Ann. Inst. H. Poincaré
Anal. Non Linéaire 39 (2022), 171–200
DOI 10.4171/AIHPC/5

Global existence of weak solutions to the Navier–Stokes–Korteweg equations

Paolo Antonelli and Stefano Spirito

Abstract. In this paper we consider the Navier–Stokes–Korteweg equations for a viscous compressible fluid with capillarity effects in three space dimensions. We prove global existence of finite energy weak solutions for large initial data. Contrary to previous results regarding this system, vacuum regions are considered in the definition of weak solutions and no additional damping terms are considered. The convergence of the approximating solutions is obtained by introducing suitable truncations of the velocity field and the mass density at different scales in the momentum equations and use only the a priori bounds obtained by the energy and the Bresch–Desjardins entropy. Moreover, the approximating solutions enjoy only a limited amount of regularity, and the derivation of the truncations of the velocity and the density is performed by a suitable regularization procedure.

1. Introduction

The aim of this paper is to prove global existence of finite energy weak solutions of the following Navier–Stokes–Korteweg system in $(0,T)\times\mathbb{T}^3$:

$$\partial_t\rho+\operatorname{div}(\rho u)=0,\qquad \rho\geq 0, \tag{1.1}$$

$$\partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)+\nabla\rho^\gamma-\operatorname{div}(\rho Du)-\rho\nabla\Delta\rho=0, \tag{1.2}$$

with initial data

$$\begin{aligned}\rho(0,x)&=\rho^0(x),\\ (\rho u)(0,x)&=\rho^0(x)u^0(x).\end{aligned} \tag{1.3}$$

Here, $\mathbb{T}^3$ denotes the three-dimensional flat torus, the function ρ represents the density of the fluid and the three-dimensional vector u is the velocity field.

More generally, the class of Navier–Stokes–Korteweg systems denotes a family of compressible viscous capillary fluids whose general form is determined by

$$\begin{aligned}&\partial_t\rho+\operatorname{div}(\rho u)=0,\\ &\partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)+\nabla p=\operatorname{div}\mathbb{S}+\operatorname{div}\mathbb{K},\end{aligned} \tag{1.4}$$

2020 Mathematics Subject Classification. Primary 35Q35; Secondary 35D05, 76N10.
Keywords. Compressible fluids, Navier–Stokes–Korteweg, capillarity, vacuum, existence.

where

$$\mathbb{S} = h(\rho)Du + g(\rho)\operatorname{div} u\mathbb{I} \tag{1.5}$$

is the viscosity stress tensor and

$$\mathbb{K} = \Big(\rho \operatorname{div}(k(\rho)\nabla\rho) - \frac{1}{2}(\rho k'(\rho) - k(\rho))|\nabla\rho|^2\Big)\mathbb{I} - k(\rho)\nabla\rho \otimes \nabla\rho \tag{1.6}$$

denotes the capillarity term. Here h and g are the viscosity coefficients satisfying

$$h \geq 0, \quad h + 3g \geq 0,$$

and k is the capillarity coefficient.

System (1.1)–(1.2) can be recast from (1.4)–(1.6) by choosing $k(\rho) = 1$, $h(\rho) = \rho$ and $g(\rho) = 0$. The tensor $\mathbb{K}$ is called the Korteweg tensor ([31]) and describes capillary effects in a fluid. It was rigorously derived in [21]; see also [29] for an alternative approach that is not based on the concept of interstitial work. Moreover, Korteweg tensors not only describe capillarity effects, but also may determine admissibility criteria in liquid–vapour phase transitions ([8, 26]).

Similar systems also appear in other contexts: for example, when $\gamma = 2$ then the two-dimensional version of (1.1)–(1.2) is the shallow water model studied in [11]; see also [41] for a derivation of an augmented model with drag forces.

In [11] the authors prove existence of arbitrarily large, global-in-time finite energy weak solutions to (1.1)–(1.2) by considering test functions of the form $\rho\phi$, with ϕ smooth and compactly supported. Roughly speaking, this is somehow equivalent to considering test functions that are supported where the mass density is positive. In the present paper we improve the result in [11] by removing the requirement on the test functions and by considering a more natural definition of weak solutions; see Definition 2.1.

Our result is achieved by considering a suitable approximate system for (1.1)–(1.2), namely (1.10), and by using a truncation argument in order to infer sufficient convergence towards global-in-time finite energy weak solutions to (1.1)–(1.2).

Let us discuss some of the mathematical difficulties in studying (1.1)–(1.2) and present some existing results. Local existence of smooth solutions and global existence with small initial perturbations of the constant solution $(\rho, u) = (1, 0)$ were obtained in [27, 28] by using a fixed point argument. Regarding weak solutions, the analysis presents various mathematical difficulties due to the presence of vacuum regions, namely the set where the mass density vanishes. Indeed, the natural bounds, for example given by the total energy of the system, yield a control on the velocity field only where the mass density is positive. For this reason in [11] the authors provide a global existence result for weak solutions to (1.1)–(1.2) by considering test functions of the form $\rho\phi$. On the other hand, in [11] the authors introduce a new entropy, nowadays known as BD entropy, that gives suitable Sobolev control on the mass density and hence on the Korteweg tensor. The analysis of the BD entropy was then generalized in the literature (see for instance [12]) to other systems and it can be seen to hold true for more general viscous stress tensors whose coefficients satisfy the relation

$$g(\rho) = \rho h'(\rho) - h(\rho). \tag{1.7}$$

The BD entropy estimate was also exploited in [30] to study the quantum Navier–Stokes equations, namely system (1.4) with $h(\rho) = \rho$, $g(\rho) = 0$, $k(\rho) = 1/\rho$. In this paper the authors show the existence of global-in-time finite energy weak solutions in the same framework as in [11]. Roughly speaking, BD entropy can be regarded as an energy estimate for the auxiliary system satisfied by the unknowns (ρ, w), where w is an auxiliary velocity field defined by

$$w = u + c\nabla \log \rho, \tag{1.8}$$

for some suitable constant $c > 0$; see also [32].

Some of the mathematical problems arising in the study of (1.1)–(1.2) are also shared with the Navier–Stokes equations with degenerate viscosity, i.e., system (1.4) with $k(\rho) = 0$. Also in this case, the analysis of weak solutions with vacuum region is of great importance. Indeed, it was shown in [38] that, while the Navier–Stokes equations with constant viscosity are ill posed for initial data with vacuum, the equations with density-dependent viscosity coefficients are better behaved. The interest for such models can also be motivated by the fact that, in their derivation from the Boltzmann equations, the viscosity depends on the temperature. Thus, for isentropic flow it is natural to translate this into a dependence on ρ. In the context of finite energy weak solutions, the degeneracy of the viscosity coefficient also prevents satisfactory bounds on the gradient of the velocity inferred from the energy dissipation functional. Indeed, for arbitrary solutions only a weaker bound is available; see for example the discussion related to identity (2.3) in Definition 2.1. This was noticed in [34] for the one-dimensional problem and in [25] for spherically symmetric solutions; see also [35, Rem. 1.5] and the discussion in [40]. Regarding the existence of weak solutions, important progress was made at around the same time in [35] and [43]. In those papers the authors, by using different strategies, showed the global existence of finite energy weak solutions to the compressible barotropic Navier–Stokes equations with degenerate viscosity. One of the main ideas used there consists in obtaining suitable compactness properties for the sequence of approximating solutions by using the Mellet–Vasseur estimate ([42]). Indeed, this further bound improves integrability for the quantity $\sqrt{\rho}u$, yielding compactness for the sequence of approximating solutions. In the case of Navier–Stokes–Korteweg system (1.4), in general it is not possible to infer a Mellet–Vasseur-type estimate. However, in the special case when the viscosity and capillarity coefficients satisfy both the relations (1.7) and

$$\rho k(\rho) = h'(\rho)^2, \tag{1.9}$$

then by choosing a specific value for the constant c in (1.8) we see that (ρ, w) satisfies a Navier–Stokes system – hence, the Korteweg tensor vanishes and it is possible to infer a Mellet–Vasseur-type estimate on (ρ, w). This, together with the bounds provided by the BD entropy, yields sufficient compactness for the unknowns. In [4, 5] this strategy was adopted in order to prove global existence of finite energy weak solutions to the quantum Navier–Stokes system with the standard notion of weak solutions;, see also [14, 33, 40].

Let us stress that for system (1.1)–(1.2) the relation (1.9) does not hold true and consequently it is not possible to derive a Mellet–Vasseur-type estimate. For this reason in this paper we adopt a different strategy. We only exploit the bounds from the energy and BD entropy estimates; the global existence is then proved by considering a suitable truncation and regularization argument for both the velocity field and the mass density, in the spirit of DiPerna–Lions ([17]) for linear continuity equations, which is also exploited in [33]. More precisely, we are going to use two different truncations, for the velocity field and the mass density, performed at different scales, which at the end will be optimized in order to prove the convergence of both the third-order term and the convective term. For a formal explanation of the main idea we refer to [6].

In order to construct approximate solutions to (1.1)–(1.2) we consider the following approximation system:

$$\begin{aligned} &\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0, \\ &\partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) - \operatorname{div}(\rho_\varepsilon D u_\varepsilon) + \nabla \rho_\varepsilon^\gamma + \varepsilon \rho_\varepsilon |u_\varepsilon|^2 u_\varepsilon + \varepsilon u_\varepsilon \\ &\qquad = \rho_\varepsilon \nabla \Delta \rho_\varepsilon + \varepsilon \rho_\varepsilon \nabla \Big(\frac{\Delta \sqrt{\rho_\varepsilon}}{\sqrt{\rho_\varepsilon}} \Big). \end{aligned} \tag{1.10}$$

Notice that following the argument in [44] and [10] it is possible to prove global existence of weak solutions. Unfortunately, due to the limited amount of regularity, see Definition 3.1, it is not possible to justify the truncations of the velocity field u_ε and the density ρ_ε. In this regard, we perform suitable regularization of the weak solutions of (1.10) which allows us to justify the formal argument in [6]; see the proof of Theorem 3.2 for more details. Roughly speaking, to gain regularity in the velocity we truncate it close to vacuum and we derive an equation for the regularized velocity. This produces several errors in the equation. In order to control the one involving the third order we need a further truncation of the density at infinity. We conclude by pointing out that it would be interesting to provide an approximating system as in [4, 35], as it would provide smooth approximating solutions for system (1.1)–(1.2).

Finally, we give a brief account of the state of art of the general system (1.4)–(1.6). In the case $\kappa = 0$, (1.4) reduces to the system of compressible Navier–Stokes equations. When the viscosity coefficient $h(\rho)$ is chosen degenerating in the vacuum region $\{\rho = 0\}$ the Lions–Feireisl theory ([23, 37]) and the recent approach in [13] cannot be used since they rely on the Sobolev bound of the velocity field. As already mentioned, finite energy weak solutions are studied in [14, 33, 35, 43]. Well-posedness of regular solutions with vacuum are also studied; see [45, 46] and also [36] where the shallow water equations are considered.

When the viscosity $\nu = 0$, system (1.4) is called Euler–Korteweg. In [9], local well-posedness for smooth, small perturbations of the reference solution $\rho = 1, u = 0$ has been proved, while in [7] the result was extended to global irrotational solutions in the same framework. Moreover, when $k(\rho) = 1/\rho$ system (1.4) is called the quantum hydrodynamic system (QHD) and arises for example in the description of quantum fluids. The global existence of finite energy weak solutions for the QHD system has been proved in [2, 3] without restrictions on the regularity or the size of the initial data. Non-uniqueness results by using convex integration methods have been proved in [18]. Relative entropy methods to study singular limits for equations (1.4)–(1.6) have been exploited in [12, 15, 18, 20, 24]; in particular we mention the incompressible limit in [1] in the quantum case, the quasineutral limit in [19] for the constant capillarity case and the vanishing viscosity limit in [12]. The analysis of the long time behaviour for the isothermal quantum Navier–Stokes equations has been performed in [16]. In [39] the authors, by using a strategy similar to [35], study the existence of global-in-time finite energy weak solutions to the compressible primitive equations with degenerate viscosity.

Organization of the paper

The paper is organized as follows. In Section 2 we fix the notation, give the precise definition of weak solutions of (1.1)–(1.2) and we recall some of the main tools used in the proofs. In Section 3 we give the definition of weak solutions of the approximating system and we prove the truncated formulation of the momentum equation. In Section 4 we prove Theorem 2.3.

2. Preliminaries

2.1. Notation

Let $\mathbb{T}^3$ be the three-dimensional flat torus $[0,1]^3$ and the space of periodic smooth functions with values in $\mathbb{R}^d$ compactly supported in $[0,T)\times\mathbb{T}^3$ will be $C_c^\infty((0,T)\times\mathbb{T}^3;\mathbb{R}^d)$. We will denote by $L^p(\mathbb{T}^3)$ the standard Lebesgue spaces and by $\|\cdot\|_{L^p}$ their norm. The Sobolev space of functions with k distributional derivatives in $L^p(\mathbb{T}^3)$ is $W^{k,p}(\mathbb{T}^3)$ and in the case $p=2$ we will write $H^k(\mathbb{T}^3)$. The spaces $W^{-k,p}(\mathbb{T}^3)$ and $H^{-k}(\mathbb{T}^3)$ denote the dual spaces of $W^{k,p'}(\mathbb{T}^3)$ and $H^k(\mathbb{T}^3)$ where p' is the Hölder conjugate of p. Given a Banach space X we use the classical Bochner space for time-dependent functions with values in X, namely $L^p(0,T;X)$, $W^{k,p}(0,T;X)$ and $W^{-k,p}(0,T;X)$ and when $X=L^p(\Omega)$, the norm of the space $L^q(0,T;L^p(\Omega))$ is denoted by $\|\cdot\|_{L^q_tL^p_x}$. Then the space $C(0,T;X_w)$ is the space of continuous functions with values in the space X endowed with the weak topology. Next, we denote by $Du=(\nabla u+(\nabla u)^T)/2$ the symmetric part of the gradient and by $Au=(\nabla u-(\nabla u)^T)/2$ the antisymmetric part. Given a matrix $C\in\mathbb{R}^{3\times3}$ we denote by C^s, the symmetric part of C and by C^a the antisymmetric part.

