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Research article

Polyconvex functionals and maximum principle†

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Abstract: Let us consider continuous minimizers $u : \bar{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$ of

$$\mathcal{F}(v) = \int_\Omega [|Dv|^p + |\det Dv|^r] dx,$$

with $p > 1$ and $r > 0$; then it is known that every component u^α of $u = (u^1, ..., u^n)$ enjoys maximum principle: the set of interior points x, for which the value $u^\alpha(x)$ is greater than the supremum on the boundary, has null measure, that is, $\mathcal{L}^n(\{x \in \Omega : u^\alpha(x) > \sup_{\partial\Omega} u^\alpha\}) = 0$. If we change the structure of the functional, it might happen that the maximum principle fails, as in the case

$$\mathcal{F}(v) = \int_\Omega [\max\{(|Dv|^p - 1); 0\} + |\det Dv|^r] dx,$$

with $p > 1$ and $r > 0$. Indeed, for a suitable boundary value, the set of the interior points x, for which the value $u^\alpha(x)$ is greater than the supremum on the boundary, has a positive measure, that is $\mathcal{L}^n(\{x \in \Omega : u^\alpha(x) > \sup_{\partial\Omega} u^\alpha\}) > 0$. In this paper we show that the measure of the image of these bad points is zero, that is $\mathcal{L}^n(u(\{x \in \Omega : u^\alpha(x) > \sup_{\partial\Omega} u^\alpha\})) = 0$, provided $p > n$. This is a particular case of a more general theorem.

Keywords: polyconvex functionals; minimizers; regularity

Dedicated to our friend Giuseppe (Rosario) Mingione on his 50th birthday.

1. Introduction

Let us consider the functional

$$\mathcal{F}(v) = \int_\Omega \Big[|Dv|^p + |\det Dv|^r\Big] dx,$$

where $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, Ω a bounded open set, $p > 1$, $r > 0$.

It is well known that, if u is a minimizer for $\mathcal{F}(v)$, the maximum principle holds, namely, each component u^α of $u = (u^1, ..., u^n)$ satisfies the following condition

$$u^\alpha(x) \leq \sup_{\partial\Omega} u^\alpha, \qquad \alpha \in \{1, 2, \dots, n\}.$$

Indeed, maximum principle holds true, in general, for minimizers of the class of functionals

$$\mathcal{F}(v) = \int_\Omega \Psi(|Dv|, \, |\det Dv|) \, dx, \tag{1.1}$$

where the integrand $\Psi(s, \, t)$ is such that $s \to \Psi(s, \, t)$ strictly increases, and $t \to \Psi(s, \, t)$ is increasing (see [39]).

What happens when we only have that $s \to \Psi(s, \, t)$ is increasing and not necessarily strictly increasing? Two examples are $\Psi(s, \, t) = |t|$ that gives

$$\mathcal{F}(v) = \int_\Omega |\det Dv| \, dx \,, \tag{1.2}$$

and $\Psi(s, \, t) = \max\{|s|^p - 1; \, 0\} + |t|^r$ that gives

$$\mathcal{F}(v) = \int_\Omega \big(\max\{|Dv|^p - 1; \, 0\} + |\det Dv|^r \big) \, dx, \tag{1.3}$$

with $p > 1$ and $r > 0$. Maximum principle fails. Namely, consider $n = 2$, $\Omega \subset \mathbb{R}^2$ is the ball $B(0; \pi)$ centered in the origin and with radius π.

The map $u := (1, \, 1 + \sin|x|)$ has gradient

$$Du = \begin{bmatrix} 0 & 0 \\ \frac{x_1}{|x|} \cos|x| & \frac{x_2}{|x|} \cos|x| \end{bmatrix},$$

$\det Du = 0$, and $|Du|^2 = \cos^2|x| \leq 1$. It minimizes both the functionals (1.2) and (1.3). Moreover, the second component $u^2 = 1 + \sin|x|$ equals 1 on the boundary of Ω, and is strictly greater than 1 inside. Therefore, the second component of the minimizer u does not satisfy the maximum principle. This example was given to the last author by V. Sverak a few years ago. F. Leonetti gladly takes the opportunity to thank V. Sverak for his kindness.

