}_2$, the cyclic group of order 2, if it can be decomposed into a direct sum of subspaces $A = A_0 + A_1$ satisfying the conditions $A_0A_0 + A_1A_1 \subseteq A_0$ and $A_0A_1 + A_1A_0 \subseteq A_1$. The elements of A_0 and A_1 are called homogeneous of degree 0 (even elements) and of degree 1 (odd elements), respectively. A $\mathbb{Z}_2$-graded algebra is simply called a superalgebra. We will denote by (A_0, A_1) the grading of a superalgebra $A = A_0 + A_1$.

Recall that an involution on A is a linear map $* : A \to A$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$, for all $a, b \in A$. Given a superalgebra $A = A_0 + A_1$ endowed with an involution $*$, we say that $*$ is a graded involution if it preserves the $\mathbb{Z}_2$-grading of the algebra, i.e. if $A_i^* = A_i$, $i = 0, 1$.

In this paper we deal with superalgebras $A = A_0 + A_1$ endowed with a graded involution $*$ and, for simplicity, we shall refer to this kind of algebras as $*$-superalgebras. Since F has characteristic zero, a $*$-superalgebra A can be decomposed in the following way:

$$A = A_0^+ + A_1^+ + A_0^- + A_1^-,$$

where $A_i^+ = \{a \in A_i : a^* = a\}$ and $A_i^- = \{a \in A_i : a^* = -a\}$, for $i = 0, 1$.

Next, we collect some results concerning the Wedderburn-Malcev decomposition of a finite dimensional $*$-superalgebra. We start by recalling some definitions. An ideal (subalgebra) I of a $*$-superalgebra A is a $*$-graded ideal (subalgebra) of A if it is a graded ideal (subalgebra) and $I^* = I$. The algebra A is a simple $*$-superalgebra if $A^2 \neq 0$ and A has no non-trivial $*$-graded ideals.

Theorem 2.1 *([7, Theorem 7.3]). Let A be a finite dimensional $*$-superalgebra over an algebraically closed field F of characteristic 0 and let $J(A)$ be its Jacobson radical. Then* *there exists a semisimple $*$-graded subalgebra B such that $A = B + J(A)$ and $J(A)$ is a $*$-graded ideal of A. Moreover $B = B_1 \oplus \cdots \oplus B_k$, where $B_1, \dots, B_k$ are simple $*$-superalgebras.*

Now let us focus our attention on finite dimensional algebras of kind $A = F + J$, where $J = J(A)$ is its Jacobson radical. In [10, Lemma 2], the authors proved that J can be decomposed into the direct sum of F-bimodules

$$J = J_{00} + J_{01} + J_{10} + J_{11}, \tag{2.1}$$

where for $i \in \{0, 1\}$, J_{ik} is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively. In a similar way, J_{ik} is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively. Moreover, for $i, k, r, s \in \{0, 1\}$, $J_{ir}J_{rs} \subseteq J_{is}$, $J_{ik}J_{rs} = 0$ for $k \neq r$ and $J_{11} = FN$ for some nilpotent subalgebra N of A commuting with F.

We also have that the given modules are graded and if the algebra A is endowed with an involution $*$, then J_{00} and J_{11} are stable under the involution whereas $J_{01}^* = J_{10}$.

Now let us consider the finite dimensional $*$-superalgebra $A = F + J$. Hence:

$$A = F + J_0^+ + J_1^+ + J_0^- + J_1^-.$$

Moreover, it follows that:

$$\begin{aligned}
A_0^+ &= F + (J_{10} + J_{01})_0^+ + (J_{00})_0^+ + (J_{11})_0^+, \\
A_0^- &= (J_{10} + J_{01})_0^- + (J_{00})_0^- + (J_{11})_0^-, \\
A_1^+ &= (J_{10} + J_{01})_1^+ + (J_{00})_1^+ + (J_{11})_1^+, \\
A_1^- &= (J_{10} + J_{01})_1^- + (J_{00})_1^- + (J_{11})_1^-.
\end{aligned}$$

Let $X = \{x_1, x_2, \dots\}$ be a countable set of non-commuting variables. We write X as a disjoint union of four subsets $X = Y_0 \cup Y_1 \cup Z_0 \cup Z_1$ and we denote by $\mathcal{F} = F\langle X|\mathbb{Z}_2, *\rangle$ the free $*$-superalgebra (the free superalgebra with graded involution), where $Y_0 = \{y_{1,0}, y_{2,0}, \dots\}$ is the set of symmetric variables of degree 0, $Y_1 = \{y_{1,1}, y_{2,1}, \dots\}$ is the set of symmetric variables of degree 1, $Z_0 = \{z_{1,0}, z_{2,0}, \dots\}$ is the set of skew variables of degree 0 and $Z_1 = \{z_{1,1}, z_{2,1}, \dots\}$ is the set of skew variables of degree 1. The elements of $\mathcal{F}$ are called $(\mathbb{Z}_2, *)$-polynomials.

We say that

$$f = f(y_{1,0}, \dots, y_{m,0}, y_{1,1}, \dots, y_{n,1}, z_{1,0}, \dots, z_{p,0}, z_{1,1}, \dots, z_{q,1}) \in \mathcal{F}$$

is a $(\mathbb{Z}_2, *)$-identity of a $*$-superalgebra A if

$$f(a_{1,0}, \dots, a_{m,0}, a_{1,1}, \dots, a_{n,1}, b_{1,0}, \dots, b_{p,0}, b_{1,1}, \dots, b_{q,1}) = 0,$$

for all $a_{1,0}, \ldots, a_{m,0} \in A_0^+$, $a_{1,1}, \ldots, a_{n,1} \in A_1^+$, $b_{1,0}, \ldots, b_{p,0} \in A_0^-$ and $b_{1,1}, \ldots, b_{q,1} \in A_1^-$. The set of all $(\mathbb{Z}_2, *)$-polynomial identities satisfied by the $*$-superalgebra A

$$\mathrm{Id}^{\mathrm{gri}}(A) = \{f \in \mathcal{F} : f \equiv 0 \text{ on } A\}$$

is a T_2^*-ideal of $\mathcal{F}$, i.e. an ideal of $\mathcal{F}$ invariant under all graded endomorphisms of $\mathcal{F}$ commuting with the graded involution $*$.

Given polynomials $f_1, \ldots, f_n \in \mathcal{F}$, we shall denote by $\langle f_1, \ldots, f_n \rangle_{T_2^*}$ the T_2^*-ideal generated by $f_1, \ldots, f_n$.

Since $\mathrm{char}(F) = 0$, each polynomial in $\mathrm{Id}^{\mathrm{gri}}(A)$ is equivalent to a system of multilinear $(\mathbb{Z}_2, *)$-identities. We denote by

$$P_n^{\mathrm{gri}} = \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} : w_i \in \{y_{i,0}, y_{i,1}, z_{i,0}, z_{i,1}\}, \sigma \in S_n\},$$

the space of multilinear $(\mathbb{Z}_2, *)$-polynomials of degree n in the variables $y_{1,0}, \ldots, y_{n,0}$, $y_{1,1}, \ldots, y_{n,1}, z_{1,0}, \ldots, z_{n,0}, z_{1,1}, \ldots, z_{n,1}$. Notice that $\dim_F P_n^{\mathrm{gri}} = 4^n n!$. The non-negative integer

$$c_n^{\mathrm{gri}}(A) = \dim_F \frac{P_n^{\mathrm{gri}}}{P_n^{\mathrm{gri}} \cap \mathrm{Id}^{\mathrm{gri}}(A)}, \ n \geq 1,$$

is called the n-th $*$-graded codimension of A.

The sequence $c_n(A)$ of the ordinary codimensions was introduced by Regev in [20] where it was proved that if A satisfies a non-trivial polynomial identity, then $c_n(A)$ is exponentially bounded. An analogue result holds for $*$-superalgebras.

Theorem 2.2 *([7, Corollary 3.2]). If A is a $*$-superalgebra satisfying a non-trivial ordinary identity, then $c_n^{\mathrm{gri}}(A)$, $n \geq 1$, is exponentially bounded.*

If A is an algebra with involution (notice that any algebra with involution $*$ can always be viewed as a $*$-superalgebra with trivial grading), we can define the space of multilinear $*$-polynomials, $*$-identities and the $*$-codimension sequence $c_n^*(A)$. The relation between codimensions, $*$-codimensions and $*$-graded codimensions is given in the following lemma.

Lemma 2.3 *([7, Lemma 3.1]). Let A be a $*$-superalgebra. Then $c_n(A) \leq c_n^*(A) \leq c_n^{\mathrm{gri}}(A)$.*

Let $n \geq 1$ and write $n = n_1 + n_2 + n_3 + n_4$ as a sum of four non-negative integers. We denote by $P_{n_1,\ldots,n_4} \subseteq P_n^{\mathrm{gri}}$ the vector space of multilinear $(\mathbb{Z}_2, *)$-polynomials in which the first n_1 variables are symmetric of degree 0, the next n_2 variables are symmetric of degree 1, the next n_3 variables are skew of degree 0 and the next n_4 variables are skew of degree 1. The group $S_{n_1} \times \cdots \times S_{n_4}$ acts on the left on the vector space $P_{n_1,\ldots,n_4}$ by permuting the respective variables and in this way $P_{n_1,\ldots,n_4}$ becomes a $S_{n_1} \times \cdots \times S_{n_4}$-left module. Now, $P_{n_1,\ldots,n_4} \cap \mathrm{Id}^{\mathrm{gri}}(A)$ is invariant under this action and so the vector space

$$P_{n_1,\dots,n_4}(A) = \frac{P_{n_1,\dots,n_4}}{P_{n_1,\dots,n_4} \cap \mathrm{Id}^{\mathrm{gri}}(A)}$$

is a $S_{n_1} \times \cdots \times S_{n_4}$-left module with induced action. We denote by $\chi_{n_1,\dots,n_4}(A)$ its character and we call it the $(n_1, \dots, n_4)$-cocharacter of A.

Moreover, if we set $c_{n_1,\dots,n_4}(A) = \dim_F P_{n_1,\dots,n_4}(A)$ it is immediate to see that

$$c_n^{\mathrm{gri}}(A) = \sum_{n_1+\cdots+n_4=n} \binom{n}{n_1, \dots, n_4} c_{n_1,\dots,n_4}(A), \tag{2.2}$$

where $\binom{n}{n_1, \dots, n_4} = \frac{n!}{n_1! \cdots n_4!}$ stands for the multinomial coefficient.

If $\lambda = (\lambda_1, \dots, \lambda_r)$ is a partition of n, we write $\lambda \vdash n$. It is well-known that there is a one-to-one correspondence between partitions of n and irreducible S_n-characters. Hence if $\lambda \vdash n$, we denote by χ_λ the corresponding irreducible S_n-character. If $\lambda(1) \vdash n_1, \dots, \lambda(4) \vdash n_4$ are partitions we write $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(4)) \vdash (n_1, \dots, n_4)$ and we say that $\langle\lambda\rangle$ is a multipartition of $n = n_1 + \cdots + n_4$. Since $\mathrm{char}(F) = 0$, by complete reducibility, $\chi_{n_1,\dots,n_4}(A)$ can be written as a sum of irreducible characters

$$\chi_{n_1,\dots,n_4}(A) = \sum_{\langle\lambda\rangle \vdash (n_1,\dots,n_4)} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}, \tag{2.3}$$

where $m_{\langle\lambda\rangle} \geq 0$ is the multiplicity of $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ in $\chi_{n_1,\dots,n_4}(A)$.

Recall that the degree $d_{\lambda(i)}$ of the S_{n_i}-irreducible character $\chi_{\lambda(i)}$, $i = 1, \dots, 4$, is given by the hook formula (see, for instance, [21, Theorem 3.10.2]). Therefore, $d_{\lambda(1)} \cdots d_{\lambda(4)}$ is the degree of the irreducible $S_{n_1} \times \cdots \times S_{n_4}$-character $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ and

$$c_{n_1,\dots,n_4}(A) = \sum_{\langle\lambda\rangle \vdash (n_1,\dots,n_4)} m_{\langle\lambda\rangle} d_{\lambda(1)} \cdots d_{\lambda(4)}. \tag{2.4}$$

Another numerical sequence that can be attached to a $*$-superalgebra A is the sequence of $*$-graded colengths. If the decomposition of the $(n_1, \dots, n_4)$-cocharacter of A is that one given in (2.3), then the n-th $*$-graded colength of A is defined as

$$l_n^{\mathrm{gri}}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\dots,n_4) \\ n_1+\cdots+n_4=n}} m_{\langle\lambda\rangle}.$$

Let A and B be two $*$-superalgebras such that

$$\chi_{n_1,\dots,n_4}(A) = \sum_{\langle\lambda\rangle \vdash (n_1,\dots,n_4)} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)} \quad \text{and}$$

$$\chi_{n_1,\dots,n_4}(B) = \sum_{\langle\lambda\rangle \vdash (n_1,\dots,n_4)} m'_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}.$$

The direct sum $A \oplus B$ is also a $*$-superalgebra, with graded involution induced by the graded involutions defined on A and B. Let

$$\chi_{n_1,\dots,n_4}(A \oplus B) = \sum_{\langle\lambda\rangle\vdash(n_1,\dots,n_4)} \overline{m}_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$$

be the decomposition of the $(n_1,\dots,n_4)$-cocharacter of $A \oplus B$.

