Nonlinear Analysis: Hybrid Systems 52 (2024) 101463

Contents lists available at ScienceDirect

Nonlinear Analysis: Hybrid Systems

journal homepage: www.elsevier.com/locate/nahs

On the robustification of digital event-based stabilizers for nonlinear time-delay systems

M. Di Ferdinando [a,b,*], S. Di Gennaro [a,b], A. Borri [b,c], G. Pola [a,b], P. Pepe [a,b]

[a] Department of Information Engineering, Computer Science, and Mathematics, University of L'Aquila, Via Vetoio, Loc. Coppito, 67100 L'Aquila, Italy
[b] Center of Excellence for Research DEWS, University of L'Aquila, Via Vetoio, Loc. Coppito, 67100 L'Aquila, Italy
[c] Institute of Systems Analysis and Computer Science, National Research Council, Rome, Italy

ARTICLE INFO

Keywords:
Quantized sampled-data controllers
Event-triggered control
Spline approximation
Nonlinear time-delay systems
Stabilization in the sample-and-hold sense

ABSTRACT

In this paper, the robust stabilization problem by means of quantized sampled-data event-based (QSE) controllers is investigated for nonlinear systems affected by state delays and unknown disturbances. In particular, a methodology for the design of robust QSE stabilizers is provided for control-affine nonlinear systems affected by unknown actuation disturbances and unknown measurement errors. Firstly, the notion of Steepest Descent Feedback (SDF), continuous or not, is suitably revised in order to deal with the robustification of event-based controllers. Then, Input-to-State Stability (ISS) redesign methodologies are used to provide the robustification term which is added to the SDF at hand in order to arbitrarily attenuate the effects of unknown external disturbances affecting the considered control scheme. A spline approximation approach is used in order to cope with the problem of the possible non-availability in the buffer of suitable past values of the system state required for the correct application of the proposed robust QSE controller. It is proved that there exist a suitably fast sampling and an accurate quantization of the input/output channels such that: the robust QSE implementation of SDFs, continuous or not, ensures the semi-global practical stability of the related closed-loop system, regardless of the above disturbances, provided that the observation errors affects marginally the new added control term. The stabilization in the sample-and-hold sense theory is used as a tool to prove the results. The provided results include the case of non-uniform quantization of the input/output channels and the case of aperiodic sampling. Applications are presented in order to validate the results.

1. Introduction

In the last years, the study of quantized sampled-data control systems has received a growing attention by the researchers because of the huge utilization of digital devices in many practical engineering applications. Many approaches have been proposed in the literature concerning the stabilization problem of nonlinear delay-free/time-delay systems by means of quantized sampled-data controllers (see, for instance, [1–8]).

A popular approach for the design of sampled-data stabilizers is the one based on the event-triggered control, which has been proved to be successful in properly managing shared computation and communication resources in the digital world [9,10]. The main

* Corresponding author at: Department of Information Engineering, Computer Science, and Mathematics, University of L'Aquila, Via Vetoio, Loc. Coppito, 67100 L'Aquila, Italy.

E-mail addresses: mario.diferdinando@univaq.it (M. Di Ferdinando), stefano.digennaro@univaq.it (S. Di Gennaro), alessandro.borri@iasi.cnr.it (A. Borri), giordano.pola@univaq.it (G. Pola), pierdomenico.pepe@univaq.it (P. Pepe).

https://doi.org/10.1016/j.nahs.2023.101463
Received 28 September 2023; Received in revised form 20 November 2023; Accepted 21 December 2023
Available online 28 December 2023
1751-570X/

idea behind such an approach is to control the system whenever it really needs attention, by avoiding continuous-time state/output monitoring and control updates unless they are necessary to satisfy a certain property, such as a stability one. Many methodologies for the design of event-triggered controllers for nonlinear delay-free/time-delay systems have been provided in the literature (see, for instance, [11–25] and references therein). In particular, as far as event-triggered control scheme are concerned, a handful of results have been proposed in the literature for various class of nonlinear time-delay systems also in the sampled-data context (see, among the others, [14,19,24,26–34]). In [32], an event-based controller is provided for nonlinear systems with state delays ensuring the global asymptotic stability property of the related closed-loop system. The considered event-based mechanism is checked in continuous-time basis and a proof of the avoidance of Zeno behaviors is provided. In [14,19,26], sampled-data event-based stabilizers are designed for nonlinear systems with time-delay and results concerning the semi-global practical stability of the related closed-loop systems are provided. The triggering conditions proposed in [14,19,26] are only examined at a sequence of state-sampling instants so that a minimum inter-event time can be naturally guaranteed (i.e., Zeno behaviors are avoided). However, quantization in the input/output channels and disturbances affecting the controller and the measurements of the system state are not considered in [14,19,26].

To our best knowledge, results concerning event-based controllers for nonlinear systems with state delays have never been provided in the literature taking simultaneously into account: (i) the presence of sampling (also aperiodic); (ii) the presence of quantization (also non-uniform) in both input/output channels; (iii) the arbitrary reduction of the effects of unknown actuation disturbances and unknown observation errors affecting the event-based digital controller at hand; (iv) possible discontinuities in the function describing the controller; (v) problems related to the possible non-availability in the buffer of suitable past values of the system state required for the implementation of the controller at hand.

In this paper, we fill this gap by providing a methodology for the design of robust quantized sampled-data event-based (QSE) controllers for the important class of control-affine nonlinear systems with state delays affected by unknown actuation disturbances and unknown observation errors. Firstly, the notion of Steepest Descent Feedback (SDF), induced by a class of Lyapunov–Krasovskii functionals, is used in order to design a QSE controller. Then, the proposed QSE controller is robustified with respect to arbitrarily large unknown actuation disturbances and suitably small unknown observation errors. In particular, the robustification of the proposed QSE controller is performed by adding a new control term built up via the ISS redesign methodologies. It is assumed that the bounds of the involved unknown disturbances are *a-priori* known and that the observation errors do not affect or affect marginally the new added control term. Implementation problems related to the possible non-availability in the buffer of suitable past values of the system state required for the correct implementation of the proposed robust QSE controller are also taken into account. In particular, such a drawback is overcome by exploiting a spline approximation approach. Then, it is proved that there exist a suitably fast sampling and an accurate quantization of the input/output channels such that a proposed robust QSE implementation of SDFs, continuous or not, ensures the semi-global practical stability property of the related closed-loop system with arbitrarily small final target ball of the origin and regardless of the above disturbances. The stabilization in the sample-and-hold sense theory (see, for instance, [8,14,19,26,35–42]) is used as a tool to prove the results. We highlight here that, in the proposed design procedure, discontinuities in the function describing the SDF at hand are allowed. Furthermore, the case of time-varying sampling periods and the case of non-uniform quantization of the input/output channels are included in the theory here developed. To our best knowledge, it is the first time in the literature that theoretical results concerning the arbitrary reduction of the effects of arbitrarily large unknown actuation disturbances and of suitably small unknown observation errors are provided in the context of the QSE control of nonlinear systems with state-delays. The proposed results are validated through applications concerning: (i) a single-link flexible joint robot arm with time delays; (ii) a particular class of nonlinear time-delay systems.

Notation $\mathbb{N}$ denotes the set of nonnegative integer numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^\star$ denotes the extended real line $[-\infty, +\infty]$, $\mathbb{R}^+$ denotes the set of nonnegative reals $[0, +\infty)$. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. For a given positive integer n and for a symmetric, positive definite matrix $P \in \mathbb{R}^{n\times n}$, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and the minimum eigenvalue of P, respectively. For a given positive integer n and a given positive real H, the symbol $\mathcal{B}^n_H$ denotes the subset $\{x \in \mathbb{R}^n \mid |x| \leq H\}$. The essential supremum norm of an essentially bounded function is indicated with the symbol $\|\cdot\|_\infty$. For a positive integer n, for a positive real Δ (maximum involved time-delay): $\mathcal{C}^n$ and $\mathcal{W}^{1,\infty}_n$ denote the space of the continuous functions mapping $[-\Delta, 0]$ into $\mathbb{R}^n$ and the space of the absolutely continuous functions, with essentially bounded derivative, mapping $[-\Delta, 0]$ into $\mathbb{R}^n$, respectively; $\mathcal{Q}^n$ denotes the space of bounded, right-continuous functions, with possibly a finite number of points with jump-type discontinuity, mapping $[-\Delta, 0)$ into $\mathbb{R}^n$. For $\phi \in \mathcal{C}^n$, $\phi_{[-\Delta,0)}$ is the function in $\mathcal{Q}^n$ defined, for $\tau \in [-\Delta, 0)$, as $\phi_{[-\Delta,0)}(\tau) = \phi(\tau)$. For a positive real H, for $\phi \in \mathcal{C}^n$, $\mathcal{C}^n_H(\phi) = \{\psi \in \mathcal{C}^n \mid \|\psi - \phi\|_\infty \leq H\}$. The symbol $\mathcal{C}^n_H$ denotes $\mathcal{C}^n_H(0)$. For a continuous function $x : [-\Delta, c) \to \mathbb{R}^n$, with $0 < c \leq +\infty$, for any real $t \in [0, c)$, x_t is the function in $\mathcal{C}^n$ defined as $x_t(\tau) = x(t + \tau),\ \tau \in [-\Delta, 0]$. For a positive integer n and for $x \in \mathbb{R}^n$, the symbol $\mathcal{Q}^n_x$ denotes a finite subset of $\mathbb{R}^n$. For a positive integer n, for $\mathbb{S} = \mathbb{R}^n$ (or $\mathbb{R}^+$), $C^1(\mathbb{S}; \mathbb{R}^+)$ denotes the space of the continuous functions from $\mathbb{S}$ to $\mathbb{R}^+$, admitting continuous (partial) derivatives; $C^1_L(\mathbb{S}; \mathbb{R}^+)$ denotes the subset of the functions in $C^1(\mathbb{S}; \mathbb{R}^+)$ admitting locally Lipschitz (partial) derivatives. A continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{P}_0$ if $\gamma(0) = 0$; of class $\mathcal{P}$ if it is of class $\mathcal{P}_0$ and $\gamma(s) > 0,\ s > 0$; of class $\mathcal{K}$ if it is of class $\mathcal{P}$ and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. The symbol $\circ$ denotes composition (of functions). For positive integers n, m, for a function $F : \mathcal{C}^n \times \mathbb{R}^m \to \mathbb{R}^n$ Lipschitz on bounded subsets of $\mathcal{C}^n \times \mathbb{R}^m$, and for a functional $V : \mathcal{C}^n \to \mathbb{R}^+$ Lipschitz on bounded subsets of $\mathcal{C}^n$, the derivative in Driver's form (see [43] and the references therein) $D^+V : \mathcal{C}^n \times \mathbb{R}^m \to \mathbb{R}^\star$, of the functional V, is defined, for $\phi \in \mathcal{C}^n$ and $u \in \mathbb{R}^m$ as

$$D^+V(\phi, u) = \limsup_{h\to 0^+} \frac{V(\phi_{h,u}) - V(\phi)}{h} \tag{1}$$

where for $0 \le h < \Delta$, $\phi_{h,u} \in \mathcal{C}^n$ is defined, for $s \in [-\Delta, 0]$, as

$$\phi_{h,u}(s) = \begin{cases} \phi(s+h), & s \in [-\Delta, -h) \\ \phi(0) + (s+h)F(\phi,u), & s \in [-h, 0]. \end{cases}$$

2. Preliminaries

Let us consider a control-affine nonlinear system (the plant) described by the following retarded functional differential equation (RFDE) [44,45],

$$\begin{aligned} \dot{x}(t) &= f(x_t) + g(x_t)u(t), \quad t \ge 0 \text{ a.e.} \\ x(\tau) &= x_0(\tau), \quad \tau \in [-\Delta, 0] \end{aligned} \tag{2}$$

where: $x(t) \in \mathbb{R}^n$, $x_0, x_t \in \mathcal{C}^n$; $\Delta > 0$ is the maximum involved time delay, assumed to be known; $u(t) \in \mathbb{R}^m$ is the input signal; $f : \mathcal{C}^n \to \mathbb{R}^n$ is a function, Lipschitz on bounded subsets of $\mathcal{C}^n$; $g : \mathcal{C}^n \to \mathbb{R}^{n\times m}$ is a function, Lipschitz on bounded subsets of $\mathcal{C}^n$; n and m are positive integers. It is assumed that the initial state $x_0 \in W_n^{1,\infty}$ (see [19,46], and, for a detailed discussion, Remark 6 in [46]).

For the reader's convenience, we recall here classes of Lyapunov–Krasovskii functionals very helpful in the context of the robust stabilization problem of nonlinear time-delay systems by means of quantized sampled-data controllers. In particular, we recall the definition of smoothly separable functionals and of invariantly differentiable functionals [19,41,47,48]. Such notions will be the key tools to introduce a suitable class of candidate Lyapunov–Krasovskii functionals by which the proposed robust QSE controller is derived (see forthcoming items **(a)–(d)**, Remarks 1, 2, Definition 3, (12), (13) and (15)).

Definition 1. A functional $V : \mathcal{C}^n \to \mathbb{R}^+$ is said to be smoothly separable if there exist a function $V_1 \in C^1_L(\mathbb{R}^n; \mathbb{R}^+)$, a locally Lipschitz functional $V_2 : \mathcal{C}^n \to \mathbb{R}^+$, functions $\beta_i \in \mathcal{K}_\infty$, $i = 1, 2$, such that, for any $\phi \in \mathcal{C}^n$, the following hold

$$\begin{aligned} V(\phi) &= V_1(\phi(0)) + V_2(\phi) \\ \beta_1(|\phi(0)|) &\le V_1(\phi(0)) \le \beta_2(|\phi(0)|). \end{aligned} \tag{3}$$

As in [48], the formalism used in the classical definition of invariantly differentiable functional [47], is here suitably modified for the purpose of formalism uniformity over the paper. For any given $x \in \mathbb{R}^n$, $\phi \in \mathcal{Q}^n$ and for any given continuous function $\mathcal{Y} : [0, \Delta] \to \mathbb{R}^n$ with $\mathcal{Y}(0) = x$, let $\psi_h^{(x,\phi,\mathcal{Y})} \in \mathcal{Q}^n$, $h \in [0, \Delta)$, be defined as $\psi_0^{(x,\phi,\mathcal{Y})} = \phi$ and, for $h > 0$,

$$\psi_h^{(x,\phi,\mathcal{Y})}(s) = \begin{cases} \phi(s+h), & s \in [-\Delta, -h), \\ \mathcal{Y}(s+h), & s \in [-h, 0). \end{cases}$$

Definition 2. A functional $V : \mathbb{R}^n \times \mathcal{Q}^n \to \mathbb{R}^+$ is said to be invariantly differentiable if, at any point $(x, \phi) \in \mathbb{R}^n \times \mathcal{Q}^n$:

- for any continuous function $\mathcal{Y} : [0, \Delta] \to \mathbb{R}^n$ with $\mathcal{Y}(0) = x$, there exists the right-hand derivative $\left.\frac{\partial V\left(x, \psi_h^{(x,\phi,\mathcal{Y})}\right)}{\partial h}\right|_{h=0}$ and such derivative is invariant with respect to the function $\mathcal{Y}$;
- there exists the derivative $\frac{\partial V(x,\phi)}{\partial x}$;
- for any continuous function $\mathcal{Y} : [0, \Delta] \to \mathbb{R}^n$ with $\mathcal{Y}(0) = x$, the following equality holds for any $z \in \mathbb{R}^n$, for any $h \in [0, \Delta)$, $V\left(x + z, \psi_h^{(x,\phi,\mathcal{Y})}\right) - V(x, \phi) = \frac{\partial V(x,\phi)}{\partial x} z + \left.\frac{\partial V\left(x, \psi_l^{(x,\phi,\mathcal{Y})}\right)}{\partial l}\right|_{l=0} h + o(\sqrt{|z|^2 + h^2})$, with $\lim_{s\to 0^+} \frac{o(\sqrt{s})}{\sqrt{s}} = 0$.