Furthermore, regarding the previous example, it is worth pointing out that the level set $\{x \in \Omega : u^2(x) > 1 = u^2_{\partial\Omega}\}$ has positive measure

$$\mathcal{L}^2(\{x \in \Omega : u^2(x) > 1 = u^2_{\partial\Omega}\}) = \mathcal{L}^2(\Omega) > 0\,, \tag{1.4}$$

on the other hand, the measure of the image of the same level set, by means of u, is zero

$$\mathcal{L}^2(u(\{x \in \Omega : u^2(x) > 1 = u^2_{\partial\Omega}\})) = 0\,, \tag{1.5}$$

see Figure 1.

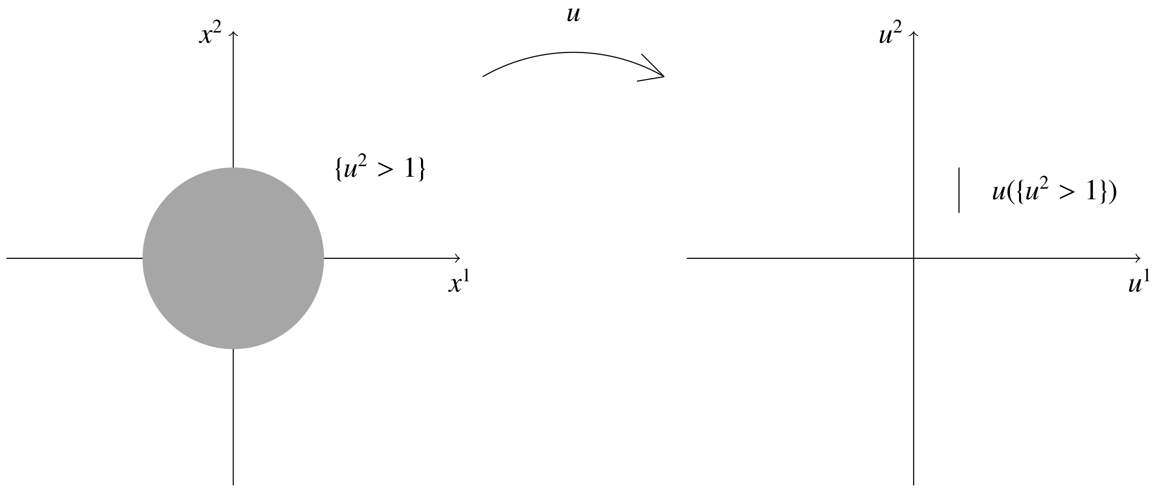

Figure 1. Image of the level set.

We ask ourselves whether the previous example shows a common feature to all minimizers when $t \to \Psi(s, t)$ strictly increases.

In this paper, we give a positive answer to previous question obtaining a modified version of maximum principle in the case the integrand $\Psi(s, t)$ of the functional (1.1) strictly increases only with respect to the second variable t.

We will suppose $p > n$ in order to ensure semicontinuity property and consequent existence of minimizers (see [17]), and also to apply the area formula, that reveals to be a key tool in our proof.

In addition, we can still get a similar maximum principle by using a version of the area formula for $u \in W^{1,1}(\Omega, \mathbb{R}^n)$, see [34, 35], provided a suitable negligible set $S = \Omega \setminus \mathcal{A}_{\mathcal{D}}$ is removed (see definition 2.1).