Remark 2.4. The following facts hold:

(1) $c_n^{\mathrm{gri}}(A), c_n^{\mathrm{gri}}(B) \leq c_n^{\mathrm{gri}}(A \oplus B) \leq c_n^{\mathrm{gri}}(A) + c_n^{\mathrm{gri}}(B)$;
(2) If $\mathrm{Id}^{\mathrm{gri}}(A) \subseteq \mathrm{Id}^{\mathrm{gri}}(B)$, then $m'_{\langle\lambda\rangle} \leq m_{\langle\lambda\rangle}$, for all $\langle\lambda\rangle \vdash (n_1,\dots,n_4)$ and $l_n^{\mathrm{gri}}(B) \leq l_n^{\mathrm{gri}}(A)$, for all n;
(3) $\overline{m}_{\langle\lambda\rangle} \leq m_{\langle\lambda\rangle} + m'_{\langle\lambda\rangle}$, for all $\langle\lambda\rangle \vdash (n_1,\dots,n_4)$.

We conclude this section by recalling some results concerning the representation theory of the general linear group GL_m (see, for instance, [2]) in order to provide a way to calculate the multiplicities $m_{\langle\lambda\rangle}$ appearing in the decomposition of $\chi_{n_1,\dots,n_4}(A)$.

Let F_m be the space of multilinear polynomials in the variables $y_{1,0},\dots,y_{m,0}$, $y_{1,1},\dots,y_{m,1}$, $z_{1,0},\dots,z_{m,0}$, $z_{1,1},\dots,z_{m,1}$. Consider $U_1 = \mathrm{span}_F\{y_{1,0},\dots,y_{m,0}\}$, $U_2 = \mathrm{span}_F\{y_{1,1},\dots,y_{m,1}\}$, $U_3 = \mathrm{span}_F\{z_{1,0},\dots,z_{m,0}\}$ and $U_4 = \mathrm{span}_F\{z_{1,1},\dots,z_{m,1}\}$.

The group $GL(U_1)\times GL(U_2)\times GL(U_3)\times GL(U_4) \cong GL_m\times GL_m\times GL_m\times GL_m = GL_m^4$ acts naturally on the left on the subspace $U_1\oplus U_2\oplus U_3\oplus U_4$ of F_m. Moreover, this action can be diagonally extended to an action on F_m. Also, for all $*$-superalgebra A, we have that $F_m \cap \mathrm{Id}^{\mathrm{gri}}(A)$ is invariant under this action.

Now let us consider F_m^n, the space of homogeneous polynomials of F_m of degree $n \geq m$. The group GL_m^4 acts diagonally on F_m^n and so F_m^n is a GL_m^4-module. Since $F_m^n \cap \mathrm{Id}^{\mathrm{gri}}(A)$ is invariant under this action, it follows that the space

$$F_m^n(A) = \frac{F_m^n}{F_m^n \cap \mathrm{Id}^{\mathrm{gri}}(A)}$$

is a GL_m^4-module. We denote by $\Psi_n^{\mathrm{gri}}(A)$ its GL_m^4-character, called GL_m^4-cocharacter of A.

The representation theory of GL_m shows that there exists a one-to-one correspondence between the irreducible GL_m^4-modules and the multipartitions $\langle\lambda\rangle = (\lambda(1),\dots,\lambda(4))$ of n, where the $\lambda(i)$'s are partitions with at most m parts (see [2, Theorem 12.4.4]). Denote by $\Psi_{\langle\lambda\rangle}$ the irreducible GL_m^4-character corresponding to the multipartition $\langle\lambda\rangle$. We have that

$$\Psi_n^{\mathrm{gri}}(A) = \sum_{\substack{\langle\lambda\rangle\vdash(n_1,\dots,n_4)\\ h(\langle\lambda\rangle)\leq m}} \widetilde{m}_{\langle\lambda\rangle}\Psi_{\langle\lambda\rangle}, \tag{2.5}$$

where $\widetilde{m}_{\langle\lambda\rangle} \geq 0$ is the multiplicity of $\Psi_{\langle\lambda\rangle}$, $h(\langle\lambda\rangle) = \max\{h(\lambda(i)), i = 1, \ldots, 4\}$ and $h(\lambda(i))$ denotes the height of the Young diagram corresponding to the partition $\lambda(i) \vdash n_i$, $i = 1, \ldots, 4$.

Moreover, we have that all the irreducible GL_m^4-modules are generated by a non-zero polynomial $f_{\langle\lambda\rangle}$ called highest weight vector associated to the multipartition $\langle\lambda\rangle$ (see [2, Theorem 12.4.12]).

A multitableau $T_{\langle\lambda\rangle} = (T_{\lambda(1)}, T_{\lambda(2)}, T_{\lambda(3)}, T_{\lambda(4)})$ is a 4-tuple of Young tableaux $T_{\lambda(i)}$, where $\lambda(i) \vdash n_i$, $1 \leq i \leq 4$. We say that the multitableau $\widetilde{T}_{\langle\lambda\rangle}$ is standard if the integers $1, \ldots, n$ are inserted, in this order, from top to bottom, from left to right, column by column, from the tableau $\widetilde{T}_{\lambda(1)}$ to the tableau $\widetilde{T}_{\lambda(4)}$. The highest weight vector associated to a standard multitableau is called standard highest weight vector and it is given by

$$\begin{aligned} f_{\widetilde{T}_{\langle\lambda\rangle}} = & \prod_{i=1}^{\lambda(1)_1} St_{h_i(\lambda(1))}(y_{1,0}, \ldots, y_{h_i(\lambda(1)),0}) \prod_{i=1}^{\lambda(2)_1} St_{h_i(\lambda(2))}(y_{1,1}, \ldots, y_{h_i(\lambda(2)),1}) \\ & \prod_{i=1}^{\lambda(3)_1} St_{h_i(\lambda(3))}(z_{1,0}, \ldots, z_{h_i(\lambda(3)),0}) \prod_{i=1}^{\lambda(4)_1} St_{h_i(\lambda(4))}(z_{1,1}, \ldots, z_{h_i(\lambda(4)),1}), \end{aligned}$$

where $St_r(x_1, \ldots, x_r) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(r)}$ is the standard polynomial of degree r.

For a multitableau $T_{\langle\lambda\rangle}$, denote by $f_{T_{\langle\lambda\rangle}}$ the highest weight vector $f_{\widetilde{T}_{\langle\lambda\rangle}} \sigma^{-1}$, where σ is the only element of S_n transforming the standard multitableau $\widetilde{T}_{\langle\lambda\rangle}$ in the multitableau $T_{\langle\lambda\rangle}$, where S_n acts on the right on F_m^n by permuting places in which the variables occur.

As in the case of algebras with involution (see [4, Theorem 3]), there is a relation between the $(n_1, \ldots, n_4)$-cocharacter and the GL_m^4-cocharacter of a $*$-superalgebra A.

Theorem 2.5. *Let A be a $*$-superalgebra. If*

$$\chi_{n_1,\ldots,n_4}(A) = \sum_{\langle\lambda\rangle \vdash (n_1,\ldots,n_4)} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)} \quad \text{and} \quad \Psi_n^{\mathrm{gri}}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\ldots,n_4) \\ h(\langle\lambda\rangle) \leq m}} \widetilde{m}_{\langle\lambda\rangle} \Psi_{\langle\lambda\rangle},$$

then $m_{\langle\lambda\rangle} = \widetilde{m}_{\langle\lambda\rangle}$, for all multipartition $\langle\lambda\rangle \vdash (n_1, \ldots, n_4)$ such that $h(\langle\lambda\rangle) \leq m$.

Now we are ready to present the following result.

Theorem 2.6 *([2, Theorem 12.4.4]). In the decomposition (2.5), the multiplicity $\widetilde{m}_{\langle\lambda\rangle} \neq 0$ if and only if there exists a multitableau $T_{\langle\lambda\rangle}$ such that $f_{T_{\langle\lambda\rangle}} \notin \mathrm{Id}^{\mathrm{gri}}(A)$. Moreover, $\widetilde{m}_{\langle\lambda\rangle}$ is equal to the maximal number of highest weigh vectors $f_{T_{\langle\lambda\rangle}} \notin \mathrm{Id}^{\mathrm{gri}}(A)$ which are linearly independent in $F_m^n(A)$.*

By putting together the two previous theorems we get the following remark.

Remark 2.7. In the decomposition (2.3), for a multipartition $\langle\lambda\rangle \vdash (n_1,\dots,n_4)$ such that $h(\langle\lambda\rangle) \leq m$, we have that $m_{\langle\lambda\rangle} \neq 0$ if and only if there exists a multitableau $T_{\langle\lambda\rangle}$ such that $f_{T_{\langle\lambda\rangle}} \notin \mathrm{Id}^{\mathrm{gri}}(A)$.

3. Varieties of $*$-superalgebras of polynomial growth

The first goal of this section is to introduce the notion of variety of $*$-superalgebras. Analogously to the corresponding definition in the ordinary case (associative algebras), the class of all $*$-superalgebras that satisfy the $(\mathbb{Z}_2,*)$-identities of a given $*$-superalgebra A is called the variety of $*$-superalgebras (or $*$-supervariety) generated by A and denoted by $\mathrm{var}^{\mathrm{gri}}(A)$.

In case $\mathcal{V} = \mathrm{var}^{\mathrm{gri}}(A)$, we write $c_n^{\mathrm{gri}}(\mathcal{V}) = c_n^{\mathrm{gri}}(A)$, i.e. the $*$-graded codimensions of a $*$-supervariety coincide with those of a generating $*$-superalgebra. We shall say that $\mathcal{V}$ has polynomial growth if there exist $k,t \geq 0$ such that $c_n^{\mathrm{gri}}(\mathcal{V}) \leq kn^t$ and that $\mathcal{V}$ has almost polynomial growth if $c_n^{\mathrm{gri}}(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

In [7], the authors characterized $*$-supervarieties of polynomial growth. In order to present such a result, we shall introduce five $*$-superalgebras generating $*$-supervarieties of almost polynomial growth.

We denote by D_* the commutative algebra $D = F \oplus F$ with trivial grading and exchange graded involution ex given by $(a,b)^{ex} = (b,a)$, for all $(a,b) \in D$. Considering a non-trivial grading on D, we denote by D^{gr} and D^{gri} the superalgebra $D = (F(1,1), F(1,-1))$ with trivial and exchange graded involution, respectively.

Let now

$$M = F(e_{11}+e_{44}) + F(e_{22}+e_{33}) + Fe_{12} + Fe_{34}$$

be a subalgebra of UT_4, the algebra of 4×4 upper-triangular matrices, endowed with the reflection involution, i.e. the involution obtained by reflecting a matrix along its secondary diagonal. If we regard M endowed with trivial grading, then the above involution is a graded involution on M and we denote such a $*$-superalgebra by M_*. Next we consider a non-trivial grading on M: we denote by M^{gri} the algebra M with grading

$$(F(e_{11}+e_{44}) + F(e_{22}+e_{33}), Fe_{12} + Fe_{34})$$

and reflection graded involution.

A characterization of $*$-supervarieties of polynomial growth is given in the next theorem.

Theorem 3.1 *([7, Theorem 8.6]). Let A be a finite dimensional $*$-superalgebra. Then $\mathrm{var}^{\mathrm{gri}}(A)$ has polynomial growth if and only if $D_*, D^{\mathrm{gr}}, D^{\mathrm{gri}}, M_*, M^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$.*

Here we want to highlight that the previous result was extended to the general case in [5]. In fact the authors proved the following theorem.

Theorem 3.2 *([5, Theorems 5.1, 5.2 and 5.3]). Let $\mathcal{V}$ be a $*$-supervariety such that $D_* \notin \mathcal{V}$. Then $\mathcal{V} = \mathrm{var}^{\mathrm{gri}}(A)$, for some finite dimensional $*$-superalgebra A.*

Other characterizations of $*$-supervarieties of polynomial growth were given in [3, Theorems 3.5 and 4.5] and [12, Theorem 5.3]. Putting these results together, in light of Theorem 3.2, we have the following theorem.

Theorem 3.3. *Let A be a $*$-superalgebra. The following conditions are equivalent:*

(1) $c_n^{\mathrm{gri}}(A)$ *is polynomially bounded;*
(2) $\mathrm{var}^{\mathrm{gri}}(A) = \mathrm{var}^{\mathrm{gri}}(B_1 \oplus \cdots \oplus B_m)$, *where the B_i's are finite dimensional $*$-superalgebras such that*

$$\dim_F \frac{B_i}{J(B_i)} \leq 1;$$

(3) *there exists a constant q such that for every n_1, n_2, n_3, n_4 with $n_1+n_2+n_3+n_4 = n$ it holds*

$$\chi_{n_1,\ldots,n_4}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\ldots,n_4) \\ n-\lambda(1)_1 < q}} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}.$$

If A is a finite dimensional algebra then q is such that $J(A)^q = 0$;
(4) *there exists a constant h such that, for all $n \geq 1$, we have that $l_n^{\mathrm{gri}}(A) \leq h$;*
(5) $D_*, D^{\mathrm{gr}}, D^{\mathrm{gri}}, M_*, M^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$.

Remark 3.4. If there exists a constant h such that, for any $n \geq 1$, $l_n^{\mathrm{gri}}(A) \leq h$, then $m_{\langle\lambda\rangle} \leq h$, for any multipartition $\langle\lambda\rangle \vdash (n_1, \ldots, n_4)$ where $(n_1, \ldots, n_4)$ runs over all the possible sums of non-negative integers.

In this paper, we are interested in the study of minimal $*$-supervarieties of polynomial growth, that is, $*$-supervarieties $\mathcal{V}$ such that $c_n^{\mathrm{gri}}(\mathcal{V}) \approx an^t$, for some $a > 0$, but $c_n^{\mathrm{gri}}(\mathcal{U}) \approx bn^k, k < t$, for any proper subvariety $\mathcal{U}$ of $\mathcal{V}$. In the next sections, we present results that will allow us to classify all minimal $*$-supervarieties of quadratic growth.