2.2. Definition of weak solutions and statement of the main result

The definition of a weak solution for system (1.1)–(1.2) is the following:

Definition 2.1. A triple $(\rho,u,\mathcal{T})$ with $\rho\geq0$ is said to be a weak solution of (1.1)–(1.2)–(1.3) if the following conditions are satisfied:

(1) Integrability conditions:

$$\rho \in L^\infty(0,T;H^1(\mathbb{T}^3)) \cap L^2(0,T;H^2(\mathbb{T}^3)), \quad \sqrt{\rho}u \in L^\infty(0,T;L^2(\mathbb{T}^3)),$$
$$\rho^{\frac{\gamma}{2}} \in L^\infty(0,T;L^2(\mathbb{T}^3)) \cap L^2(0,T;H^1(\mathbb{T}^3)), \quad \nabla\sqrt{\rho} \in L^\infty(0,T;L^2(\mathbb{T}^3)),$$
$$\mathcal{T} \in L^2(0,T;L^2(\mathbb{T}^3)), \quad \rho u \in C([0,T);L^{\frac{3}{2}}_w(\mathbb{T}^3)).$$

(2) Continuity equation:
For any $\phi \in C_c^\infty([0,T)\times\mathbb{T}^3;\mathbb{R})$,

$$\int \rho^0\phi(0)\,dx + \iint \rho\phi_t + \sqrt{\rho}\sqrt{\rho}u\nabla\phi\,dx\,dt = 0. \tag{2.1}$$

(3) Momentum equation:
For any fixed $l = 1,2,3$ and $\psi \in C_c^\infty([0,T)\times\mathbb{T}^3;\mathbb{R}^3)$,

$$\begin{aligned}&\int \rho^0u^{0,l}\psi(0)\,dx + \iint \sqrt{\rho}(\sqrt{\rho}u^l)\psi_t\,dx\,dt + \iint \sqrt{\rho}u^l\sqrt{\rho}u\cdot\nabla\psi\,dx\,dt\\ &- \iint \sqrt{\rho}\mathcal{T}^s_{\cdot,l}\nabla\psi - 2\iint \nabla\rho^{\frac{\gamma}{2}}\rho^{\frac{\gamma}{2}}\cdot\psi\,dx\,dt - \iint \nabla_l\rho\Delta\rho\psi\,dx\,dt\\ &- \iint \rho\Delta\rho\nabla_l\psi\,dx\,dt = 0.\end{aligned} \tag{2.2}$$

(4) Energy dissipation:
For any $\varphi \in C_c^\infty([0,T)\times\mathbb{T}^3;\mathbb{R})$,

$$\iint \sqrt{\rho}\mathcal{T}_{i,j}\varphi\,dx\,dt = -\iint \rho u_i\nabla_j\varphi\,dx\,dt - \iint 2\sqrt{\rho}u_i\otimes\nabla_j\sqrt{\rho}\varphi\,dx\,dt. \tag{2.3}$$

(5) Energy inequality:
The following energy inequality holds:

$$\begin{aligned}&\sup_{t\in(0,T)}\int_{\mathbb{T}^3}\frac{\rho|u|^2}{2} + \frac{\rho(t,x)^\gamma}{\gamma-1} + \frac{|\nabla\rho(t,x)|^2}{2}\,dx + \iint |\mathcal{T}^s(t,x)|^2\,dx\,dt\\ &\le \int_{\mathbb{T}^3}\rho^0(x)|u^0(x)|^2 + \frac{\rho^0(x)^\gamma}{\gamma-1} + \frac{|\nabla\rho^0(x)|^2}{2}\,dx.\end{aligned} \tag{2.4}$$

Remark 2.2. Let us remark that for smooth solutions for system (1.1)–(1.2), the energy inequality[1] reads

$$E(t) + \int_0^t\int_{\mathbb{T}^3}\rho|Du|^2\,dx\,dt' \le E(0), \tag{2.5}$$

where

$$E(t) = \int_{\mathbb{T}^3}\frac{1}{2}\rho|u|^2 + \frac{1}{\gamma-1}\rho^\gamma + \frac{1}{2}|\nabla\rho|^2\,dx.$$

[1] Actually, for smooth solutions this becomes an equality.

At present it is not clear whether arbitrary finite energy weak solutions satisfy inequality (2.5). In particular, the current analysis does not allow us to conclude that the weak limit in $L^2_{t,x}$ of $\sqrt{\rho_n} Du_n$ is $\sqrt{\rho} Du$. Thus in general we are only able to infer

$$\sqrt{\rho} Du \rightharpoonup \mathcal{T}^s \quad \text{in } L^2_{t,x},$$

where $\mathcal{T}^s$ is the symmetric part of the tensor $\mathcal{T}$ defined by

$$\sqrt{\rho}\mathcal{T}^{jk} = \partial_j(\rho u_k) - 2\partial_j \sqrt{\rho}(\sqrt{\rho} u_k).$$

For this reason the energy inequality (2.4) for arbitrary weak solutions holds with the L^2 norm of the symmetric part of $\mathbb{T}^3$, which is defined in (2.3).

In order to state our main result, we first specify the assumptions on the initial data. We assume that

$$\rho^0 \geq 0, \quad \rho^0 \in L^1 \cap L^\gamma(\mathbb{T}^3), \quad \nabla\sqrt{\rho^0} \in L^2(\mathbb{T}^3), \quad \log \rho^0 \in L^1(\mathbb{T}^3). \tag{2.6}$$

We point out that the assumption on the summability of $\log \rho^0$ is made only to avoid the technicalities in approximating the initial data. Regarding the initial velocity, we assume that u^0 is a measurable vector field, finite almost everywhere such that

$$\sqrt{\rho^0} u^0 \in L^2(\mathbb{T}^3), \quad \rho^0 u^0 \in L^p(\mathbb{T}^3) \text{ with } p < 2. \tag{2.7}$$

The main theorem of our paper is the following.

Theorem 2.3. *Assume ρ^0 and $\rho^0 u^0$ satisfy (2.6) and (2.7). Then there exists at least a weak solution $(\rho, u, \mathcal{T})$ of (1.1)–(1.3) in the sense of Definition 2.1.*

Remark 2.4. We stress that the velocity field is not uniquely defined in the vacuum region $\{\rho = 0\}$.

2.3. The truncations

Let $\bar{\beta}\colon \mathbb{R} \to \mathbb{R}$ be an even, positive, compactly supported smooth function such that

$$\bar{\beta}(z) = 1 \quad \text{for } z \in [-1, 1],$$

$\operatorname{supp} \bar{\beta} \subset (-2, 2)$ and $0 \leq \bar{\beta} \leq 1$. We also define $\tilde{\beta}\colon \mathbb{R} \to \mathbb{R}$ as the antiderivative of $\bar{\beta}$, namely

$$\tilde{\beta}(z) = \int_0^z \bar{\beta}(s)\, ds.$$

For any $\delta > 0$ we define $\bar{\beta}_\delta(z) = \bar{\beta}(\delta z)$, $\tilde{\beta}_\delta(z) = \tilde{\beta}(\delta z)$. If $y \in \mathbb{R}^3$ then

$$\hat{\beta}_\delta(y) := \prod_{\ell=1}^{3} \bar{\beta}_\delta(y_\ell)$$

and finally for any fixed $\ell = 1, 2, 3$ let us define

$$\beta_\delta^l(y) = \frac{1}{\delta}\tilde{\beta}_\delta(y_l)\prod_{i\neq\ell}\bar{\beta}_\delta(y_i).$$

We notice that for fixed $l = 1, 2, 3$ the function $\beta_\delta^l : \mathbb{R}^3 \to \mathbb{R}$ is a truncation of the function $f(y) = y_l$. In the next lemma we collect some elementary properties of β_δ^l, $\hat{\beta}_\delta$ and $\bar{\beta}_\lambda$, which can be deduced directly from the definitions.

Lemma 2.5. *Let $\lambda, \delta > 0$ and $K := \|\bar{\beta}\|_{W^{2,\infty}}$. Then there exists $C = C(K)$ such that the following bounds hold:*

(1) *For any $\delta > 0$ and $l = 1, 2, 3$,*

$$\|\beta_\delta^l\|_{L^\infty} \leq \frac{C}{\delta}, \quad \|\nabla\beta_\delta^l\|_{L^\infty} \leq C, \quad \|\nabla^2\beta_\delta^l\|_{L^\infty} \leq C\delta. \tag{2.8}$$

(2) *For any $\lambda > 0$,*

$$\|\bar{\beta}_\lambda\|_{L^\infty} \leq 1, \quad \|\bar{\beta}'_\lambda\|_{L^\infty} \leq C\lambda, \quad \sqrt{|s|}\bar{\beta}_\lambda(s) \leq \frac{C}{\sqrt{\lambda}}. \tag{2.9}$$

(3) *For any $\delta > 0$,*

$$\|\hat{\beta}_\delta\|_{L^\infty} \leq 1, \quad \|\nabla\hat{\beta}_\delta\|_{L^\infty} \leq C\delta, \quad |y|\,|\hat{\beta}_\delta(y)| \leq \frac{C}{\delta}. \tag{2.10}$$

(4) *The following convergences hold for $l = 1, 2, 3$, pointwise on $\mathbb{R}^3$, as $\delta \to 0$:*

$$\beta_\delta^l(y) \to y_l, \quad (\nabla_y\beta_\delta^l)(y) \to \nabla_{y_l} y, \quad \hat{\beta}_\delta(y) \to 1. \tag{2.11}$$

(5) *The following convergence holds pointwise on $\mathbb{R}$ as $\lambda \to 0$:*

$$\bar{\beta}_\lambda(s) \to 1. \tag{2.12}$$

2.4. DiPerna–Lions commutator estimate

In this subsection we recall the commutator estimate for convolutions of DiPerna–Lions ([17]). First, for any function f we denote by $\bar{f}_r$ the time-space convolution of f with a smooth sequence of even mollifiers $\{\Psi_r\}_r$, namely

$$\bar{f}_r = \Psi_r * f(t, x), \quad t > r,$$

where

$$\Psi_r(t, x) = \frac{1}{r^4}\Psi\Big(\frac{t}{r}, \frac{x}{r}\Big)$$

and Ψ is a smooth nonnegative even function such that $\operatorname{supp}\Psi \subset B_1(0)$ and

$$\iint \Psi\, dx\, dt = 1.$$

Then the following lemma holds true.

Lemma 2.6. *Let $p_1, p_2 \in [1,\infty]$ and $p_3 < \infty$.*

(1) *Let $B \in L^{p_1}((0,T)\times\mathbb{T}^3;\mathbb{R}^3)$ such that $\nabla B \in L^{p_1}((0,T)\times\mathbb{T}^3;\mathbb{R})$ and let $f \in L^{p_2}((0,T)\times\mathbb{T}^3;\mathbb{R})$; then*

$$\|\operatorname{div}\overline{(Bf)_r} - \operatorname{div}(B\bar{f}_r)\|_{L^{p_3}_{t,x}} \quad \to 0 \text{ as } r \to 0$$

provided $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$.

(2) *Let $g \in L^{p_1}((0,T)\times\mathbb{T}^3;\mathbb{R})$ such that $\partial_t g \in L^{p_1}((0,T)\times\mathbb{T}^3;\mathbb{R})$ and let $f \in L^{p_2}((0,T)\times\mathbb{T}^3;\mathbb{R})$; then*

$$\|\overline{\partial_t(gf)_r} - \partial_t(g\bar{f}_r)\|_{L^{p_3}_{t,x}} \quad \to 0 \text{ as } r \to 0,$$

provided $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$.

We omit the proof of the lemma. Notice that part (1) can be easily deduced from [17, Lem. II.1] and part (2) is a simple corollary.

3. Weak solutions of approximating system and their properties

The proof of the main Theorem 2.3 goes through the analysis of weak solutions to the approximating system

$$\begin{aligned}
&\partial_t\rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0,\\
&\partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon\otimes u_\varepsilon) - \operatorname{div}(\rho_\varepsilon Du_\varepsilon) + \nabla\rho_\varepsilon^\gamma + \varepsilon\rho_\varepsilon|u_\varepsilon|^2u_\varepsilon + \varepsilon u_\varepsilon\\
&\qquad = \rho_\varepsilon\nabla\Delta\rho_\varepsilon + \varepsilon\rho_\varepsilon\nabla\Big(\frac{\Delta\sqrt{\rho_\varepsilon}}{\sqrt{\rho_\varepsilon}}\Big),
\end{aligned} \tag{3.1}$$

with initial data

$$\begin{aligned}
\rho_\varepsilon(0,x) &= \rho^0(x),\\
(\rho_\varepsilon u_\varepsilon)(0,x) &= \rho^0(x)u^0(x),
\end{aligned} \tag{3.2}$$

satisfying the hypothesis (2.6) and (2.7).

More specifically, the aim of this section is to show a truncated formulation for solutions to (3.1); see Theorem 3.2 below for a more precise statement.

Before introducing the main result of this section, we provide the definition of weak solutions to system (3.1).

Definition 3.1. A triple $(\rho_\varepsilon, u_\varepsilon, \mathcal{T}_\varepsilon)$ is a weak solution of (3.1)–(3.2) provided the following properties hold.

(1) Integrability hypothesis:

$$\begin{aligned}
&\rho_\varepsilon \in L^\infty(0,T;H^1(\mathbb{T}^3))\cap L^2(0,T;H^2(\mathbb{T}^3)), \quad \sqrt{\rho_\varepsilon}u_\varepsilon \in L^\infty(0,T;L^2(\mathbb{T}^3)),\\
&\rho_\varepsilon^{\frac{\gamma}{2}} \in L^\infty(0,T;L^2(\mathbb{T}^3))\cap L^2(0,T;H^1(\mathbb{T}^3)), \quad \mathcal{T}_\varepsilon \in L^\infty(0,T;L^2(\mathbb{T}^3)),
\end{aligned}$$

$$\sqrt{\rho_\varepsilon} \in L^\infty(0,T;H^1(\mathbb{T}^3)) \cap L^2(0,T;H^2(\mathbb{T}^3)), \quad \rho_\varepsilon^{\frac{1}{4}} u_\varepsilon \in L^4((0,T)\times(\mathbb{T}^3)),$$
$$u_\varepsilon \in L^2((0,T)\times(\mathbb{T}^3)), \quad \rho_\varepsilon u_\varepsilon \in C([0,T);L_w^{\frac{3}{2}}(\mathbb{T}^3)).$$

(2) Continuity equation:
For any $\phi \in C_c^\infty([0,T)\times\mathbb{T}^3;\mathbb{R})$,

$$\iint \rho_\varepsilon \partial_t \phi + \rho_\varepsilon u_\varepsilon \nabla\phi \, dx\, dt + \int \rho^0 \phi(0)\, dx = 0. \tag{3.3}$$

(3) Momentum equation:
For any $\psi \in C_c^\infty([0,T)\times\mathbb{T}^3;\mathbb{R}^3)$,

$$\begin{aligned}
&\iint \rho_\varepsilon u_\varepsilon \partial_t \psi + \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon \nabla\psi - \sqrt{\rho_\varepsilon}\mathcal{T}_\varepsilon^s \nabla\psi - \varepsilon\rho_\varepsilon |u_\varepsilon|^2 u_\varepsilon \psi - \varepsilon u_\varepsilon \psi \, dx\, dt \\
&\quad - \iint 2\rho_\varepsilon^{\frac{\gamma}{2}} \nabla \rho_\varepsilon^{\frac{\gamma}{2}} \psi - \varepsilon\sqrt{\rho_\varepsilon}\nabla^2\sqrt{\rho_\varepsilon}\nabla\psi + \varepsilon\nabla\sqrt{\rho_\varepsilon}\otimes\nabla\sqrt{\rho_\varepsilon}\nabla\psi \, dx\, dt \\
&\quad - \iint \nabla\rho_\varepsilon \Delta\rho_\varepsilon \psi + \rho_\varepsilon\Delta\rho_\varepsilon \operatorname{div}\psi \, dx\, dt + \int \rho^0 u^0 \psi(0)\, dx = 0.
\end{aligned} \tag{3.4}$$