Let us come back to the functional (1.3): coercivity holds true with exponent p and growth from above with exponent $q =: nr$ that could be different from p. When we deal with functionals with different growth, regularity for minimizers is usually obtained when the two exponents of growth and coercivity are not too far apart, see [3,6,10–13,18,32,49,50]. In our case, we do not assume anything on the distance between the two exponents p and q. This is not in contradiction with the counterexamples in the double phase case [22, 25], since our functional (1.3) is autonomous, neither is in contrast with counterexamples in the autonomous case [33, 38, 47, 48], since they show blow up along a line that intersects the boundary of Ω while, in our case, minimizers are bounded on $\partial\Omega$.

With regard to the regularity of minimizers u of (1.1), let us mention partial regularity results in [9, 23, 26–28, 30, 36, 52]. Everywhere regularity results can be found in [7, 19, 29, 31], for $n = 2$. We also mention global L^∞ bounds in [4, 5, 21, 39–44], and local L^∞ regularity in [8, 14–16, 20]. Furthermore, concerning nonlinear elasticity, we cite, in particular, the results in [1, 37, 45, 46, 51].

In the next section 2 we write some preliminaries. In section 3 we state our result and we give the proof.

2. Preliminaries

In order to obtain our result, we need that the area formula holds. Therefore, let us recall the following

Definition 2.1. *Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a map which is almost everywhere approximately differentiable and let A be a measurable subset of $\mathbb{R}^n$. We define the Banach indicatrix of u by*

$$N(u, A, y) := \sharp\{x : x \in A \cap \mathcal{A}_D(u),\ u(x) = y\}$$

where

$$\mathcal{A}_D(u) = \{x : u \textit{ is approximately differentiable at } x\},$$

and the theorem

Theorem 2.2. (see Theorem 1 in section 1.5, chapter 3, at page 220 of [35]) *Let Ω be an open subset of $\mathbb{R}^n$ and u be an almost everywhere approximately differentiable map, in particular let $u \in W^{1,1}(\Omega; \mathbb{R}^n)$. Then for any measurable subset A of Ω we have that $N(u, A, \cdot)$ is measurable and*

$$\int_A |\det Du(x)| dx = \int_{\mathbb{R}^n} N(u, A, y) dy \tag{2.1}$$

holds.

Furthermore, a related condition we will refer to is the Lusin property (N) that is so defined

Definition 2.3. (Lusin property (N)) *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ a mapping. We say that f satisfies Lusin property (N) if the implication*

$$\mathcal{L}^n(E) = 0 \quad \Longrightarrow \quad \mathcal{L}^n(f(E)) = 0$$

holds for each subset $E \subset \Omega$.

3. Main results

Let $\Psi : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ be a continuous non negative function such that

$$s \to \Psi(s, t) \qquad \text{is increasing for every } t \in [0, +\infty) \tag{H1}$$

$$t \to \Psi(s, t) \qquad \text{is strictly increasing for every } s \in [0, +\infty), \tag{H2}$$

and let us denote $\Omega \subset \mathbb{R}^n$ a bounded open set. We will consider integral functional of the type

$$\mathcal{F}(u) := \int_\Omega \Psi(|Du|, |\det Du|)\, dx. \tag{3.1}$$

Definition 3.1. *Let $p \geq 1$ and $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ such that $\mathcal{F}(u) < \infty$. We will say that u is a minimizer of $\mathcal{F}$ in Ω, if and only if*

$$\mathcal{F}(u) \leq \mathcal{F}(v) \qquad\qquad \forall v \in u + W_0^{1,p}(\Omega;\mathbb{R}^n). \tag{3.2}$$

The main result is the following

Theorem 3.2. *Let $u \in W^{1,p}(\Omega;\mathbb{R}^n)$, $p > n$, be the continuous representative of a minimizer of the functional* (3.1)*, under assumptions* ($H1$) *and* ($H2$)*. Fix $\alpha \in \{1,\dots,n\}$, and let us denote*

$$L_\alpha := \sup_{x\in\partial\Omega} u^\alpha(x) < +\infty, \quad B_{L_\alpha} := \{x \in \Omega : u^\alpha(x) > L_\alpha\},$$