4. Constructing $*$-superalgebras with quadratic growth

In this section, we will present a list of 37 $*$-superalgebras having quadratic growth of the $*$-graded codimension sequence. We start by considering G_2, the finite dimensional subalgebra of the infinite dimensional Grassmann algebra, defined as

$$G_2 = \langle 1, e_1, e_2 : e_1e_2 = -e_2e_1 \rangle.$$

On such an algebra it is possible to define the following involutions:

$$\tau : e_i \mapsto -e_i, \quad \psi : e_i \mapsto e_i \quad \text{and} \quad \rho : e_i \mapsto (-1)^i e_i.$$

Now let us consider the following 6 $*$-superalgebras:

- $G_{2,0,\tau}$, the algebra G_2 with trivial grading and involution τ;
- $G_{2,1,\tau}$, the algebra G_2 with grading $(F1 + Fe_1e_2, Fe_1 + Fe_2)$ and involution τ;
- $G_{2,2,\tau}$, the algebra G_2 with grading $(F1 + Fe_1, Fe_1e_2 + Fe_2)$ and involution τ;
- $G_{2,1,\rho}$, the algebra G_2 with grading $(F1 + Fe_1e_2, Fe_1 + Fe_2)$ and involution ρ;
- $G_{2,2,\rho}$, the algebra G_2 with grading $(F1 + Fe_1, Fe_1e_2 + Fe_2)$ and involution ρ;
- $G_{2,1,\psi}$, the algebra G_2 with grading $(F1 + Fe_1e_2, Fe_1 + Fe_2)$ and involution ψ.

In the next result, we present the $*$-graded codimensions of the $*$-superalgebras defined above.

Lemma 4.1 *([19, Lemmas 7.1, 7.2 and 7.3]).*

(1) $c_n^{\mathrm{gri}}(G_{2,0,\tau}) = 1 + n + \dfrac{n^2 - n}{2}$;

(2) $c_n^{\mathrm{gri}}(G_{2,1,\tau}) = c_n^{\mathrm{gri}}(G_{2,2,\tau}) = c_n^{\mathrm{gri}}(G_{2,1,\rho}) = c_n^{\mathrm{gri}}(G_{2,2,\rho}) = c_n^{\mathrm{gri}}(G_{2,1,\psi}) = 1 + 2n + \dfrac{n^2 - n}{2}$.

Now we consider the following subalgebras of UT_n, the algebra of $n \times n$ upper triangular matrices.

- $N_3 = F(e_{11} + e_{22} + e_{33} + e_{44} + e_{55} + e_{66}) + F(e_{23} + e_{45}) + F(e_{12} - e_{56}) + Fe_{13} + Fe_{46} \subseteq UT_6$;
- $U_3 = F(e_{11} + e_{22} + e_{33} + e_{44} + e_{55} + e_{66}) + F(e_{23} + e_{45}) + F(e_{12} + e_{56}) + Fe_{13} + Fe_{46} \subseteq UT_6$;
- $C_3 = F(e_{11} + e_{22} + e_{33}) + F(e_{12} + e_{23}) + Fe_{13} \subseteq UT_3$.

Here we recall that if $\mathbf{g} = (g_1, \dots, g_{2k}) \in \mathbb{Z}_2^{2k}$ is an arbitrary $2k$-tuple of elements of $\mathbb{Z}_2$, then $\mathbf{g}$ defines an elementary $\mathbb{Z}_2$-grading on UT_{2k} by setting:

$$(UT_{2k})_0 = \mathrm{span}_F\{e_{ij} : g_i + g_j = 0 \ (\mathrm{mod}\ 2)\} \quad \text{and}$$
$$(UT_{2k})_1 = \mathrm{span}_F\{e_{ij} : g_i + g_j = 1 \ (\mathrm{mod}\ 2)\}.$$

If UT_{2k} is endowed with an elementary grading and A is a graded subalgebra of UT_{2k}, then the induced grading on A is also called elementary.

From the algebras N_3, U_3 and C_3, we construct the following 7 $*$-superalgebras:

- $N_{3,*}$, the algebra N_3 with trivial grading and reflection involution;
- N_3^{gri}, the algebra N_3 with elementary grading defined by $(0,1,1,0,0,1)$ and reflection involution;
- $U_{3,*}$, the algebra U_3 with trivial grading and reflection involution;
- U_3^{gri}, the algebra U_3 with elementary grading defined by $(0,1,1,0,0,1)$ and reflection involution;
- $C_{3,*}$, the algebra C_3 with trivial grading and involution $*$ defined by

$$\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & -b & c \\ 0 & a & -b \\ 0 & 0 & a \end{pmatrix};$$

- C_3^{gri}, the algebra C_3 with grading $(F(e_{11}+e_{22}+e_{33})+Fe_{13}, F(e_{12}+e_{23}))$ and involution $*$;
- C_3^{gr}, the algebra C_3 with grading $(F(e_{11}+e_{22}+e_{33})+Fe_{13}, F(e_{12}+e_{23}))$ and trivial involution.

We summarize the information about the $*$-graded codimensions of the $*$-superalgebras defined above in the next lemma.

Lemma 4.2.

(1) $c_n^{\mathrm{gri}}(N_{3,*}) = n^2+1$ *[16, Lemma 2].*
(2) $c_n^{\mathrm{gri}}(N_3^{\mathrm{gri}}) = c_n^{\mathrm{gri}}(U_3^{\mathrm{gri}}) = n^2+n+1$ *[13, Theorems 4.4 and 4.5].*
(3) $c_n^{\mathrm{gri}}(U_{3,*}) = c_n^{\mathrm{gri}}(C_{3,*}) = \dfrac{n^2+n+2}{2}$ *[16, Lemmas 3 and 9].*
(4) $c_n^{\mathrm{gri}}(C_3^{\mathrm{gri}}) = c_n^{\mathrm{gri}}(C_3^{\mathrm{gr}}) = \dfrac{n^2+n+2}{2}$ *[13, Lemma 6.1], [15, Theorem 8.1].*

As a consequence of the last two lemmas we get the following.

Corollary 4.3. *All the 13 $*$-superalgebras $G_{2,0,\tau}$, $G_{2,1,\tau}$, $G_{2,2,\tau}$, $G_{2,1,\rho}$, $G_{2,2,\rho}$, $G_{2,1,\psi}$, $N_{3,*}$, N_3^{gri}, $U_{3,*}$, U_3^{gri}, $C_{3,*}$, C_3^{gr}, C_3^{gri} have quadratic growth of the $*$-graded codimensions.*

Also it is worth to take into account the next remark (see, for instance, [13], [15] and [16]).

Remark 4.4. We have $C_{3,*} \in \mathrm{var}^{\mathrm{gri}}(D_*)$, $C_3^{\mathrm{gr}} \in \mathrm{var}^{\mathrm{gri}}(D^{\mathrm{gr}})$, $C_3^{\mathrm{gri}} \in \mathrm{var}^{\mathrm{gri}}(D^{\mathrm{gri}})$, $N_{3,*} \in \mathrm{var}^{\mathrm{gri}}(M_*)$ and $N_3^{\mathrm{gri}} \in \mathrm{var}^{\mathrm{gri}}(M^{\mathrm{gri}})$.

In order to construct the remaining 24 $*$-superalgebras, let us consider the following subalgebras of UT_n:

- $M_4 = F(e_{11}+e_{33}) + Fe_{12} + Fe_{13} + Fe_{23} \subseteq UT_3$;

- $M_5 = Fe_{12} + Fe_{13} + Fe_{22} + Fe_{23} \subseteq UT_3$;
- $M_6 = F(e_{11} + e_{44}) + Fe_{12} + Fe_{13} + Fe_{14} + Fe_{24} + Fe_{34} \subseteq UT_4$;
- $M_7 = F(e_{22} + e_{33}) + Fe_{12} + Fe_{13} + Fe_{14} + Fe_{24} + Fe_{34} \subseteq UT_4$;
- $M_8 = F(e_{11} + e_{66}) + Fe_{12} + Fe_{13} + F(e_{23} + e_{45}) + Fe_{46} + Fe_{56} \subseteq UT_6$;
- $M_9 = F(e_{11} + e_{66}) + Fe_{12} + Fe_{13} + F(e_{23} - e_{45}) + Fe_{46} + Fe_{56} \subseteq UT_6$;
- $M_{10} = F(e_{11} + e_{22} + e_{55} + e_{66}) + F(e_{12} - e_{56}) + Fe_{13} + Fe_{23} + Fe_{45} + Fe_{46} \subseteq UT_6$;
- $M_{11} = F(e_{11} + e_{22} + e_{55} + e_{66}) + F(e_{12} + e_{56}) + Fe_{13} + Fe_{23} + Fe_{45} + Fe_{46} \subseteq UT_6$.

Finally, we can consider the following $*$-superalgebras:

- $M_{4,*}$, the algebra M_4 with trivial grading and reflection involution;
- M_4^{gri}, the algebra M_4 with grading $(F(e_{11} + e_{33}) + Fe_{13}, Fe_{12} + Fe_{23})$ and reflection involution;
- $M_{5,*}$, the algebra M_5 with trivial grading and reflection involution;
- M_5^{gri}, the algebra M_5 with grading $(Fe_{22} + Fe_{13}, Fe_{12} + Fe_{23})$ and reflection involution;
- $M_{6,*}$, the algebra M_6 with trivial grading and involution $\dagger$ defined by

$$\begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & a \end{pmatrix}^{\dagger} = \begin{pmatrix} a & -f & e & -d \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & a \end{pmatrix};$$

- $M_{6,1}^{\mathrm{gri}}$, the algebra M_6 with elementary grading defined by $(0, 1, 0, 1)$ and involution $\dagger$;
- $M_{6,2}^{\mathrm{gri}}$, the algebra M_6 with elementary grading defined by $(0, 1, 1, 0)$ and involution $\dagger$;
- $M_{7,*}$, the algebra M_7 with trivial grading and involution $\ddagger$ defined by

$$\begin{pmatrix} 0 & b & c & d \\ 0 & a & 0 & e \\ 0 & 0 & a & f \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\ddagger} = \begin{pmatrix} 0 & -f & e & -d \\ 0 & a & 0 & c \\ 0 & 0 & a & -b \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

- $M_{7,1}^{\mathrm{gri}}$, the algebra M_7 with elementary grading defined by $(0, 1, 0, 1)$ and involution $\ddagger$;
- $M_{7,2}^{\mathrm{gri}}$, the algebra M_7 with elementary grading defined by $(0, 1, 1, 0)$ and involution $\ddagger$;
- $M_{8,*}$, the algebra M_8 with trivial grading and reflection involution;
- $M_{8,1}^{\mathrm{gri}}$, the algebra M_8 with elementary grading defined by $(0, 1, 1, 1, 1, 0)$ and reflection involution;
- $M_{8,2}^{\mathrm{gri}}$, the algebra M_8 with elementary grading defined by $(0, 1, 0, 0, 1, 0)$ and reflection involution;

- $M_{8,3}^{\mathrm{gri}}$, the algebra M_8 with elementary grading defined by $(0,0,1,1,0,0)$ and reflection involution;
- $M_{9,*}$, the algebra M_9 with trivial grading and reflection involution;
- $M_{9,1}^{\mathrm{gri}}$, the algebra M_9 with elementary grading defined by $(0,1,1,1,1,0)$ and reflection involution;
- $M_{9,2}^{\mathrm{gri}}$, the algebra M_9 with elementary grading defined by $(0,1,0,0,1,0)$ and reflection involution;
- $M_{9,3}^{\mathrm{gri}}$, the algebra M_9 with elementary grading defined by $(0,0,1,1,0,0)$ and reflection involution;
- $M_{10,*}$, the algebra M_{10} with trivial grading and reflection involution;
- $M_{10,1}^{\mathrm{gri}}$, the algebra M_{10} with elementary grading defined by $(0,1,1,1,1,0)$ and reflection involution;
- $M_{10,2}^{\mathrm{gri}}$, the algebra M_{10} with elementary grading defined by $(0,0,1,1,0,0)$ and reflection involution;
- $M_{10,3}^{\mathrm{gri}}$, the algebra M_{10} with elementary grading defined by $(0,1,0,0,1,0)$ and reflection involution;
- $M_{11,1}^{\mathrm{gri}}$, the algebra M_{11} with elementary grading defined by $(0,1,1,1,1,0)$ and reflection involution;
- $M_{11,2}^{\mathrm{gri}}$, the algebra M_{11} with elementary grading defined by $(0,1,0,0,1,0)$ and reflection involution.

We remark that each one of these 24 $*$-superalgebras is of type $F+J$ with $J^3=0$, where J denotes the respective Jacobson radical. Moreover, by Theorem 3.3, they have polynomial growth of the $*$-graded codimensions. Now, our goal is to prove that all these 24 $*$-superalgebras have quadratic growth of the $*$-graded codimension sequence. To this end, we start by proving the following lemmas.

Lemma 4.5. *The $*$-superalgebras $M_{4,*}, M_{5,*}, M_{6,*}, M_{7,*}, M_{8,*}, M_{9,*}, M_{10,*}$ have at least quadratic growth of the $*$-graded codimensions.*

Proof. Notice that all the algebras considered here have trivial grading. Hence, if A is one of them, it follows that $c_n^{\mathrm{gri}}(A) = c_n^*(A)$, where $c_n^*(A)$ denote the $*$-codimension sequence of the algebra A viewed as an algebra with involution. Now, in order to prove the lemma, we can use the corresponding results given in the setting of algebras with involution.