In the following, the proposed procedure for the design of robust QSE stabilizers is presented. In particular, the proposed design methodology is based on the Artstein's approaches (see, for instance, [19,35,36,38,42,49–51]) making use of control Lyapunov–Krasovskii functionals for the design of stabilizers. According to such approaches, as a first step of the proposed design procedure, in the following a class of candidate Lyapunov–Krasovskii functionals is introduced. In particular, let $V_1 : \mathbb{R}^n \to \mathbb{R}^+$ and $V_2 : \mathcal{Q}^n \to \mathbb{R}^+$ be two Lipschitz on bounded subsets functions. Then, we denote here with $\mathcal{V}$ the set of candidate Lyapunov–Krasovskii functionals $V : \mathcal{C}^n \to \mathbb{R}^+$ defined for $\phi \in \mathcal{C}^n$, as

$$V(\phi) = V_1(\phi(0)) + \widetilde{V}_2(\phi), \tag{4}$$

where $\widetilde{V}_2 : \mathcal{C}^n \to \mathbb{R}^+$ is defined for $\phi \in \mathcal{C}^n$ as $\widetilde{V}_2(\phi) = V_2(\phi_{[-\Delta,0)})$ and satisfying the following properties:

- **(a)** the functional V is smoothly separable with related functions β_1, β_2 as in (3);
- **(b)** the function $(\phi, u) \to D^+\widetilde{V}_2(\phi, u)$, $\phi \in \mathcal{C}^n$, $u \in \mathbb{R}^m$, is Lipschitz on bounded subsets of $\mathcal{C}^n \times \mathbb{R}^m$ where the derivative in Driver's form (see (1)) of the functional $\widetilde{V}_2$ is computed with respect to the function $F(\phi, u) = f(\phi) + g(\phi)u$ with f and g in (2);
- **(c)** the functional $\bar{V} : \mathbb{R}^n \times \mathcal{Q}^n \to \mathbb{R}^+$ defined, for $x \in \mathbb{R}^n$, $\phi \in \mathcal{Q}^n$, as $\bar{V}(x, \phi) = V_1(x) + V_2(\phi)$, is invariantly differentiable;
- **(d)** there exist functions γ_i, $i = 1, 2$, of class $\mathcal{K}_\infty$, such that, for any $\phi \in \mathcal{C}^n$,

$$\gamma_1(|\phi(0)|) \le V(\phi) \le \gamma_2(\|\phi\|_\infty). \tag{5}$$

Remark 1. Notice that, the items **(a)–(d)** are satisfied by a very large class of Lyapunov–Krasovskii functionals, including standard complete quadratic ones (see, for instance, [48,52–56]). For instance, the following standard functional

$$V(\phi) = \phi^T(0)P\phi(0) + \int_{-\Delta}^{0} \phi^T(\tau)Q\phi(\tau)d\tau, \ \phi \in C^n, \tag{6}$$

fulfills items **(a)–(d)** with functions $\beta_1(s) = \gamma_1(s) = \lambda_{\min}(P)s^2$, $\beta_2(s) = \lambda_{\max}(P)s^2$, $\gamma_2(s) = (\lambda_{\max}(P) + \Delta\lambda_{\max}(Q))s^2$. The invariant differentiability property, as here connected with the smooth separability one (see items **(a)** and **(c)**), has been proved to be very helpful in order to apply ISS redesign methodologies for the robustification of stabilizers for control-affine nonlinear time-delay systems (see [40,48] and the references therein). In particular, from a technical point view, as shown in forthcoming Lemma 2, the smooth separability property turns out to be very helpful to derive suitable Lyapunov–Krasovskii functionals involving lower and upper bounds in terms of the supremum norm (see forthcoming points **(f.1)**, **(f.2)**, **(f.3)** and, in Lemma 2, points **(c.1)**, **(c.3)** and **(c.4)**). Such Lyapunov–Krasovskii functionals will be exploited for the stability analysis of the considered QSE control system. The invariant differentiability property (see the item **(c)**) is here introduced to ensure that the derivative related to the candidate Lyapunov–Krasovskii functional $\widetilde{V}_2(\phi)$ evaluated along the solution of system (2) does not involve the control input u. As shown in forthcoming Lemmas 1 and 2, in the context of systems with state delays, such a requirement turns out to be very helpful from a robustification point of view. For more details, the reader is referred to the proof of Lemma 1 reported in [40] where only the case of robust sampled-data controllers is investigated without taking into account the presence of quantization, spline approximation strategies and event-based updates. We highlight also that, in the forthcoming Section 5, it is shown how a standard functional of the form (6) can be easily used to apply the control design methodology proposed in this paper. In particular, the proposed results are applied to a mechanical system with state delays (see, for instance, [57]) and to a class of nonlinear time-delay systems.

In the following, the well-known notion of Steepest Descent Feedback (SDF) (see, [35,36,42] for the delay-free case and [19,40] for the delayed case) is revised in order to deal with the design of robust QSE controllers.

Definition 3. Let $V \in \mathcal{V}$. A locally bounded function $k : C^n \to \mathbb{R}^m$, continuous or not, is said to be a SDF for the system described by (2), induced by V, if there exist positive reals η, μ, $\bar{p}$, a function p in $C^1_L(\mathbb{R}^+;\mathbb{R}^+)$, of class $\mathcal{K}_\infty$ and satisfying $\frac{dp(s)}{ds} \le \bar{p}$, a function $\overline{\alpha}$ of class $\mathcal{P}_0$ such that $I_d - \overline{\alpha}$ is of class $\mathcal{K}_\infty$, a real $\nu \in \{0,1\}$, such that, for any $\phi \in C^n$, the following conditions hold

$$\nu D^+V(\phi, k(\phi)) + \eta \max\left\{0, D^+p\circ V_1\left(\phi, k(\phi)\right) + \mu p\circ V_1\left(\phi(0)\right)\right\} \le \overline{\alpha}(\eta\mu e^{-\mu\Delta}p\circ\beta_1(\|\phi\|_\infty)), \tag{7}$$

$$f(0) + g(0)k(0) = 0, \tag{8}$$

where: β_1 is the function of class $\mathcal{K}_\infty$ in Definition 1; the derivative in Driver's form (1) of the functional V is computed with respect to the function $F(\phi,u) = f(\phi) + g(\phi)u$ with f and g in (2).

Let us introduce the following assumption.

Assumption 1. There exist a functional $V \in \mathcal{V}$ and a related SDF k for the system described by (2) (see Definition 3).

Remark 2. Notice that, inspired by the well-known Artstein's approaches proposed in the literature (see, for instance, [19,35,36,38,42,49–51]), from a practical point of view, Assumption 1 can be checked by exploiting the procedure consisting in the following steps:

s.1 define a candidate Lyapunov–Krasovskii functional $V \in \mathcal{V}$ (see, for instance, (6) in Remark 1);
s.2 try to find a locally bounded function k, continuous or not, satisfying (7) (i.e., a SDF k according to Definition 3).

It is here highlighted that, inequality (7) concerns robustness of negative definiteness, with respect to a small perturbation term, of the functional derivative (see Remark 2 in [41]). Taking into account that in Definition 3, discontinuities in the function describing the SDF at hand are allowed, no kind of stability property is ensured for the continuous-time closed-loop system described by (2) with $u(t) = k(x_t)$. On the other hand, in the particular case of SDFs described by Lipschitz on bounded subsets functions, from Theorem 3.5 in [41], inequality (7) implies the global asymptotic stability property of the related continuous-time closed-loop system.

3. Digital event-based implementation

In this section, the proposed robust QSE implementation of SDFs is presented. Firstly, for the presentation of the proposed digital event-based controller, the notions of quantizer (see, for instance, [2]), of partition (see [36,41]) and of spline approximation (see [19]) are recalled. Such notions will be used for the characterization of the digital framework under study (see forthcoming Fig. 1).

For a given positive integer N, a quantizer is a function $q_y : \mathbb{R}^N \to Q_y^N$ (see the Notation Section) such that, for some given positive real E (range of the quantizer) and μ_y (error bound of the quantizer), the following implication holds (see, for instance, [2]):

$$|y| \le E \quad \to \quad |q_y(y) - y| \le \mu_y, \ y \in \mathbb{R}^N. \tag{9}$$

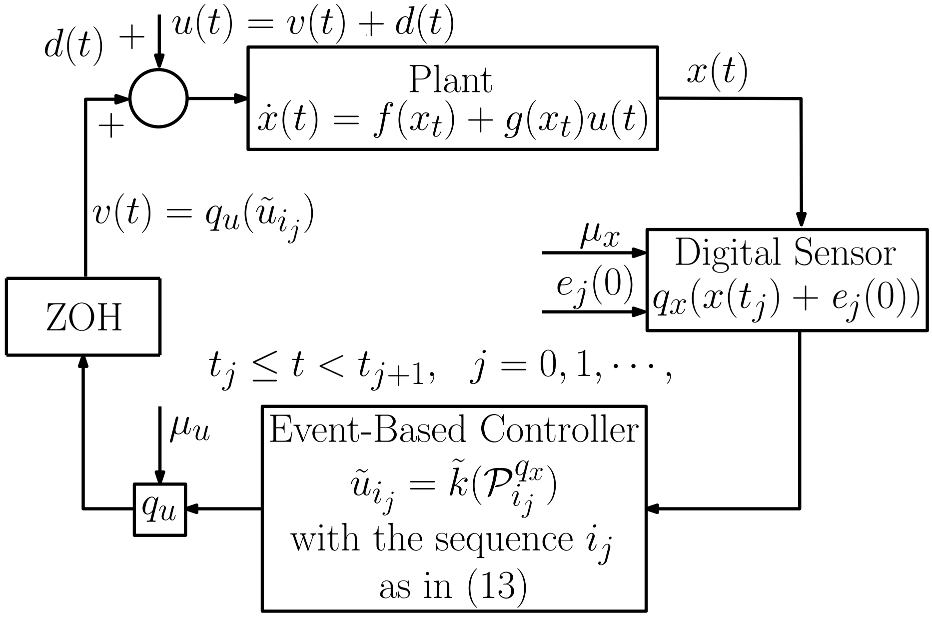

Fig. 1. Control scheme.

Definition 4. For a positive integer l, a partition $\pi = \{t_j,\ j = -l, -l+1, \dots\}$ of $[-l\Delta, \infty)$ is a countable, strictly increasing sequence $t_j \in [-l\Delta, \infty)$, with $t_0 = 0$, such that $t_j \to \infty$ as $j \to \infty$. The diameter of π, denoted $\mathrm{diam}(\pi)$, is defined as $\sup_{j \ge -l} t_{j+1} - t_j$. The dwell time of π, denoted $\mathrm{dwell}(\pi)$, is defined as $\inf_{j \ge -l} t_{j+1} - t_j$. For a given $a \in (0, 1]$, $\delta > 0$, $\pi_{a,\delta}$ is any partition π with $a\delta \le \mathrm{dwell}(\pi) \le \mathrm{diam}(\pi) \le \delta$.

Remark 3. Notice that, Definition 4 aims at characterizing the sampled-data framework here considered by partitioning the time axis into sampling intervals $[t_j, t_{j+1})$, $j = -l, -l+1, \dots$. We highlight that, in Definition 4, the positive real $a \in (0, 1]$ is introduced in order to consider the case of non-uniform sampling in which, for $j = -l, -l+1, \dots$, $a\delta \le t_{j+1} - t_j \le \delta$, with δ representing the upper bound for the sampling period.

For given $\delta < \Delta$ ($\Delta > 0$), $a \in (0, 1]$, let l be the smallest positive integer such that $la\delta \ge \Delta$. Let $\mathcal{T}_{l,a,\delta} \subset \mathbb{R}^{l+1}$ be the set defined as follows (see [19])

$$\mathcal{T}_{l,a,\delta} = \Big\{ w = \begin{pmatrix} w_0 & \cdots & w_l \end{pmatrix}^T \in \mathbb{R}^{l+1},\ w_k \in [-l\delta, 0],\ k = 0, 1, \dots, l, \\ w_0 = 0,\ w_0 - w_l \ge \Delta,\ \delta \ge w_k - w_{k+1} \ge a\delta, k = 0, 1, \dots, l-1 \Big\}. \tag{10}$$

Let $P_{l,a,\delta} : \mathbb{R}^{n(l+1)} \times \mathcal{T}_{l,a,\delta} \to C^n$ be the function defined (see [19]), for $z = \begin{pmatrix} z_0^T & \cdots & z_l^T \end{pmatrix}^T \in \mathbb{R}^{n(l+1)}$, $w = \begin{pmatrix} w_0 & \cdots & w_l \end{pmatrix}^T \in \mathcal{T}_{l,a,\delta}$ and $\tau \in [-\Delta, 0]$, as follows

$$(P_{l,a,\delta}(z, w))(\tau) = z_{k+1} + \frac{\tau - w_{k+1}}{w_k - w_{k+1}}(z_k - z_{k+1}), \tag{11}$$

where k is the smallest integer in $\{0, 1, \dots, l-1\}$ such that $w_k \ge \tau \ge w_{k+1}$. Fig. 1 illustrates the considered QSE control scheme in presence of actuation disturbances and measurement errors. In the following points (i) and (ii), the actuation disturbances and measurement errors under investigation (see the control scheme in Fig. 1) are described. In particular, for a given partition $\pi_{a,\delta}$ (see Definition 4),

(i) the actuation disturbances are characterized by an unknown function $d(t)$:

  (i.a) assumed to be continuous in any interval $[t_j, t_{j+1})$ with possible discontinuities in the sampling instants t_j, $j \in \mathbb{N}$;
  (i.b) satisfying $|d(t)| \le \bar{d}$, $\forall t \in \mathbb{R}^+$, with $\bar{d}$ a known positive real;
  (i.c) such that there exists finite $\lim_{t \to t_{j+1}^-} d(t)$, $j = 0, 1, \dots$.

(ii) the measurement errors affecting the quantized sampled-data output channel are characterized by an unknown sequence $e : \mathbb{N} \to C^n$, satisfying $\|e_j\|_\infty \le \bar{e}$, $j = 0, 1, \dots$, with $\bar{e}$ a known positive real.

In the following, for the first time in the literature of nonlinear time-delay systems, the ISS redesign methodologies (see, for instance, [40,42,58–60]) are used for the robustification of QSE stabilizers, induced by SDFs, with respect to actuation disturbances and measurement noises (see Fig. 1). In particular, a new control term is designed and added to the SDF at hand in order to arbitrarily attenuate the effects of the disturbances considered in points (i) and (ii). To such an aim, under Assumption 1, let:

- $S : C^n \to \mathbb{R}^m$ be the function defined, for $\phi \in C^n$, as follows

$$S(\phi) = \left(\frac{\partial V_1(x)}{\partial x}\Big|_{x=\phi(0)} g(\phi) \right)^T, \tag{12}$$

  where V_1 is the function related to the SDF at hand (see Definition 3);
- $\tilde{k} : C^n \to \mathbb{R}^m$ be the function defined, for $\phi \in C^n$, as follows

$$\tilde{k}(\phi) = k(\phi) - \omega S(\phi), \tag{13}$$

  where: $\omega > 0$ is a control tuning parameter to be chosen (see forthcoming Theorem 1); k is the SDF in Assumption 1 (see also Definition 3).

Remark 4. Notice that, in order to cope with nonlinear systems affected by state delays, the robustification term $-\omega S(\phi)$ (see (12) and (13)) has been here designed by the introduction of suitable Lyapunov–Krasovskii functionals which are invariantly differentiable and smoothly separable (see, for instance, [40] for the case of sampled-data controllers without quantization, spline approximation strategies and event-triggered updates). In particular, the robustification term $-\omega S(\phi)$ (see (12)) is here designed by exploiting the Lie derivative of the function V_1 along the vector field g. In the literature concerning nonlinear systems, the term $-\omega S(\phi)$ (see (12)) is commonly called $L_g V$ control term (firstly introduced in [60]) and it has been widely used to solve robust control problems in many contexts (see, for instance, [40,42,58–64] and the references therein) which, however, do not include the framework here considered (see Fig. 1). We highlight that, to our best knowledge, it is the first time in the literature that a robustification approach based on $L_g V$ control terms is successfully applied to QSE stabilizers induced by (continuous or not) SDFs.