(4) Dissipation:
For any $\varphi \in C_c^\infty([0,T)\times\mathbb{T}^3;\mathbb{R})$,

$$\iint \sqrt{\rho_\varepsilon}\mathcal{T}_{\varepsilon,i,j}\varphi \, dx\, dt = -\iint \rho_\varepsilon u_{\varepsilon,i}\nabla_j \varphi \, dx\, dt - 2\iint \sqrt{\rho_\varepsilon} u_{\varepsilon,i} \otimes \nabla_j \sqrt{\rho_\varepsilon}\varphi \, dx\, dt. \tag{3.5}$$

(5) Energy estimate:

$$\begin{aligned}
&\sup_{t\in(0,T)}\left(\int \rho_\varepsilon \frac{|u_\varepsilon|^2}{2} + \frac{\rho_\varepsilon^\gamma}{\gamma-1} + \frac{|\nabla\rho_\varepsilon|^2}{2} + \varepsilon|\nabla\sqrt{\rho_\varepsilon}|^2 \, dx\right) \\
&\qquad + \iint |\mathcal{T}_\varepsilon^s|^2 \, dx\, dt + \varepsilon\iint \rho_\varepsilon |u_\varepsilon|^4 \, dx\, dt + \varepsilon\iint |u_\varepsilon|^2 \, dx\, dt \\
&\qquad \le \int \rho^0 \frac{|u^0|^2}{2} + \frac{(\rho^0)^\gamma}{\gamma-1} + \frac{|\nabla\rho^0|^2}{2} + \varepsilon|\nabla\sqrt{\rho^0}|^2 \, dx.
\end{aligned} \tag{3.6}$$

(6) BD entropy:
By defining $w_\varepsilon = u_\varepsilon + \nabla\log\rho_\varepsilon$ we have

$$\begin{aligned}
&\sup_{t\in(0,T)}\left(\int \rho_\varepsilon \frac{|w_\varepsilon|^2}{2} + \frac{\rho_\varepsilon^\gamma}{\gamma-1} + \frac{|\nabla\rho_\varepsilon|^2}{2} + (\rho_\varepsilon - \varepsilon\log\rho_\varepsilon) + \varepsilon|\nabla\sqrt{\rho_\varepsilon}|^2 \, dx\right) \\
&\qquad + \frac{4}{\gamma}\iint |\nabla\rho_\varepsilon^{\frac{\gamma}{2}}|^2 \, dx\, dt + \frac{1}{2}\iint |\mathcal{T}_\varepsilon^a|^2 \, dx\, dt + \iint |\Delta\rho_\varepsilon|^2 \, dx\, dt \\
&\qquad + \varepsilon\iint |\nabla^2\sqrt{\rho_\varepsilon}|^2 + |\nabla\rho_\varepsilon^{\frac{1}{4}}|^4 \, dx\, dt + \varepsilon\iint \rho_\varepsilon|u_\varepsilon|^4 + |u_\varepsilon|^2 \, dx\, dt \\
&\quad \le \int \rho^0 \frac{|u^0|^2}{2} + \frac{2\rho^{0\gamma}}{\gamma-1} + \frac{2|\nabla\rho^0|^2}{2} + \varepsilon|\nabla\sqrt{\rho^0}|^2 \, dx + \int \rho^0 \frac{|w^0|^2}{2} \, dx \\
&\qquad + \int (\rho^0 - \varepsilon\log\rho^0)\, dx.
\end{aligned} \tag{3.7}$$

By using the energy and BD entropy estimates we can list some bounds satisfied by solutions considered in Definition 3.1, namely

$$
\begin{gathered}
\|\rho_\varepsilon\|_{L_t^\infty(L_x^1\cap L_x^\gamma)} \le C, \quad \|\nabla\rho_\varepsilon\|_{L_t^\infty L_x^2} \le C, \quad \|\nabla\sqrt{\rho_\varepsilon}\|_{L_t^\infty L_x^2} \le C, \\
\|\nabla\rho_\varepsilon^{\frac{\gamma}{2}}\|_{L^2_{t,x}} \le C, \quad \|\varepsilon^{\frac12}\nabla^2\sqrt{\rho_\varepsilon}\|_{L^2_{t,x}} \le C, \quad \|\varepsilon^{\frac14}\nabla\rho_\varepsilon^{\frac14}\|_{L^4_{t,x}} \le C, \\
\|\varepsilon^{\frac14}\rho_\varepsilon^{\frac14}u_\varepsilon\|_{L^4_{t,x}} \le C, \quad \|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L_t^\infty L_x^2} \le C, \\
\|\mathcal{T}_\varepsilon\|_{L^2_{t,x}} \le C, \quad \|\sqrt{\varepsilon}u_\varepsilon\|_{L^2_{t,x}} \le C, \quad \|\nabla^2\rho_\varepsilon\|_{L^2_{t,x}} \le C,
\end{gathered}
\tag{3.8}
$$

where C depends only on the initial data (1.3). Notice that, by using a combination of bounds in (3.8) and standard interpolation inequalities, we can also infer the following estimates for fixed $\varepsilon > 0$:

$$
\begin{gathered}
\|\partial_t\rho_\varepsilon\|_{L^{\frac43}_{t,x}} \le C_\varepsilon, \quad \|\nabla(\rho_\varepsilon u_\varepsilon)\|_{L^{\frac43}_{t,x}} \le C_\varepsilon, \\
\|\nabla\rho_\varepsilon\Delta\rho_\varepsilon\|_{L^{\frac54}_{t,x}} \le C_\varepsilon, \quad \|\rho_\varepsilon\Delta\rho_\varepsilon\|_{L^{\frac43}_{t,x}} \le C_\varepsilon.
\end{gathered}
\tag{3.9}
$$

Following the arguments in [44] and [10] it is easy to prove that there exists at least a weak solution in the sense of Definition 3.1, so we omit the proof. On the other hand, in what follows we are going to show that weak solutions to (3.1) also satisfy a truncated formulation; see (3.11) below. Formally, (3.11) can be obtained by taking $\nabla_y\beta_\delta^\ell(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\psi$ as a test function in (3.4). However, weak solutions to (3.1) do not have the necessary regularity needed in order to rigorously justify all the passages. For this reason we are going to use several layers of approximations and truncations in order to infer the desired formulas (3.11) and (3.14) below. One of the main difficulties encountered in proving Theorem 3.2 below is the lack of good Sobolev bounds for the velocity field; let us recall that the only available estimate is $\sqrt{\rho_\varepsilon}\nabla u_\varepsilon \in L^2_{t,x}$. To overcome this difficulty, we first consider a truncated velocity field, as in [33]. Let $\phi_m(y)$ be the function defined as

$$
\phi_m(y) = \begin{cases}
0 & \text{for } 0 < y \le \frac{1}{2m}\,, \\
2my - 1 & \text{for } \frac{1}{2m} \le y \le \frac{1}{m}\,, \\
1 & \text{for } \frac{1}{m} \le y \le m\,, \\
2 - \frac{y}{m} & \text{for } m \le y \le 2m\,, \\
0 & \text{for } 2m \le y.
\end{cases}
$$

We define $v_{\varepsilon,m} := \phi_m(\rho_\varepsilon)u_\varepsilon$. By using (3.8) and the definition of ϕ_m we have

$$
\|v_{\varepsilon,m}\|_{L^4_{t,x}} \le C_{m,\varepsilon}, \quad \|\nabla v_{\varepsilon,m}\|_{L^2_{t,x}} \le C_{m,\varepsilon}. \tag{3.10}
$$

However, this will not be sufficient, due to the capillarity coefficient. Consequently, we will need to exploit a further truncation for the approximated mass density.

The main result of this section is the following.

Theorem 3.2. *Let $(\rho_\varepsilon, u_\varepsilon, \mathcal{T}_\varepsilon)$ be a weak solution of the system (3.1)–(3.2) in the sense of Definition 3.1. Let β_δ^l and $\bar{\beta}_\lambda$ the truncation defined in the Section 2.3. Then the following equalities hold:*

(1) *For any $\psi \in C_c^\infty([0,T) \times \mathbb{T}^3; \mathbb{R})$,*

$$\begin{aligned}\int \rho^0 \beta_\delta^l(u^0)\bar{\beta}_\lambda(\rho^0)\psi(0,x)\,dx + \iint \rho_\varepsilon \beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\partial_t\psi \\ - \iint \rho_\varepsilon u_\varepsilon \beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\cdot\nabla\psi\,dx\,dt - \iint \sqrt{\rho_\varepsilon}\mathcal{T}_\varepsilon^s : \nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\otimes\nabla\psi\,dx\,dt \\ -2\iint \rho_\varepsilon^{\frac{\gamma}{2}}\nabla\rho_\varepsilon^{\frac{\gamma}{2}}\cdot\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\psi\,dx\,dt - \iint \nabla\rho_\varepsilon\Delta\rho_\varepsilon\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\psi\,dx\,dt \\ - \iint \rho_\varepsilon\Delta\rho_\varepsilon\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\nabla\psi\,dx\,dt + \iint R_\varepsilon^{\delta,\lambda}\psi\,dx\,dt \\ + \iint \widetilde{R}_\varepsilon^{\delta,\lambda}\psi\,dx\,dt = 0,\end{aligned} \tag{3.11}$$

where the remainders are given by

$$\begin{aligned}R_\varepsilon^{\delta,\lambda} = \sum_{i=1}^{6} R_{\varepsilon,i}^{\delta,\lambda} &= \rho_\varepsilon\beta_\delta^l(u_\varepsilon)\bar{\beta}'_\lambda(\rho_\varepsilon)\partial_t\rho_\varepsilon + \rho_\varepsilon u\beta_\delta^l(u_\varepsilon)\bar{\beta}'_\lambda(\rho_\varepsilon)\nabla\rho_\varepsilon \\ &\quad - \sqrt{\rho_\varepsilon}\mathcal{T}_\varepsilon^s : \nabla_y\beta_\delta^l(u_\varepsilon)\otimes\nabla\rho_\varepsilon\bar{\beta}'_\lambda(\rho_\varepsilon) \\ &\quad + \sqrt{\rho_\varepsilon}\Delta\rho_\varepsilon\nabla_y^2\beta_\delta^l(u_\varepsilon)\mathcal{T}_\varepsilon\bar{\beta}_\lambda(\rho_\varepsilon) \\ &\quad + \rho_\varepsilon\Delta\rho_\varepsilon\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}'_\lambda(\rho_\varepsilon)\nabla\rho_\varepsilon - \mathcal{T}_\varepsilon^s\mathcal{T}_\varepsilon\nabla_y^2\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon),\end{aligned} \tag{3.12}$$

$$\begin{aligned}\widetilde{R}_\varepsilon^{\delta,\lambda} = \sum_{i=1}^{6} \widetilde{R}_{\varepsilon,i}^{\delta,\lambda} &= -\varepsilon\nabla^2\sqrt{\rho_\varepsilon}\mathcal{T}_\varepsilon\nabla_y^2\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon) \\ &\quad + 4\varepsilon\nabla\rho_\varepsilon^{\frac{1}{4}}\otimes\nabla\rho_\varepsilon^{\frac{1}{4}}\mathcal{T}_\varepsilon\nabla_y^2\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon) \\ &\quad - \varepsilon\sqrt{\rho_\varepsilon}\nabla^2\sqrt{\rho_\varepsilon}\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}'_\lambda(\rho_\varepsilon)\nabla\rho_\varepsilon \\ &\quad + 4\varepsilon\sqrt{\rho_\varepsilon}\nabla\rho_\varepsilon^{\frac{1}{4}}\otimes\nabla\rho_\varepsilon^{\frac{1}{4}}\beta_\delta^l(u_\varepsilon)\bar{\beta}'_\lambda(\rho_\varepsilon)\nabla\rho_\varepsilon \\ &\quad - \varepsilon\rho|u|^2u\nabla_y\beta_\delta^l(u)\bar{\beta}_\lambda(\rho) - \varepsilon u\nabla_y\beta_\delta^l(u)\bar{\beta}_\lambda(\rho).\end{aligned} \tag{3.13}$$

(2) *For any $\varphi \in C_c^\infty((0,T)\times\mathbb{T}^3;\mathbb{R})$ the following (tensor) equality holds:*

$$\begin{aligned}\iint \sqrt{\rho_\varepsilon}\mathcal{T}_\varepsilon\hat{\beta}_\delta(u_\varepsilon)\varphi\,dx\,dt &= -\iint \hat{\beta}_\delta(u_\varepsilon)\rho_\varepsilon u_\varepsilon\otimes\nabla\varphi\,dx\,dt \\ &\quad - \iint \sqrt{\rho_\varepsilon}u_\varepsilon\varphi\nabla_y\hat{\beta}_\delta(u_\varepsilon)\mathcal{T}_\varepsilon\,dx\,dt \\ &\quad - 2\iint \sqrt{\rho_\varepsilon}u_\varepsilon\otimes\nabla\sqrt{\rho_\varepsilon}\varphi\hat{\beta}_\delta(u_\varepsilon)\,dx\,dt.\end{aligned} \tag{3.14}$$

Proof. In order to simplify the notation, in what follows we drop the subscripts ε. Let us define the quantities

$$\begin{aligned} M &:= \sqrt{\rho}\mathcal{T}^s + \varepsilon\sqrt{\rho}\nabla^2\sqrt{\rho} - \varepsilon\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}, \\ N &:= \varepsilon\rho|u|^2 u + \varepsilon u + 2\rho^{\frac{\gamma}{2}}\nabla\rho^{\frac{\gamma}{2}}. \end{aligned}$$

Consider the weak formulation of the momentum equation (3.4) in Definition 3.1, namely

$$\begin{aligned} &\iint \rho u\partial_t\psi + \rho u\otimes u\nabla\psi - M\nabla\psi - N\psi - \nabla\rho\Delta\rho\psi\,dx\,dt \\ &\quad - \iint \rho\Delta\rho\nabla\psi\,dx\,dt = 0, \end{aligned} \tag{3.15}$$

with $\psi \in C_c^\infty((0,T)\times\mathbb{T}^3;\mathbb{R})$. Note that the initial datum disappears because of the choice of the test function. We claim that for the truncated velocity field v_m the following equation holds:

$$\begin{aligned} &\iint \rho v_m\partial_t\psi + \rho u\otimes v_m\nabla\psi - \psi\sqrt{\rho}\,\mathrm{tr}(\mathcal{T})\phi_m'(\rho)\rho u\,dx\,dt \\ &\quad - \iint \phi_m(\rho)M\nabla\psi\,dx\,dt - \iint M\phi_m'(\rho)\nabla\rho\psi\,dx\,dt \\ &\quad - \iint N\phi_m(\rho)\psi\,dx\,dt - \iint \nabla\rho\Delta\rho\phi_m(\rho)\psi\,dx\,dt \\ &\quad - \iint \rho\Delta\rho\phi_m'(\rho)\nabla\rho\psi\,dx\,dt - \iint \rho\Delta\rho\phi_m(\rho)\nabla\psi\,dx\,dt = 0. \end{aligned} \tag{3.16}$$