For the algebras $M_{4,*}, M_{5,*}, M_{6,*}, M_{7,*}, M_{9,*}, M_{10,*}$ we refer to [17, Lemmas 20–23, 25, 26], respectively. For the algebra $M_{8,*}$ the result follows by [18, Lemma 3.10], where the authors proved that

$$c_n^{\mathrm{gri}}(M_{8,*}) = 4n^2 - 2n - 1. \quad \square$$

Lemma 4.6. *The $*$-superalgebras $M_{11,1}^{\mathrm{gri}}, M_{11,2}^{\mathrm{gri}}$ have at least quadratic growth of the $*$-graded codimensions.*

Proof. We start by proving the result for the algebra $M_{11,1}^{\mathrm{gri}}$. First, notice that

- $(M_{11,1}^{\mathrm{gri}})_0^+ = F(e_{11}+e_{22}+e_{55}+e_{66}) + F(e_{23}+e_{45})$;
- $(M_{11,1}^{\mathrm{gri}})_1^+ = F(e_{12}+e_{56}) + F(e_{13}+e_{46})$;
- $(M_{11,1}^{\mathrm{gri}})_0^- = F(e_{23}-e_{45})$;
- $(M_{11,1}^{\mathrm{gri}})_1^- = F(e_{13}-e_{46})$.

Now, let $f = y_{1,0}^{n-2} y_{1,1} z_{1,0}$ be the highest weight vector corresponding to the multitableau

$$T_{((n-2),(1),(1),\varnothing)} = \left(\boxed{1}\boxed{\cdots}\boxed{n-2} \;,\; \boxed{n-1} \;,\; \boxed{n} \;,\; \varnothing \right).$$

Considering the evaluation $y_{1,0} = e_{11}+e_{22}+e_{55}+e_{66}$, $y_{1,1} = e_{12}+e_{56}$ and $z_{1,0} = e_{23}-e_{45}$, it follows that $f = e_{13} \neq 0$ and so it does not vanish on $M_{11,1}^{\mathrm{gri}}$.

Hence, by Remark 2.7 we get that the irreducible character $\chi_{((n-2),(1),(1),\varnothing)}$ appears with non-zero multiplicity in the decomposition of the $(n-2,1,1,0)$-cocharacter of $M_{11,1}^{\mathrm{gri}}$ as given in (2.3). So, by (2.2) and (2.4) we get that

$$c_n^{\mathrm{gri}}(M_{11,1}^{\mathrm{gri}}) \geq \binom{n}{n-2,1,1,0} d_{(n-2)} d_{(1)} d_{(1)} d_{\varnothing} = n(n-1).$$

In conclusion $M_{11,1}^{\mathrm{gri}}$ has at least quadratic growth of the $*$-graded codimensions.

With a similar proof the result also follows for the algebra $M_{11,2}^{\mathrm{gri}}$. □

We are ready to prove the following corollary.

Corollary 4.7. *The $*$-superalgebras*

$$M_4^{\mathrm{gri}},\ M_5^{\mathrm{gri}},\ M_{6,1}^{\mathrm{gri}},\ M_{6,2}^{\mathrm{gri}},\ M_{7,1}^{\mathrm{gri}},\ M_{7,2}^{\mathrm{gri}},\ M_{8,1}^{\mathrm{gri}},\ M_{8,2}^{\mathrm{gri}},$$
$$M_{8,3}^{\mathrm{gri}},\ M_{9,1}^{\mathrm{gri}},\ M_{9,2}^{\mathrm{gri}},\ M_{9,3}^{\mathrm{gri}},\ M_{10,1}^{\mathrm{gri}},\ M_{10,2}^{\mathrm{gri}},\ M_{10,3}^{\mathrm{gri}}$$

have at least quadratic growth of the $*$-graded codimensions.

Proof. First, notice that if A is a $*$-superalgebra with a nontrivial grading, then we can consider the $*$-superalgebra A_*, the algebra A with trivial grading and involution $*$. In this case, we have that $c_n^*(A_*) = c_n^{\mathrm{gri}}(A_*)$ and, by Lemma 2.3, $c_n^{\mathrm{gri}}(A_*) \leq c_n^{\mathrm{gri}}(A)$.

Let us consider the case of M_4^{gri}. By the above, we have that

$$c_n^{\mathrm{gri}}(M_{4,*}) \leq c_n^{\mathrm{gri}}(M_4^{\mathrm{gri}}).$$

Now the result follows by Lemma 4.5, where we have showed that $M_{4,*}$ has at least quadratic growth of the $*$-graded codimensions. The same proof holds for all the other $*$-superalgebras listed in the statement. □

Here we want to highlight that for the $*$-superalgebra $M_{8,1}^{\mathrm{gri}}$ we have the exact value of the $*$-graded codimensions, given in [13, Theorem 5.1]: $c_n^{\mathrm{gri}}(M_{8,1}^{\mathrm{gri}}) = 4n^2 + 1$.

The following result will help us to prove that all these 24 $*$-superalgebras have quadratic growth of the $*$-graded codimensions. We denote by $J = J(A)$ the Jacobson radical of A.

Proposition 4.8. *Let A be a finite dimensional $*$-superalgebra such that $c_n^{\mathrm{gri}}(A)$ is polynomially bounded. If $J^q = 0$ then $c_n^{\mathrm{gri}}(A) \leq \alpha n^{q-1}$, for some constant α.*

Proof. Since A is a finite dimensional $*$-superalgebra and $c_n^{\mathrm{gri}}(A)$ is polynomially bounded, by Theorem 3.3 (items 3 and 4), for each n_1, n_2, n_3, n_4 with $n_1+n_2+n_3+n_4 = n$ we obtain the following decomposition of the $(n_1, \ldots, n_4)$-cocharacter of A

$$\chi_{n_1,\ldots,n_4}(A) = \sum_{\substack{\langle\lambda\rangle\vdash(n_1,\ldots,n_4)\\ n-\lambda(1)_1<q}} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)},$$

where, by Remark 3.4, $m_{\langle\lambda\rangle} \leq h$, for some constant h and all $\langle\lambda\rangle \vdash (n_1, \ldots, n_4)$.

Thus, by (2.4), for any $(n_1, \ldots, n_4)$ we have that

$$c_{n_1,\ldots,n_4}(A) = \sum_{\substack{\langle\lambda\rangle\vdash(n_1,\ldots,n_4)\\ n-\lambda(1)_1<q}} m_{\langle\lambda\rangle} d_{\lambda(1)} \cdots d_{\lambda(4)} \leq \sum_{\substack{\langle\lambda\rangle\vdash(n_1,\ldots,n_4)\\ n-\lambda(1)_1<q}} h d_{\lambda(1)} \cdots d_{\lambda(4)}.$$

We observe that if $\lambda(1)_1 = n_1 - r$, then by the hook formula we obtain that $d_{\lambda(1)} \leq \frac{n_1!}{(n_1-r)!} \leq c n_1^r \leq d n^r$ for some constants c and d. Also, $d_{\lambda(i)}$ is constant for all partition $\lambda(i) \vdash n_i$, $i = 2, 3, 4$. Furthermore, denoting by $t = n_2+n_3+n_4$, we have that $\binom{n}{n_1,\ldots,n_4} \leq b n^t$ for some constant b.

Now, since $n - \lambda(1)_1 < q$, it follows that $\lambda(1)_1 \in \{n-(q-1), \ldots,\ n-1,\ n\}$. So if we consider the set

$$\mathcal{S} = \{(n_1, \ldots, n_4)\ :\ n-(q-1) \leq n_1 \leq n \text{ and } 0 \leq n_i \leq q-1,\ i \neq 1\},$$

by putting together all the information above, for all $(n_1, \ldots, n_4) \in \mathcal{S}$, we have that

$$\binom{n}{n_1, \ldots, n_4} c_{n_1,\ldots,n_4}(A) \leq k n^{r+t} = k n^{n-\lambda(1)_1} \leq k n^{q-1},$$

where k is a constant.

Hence, by (2.2), we get that

$$c_n^{\mathrm{gri}}(A) = \sum_{(n_1,\ldots,n_4)\in\mathcal{S}} \binom{n}{n_1, \ldots, n_4} c_{n_1,\ldots,n_4}(A) \leq \alpha n^{q-1},$$

for some constant α and the proof is complete. □

Theorem 4.9. *All the 24 $*$-superalgebras*

$$M_{4,*},\ M_4^{\mathrm{gri}},\ M_{5,*},\ M_5^{\mathrm{gri}},\ M_{6,*},\ M_{6,1}^{\mathrm{gri}},\ M_{6,2}^{\mathrm{gri}},\ M_{7,*},\ M_{7,1}^{\mathrm{gri}},\ M_{7,2}^{\mathrm{gri}},\ M_{8,*},\ M_{8,1}^{\mathrm{gri}},\ M_{8,2}^{\mathrm{gri}},$$
$$M_{8,3}^{\mathrm{gri}},\ M_{9,*},\ M_{9,1}^{\mathrm{gri}},\ M_{9,2}^{\mathrm{gri}},\ M_{9,3}^{\mathrm{gri}},\ M_{10,*},\ M_{10,1}^{\mathrm{gri}},\ M_{10,2}^{\mathrm{gri}},\ M_{10,3}^{\mathrm{gri}},\ M_{11,1}^{\mathrm{gri}},\ M_{11,2}^{\mathrm{gri}}$$

have quadratic growth of the $*$-graded codimensions.

Proof. By Lemmas 4.5, 4.6 and Corollary 4.7, we have that all these $*$-superalgebras have at least quadratic growth of the $*$-graded codimensions. Moreover, as previously mentioned, the Jacobson radical of each one of these $*$-superalgebras has nilpotency index equal to 3. Thus, they satisfy the hypotheses of Proposition 4.8 with $q = 3$ and so we get that the growth is at most quadratic. The proof is complete. □

By the previous theorem and Corollary 4.3, we have that the 37 $*$-superalgebras presented in this section have quadratic growth of the $*$-graded codimensions. Before ending this section we shall prove the following result concerning the algebras $M_{6,*}$ and $U_{3,*}$.

Lemma 4.10. $\mathrm{Id}^{\mathrm{gri}}(M_{6,*}) \subsetneq \mathrm{Id}^{\mathrm{gri}}(U_{3,*})$.

Proof. Let us suppose, by contradiction, that $\mathrm{Id}^{\mathrm{gri}}(M_{6,*}) \not\subseteq \mathrm{Id}^{\mathrm{gri}}(U_{3,*})$. Then there exists a multilinear polynomial f of a certain degree, say n, such that

$$f \in \mathrm{Id}^{\mathrm{gri}}(M_{6,*}) \quad \text{and} \quad f \notin \mathrm{Id}^{\mathrm{gri}}(U_{3,*}).$$

Recall that $\mathrm{Id}^{\mathrm{gri}}(U_{3,*}) = \langle y_{1,1},\ z_{1,1},\ [z_{1,0}, y_{2,0}],\ z_{1,0}z_{2,0}\rangle_{T_2^*}$ [16, Lemma 3]. We want to highlight that also $[y_{1,0}, y_{2,0}, y_{3,0}] \equiv 0$ and $[y_{1,0}, y_{2,0}][y_{3,0}, y_{4,0}] \equiv 0$ are identities of $U_{3,*}$. By taking into account all these identities, it is easy to see that, modulo $\mathrm{Id}^{\mathrm{gri}}(U_{3,*})$, f must be a polynomial of type

$$f = \alpha y_{1,0} \cdots y_{n,0} + \sum_{i=1}^{n} \beta_i y_{i_1,0} \cdots y_{i_{n-1},0} z_{i,0} + \sum_{h=1}^{n-1} \left(\sum_{k=h+1}^{n} \gamma_{k,h} y_{l_1,0} \cdots y_{l_{n-2},0} [y_{k,0}, y_{h,0}] \right),$$

where $i_1 < \cdots < i_{n-1}$, $l_1 < \cdots < l_{n-2}$ and $k > h$. In the last term of f we have exactly $\frac{n(n-1)}{2}$ polynomials.

Recall that

$$\begin{aligned}
(M_{6,*})_0^+ &= F(e_{11} + e_{44}) + F(e_{13} + e_{24}) + F(e_{12} - e_{34}),\\
(M_{6,*})_0^- &= F(e_{12} + e_{34}) + F(e_{13} - e_{24}) + Fe_{14},\\
(M_{6,*})_1^+ &= (M_{6,*})_1^- = \{0\}.
\end{aligned}$$

In order to reach a contradiction, we shall prove that f is actually the zero polynomial. To this end, we consider the following evaluations in elements of $M_{6,*}$.

1. For any $a, b \in \{1, \dots, n\}$, let $y_{a,0} = e_{11} + e_{44}$ and $z_{b,0} = 0$. We get $f = \alpha(e_{11} + e_{44}) = 0$, so $\alpha = 0$.
2. For a fixed $i \in \{1, \dots, n\}$, let $z_{i,0} = e_{14}$, $z_{j,0} = 0$, for any $j \neq i$ and $y_{c,0} = e_{11} + e_{44}$, for any $c \in \{1, \dots, n\}$. We get $f = \beta_i e_{14} = 0$ and so $\beta_i = 0$. At the same way we can deal with all the β_j's.
3. Let us consider first the cases in which $k < n - 1$ (recall that $k > h$). With the evaluation

$$y_{k,0} = e_{12} - e_{34}, \quad y_{h,0} = e_{13} + e_{24}, \quad y_{l_a,0} = e_{11} + e_{44} \text{ for any } a \in \{1, \dots, n\} \setminus \{k, h\},$$

we get $f = 2\gamma_{k,h} e_{14} = 0$ and so $\gamma_{k,h} = 0$. Here the trick is that all the other monomials do not have y_k or y_h (you cannot put both of them outside the commutator) just on the left of the commutator. This happens because $h < k < n - 1$ (recall that the variables outside the commutator are ordered). In conclusion we have proved that $\gamma_{k,h} = 0$ whenever $k < n - 1$.