In the following, some useful functionals are introduced for the presentation of the event-based mechanism which will be exploited for the update of the controller at hand. In particular, under Assumption 1 and taking into account the positive reals η, μ, ν and the functions p and V related to Definition 3, let:

(f.1) $V_3 : C^n \to \mathbb{R}^+$ be the functional defined, for $\phi \in C^n$, as $V_3(\phi) = \sup_{\theta \in [-\Delta, 0]} e^{\mu\theta} p \circ V_1(\phi(\theta))$;
(f.2) $V_\infty : C^n \to \mathbb{R}^+$ be the functional defined, for $\phi \in C^n$, as $V_\infty(\phi) = \nu V(\phi) + \eta V_3(\phi)$;
(f.3) $\mathcal{D}_\infty : C^n \times \mathbb{R}^m \to \mathbb{R}$ be the functional defined, for $\phi \in C^n$, $u \in R^m$, as follows

$$\begin{aligned}\mathcal{D}_\infty(\phi, u) = {} & \nu D^+ V(\phi, u) - \eta\mu V_3(\phi) \\ & + \eta \max\{0, D^+ p \circ V_1(\phi, u) + \mu p \circ V_1(\phi(0))\}.\end{aligned} \tag{14}$$

In the following, the proposed QSE controller is provided. For a given positive real $\sigma \in (0,1)$, for given positive reals $\bar{\mu}$ and $\tilde{\mu}$ in $(0,1]$, for a given partition $\pi_{a,\delta}$ (see Definition 4), for given quantizers $q_x : \mathbb{R}^n \to Q_x^n$ and $q_u : \mathbb{R}^m \to Q_u^m$ (satisfying (9)), the proposed robust QSE controller for the system (2) when affected by measurement errors and actuation disturbances (see Fig. 1 and the considered disturbances in points (i) and (ii) above) is described by

$$\begin{aligned} & u(t) = q_u(\tilde{u}_{i_j}) + d(t) = q_u\left(\tilde{k}\left(\mathcal{P}_{i_j}^{q_x}\right)\right) + d(t), \\ & t \in [t_j, t_{j+1}), \quad j = 0, 1, \ldots, \quad t_j, t_{j+1} \in \pi_{a,\delta}, \end{aligned} \tag{15}$$

where:

- $\tilde{k}$ is the function in (13);
- $\mathcal{P}_j^{q_x} = P_{l,a,\delta}(B_S^{q_x}(j), B_{\mathcal{T}}(j))$, $j = 0, 1, \ldots,$ with $P_{l,a,\delta}$ the function defined in (11) (see Fig. 2)
- $B_S^{q_x} : \mathbb{N} \to \mathbb{R}^{n(l+1)}$ and $B_{\mathcal{T}} : \mathbb{N} \to \mathbb{R}^{l+1}$ are defined (recursively) as

$$\begin{aligned}
& B_S^{q_x}(0) = \begin{pmatrix} q_x(\bar{x}_0(0) + \bar{e}_0(0)) \\ \vdots \\ q_x(\bar{x}_0(t_{-l}) + \bar{e}_0(t_{-l})) \end{pmatrix}, \\
& q_x(\bar{x}_0(\tau) + \bar{e}_0(\tau)) = \begin{cases} q_x(x_0(\tau) + e_0(\tau)) & \tau \in [-\Delta, 0] \\ q_x(x_0(-\Delta) + e_0(-\Delta)) & \tau \in [t_{-l}, -\Delta] \end{cases} \\
& B_S^{q_x}(j) = \begin{pmatrix} q_x(x(t_j) + e_j(0)) \\ 0_{ln\times 1} \end{pmatrix} + \begin{pmatrix} 0_{n\times ln} & 0_n \\ I_{ln} & 0_{ln\times n} \end{pmatrix} B_S^{q_x}(j-1), \\
& B_{\mathcal{T}}(0) = \begin{pmatrix} 0 \\ t_{-1} \\ \vdots \\ t_{-l} \end{pmatrix}, \; B_{\mathcal{T}}(j) = \begin{pmatrix} 0_{1\times l} & 0 \\ I_l & 0 \end{pmatrix} \left(B_{\mathcal{T}}(j-1) - (t_j - t_{j-1}) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right), \\
& j = 1, 2, \ldots;
\end{aligned} \tag{16}$$

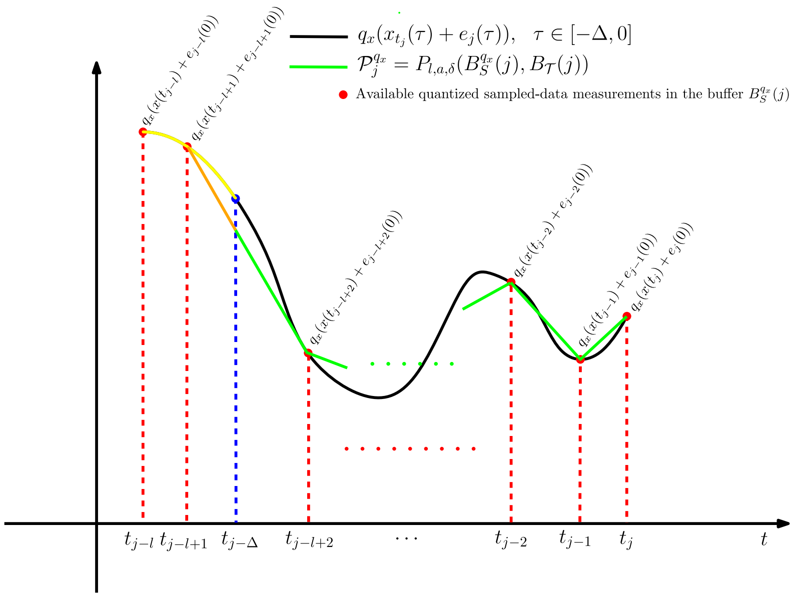

Fig. 2. An example of the spline approximation method here used (see $\mathcal{P}_j^{q_x}$ in (15)).

- the sequence i_j, $j = 0, 1, \dots$, is defined as $i_0 = 0$ and, for $j \geq 1$, $i_j = j$ in the event that (see **(f.3)**)

$$-\mathcal{D}_\infty(\mathcal{P}_j^{q_x}, q_u(\tilde{u}_{i_{j-1}})) + \sigma\mathcal{D}_\infty(\mathcal{P}_j^{q_x}, q_u(\tilde{u}_j)) \leq H(\mathcal{P}_j^{q_x}), \tag{17}$$

  and $i_j = i_{j-1}$ otherwise;
- the function $H : \mathcal{C}^n \to \mathbb{R}^+$ is defined, for $\phi \in \mathcal{C}^n$, as follows

$$H(\phi) = \bar{d}\big(\nu(1-\sigma) + 4\eta\bar{p}(1+\sigma)\big)\left(|S(\phi)| + \frac{\bar{e} + \bar{\mu} + \tilde{\mu}}{\omega}\right) + 3(1+\sigma)L_D\bar{e}, \tag{18}$$

- η, ν and $\bar{p}$ are the positive reals in Definition 3;
- ω is the control tuning parameter in (13);
- L_D is a suitable control tuning parameter to be chosen (see forthcoming Theorem 1);
- $\bar{e}$ and $\bar{d}$ are the bounds of the involved observation errors and actuation disturbances, respectively (see points (i) and (ii) above).

Remark 5. Notice that, the knowledge of infinite dimensional measurements $q_x(x_{t_j}(\tau) + e_j(\tau))$, $\tau \in [-\Delta, 0]$ is not needed for the correct implementation of the proposed robust QSE controller (15). Indeed, spline approximation methodologies (see [19]) are here used in order to obtain an approximation of the infinite dimensional variable $q_x(x_{t_j}(\tau)+e_j(\tau))$, $\tau \in [-\Delta, 0]$, by interpolating the available quantized sampled-data measurements $q_x(x(t_j)+e_j(0))$ (see (11) and $\mathcal{P}_j^{q_x}$ in (15)). In Fig. 2, an example of the interpolation method here considered is reported.

Remark 6. Notice that, the proposed triggering condition (17) is checked just at times t_j, $j = 0, 1, \dots$, guaranteeing a minimum dwell-time $a\delta$ between two consecutive sampling instants (see Definition 4). Hence, no continuous-time monitoring of the state variables is needed and possible Zeno behaviors are avoided by sampling with dwell-time.

Remark 7. Notice that, in (16), $B_S^{q_x}$ and $B_\mathcal{T}$ describe buffers of length $(l+1)n$ and $l+1$ collecting, respectively, the quantized sampled-data state measurements affected by the noises e_j, $j = 0, 1, \dots$, and the times elapsed between a sampling and the following. The informations in $B_S^{q_x}$ and $B_\mathcal{T}$ (see (16)) are used in order to obtain an approximation of suitable past values of the system variables via (11) that are not available in the buffer and which are needed for the implementation of the controller.

In the next section, semi-global practical stability results will be provided for the QSE closed-loop system described by (2)–(15) (see Fig. 1). For the reader's convenience, in the following, for the first in the literature, the notion of semi-global practical stability is provided in the context of nonlinear time-delay systems exploiting QSE controllers and affected by actuation disturbances and measurement errors. In the forthcoming Definition 5, for a given sequence $e : \mathbb{N} \to \mathcal{C}^n$ (see point (ii) in Section 3), we will consider the function $B_S^e : \mathbb{N} \to \mathbb{R}^{n(l+1)}$ defined (recursively) as:

$$\begin{aligned} B_S^e(0) &= \begin{pmatrix} \bar{e}_0(0) \\ \vdots \\ \bar{e}_0(t_{-l}) \end{pmatrix}, \ \bar{e}_0(\tau) = \begin{cases} e_0(\tau) & \tau \in [-\Delta, 0] \\ e_0(-\Delta) & \tau \in [t_{-l}, -\Delta] \end{cases} \\ B_S^e(j) &= \begin{pmatrix} e_j(0) \\ 0_{ln\times 1} \end{pmatrix} + \begin{pmatrix} 0_{n\times ln} & 0_n \\ I_{ln} & 0_{ln\times n} \end{pmatrix} B_S^e(j-1), \ j = 1, 2, \dots . \end{aligned} \tag{19}$$

Definition 5. The QSE closed-loop system described by (2)–(15) is said to be semi-globally practically stable by fast sampling and accurate quantization if, for any positive reals $\bar{e}$ and $\bar{d}$ (bounds of the involved measurement errors and actuation disturbances), for any positive reals R (radius of the ball of the initial states), r (radius of the final target ball) and q, with $0 < r < R$, for any positive reals a, $\bar{\mu}$, $\tilde{\mu}$ in $(0, 1]$ and $\sigma \in (0, 1)$, there exist positive control tuning parameters δ (sampling period), E_1, μ_x (range and quantization error bound of the output quantizer q_x), U, μ_u (range and quantization error bound of the input quantizer q_u), ω (control parameter in (13)), L_D (control parameter in (18)) and positive reals E (overshoot) and T (settling time) such that: for any initial condition $x_0 \in C_R^n$, satisfying $\operatorname{ess\,sup}_{\theta\in[-\Delta,0]}\left|\frac{dx_0(\theta)}{d\theta}\right| \le q$, for any unknown actuation disturbance as in point (i), for any unknown observation error as in point (ii) and satisfying

$$\sup_{\tilde{x} \in \mathcal{B}^{n(l+1)}_{E\sqrt{l+1}}} |S(P_{l,a,\delta}(\tilde{x} + B_S^e(j), w)) - S(P_{l,a,\delta}(\tilde{x}, w))| \le \frac{\bar{e}}{\omega}, \ \forall w \in \mathcal{T}_{l,a,\delta}, \tag{20}$$

the corresponding unique, locally absolutely continuous solution exists $\forall t \ge 0$ and, furthermore, satisfies

$$\|x_t\|_\infty \le E, \ \forall t \ge 0, \quad \|x_t\|_\infty \le r, \ \forall t \ge T. \tag{21}$$

4. Main results

In the following, the main results of the paper are provided. In particular, we will show that, under Assumption 1, there exist suitable control tuning parameters ω and L_D (see (15)–(17)), a suitably fast sampling δ and an accurate quantization of the input/output channels (i.e., ranges and error bounds for the quantizers q_x and q_u in (15)) such that the semi-global practical stability property of the closed-loop system (2)–(15) (see Definition 5) is ensured regardless of the unknown actuation disturbances d (see point (i) and Fig. 1 in Section 3) and of the unknown observation errors e (see Fig. 1 and point (ii) in Section 3). In the following, we will consider functions α_i, $i = 1, 2$, of class $\mathcal{K}_\infty$, defined for $s \in \mathbb{R}^+$ as follows

$$\alpha_1(s) = \eta e^{-\mu\Delta} p(\beta_1(s)), \quad \alpha_2(s) = \nu\gamma_2(s) + \eta p(\beta_2(s)), \tag{22}$$

blue where: η, μ and ν are the positive reals in Definition 3; p is the function in Definition 3; γ_2 and β_i, $i = 1, 2$, are the functions related to the Lyapunov–Krasovskii functional $V \in \mathcal{V}$ (see items **(a)** and **(d)**).

Theorem 1. *Let Assumption 1 hold. Let a, $\bar{\mu}$ and $\tilde{\mu}$ be arbitrary reals in $(0, 1]$. Let σ be an arbitrary real in $(0, 1)$. Then, for any positive reals r, R, q, $\bar{d}$ and $\bar{e}$ with $0 < r < R$, for any positive real $E > R$ with $\alpha_1(E) > \alpha_2(R)$, there exists a positive real $\bar{\omega}$ such that for any $\omega \ge \bar{\omega}$, there exist positive reals δ, T, U, L_D, E_1, μ_x and μ_u, such that: for any state quantizer q_x with error bound μ_x and range E_1, for any input quantizer q_u with error bound μ_u and range U, for any initial state $x_0 \in W_n^{1,\infty} \cap C_R^n$, and satisfying $\operatorname{ess\,sup}_{\theta\in[-\Delta,0]}\left|\frac{dx_0(\theta)}{d\theta}\right| \le q$, for any partition $\pi_{a,\delta} = \{t_j, \ j = -l, -l+1, \dots\}$, where l is the smallest (nonnegative) integer such that $la\delta \ge \Delta$ and $\{t_{-l}, t_{-l+1}, \dots, 0\} \in \mathcal{T}_{l,a,\delta}$, for any signal $d : \mathbb{R}^+ \to \mathbb{R}^m$ of unknown actuation disturbance as in point (i), for any signal $e : \mathbb{N} \to C^n$ of unknown observation error as in point (ii) and satisfying (20) the corresponding unique locally absolutely continuous solution of the QSE closed-loop system, described by (2)–(15), exists $\forall t \ge 0$ and, furthermore, satisfies:*

$$x_t \in C_E^n, \ \forall t \ge 0, \quad x_t \in C_r^n, \ \forall t \ge T, \tag{23}$$

i.e., the QSE closed-loop system described by (2)–(15) is semi-globally practically stable by fast sampling and accurate quantization (see Definition 5).

Proof. The proof of Theorem 1 is reported in Appendix.

Remark 8. Notice that, the main challenge addressed in this paper and overcome with the results provided in Theorem 1 concerns: how to design stabilizers for the system (2) taking simultaneously into account the following aspects (see Fig. 1): (a.1) the presence of sampling (possibly aperiodic); (a.2) the presence of quantization (possibly non–uniform) in both input/output channels; (a.3) the arbitrary reduction of the effects of unknown actuation disturbances and unknown observation errors affecting the event-based digital controller at hand; (a.4) possible discontinuities in the function describing the controller; (a.5) problems related to the possible non-availability in the buffer of suitable past values of the system state required for the implementation of the controller at hand. To our best knowledge the framework reported in Fig. 1 has never been investigated in the literature of nonlinear time-delay systems. Theorem 1 provides, for the first time in the literature, stability results for nonlinear systems with state delays affected by actuation disturbances and measurement errors and making use of QSE controllers based on (continuous or not) SDFs (i.e., taking simultaneously into account the aspects (a.1)-(a.5)). The simultaneous consideration of the aspects (a.1)-(a.5) introduces several difficulties in the stability analysis of the related closed-loop system which are addressed and solved by suitably reformulating the stabilization in the sample-and-hold sense theory [8,14,19,26,35–42], here used as a tool to prove the proposed results (see the proof of Theorem 1 reported in the Appendix). For instance, differently from the frameworks commonly studied in the literature (see, for instance, [10,14,19,20,22,26,65,66] and the references therein), the following difficulties are here addressed and overcome: (d.1) how to cope with sampled-data event-based controllers based on (continuous or not) SDFs in presence of quantization and unknown external disturbances; (d.2) ensure the efficacy of the added $L_g V$ control term against unknown external disturbances when implemented in presence of sampling, quantization, spline approximation strategies, and event-triggered updates. Indeed, as far as (d.1) is concerned, it is well-known that, event-triggered mechanisms based on Lyapunov functions require the evaluation of the related Lyapunov derivatives along the solution of the closed-loop system under study in order to be correctly applied (see, for instance, [10,14,19,20,22,26,65,66] and the references therein). In presence of unknown actuation disturbances and unknown measurement errors, the exact evaluation of the related Lyapunov derivatives is prevented because of the required knowledge of the signals describing the involved uncertainties. Moreover, in the case of nonlinear systems with state delays, such an evaluation requires the knowledge of the infinite dimensional variable x_t which, in a real practice, is unavailable due to technological constraints. The Lyapunov–Krasovskii event-triggered mechanism (17) overcomes these drawbacks by exploiting, in the evaluation of the related derivatives, only the quantized sampled-data measurements acquired at sampling instants t_j (i.e. $q_x(x(t_j)+e_j(0))$) and the bounds of the involved disturbances (i.e. $\bar{d}$ and $\bar{e}$) together with a spline interpolation methodology to obtain an approximation of the required infinite dimensional variables (see (11), $\mathcal{P}_j^{q_x}$ in (15), (17), Fig. 2 and Remark 5). As far as (d.2) is concerned, we highlight here that, in the literature of nonlinear systems with state delays, the robustification property of the added $L_g V$ control term (see (12), (13)) has been proved just in the context of sampled-data control (see [40]). On the other hand, the efficacy of the robustification term (12) has never been proved by taking into account the simultaneous presence of sampling, quantization, spline approximations and with an event-triggered strategy exploited for its updates. In Theorem 1, for the first time in the literature, theoretical results concerning the arbitrary reduction of the effects of arbitrarily large unknown actuation disturbances and of suitably small unknown observation errors are provided in the context of the QSE control of nonlinear systems with state-delays. We highlight also that, results concerning the quantized sampled-data implementation of stabilizers possibly described by discontinuous functions have never been provided in the literature of nonlinear systems with state delays, not even for the case without event-based update strategies (see Definition 3 and Theorem 1).