A similar analysis is also performed in [33, Sect. 3.2]. In order to show (3.16), first consider (3.3) with $\overline{\phi_m'(\overline{\rho}_r)\psi_r}$ as test function. After integrating by parts and passing to the limit as r goes to 0, one obtains

$$\iint \phi_m(\rho)\partial_t\psi - \phi_m'(\rho)\big(\mathrm{tr}(\sqrt{\rho}\mathcal{T}) + 2\sqrt{\rho}u\cdot\nabla\sqrt{\rho}\big)\psi\,dx\,dt = 0.$$

By using the bounds (3.8) it follows that a.e. on $(0,T)\times\mathbb{T}^3$,

$$\partial_t\phi_m(\rho) + (\mathrm{tr}(\sqrt{\rho}\mathcal{T}) + 2\sqrt{\rho}u\cdot\nabla\sqrt{\rho})\phi_m'(\rho) = 0. \tag{3.17}$$

Then (3.16) follows by considering $\overline{\phi_m(\rho)\psi_r}$ as test function in (3.4), by sending r to 0 and by using (3.17). Let us just outline how to deal with the capillarity term. We have

$$\begin{aligned} &\iint \nabla\rho\Delta\rho\overline{\phi_m(\rho)\psi_r} + \rho\Delta\rho\nabla[\overline{\phi_m(\rho)\psi_r}]\,dx\,dt \\ &\quad = \iint \overline{\nabla\rho\Delta\rho_r}\phi_m(\rho)\psi + \overline{\rho\Delta\rho_r}\phi_m'(\rho)\nabla\rho\psi + \overline{\rho\Delta\rho_r}\phi_m(\rho)\psi\,dx\,dt. \end{aligned} \tag{3.18}$$

Then, by using the bounds in (3.9) and the definition of ϕ_m, it follows that

$$\iint \overline{\nabla\rho\Delta\rho}_r\phi_m(\rho)\psi\,dx\,dt \to \iint \nabla\rho\Delta\rho\phi_m(\rho)\psi\,dx\,dt,$$
$$\iint \overline{\rho\Delta\rho}_r\phi_m'(\rho)\nabla\rho\psi\,dx\,dt \to \iint \rho\Delta\rho\phi_m'(\rho)\nabla\rho\psi\,dx\,dt,$$
$$\iint \overline{\rho\Delta\rho}_r\phi_m(\rho)\psi\,dx\,dt \to \iint \rho\Delta\rho\phi_m(\rho)\psi\,dx\,dt.$$

The other terms in (3.16) are dealt with analogously. Once we obtain (3.16), we derive the truncated formulation for v_m. We claim

$$\begin{aligned}
&\iint \rho\beta_\delta^l(v_m)\partial_t\psi + \rho u\beta_\delta^l(v_m)\nabla\psi - \sqrt{\rho}\,\mathrm{tr}(\mathcal{T})\phi_m'(\rho)\rho u\nabla_y\beta_\delta^l(v_m)\psi\,dx\,dt\\
&\quad - \iint \phi_m(\rho)M\nabla_y^2\beta_\delta^l(v_m)\nabla v_m\psi\,dx\,dt - \iint \phi_m(\rho)M\nabla_y\beta_\delta^l(v_m)\nabla\psi\,dx\,dt\\
&\quad - \iint M\phi_m'(\rho)\nabla\rho\nabla_y\beta_\delta^l(v_m)\psi\,dx\,dt - \iint N\phi_m(\rho)\nabla_y\beta_\delta^l(v_m)\psi\,dx\,dt\\
&\quad + \iint \nabla\rho\Delta\rho\phi_m(\rho)\nabla_y\beta_\delta^l(v_m)\psi\,dx\,dt + \iint \rho\Delta\rho\phi_m'(\rho)\nabla\rho\nabla_y\beta_\delta^l(v_m)\psi\,dx\,dt\\
&\quad + \iint \rho\Delta\rho\phi_m(\rho)\nabla_y^2\beta_\delta^l(v_m)\nabla v_m\psi\,dx\,dt + \iint \rho\Delta\rho\phi_m(\rho)\nabla_y\beta_\delta^l(v_m)\nabla\psi\,dx\,dt\\
&= 0.
\end{aligned} \tag{3.19}$$

This is proved by considering $\overline{\nabla_y\beta_\delta^l(\overline{v_m}_r)\psi}_r$ as test function in (3.16), with $\psi \in C_c^\infty((0,T)\times\mathbb{T}^3;\mathbb{R})$. Let us first focus on the transport terms. By using standard properties of the convolutions we have

$$\begin{aligned}
&-\iint \rho v_m\partial_t\big[\overline{\nabla_y\beta_\delta^l(\overline{v_m}_r)\psi}_r\big] + \rho u\otimes v_m\nabla\big[\overline{\nabla_y\beta_\delta^l(\overline{v_m}_r)\psi}_r\big]\,dx\,dt\\
&\quad = \iint [\partial_t(\overline{\rho v_m}_r) + \mathrm{div}(\overline{\rho u\otimes v_m}_r)]\nabla_y\beta_\delta^l(\overline{v_m}_r)\psi\,dx\,dt\\
&\quad = \iint [\partial_t(\overline{\rho v_m}_r) - \partial_t(\rho\overline{v_m}_r) + \mathrm{div}(\overline{\rho u\otimes v_m}_r) - \mathrm{div}(\rho u\otimes\overline{v_m}_r)]\nabla_y\beta_\delta^l(\overline{v_m}_r)\psi\,dx\,dt\\
&\qquad + \iint [\partial_t(\rho\overline{v_m}_r) + \mathrm{div}(\rho u\otimes\overline{v_m}_r)]\nabla_y\beta_\delta^l(\overline{v_m}_r)\psi\,dx\,dt = \mathrm{I}_r + \mathrm{II}_r.
\end{aligned}$$

The term I_r is treated by using the commutator estimate of DiPerna–Lions ([17]). Indeed, by Lemma 2.6 (2) with $g = \rho$ and $f = v_{i,m}$ and Lemma 2.6 with $B = \rho u$ and $f = v_{i,m}$, by using the bounds in (3.9) and (3.10) we have

$$I_r \to 0 \quad \text{as } r \to 0.$$

Regarding the second term, we first note that by (3.9) the continuity equation holds a.e. in $(0,T)\times\mathbb{T}^3$; then, since $\overline{v_m}_r$ is smooth, we have

$$\begin{aligned}\mathrm{II}_r &= \iint \big(\partial_t(\rho\beta_\delta^l(\overline{v_m}_r)) + \operatorname{div}(\rho u\beta_\delta^l\overline{v_m}_r))\big)\psi\,dx\,dt\\ &\quad - \iint \big(\rho\beta_\delta^l(\overline{v_m}_r)\partial_t\psi + \rho u\beta_\delta^l(\overline{v_m}_r)\nabla\psi\big)\,dx\,dt,\end{aligned}$$

which by using standard properties of convolutions and (3.9) converges to

$$-\iint \big(\rho\beta_\delta^l(v_m)\partial_t\psi + \rho u\beta_\delta^l(v_m)\nabla\psi\big)\,dx\,dt.$$

Again, we deal with the capillarity terms: by standard properties of convolutions we have

$$\begin{aligned}&\iint \nabla\rho\Delta\rho\phi_m(\rho)\overline{\nabla_y\beta_\delta^l(\overline{v_m}_r)\psi}_r\,dx\,dt + \iint \rho\Delta\rho\phi_m'(\rho)\nabla\rho\overline{\nabla_y\beta_\delta^l(\overline{v_m}_r)\psi}_r\,dx\,dt\\ &+ \iint \rho\Delta\rho\phi_m(\rho)\nabla[\overline{\nabla_y\beta_\delta^l(\overline{v_m}_r)\psi}_r]\,dx\,dt = \iint \overline{\nabla\rho\Delta\rho\phi_m(\rho)}_r\nabla_y\beta_\delta^l(\overline{v_m}_r)\psi\,dx\,dt\\ &+ \iint \overline{\rho\Delta\rho\phi_m'(\rho)\nabla\rho}_r\nabla_y\beta_\delta^l(\overline{v_m}_r)\psi\,dx\,dt + \iint \overline{\rho\Delta\rho\phi_m(\rho)}_r\nabla_y^2\beta_\delta^l(\overline{v_m}_r)\nabla\overline{v_m}_r\psi\,dx\,dt\\ &+ \iint \overline{\rho\Delta\rho\phi_m(\rho)}_r\nabla_y\beta_\delta^l(\overline{v_m}_r)\nabla\psi\,dx\,dt.\end{aligned}$$

Then, by using (3.9)–(3.10), the definition of ϕ_m and the fact that $\beta_\delta^l \in W^{2,\infty}(\mathbb{R})$, the following convergences as $r\to 0$ follow easily:

$$\begin{aligned}\iint \overline{\nabla\rho\Delta\rho\phi_m(\rho)}_r\nabla_y\beta_\delta^l(\overline{v_m}_r)\psi\,dx\,dt &\to \iint \nabla\rho\Delta\rho\phi_m(\rho)\nabla_y\beta_\delta^l(v_m)\psi\,dx\,dt,\\ \iint \overline{\rho\Delta\rho\phi_m'(\rho)\nabla\rho}_r\nabla_y\beta_\delta^l(\overline{v_m}_r)\psi\,dx\,dt &\to \iint \rho\Delta\rho\phi_m'(\rho)\nabla\rho\nabla_y\beta_\delta^l(v_m)\psi\,dx\,dt,\\ \iint \overline{\rho\Delta\rho\phi_m(\rho)}_r\nabla_y^2\beta_\delta^l(\overline{v_m}_r)\nabla\overline{v_m}_r\psi\,dx\,dt &\to \iint \rho\Delta\rho\phi_m(\rho)\nabla_y^2\beta_\delta^l(v_m)\nabla v_m\psi\,dx\,dt,\\ \iint \overline{\rho\Delta\rho\phi_m(\rho)}_r\nabla_y\beta_\delta^l(\overline{v_m}_r)\nabla\psi\,dx\,dt &\to \iint \rho\Delta\rho\phi_m(\rho)\nabla_y\beta_\delta^l(v_m)\nabla\psi\,dx\,dt,\end{aligned}$$

where in third limit the dominated convergence theorem and a possible passage to subsequence is needed. At this point we would like to pass in the limit as $m\to\infty$, but we cannot deal with the term

$$\iint \rho\Delta\rho\phi_m(\rho)\nabla_y^2\beta_\delta^l(v_m)\nabla v_m\psi\,dx\,dt,$$

because we lack uniform bounds for weak solutions to (3.1). For example, a bound like $\sqrt{\rho}\in L^\infty_{t,x}$ would be sufficient to perform the limit. To overcome this problem, we introduce a further truncation for the mass density with an additional parameter $\lambda > 0$.

We consider $\bar\beta_\lambda(\bar\rho_r)\psi$ as test function in (3.19):

$$
\begin{aligned}
&\iint \rho\beta^l_\delta(v_m)\bar\beta'_\lambda(\bar\rho_r)\partial_t\bar\rho_r\psi\,dx\,dt + \iint \rho\beta^l_\delta(v_m)\bar\beta_\lambda(\bar\rho_r)\partial_t\psi\,dx\,dt\\
&+\iint \rho u\beta^l_\delta(v_m)\bar\beta'_\lambda(\bar\rho_r)\nabla\bar\rho_r\psi\,dx\,dt + \iint \rho u\beta^l_\delta(v_m)\bar\beta_\lambda(\bar\rho_r)\nabla\psi\,dx\,dt\\
&-\iint \sqrt{\rho}\,\mathrm{tr}(\mathcal{T})\phi'_m(\rho)\rho u\nabla_y\beta^l_\delta(v_m)\bar\beta_\lambda(\bar\rho_r)\psi\,dx\,dt\\
&-\iint \phi_m(\rho)M\nabla^2_y\beta^l_\delta(v_m)\nabla v_m\bar\beta_\lambda(\bar\rho_r)\psi\,dx\,dt\\
&-\iint \phi_m(\rho)M\nabla_y\beta^l_\delta(v_m)\bar\beta'_\lambda(\bar\rho_r)\nabla\bar\rho_r\psi\,dx\,dt - \iint \phi_m(\rho)M\nabla_y\beta^l_\delta(v_m)\bar\beta_\lambda(\bar\rho_r)\nabla\psi\,dx\,dt\\
&-\iint M\phi'_m(\rho)\nabla\rho\nabla_y\beta^l_\delta(v_m)\bar\beta_\lambda(\bar\rho_r)\psi\,dx\,dt - \iint N\phi_m(\rho)\nabla_y\beta^l_\delta(v_m)\bar\beta_\lambda(\bar\rho_r)\psi\,dx\,dt\\
&+\iint \nabla\rho\Delta\rho\phi_m(\rho)\nabla_y\beta^l_\delta(v_m)\bar\beta_\lambda(\bar\rho_r)\psi\,dx\,dt\\
&+\iint \rho\Delta\rho\phi'_m(\rho)\nabla\rho\nabla_y\beta^l_\delta(v_m)\bar\beta_\lambda(\bar\rho_r)\psi\,dx\,dt\\
&+\iint \rho\Delta\rho\phi_m(\rho)\nabla^2_y\beta^l_\delta(v_m)\nabla v_m\bar\beta_\lambda(\bar\rho_r)\psi\,dx\,dt\\
&+\iint \rho\Delta\rho\phi_m(\rho)\nabla_y\beta^l_\delta(v_m)\bar\beta'_\lambda(\bar\rho_r)\nabla\bar\rho_r\psi\,dx\,dt\\
&+\iint \rho\Delta\rho\phi_m(\rho)\nabla_y\beta^l_\delta(v_m)\bar\beta_\lambda(\bar\rho_r)\nabla\psi\,dx\,dt = 0.
\end{aligned}
$$