B_{L_α} is the set of points in Ω where the maximum principle is violated, then

$$\mathcal{L}^n(u(B_{L_\alpha})) = 0. \tag{3.3}$$

Proof. Let us define

$$v^\beta(x) := \begin{cases} u^\beta(x) & \text{if} \quad \beta \neq \alpha \\ \min\{u^\alpha(x); L_\alpha\} & \text{if} \quad \beta = \alpha. \end{cases}$$

It results that v is a good test function in (3.2), namely $u - v \in W_0^{1,p}(\Omega;\mathbb{R}^n)$, then we deduce that

$$\mathcal{F}(u) = \int_\Omega \Psi(|Du|, |\det Du|)\, dx \leq \int_\Omega \Psi(|Dv|, |\det Dv|)\, dx = \mathcal{F}(v). \tag{3.4}$$

Let us denote

$$G_{L_\alpha} := \{x \in \Omega : u^\alpha(x) \leq L_\alpha\}, \text{ then } \quad B_{L_\alpha} = \Omega \setminus G_{L_\alpha} = \{x \in \Omega : u^\alpha(x) > L_\alpha\},$$

and let us split the integrals in (3.4) on the sets G_{L_α} and B_{L_α}. Observing that $Du \equiv Dv$ on the set G_{L_α} we can get rid of the common part in (3.4) thus obtaining

$$\int_{B_{L_\alpha}} \Psi(|Du|, |\det Du|)\, dx \leq \int_{B_{L_\alpha}} \Psi(|Dv|, |\det Dv|)\, dx.$$

Now we observe that on B_{L_α}, $Dv^\alpha = 0$ and $\det Dv = 0$, then

$$\int_{B_{L_\alpha}} \Psi(|Du|, |\det Du|)\, dx \leq \int_{B_{L_\alpha}} \Psi(|Dv|, 0)\, dx$$

Now, argue by contradiction, by assuming that

$$\mathcal{L}^n(B_{L_\alpha} \cap \{|\det Du| > 0\}) > 0. \tag{3.5}$$

At this stage, we recall that $|Dv| \leq |Du|$ on B_{L_α}, and we use the strict monotonicity of Ψ with respect to the second argument ($H2$), and hypothesis ($H1$), to deduce

$$\int_{B_{L_\alpha}} \Psi(|Du|, |\det Du|)\, dx \leq \int_{B_{L_\alpha}} \Psi(|Dv|, 0)\, dx \tag{3.6}$$

$$< \int_{B_{L_\alpha}} \Psi(|Dv|, |\det Du|)\, dx \le \int_{B_{L_\alpha}} \Psi(|Du|, |\det Du|)\, dx,$$

thus reaching a contradiction. The previous argument shows that

$$\mathcal{L}^n(B_{L_\alpha} \cap \{|\det Du| > 0\}) = 0.$$

Using the area formula (2.1) we conclude

$$\begin{aligned} \mathcal{L}^n(u(B_{L_\alpha} \cap \mathcal{A}_D(u))) = \int_{u(B_{L_\alpha} \cap \mathcal{A}_D(u))} 1\, dy \le \int_{u(B_{L_\alpha} \cap \mathcal{A}_D(u))} N(u, B_{L_\alpha}, y)\, dy \le \\ \int_{\mathbb{R}^n} N(u, B_{L_\alpha}, y)\, dy = \int_{B_{L_\alpha}} |\det Du|\, dx = 0. \end{aligned} \tag{3.7}$$

To conclude the proof we recall that the condition $p > n$ ensures that $u : \Omega \to \mathbb{R}^n$ satisfies the Lusin property (N), that is $\mathcal{L}^n(u(E)) = 0$ whenever $E \subset \Omega$ and $\mathcal{L}^n(E) = 0$. In particular $\mathcal{L}^n(B_{L_\alpha} \setminus \mathcal{A}_D(u)) = 0$ and this implies that

$$\mathcal{L}^n(u(B_{L_\alpha} \setminus \mathcal{A}_D(u))) = 0. \tag{3.8}$$

Connecting (3.7) and (3.10) we get (3.3). □

It is worth pointing out some comments concerning the hypotheses in Theorem 3.2.