Remark 9. Notice that, in Theorem 1, the signals d and e are unknown and characterize the involved actuation disturbances and measurement errors, which affect the proposed QSE controller (15) when closed in the loop with system (2) (see Fig. 1 and points (i), (ii) in Section 3). In order to arbitrarily attenuate the effects of such disturbances, the term $-\omega S(\phi)$ (see (12)) has been added to the SDF at hand (see (13)). In particular, the results provided in Theorem 1 are valid for any actuation disturbance with arbitrarily large bound as described in point (i) of Section 3. On the other hand, in Theorem 1, the involved measurement errors are supposed to be suitably small so that the robustification term $-\omega S(\phi)$ (see (12)) is marginally affected by these errors (see (20)). This fact can happen, for instance, when the robustification term depends on state variables that can be measured better than other state variables, or when the variation of the robustification term is sufficiently slow with respect to measurement errors. We highlight that, taking into account that discontinuities in the function describing the SDF at hand are allowed (see Definition 3), even small measurement errors may turn in serious performances deterioration of the feedback control law. Then, the robustification of quantized sampled-data event-based SDFs is significant also in the case of suitably small measurement errors (see (20)).

Remark 10. Notice that, in the proof of Theorem 1 (see the Appendix), a methodology for the computation of an upper bound for the sampling period δ, of upper bounds for the quantization errors μ_x, μ_u, of quantizers ranges E_1, U, of control parameters ω, L_D, and of a settling time T is provided (see Steps (1)–(10) soon after Lemma 2 and (A.29)). According to our experience, such steps may well provide a conservative upper bound for the sampling period as well as a conservative quantization of the input/output channels. The source of such conservatism may be the use of Lipschitz constants of many involved functions as well as lower and upper bounds of Lyapunov–Krasovskii functionals and derivatives. On the other hand, the results provided in Theorem 1 are of the existence type, and the study of the conservativeness of the sampling frequency as well as of the quantization in the input/output channels is beyond the aim of this work, and is left for future investigations. We highlight here that, to our best knowledge, it is the first time in the literature of nonlinear systems with state-delays that a methodology for the design of QSE controllers, robustified with respect to arbitrarily large unknown actuation disturbances and suitably bounded unknown observation errors and ensuring the semi-global stability property of the related closed-loop system regardless of the mentioned disturbances, is provided.

5. Applications

5.1. Application to a single-link robot arm

In the following, the results stated in Theorem 1 are applied to a single-link flexible joint robot arm with time delays. Let us consider a robot arm described by the following nonlinear time-delay system [57,67]:

$$\begin{aligned} \dot{q}_{1,1}(t) &= q_{1,2}(t), \\ \dot{q}_{1,2}(t) &= -p_1 \sin(q_{1,1}(t)) + p_2(q_{2,1}(t) - q_{1,1}(t)) \\ &\quad - p_3 q_{1,1}(t-\Delta_1)\cos(q_{1,2}(t-\Delta_1)), \\ \dot{q}_{2,1}(t) &= q_{2,2}(t), \\ \dot{q}_{2,2}(t) &= -p_4 q_{2,2}(t) + p_5(q_{1,1}(t) - q_{2,1}(t)) - p_6(q_{2,2}(t-\Delta_2)\sin(q_{1,1}(t-\Delta_2)) \\ &\quad + 1.2q_{1,2}^2(t-\Delta_2)q_{2,1}(t-\Delta_2) - u(t)), \end{aligned} \tag{24}$$

where: $q_{i,1}$, $q_{i,2} \in \mathbb{R}$, $i = 1, 2$, are the positions and velocities of the link and the actuator, respectively; $p_1 = \frac{mgl}{M}$, $p_2 = \frac{K}{M}$, $p_3 = \frac{1}{M}$, $p_4 = \frac{B}{J}$, $p_5 = \frac{K}{J}$, $p_6 = \frac{1}{J}$ are the involved parameters; $\Delta_1 = 1[s]$ and $\Delta_2 = 2[s]$ are the involved time-delays related to the interconnection between the link-side subsystem and the actuator-side subsystem which is often accompanied by energy transfer that leads to the time-delay phenomenon. See [57,67] for more details concerning the model (24) and related parameters. Let $\chi_1 \in \mathbb{R}$ and $\chi_3(t)$ be the desired position for the link and actuator, respectively. Let us define $x_1(t) = q_{1,1}(t) - \chi_1$, $x_2(t) = q_{1,2}(t)$, $x_3(t) = q_{2,1}(t) - \chi_3(t)$, $x_4(t) = q_{2,2}(t) - \dot{\chi}_3(t)$. From (24), we obtain the following error system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -p_1 \sin(x_1(t) + \chi_1) + p_2(x_3(t) + \chi_3(t) - x_1(t) - \chi_1) \\ &\quad - p_3(x_1(t-\Delta_1) + \chi_1)\cos(x_2(t-\Delta_1)), \\ \dot{x}_3(t) &= x_4(t), \\ \dot{x}_4(t) &= -p_4(x_4(t) + \dot{\chi}_3(t)) + p_5(x_1(t) + \chi_1 - x_3(t) - \chi_3(t)) \\ &\quad - p_6((x_4(t-\Delta_2) + \dot{\chi}_3(t-\Delta_2))\sin(x_1(t-\Delta_2) + \chi_1) \\ &\quad + 1.2x_2^2(t-\Delta_2)(x_3(t-\Delta_2) + \chi_3(t-\Delta_2))) - \ddot{\chi}_3(t) + p_6 u(t). \end{aligned} \tag{25}$$

Let $\bar{k} : C^n \to \mathbb{R}$ be the function, defined for any $\phi \in C^n$, as follows

$$\begin{aligned} \bar{k}(\phi) &= \phi_1(0) + \chi_1 + \frac{1}{p_2}(-k_1\phi_1(0) - k_2\phi_2(0) + p_1 \sin(\phi_1(0) + \chi_1) \\ &\quad + p_3(\phi_1(-\Delta_1) + \chi_1)\cos(\phi_2(-\Delta_1))), \qquad k_1, k_2 > 0. \end{aligned} \tag{26}$$

By choosing $\chi_3(t) = \bar{k}(x_t)$, from (25) we obtain

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \overline{A}x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_4(x_t) + p_6 u(t) \end{bmatrix}, \quad \overline{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k_1 & -k_2 & p_2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{27}$$

where:

$$\begin{aligned} f_4(x_t) &= -p_4(x_4(t) + \bar{k}_1(x_t)) + p_5(x_1(t) + \chi_1 - x_3(t) - \bar{k}(x_t)) \\ &\quad - p_6((x_4(t-\Delta_2) + \bar{k}_2(x_t))\sin(x_1(t-\Delta_2) + \chi_1) \\ &\quad + 1.2x_2^2(t-\Delta_2)(x_3(t-\Delta_2) + \bar{k}_3(x_t))) - \bar{k}_4(x_t); \\ \bar{k}_1(x_t) &= \dot{\chi}_3(t); \quad \bar{k}_2(x_t) = \dot{\chi}_3(t-\Delta_2); \quad \bar{k}_3(x_t) = \chi_3(t-\Delta_2); \quad \bar{k}_4(x_t) = \ddot{\chi}_3(t); \end{aligned} \tag{28}$$

the functions $\bar{k}_i : C^n \to \mathbb{R}$, $i = 1, \dots, 4$, are readily defined in (28). Notice that system (27) is in the form (2) with $x_t \in C^4$ and $\Delta = 2\Delta_1 + 2\Delta_2$.

According to the proposed design procedure, let $V_1 : \mathbb{R}^4 \to \mathbb{R}^+$ and $V_2 : Q^4 \to \mathbb{R}^+$ be the functions defined, for $x \in \mathbb{R}^4$ and $\phi \in Q^4$ as follows

$$V_1(x) = x^T P x, \quad V_2(\phi) = 0, \tag{29}$$

where P is the symmetric positive definite matrix satisfying $A^T P + PA = -I_4$ with $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k_1 & -k_2 & p_2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -k_3 & -k_4 \end{pmatrix}$, $k_3, k_4 > 0$. Notice that, functions V_1 and V_2 in (29) satisfy points **(a)–(d)** in Section 2 with functions $\beta_1(s) = \gamma_1(s) = \lambda_{\min}(P)s^2$, $\beta_2(s) = \gamma_2(s) = \lambda_{\max}(P)s^2$. Now, taking into account the considered candidate Lyapunov–Krasovskii functional V (see (4)) obtained from (29), in the following, we try to find a SDF according to Definition 3. In particular, taking into account (27) and the functions V_1, V_2 in (29), for any $\phi \in C^4$, the following equality holds:

$$D^+V(\phi,u) = 2\phi(0)^T P\overline{A}\phi(0) + 2\phi(0)^T P \begin{bmatrix} 0 & 0 & 0 & f_4(\phi) + p_6 u \end{bmatrix}^T. \tag{30}$$

From (30), by choosing $u = k(\phi)$, where $k : C^n \to \mathbb{R}$ is the function defined, for $\phi \in C^n$, as follows

$$k(\phi) = \frac{1}{p_6}(-f_4(\phi) - k_3\phi_3(0) - k_4\phi_4(0)), k_3,\ k_4 > 0, \tag{31}$$

we obtain

$$D^+V(\phi, k(\phi)) \leq -|\phi(0)|^2. \tag{32}$$

Then, taking into account (32) and by choosing, for instance, $p = I_d$, $\nu = 1$, $\eta > 0$, $\mu = \frac{1}{\lambda_{\max}(P)}$, $\bar{\alpha} = 0$, inequality (7) is satisfied. It follows that the function k in (31) is a SDF for the system (27) according to Definition 3 (i.e. Assumption 1 is satisfied in this case). It follows that all the conditions to apply Theorem 1 are satisfied for this case.

In the performed simulations: the initial state of system (24) has been chosen equal to $\begin{pmatrix} q_{1,1}(t+\tau) & q_{1,2}(t+\tau) & q_{2,1}(t+\tau) & q_{2,2}(t+\tau) \end{pmatrix}^T = 0$, $\tau \in [-\Delta, 0]$; the desired position of the link χ_1 has been chosen equal to 1; a uniform sampling period (i.e. $a = 1$) equal to $\delta = 0.01[s]$ and quantizers based on the round-to-nearest method with $Q_x^4 = \{x \in \mathbb{R}^4 | \ x_i = \pm 0.0001j,\ i = 1,\dots,4,\ j = 0,1,\dots,10^5\}$ and $Q_u = \{u \in \mathbb{R} | \ u = \pm 0.01j,\ j = 0,1,\dots,1000\}$ have been chosen; the controller parameters in (31) are chosen equal to $k_i = 5$, $i = 1,\dots,4$; an actuation disturbance $d(t) = \sin(t) + d_r(t)$ has been considered where $d_r(t) = \tilde{d}_r(j)$, $t \in [t_j, t_{j+1})$, $j = 0,1,\dots$, with $\tilde{d}_r(j)$ taken from the interval $[-0.5, 0.5]$ by emulation of the uniform probability density function; observation errors $e_1(j) = 10^{-5}\sin(t_j)$, $e_2(j) = 10^{-5}\cos(t_j)$, $e_3(j) = 10^{-5}\sin(t_j)$ and $e_4(j) = 10^{-8}\cos(t_j)$, $j = 0,1,\dots$, have been considered; the parameters related to the robustification term (see (12)) and to the triggering mechanism (see (17)) have been chosen equal to $\omega = 20$, $\mu = 10^{-5}$, $\tilde{\mu} = 10^{-5}$, $\bar{\mu} = 10^{-5}$ and $L_D = 200$; different values of the parameter σ, related to the triggering mechanism (see (17)), have been considered. The simulation results, in the case $\sigma = 0.1$, are shown in Fig. 3. In particular, Fig. 3 compares the performances of the proposed robustified digital event-based controller with the ones related to the robustified digital time-triggered controller, to the non-robustified digital event-based controller (i.e., $\omega = 0$) and to the non-robustified digital time-triggered controller (i.e., $\omega = 0$). The robust event-triggered solution with $\delta = 0.01[s]$ achieves very good performances similar to the ones of the robust time-triggered solution, in spite of lower average frequency of control updates with respect to the robust quantized sampled-data time-triggered controller with the same sampling interval (around 11.9% of the sampling intervals). Moreover, it is clear from Figures 3 that the proposed robustified digital event-based controller can drastically reduce the effects of actuator disturbances and observation errors, forcing the state variables to a neighborhood of the origin which is much smaller than the one with the non-robustified digital event-based/time-based controller. Simulations fully validate the theoretical results.

5.2. Application to a particular class of nonlinear time-delay systems

In this subsection, the proposed results are applied to a particular class of nonlinear time-delay systems in control-affine form. Let us consider the nonlinear time-delay system described by the following RFDEs:

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_t), \\ \dot{x}_2(t) &= f_2(x_t) + g(x_t)u(t), \end{aligned} \tag{33}$$

where: $x_t = \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} \in C^{n+1}$, $x_{1,t} \in C^n$, $x_{2,t} \in C$ is the system state; $f_1 : C^{n+1} \to \mathbb{R}^n$ is a function Lipschitz on bounded subsets of C^{n+1} and, such that there exists a symmetric positive definite matrix $\tilde{P} \in \mathbb{R}^{n \times n}$ satisfying, for any $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in C^{n+1}$, $\phi_1 \in C^n$, $\phi_2 \in C$,

$$2\phi_1(0)^T \tilde{P} f_1(\phi) \leq -\sigma_1 |\phi_1(0)|^2 + |\phi_2(0)|\sigma_2(\phi), \tag{34}$$

with σ_1 a positive real and $\sigma_2 : C^{n+1} \to \mathbb{R}$ a known function Lipschitz on bounded subsets of C^{n+1}; $f_2 : C^{n+1} \to \mathbb{R}$ is a known function Lipschitz on bounded subsets of C^{n+1}; $g : C^{n+1} \to \mathbb{R}$ is a known function Lipschitz on bounded subsets of C^{n+1} and satisfying $g_{\min} < g(\phi) < g_{\max}$, $\forall \phi \in C^{n+1}$, where $g_{\min}$ and $g_{\max}$ are positive reals; $u(t) \in \mathbb{R}$ is the control input. According to the proposed design procedure, let $V_1 : \mathbb{R}^{n+1} \to \mathbb{R}^+$ and $V_2 : Q^{n+1} \to \mathbb{R}^+$ be the functions defined, for $x \in \mathbb{R}^{n+1}$ and $\phi \in Q^{n+1}$ as follows

$$V_1(x) = x^T P x, \quad V_2(\phi) = 0, \tag{35}$$

where, $P = \begin{pmatrix} \tilde{P} & \bar{0} \\ \bar{0}^T & 1 \end{pmatrix}$, with $\tilde{P}$ the matrix in (34) and $\bar{0} \in \mathbb{R}^n$ a vector of zeros. Notice that, the functions V_1 and V_2 in (35) satisfy points **(a)–(d)** in Section 2 with functions $\beta_1(s) = \gamma_1(s) = \lambda_{\min}(P)s^2$, $\beta_2(s) = \gamma_2(s) = \lambda_{\max}(P)s^2$. Now, taking into account the considered candidate Lyapunov–Krasovskii functional V (see (4)) obtained from (35), in the following, we try to find a SDF according to Definition 3. In particular, taking into account (34) and the functions V_1, V_2 in (35), for any $\phi \in C^{n+1}$ and for any $u \in \mathbb{R}$, the following equalities/inequality hold:

$$\begin{aligned} D^+V(\phi,u) &= 2\phi(0)^T P \begin{pmatrix} f_1(\phi) \\ f_2(\phi) + g(\phi)u \end{pmatrix} \\ &= 2\phi_1(0)^T \tilde{P} f_1(\phi) + 2\phi_2(0)(f_2(\phi) + g(\phi)u) \\ &\leq -\sigma_1|\phi_1(0)|^2 + |\phi_2(0)|\sigma_2(\phi) + 2\phi_2(0)(f_2(\phi) + g(\phi)u). \end{aligned} \tag{36}$$