By using (3.8), the definition of ϕ_m, the bounds (3.9)–(3.10) and the fact that for fixed δ and fixed λ we have that β^l and $\bar\beta_\delta$ are smooth and satisfy the bounds (2.8) and (2.9), we have that as $r\to 0$, after a possible passage to subsequence, by using the dominated convergence theorem the following identity holds true:

$$
\begin{aligned}
&\iint \rho\beta^l_\delta(v_m)\bar\beta'_\lambda(\rho)\partial_t\rho\psi\,dx\,dt + \iint \rho\beta^l_\delta(v_m)\bar\beta_\lambda(\rho)\partial_t\psi\,dx\,dt\\
&+\iint \rho u\beta^l_\delta(v_m)\bar\beta'_\lambda(\rho)\nabla\rho\psi\,dx\,dt + \iint \rho u\beta^l_\delta(v_m)\bar\beta_\lambda(\rho)\nabla\psi\,dx\,dt\\
&-\iint \sqrt{\rho}\,\mathrm{tr}(\mathcal{T})\phi'_m(\rho)\rho u\nabla_y\beta^l_\delta(v_m)\bar\beta_\lambda(\rho)\psi\,dx\,dt\\
&-\iint \phi_m(\rho)M\nabla^2_y\beta^l_\delta(v_m)\nabla v_m\bar\beta_\lambda(\rho)\psi\,dx\,dt\\
&-\iint \phi_m(\rho)M\nabla_y\beta^l_\delta(v_m)\bar\beta'_\lambda(\rho)\nabla\rho\psi\,dx\,dt - \iint \phi_m(\rho)M\nabla_y\beta^l_\delta(v_m)\bar\beta_\lambda(\rho)\nabla\psi\,dx\,dt\\
&-\iint M\phi'_m(\rho)\nabla\rho\nabla_y\beta^l_\delta(v_m)\bar\beta_\lambda(\rho)\psi\,dx\,dt - \iint N\phi_m(\rho)\nabla_y\beta^l_\delta(v_m)\bar\beta_\lambda(\rho)\psi\,dx\,dt
\end{aligned}
$$

$$+\iint \nabla\rho\Delta\rho\phi_m(\rho)\nabla_y\beta_\delta^l(v_m)\bar{\beta}_\lambda(\rho)\psi\,dx\,dt$$
$$+\iint \rho\Delta\rho\phi_m'(\rho)\nabla\rho\nabla_y\beta_\delta^l(v_m)\bar{\beta}_\lambda(\rho)\psi\,dx\,dt$$
$$+\iint \rho\Delta\rho\phi_m(\rho)\nabla_y^2\beta_\delta^l(v_m)\nabla v_m\bar{\beta}_\lambda(\rho)\psi\,dx\,dt$$
$$+\iint \rho\Delta\rho\phi_m(\rho)\nabla_y\beta_\delta^l(v_m)\bar{\beta}_\lambda'(\rho)\nabla\rho\psi\,dx\,dt$$
$$+\iint \rho\Delta\rho\phi_m(\rho)\nabla_y\beta_\delta^l(v_m)\bar{\beta}_\lambda(\rho)\nabla\psi\,dx\,dt = 0. \tag{3.20}$$

Now we are able to perform the limit $m \to \infty$. From the BD entropy estimate (3.7) we can infer that $\log\rho \in L^1_{t,x}$ and consequently the set $\{\rho = 0\}$ has zero measure. Thus the following convergences hold:

$$\begin{aligned}
\phi_m(\rho) &\to 1 && \text{a.e. in } (0,T)\times\mathbb{T}^3 \text{ and } |\phi_m(\rho)| \le 1,\\
v_m &\to u && \text{a.e. in } (0,T)\times\mathbb{T}^3,\\
\rho\phi_m'(\rho) &\to 0 && \text{a.e. in } (0,T)\times\mathbb{T}^3 \text{ and } |\rho\phi_m'(\rho)| \le 2,\\
\sqrt{\rho}\nabla v_m &\to \mathcal{T} && \text{strongly in } L^2_{t,x}.
\end{aligned} \tag{3.21}$$

We only prove that last convergence, since the others are obtained directly from the definition of ϕ_m. By the definitions of v_m and ϕ_m we have

$$\sqrt{\rho}\nabla v_m = \nabla\Big(\frac{\phi_m(\rho)}{\sqrt{\rho}}\rho u\Big) - \phi_m(\rho)\nabla\rho\otimes u.$$

By using that

$$\nabla(\rho u) = \sqrt{\rho}\mathcal{T} + 2\nabla\sqrt{\rho}\otimes\sqrt{\rho}u,$$

we have

$$\sqrt{\rho}\nabla v_m = \phi_m(\rho)\mathcal{T} + 4\phi_m'(\rho)\rho\nabla\rho^{\frac14}\otimes\rho^{\frac14}u.$$

Then

$$\|\sqrt{\rho}\nabla v_m - \mathcal{T}\|_{L^2_{t,x}} \le \|(\phi_m(\rho)-1)\mathcal{T}\|_{L^2_{t,x}} + 4\|\phi_m'(\rho)\rho\nabla\rho^{\frac14}\otimes\rho^{\frac14}u\|_{L^2_{t,x}}, \tag{3.22}$$

and the right-hand side goes to zero by the dominated convergence theorem and (3.21). Next we start to analyze the terms in (3.20). By using the definition of M we have

$$\begin{aligned}
&\iint \phi_m(\rho)M\nabla_y^2\beta_\delta^l(v_m)\nabla v_m\bar{\beta}_\lambda(\rho)\psi\,dx\,dt\\
&\quad= \iint \phi_m(\rho)\sqrt{\rho}\mathcal{T}^s\nabla_y^2\beta_\delta^l(v_m)\nabla v_m\bar{\beta}_\lambda(\rho)\psi\,dx\,dt\\
&\qquad+\varepsilon\iint \phi_m(\rho)\sqrt{\rho}\nabla^2\sqrt{\rho}\nabla_y^2\beta_\delta^l(v_m)\nabla v_m\bar{\beta}_\lambda(\rho)\psi\,dx\,dt\\
&\qquad-4\varepsilon\iint \nabla\rho^{\frac14}\otimes\nabla\rho^{\frac14}\nabla_y^2\beta_\delta^l(v_m)\sqrt{\rho}\nabla v_m\bar{\beta}_\lambda(\rho)\psi\,dx\,dt,
\end{aligned}$$

which thanks to (3.8), the dominated convergence theorem and (3.21) converges to

$$\iint \mathcal{T}^s\mathcal{T}\nabla_y^2\beta_\delta^l(u)\bar{\beta}_\lambda(\rho)\psi\,dx\,dt + \varepsilon\iint \nabla^2\sqrt{\rho}\nabla_y^2\beta_\delta^l(u)\mathcal{T}\bar{\beta}_\lambda(\rho)\psi\,dx\,dt$$
$$- 4\varepsilon\iint \nabla\rho^{\frac14}\otimes\nabla\rho^{\frac14}\mathcal{T}\nabla_y^2\beta_\delta^l(u)\bar{\beta}_\lambda(\rho)\psi\,dx\,dt.$$

Then, by using the definition of M, we have

$$\begin{aligned}
&\iint \phi_m(\rho)M\nabla_y\beta_\delta^l(v_m)\bar{\beta}'_\lambda(\rho)\nabla\rho\psi\,dx\,dt\\
&\quad= \iint \phi_m(\rho)\sqrt{\rho}\mathcal{T}^s\nabla_y\beta_\delta^l(v_m)\bar{\beta}'_\lambda(\rho)\nabla\rho\psi\,dx\,dt\\
&\qquad+ \varepsilon\iint \phi_m(\rho)\sqrt{\rho}\nabla^2\sqrt{\rho}\nabla_y\beta_\delta^l(v_m)\bar{\beta}'_\lambda(\rho)\nabla\rho\psi\,dx\,dt\\
&\qquad- \varepsilon\iint \phi_m(\rho)\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}\nabla_y\beta_\delta^l(v_m)\bar{\beta}'_\lambda(\rho)\nabla\rho\psi\,dx\,dt,
\end{aligned}$$

which by (3.8) and the dominated convergence theorem converges to

$$\iint \sqrt{\rho}\mathcal{T}^s\nabla_y\beta_\delta^l(u)\bar{\beta}'_\lambda(\rho)\nabla\rho\psi\,dx\,dt + \varepsilon\iint \sqrt{\rho}\nabla^2\sqrt{\rho}\nabla_y\beta_\delta^l(u)\bar{\beta}'_\lambda(\rho)\nabla\rho\psi\,dx\,dt$$
$$- \varepsilon\iint \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}\nabla_y\beta_\delta^l(u)\bar{\beta}'_\lambda(\rho)\nabla\rho\psi\,dx\,dt.$$

Next, again by the definition of M, we have

$$\begin{aligned}
&\iint M\phi'_m(\rho)\nabla\rho\nabla_y\beta_\delta^l(v_m)\bar{\beta}_\lambda(\rho)\psi\,dx\,dt = \iint \sqrt{\rho}\mathcal{T}^s\phi'_m(\rho)\nabla\rho\nabla_y\beta_\delta^l(v_m)\bar{\beta}_\lambda(\rho)\psi\,dx\,dt\\
&\qquad+ \varepsilon\iint \sqrt{\rho}\nabla^2\sqrt{\rho}\phi'_m(\rho)\nabla\rho\nabla_y\beta_\delta^l(v_m)\bar{\beta}_\lambda(\rho)\psi\,dx\,dt\\
&\qquad- \varepsilon\iint \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}\phi'_m(\rho)\nabla\rho\nabla_y\beta_\delta^l(v_m)\bar{\beta}_\lambda(\rho)\psi\,dx\,dt\\
&\quad= 2\iint \mathcal{T}^s\rho\phi'_m(\rho)\nabla\sqrt{\rho}\nabla_y\beta_\delta^l(v_m)\bar{\beta}_\lambda(\rho)\psi\,dx\,dt\\
&\qquad+ 2\varepsilon\iint \nabla^2\sqrt{\rho}\rho\phi'_m(\rho)\nabla\sqrt{\rho}\nabla_y\beta_\delta^l(v_m)\bar{\beta}_\lambda(\rho)\psi\,dx\,dt\\
&\qquad- 4\varepsilon\iint \nabla\rho^{\frac14}\otimes\nabla\rho^{\frac14}\rho\phi'_m(\rho)\nabla\sqrt{\rho}\nabla_y\beta_\delta^l(v_m)\bar{\beta}_\lambda(\rho)\psi\,dx\,dt,
\end{aligned}$$

which thanks to (3.8), (3.21) and the dominated convergence theorem converges to 0. Finally, we consider the term

$$\iint \rho\Delta\rho\phi_m(\rho)\nabla_y^2\beta_\delta^l(v_m)\nabla v_m\bar{\beta}_\lambda(\rho)\psi\,dx\,dt.$$

In order to show the convergence of this term we need to use the additional truncation of ρ at height λ. We have

$$\begin{aligned}\iint \rho\Delta\rho\phi_m(\rho)\nabla_y^2\beta_\delta^l(v_m)\nabla v_m\bar{\beta}_\lambda(\rho)\psi\,dx\,dt \\ = \iint \sqrt{\rho}\Delta\rho\phi_m(\rho)\nabla_y^2\beta_\delta^l(v_m)\mathcal{T}\bar{\beta}_\lambda(\rho)\psi\,dx\,dt \\ + \iint (\sqrt{\rho}\Delta\rho\nabla_y^2\beta_\delta^l(v_m)(\sqrt{\rho}\nabla v_m - \mathcal{T})\bar{\beta}_\lambda(\rho)\psi\,dx\,dt.\end{aligned}$$

Regarding the first term, by noticing that

$$|\sqrt{\rho}\Delta\rho\phi_m(\rho)\nabla_y^2\beta_\delta^l(v_m)\mathcal{T}\bar{\beta}_\lambda(\rho)\psi| \le \frac{C}{\sqrt{\lambda}}(|\Delta\rho|^2 + |\mathcal{T}|^2),$$

we have that the dominated convergence theorem implies that

$$\iint \sqrt{\rho}\Delta\rho\phi_m(\rho)\nabla_y^2\beta_\delta^l(v_m)\mathcal{T}\bar{\beta}_\lambda(\rho)\psi\,dx\,dt \to \iint \sqrt{\rho}\Delta\rho\nabla_y^2\beta_\delta^l(u)\mathcal{T}\bar{\beta}_\lambda(\rho)\psi\,dx\,dt.$$

Concerning the second one we have

$$\begin{aligned}\left|\iint \sqrt{\rho}\Delta\rho\nabla_y^2\beta_\delta^l(v_m)(\sqrt{\rho}\nabla v_m - \mathcal{T})\bar{\beta}_\lambda(\rho)\psi\,dx\,dt\right| \\ \le C_{\lambda,\delta,\varepsilon}\|\Delta\rho\|_{L^2_{t,x}}\|\sqrt{\rho}\nabla v_m - \mathcal{T}\|_{L^2_{t,x}} \to 0,\end{aligned}$$

where the limit yields from (3.22). For all the other terms in (3.20), the analysis of the limit as $m \to \infty$ for fixed ε is a consequence of the convergences of ϕ_m and v_m, a combination of the estimates in (3.8) and the dominated convergence theorem. Equality (3.11) is then proved for any $\psi \in C_c^\infty((0,T)\times\mathbb{T}^3;\mathbb{R})$. The initial data can be recovered by using the weak continuity of ρu in Definition 3.1, and by considering $\chi_n(t)\psi(t,x)$ as a test function, with $\psi \in C^\infty([0,T);C_c^\infty(\mathbb{T}^3))$ and χ_n being an approximation of the Dirac delta in $t=0$.

Now we are going to prove identity (3.14). Let us multiply (3.5) by $\overline{\phi_m(\rho)\hat{\beta}_\delta(\overline{v_m}_r)}_r\varphi$ with $\varphi \in C_c^\infty((0,T)\times\mathbb{T}^3;\mathbb{R})$ to obtain

$$\begin{aligned}\iint \sqrt{\rho}\mathcal{T}\overline{\phi_m(\rho)\hat{\beta}_\delta(\overline{v_m}_r)}_r\varphi\,dx\,dt = &-\iint \phi_m(\rho)\hat{\beta}_\delta(\overline{v_m}_r)\overline{\rho u\otimes\nabla\varphi}_r\,dx\,dt \\ &-\iint \overline{\rho u\varphi}_r\phi_m'(\rho)\nabla\rho\hat{\beta}_\delta(\overline{v_m}_r)\,dx\,dt \\ &-\iint \overline{\rho u\varphi}_r\phi_m(\rho)\nabla_y\hat{\beta}_\delta(\overline{v_m}_r)\nabla\overline{v_m}_r\,dx\,dt \\ &-2\iint \overline{\sqrt{\rho}u\otimes\nabla\sqrt{\rho}\varphi}_r\phi_m(\rho)\hat{\beta}_\delta(\overline{v_m}_r)\,dx\,dt.\end{aligned}$$

By sending $r \to 0$ and using (3.8) we easily get

$$\begin{aligned}\iint \sqrt{\rho}\mathcal{T}\phi_m(\rho)\hat{\beta}_\delta(v_m)\varphi\,dx\,dt &= -\iint \phi_m(\rho)\hat{\beta}_\delta(v_m)\rho u\otimes\nabla\varphi\,dx\,dt\\ &\quad-\iint \rho u\varphi\phi_m'(\rho)\nabla\rho\hat{\beta}_\delta(v_m)\,dx\,dt\\ &\quad-\iint \rho u\varphi\phi_m(\rho)\nabla_y\hat{\beta}_\delta(v_m)\nabla v_m\,dx\,dt\\ &\quad-2\iint \sqrt{\rho}u\otimes\nabla\sqrt{\rho}\varphi\phi_m(\rho)\hat{\beta}_\delta(v_m)\,dx\,dt.\end{aligned}$$

Then, by sending $m \to \infty$ and using (3.8) and (3.21), we easily get (3.14). ■

4. Global existence of weak solutions

In this section we are going to prove the main result of our paper.