In practice, the polynomial f is now of this form:

$$f = \sum_{h=1}^{n-1} \gamma_{n,h} y_{l_1,0} \cdots y_{l_{n-2},0} [y_{n,0}, y_{h,0}] + \sum_{h=1}^{n-2} \gamma_{n-1,h} y_{l_1,0} \cdots y_{l_{n-2},0} [y_{n-1,0}, y_{h,0}].$$

Let us consider the cases in which $h \leq n - 3$. We make the evaluations:

a) $y_{h,0} = e_{13} + e_{24}$, $y_{n-1,0} = e_{12} - e_{34}$, $y_{l,0} = e_{11} + e_{44}$, for all $l \in \{1, \dots, n\} \setminus \{n-1, h\}$,

b) $y_{h,0} = e_{13} + e_{24}$, $y_{l,0} = e_{11} + e_{44}$, for all $l \in \{1, \dots, n\} \setminus \{h\}$.

We get, respectively,

$$(2\gamma_{n-1,h} - \gamma_{n,h}) e_{14} = 0 \quad \text{and} \quad (\gamma_{n-1,h} + \gamma_{n,h}) e_{13} = 0.$$

It follows clearly that $\gamma_{n-1,h} = \gamma_{n,h} = 0$.

Finally, the polynomial f is of the form (here y_i means $y_{i,0}$):

$$\begin{aligned} f = {} & \gamma_{n,n-1} y_1 \cdots y_{n-2} [y_n, y_{n-1}] + \gamma_{n,n-2} y_1 \cdots y_{n-3} y_{n-1} [y_n, y_{n-2}] \\ & + \gamma_{n-1,n-2} y_1 \cdots y_{n-3} y_n [y_{n-1}, y_{n-2}]. \end{aligned}$$

We consider the following evaluations:

a) $y_{n-2} = e_{13} + e_{24}$, $y_{n-1} = e_{12} - e_{34}$, $y_l = e_{11} + e_{44}$, for all $l \in \{1, \dots, n\} \setminus \{n-1, n-2\}$,

b) $y_{n-2} = e_{13} + e_{24}$, $y_l = e_{11} + e_{44}$, for all $l \in \{1, \dots, n\} \setminus \{n - 2\}$,

c) $y_{n-1} = e_{13} + e_{24}$, $y_l = e_{11} + e_{44}$, for all $l \in \{1, \dots, n\} \setminus \{n - 1\}$.

We get, respectively:

a) $(\gamma_{n,n-1} - \gamma_{n,n-2} + 2\gamma_{n-1,n-2}) e_{14} = 0$,

b) $(\gamma_{n-1,n-2} + \gamma_{n,n-2}) e_{13} = 0$,

c) $(\gamma_{n-1,n-2} - \gamma_{n,n-1}) e_{13} = 0$.

In conclusion, we obtain that $\gamma_{n,n-1} = \gamma_{n,n-2} = \gamma_{n-1,n-2} = 0$. Hence f is the zero polynomial as desired.

Now, it is easy to see that $z_{1,0}z_{2,0} \equiv 0$ on $U_{3,*}$ and that $z_{1,0}z_{2,0} \not\equiv 0$ on $M_{6,*}$. The proof is now complete. $\square$

In light of the result above we have that the $*$-superalgebra $M_{6,*}$ does not generate a minimal variety of quadratic growth. Hence we will focus our attention to the 36 $*$-superalgebras in the following set

$$\begin{aligned}\mathcal{L} = \{&G_{2,0,\tau}, G_{2,1,\tau}, G_{2,2,\tau}, G_{2,1,\psi}, G_{2,1,\rho}, G_{2,2,\rho}, C_{3,*}, C_3^{\mathrm{gri}}, C_3^{\mathrm{gr}}, N_{3,*}, N_3^{\mathrm{gri}}, U_{3,*}, U_3^{\mathrm{gri}},\\ &M_{4,*}, M_4^{\mathrm{gri}}, M_{5,*}, M_5^{\mathrm{gri}}, M_{6,1}^{\mathrm{gri}}, M_{6,2}^{\mathrm{gri}}, M_{7,*}, M_{7,1}^{\mathrm{gri}}, M_{7,2}^{\mathrm{gri}}, M_{8,*}, M_{8,1}^{\mathrm{gri}}, M_{8,2}^{\mathrm{gri}}, M_{8,3}^{\mathrm{gri}},\\ &M_{9,*}, M_{9,1}^{\mathrm{gri}}, M_{9,2}^{\mathrm{gri}}, M_{9,3}^{\mathrm{gri}}, M_{10,*}, M_{10,1}^{\mathrm{gri}}, M_{10,2}^{\mathrm{gri}}, M_{10,3}^{\mathrm{gri}}, M_{11,1}^{\mathrm{gri}}, M_{11,2}^{\mathrm{gri}}\}.\end{aligned}$$

The final goal of this paper is to prove that the $*$-superalgebras in $\mathcal{L}$ generate the only minimal varieties of quadratic growth of the $*$-graded codimensions.

Remark 4.11. A long and arduous calculation proves that for distinct $*$-superalgebras A and B in $\mathcal{L}$, we have $\mathrm{Id}^{\mathrm{gri}}(A) \nsubseteq \mathrm{Id}^{\mathrm{gri}}(B)$.

5. Wedderburn-Malcev decomposition of $*$-superalgebras

In this section we will prove that the Jacobson radical of a finite dimensional $*$-superalgebra A with Wedderburn-Malcev decomposition of type $F+J$ as in Theorem 2.1 has particular properties when we exclude some $*$-superalgebras of quadratic growth from $\mathrm{var}^{\mathrm{gri}}(A)$.

Lemma 5.1.

(1) *If* $M_{4,*}, M_{5,*}, M_{6,*}, M_{7,*} \notin \mathrm{var}^{\mathrm{gri}}(A)$ *then* $(J_{01})_0(J_{10})_0 = (J_{10})_0(J_{01})_0 = 0$.
(2) *If* $M_{4,*}, M_4^{\mathrm{gri}}, M_{5,*}, M_5^{\mathrm{gri}}, M_{6,1}^{\mathrm{gri}}, M_{6,2}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$ *then* $(J_{10})_1J_{01} = (J_{10})_0(J_{01})_1 = 0$.
(3) *If* $M_{4,*}$, M_4^{gri}, $M_{5,*}$, M_5^{gri}, $M_{7,1}^{\mathrm{gri}}, M_{7,2}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$ *then* $(J_{01})_1J_{10} = (J_{01})_0(J_{10})_1 = 0$.

Proof. The first item is proved in [17, Lemmas 30, 31].

In order to prove item (2), we suppose $M_{4,*}, M_4^{\mathrm{gri}}, M_{5,*}, M_5^{\mathrm{gri}}, M_{6,1}^{\mathrm{gri}}, M_{6,2}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$. First we notice that $a^*a = aa^* = 0$, for all $a \in (J_{10})_1$. In fact, suppose that there exists $a \in (J_{10})_1$ such that $aa^* \neq 0$. Consider B the $*$-graded subalgebra generated by $1_F, a, a^*$ and I the $*$-graded ideal generated by a^*a. It is not difficult to see that $\varphi : B/I \to M_4^{\mathrm{gri}}$ defined by

$$\varphi(\overline{1_F}) = e_{11} + e_{33}, \quad \varphi(\overline{a}) = e_{12}, \quad \varphi(\overline{a^*}) = e_{23}, \quad \varphi(\overline{aa^*}) = e_{13}$$

is an isomorphism of $*$-superalgebras and since $M_4^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$, we have a contradiction. In an analogous way, since $M_5^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$, we also have $aa^* = 0$, for all $a \in (J_{01})_1$.

Now suppose that there exist $a \in (J_{10})_1$ and $b \in (J_{01})_1$ such that $ab \neq 0$. In this case, $a^*a = aa^* = b^*b = bb^* = a^2 = b^2 = 0$. Notice that $(a+b^*) \in (J_{10})_1$, so $(a+b^*)(a+b^*)^* = (a+b^*)(a^*+b) = 0$. We get, $ab + b^*a^* = 0$ and $(ab)^* = b^*a^* = -ab$. As a consequence $ab \in (J_{11})_0^-$.

Also, since $ab = -b^*a^*$, we have $aba = -b^*a^*a = 0$ and $bab = -bb^*a^* = 0$. Let B' be the $*$-graded subalgebra of A generated by $1_F, a, a^*, b, b^*$ and notice that $B' = \text{span}\{1, a, a^*, b, b^*, ab, ba\}$.

Let I' be the $*$-graded ideal generated by ba. The map $\varphi' : B'/I' \to M_{6,2}^{\text{gri}}$ given by

$$\overline{1_F} \mapsto e_{11} + e_{44}, \quad \overline{a} \mapsto e_{12}, \quad \overline{a^*} \mapsto -e_{34}, \quad \overline{b} \mapsto e_{24}, \quad \overline{b^*} \mapsto e_{13}, \quad \overline{ab} \mapsto e_{14}$$

is an isomorphism of $*$-superalgebras and so $M_{6,2}^{\text{gri}} \in \text{var}^{\text{gri}}(A)$, a contradiction. As a consequence, we get $(J_{10})_1(J_{01})_1 = 0$.

Next, suppose that there exist $a \in (J_{10})_0$ and $b \in (J_{01})_1$ such that $ab \neq 0$. Since $M_{4,*}, M_{5,*} \notin \text{var}^{\text{gri}}(A)$, by [17, Lemma 29] we have $a^*a = aa^* = 0$. Also since M_4^{gri}, $M_5^{\text{gri}} \notin \text{var}^{\text{gri}}(A)$, we have proved in the previous item that $b^*b = bb^* = 0$. Using the same strategy as above we get $ab \in (J_{11})_1^-$ and again we can define the $*$-superalgebra $B'' = \text{span}\{1, a, a^*, b, b^*, ab, ba\}$ and the $*$-graded ideal I'' generated by ba so that we have an isomorphism of $*$-superalgebras $\varphi'' : B''/I'' \to M_{6,1}^{\text{gri}}$ given by

$$\overline{1_F} \mapsto e_{11} + e_{44}, \quad \overline{a} \mapsto e_{13}, \quad \overline{a^*} \mapsto e_{24}, \quad \overline{b} \mapsto -e_{34}, \quad \overline{b^*} \mapsto e_{12}, \quad \overline{ab} \mapsto -e_{14},$$

a contradiction. This proves that $(J_{10})_0(J_{01})_1 = (J_{10})_1(J_{01})_0 = 0$.

The last item can be proved in a similar way. The proof is complete. □

Corollary 5.2. *If* $M_{4,*}$, M_4^{gri}, $M_{5,*}$, M_5^{gri}, $M_{6,*}$, $M_{6,1}^{\text{gri}}$, $M_{6,2}^{\text{gri}}$, $M_{7,*}$, $M_{7,1}^{\text{gri}}$, $M_{7,2}^{\text{gri}} \notin \text{var}^{\text{gri}}(A)$ *then* $J_{01}J_{10} = J_{10}J_{01} = 0$. *As a consequence,* $F + J_{10} + J_{01} + J_{11}$ *is a $*$-graded subalgebra of* A.

In the following, we establish conditions to have a direct sum between $F+J_{10}+J_{01}+J_{11}$ and J_{00}.

Lemma 5.3. *If* $M_{8,*}, M_{8,1}^{\text{gri}}, M_{8,2}^{\text{gri}}, M_{8,3}^{\text{gri}} \notin \text{var}^{\text{gri}}(A)$ *then* $J_{10}J_{00}^+ = J_{00}^+J_{01} = 0$.

Proof. Suppose that $M_{8,*}, M_{8,1}^{\text{gri}}, M_{8,2}^{\text{gri}}, M_{8,3}^{\text{gri}} \notin \text{var}^{\text{gri}}(A)$. If $(J_{10})_0(J_{00})_0^+ \neq 0$ then there exist $a \in (J_{10})_0$ and $b \in (J_{00})_0^+$ such that $ab \neq 0$. Considering B to be the $*$-graded subalgebra generated by $1_F, a, a^*, b$ and I to be the $*$-graded ideal generated by aa^*, a^*a, b^2, aba^*, we have that B/I is a $*$-superalgebra isomorphic to $M_{8,*}$ via the map defined by

$$\overline{1_F} \mapsto e_{11} + e_{66}, \quad \overline{a} \mapsto e_{12}, \quad \overline{a^*} \mapsto e_{56}, \quad \overline{b} \mapsto e_{23} + e_{45}, \quad \overline{ab} \mapsto e_{13}, \quad \overline{ba^*} \mapsto e_{46}. \tag{5.1}$$

So $M_{8,*} \in \text{var}^{\text{gri}}(A)$, a contradiction. Then $(J_{10})_0(J_{00})_0^+ = (J_{00})_0^+(J_{01})_0 = 0$.

In an analogous way, if there exist $a \in (J_{10})_1$ and $b \in (J_{00})_0^+$ such that $ab \neq 0$, it is possible to construct a $*$-superalgebra B' and a $*$-graded ideal I' exactly as above and notice that the same rule given in (5.1) defines an isomorphism of $*$-superalgebras from B'/I' to $M_{8,1}^{\mathrm{gri}}$. Again, we get a contradiction and so $(J_{10})_1(J_{00})_0^+ = (J_{00})_0^+(J_{01})_1 = 0$. We conclude that $M_{8,*}, M_{8,1}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$ implies $J_{10}(J_{00})_0^+ = (J_{00})_0^+ J_{01} = 0$.