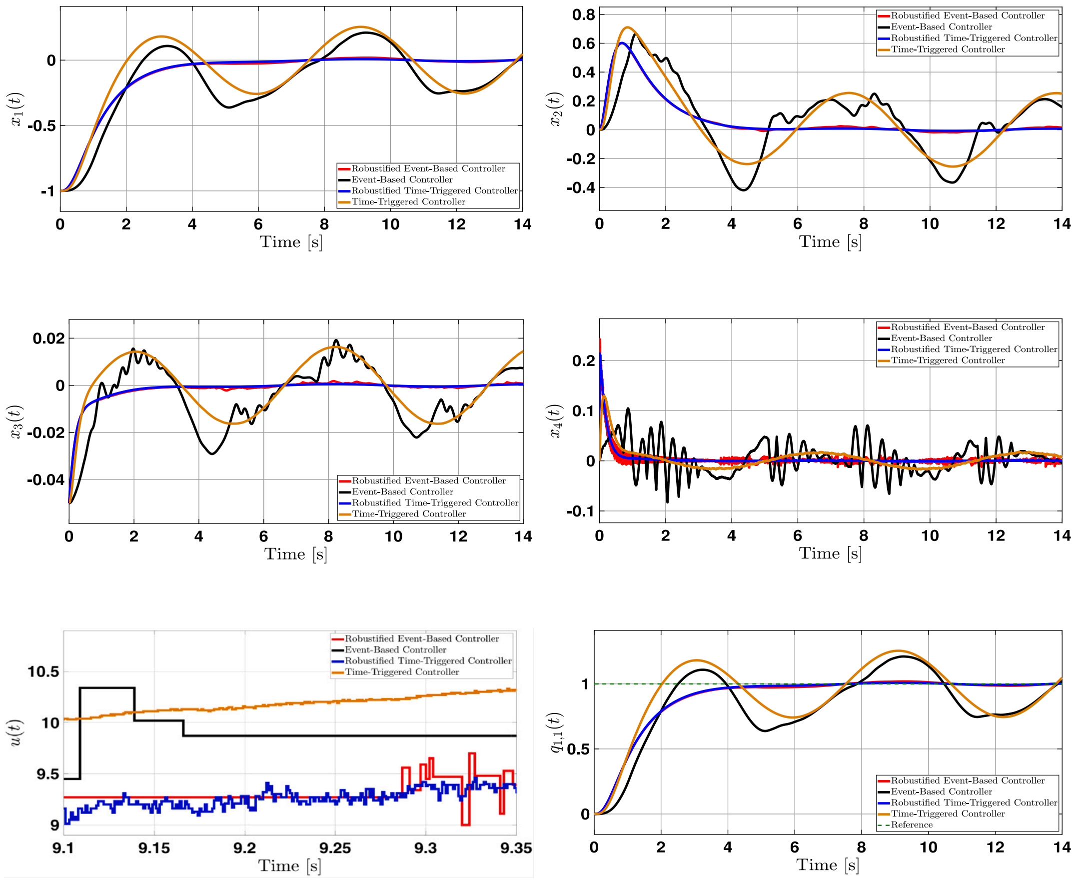

Fig. 3. In the first four panels, the evolutions of $x_i(t)$, $i = 1, 2, 3, 4$ are reported. In the last two panels, the evolution of the state variable $q_{1,1}(t)$ and a zoom of the control input $u(t)$ is reported.

From (36), by choosing $u = k(\phi)$, where $k : C^{n+1} \to \mathbb{R}$ is the function defined, for $\phi \in C^{n+1}$, as follows

$$k(\phi) = -\frac{0.5\text{sign}(\phi_2(0))\sigma_2(\phi) + f_2(\phi) + \sigma_3\phi_2(0)}{g(\phi)}, \tag{37}$$

with $\sigma_3 > 0$ a positive tuning parameter, we obtain

$$D^+V(\phi, k(\phi)) \leq -\sigma_1|\phi_1(0)|^2 - 2\sigma_3|\phi_2(0)|^2 \leq -\min\{\sigma_1, \sigma_3\}|\phi(0)|^2. \tag{38}$$

Then, as far as inequality (7) is concerned, taking into account (38) and by choosing, for instance, $p = I_d$, $\nu = 1$, $\eta > 0$, $\mu = \frac{\min\{\sigma_1, \sigma_3\}}{\lambda_{\max}(P)}$, $\bar{\alpha} = 0$, then, for any $\phi \in C^{n+1}$, the following inequality holds

$$\begin{aligned} &\nu D^+V(\phi, k(\phi)) + \eta \max\left\{0, D^+p\circ V_1\left(\phi, k(\phi)\right) + \mu p\circ V_1\left(\phi(0)\right)\right\} \leq \\ &\qquad - \min\{\sigma_1, \sigma_3\}|\phi(0)|^2 \leq 0. \end{aligned} \tag{39}$$

It follows that the function k in (37) is a SDF for the system (33) according to Definition 3 (i.e. Assumption 1 is satisfied in this case). It follows that all the conditions to apply Theorem 1 are satisfied for this case. Notice that, the proposed SDF k in (37) is discontinuous.

Example Let us consider a nonlinear time-delay system described by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -x_1(t) - x_2(t) + x_2^2(t)x_3(t) + x_1^2(t-\Delta)x_3(t) + x_3^3(t) - x_3(t)x_3(t-\Delta), \\ \dot{x}_3(t) &= x_2^2(t)x_3(t) + 3x_1(t) + x_2(t-\Delta) + u(t), \end{aligned} \tag{40}$$

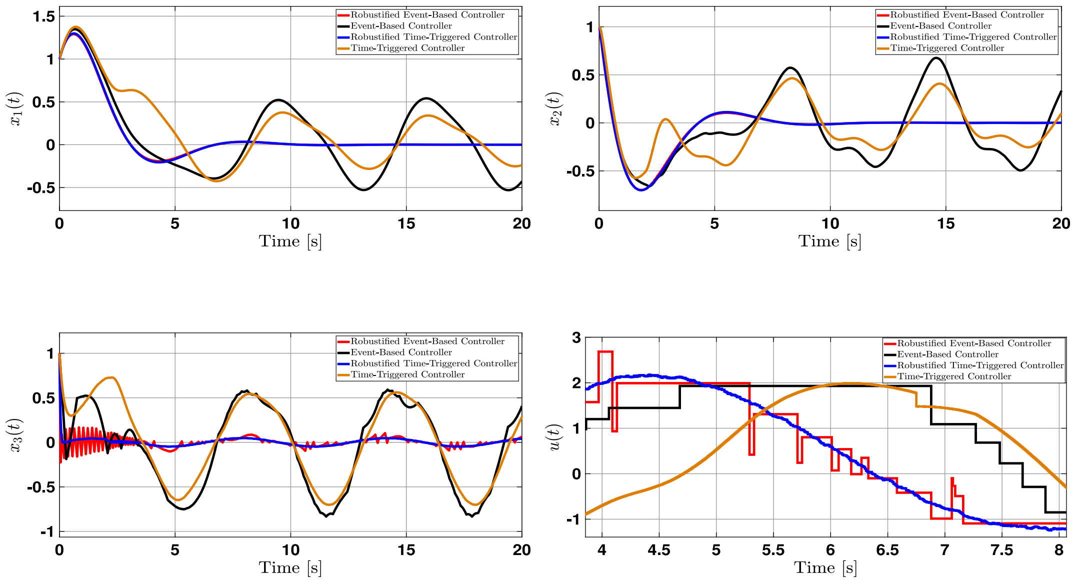

Fig. 4. In the first three panels, the evolutions of $x_i(t)$, $i = 1, 2, 3$ are reported. In the last panel, a zoom of the control input $u(t)$ is reported.

where: $x_t = \begin{pmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{pmatrix} \in C^3$, $x_{1,t}$, $x_{2,t}$, $x_{3,t} \in C$ is the system state; $u(t)$ is the control input; Δ is the involved time-delay. Notice that, system (40) is in the form (33) with the functions $f_1 : C^3 \to \mathbb{R}^2$, $f_2 : C^3 \to \mathbb{R}$ and $g : C^3 \to \mathbb{R}$ defined, for $\phi \in C^3$, as follows

$$\begin{aligned} f_1(\phi) &= \begin{pmatrix} \phi_2(0) \\ -\phi_1(0) - \phi_2(0) + \phi_2^2(0)\phi_3(0) + \phi_1^2(-\Delta)\phi_3(0) + \phi_3^3(0) - \phi_3(0)\phi_3(-\Delta) \end{pmatrix}, \\ f_2(\phi) &= \phi_2^2(0)\phi_3(0) + 3\phi_1(0) + \phi_2(-\Delta), \;\; g(\phi) = 1. \end{aligned} \tag{41}$$

Moreover, system (40) satisfies the condition (34) by choosing, for instance, $\tilde{P} = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{pmatrix}$. Indeed, as far as condition (34) is concerned, taking into account, for any $\phi \in C^3$, the following equalities/inequality hold

$$\begin{aligned} & 2\phi_1(0)^T \tilde{P} f_1(\phi) = \begin{pmatrix} 3\phi_1(0) + \phi_2(0) & \phi_1(0) + 2\phi_2(0) \end{pmatrix} f_1(\phi) = -|\phi_1(0)|^2 \\ & - |\phi_2(0)|^2 + \phi_3(0)(\phi_1(0) + 2\phi_2(0))(\phi_2^2(0) + \phi_1^2(-\Delta) + \phi_3^2(0) - \phi_3(-\Delta)) \leq \\ & - |\phi_1(0)|^2 - |\phi_2(0)|^2 + |\phi_3(0)|\sigma_2(\phi), \end{aligned} \tag{42}$$

where, in this case, $\sigma_2 : C^3 \to \mathbb{R}$ is the function defined, for $\phi \in C^3$, as follows

$$\sigma_2(\phi) = |(\phi_1(0) + 2\phi_2(0))(\phi_2^2(0) + \phi_1^2(-\Delta) + \phi_3^2(0) - \phi_3(-\Delta))|.$$

In the performed simulations: the initial state has been chosen equal to $x_0(\tau) = 1$, $\tau \in [-\Delta, 0]$; a uniform sampling period (i.e. $a = 1$) equal to $\delta = 0.01$[s] and quantizers based on the round-to-nearest method with $Q_x^3 = \{x \in \mathbb{R}^3 | \, x_i = \pm 10^{-5} j, \, i = 1, 2, 3, \, j = 0, 1, \ldots, 10^6\}$ and $Q_u = \{u \in \mathbb{R} | \, u = \pm 0.001 j, \, j = 0, 1, \ldots, 10^4\}$ have been chosen; the controller parameter in (37) is chosen as $\sigma_3 = 2$; an actuation disturbance $d(t) = \sin(t) + d_r(t)$ has been considered where $d_r(t) = \tilde{d}_r(j)$, $t \in [t_j, t_{j+1})$, $j = 0, 1, \ldots$, with $\tilde{d}_r(j)$ taken from the interval $[-0.1, 0.1]$ by emulation of the uniform probability density function; observation errors $e_1(j) = 10^{-5} \sin(t_j)$, $e_2(j) = 10^{-5} \cos(t_j)$ and $e_3(j) = 10^{-8} \cos(t_j)$, $j = 0, 1, \ldots$, have been considered; the parameters related to the robustification term (see (12)) and to the triggering mechanism (see (17)) have been chosen equal to $\omega = 10$, $\mu = 10^{-5}$, $\tilde{\mu} = 10^{-5}$, $\bar{\mu} = 10^{-5}$ and $L_D = 20$; different values of the parameter σ, related to the triggering mechanism (see (17)), have been considered. The simulation results, in the case $\sigma = 0.3$ and $\delta = 0.01$[s], are shown in Fig. 4. As for the precedent case (see Section 5.1), simulations show the very good performances of the proposed robust event-triggered solution and its efficacy in drastically reducing the effects of actuator disturbances and observation errors with a lower average frequency of control updates with respect to the robust time-triggered solution (around 12.2% of the sampling intervals). Simulations fully validate the theoretical results.

6. Conclusions

In this paper, the robust quantized sampled-data event-based (QSE) control problem for nonlinear systems with state delays has been studied. In particular, a methodology for the design of robust QSE stabilizers for nonlinear systems affected by state-delays, actuation disturbances and observation errors has been provided. The stabilization in the sample-and-hold sense theory has been used as a tool in order to prove the semi-global practical stability of the related QSE closed-loop system. The proposed theoretical results have been validated through applications concerning: (i) a single-link flexible joint robot arm with time delays; (ii) a particular class of nonlinear time-delay systems. Numerical simulations fully validates the results by showing: (i) the good performances of the proposed robust QSE controller comparable with the ones of the robust time–triggered solution, in spite of lower average frequency of control updates; (ii) the efficacy of the proposed robust QSE controller in the rejection of unknown actuation disturbances and unknown measurement errors.

CRediT authorship contribution statement

M. Di Ferdinando: Conceptualization, Investigation, Methodology, Software, Visualization, Writing – original draft, Writing – review & editing, Formal analysis. **S. Di Gennaro:** Supervision. **A. Borri:** Visualization, Investigation. **G. Pola:** Visualization. **P. Pepe:** Conceptualization, Methodology, Visualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix. Proof of Theorem 1

Firstly, two useful lemmas are provided which will be helpful for proving Theorem 1. In particular, Lemma 13 and Lemma 14 in [40] are here recalled and adapted to the notation used in the present paper.

Lemma 1 (*See Lemma 13 in [40]*). *Let Assumption 1 hold. Let $\tilde{k}$ be the function defined in (13) with given positive real ω. Let γ be an arbitrary positive real and $D = (-\gamma, \gamma)^m \subset \mathbb{R}^m$. Let α_4 be the function of class $\mathcal{K}$ defined, for $s \in \mathbb{R}^+$, as follows*

$$\alpha_4(s) = \frac{(\nu + \eta\bar{p})s^2}{4\omega},$$

where η, ν and $\bar{p}$ are the positive reals in Definition 3. Then, for any $\phi \in C^n$ and for any $d \in D$ the following inequality holds:

$$\nu D^+ V(\phi, \tilde{k}(\phi) + d) + \eta \max\{0, D^+ p \circ V_1(\phi, \tilde{k}(\phi) + d) + \mu p \circ V_1(\phi(0))\} \leq \alpha(\eta\mu e^{-\mu\Delta} p \circ \beta_1(\|\phi\|_\infty)) + \alpha_4(|d|). \tag{A.1}$$

Lemma 2 (*See Lemma 14 in [40]*). *Let Assumption 1 hold. Let the functional V_3, V_∞ and $\mathcal{D}_\infty$ as defined in points* **(f.1)**, **(f.2)** *and* **(f.3)** *of Section 3. Let α_i, $i = 1, 2$, be the functions of class $\mathcal{K}_\infty$ defined in (22). Let α_3 be a function of class $\mathcal{K}_\infty$, for $s \in \mathbb{R}^+$, as*

$$\alpha_3(s) = (I_d - \bar{\alpha})(\eta\mu e^{-\mu\Delta} p \circ \beta_1(s)),$$

where β_1 is the function of class $\mathcal{K}_\infty$ related to the smooth separability property of the functional V and $\bar{\alpha}$ is the function in Definition 3. Let γ be an arbitrary positive real and $D = (-\gamma, \gamma)^m \subset \mathbb{R}^m$. Let α_4 be the function of class $\mathcal{K}_\infty$ in Lemma 1. Then, the following conditions hold:

- **(c.1)** $\alpha_1(\|\phi\|_\infty) \leq V_\infty(\phi) \leq \alpha_2(\|\phi\|_\infty)$, $\forall \phi \in C^n$;
- **(c.2)** *the function $(\phi, u) \to \mathcal{D}_\infty(\phi, u)$ is Lipschitz on bounded subsets of $C^n \times \mathbb{R}^m$;*
- **(c.3)** $D^+ V_\infty(\phi, u) \leq \mathcal{D}_\infty(\phi, u)$, $\forall \phi \in C^n$, $\forall u \in \mathbb{R}^m$;
- **(c.4)** $\mathcal{D}_\infty\left(\phi, \tilde{k}(\phi) + d\right) \leq -\alpha_3(\|\phi\|_\infty) + \alpha_4(|d|)$, $\forall \phi \in C^n$, $\forall d \in D$.