4.1. Bounds independent of ε

We collect the ε-independent bounds from (3.6) and (3.7), which we will use in the sequence. First, we have that, for a generic constant $C > 0$ independent of ε, the following bounds hold true:

$$\begin{gathered}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^\infty_t L^2_x} \le C, \quad \|\nabla\rho_\varepsilon\|_{L^\infty_t L^2_x} \le C, \quad \|\rho_\varepsilon\|_{L^\infty_t(L^1_x\cap L^\gamma_x)} \le C,\\ \|\mathcal{T}_\varepsilon\|_{L^2_{t,x}} \le C, \quad \|\nabla\rho_n^{\gamma/2}\|_{L^2_{t,x}} \le C, \quad \|\Delta\rho_\varepsilon\|_{L^2_{t,x}} \le C,\\ \|\nabla\sqrt{\rho_\varepsilon}\|_{L^\infty_t L^2_x} \le C.\end{gathered} \tag{4.1}$$

Moreover,

$$\|\rho_\varepsilon\|_{L^2_t L^\infty_x} \le C, \quad \|\nabla\rho_\varepsilon\|_{L^{\frac{10}{3}}_{t,x}} \le C, \quad \|\rho_\varepsilon^{\frac{\gamma}{2}}\|_{L^{\frac{10}{3}}_{t,x}} \le C. \tag{4.2}$$

By using (2.3), (4.1), (4.2) we have

$$\|\rho_\varepsilon u_\varepsilon\|_{L^2_{t,x}} \le C, \quad \|\nabla(\rho_\varepsilon u_\varepsilon)\|_{L^2_t L^1_x} \le C. \tag{4.3}$$

By using the continuity equation (2.1) and (4.3) we have

$$\|\partial_t\rho_\varepsilon\|_{L^2_t L^1_x} \le C. \tag{4.4}$$

Finally, from (3.6) we also have

$$\begin{gathered}\|\varepsilon^{\frac12}\nabla^2\sqrt{\rho_\varepsilon}\|_{L^2_{t,x}} \le C, \quad \|\varepsilon^{\frac14}\nabla\rho_\varepsilon^{\frac14}\|_{L^4_{t,x}} \le C,\\ \|\varepsilon^{\frac14}\rho_\varepsilon^{\frac14}u_\varepsilon\|_{L^4_{t,x}}, \quad \|\sqrt{\varepsilon}u_\varepsilon\|_{L^2_{t,x}} \le C.\end{gathered} \tag{4.5}$$

4.2. Convergence lemma

By using the above uniform bounds we prove the following convergences.

Lemma 4.1. *Let* $\{(\rho_\varepsilon, u_\varepsilon, \mathcal{T}_\varepsilon)\}_\varepsilon$ *be a sequence of weak solutions of* (3.1)–(3.2).

(1) *Up to subsequences there exist,* ρ, m, $\mathcal{T}$ *and* Λ *such that*

$$\rho_\varepsilon \to \rho \quad \text{strongly in } L^2(0,T;H^1(\mathbb{T}^3)), \tag{4.6}$$

$$\rho_\varepsilon u_\varepsilon \to m \quad \text{strongly in } L^p(0,T;L^p(\mathbb{T}^3)) \text{ with } p \in [1,2), \tag{4.7}$$

$$\mathcal{T}_\varepsilon \rightharpoonup \mathcal{T} \quad \text{weakly in } L^2((0,T)\times\mathbb{T}^3), \tag{4.8}$$

$$\sqrt{\rho_\varepsilon}u_\varepsilon \overset{*}{\rightharpoonup} \Lambda \quad \text{weakly* in } L^\infty(0,T;L^2(\mathbb{T}^3)). \tag{4.9}$$

Moreover, Λ *is such that* $\sqrt{\rho}\Lambda = m$.

(2) *The following additional convergences hold true for the density:*

$$\nabla\sqrt{\rho_\varepsilon} \rightharpoonup \nabla\sqrt{\rho} \quad \text{weakly in } L^2((0,T)\times\mathbb{T}^3), \tag{4.10}$$

$$\Delta\rho_\varepsilon \rightharpoonup \Delta\rho \quad \text{weakly in } L^2((0,T)\times\mathbb{T}^3), \tag{4.11}$$

$$\rho_\varepsilon^\gamma \to \rho^\gamma \quad \text{strongly in } L^1((0,T)\times\mathbb{T}^3), \tag{4.12}$$

$$\nabla\rho_\varepsilon^{\frac{\gamma}{2}} \rightharpoonup \nabla\rho^{\frac{\gamma}{2}} \quad \text{weakly in } L^2((0,T)\times\mathbb{T}^3). \tag{4.13}$$

Proof. By using (1.1) and (4.2), we have

$$\{\partial_t\rho_\varepsilon\}_\varepsilon \text{ is uniformly bounded in } L^2(0,T;H^{-1}(\mathbb{T}^3)).$$

Then, since $\{\rho_\varepsilon\}_\varepsilon$ is uniformly bounded in $L^2(0,T;H^2(\mathbb{T}^3))$, by using the Aubin–Lions lemma we get (4.6). Next, by using the momentum equations and the bounds (4.1)–(4.2), it is easy to prove that

$$\{\partial_t(\rho_\varepsilon u_\varepsilon)\}_\varepsilon \text{ is uniformly bounded in } L^2(0,T;W^{-2,\frac{3}{2}}(\mathbb{T}^3)).$$

Then, by using (4.2), (4.3) and the Aubin–Lions lemma, (4.7) follows. The convergences (4.8) and (4.9) follow by standard weak compactness theorems and the equality $\sqrt{\rho}\Lambda = m$ follows easily from (4.6) and (4.9). Next, the convergences (4.10), (4.11) follow from the uniform bounds (4.1) and standard weak compactness arguments. Finally, the convergence (4.12) is obtained by using (4.6) and the bound (4.1), and the convergence (4.13) follows by (4.1) and (4.6). ∎

Lemma 4.2. *Let* $f \in C \cap L^\infty(\mathbb{R}^3;\mathbb{R})$ *and* $(\rho_\varepsilon, u_\varepsilon)$ *be a solution of* (1.1)–(1.2) *and let* u *be defined as*

$$u = \begin{cases} \dfrac{m(t,x)}{\rho(t,x)} = \dfrac{\Lambda(t,x)}{\sqrt{\rho(t,x)}}, & (t,x) \in \{\rho > 0\}, \\ 0, & (t,x) \in \{\rho = 0\}. \end{cases} \tag{4.14}$$

Then the following convergences hold:

$$\rho_\varepsilon f(u_\varepsilon) \to \rho f(u) \quad \text{strongly in } L^p((0,T)\times\mathbb{T}^3) \text{ for any } p<6, \tag{4.15}$$

$$\nabla\rho_\varepsilon f(u_\varepsilon) \to \nabla\rho f(u) \quad \text{strongly in } L^p((0,T)\times\mathbb{T}^3) \text{ for any } p<\frac{10}{3}, \tag{4.16}$$

$$\rho_\varepsilon u_\varepsilon f(u_\varepsilon) \to \rho u f(u) \quad \text{strongly in } L^p((0,T)\times\mathbb{T}^3) \text{ for any } p<2, \tag{4.17}$$

$$\rho_\varepsilon^{\frac{\gamma}{2}} f(u_\varepsilon) \to \rho^{\frac{\gamma}{2}} f(u) \quad \text{strongly in } L^p((0,T)\times\mathbb{T}^3) \text{ for any } p<\frac{10}{3}. \tag{4.18}$$

Proof. We first note that, up to a subsequence not relabelled, (4.6) and (4.7) imply

$$\begin{aligned} \rho_\varepsilon &\to \rho && \text{a.e. in } (0,T)\times\mathbb{T}^3,\\ \rho_\varepsilon u_\varepsilon &\to m && \text{a.e. in } (0,T)\times\mathbb{T}^3,\\ \nabla\rho_\varepsilon &\to \nabla\rho && \text{a.e. in } (0,T)\times\mathbb{T}^3. \end{aligned} \tag{4.19}$$

Moreover, by the Fatou lemma we have

$$\iint \liminf_{\varepsilon\to 0} \frac{m_\varepsilon^2}{\rho_\varepsilon}\,dx\,dt \le \liminf_{\varepsilon\to 0} \iint \frac{m_\varepsilon^2}{\rho_\varepsilon} < \infty, \tag{4.20}$$

which implies that $m=0$ on $\{\rho=0\}$ and

$$\sqrt{\rho}u \in L^\infty(0,T;L^2(\mathbb{T}^3)).$$

Moreover, $m=\rho u=\sqrt{\rho}\Lambda$. Let us prove (4.15). On $\{\rho>0\}$, by using (4.19) we have

$$\rho_\varepsilon f(u_\varepsilon) \to \rho f(u) \quad \text{a.e. in } \{\rho>0\}.$$

On the other hand, since $f\in L^\infty(\mathbb{R}^3;\mathbb{R})$ we have

$$|\rho_\varepsilon f(u_\varepsilon)| \le |\rho_\varepsilon|\|f\|_\infty \to 0 \quad \text{a.e. in } \{\rho=0\}.$$

Then $\rho_\varepsilon f(u_\varepsilon)\to\rho f(u)$ a.e. in $(0,T)\times\mathbb{T}^3$ and the convergence in (4.15) follows by the uniform bound

$$\|\rho_\varepsilon\|_{L^6_{t,x}} \le C$$

and Vitali's theorem. Regarding (4.16), from Lemma 4.1 we have that ρ is a Sobolev function; then (see [22])

$$\nabla\rho = 0 \quad \text{a.e. in } \{\rho=0\}.$$

From (4.19) we have

$$\begin{aligned} &\nabla\rho_\varepsilon f(u_\varepsilon) \to \nabla\rho f(u) && \text{a.e. in } \{\rho>0\},\\ &|\nabla\rho_\varepsilon f(u_\varepsilon)| \le |\nabla\rho_\varepsilon|\|f\|_\infty \to 0 && \text{a.e. in } \{\rho=0\}. \end{aligned}$$

Then $\nabla\rho_\varepsilon f(u_\varepsilon)\to\nabla\rho f(u)$ a.e. in $(0,T)\times\mathbb{T}^3$ and (4.16) follows from the uniform bound (4.2) and Vitali's theorem. Concerning (4.17), again (4.19) implies the convergences

$$\begin{aligned} &\rho_\varepsilon u_\varepsilon f(u_\varepsilon) \to m f(u) && \text{a.e. in } \{\rho>0\},\\ &|\rho_\varepsilon u_\varepsilon f(u_\varepsilon)| \le |\rho_\varepsilon u_\varepsilon|\|f\|_\infty \to 0 && \text{a.e. in } \{\rho=0\}, \end{aligned}$$

which, together with (4.2) and Vitali's theorem, imply (4.17). Finally, (4.18) follows by the same arguments used to prove (4.15) and the uniform bounds on the pressure in (4.1). ∎

4.3. Proof of the main theorem

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. Let $\{(\rho_\varepsilon, u_\varepsilon, \mathcal{T}_\varepsilon)\}_\varepsilon$ be a sequence of weak solutions of (3.1)–(3.2). By Lemma 4.1 there exist ρ, m, Λ and $\mathcal{T}$ such that the convergences (4.6), (4.7) and (4.9) hold. Moreover, by defining the velocity u as in Lemma 4.2 we have

$$\sqrt{\rho}u \in L^\infty(0,T;L^2(\mathbb{T}^3)), \quad \mathcal{T} \in L^2((0,T)\times\mathbb{T}^3), \quad m = \sqrt{\rho}\Lambda = \rho u.$$

By using (4.6), (4.7) and (2.6) it is straightforward to prove that

$$\int \rho_\varepsilon^0 \phi(0,x)\,dx + \iint \rho_\varepsilon \phi_t \,dx\,dt + \iint \rho_\varepsilon u_\varepsilon \nabla\phi\,dx\,dt$$

converges to

$$\int \rho^0 \phi(0,x)\,dx + \iint \rho \phi_t \,dx\,dt + \iint \rho u \nabla\phi\,dx\,dt,$$

for any $\phi \in C_c^\infty([0,T)\times\mathbb{T}^3)$. Let us consider the momentum equations. Let $l \in \{1,2,3\}$ be fixed; by using Theorem 3.2 we have that for any $\psi \in C_c^\infty([0,T)\times\mathbb{T}^3;\mathbb{R})$ the following equality holds:

$$\begin{aligned}
&\int \rho^0 \beta_\delta^l(u^0)\bar{\beta}_\lambda(\rho^0)\psi(0,x)\,dx + \iint \rho_\varepsilon \beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\partial_t\psi \\
&\quad - \iint \rho_\varepsilon u_\varepsilon \beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\cdot\nabla\psi\,dx\,dt - \iint \sqrt{\rho_\varepsilon}\mathcal{T}_\varepsilon^s : \nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\otimes\nabla\psi\,dx\,dt \\
&\quad - 2\iint \rho_\varepsilon^{\frac{\gamma}{2}}\nabla\rho_\varepsilon^{\frac{\gamma}{2}}\cdot\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\psi\,dx\,dt - \iint \nabla\rho_\varepsilon\Delta\rho_\varepsilon\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\psi\,dx\,dt \\
&\quad - \iint \rho_\varepsilon\Delta\rho_\varepsilon\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\nabla\psi\,dx\,dt + \iint R_\varepsilon^{\delta,\lambda}\psi\,dx\,dt + \iint \tilde{R}_\varepsilon^{\delta,\lambda}\psi\,dx\,dt \\
&= 0,
\end{aligned} \tag{4.21}$$

where the remainders are

$$\begin{aligned}
R_\varepsilon^{\delta,\lambda} = \sum_{i=1}^{6} R_{\varepsilon,i}^{\delta,\lambda} &= \rho_\varepsilon\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda'(\rho_\varepsilon)\partial_t\rho_\varepsilon + \rho_\varepsilon u_\varepsilon\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda'(\rho_\varepsilon)\nabla\rho_\varepsilon \\
&\quad - \sqrt{\rho_\varepsilon}\mathcal{T}_e^s : \nabla_y\beta_\delta^l(u_\varepsilon)\otimes\nabla\rho_\varepsilon\bar{\beta}_\lambda'(\rho_\varepsilon) + \sqrt{\rho_\varepsilon}\Delta\rho_\varepsilon\nabla_y^2\beta_\delta^l(u_\varepsilon)\mathcal{T}_\varepsilon\bar{\beta}_\lambda(\rho_\varepsilon) \\
&\quad + \rho_\varepsilon\Delta\rho_\varepsilon\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda'(\rho_\varepsilon)\nabla\rho_\varepsilon - \mathcal{T}_\varepsilon^s\mathcal{T}_\varepsilon\nabla_y^2\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon),
\end{aligned}$$

$$\begin{aligned}\widetilde{R}_\varepsilon^{\delta,\lambda} = \sum_{i=1}^4 \widetilde{R}_{\varepsilon,i}^{\delta,\lambda} &= -\varepsilon\nabla^2\sqrt{\rho_\varepsilon}\mathcal{T}_\varepsilon\nabla_y^2\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon) + 4\varepsilon\nabla\rho_\varepsilon^{\frac14}\otimes\nabla\rho_\varepsilon^{\frac14}\mathcal{T}\nabla_y^2\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\\ &- \varepsilon\sqrt{\rho_\varepsilon}\nabla^2\sqrt{\rho_\varepsilon}\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda'(\rho_\varepsilon)\nabla\rho_\varepsilon\\ &+ 4\varepsilon\sqrt{\rho_\varepsilon}\nabla\rho_\varepsilon^{\frac14}\otimes\nabla\rho_\varepsilon^{\frac14}\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda'(\rho_\varepsilon)\nabla\rho_\varepsilon\\ &- \varepsilon\rho_\varepsilon|u_\varepsilon|^2u_\varepsilon\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon) - \varepsilon u_\varepsilon\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon).\end{aligned}$$