We finish the proof by following the same reasoning as above to guarantee that $M_{8,2}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$ implies $(J_{10})_1(J_{00})_1^+ = (J_{00})_1^+(J_{01})_1 = 0$ and $M_{8,3}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$ implies $(J_{10})_0(J_{00})_1^+ = (J_{00})_1^+(J_{01})_0 = 0$. This means that $J_{10}(J_{00})_1^+ = (J_{00})_1^+ J_{01} = 0$ and we are done. □

In a similar way we get the following lemma.

Lemma 5.4. *If* $M_{9,*}, M_{9,1}^{\mathrm{gri}}, M_{9,2}^{\mathrm{gri}}, M_{9,3}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$ *then* $J_{10}J_{00}^- = J_{00}^- J_{01} = 0$.

Corollary 5.5. *If* $M_{8,*}$, $M_{8,1}^{\mathrm{gri}}$, $M_{8,2}^{\mathrm{gri}}$, $M_{8,3}^{\mathrm{gri}}$, $M_{9,*}$, $M_{9,1}^{\mathrm{gri}}$, $M_{9,2}^{\mathrm{gri}}$, $M_{9,3}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$ *then* $J_{10}J_{00} = J_{00}J_{01} = 0$. *As a consequence, under the conditions of Corollary 5.2, we have a direct sum of algebras*

$$A = (F + J_{10} + J_{01} + J_{11}) \oplus J_{00}.$$

Lemma 5.6.

(1) *If* $M_{10,*}, M_{10,2}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$ *then* $J_{01}(J_{11})_0^- = (J_{11})_0^- J_{10} = 0$.
(2) *If* $M_{10,1}^{\mathrm{gri}}, M_{10,3}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$ *then* $J_{01}(J_{11})_1^- = (J_{11})_1^- J_{10} = 0$.
(3) *If* $M_{11,1}^{\mathrm{gri}}, M_{11,2}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$ *then* $J_{01}(J_{11})_1^+ = (J_{11})_1^+ J_{10} = 0$.

Proof. Let us prove item (1). For that, we suppose $M_{10,*}, M_{10,2}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$ and $(J_{01})_0(J_{11})_0^- \neq 0$. Consider $a \in (J_{01})_0$ and $b \in (J_{11})_0^-$ such that $ab \neq 0$. Let B be the $*$-graded subalgebra generated by $1_F, a, a^*, b$ and I to be the $*$-graded ideal generated by aa^*, a^*a, b^2, aba^*. Hence we have a map $\varphi : B/I \to M_{10,*}$ defined by

$$\overline{1_F} \mapsto e_{11} + e_{22} + e_{55} + e_{66}, \quad \overline{a} \mapsto e_{45}, \quad \overline{a^*} \mapsto e_{23}, \quad \overline{b} \mapsto e_{12} - e_{56}, \quad \overline{ab} \mapsto -e_{46}, \quad \overline{ba^*} \mapsto e_{13},$$

which is an isomorphism of $*$-superalgebras and this implies that $M_{10,*} \in \mathrm{var}^{\mathrm{gri}}(A)$. This contradiction proves that $(J_{01})_0(J_{11})_0^- = (J_{11})_0^-(J_{10})_0 = 0$.

In the same way, if $(J_{01})_1(J_{11})_0^- \neq 0$, we can construct a $*$-superalgebra isomorphic to $M_{10,2}^{\mathrm{gri}}$ in $\mathrm{var}^{\mathrm{gri}}(A)$. Hence, $(J_{01})_1(J_{11})_0^- = (J_{11})_0^-(J_{10})_1 = 0$. So the exclusion of the $*$-superalgebras $M_{10,*}$ and $M_{10,2}^{\mathrm{gri}}$ from the variety $\mathrm{var}^{\mathrm{gri}}(A)$ guarantees that $J_{01}(J_{11})_0^- = (J_{11})_0^- J_{10} = 0$. The item (1) is proved.

The items (2) and (3) can be proved by using similar arguments as above. □

Corollary 5.7. *If* $M_{10,*}$, $M_{10,1}^{\mathrm{gri}}$, $M_{10,2}^{\mathrm{gri}}$, $M_{10,3}^{\mathrm{gri}}$, $M_{11,1}^{\mathrm{gri}}$, $M_{11,2}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$ *then* $J_{01}(J_{11})_1^+ = J_{01}(J_{11})_0^- = J_{01}(J_{11})_1^- = (J_{11})_1^+ J_{10} = (J_{11})_0^- J_{10} = (J_{11})_1^- J_{10} = 0$.

6. $*$-superalgebras with particular decompositions

Let $A = F + J$ be a finite dimensional $*$-superalgebra such that $J_{01}J_{10} = 0$. So $B = F + J_{01} + J_{10} + J_{11}$ is a $*$-graded subalgebra of A and in this case, the following situations are possible:

Case (1) $J_{10} = 0$ and $J_{11} \neq 0$, that is, $B = F + J_{11}$;
Case (2) $J_{11} = 0$ and $J_{10} \neq 0$, that is, $B = F + J_{01} + J_{10}$;
Case (3) Both J_{11} and J_{10} are non-zero.

We will analyze some properties of finite dimensional $*$-superalgebras in each one of the situations above. In the next three lemmas we start with Case (1). So we consider $B = F + J_{11}$, with $J_{11} \neq 0$.

Lemma 6.1 *([19, Lemma 8.5]). If* $G_{2,1,\rho}, G_{2,2,\rho}, G_{2,2,\tau} \notin \mathrm{var}^{\mathrm{gri}}(B)$ *then* $(J_{11})_1^+(J_{11})_1^- = (J_{11})_1^-(J_{11})_1^+ = (J_{11})_0^-(J_{11})_1^+ = (J_{11})_1^+(J_{11})_0^- = (J_{11})_0^-(J_{11})_1^- = (J_{11})_1^-(J_{11})_0^- = 0$.

Lemma 6.2. *If* $G_{2,0,\tau}$, $G_{2,1,\tau}$, $G_{2,1,\psi}$, $C_{3,*}$, C_3^{gr}, $C_3^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(B)$ *then* $(J_{11})_1^+(J_{11})_1^+ = (J_{11})_0^-(J_{11})_0^- = (J_{11})_1^-(J_{11})_1^- = 0$.

Proof. Since $G_{2,0,\tau}$, $G_{2,1,\tau}$, $G_{2,1,\psi} \notin \mathrm{var}^{\mathrm{gri}}(B)$, by [19, Lemma 8.4], we have $[z_{1,0}, z_{2,0}] \equiv 0$, $[z_{1,1}, z_{2,1}] \equiv 0$ and $[y_{1,1}, y_{2,1}] \equiv 0$ on B and this implies $[(J_{11})_0^-, (J_{11})_0^-] = [(J_{11})_1^-, (J_{11})_1^-] = [(J_{11})_1^+, (J_{11})_1^+] = 0$. Furthermore using that $C_{3,*}$, C_3^{gr}, $C_3^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(B)$, by linearizing the identities $z_{1,0}^2 \equiv 0$, $y_{1,1}^2 \equiv 0$ and $z_{1,1}^2 \equiv 0$ obtained in [19, Lemma 8.2], we get $z_{1,0} \circ z_{2,0} \equiv 0$, $y_{1,1} \circ y_{2,1} \equiv 0$ and $z_{1,1} \circ z_{2,1} \equiv 0$ on B. In conclusion we have $(J_{11})_0^-(J_{11})_0^- = (J_{11})_1^+(J_{11})_1^+ = (J_{11})_1^-(J_{11})_1^- = 0$. □

Lemma 6.3.

(1) *If* $N_{3,*}, N_3^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(B)$ *then* $(J_{11})_0^+ J_{11}^- = J_{11}^-(J_{11})_0^+ = 0$.
(2) *If* $U_{3,*}, U_3^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(B)$ *then* $(J_{11})_0^+ J_{11}^+ = J_{11}^+(J_{11})_0^+ = 0$.

Proof. We only present the proof of item (1) since the proof of the second item follows in a very similar way. Suppose that $N_{3,*}, N_3^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(B)$ and $(J_{11})_0^+(J_{11})_0^- \neq 0$. Consider $a \in (J_{11})_0^+$ and $b \in (J_{11})_0^-$ such that $ab \neq 0$. Notice that $ba \neq 0$. Let R be the $*$-graded subalgebra generated by $1_F, a, b$ and I to be the $*$-graded ideal generated by a^2, b^2, aba, bab. We have that $\overline{R} = R/I$ is a $*$-superalgebra with trivial grading isomorphic to $N_{3,*}$ via $\varphi : \overline{R} \to N_{3,*}$ given by

$$\overline{1_F} \mapsto e_{11}+\cdots+e_{66},\ \ \overline{a} \mapsto e_{23}+e_{45},\ \ \overline{b} \mapsto e_{12}-e_{56},\ \ \overline{a}\circ\overline{b} \mapsto e_{13}-e_{46},\ \ [\overline{a},\overline{b}] \mapsto -(e_{13}+e_{46}).$$

Since $N_{3,*} \notin \mathrm{var}^{\mathrm{gri}}(B)$ we get a contradiction and this proves that $(J_{11})_0^+(J_{11})_0^- = (J_{11})_0^-(J_{11})_0^+ = 0$.

Now suppose that $(J_{11})_0^+(J_{11})_1^- \neq 0$ and consider $a \in (J_{11})_0^+$ and $b \in (J_{11})_1^-$ such that $ab \neq 0$. If R' and I' are, respectively, the $*$-superalgebra and the $*$-graded ideal defined exactly as in the previous situation, we observe that R'/I' is isomorphic to N_3^{gri} via an isomorphism given as above. So, $N_3^{\mathrm{gri}} \in \mathrm{var}^{\mathrm{gri}}(B)$. With this contradiction we finish the proof. □

Corollary 6.4. *If* $C_{3,*}, C_3^{\mathrm{gr}}, C_{3,*}^{\mathrm{gri}}, N_{3,*}, N_3^{\mathrm{gri}}, U_{3,*}, U_3^{\mathrm{gri}}, G_{2,1,\rho}, G_{2,1,\tau}, G_{2,1,\psi}, G_{2,0,\tau}, G_{2,1,\tau}, G_{2,1,\psi} \notin \mathrm{var}^{\mathrm{gri}}(B)$ *then* $J_{11}^2 = 0$.

At this point, it is important to consider the following subalgebras of UT_2 and UT_4, respectively:

- $C_2 = F(e_{11} + e_{22}) + Fe_{12}$,
- $A_2 = F(e_{11} + e_{44}) + Fe_{12} + Fe_{34}$.

We construct the $*$-superalgebras:

- C_2^{gr} the algebra C_2 with grading $(F(e_{11} + e_{22}), Fe_{12})$ and trivial involution;
- $C_{2,*}$ the algebra C_2 with trivial grading and involution $\circ$ defined as:

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^{\circ} = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix};$$

- C_2^{gri} the algebra C_2 with grading $(F(e_{11} + e_{22}), Fe_{12})$ and involution $\circ$;
- $A_{2,*}$ the algebra A_2 with trivial grading and reflection involution;
- A_2^{gri} the algebra A_2 with grading $(F(e_{11}+e_{44}), Fe_{12}+Fe_{34})$ and reflection involution.

The information about the $*$-graded codimensions of the $*$-superalgebras defined above is summarized in the next lemma.

Lemma 6.5.

(1) $c_n^{\mathrm{gri}}(A_2^{\mathrm{gri}}) = 4n + 1$ *and* $c_n^{\mathrm{gri}}(C_2^{\mathrm{gri}}) = n + 1$ *([13, Theorems 5.1 and 6.1]).*
(2) $c_n^{\mathrm{gri}}(C_{2,*}) = c_n^{\mathrm{gri}}(C_2^{\mathrm{gr}}) = n + 1$ *([16, Lemma 9]), ([15, Theorem 8.1]).*
(3) $c_n^{\mathrm{gri}}(A_{2,*}) = 4n - 1$ *([18, Lemma 3.10]).*

In the next two lemmas we still consider $B = F + J_{11}$.

Lemma 6.6 *([19, Lemma 8.7]). If* $B \in \mathrm{var}^{\mathrm{gri}}(D_* \oplus D^{\mathrm{gr}} \oplus D^{\mathrm{gri}})$ *then* $B \sim_{T_2^*} B_1 \oplus B_2 \oplus B_3$, *where* $B_1 \in \mathrm{var}^{\mathrm{gri}}(D_*)$, $B_2 \in \mathrm{var}^{\mathrm{gri}}(D^{\mathrm{gr}})$ *and* $B_3 \in \mathrm{var}^{\mathrm{gri}}(D^{\mathrm{gri}})$.

Lemma 6.7. *If* $J_{11}^2 = 0$ *then either* $B \sim_{T_2^*} C$ *or* $B \sim_{T_2^*} C_2^{\mathrm{gri}}$ *or* $B \sim_{T_2^*} C_2^{\mathrm{gr}}$ *or* $B \sim_{T_2^*} C_{2,*}$ *or* $B \sim_{T_2^*} C_2^{\mathrm{gri}} \oplus C_2^{\mathrm{gr}}$ *or* $B \sim_{T_2^*} C_2^{\mathrm{gri}} \oplus C_{2,*}$ *or* $B \sim_{T_2^*} C_2^{\mathrm{gr}} \oplus C_{2,*}$ *or* $B \sim_{T_2^*}$

$C_2^{\mathrm{gri}} \oplus C_{2,*} \oplus C_2^{\mathrm{gr}}$, where C is a commutative $*$-superalgebra with trivial grading and trivial involution.