Let:
(1) r, R, be any positive reals, $0 < r < R$;
(2) a, $\bar{\mu}$, $\tilde{\mu} \in (0, 1]$ and $\sigma \in (0, 1)$ be arbitrarily fixed;
(3) q be any positive real and $x_0 \in W_n^{1,\infty} \cap C_R^n$ satisfying $\operatorname{ess\,sup}_{\theta \in [-\Delta, 0]} \left| \frac{dx_0(\theta)}{d\theta} \right| \leq q$;
(4) e_1, e_2, E be positive reals satisfying:

$$0 < e_2 < e_1 < r < R < E, \ \alpha_1(E) > \alpha_2(R), \ \alpha_1(r) > \alpha_2(e_1); \tag{A.2}$$

(5)

$$\begin{aligned} & E_1 = E + e, \ E_2 = E_1 + 1, \ \tilde{L} = \sup_{\phi_1 \in C_{E_2}^n, \phi_2 \in C_{E_1}^n} \left| k(\phi_1) - k(\phi_2) \right|, \\ & \bar{L} = \sup_{\phi_1 \in C_E^n, \phi_2 \in C_E^n} \left| k(\phi_1) - k(\phi_2) \right|, \ L = \sup_{\phi_1 \in C_{E_1}^n, \phi_2 \in C_E^n} \left| k(\phi_1) - k(\phi_2) \right|, \\ & \omega \geq \bar{\omega} = \max\left\{ 1, \frac{(\nu + \eta\bar{p})(\bar{d} + \bar{e} + \bar{\mu} + \tilde{\mu} + \bar{L} + \tilde{L} + L)^2}{\sigma\alpha_3(e_2)} \right\}, \\ & U = \sup_{\phi \in C_{E_2}^n} \left| \tilde{k}(\phi) \right|, \ \overline{U} = U + 1 + \bar{d}. \end{aligned} \tag{A.3}$$

(6) M, L_D, L_S be positive reals such that the following inequalities hold, $\forall \phi_1, \phi_2 \in C_{E_2^n}$ and $\forall u_1, u_2 \in \mathcal{B}^m_{\overline{U}}$:

$$\begin{aligned} & |f(\phi_1, u_1)| \leq M, \\ & |S(\phi_1) - S(\phi_2)| \leq L_S \|\phi_1 - \phi_2\|_\infty, \\ & |\mathcal{D}_\infty(\phi_1, u_1) - \mathcal{D}_\infty(\phi_2, u_2)| \leq L_D(\|\phi_1 - \phi_2\|_\infty + |u_1 - u_2|); \end{aligned} \tag{A.4}$$

(7) $\tilde{q} = \max\{q, M\}$ with M the positive real in (A.4);

(8) $\beta = \omega\sigma\alpha_3(e_2) - \dfrac{(\nu + \eta\bar{p})(\bar{d} + \bar{e} + \bar{\mu} + \tilde{\mu} + \bar{L} + \tilde{L} + L)^2}{2}$;

(9) δ, μ_x, μ_u be positive reals and q_x be a state quantizer with range E_1 and error bound μ_x such that:

$$\begin{aligned} & \delta < \min\{1, \Delta\}, \quad 0 < \mu_x \leq 1, \quad 0 < \mu_u \leq 1, \quad e_2 + \delta M < e_1, \quad R + \delta M < E, \\ & \alpha_1(r) > \alpha_2(e_1) + \frac{2}{3}\frac{\beta\delta}{\omega}, \quad 2L_S\tilde{q}\delta \leq \frac{\bar{\mu}}{\omega}, \quad \frac{\beta}{3} > 2\omega L_D(2+\sigma)(2\tilde{q}\delta + \mu_u + 3\mu_x), \\ & \sup_{\tilde{x}_0,\dots,\tilde{x}_l \in \mathcal{B}^n_{E_1}} \left| S(P_{l,a,\delta}(\begin{pmatrix} q_x(\tilde{x}_0) \\ \vdots \\ q_x(\tilde{x}_l) \end{pmatrix}, w)) - S(P_{l,a,\delta}(\begin{pmatrix} \tilde{x}_0 \\ \vdots \\ \tilde{x}_l \end{pmatrix}, w)) \right| \leq \frac{\tilde{\mu}}{\omega}, \ \forall w \in \mathcal{T}_{l,a,\delta}. \end{aligned} \tag{A.5}$$

(10) q_u be an input quantizer with range U and error bound μ_u.

Let us consider a partition $\pi_{a,\delta}$. Let $B^{x+e}_S : \mathbb{N} \to \mathbb{R}^{n(l+1)}$ and $B^x_S : \mathbb{N} \to \mathbb{R}^{n(l+1)}$ be defined (recursively) as

$$\begin{aligned} & B^{x+e}_S(0) = \begin{pmatrix} \bar{x}_0(0) + \bar{e}_0(0) \\ \vdots \\ \bar{x}_0(t_{-l}) + \bar{e}_0(t_{-l}) \end{pmatrix}, \quad B^x_S(0) = \begin{pmatrix} \bar{x}_0(0) \\ \vdots \\ \bar{x}_0(t_{-l}) \end{pmatrix}, \\ & \bar{x}_0(\tau) + \bar{e}_0(\tau) = \begin{cases} x_0(\tau) + e_0(\tau) & \tau \in [-\Delta, 0] \\ x_0(-\Delta) + e_0(-\Delta) & \tau \in [t_{-l}, -\Delta], \end{cases} \\ & \bar{x}_0(\tau) = \begin{cases} x_0(\tau) & \tau \in [-\Delta, 0] \\ x_0(-\Delta) & \tau \in [t_{-l}, -\Delta] \end{cases} \\ & B^{x+e}_S(j) = \begin{pmatrix} x(t_j) + e_j(0) \\ 0_{ln\times 1} \end{pmatrix} + \begin{pmatrix} 0_{n\times ln} & 0_n \\ I_{ln} & 0_{ln\times n} \end{pmatrix} B^{x+e}_S(j-1), \\ & B^x_S(j) = \begin{pmatrix} x(t_j) \\ 0_{ln\times 1} \end{pmatrix} + \begin{pmatrix} 0_{n\times ln} & 0_n \\ I_{ln} & 0_{ln\times n} \end{pmatrix} B^x_S(j-1), \quad j = 1, \dots \end{aligned} \tag{A.6}$$

In the following, we denote with: P^x_j the function $P_{l,a,\delta}(B^x_S(j), B_{\mathcal{T}}(j))$; P^{x+e}_j the function $P_{l,a,\delta}(B^{x+e}_S(j), B_{\mathcal{T}}(j))$;

Firstly, we notice that, for any $\tilde{x} = \begin{pmatrix} \tilde{x}_0^T & \cdots & \tilde{x}_l^T \end{pmatrix}^T$, $\tilde{x}_i \in \mathcal{B}^n_E$, $i = 0, \dots, l$ and for any $\tilde{e} = \begin{pmatrix} \tilde{e}_0^T & \cdots & \tilde{e}_l^T \end{pmatrix}^T$, $\tilde{e}_i \in \mathcal{B}^n_{\bar{e}}$, $i = 0, \dots, l$, we have $|\tilde{x}_i + \tilde{e}_i| \leq E_1$, $i = 0, \dots, l$ and, consequently, for any $w \in \mathcal{T}_{l,a,\delta}$, $\|P_{l,a,\delta}(\tilde{x} + \tilde{e}, w)\|_\infty \leq E_1$. Moreover, for any $\tilde{x} = \begin{pmatrix} \tilde{x}_0^T & \cdots & \tilde{x}_l^T \end{pmatrix}^T$, $\tilde{x}_i \in \mathcal{B}^n_{E_1}$, $i = 0, \dots, l$, we have $|q_x(\tilde{x}_i)| \leq E_1 + 1 = E_2$ and, consequently, for any $w \in \mathcal{T}_{l,a,\delta}$, $\|P_{l,a,\delta}(\tilde{x}, w)\|_\infty \leq E_2$. From such considerations, it follows that $\|\mathcal{P}^{x+e}_0\|_\infty \leq E_1$ and $\|\mathcal{P}^{q_x}_0\|_\infty \leq E_2$. Then, for any $d \in \mathcal{B}^m_{\bar{d}}$, $q_u(\tilde{u}_0) + d \in \mathcal{B}^m_{\overline{U}}$. Let us consider the solution of the QSE closed-loop system (2)–(15). We show first that the solution exists in $[0, t_1]$. Otherwise, by contradiction, if the solution blows up, there exists a time $\tau \in [0, t_1)$ such that $|x(t)| < E$, $t \in [0, \tau)$, and $|x(\tau)| = E$. But, from (A.4), (A.5), for $t \in [0, \tau]$, the inequalities hold:

$$|x(t)| \leq |x_0(0)| + \int_0^t |f(x_\theta) + g(x_\theta)(q_u(\tilde{u}_0) + d(\theta))| d\theta \leq R + \delta M < E. \tag{A.7}$$

Thus, taking $t = \tau$, the absurd inequality arises $E < E$. Therefore, the solution exists in $[0, t_1]$ and, by (A.7), it follows that $x_t \in C^n_E$, $t \in [0, t_1]$. Let $W(t) = \omega V_\infty(x_t)$, $t \in [0, t_1]$, with $V_\infty : C^n \to \mathbb{R}^+$ provided in Lemma 2. Taking into account point (c.3) in Lemma 2 and Steps (6), (9), (10), for any fixed $t \in (0, t_1]$, for some $t^* \in [0, t]$, the following equalities/inequalities hold:

$$\begin{aligned} & W(t) - W(0) = \int_0^t \omega D^+ V_\infty(x_\tau, q_u(\tilde{u}_0) + d(\tau)) d\tau \leq \\ & t\left(\frac{1}{t}\int_0^t \omega \mathcal{D}_\infty(x_\tau, q_u(\tilde{u}_0) + d(\tau)) d\tau\right) = t\omega \mathcal{D}_\infty(x^*_t, q_u(\tilde{u}_0) + \tilde{d}(t^*)) = \\ & t\omega \mathcal{D}_\infty(x^*_t, q_u(\tilde{u}_0) + \tilde{d}(t^*)) - t\omega \mathcal{D}_\infty(x_0, \tilde{u}_0 + \tilde{d}(t^*)) + \\ & t\omega \mathcal{D}_\infty(x_0, \tilde{u}_0 + \tilde{d}(t^*)) - t\omega\sigma \mathcal{D}_\infty(x_0, \tilde{u}_0 + \tilde{d}(t^*)) + t\omega\sigma \mathcal{D}_\infty(x_0, \tilde{u}_0 + \tilde{d}(t^*)) \leq \\ & t\omega L_D(2\tilde{q}\delta + \mu_u) + t\omega(1-\sigma)\mathcal{D}_\infty(x_0, \tilde{u}_0 + \tilde{d}(t^*)) + t\omega\sigma \mathcal{D}_\infty(x_0, \tilde{u}_0 + \tilde{d}(t^*)), \end{aligned} \tag{A.8}$$

where, $\tilde{d}(t^*) = d(t^*)$ if $t^* < t_1$ and $\tilde{d}(t^*) = \lim_{t \to t_1^-} d(t)$ if $t^* = t_1$ and, by suitably repeating the reasoning in [8] (see, also, [41]), $\|x_{t^*} - x_0\|_\infty \leq 2\tilde{q}\delta$. Now, we notice that, $\mathcal{P}^x_0 \in C^n_E$, $\mathcal{P}^{x+e}_0 \in C^n_{E_1}$ and $\mathcal{P}^{q_x}_0 \in C^n_{E_2}$. Then, taking into account (A.4) and (A.5), the following equality/inequalities hold:

$$\begin{aligned} & |S(\mathcal{P}^{q_x}_0) - S(\mathcal{P}^{x+e}_0)| = \\ & |S(P_{l,a,\delta}(B^{q_x}_S(0), B_{\mathcal{T}}(0))) - S(P_{l,a,\delta}(B^{x+e}_S(0), B_{\mathcal{T}}(0)))| \leq \frac{\tilde{\mu}}{\omega} \\ & |S(\mathcal{P}^x_0) - S(x_0)| \leq L_S \|P^x_0 - x_0\|_\infty \leq 2L_S\tilde{q}\delta \leq \frac{\bar{\mu}}{\omega}, \end{aligned} \tag{A.9}$$

where, by suitably repeating the reasoning in [8], $\|P_0^x - x_0\|_\infty \leq 2\tilde{q}\delta$. Taking into account (A.3) and (A.9), let $v_i \in \mathcal{B}_1^m$, $i = 1, \ldots, 6$, be such that:

$$\begin{aligned} & k(\mathcal{P}_0^{q_x}) = k(\mathcal{P}_0^{x+e}) + \tilde{L}v_1,\ k(\mathcal{P}_0^{x+e}) = k(\mathcal{P}_0^x) + Lv_2,\ k(\mathcal{P}_0^x) = k(x_0) + \bar{L}v_3, \\ & S(\mathcal{P}_0^{q_x}) = S(\mathcal{P}_0^{x+e}) + \frac{\tilde{\mu}}{\omega}v_4,\ S(\mathcal{P}_0^{x+e}) = S(\mathcal{P}_0^x) + \frac{\bar{e}}{\omega}v_5,\ S(\mathcal{P}_0^x) = S(x_0) + \frac{\bar{\mu}}{\omega}v_6. \end{aligned} \tag{A.10}$$

Then, taking into account point (c.4) in Lemma 2 and (A.10), the following equalities/inequality hold:

$$\begin{aligned} & \mathcal{D}_\infty(x_0, \tilde{u}_0 + \tilde{d}(t^*)) = \\ & \mathcal{D}_\infty(x_0, \tilde{k}(\mathcal{P}_0^{q_x}) + \tilde{d}(t^*)) = \mathcal{D}_\infty(x_0, k(\mathcal{P}_0^{q_x}) - \omega S(\mathcal{P}_0^{q_x}) + \tilde{d}(t^*)) = \\ & \mathcal{D}_\infty(x_0, k(x_0) - \omega S(x_0) + \tilde{L}v_1 + Lv_2 + \bar{L}v_3 - \tilde{\mu}v_4 - \mu v_5 - \bar{\mu}v_6 + \tilde{d}(t^*)) \leq \\ & -\alpha_3(\|x_0\|_\infty) + \frac{(\nu + \eta\bar{p})(\bar{d} + \bar{e} + \bar{\mu} + \tilde{\mu} + \bar{L} + \tilde{L} + L)^2}{4\omega}. \end{aligned} \tag{A.11}$$

From (A.8), taking into account (A.5) and (A.11), for $t \in [0, t_1]$, the following inequality holds

$$\begin{aligned} & W(t) - W(0) \leq t\omega L_\mathcal{D}(2\tilde{q}\delta + \mu_u) - t\omega\sigma\alpha_3(\|x_0\|_\infty) + \\ & t\frac{(\nu + \eta\bar{p})(\bar{d} + \bar{e} + \bar{\mu} + \tilde{\mu} + \bar{L} + \tilde{L} + L)^2}{4} \leq \\ & \frac{\beta}{3}t - t\omega\sigma\alpha_3(\|x_0\|_\infty) + t\frac{(\nu + \eta\bar{p})(\bar{d} + \bar{e} + \bar{\mu} + \tilde{\mu} + \bar{L} + \tilde{L} + L)^2}{4}. \end{aligned} \tag{A.12}$$

Let us now consider the following two cases: (1) $\|x_0\|_\infty \leq e_2$; (2) $\|x_0\|_\infty > e_2$. As far as case (1) is concerned, by using again the first inequality in (A.7) and from (A.5), the following inequality holds, for any $t \in [0, t_1]$,

$$|x(t)| \leq e_2 + \delta M < e_1.$$

From point (c.1) in Lemma 2, it follows

$$W(t) \leq \omega\alpha_2(e_1),\ t \in [0, t_1].$$

As far as case (2) is concerned, we have that

$$-\beta > -\omega\sigma\alpha_3(\|x_0\|_\infty) + \frac{(\nu + \eta\bar{p})(\bar{d} + \bar{e} + \bar{\mu} + \tilde{\mu} + \bar{L} + \tilde{L} + L)^2}{4}.$$

Therefore, from (A.5), we have, for any $t \in [0, t_1]$,

$$W(t) \leq W(0) + \frac{\beta}{3}t - \beta t = W(0) - \frac{2}{3}\beta t.$$

Let us introduce the following claim, which will be proved later.