We first perform the limit as ε goes to 0 for δ and λ fixed. Notice that, since $\bar{\beta}_\lambda \in L^\infty(\mathbb{R})$, and $\{\rho_\varepsilon\}_\varepsilon$ converges almost everywhere, by dominated convergence we have

$$\bar{\beta}_\lambda(\rho_\varepsilon) \to \bar{\beta}_\lambda(\rho) \quad \text{strongly in } L^q((0,T)\times\mathbb{T}^3) \text{ for any } q<\infty. \tag{4.22}$$

By using (4.15) with $p=2$ and choosing $q=2$ in (4.22) we have

$$\iint \rho_\varepsilon\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\partial_t\psi\,dx\,dt \to \iint \rho\beta_\delta^l(u)\bar{\beta}_\lambda(\rho)\partial_t\psi\,dx\,dt.$$

Next, by (4.17) with $p=3/2$ and choosing $q=3$ in (4.22) we get

$$\iint \rho_\varepsilon u_\varepsilon\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\cdot\nabla\psi\,dx\,dt \to \iint \rho u\beta_\delta^l(u)\bar{\beta}_\lambda(\rho)\cdot\nabla\psi\,dx\,dt.$$

By using (4.8), (4.15) with $p=4$ and (4.22) with $q=4$ it follows that

$$\iint \mathcal{T}_\varepsilon^s : \sqrt{\rho_\varepsilon}\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\otimes\nabla\psi\,dx\,dt \to \iint \sqrt{\rho}\mathcal{T} : \nabla_y\beta_\delta^l(u)\bar{\beta}_\lambda(\rho)\otimes\nabla\psi\,dx\,dt.$$

By using (4.13), (4.18) with $p=3$ and (4.22) with $q=6$ it follows that

$$\iint \rho_\varepsilon^{\frac{\gamma}{2}}\nabla\rho_\varepsilon^{\frac{\gamma}{2}}\cdot\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\psi\,dx\,dt \to \iint \rho^{\frac{\gamma}{2}}\nabla\rho^{\frac{\gamma}{2}}\cdot\nabla_y\beta_\delta^l(u)\bar{\beta}_\lambda(\rho)\psi\,dx\,dt.$$

By using (4.11), (4.16) with $p=3$ and (4.22) with $q=6$ it follows that

$$\iint \nabla\rho_\varepsilon\Delta\rho_\varepsilon\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\psi\,dx\,dt \to \iint \nabla\rho\Delta\rho\nabla_y\beta_\delta^l(u)\bar{\beta}_\lambda(\rho)\psi\,dx\,dt.$$

Next, by using (4.11), (4.15) with $p=3$ and (4.22) with $q=6$, it follows that

$$\iint \rho_\varepsilon\Delta\rho_\varepsilon\nabla_y\beta_\delta^l(u_\varepsilon)\bar{\beta}_\lambda(\rho_\varepsilon)\nabla\psi\,dx\,dt \to \iint \rho\Delta\rho\nabla_y\beta_\delta^l(u)\bar{\beta}_\lambda(\rho)\nabla\psi\,dx\,dt.$$

It remains to study the remainders $\widetilde{R}_\varepsilon^{\delta,\lambda}$ and $R_\varepsilon^{\delta,\lambda}$. Regarding $\widetilde{R}_\varepsilon^{\delta,\lambda}$ we prove the following convergence:

$$\widetilde{R}_\varepsilon^{\delta,\lambda} \to 0 \text{ in } L^1_{t,x}.$$

Indeed, by considering term by term and using the uniform bounds (4.1) and (4.5) we have

$$
\begin{aligned}
\|\widetilde{R}^{\delta,\lambda}_{\varepsilon,1}\|_{L^1_{t,x}} &\le C_{\delta,\lambda}\sqrt{\varepsilon}\|\sqrt{\varepsilon}\nabla^2\sqrt{\rho_\varepsilon}\|_{L^2_{t,x}}\|\mathcal{T}_\varepsilon\|_{L^2_{t,x}} \le C_{\delta,\lambda}\sqrt{\varepsilon},\\
\|\widetilde{R}^{\delta,\lambda}_{\varepsilon,2}\|_{L^1_{t,x}} &\le C_{\delta,\lambda}\sqrt{\varepsilon}\|\varepsilon^{\frac14}\nabla\rho_\varepsilon^{\frac14}\|^2_{L^4_{t,x}}\|\mathcal{T}_\varepsilon\|_{L^2_{t,x}} \le C_{\delta,\lambda}\sqrt{\varepsilon},\\
\|\widetilde{R}^{\delta,\lambda}_{\varepsilon,3}\|_{L^1_{t,x}} &\le C_{\delta,\lambda}\sqrt{\varepsilon}\|\sqrt{\varepsilon}\nabla^2\sqrt{\rho_\varepsilon}\|_{L^2_{t,x}}\|\nabla\rho_\varepsilon\|_{L^2_{t,x}} \le C_{\delta,\lambda}\sqrt{\varepsilon},\\
\|\widetilde{R}^{\delta,\lambda}_{\varepsilon,4}\|_{L^1_{t,x}} &\le C_{\delta,\lambda}\sqrt{\varepsilon}\|\varepsilon^{\frac14}\nabla\rho_\varepsilon^{\frac14}\|^2_{L^4_{t,x}}\|\nabla\rho_\varepsilon\|_{L^2_{t,x}} \le C_{\delta,\lambda}\sqrt{\varepsilon},\\
\|\widetilde{R}^{\delta,\lambda}_{\varepsilon,5}\|_{L^1_{t,x}} &\le C_{\delta,\lambda}\varepsilon^{\frac14}\|\rho\|^{\frac14}_{L^1_{t,x}}\|\varepsilon^{\frac14}\rho_\varepsilon^{\frac14}u_\varepsilon\|^3_{L^4_{t,x}} \le C_{\delta,\lambda}\varepsilon^{\frac14},\\
\|\widetilde{R}^{\delta,\lambda}_{\varepsilon,6}\|_{L^1_{t,x}} &\le C_{\delta,\lambda}\sqrt{\varepsilon}\|\sqrt{\varepsilon}u_\varepsilon\|_{L^2_{t,x}} \le C_{\delta,\lambda}\sqrt{\varepsilon}.
\end{aligned}
$$

Now we consider $R^{\delta,\lambda}_\varepsilon$. We claim that there exists a $C > 0$ independent of ε, δ and λ such that

$$
\|R^{\delta,\lambda}_\varepsilon\|_{L^1_{t,x}} \le C\Big(\frac{\delta}{\sqrt{\lambda}} + \frac{\lambda}{\delta} + \lambda + \delta\Big). \tag{4.23}
$$

In order to prove (4.23) we estimate all the terms in (3.12) separately. By using the uniform bounds (4.1), (4.2), (4.4) and the bounds on the truncations (2.8) and (2.9) we have

$$
\begin{aligned}
\|R^{\delta,\lambda}_{\varepsilon,1}\|_{L^1_{t,x}} &\le \|\rho_\varepsilon\|_{L^2(L^\infty)}\|\partial_t\rho_\varepsilon\|_{L^2(L^1)}\|\beta^l_\delta(u_\varepsilon)\|_{L^\infty_{t,x}}\|\bar\beta'_\lambda(\rho_\varepsilon)\|_{L^\infty_{t,x}} \le C\frac{\lambda}{\delta},\\
\|R^{\delta,\lambda}_{\varepsilon,2}\|_{L^1_{t,x}} &\le \|\rho_\varepsilon u_\varepsilon\|_{L^2_{t,x}}\|\nabla\rho_\varepsilon\|_{L^2_{t,x}}\|\beta^l_\delta(u_\varepsilon)\|_{L^\infty_{t,x}}\|\bar\beta'_\lambda(\rho_\varepsilon)\|_{L^\infty_{t,x}} \le C\frac{\lambda}{\delta},\\
\|R^{\delta,\lambda}_{\varepsilon,3}\|_{L^1_{t,x}} &\le \|\rho_\varepsilon\|_{L^2(L^\infty)}\|\mathcal{T}^s_\varepsilon\|_{L^2_{t,x}}\|\nabla\rho_\varepsilon\|_{L^\infty(L^2)}\|\nabla_y\beta^l_\delta(u_\varepsilon)\|_{L^\infty_{t,x}}\|\bar\beta'_\lambda(\rho_\varepsilon)\|_{L^\infty_{t,x}} \le C\lambda,\\
\|R^{\delta,\lambda}_{\varepsilon,4}\|_{L^1_{t,x}} &\le \|\Delta\rho_\varepsilon\|_{L^2_{t,x}}\|\mathcal{T}^s_\varepsilon\|_{L^2_{t,x}}\|\nabla^2_y\beta^l_\delta(u_\varepsilon)\|_{L^\infty_{t,x}}\|\sqrt{\rho_\varepsilon}\bar\beta_\lambda(\rho_\varepsilon)\|_{L^\infty_{t,x}} \le C\frac{\delta}{\sqrt{\lambda}},\\
\|R^{\delta,\lambda}_{\varepsilon,5}\|_{L^1_{t,x}} &\le \|\rho_\varepsilon\|_{L^2(L^\infty)}\|\Delta\rho_\varepsilon\|_{L^2_{t,x}}\|\nabla\rho_\varepsilon\|_{L^\infty(L^2)}\|\nabla_y\beta^l_\delta(u_\varepsilon)\|_{L^\infty_{t,x}}\|\bar\beta'_\lambda(\rho_\varepsilon)\|_{L^\infty_{t,x}} \le C\lambda,\\
\|R^{\delta,\lambda}_{\varepsilon,6}\|_{L^1_{t,x}} &\le \|\mathcal{T}_\varepsilon\|^2_{L^2_{t,x}}\|\nabla^2_y\beta^l_\delta(u_\varepsilon)\|_{L^\infty_{t,x}}\|\bar\beta_\lambda(\rho_\varepsilon)\|_{L^\infty_{t,x}} \le C\delta.
\end{aligned}
$$

Then (4.23) is proved and, when ε goes to 0, we have that $(\rho, u, \mathcal{T})$ satisfies the following integral equality:

$$
\begin{aligned}
&\iint \rho u\beta^l_\delta(u)\bar\beta_\lambda(\rho)\cdot\nabla\psi\,dx\,dt - 2\nu\iint\sqrt{\rho}\mathcal{S} : \nabla_y\beta^l_\delta(u)\bar\beta_\lambda(\rho)\otimes\nabla\psi\,dx\,dt\\
&\quad - \iint \rho^{\frac{\gamma}{2}}\nabla\rho^{\frac{\gamma}{2}}\cdot\nabla_y\beta^l_\delta(u)\bar\beta_\lambda(\rho)\psi\,dx\,dt\\
&\quad - 2\kappa^2\iint\nabla\rho\Delta\rho\nabla_y\beta^l_\delta(u)\bar\beta_\lambda(\rho)\psi\,dx\,dt - 2\kappa^2\iint\rho\Delta\rho\nabla_y\beta^l_\delta(u)\bar\beta_\lambda(\rho)\nabla\psi\,dx\,dt\\
&\quad - \int\rho^0\beta^l_\delta(u^0)\bar\beta_\lambda(\rho^0)\psi(0,x)\,dx + \langle\mu^{\delta,\lambda},\psi\rangle = 0,
\end{aligned} \tag{4.24}
$$

where $\mu^{\delta,\lambda}$ is a measure such that as $\varepsilon \to 0$,

$$\iint R_\varepsilon^{\delta,\lambda}\psi\,dx\,dt \to \langle \bar{\mu}^{\delta,\lambda}, \psi \rangle,$$

and its total variation satisfies

$$|\mu^{\delta,\lambda}|(\mathbb{T}^3) \leq C\Big(\frac{\delta}{\sqrt{\lambda}} + \frac{\lambda}{\delta} + \lambda + \delta\Big). \tag{4.25}$$

Let $\delta = \lambda^\alpha$ with $\alpha \in (1/2, 1)$; then when $\lambda \to 0$ we have

$$|\mu^{\lambda^\alpha,\lambda}|(\mathbb{T}^3) \to 0$$

and by (2.11), (2.12) and the dominated convergence theorem we have that (4.24) converges to

$$\begin{aligned} \int \rho^0 u^{l,0}\psi(0,x)\,dx + \iint \rho u^l \partial_t\psi + \iint \rho u u^l \cdot \nabla\psi\,dx\,dt - \iint \sqrt{\rho}\mathcal{T}_{lj}^s \nabla_j\psi\,dx\,dt \\ - \iint \rho^{\frac{\gamma}{2}}\nabla_l \rho^{\frac{\gamma}{2}}\psi\,dx\,dt - \iint \nabla_l\rho\Delta\rho\psi\,dx\,dt - \iint \rho\Delta\rho\nabla_l\psi\,dx\,dt = 0. \end{aligned} \tag{4.26}$$

It remains to prove (2.3). By using Theorem 3.2 (2) we have that for any $\varphi \in C_c^\infty((0,T)\times \mathbb{T}^3;\mathbb{R})$ it holds that

$$\begin{aligned} \iint \sqrt{\rho_\varepsilon}\mathcal{T}_\varepsilon \hat{\beta}_\delta(u_\varepsilon)\varphi\,dx\,dt = &-\iint \hat{\beta}_\delta(u_\varepsilon)\rho_\varepsilon u_\varepsilon \otimes \nabla\varphi\,dx\,dt \\ &- \iint \sqrt{\rho_\varepsilon}u_\varepsilon\varphi\nabla_y\hat{\beta}_\delta(u_\varepsilon)\mathcal{T}_\varepsilon\,dx\,dt \\ &- 2\iint \sqrt{\rho_\varepsilon}u_\varepsilon \otimes \nabla\sqrt{\rho_\varepsilon}\varphi\hat{\beta}_\delta(u_\varepsilon)\,dx\,dt. \end{aligned} \tag{4.27}$$