As a matter of fact, assuming $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ for $p > n$ ensures some fundamental conditions.

The first point concerns the existence of minimizers of the functional (3.1). Assuming that $p > n$ guarantees not only that $\det Du \in L^1$, but more that the map

$$u \in W^{1,p}(\Omega; \mathbb{R}^n) \to \det Du \in L^{\frac{p}{n}}$$

is sequentially continuous with respect to the weak topology (see Theorem 8.20 in [17]). The aforementioned property, that is no longer true for $p < n$, see [2], is one of the main ingredients to prove the lower semicontinuity of the functional (3.1). The second main ingredient to deduce the existence of minimizers of the functional (3.1) is a kind of convexity assumption on the function Ψ. Precisely, we have that if the function

$$(X, \det X) \in \mathbb{R}^{n\times n} \times \mathbb{R} \to \Psi(|X|, |\det X|) \in \mathbb{R}$$

is convex and

$$C|X|^p \le \Psi(|X|, |\det X|) \quad \forall X \in \mathbb{R}^{n\times n},$$

then the functional (3.1) is weakly lower semicontinuous and coercive in $W^{1,p}(\Omega; \mathbb{R}^n)$. The existence of minimizers of the functional (3.1) follows for any fixed boundary datum $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ such that $\mathcal{F}(u) < +\infty$ (see Theorem 8.31 in [17]; see also [24]).

The second main point, where the assumption $p > n$ is crucial, concerns the Lusin property (N) quoted in the Definition 2.3. It is known that the Lusin property (N) still holds true for $u \in W^{1,n}(\Omega; \mathbb{R}^n)$, if u is a homeomorphism. Moreover, there are also other results about the validity of the Lusin property (N) for suitable $p < n$, or with integrability rate close to n under particular assumptions, but, beyond that, the Lusin property (N) is no longer true, in general, for $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ with $p \le n$. In this case we can carry on the proof of Theorem 3.2 as before, but we can not conclude in the same way because we do not have any information regarding the set $\mathcal{L}^n(u(B_{L_\alpha} \setminus \mathcal{A}_D(u)))$. Nevertheless we can state the Theorem 3.2 in a weaker form. We need to stress the dependence of the level set $B_{L_\alpha} = \{x \in \Omega : u^\alpha(x) > L_\alpha\} = B_{L_\alpha}(u)$ on the considered representative u of the minimizer.

Theorem 3.3. *Let $u \in W^{1,p}(\Omega; \mathbb{R}^n)$, $p \geq 1$, be a minimizer of the functional* (3.1) *under assumptions* ($H1$) *and* ($H2$). *Fix $\alpha \in \{1, \dots, n\}$, then*

$$\mathcal{L}^n(u(B_{L_\alpha}(u) \cap \mathcal{A}_D(u))) = 0. \tag{3.9}$$

Remark 3.4. *We note that (3.9) holds true for every representative u of a $W^{1,p}$- minimizer (see section 1.5, chapter 3 of [35]). Moreover, in accordance with Corollary 1, chapter 3 of [35], if we consider a Lusin representative u, it satisfies Lusin property (N) in whole Ω so that*

$$\mathcal{L}^n(u(B_{L_\alpha}(u) \setminus \mathcal{A}_D(u))) = 0 \tag{3.10}$$

holds, and for such a representative we come to the conclusion that

$$\mathcal{L}^n(u(B_{L_\alpha}(u))) = 0. \tag{3.11}$$

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Conflict of interest

The authors declare no conflict of interest.

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