Claim 1. *The solution $x(t)$ of (2)–(15), exists in $[0, +\infty)$ and, furthermore, $x_t \in C_E^n$, $\forall t \geq 0$.*

Notice that, taking into account the control input in (15), Claim 1 and the same reasoning used in the first interval $[0, t_1]$, for any $d \in \mathcal{B}_{\bar{d}}^m$, $q_u(\tilde{u}_{i_j}) + d \in \mathcal{B}_{\bar{U}}^m$, $j = 1, \ldots$. Let $W(t) = \omega V_\infty(x_t)$, $t \in \mathbb{R}^+$. Taking into account the reasoning used in the interval $[0, t_1]$, points (c.3) in Lemma 2 and Steps (6), (9), (10), for any fixed $t \in (t_j, t_{j+1}]$, $j \geq 1$, for some $t^* \in [t_j, t]$, the following inequalities hold:

$$\begin{aligned} & W(t) - W(t_j) \leq \omega(t - t_j)\mathcal{D}_\infty(x_t^*, q_u(\tilde{u}_{i_j}) + \tilde{d}(t^*)) \leq \\ & \omega(t - t_j)\Big(\mathcal{D}_\infty(x_t^*, q_u(\tilde{u}_{i_j}) + \tilde{d}(t^*)) - \mathcal{D}_\infty(x_{t_j}, \tilde{u}_{i_j} + \tilde{d}(t^*)) + \mathcal{D}_\infty(x_{t_j}, \tilde{u}_{i_j} + \tilde{d}(t^*)) \\ & -\sigma\mathcal{D}_\infty(x_{t_j}, \tilde{u}_j + \tilde{d}(t^*)) + \sigma\mathcal{D}_\infty(x_{t_j}, \tilde{u}_j + \tilde{d}(t^*))\Big) \leq \\ & \omega(t - t_j)L_\mathcal{D}(2\tilde{q}\delta + \mu_u) + \omega(t - t_j)\Big(\mathcal{D}_\infty(x_{t_j}, \tilde{u}_{i_j} + \tilde{d}(t^*)) - \sigma\mathcal{D}_\infty(x_{t_j}, \tilde{u}_j + \tilde{d}(t^*)) \\ & +\sigma\mathcal{D}_\infty(x_{t_j}, \tilde{u}_j + \tilde{d}(t^*))\Big), \end{aligned} \tag{A.13}$$

where, $\tilde{d}(t^*) = d(t^*)$ if $t^* < t_j$ and $\tilde{d}(t^*) = \lim_{t \to t_j^-} d(t)$ if $t^* = t_j$ and, by suitably repeating the reasoning in [8] (see, also, [41]), $\|x_{t^*} - x_{t_j}\|_\infty \leq 2\tilde{q}\delta$. Taking into account Claim 1, we notice that, $\mathcal{P}_j^x \in C_E^n$, $\mathcal{P}_j^{x+e} \in C_{E_1}^n$ and $\mathcal{P}_j^{q_x} \in C_{E_2}^n$, $j = 0, 1, \ldots$. Then, taking into account (A.4) and (A.5), the following equality/inequalities hold:

$$\begin{aligned} & |S(\mathcal{P}_j^{q_x}) - S(\mathcal{P}_j^{x+e})| = \\ & |S(P_{l,a,\delta}(B_S^{q_x}(j), B_\mathcal{T}(j))) - S(P_{l,a,\delta}(B_S^{x+e}(j), B_\mathcal{T}(j)))| \leq \frac{\tilde{\mu}}{\omega} \\ & |S(\mathcal{P}_j^x) - S(x_j)| \leq L_S\|P_j^x - x_{t_j}\|_\infty \leq 2L_S\tilde{q}\delta \leq \frac{\bar{\mu}}{\omega}, \end{aligned} \tag{A.14}$$

where, by suitably repeating the reasoning in [8], $\|P_0^x - x_0\|_\infty \leq 2\tilde{q}\delta$. Taking into account (A.3) and (A.14), let $v_i \in \mathcal{B}_1^m$, $i = 1, \ldots, 6$, be such that:

$$\begin{aligned} & k(\mathcal{P}_j^{q_x}) = k(\mathcal{P}_j^{x+e}) + \tilde{L}v_1,\ k(\mathcal{P}_j^{x+e}) = k(\mathcal{P}_j^x) + Lv_2,\ k(\mathcal{P}_j^x) = k(x_{t_j}) + \bar{L}v_3, \\ & S(\mathcal{P}_j^{q_x}) = S(\mathcal{P}_j^{x+e}) + \frac{\tilde{\mu}}{\omega}v_4,\ S(\mathcal{P}_j^{x+e}) = S(\mathcal{P}_j^x) + \frac{\bar{e}}{\omega}v_5,\ S(\mathcal{P}_j^x) = S(x_{t_j}) + \frac{\bar{\mu}}{\omega}v_6. \end{aligned} \tag{A.15}$$

Then, taking into account point (c.4) in Lemma 2 and (A.15), the following equalities/inequality hold:

$$\begin{aligned}
&\mathcal{D}_\infty(x_{t_j},\tilde{u}_j+\tilde{d}(t^*)) = \\
&\mathcal{D}_\infty(x_{t_j},\tilde{k}(\mathcal{P}_j^{q_x})+\tilde{d}(t^*)) = \mathcal{D}_\infty(x_{t_j},k(\mathcal{P}_j^{q_x})-\omega S(\mathcal{P}_j^{q_x})+\tilde{d}(t^*)) = \\
&\mathcal{D}_\infty(x_{t_j},k(x_{t_j})-\omega S(x_{t_j})+\tilde{L}v_1+Lv_2+\bar{L}v_3-\tilde{\mu}v_4-\mu v_5-\bar{\mu}v_6+\tilde{d}(t^*)) \leq \\
&-\alpha_3(\|x_{t_j}\|_\infty)+\frac{(\nu+\eta\bar{p})(\bar{d}+\bar{e}+\bar{\mu}+\tilde{\mu}+\bar{L}+\tilde{L}+L)^2}{4\omega}.
\end{aligned} \tag{A.16}$$

Moreover, taking into account (15), (17) and (A.13), we have that

$$\begin{aligned}
&\omega\Big(\mathcal{D}_\infty(x_{t_j},\tilde{u}_{i_j}+\tilde{d}(t^*))-\sigma\mathcal{D}_\infty(x_{t_j},\tilde{u}_j+\tilde{d}(t^*))\Big) = \\
&\begin{cases}\omega(1-\sigma)\mathcal{D}_\infty(x_{t_j},\tilde{u}_j+\tilde{d}(t^*)) & i_j=j,\\ \omega\Big(\mathcal{D}_\infty(x_{t_j},\tilde{u}_{i_{j-1}}+\tilde{d}(t^*))-\sigma\mathcal{D}_\infty(x_{t_j},\tilde{u}_j+\tilde{d}(t^*))\Big) & i_j=i_{j-1}.\end{cases}
\end{aligned} \tag{A.17}$$

Taking into account (A.16), if $i_j = j$ (trigger), the following inequality holds:

$$\begin{aligned}
&\omega(1-\sigma)\mathcal{D}_\infty(x_{t_j},\tilde{u}_j+\tilde{d}(t^*)) \leq \\
&-\omega(1-\sigma)\alpha_3(\|x_{t_j}\|_\infty)+(1-\sigma)\frac{(\nu+\eta\bar{p})(\bar{d}+\bar{e}+\bar{\mu}+\tilde{\mu}+\bar{L}+\tilde{L}+L)^2}{4}.
\end{aligned} \tag{A.18}$$

In the case that $i_j = i_{j-1}$ (no trigger), the triggering condition (17) is false and, consequently, the following inequality holds:

$$\mathcal{D}_\infty(\mathcal{P}_j^{q_x},q_u(\tilde{u}_{i_{j-1}}))-\sigma\mathcal{D}_\infty(\mathcal{P}_j^{q_x},q_u(\tilde{u}_j)) \leq -H(\mathcal{P}_j^{q_x}). \tag{A.19}$$

For simplicity in the notation, in the following we will call with: $\Psi_1(\phi,u)$ the function $D^+p\circ V_1(\phi,u)+\mu p\circ V_1(\phi(0))$ and with $\Psi_2(\phi,d)$ the function $\left.\frac{dp}{ds}\right|_{s=V_1(\phi(0))}\left.\frac{\partial V_1}{\partial x}\right|_{x=\phi(0)} g(\phi)\,d$. Taking into account (14), the following equalities hold

$$\begin{aligned}
&\mathcal{D}_\infty(x_{t_j},\tilde{u}_{i_{j-1}}+\tilde{d}(t^*)) = \nu D^+V(x_{t_j},\tilde{u}_{i_{j-1}}+\tilde{d}(t^*))-\eta\mu V_3(x_{t_j}) \\
&\quad+\eta\max\{0,\Psi_1(x_{t_j},\tilde{u}_{i_{j-1}}+\tilde{d}(t^*))\} = \\
&\nu D^+V(x_{t_j},\tilde{u}_{i_{j-1}})+\nu\left.\frac{\partial V_1}{\partial x}\right|_{x=x(t_j)} g\left(x_{t_j}\right)\tilde{d}(t^*)-\eta\mu V_3(x_{t_j}) \\
&\quad+\eta\max\{0,\Psi_1(x_{t_j},\tilde{u}_{i_{j-1}}+\tilde{d}(t^*))\}+\eta\max\{0,\Psi_1(x_{t_j},\tilde{u}_{i_{j-1}})\} \\
&\quad-\eta\max\{0,\Psi_1(x_{t_j},\tilde{u}_{i_{j-1}})\} = \\
&\mathcal{D}_\infty(x_{t_j},\tilde{u}_{i_{j-1}})+\nu\left.\frac{\partial V_1}{\partial x}\right|_{x=x(t_j)} g\left(x_{t_j}\right)\tilde{d}(t^*) \\
&\quad+\eta\max\{0,\Psi_1(x_{t_j},\tilde{u}_{i_{j-1}})+\Psi_2(x_{t_j},\tilde{d}(t^*))\}-\eta\max\{0,\Psi_1(x_{t_j},\tilde{u}_{i_{j-1}})\}.
\end{aligned} \tag{A.20}$$

Moreover, by exploiting the same reasoning used in (A.20), we have that

$$\begin{aligned}
&\mathcal{D}_\infty(x_{t_j},\tilde{u}_j+\tilde{d}(t^*)) = \mathcal{D}_\infty(x_{t_j},\tilde{u}_j)+\nu\left.\frac{\partial V_1}{\partial x}\right|_{x=x(t_j)} g\left(x_{t_j}\right)\tilde{d}(t^*) \\
&+\eta\max\{0,\Psi_1(x_{t_j},\tilde{u}_j)+\Psi_2(x_{t_j},\tilde{d}(t^*))\}-\eta\max\{0,\Psi_1(x_{t_j},\tilde{u}_j)\}.
\end{aligned} \tag{A.21}$$

Then, taking into account (A.20) and (A.21), the following equality/inequality hold:

$$\begin{aligned}
&\omega\Big(\mathcal{D}_\infty(x_{t_j},\tilde{u}_{i_{j-1}}+\tilde{d}(t^*))-\sigma\mathcal{D}_\infty(x_{t_j},\tilde{u}_j+\tilde{d}(t^*))\Big) = \\
&\omega\Big(\mathcal{D}_\infty(x_{t_j},\tilde{u}_{i_{j-1}})-\sigma\mathcal{D}_\infty(x_{t_j},\tilde{u}_j)\Big)+\omega\nu(1-\sigma)\left.\frac{\partial V_1}{\partial x}\right|_{x=x(t_j)} g\left(x_{t_j}\right)\tilde{d}(t^*) \\
&+\omega\eta\Big(\max\{0,\Psi_1(x_{t_j},\tilde{u}_{i_{j-1}})+\Psi_2(x_{t_j},\tilde{d}(t^*))\}-\max\{0,\Psi_1(x_{t_j},\tilde{u}_{i_{j-1}})\}\Big) \\
&+\omega\sigma\eta\Big(\max\{0,\Psi_1(x_{t_j},\tilde{u}_j)\}-\max\{0,\Psi_1(x_{t_j},\tilde{u}_j)+\Psi_2(x_{t_j},\tilde{d}(t^*))\}\Big) \leq \\
&\omega\Big(\mathcal{D}_\infty(x_{t_j},\tilde{u}_{i_{j-1}})-\sigma\mathcal{D}_\infty(x_{t_j},\tilde{u}_j)\Big)+\omega\nu(1-\sigma)\left.\frac{\partial V_1}{\partial x}\right|_{x=x(t_j)} g\left(x_{t_j}\right)\tilde{d}(t^*) \\
&+4\omega(1+\sigma)\eta\bar{p}\bar{d}\left|\left.\frac{\partial V_1}{\partial x}\right|_{x=x(t_j)} g\left(x_{t_j}\right)\right|.
\end{aligned} \tag{A.22}$$

From (12) and (A.15), we notice that

$$\left.\frac{\partial V_1}{\partial x}\right|_{x=x(t_j)} g\left(x_{t_j}\right) = S(x_{t_j})^T = S(\mathcal{P}_j^{q_x})^T-\frac{\tilde{\mu}v_4^T+\bar{e}v_5^T+\bar{\mu}v_6^T}{\omega}. \tag{A.23}$$

Taking into account (A.23), from (A.22), the following inequality holds:

$$\begin{aligned}&\omega\Big(\mathcal{D}_\infty(x_{t_j},\tilde{u}_{i_{j-1}}+\tilde{d}(t^*))-\sigma\mathcal{D}_\infty(x_{t_j},\tilde{u}_j+\tilde{d}(t^*))\Big)\leq\\&\omega\Big(\mathcal{D}_\infty(x_{t_j},\tilde{u}_{i_{j-1}})-\sigma\mathcal{D}_\infty(x_{t_j},\tilde{u}_j)\Big)+\\&\omega\bigg(\bar{d}\big(\nu(1-\sigma)+4\eta\bar{p}(1+\sigma)\big)\Big(|S(\mathcal{P}_j^{q_x})|+\frac{\tilde{\mu}+\bar{e}+\bar{\mu}}{\omega}\Big)\bigg).\end{aligned}\tag{A.24}$$

Then, taking into account (18), (A.4) and (A.24), the following inequalities hold:

$$\begin{aligned}&\omega\ \Big(\ \mathcal{D}_\infty(x_{t_j},\tilde{u}_{i_{j-1}}+\tilde{d}(t^*))-\sigma\mathcal{D}_\infty(x_{t_j},\tilde{u}_j+\tilde{d}(t^*))\ \Big)\leq\\&\quad\omega\Big(\mathcal{D}_\infty(x_{t_j},\tilde{u}_{i_{j-1}})-\sigma\mathcal{D}_\infty(x_{t_j},\tilde{u}_j)\Big)+\\&\quad\omega\bigg(\bar{d}\big(\nu(1-\sigma)+4\eta\bar{p}(1+\sigma)\big)\Big(|S(\mathcal{P}_j^{q_x})|+\frac{\tilde{\mu}+\bar{e}+\bar{\mu}}{\omega}\Big)\bigg)\\&\quad+\omega\ \Big(\ \mathcal{D}_\infty(\mathcal{P}_j^{q_x},q_u(\tilde{u}_{i_{j-1}}))-\sigma\mathcal{D}_\infty(\mathcal{P}_j^{q_x},q_u(\tilde{u}_j))\\&\quad-\mathcal{D}_\infty(\mathcal{P}_j^{q_x},q_u(\tilde{u}_{i_{j-1}}))+\sigma\mathcal{D}_\infty(\mathcal{P}_j^{q_x},q_u(\tilde{u}_j))\\&\quad+\mathcal{D}_\infty(\mathcal{P}_j^{q_x},\tilde{u}_{i_{j-1}})-\mathcal{D}_\infty(\mathcal{P}_j^{q_x},\tilde{u}_{i_{j-1}})+\sigma\mathcal{D}_\infty(\mathcal{P}_j^{q_x},\tilde{u}_j)-\sigma\mathcal{D}_\infty(\mathcal{P}_j^{q_x},\tilde{u}_j)\ \Big)\leq\\&\omega\Big(\mathcal{D}_\infty(x_{t_j},\tilde{u}_{i_{j-1}})-\sigma\mathcal{D}_\infty(x_{t_j},\tilde{u}_j)\Big)-3\omega(1+\sigma)L_{\mathcal{D}}\bar{e}\\&\quad+\omega\ \Big(\ -\mathcal{D}_\infty(\mathcal{P}_j^{q_x},q_u(\tilde{u}_{i_{j-1}}))+\sigma\mathcal{D}_\infty(\mathcal{P}_j^{q_x},q_u(\tilde{u}_j))+\mathcal{D}_\infty(\mathcal{P}_j^{q_x},\tilde{u}_{i_{j-1}})\\&\quad-\mathcal{D}_\infty(\mathcal{P}_j^{q_x},\tilde{u}_{i_{j-1}})+\sigma\mathcal{D}_\infty(\mathcal{P}_j^{q_x},\tilde{u}_j)-\sigma\mathcal{D}_\infty(\mathcal{P}_j^{q_x},\tilde{u}_j)\ \Big)\leq\\&-3\omega(1+\sigma)L_{\mathcal{D}}\bar{e}+\omega(1+\sigma)L_{\mathcal{D}}\mu_u+\omega(1+\sigma)L_{\mathcal{D}}\|\mathcal{P}_j^{q_x}-x_{t_j}\|_\infty\leq\\&\omega(1+\sigma)L_{\mathcal{D}}\|\mathcal{P}_j^{q_x}-\mathcal{P}_j^{x+e}+\mathcal{P}_j^{x+e}-\mathcal{P}_j^{x}+\mathcal{P}_j^{x}-x_{t_j}\|_\infty\\&\quad-3\omega(1+\sigma)L_{\mathcal{D}}\bar{e}+\omega(1+\sigma)L_{\mathcal{D}}\mu_u\leq\\&-3\omega(1+\sigma)L_{\mathcal{D}}\bar{e}+\omega(1+\sigma)L_{\mathcal{D}}\mu_u+\omega(1+\sigma)L_{\mathcal{D}}(2\tilde{q}\delta+3\mu_x+3\bar{e})\leq\\&\omega(1+\sigma)L_{\mathcal{D}}(\mu_u+2\tilde{q}\delta+3\mu_x).\end{aligned}\tag{A.25}$$