For fixed δ, by using the convergence (4.8) and (4.15) with $p = 4$, we have

$$\iint \sqrt{\rho_\varepsilon}\hat{\beta}_\delta(u_\varepsilon)\mathcal{T}_\varepsilon\varphi\,dx\,dt \to \iint \sqrt{\rho}\hat{\beta}_\delta(u)\mathcal{T}\varphi\,dx\,dt.$$

Next, we have

$$\iint \hat{\beta}_\delta(u_\varepsilon)\rho_\varepsilon u_\varepsilon \otimes \nabla\varphi\,dx\,dt \to \iint \hat{\beta}_\delta(u)\rho u \otimes \nabla\varphi\,dx\,dt,$$

owing to (4.17) with $p = 1$. By using (2.10), (4.15) with $p = 2$ and the weak convergence of $\nabla\sqrt{\rho_\varepsilon}$ in $L^2_{t,x}$ we get

$$\iint \sqrt{\rho_\varepsilon}u_\varepsilon \otimes \nabla\sqrt{\rho_\varepsilon}\hat{\beta}_\delta(u_\varepsilon)\varphi\,dx\,dt \to \iint \sqrt{\rho}u \otimes \nabla\sqrt{\rho}\hat{\beta}_\delta(u)\varphi\,dx\,dt.$$

Let

$$\bar{R}^{\delta}_{\varepsilon} = \sqrt{\rho_{\varepsilon}} u_{\varepsilon} \varphi \nabla_y \hat{\beta}_{\delta}(u_{\varepsilon}) \mathcal{T}_{\varepsilon}; \tag{4.28}$$

by using (4.1) and (2.10) we have

$$\| \bar{R}^{\delta}_{n} \|_{L^1_{t,x}} \leq C \| \sqrt{\rho_{\varepsilon}} u_{\varepsilon} \|_{L^{\infty}(L^2_{t,x})} \| \mathcal{T}_{\varepsilon} \|_{L^2_{t,x}} \| \nabla_y \hat{\beta}_{\delta}(u_{\varepsilon}) \|_{L^{\infty}_{t,x}} \leq C\delta,$$

and then there exists a measure $\bar{\mu}^{\delta}$ such that

$$\iint \bar{R}^{\delta}_{\varepsilon} \nabla\varphi \, dx \, dt \to \langle \bar{\mu}^{\delta}, \nabla\varphi \rangle, \tag{4.29}$$

and its total variation satisfies

$$|\bar{\mu}^{\delta}|(\mathbb{T}^3) \leq C\delta.$$

Collecting the previous convergences, we have

$$\begin{aligned} \iint \sqrt{\rho_{\varepsilon}} \hat{\beta}_{\delta}(u_{\varepsilon}) \mathcal{T}_{\varepsilon} \varphi \, dx \, dt = & - \iint \hat{\beta}_{\delta}(u) \rho u \otimes \nabla\varphi \, dx \, dt \\ & - 2 \iint \sqrt{\rho} u \otimes \nabla \sqrt{\rho} \hat{\beta}_{\delta}(u) \varphi \, dx \, dt \\ & - \langle \bar{\mu}^{\delta}, \nabla\psi \rangle. \end{aligned}$$

By using (2.11), the dominated convergence theorem and (4.29) we get (2.3). Finally, the energy inequality follows from the lower semicontinuity of the norms. ∎

Acknowledgements. The authors gratefully acknowledge useful discussions with Didier Bresch.

Funding. Partially supported by the INdAM-GNAMPA project “Esistenza, limiti singolari e comportamento asintotico per equazioni Eulero/Navier–Stokes–Korteweg”.

References

[1] P. Antonelli, L. E. Hientzsch, and P. Marcati, On the low Mach number limit for quantum Navier-Stokes equations. *SIAM J. Math. Anal.* **52** (2020), no. 6, 6105–6139 Zbl 1454.35272 MR 4181094

[2] P. Antonelli and P. Marcati, On the finite energy weak solutions to a system in quantum fluid dynamics. *Comm. Math. Phys.* **287** (2009), no. 2, 657–686 Zbl 1177.82127 MR 2481754

[3] P. Antonelli and P. Marcati, The quantum hydrodynamics system in two space dimensions. *Arch. Ration. Mech. Anal.* **203** (2012), no. 2, 499–527 Zbl 1290.76165 MR 2885568

[4] P. Antonelli and S. Spirito, Global existence of finite energy weak solutions of quantum Navier-Stokes equations. *Arch. Ration. Mech. Anal.* **225** (2017), no. 3, 1161–1199 Zbl 1375.35306 MR 3667286

[5] P. Antonelli and S. Spirito, On the compactness of finite energy weak solutions to the quantum Navier-Stokes equations. *J. Hyperbolic Differ. Equ.* **15** (2018), no. 1, 133–147 Zbl 1390.35223 MR 3786772

[6] P. Antonelli and S. Spirito, On the compactness of weak solutions to the Navier-Stokes-Korteweg equations for capillary fluids. *Nonlinear Anal.* **187** (2019), 110–124 Zbl 1428.35342 MR 3942757

[7] C. Audiard and B. Haspot, Global well-posedness of the Euler-Korteweg system for small irrotational data. *Comm. Math. Phys.* **351** (2017), no. 1, 201–247 Zbl 1369.35047 MR 3613503

[8] S. Benzoni-Gavage, Stability of multi-dimensional phase transitions in a van der Waals fluid. *Nonlinear Anal.* **31** (1998), no. 1-2, 243–263 Zbl 0928.76015 MR 1487544

[9] S. Benzoni-Gavage, R. Danchin, and S. Descombes, On the well-posedness for the Euler-Korteweg model in several space dimensions. *Indiana Univ. Math. J.* **56** (2007), no. 4, 1499–1579 Zbl 1125.76060 MR 2354691

[10] D. Bresch and B. Desjardins, Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model. *Comm. Math. Phys.* **238** (2003), no. 1-2, 211–223 Zbl 1037.76012 MR 1989675

[11] D. Bresch, B. Desjardins, and C.-K. Lin, On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Comm. Partial Differential Equations* **28** (2003), no. 3-4, 843–868 Zbl 1106.76436 MR 1978317

[12] D. Bresch, M. Gisclon, and I. Lacroix-Violet, On Navier-Stokes-Korteweg and Euler-Korteweg systems: application to quantum fluids models. *Arch. Ration. Mech. Anal.* **233** (2019), no. 3, 975–1025 Zbl 1433.35217 MR 3961293

[13] D. Bresch and P.-E. Jabin, Global existence of weak solutions for compressible Navier-Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor. *Ann. of Math. (2)* **188** (2018), no. 2, 577–684 Zbl 1405.35133 MR 3862947

[14] D. Bresch, A. Vasseur, and C. Yu, Global existence of entropy-weak solutions to the compressible Navier-Stokes equations with non-linear density dependent viscosities. 2019, arXiv:1905.02701

[15] M. Caggio and D. Donatelli, High Mach number limit for Korteweg fluids with density dependent viscosity. *J. Differential Equations* **277** (2021), 1–37 Zbl 1456.76110 MR 4198204

[16] R. Carles, K. Carrapatoso, and M. Hillairet, Rigidity results in generalized isothermal fluids. *Ann. H. Lebesgue* **1** (2018), 47–85 Zbl 1428.35348 MR 3963286

[17] R. J. DiPerna and P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.* **98** (1989), no. 3, 511–547 Zbl 0696.34049 MR 1022305

[18] D. Donatelli, E. Feireisl, and P. Marcati, Well/ill posedness for the Euler-Korteweg-Poisson system and related problems. *Comm. Partial Differential Equations* **40** (2015), no. 7, 1314–1335 Zbl 1326.35253 MR 3341206

[19] D. Donatelli and P. Marcati, Quasi-neutral limit, dispersion, and oscillations for Korteweg-type fluids. *SIAM J. Math. Anal.* **47** (2015), no. 3, 2265–2282 Zbl 1320.35277 MR 3356984

[20] D. Donatelli and P. Marcati, Low Mach number limit for the quantum hydrodynamics system. *Res. Math. Sci.* **3** (2016), Paper No. 13, 13 Zbl 1344.35093 MR 3500498

[21] J. E. Dunn and J. Serrin, On the thermomechanics of interstitial working. *Arch. Rational Mech. Anal.* **88** (1985), no. 2, 95–133 Zbl 0582.73004 MR 775366

[22] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*. Stud. Adv. Math., CRC Press, Boca Raton, FL, 1992 Zbl 0804.28001 MR 1158660

[23] E. Feireisl, On compactness of solutions to the compressible isentropic Navier-Stokes equations when the density is not square integrable. *Comment. Math. Univ. Carolin.* **42** (2001), no. 1, 83–98 Zbl 1115.35096 MR 1825374

[24] J. Giesselmann, C. Lattanzio, and A. E. Tzavaras, Relative energy for the Korteweg theory and related Hamiltonian flows in gas dynamics. *Arch. Ration. Mech. Anal.* **223** (2017), no. 3, 1427–1484 Zbl 1359.35140 MR 3594360

[25] Z. Guo, Q. Jiu, and Z. Xin, Spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients. *SIAM J. Math. Anal.* **39** (2008), no. 5, 1402–1427 Zbl 1151.35071 MR 2377283

[26] R. Hagan and M. Slemrod, The viscosity-capillarity criterion for shocks and phase transitions. *Arch. Rational Mech. Anal.* **83** (1983), no. 4, 333–361 Zbl 0531.76069 MR 714979

[27] H. Hattori and D. N. Li, Solutions for two-dimensional system for materials of Korteweg type. *SIAM J. Math. Anal.* **25** (1994), no. 1, 85–98 Zbl 0817.35076 MR 1257143

[28] H. H. H. Hattori and D. Li, Global solutions of a high-dimensional system for Korteweg materials. *J. Math. Anal. Appl.* **198** (1996), no. 1, 84–97 Zbl 0858.35124 MR 1373528

[29] M. Heida and J. Málek, On compressible Korteweg fluid-like materials. *Internat. J. Engrg. Sci.* **48** (2010), no. 11, 1313–1324 Zbl 1231.76248 MR 2760987

[30] A. Jüngel, Global weak solutions to compressible Navier-Stokes equations for quantum fluids. *SIAM J. Math. Anal.* **42** (2010), no. 3, 1025–1045 Zbl 1228.35083 MR 2644915

[31] D. J. Korteweg, Sur la forme que prennent les équations du mouvement des fluides si l'on tient en compte des forces capillaires causées par des variations de densité. *Archives Néerl. Sci. Exactes Nat. Ser. II* **60** (1901), 1–24. Zbl 32.0756.02

[32] A. Jüngel, Effective velocity in compressible Navier-Stokes equations with third-order derivatives. *Nonlinear Anal.* **74** (2011), no. 8, 2813–2818 Zbl 1211.35217 MR 2776529

[33] I. Lacroix-Violet and A. Vasseur, Global weak solutions to the compressible quantum Navier-Stokes equation and its semi-classical limit. *J. Math. Pures Appl. (9)* **114** (2018), 191–210 Zbl 1392.35228 MR 3801754

[34] H.-L. Li, J. Li, and Z. Xin, Vanishing of vacuum states and blow-up phenomena of the compressible Navier-Stokes equations. *Comm. Math. Phys.* **281** (2008), no. 2, 401–444 Zbl 1173.35099 MR 2410901

[35] J. Li and Z. Xin, Global existence of weak solutions to the barotropic compressible Navier-Stokes flows with degenerate viscosities. arXiv:1504.06826

[36] Y. Li, R. Pan, and S. Zhu, On classical solutions to 2D shallow water equations with degenerate viscosities. *J. Math. Fluid Mech.* **19** (2017), no. 1, 151–190 Zbl 1369.35065 MR 3607451

[37] P.-L. Lions, *Mathematical topics in fluid mechanics. Vol. 2*. Oxford Lecture Ser. Math. Appl. 10, Clarendon Press, Oxford University Press, New York, 1998 Zbl 0908.76004 MR 1637634

[38] T.-P. Liu, Z. Xin, and T. Yang, Vacuum states for compressible flow. *Discrete Contin. Dynam. Systems* **4** (1998), no. 1, 1–32 Zbl 0970.76084 MR 1485360

[39] X. Liu and E. S. Titi, Global existence of weak solutions to the compressible primitive equations of atmospheric dynamics with degenerate viscosities. *SIAM J. Math. Anal.* **51** (2019), no. 3, 1913–1964 Zbl 1419.35150 MR 3949696

[40] B. Lü, R. Zhang, and X. Zhong, Global existence of weak solutions to the compressible quantum Navier-Stokes equations with degenerate viscosity. *J. Math. Phys.* **60** (2019), no. 12, 121502, 31 Zbl 1432.76307 MR 4037402

[41] F. Marche, Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. *Eur. J. Mech. B Fluids* **26** (2007), no. 1, 49–63 Zbl 1105.76021 MR 2281291

[42] A. Mellet and A. Vasseur, On the barotropic compressible Navier-Stokes equations. *Comm. Partial Differential Equations* **32** (2007), no. 1-3, 431–452 Zbl 1149.35070 MR 2304156

[43] A. F. Vasseur and C. Yu, Existence of global weak solutions for 3D degenerate compressible Navier-Stokes equations. *Invent. Math.* **206** (2016), no. 3, 935–974 Zbl 1354.35115 MR 3573976

[44] A. F. Vasseur and C. Yu, Global weak solutions to the compressible quantum Navier-Stokes equations with damping. *SIAM J. Math. Anal.* **48** (2016), no. 2, 1489–1511 Zbl 1343.35189 MR 3490496

[45] Z. Xin and S. Zhu, Global well-posedness of regular solutions to the three-dimensional isentropic compressible Navier-Stokes equations with degenerate viscosities and vacuum. arXiv:1806.02383

[46] Z. Xin and S. Zhu, Well-posedness of the three-dimensional isentropic compressible Navier-Stokes equations with degenerate viscosities and far field vacuum. *J. Math. Pures Appl. (9)* **152** (2021), 94–144 Zbl 07379399 MR 4280833

Received 8 November 2019; revised 7 December 2020; accepted 20 April 2021.

Paolo Antonelli
GSSI – Gran Sasso Science Institute, Viale Francesco Crispi 7, 67100 L'Aquila, Italy; paolo.antonelli@gssi.it

Stefano Spirito
DISIM – Dipartimento di Ingegneria e Science dell'Informazione e Matematica, Via Vetoio, 67100 L'Aquila, Italy; stefano.spirito@univaq.it