Proof. Notice that B satisfies the following $(\mathbb{Z}_2,*)$-identities

$$[y_{1,0},y_{2,0}],[y_{1,0},y_{2,1}],[y_{1,0},z_{2,0}],[y_{1,0},z_{2,1}],z_{1,0}z_{2,0},z_{1,0}y_{2,1},z_{1,0}z_{2,1},x_{1,1}x_{2,1},$$

where $x_i = y_i$ or $x_i = z_i$, for $i = 1,2$. Considering $\mathcal{S} = D_* \oplus D^{\mathrm{gr}} \oplus D^{\mathrm{gri}}$, by [19, Lemma 7.5] we have $\mathrm{Id}^{\mathrm{gri}}(\mathcal{S}) = \langle z_{1,0}y_{2,1}, z_{1,0}z_{2,1}, y_{1,1}z_{2,1}, [y_{1,0},y_{2,0}], [y_{1,0},y_{2,1}], [y_{1,0},z_{2,0}], [y_{1,0},z_{2,1}], [y_{1,1},y_{2,1}], [z_{1,0},z_{2,0}], [z_{1,1},z_{2,1}]\rangle_{T_2^*}$. It follows that $B \in \mathrm{var}^{\mathrm{gri}}(\mathcal{S})$ so $B \sim_{T_2^*} B_1 \oplus B_2 \oplus B_3$, where $B_1 \in \mathrm{var}^{\mathrm{gri}}(D_*)$, $B_2 \in \mathrm{var}^{\mathrm{gri}}(D^{\mathrm{gr}})$ and $B_3 \in \mathrm{var}^{\mathrm{gri}}(D^{\mathrm{gri}})$ as in Lemma 6.6.

Finally, since $C_{3,*}, C_3^{\mathrm{gr}}$ and C_3^{gri} do not satisfy the $(\mathbb{Z}_2,*)$-identities $z_{1,0}z_{2,0}$, $y_{1,1}y_{2,1}$ and $z_{1,1}z_{2,1}$, respectively we have $C_{3,*}, C_3^{\mathrm{gr}}, C_3^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(B)$. We conclude that B is T_2^*-equivalent to either C or C_2^{gri} or C_2^{gr} or $C_{2,*}$ or $C_2^{\mathrm{gri}} \oplus C_2^{\mathrm{gr}}$ or $C_2^{\mathrm{gri}} \oplus C_{2,*}$ or $C_2^{\mathrm{gr}} \oplus C_{2,*}$ or $C_2^{\mathrm{gri}} \oplus C_{2,*} \oplus C_2^{\mathrm{gr}}$, as desired. □

Now we can deal with Case (2), i.e. with finite dimensional $*$-superalgebras $B = F + J_{01} + J_{10}$, $J_{10} \neq 0$.

Lemma 6.8. *Let $B = F + J_{01} + J_{10}$, $J_{10} \neq 0$.*

1. *If $(J_{01} + J_{10})_0 \neq 0$, then $B_1 = F + (J_{01} + J_{10})_0 \sim_{T_2^*} A_{2,*}$.*
2. *If $(J_{01} + J_{10})_1 \neq 0$, then $B_2 = F + (J_{01} + J_{10})_1 \sim_{T_2^*} A_2^{\mathrm{gri}}$.*

Moreover $B \sim_{T_2^*} B_1 \oplus B_2$.

Proof. Since $(J_{01} + J_{10})^2 = 0$, we have that B_1 and B_2 are $*$-graded subalgebras of B.

Now suppose $(J_{01} + J_{10})_0 \neq 0$. This implies that $(J_{10})_0 \neq 0$ and so by [19, Lemma 8.1], we get $A_{2,*} \in \mathrm{var}^{\mathrm{gri}}(B_1)$. On the other hand, we have $\mathrm{Id}^{\mathrm{gri}}(A_{2,*}) = \langle y_{1,1}, z_{1,1}, z_{1,0}z_{2,0}, y_{1,0}z_{2,0}y_{3,0}, St_3(y_{1,0},y_{2,0},y_{3,0})\rangle_{T_2^*}$ (see [18, Lemma 3.10]). In this way, since $(J_{01} + J_{10})^2 = 0$, it is not difficult to verify that $\mathrm{Id}^{\mathrm{gri}}(A_{2,*}) \subseteq \mathrm{Id}^{\mathrm{gri}}(B_1)$. As a consequence, $B_1 \sim_{T_2^*} A_{2,*}$.

In case of $(J_{01} + J_{10})_1 \neq 0$ we get $(J_{10})_1 \neq 0$ and by using [19, Lemma 8.1] again, we obtain $\mathrm{Id}^{\mathrm{gri}}(B_2) \subseteq \mathrm{Id}^{\mathrm{gri}}(A_2^{\mathrm{gri}})$. Now it is enough to take into account the T_2^*-ideal of A_2^{gri} given in [13, Theorem 5.1] to conclude that $B_2 \sim_{T_2^*} A_2^{\mathrm{gri}}$.

Now, since B_1 and B_2 are $*$-graded subalgebras of B, we have that $\mathrm{Id}^{\mathrm{gri}}(B) \subseteq \mathrm{Id}^{\mathrm{gri}}(B_1 \oplus B_2)$. Let $f \in \mathrm{Id}^{\mathrm{gri}}(B_1 \oplus B_2)$ and assume that it is multilinear. We claim that $f \in \mathrm{Id}^{\mathrm{gri}}(B)$. In fact, let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ be a $*$-graded basis of B where $\mathcal{B}_1 = \{1_F\}$ and $\mathcal{B}_2$ is a $*$-graded basis of $J_{01} + J_{10}$. Since f is a multilinear polynomial, it is enough to evaluate f on the elements of $\mathcal{B}$ to ensure that our statement is true. In order to obtain a non-zero result under the evaluation, f must be evaluated on 1_F and in at most one element of $J_{01} + J_{10}$. This means that f is evaluated on elements of B_1 or B_2. As a consequence, $f \in \mathrm{Id}^{\mathrm{gri}}(B)$ and we are done. □

Corollary 6.9. *We have that* $B = F + J_{01} + J_{10}$ *is* T_2^**-equivalent to either* $A_{2,*}$ *or* A_2^{gri} *or* $A_{2,*} \oplus A_2^{\mathrm{gri}}$.

Finally, we consider B in Case (3). Before stating the main result about $*$-superalgebras of type $B = F + J_{01} + J_{10} + J_{11}$, with $J_{10} \neq 0$ and $J_{11} \neq 0$, let us present a technical lemma.

Lemma 6.10. *Let* f *be a multilinear* $(\mathbb{Z}_2, *)$*-polynomial of degree* n *in symmetric variables of degree 0 and consider* $I = \langle [y_{1,0}, y_{2,0}][y_{3,0}, y_{4,0}], y_{1,0}[y_{2,0}, y_{3,0}]y_{4,0}, St_3(y_{1,0}, y_{2,0}, y_{3,0})\rangle_{T_2^*}$. *Then* f *can be written, modulo* I*, as a linear combination of polynomials of type:*

$$y_{1,0} \cdots y_{n,0}, \quad [y_{j,0}, y_{1,0}]y_{2,0} \cdots \widehat{y_{j,0}} \cdots y_{n,0} \quad \textit{and} \quad y_{2,0} \cdots \widehat{y_{k,0}} \cdots y_{n,0}[y_{k,0}, y_{1,0}]. \tag{6.1}$$

Here the symbol $\widehat{y_{i,0}}$ *means that the variable* $y_{i,0}$ *is omitted.*

Proof. Since $f \in P_{n,0,0,0}$, by Poincaré-Birkhoff-Witt Theorem, f can be written as a linear combination of polynomials of type:

$$y_{i_1,0} \cdots y_{i_s,0} w_1 \cdots w_m, \tag{6.2}$$

where $i_1 < \cdots < i_s$ and $w_1, \ldots, w_m$ are left normed Lie commutators in variables $y_{i,0}$'s. Then, modulo I, we have that at most one commutator can appear in (6.2). Furthermore, using that $y_{1,0}[y_{2,0}, y_{3,0}]y_{4,0} \in I$, we have

$$[y_{r,0}, y_{j_1,0}, \ldots, y_{j_t,0}] = [y_{r,0}, y_{j_1,0}]y_{j_2,0} \cdots y_{j_t,0} \pm y_{j_t,0} \cdots y_{j_2,0}[y_{r,0}, y_{j_1,0}] \pmod{I},$$

with $r > j_1 < \cdots < j_t$.

Thus, modulo I, we have that f is a linear combination of polynomials of type

$$y_{1,0} \cdots y_{n,0}, \quad [y_{r,0}, y_{1,0}]y_{2,0} \cdots \widehat{y_{r,0}} \cdots y_{n,0} \quad \text{and} \quad y_{i_1,0} \cdots y_{i_{n-2},0}[y_{i,0}, y_{j,0}], \tag{6.3}$$

with $2 \leq r \leq n,\ 1 \leq i < j \leq n$.

Also it is possible to prove that $[y_{1,0}, y_{2,0}]m[y_{3,0}, y_{4,0}] \in I$, where m is a monomial in variables $y_{i,0}$'s. So the variables outside the commutator in the polynomials of the third type in (6.3) can be ordered. Moreover we use that $St_3(y_{1,0}, y_{2,0}, y_{3,0}) \in I$ and finally, we obtain that f can be written, modulo I, as a linear combination of polynomials of type (6.1) as desired. □

Lemma 6.11. *Let* $B = F + J_{01} + J_{10} + J_{11}$*, with* $J_{10} \neq 0$ *and* $J_{11} \neq 0$ *and suppose that* $J_{11}^2 = J_{01}J_{10} = J_{10}J_{01} = (J_{11})_0^- J_{10} = (J_{11})_1^+ J_{10} = (J_{11})_1^- J_{10} = J_{01}(J_{11})_0^- = J_{01}(J_{11})_1^+ = J_{01}(J_{11})_1^- = 0$. *Then*

$$B \sim_{T_2^*} (F + J_{11}) \oplus (F + J_{10} + J_{01}).$$

Proof. Since $(J_{01}+J_{10})^2=0$, we have that $B_1=F+J_{11}$ and $B_2=F+J_{01}+J_{10}$ are $*$-graded subalgebras of B. Thus $\mathrm{Id}^{\mathrm{gri}}(B)\subseteq \mathrm{Id}^{\mathrm{gri}}(B_1\oplus B_2)$. Now observe that

$$\begin{aligned} B_0^+ &= F+(J_{11})_0^+ +(J_{10}+J_{01})_0^+, & B_0^- &= (J_{11})_0^- +(J_{10}+J_{01})_0^-,\\ B_1^+ &= (J_{11})_1^+ +(J_{10}+J_{01})_1^+, & B_1^- &= (J_{11})_1^- +(J_{10}+J_{01})_1^-. \end{aligned}$$

Using the zero products given by hypothesis, we have that the polynomials $z_{1,0}z_{2,0}, x_{1,1}x_{2,1}, z_{1,0}y_{2,1},\ z_{1,0}z_{2,1}, St_3(y_{1,0},y_{2,0},y_{3,0})$ are $(\mathbb{Z}_2,*)$-identities of B, where $x_{i,0}=y_{i,0}$ or $x_{i,0}=z_{i,0}$, for $i=1,2$. Moreover, since $[B_0^+,B]\subseteq J_{10}+J_{01}$, it follows that $y_{1,0}[y_{2,0},x_{3,s}]y_{4,0}\in \mathrm{Id}^{\mathrm{gri}}(B)$, where $x_{3,s}\in\{y_{3,0},y_{3,1},z_{3,0},z_{3,1}\}$.

Also we have that $z_{1,0}mz_{2,0}, x_{1,1}mx_{2,1}, z_{1,0}my_{2,1}, z_{1,0}mz_{2,1}$ are $(\mathbb{Z}_2,*)$-identities of B, where m is a (eventually empty) monomial in symmetric variables of degree 0.

By taking into account all these identities, let $f\in \mathrm{Id}^{\mathrm{gri}}(B_1\oplus B_2)$ be a multilinear $(\mathbb{Z}_2,*)$-polynomial of degree n. By the multihomogeneity of T_2^*-ideals, modulo $\mathrm{Id}^{\mathrm{gri}}(B)$, we can consider that either

$$f\in P_{n,0,0,0}\quad\text{or}\quad f\in P_{n-1,1,0,0}\quad\text{or}\quad f\in P_{n-1,0,1,0}\quad\text{or}\quad f\in P_{n-1,0,0,1}.$$

Since $z_{1,0}z_{2,0}, y_{1,0}[y_{2,0},y_{3,0}]y_{4,0}, St_3(y_{1,0},y_{2,0},y_{3,0})\in \mathrm{Id}^{\mathrm{gri}}(B)$, if $f\in P_{n,0,0,0}\cap \mathrm{Id}^{\mathrm{gri}}(B_1\oplus B_2)$, we use Lemma 6.10 and write f modulo $\mathrm{Id}^{\mathrm{gri}}(B)$ as

$$f=\alpha y_{1,0}\cdots y_{n,0}+\sum_{j=2}^{n}\beta_j[y_{j,0},y_{1,0}]y_{2,0}\cdots\widehat{y_{j,0}}\cdots y_{n,0}+\sum_{k=2}^{n}\gamma_k y_{2,0}\cdots\widehat{y_{k,0}}\cdots y_{n,0}[y_{k,0},y_{1,0}].$$

Then by making the evaluation $y_{i,0}=(1_F,0)$, for all $1\le i\le n$, we obtain $\alpha=0$. Now for a fixed j, the evaluation $y_{j,0}=(0,a+a^*)$, with $0\neq a\in (J_{10})_0$ and $y_{l,0}=(0,1_F)$, for all $l\neq j$, gives us $\beta_j=\gamma_j=0$. We can repeat the argument for any j to conclude that $\beta_j=\gamma_j=0$, for all $2\le j\le n$. In this way, we get $P_{n,0,0,0}\cap \mathrm{Id}^{\mathrm{gri}}(B_1\oplus B_2)=P_{n,0,0,0}\cap \mathrm{Id}^{\mathrm{gri}}(B)$.