Then, taking into account (A.19), (A.25), from (A.17), we have that, for $j\geq 1$, the following inequality holds:

$$\begin{aligned}&\omega\Big(\mathcal{D}_\infty(x_{t_j},\tilde{u}_{i_j}+\tilde{d}(t^*))-\sigma\mathcal{D}_\infty(x_{t_j},\tilde{u}_j+\tilde{d}(t^*))\Big)\leq\\&\omega(1+\sigma)L_{\mathcal{D}}(\mu_u+2\tilde{q}\delta+3\mu_x)+(1-\sigma)\frac{(\nu+\eta\bar{p})(\bar{d}+\bar{e}+\bar{\mu}+\tilde{\mu}+\bar{L}+\tilde{L}+L)^2}{4}.\end{aligned}\tag{A.26}$$

From (A.13), and taking into account (A.5), (A.26), for $t\in[t_j,t_{j+1}]$, $j\geq 1$, the following inequality holds

$$\begin{aligned}W(t)&\leq W(t_j)+(t-t_j)\frac{\beta}{3}-(t-t_j)\sigma\omega\alpha_3(\|x_{t_j}\|_\infty)\\&\quad+(t-t_j)\frac{(\nu+\eta\bar{p})(\bar{d}+\bar{e}+\bar{\mu}+\tilde{\mu}+\bar{L}+\tilde{L}+L)^2}{4}.\end{aligned}\tag{A.27}$$

Then, taking into account of both cases $\|x_{t_j}\|_\infty\leq e_2$ and $\|x_{t_j}\|_\infty> e_2$ (see cases (1) and (2) in $[0,t_1]$), for any $t\in[t_j,\ t_{j+1}]$, $j=0,1,\ldots$, we obtain:

$$\begin{aligned}W(t)\leq&(W(t_j)-\frac{2}{3}\beta(t-t_j))H(\|x_{t_j}\|_\infty-e_2)\\&+\omega\alpha_2(e_1)H_0(e_2-\|x_{t_j}\|_\infty).\end{aligned}\tag{A.28}$$

The symbols H_0 and H denote Heaviside functions defined, for $s\in\mathbb{R}$, as follows: $H_0(s)=1$ if $s\geq 0$, $H_0(s)=0$ if $s<0$; $H(s)=1$ if $s>0$, $H(s)=0$ if $s\leq 0$.

Notice that, by induction reasoning with (A.28), for any integer $j\geq 0$, the inequality holds $W(t_j)\leq\omega\alpha_2(R)$. From here on, by suitably exploiting (A.28), the same steps used in the proof of Theorem 5.3 in [41] can be properly repeated, in order to prove that the solution $x(t)$ of the closed-loop system (2)–(15), exists for all $t\in\mathbb{R}^+$ and, furthermore, satisfies $x_t\in\mathcal{C}_E^n$, $\forall t\in\mathbb{R}^+$ (Claim 1 holds true) and $x_t\in\mathcal{C}_r^n$, $\forall t\geq T$, with

$$T=\frac{3\omega\alpha_2(R)}{\beta a}+1.\tag{A.29}$$

The reader can refer to steps from (5.15) to (5.23) in [41] with $k_2=[\frac{3\omega\alpha_2(R)}{\beta a\delta}]+1$. The proof of the theorem is complete.

References

[1] X. Han, Y. Ma, Sampled-data robust H_∞ control for T-S fuzzy time-delay systems with state quantization, Int. J. Control Autom. Syst. 17 (2019) 46–56.
[2] D. Liberzon, Quantization, time delays, and nonlinear stabilization, IEEE Trans. Automat. Control 51 (2006) 1190–1195.
[3] C. Liu, D. Dengping, Sampled-data control of nonlinear systems with quantization, Math. Prob. Eng. (2015) http://dx.doi.org/10.1155/2015/314090.
[4] D. Nesic, D. Liberzon, A unified framework for design and analysis of networked and quantized control systems, IEEE Trans. Automat. Control 54 (2009) 732–747.
[5] G. Song, J. Lam, S. Xu, Quantized feedback stabilization of continuous time-delay systems subject to actuator saturation, Nonlinear Anal. Hybrid Syst. 30 (2018) 1–13.

[6] N. Sun, Y. Zhang, G. Song, T. Li, Design of observer-based quantized control for nonlinear time-delay systems, Trans. Inst. Meas. Control 40 (2018) 4220–4232.
[7] H. Zhang, G. Feng, H. Yan, Q. Chen, Sampled-data control of nonlinear networked systems with time-delay and quantization, Internat. J. Robust Nonlinear Control 26 (2016) 919–933.
[8] M. Di Ferdinando, P. Pepe, A. Borri, On practical stability preservation under fast sampling and accurate quantization of feedbacks for nonlinear time-delay systems, IEEE Trans. Automat. Control 66 (2021) 314–321.
[9] W.P.M.H. Heemels, K.H. Johansson, P. Tabuada, An introduction to event-triggered and self-triggered control, in: 51st IEEE Conference on Decision and Control, CDC, 2012, pp. 3270–3285.
[10] P. Tabuada, Event-triggered real-time scheduling of stabilizing control tasks, IEEE Trans. Automat. Control 52 (2007) 1680–1685.
[11] J. Sun, J. Yang, W.X. Zheng, S. Li, Periodic event-triggered control for a class of nonminimum-phase nonlinear systems using dynamic triggering mechanism, IEEE Trans. Circuits Syst. I 69 (2022) 1302–1311.
[12] A. Anta, P. Tabuada, Exploiting isochrony in self-triggered control, IEEE Trans. Automat. Control 57 (2011) 950–962.
[13] M. Abdelrahim, R. Postoyan, J. Daafouz, D. Nesic, Stabilization of nonlinear systems using event-triggered output feedback controllers, IEEE Trans. Automat. Control 61 (2016) 2682–2687.
[14] A. Borri, P. Pepe, I. Di Loreto, M. Di Ferdinando, Finite-dimensional periodic event-triggered control of nonlinear time-delay systems with an application to the artificial pancreas, IEEE Control Syst. Lett. 5 (2021) 31–36.
[15] R. Dou, Q. Ling, Model-based periodic event-triggered control strategy to stabilize a scalar nonlinear system, IEEE Trans. Syst. Man Cybern. A 51 (2021) 5322–5335.
[16] A. Fu, J. Qiao, Periodic decentralized event-triggered control for nonlinear systems with asynchronous update and dynamic quantization, Nonlinear Dynam. 109 (2022) 877–890.
[17] M. Hertneck, S. Linsenmayer, F. Allgöwer, Model-based nonlinear periodic event-triggered control for continuous-time systems with sampled-data prediction, in: 2020 European Control Conference, ECC, 2020, pp. 1814–1819.
[18] S. Hu, D. Yue, X. Xie, Y. Ma, X. Yin, Stabilization of neural-network-based control systems via event-triggered control with nonperiodic sampled data, IEEE Trans. Neural Netw. Learn. Syst. 29 (2018) 573–585.
[19] P. Pepe, On stability preservation under sampling and approximation of feedbacks for retarded systems, SIAM J. Control Optim. 54 (2016) 1895–1918.
[20] R. Postoyan, P. Tabuada, D. Nešić, A. Anta, A framework for the event-triggered stabilization of nonlinear systems, IEEE Trans. Automat. Control 60 (2014) 982–996.
[21] A. Seuret, C. Prieur, N. Marchand, Stability of non-linear systems by means of event-triggered sampling algorithms, IMA J. Math. Control Inform. 31 (2014) 415–433.
[22] W. Wang, R. Postoyan, D. Nešić, W.P.M.H. Heemels, Periodic event-triggered control for nonlinear networked control systems, IEEE Trans. Automat. Control 65 (2019) 620–635.
[23] H. Yu, T. Chen, Periodic event-triggered networked control systems subject to large transmission delays, IEEE Trans. Automat. Control 68 (2021) 63–79.
[24] P. Zhang, T. Liu, Z.-P. Jiang, Event-triggered stabilization of a class of nonlinear time-delay systems, IEEE Trans. Automat. Control 66 (2021) 421–428.
[25] M. Zhao, S. Zheng, C.K. Ahn, X. Zong, C.K. Zhang, Periodic event-triggered control with multisource disturbances and quantized states, Internat. J. Robust Nonlinear Control 31 (2021) 5404–5426.
[26] A. Borri, P. Pepe, Event-triggered control of nonlinear systems with time-varying state delays, IEEE Trans. Automat. Control 66 (2021) 2846–2853.
[27] Y. Cao, S. Wang, Z. Guo, T. Huang, S. Wen, Synchronization of memoristive neural networks with leakage delay and parameters mismatch via event-triggered control, Neural Netw. 119 (2019) 178–189.
[28] J. Chen, B. Chen, Z. Zeng, P. Jiang, Event-triggered synchronization strategy for multiple neural networks with time delay, IEEE Trans. Cybern. 50 (2020) 3271–3280.
[29] Y. Dong, J.-W. Chen, J.-G. Xian, Event-triggered control for finitetime lag synchronisation of time-delayed complex networks, IET Control Theory Appl. 12 (2018) 1916–1923.
[30] S. Durand, N. Marchand, J.F. Guerrero-Castellanos, Event-based stabilization of nonlinear time-delay systems, in: Proc. 19th IFAC World Congr., 2014, pp. 6953–6958.
[31] Z. Fei, C. Guan, H. Gao, Exponential synchronization of networked chaotic delayed neural network by a hybrid event trigger scheme, IEEE Trans. Neural Netw. Learn. Syst. 29 (2017) 2558–2567.
[32] K. Zhan, B. Gharesifard, E. Braverman, Event-triggered control for nonlinear time-delay systems, IEEE Trans. Automat. Control 67 (2022) 1031–1037.
[33] K. Zhang, B. Gharesifard, Excluding Zeno behaviour in event-triggered time-delay systems by impulsive controls, arXiv:1912.02396.
[34] Q. Zhu, Stabilization of stochastic nonlinear delay systems with exogenous disturbances and the event-triggered feedback control, IEEE Trans. Automat. Control 64 (2019) 3764–3771.
[35] F.H. Clarke, Discontinuous feedback and nonlinear systems, in: Plenary Lecture at IFAC Conference on Nonlinear Control Systems, 2010.
[36] F.H. Clarke, Y.S. Ledyaev, E.D. Sontag, A.I. Subbotin, Asymptotic controllability implies feedback stabilization, IEEE Trans. Automat. Control 42 (1997) 1394–1407.
[37] M. Di Ferdinando, P. Pepe, S. Di Gennaro, On semi–global exponential stability under sampling for locally Lipschitz time–delay systems, IEEE Trans. Automat. Control http://dx.doi.org/10.1109/TAC.2022.3152726.
[38] M. Di Ferdinando, P. Pepe, S. Di Gennaro, A new approach to the design of sampled-data dynamic output feedback stabilizers, IEEE Trans. Automat. Control 67 (2022) 1038–1045.
[39] M. Di Ferdinando, P. Pepe, On emulation of observer-based stabilizers for nonlinear systems, in: 56th IEEE Conference on Decision and Control, 2017, pp. 6738–6743.
[40] M. Di Ferdinando, P. Pepe, Robustification of sample-and-hold stabilizers for control-affine time-delay systems, Automatica 83 (2017) 141–154.
[41] P. Pepe, Stabilization in the sample-and-hold sense of nonlinear retarded systems, SIAM J. Control Optim. 52 (2014) 3053–3077.
[42] P. Pepe, Robustification of nonlinear stabilizers in the sample-and-hold sense, J. Franklin Inst. B 42 (2015) 4107–4128.
[43] P. Pepe, On Lyapunov-Krasovskii functionals under caratheodory conditions, Automatica J. IFAC 43 (2007) 701–706.
[44] J.K. Hale, S.M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer Verlag, New York, 1993.
[45] V. Kolmanovskii, A. Myshkis, Introduction to the Theory and Applications of Functional Differential Equations, Kluwer Academic, Dordrecht, 1999.
[46] P. Pepe, On control Lyapunov-Razumikhin functions, nonconstant delays, nonsmooth feedbacks, and nonlinear sampled-data stabilization, IEEE Trans. Automat. Control 62 (2017) 5604–5619.
[47] A.V. Kim, On the Lyapunov's functionals method for systems with delays, Nonlinear Anal. 28 (1997) 673–687.
[48] P. Pepe, H. Ito, On saturation, discontinuities and delays, in iISS and ISS feedback control redesign, IEEE Trans. Automat. Control 57 (2012) 1125–1140.
[49] Z. Artstein, Stabilization with relaxed controls, Nonlinear Anal. TMA 7 (1983) 1163–1173.
[50] M. Di Ferdinando, S. Di Gennaro, P. Pepe, On Sontag's formula for the sampled-data observer-based stabilization of nonlinear time-delay systems, Automatica 153 (2023) 111052.
[51] E.D. Sontag, A ''universal'' construction of Artstein's theorem on nonlinear stabilization, Systems Control Lett. 13 (1989) 117–123.
[52] R.M. Esfanjani, S.K.Y. Nikravesh, Stabilising predictive control of non-linear time-delay systems using control Lyapunov-Krasovskii functionals, IET Control Theory Appl. 3 (2009) 1395–1400.

[53] K. Gu, V.L. Kharitonov, J. Chen, Stability of Time Delay Systems, Birkhauser, Boston, 2003.
[54] M. Jankovic, Extension of control Lyapunov functions to time-delay systems, in: 39th IEEE CDC, 2000, pp. 4403–4408.
[55] V.L. Kharitonov, Time-Delay Systems, Lyapunov Functionals and Matrices, 2013.
[56] S.I. Niculescu, Delay effects on stability, a robust control approach, Lecture Notes in Control and Inform. Sci. 269 (2001).
[57] Y. Chang, M. Wu, Robust tracking control for a class of flexible-joint time-delay robots using only position measurements, Internat. J. Systems Sci. 47 (2016) 3336–3349.
[58] Y.S. Ledyaev, E.D. Sontag, A Lyapunov characterization of robust stabilization, Nonlinear Anal. TMA 37 (1999) 813–840.
[59] M. Malisoff, E.D. Sontag, Asymptotic controllability and input-to-state stabilization: the effect of actuator errors, in: Optimal Control, Stabilization and Nonsmooth Analysis, in: Lecture Notes in Control and Inform. Sci., vol. 301, 2004, pp. 155–171.
[60] E.D. Sontag, Smooth stabilization implies coprime factorization, IEEE Trans. Automat. Control 34 (1989) 435–443.
[61] E.D. Sontag, Input to State Stability: Basic Concepts and Results, Springer Berlin Heidelberg, 2008, pp. 163–220.
[62] A.S. Bazanella, P.V. Kokotovic, A.S. e Silva, A dynamic extension for L_gV controllers, IEEE Trans. Automat. Control 44 (1999) 588–592.
[63] M. Krstic, J. Sun, P.V. Kokotovic, Robust control of nonlinear systems with input unmodeled dynamics, IEEE Trans. Automat. Control 41 (1996) 913–920.
[64] S. Kolathaya, J. Reher, A. Hereid, A.D. Ames, Input to state stabilizing control Lyapunov functions for robust bipedal robotic locomotion, in: 2018 Annual American Control Conference, ACC, 2018, pp. 2224–2230.
[65] N. Marchand, S. Durand, J.F.G. Castellanos, A general formula for event-based stabilization of nonlinear systems, IEEE Trans. Automat. Control 58 (2013) 1332–1337.
[66] Z.P. Jiang, T.F. Liu, A survey of recent results in quantized and event-based nonlinear control, Int. J. Autom. Comput. 12 (2015) 455–466.
[67] M. Spong, M. Vidyasagar, Robot Dynamics and Control, Wiley, New York, NY, 1989.