In case of $f\in P_{n-1,1,0,0}$, we use that $z_{1,0}y_{2,1},\ z_{1,0}my_{2,1},\ y_{1,0}[y_{2,0},y_{3,1}]y_{4,0}\in \mathrm{Id}^{\mathrm{gri}}(B)$ and write f modulo $\mathrm{Id}^{\mathrm{gri}}(B)$ as

$$f=\alpha y_{n,1}y_{1,0}\cdots y_{n-1,0}+\beta y_{1,0}y_{n,1}y_{2,0}\cdots y_{n-1,0}+\gamma y_{1,0}\cdots y_{n-1,0}y_{n,1}.$$

If we make the evaluation $y_{n,1}=(0,a+a^*)$, with $0\neq a\in (J_{10})_1$ and $y_{i,0}=(0,1_F)$, for $i\neq n$, then we obtain $\alpha=\gamma=0$. Also the evaluation $y_{n,1}=(b,0)$, with $0\neq b\in (J_{11})_1^+$ and $y_{i,0}=(1_F,0)$, for $i\neq n$, results in $\beta=0$. Thus $P_{n-1,1,0,0}\cap \mathrm{Id}^{\mathrm{gri}}(B_1\oplus B_2)=P_{n-1,1,0,0}\cap \mathrm{Id}^{\mathrm{gri}}(B)$.

Finally, we can use the same strategy as above to get $P_{n-1,0,1,0}\cap \mathrm{Id}^{\mathrm{gri}}(B_1\oplus B_2)=P_{n-1,0,1,0}\cap \mathrm{Id}^{\mathrm{gri}}(B)$ and $P_{n-1,0,0,1}\cap \mathrm{Id}^{\mathrm{gri}}(B_1\oplus B_2)=P_{n-1,0,0,1}\cap \mathrm{Id}^{\mathrm{gri}}(B)$.

In conclusion, we have $\mathrm{Id}^{\mathrm{gri}}(B_1\oplus B_2)\subseteq \mathrm{Id}^{\mathrm{gri}}(B)$ and so, $B\sim_{T_2^*}(F+J_{11})\oplus(F+J_{10}+J_{01})$. □

By putting together Lemmas 6.7 and 6.11 and Corollary 6.9 we get the last result of this section.

Corollary 6.12. *Let* $B = F + J_{01} + J_{10} + J_{11}$ *with* $J_{10} \neq 0$ *and* $J_{11} \neq 0$. *If* $J_{11}^2 = J_{01}J_{10} = J_{10}J_{01} = (J_{11})_0^- J_{10} = (J_{11})_1^+ J_{10} = (J_{11})_1^- J_{10} = J_{01}(J_{11})_0^- = J_{01}(J_{11})_1^+ = J_{01}(J_{11})_1^- = 0$ *then*

$$B \sim_{T_2^*} B_1 \oplus B_2,$$

with $B_1 \in \{C, C_{2,*}, C_2^{\mathrm{gri}}, C_2^{\mathrm{gr}}, C_{2,*} \oplus C_2^{\mathrm{gri}}, C_{2,*} \oplus C_2^{\mathrm{gr}}, C_2^{\mathrm{gri}} \oplus C_2^{\mathrm{gr}}, C_{2,*} \oplus C_2^{\mathrm{gri}} \oplus C_2^{\mathrm{gr}}\}$ *and* $B_2 \in \{A_{2,*}, A_2^{\mathrm{gri}}, A_2^{\mathrm{gri}} \oplus A_{2,*}\}$, *where* C *is a commutative* $*$*-superalgebra with trivial grading and trivial involution.*

7. Classification of minimal $*$-superalgebras with quadratic growth

By Remark 2.4, we can notice that any direct sum of distinct $*$-superalgebras in the set

$$\mathcal{T} = \{C_{2,*}, C_2^{\mathrm{gri}}, C_2^{\mathrm{gr}}, A_{2,*}, A_2^{\mathrm{gri}}\}$$

has linear growth. In fact, in [13, Theorem 7.2] Ioppolo and La Mattina proved that a finite dimensional $*$-superalgebra A has at most linear growth if and only if A is T_2^*-equivalent to $B \oplus N$ where N is a nilpotent $*$-superalgebra and either $B = 0$ or is a commutative $*$-superalgebra with trivial grading and trivial involution or is a direct sum of distinct $*$-superalgebras in $\mathcal{T}$. In particular, the $*$-superalgebras in $\mathcal{T}$ generate minimal varieties of linear growth (see [13], [19]). As a consequence they generate the only minimal varieties of linear growth of the $*$-graded codimensions.

Now we consider the set $\mathcal{L}$ composed by the 36 $*$-superalgebras defined in Section 4. We are ready to prove the main theorem of this paper, classifying the $*$-superalgebras generating minimal varieties of quadratic growth.

Theorem 7.1. *Let* A *be a* $*$*-superalgebra. The following conditions are equivalent.*

1. $B \notin \mathrm{var}^{\mathrm{gri}}(A)$, *for all* $*$*-superalgebra* $B \in \mathcal{L}$.
2. *Either* $A \sim_{T_2^*} N$ *or* $A \sim_{T_2^*} C \oplus N$ *or* $A \sim_{T_2^*} B_1 \oplus N$ *or* $A \sim_{T_2^*} B_2 \oplus N$ *or* $A \sim_{T_2^*} B_1 \oplus B_2 \oplus N$, *where* $B_1 \in \{C_{2,*},\ C_2^{\mathrm{gri}},\ C_2^{\mathrm{gr}},\ C_{2,*} \oplus C_2^{\mathrm{gri}},\ C_{2,*} \oplus C_2^{\mathrm{gr}},\ C_2^{\mathrm{gri}} \oplus C_2^{\mathrm{gr}},\ C_{2,*} \oplus C_2^{\mathrm{gri}} \oplus C_2^{\mathrm{gr}}\}$ *and* $B_2 \in \{A_{2,*},\ A_2^{\mathrm{gri}},\ A_2^{\mathrm{gri}} \oplus A_{2,*}\}$, N *is a nilpotent* $*$*-superalgebra,* C *is a commutative* $*$*-superalgebra with trivial grading and trivial involution.*

Proof. First, we observe that all the $*$-superalgebras in $\mathcal{L}$ have quadratic growth and by Remark 2.4 and Lemma 6.5, the $*$-superalgebras listed in (2) have at most linear growth. So it is obvious that (2) implies (1).

Now we suppose that (1) holds. By Remark 4.4, we have that M_*, M^{gri}, D_*, D^{gr}, $D^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$ and so, by Theorem 3.3, we get that $c_n^{\mathrm{gri}}(A)$ is polynomially bounded and

$$A \sim_{T_2^*} \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_m,$$

where each $\mathcal{A}_i$ is a finite dimensional $*$-superalgebra such that $\dim_F \mathcal{A}_i/J(\mathcal{A}_i) \leq 1$, for $i = 1, \ldots, m$. If $\mathcal{A}_i$ is nilpotent for all i, then $A \sim_{T_2^*} N$, where N is a nilpotent $*$-superalgebra and we are done.

Suppose that there exists $i \in \{1, \ldots, m\}$ such that $\dim_F \mathcal{A}_i/J(\mathcal{A}_i) = 1$. As we have observed after Theorem 2.1, this implies that $\mathcal{A}_i$ has a decomposition of type

$$\mathcal{A}_i = F + J_{11} + J_{10} + J_{01} + J_{00}.$$

Since $M_{4,*}$, M_4^{gri}, $M_{5,*}$, M_5^{gri}, $M_{7,*}$, $M_{7,1}^{\mathrm{gri}}$, $M_{7,2}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(\mathcal{A}_i)$, by Corollary 5.2, we get that $B = F + J_{10} + J_{01} + J_{11}$ is a $*$-graded subalgebra of $\mathcal{A}_i$. Moreover, since $M_{8,*}$, $M_{8,1}^{\mathrm{gri}}$, $M_{8,2}^{\mathrm{gri}}$, $M_{8,3}^{\mathrm{gri}}$, $M_{9,*}$, $M_{9,1}^{\mathrm{gri}}$, $M_{9,2}^{\mathrm{gri}}$, $M_{9,3}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(\mathcal{A}_i)$, by Corollary 5.5, we have a direct sum $\mathcal{A}_i = (F + J_{10} + J_{01} + J_{11}) \oplus J_{00}$.

From now on we will deal with algebras of type $B = F + J_{10} + J_{01} + J_{11}$.

We observe that if $J_{10} = J_{11} = 0$ then $B \sim_{T_2^*} C$, where C is a commutative $*$-superalgebra with trivial grading and trivial involution. On the other hand, if $J_{10} = 0$ and $J_{11} \neq 0$, since $C_{3,*}$, C_3^{gr}, C_3^{gri}, $N_{3,*}$, N_3^{gri}, $U_{3,*}$, U_3^{gri}, $G_{2,1,\rho}$, $G_{2,1,\tau}$, $G_{2,1,\psi}$, $G_{2,0,\tau}$, $G_{2,1,\tau}$, $G_{2,1,\psi} \notin \mathrm{var}^{\mathrm{gri}}(B)$, by Corollary 6.4, we have $J_{11}^2 = 0$. Hence by Corollary 6.7, we conclude that B is T_2^*-equivalent to one of the following $*$-superalgebras:

$$C, \quad C_2^{\mathrm{gri}}, \quad C_2^{\mathrm{gr}}, \quad C_{2,*}, \quad C_2^{\mathrm{gri}} \oplus C_2^{\mathrm{gr}}, \quad C_2^{\mathrm{gri}} \oplus C_{2,*}, \quad C_2^{\mathrm{gr}} \oplus C_{2,*}, \quad C_2^{\mathrm{gri}} \oplus C_{2,*} \oplus C_2^{\mathrm{gr}}.$$

Now suppose that $J_{10} \neq 0$. If $J_{11} = 0$ then by Corollary 6.9, we have that either

$$B \sim_{T_2^*} A_{2,*} \quad \text{or} \quad B \sim_{T_2^*} A_2^{\mathrm{gri}} \quad \text{or} \quad B \sim_{T_2^*} A_{2,*} \oplus A_2^{\mathrm{gri}}.$$

On the other hand, if $J_{11} \neq 0$, since $M_{10,*}$, $M_{10,1}^{\mathrm{gri}}$, $M_{10,2}^{\mathrm{gri}}$, $M_{10,3}^{\mathrm{gri}}$, $M_{11,1}^{\mathrm{gri}}$, $M_{11,2}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(B)$, by Corollary 5.7, it follows that

$$J_{01}(J_{11})_1^+ = J_{01}(J_{11})_0^- = J_{01}(J_{11})_1^- = (J_{11})_1^+ J_{10} = (J_{11})_0^- J_{10} = (J_{11})_1^- J_{10} = 0.$$

Also, since $U_{3,*} \notin \mathrm{var}^{\mathrm{gri}}(B)$, by Lemma 4.10, we have that $M_{6,*} \notin \mathrm{var}^{\mathrm{gri}}(B)$. With this in mind we have that $M_{4,*}$, M_4^{gri}, $M_{5,*}$, M_5^{gri}, $M_{6,*}$, $M_{6,1}^{\mathrm{gri}}$, $M_{6,2}^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(B)$. Hence, by Corollary 5.2, we have $J_{10}J_{01} = 0$.

Thus, we are under the conditions of Corollary 6.12 and so we get that $B \sim_{T_2^*} B_1 \oplus B_2$, where $B_1 \in \{C_{2,*}, C_2^{\mathrm{gri}}, C_2^{\mathrm{gr}}, C_{2,*} \oplus C_2^{\mathrm{gri}}, C_{2,*} \oplus C_2^{\mathrm{gr}}, C_2^{\mathrm{gri}} \oplus C_2^{\mathrm{gr}}, C_{2,*} \oplus C_2^{\mathrm{gri}} \oplus C_2^{\mathrm{gr}}\}$ and $B_2 \in \{A_{2,*}, A_2^{\mathrm{gri}}, A_2^{\mathrm{gri}} \oplus A_{2,*}\}$.

Recalling that $A \sim_{T_2^*} \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_m$ and putting together all pieces, we get that (2) holds and so the proof is complete. □

By using Remark 4.11 and the previous theorem, we get the following.

Corollary 7.2. *The $*$-superalgebras in $\mathcal{L}$ generate the only minimal $*$-supervarieties of quadratic growth.*

It is worth to observe that if we consider the subset of $\mathcal{L}$ formed by the $*$-superalgebras with trivial grading

$$\widetilde{\mathcal{L}} = \{C_{3,*}, N_{3,*}, U_{3,*}, M_{4,*}, M_{5,*}, M_{7,*}, M_{8,*}, M_{9,*}, M_{10,*}, G_{2,0,\tau}\}$$

we have a list of algebras with involution generating minimal varieties of quadratic growth. So as a consequence of Theorem 7.1 and Remark 4.11, we have the following classification.

Corollary 7.3. *The $*$-algebras in $\widetilde{\mathcal{L}}$ generate the only minimal varieties of algebras with involution of quadratic growth.*

Declaration of competing interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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