UNIVERSITÀ DEGLI STUDI DELL'AQUILA

DIPARTIMENTO DI INGEGNERIA E SCIENZE
DELL'INFORMAZIONE E MATEMATICA

Dottorato di Ricerca in Matematica e Modelli
XXXVI Ciclo

On the classification of the unrefinable partitions into distinct parts

Settore Scientifico-Disciplinare:
MAT/02 Algebra

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Anno Accademico 2021/2022

Introduction

A partition into distinct parts is refinable if one of its parts a can be replaced by two different positive integers which do not belong to the partition and whose sum is a, otherwise the partition is unrefinable. For example the partition $(1,2,3,5,9,10,12)$ is a refinable partition because we can substitute $10 = 4+6$ or $12 = 4+8$, while if we consider one of these two refinements we obtain an unrefinable partition, i.e., in $(1,2,3,4,5,8,9,12)$ we cannot replace any parts.

Unrefinable partitions into distinct parts were introduced in the work of Aragona et al. [ACGS22], where they were related to the generators of a chain of normalisers. The authors introduced in [ACGS21a] a chain of normalisers, which begins with the normaliser N_n^0 of the translation group T on $\mathbb{F}_2^n$ in a suitable Sylow 2-subgroup Σ_n of the symmetric group on 2^n letters $Sym\left(2^n\right)$ and whose i-th term N_n^i is defined as the normaliser in Σ_n of the previous one. They also proved [ACGS21b] that the number $\log_2\left|N_n^i \colon N_n^{i-1}\right|$ is independent of n for $1 \leq i \leq n-2$ and in particular it is equal to the $(i+2)$-th term of the sequences of the partial sums of the sequence $\{b_j\}$, where b_j is the number of partitions of j into at least two distinct parts. Finally in [ACGS22] the authors observed that the number $\log_2\left|N_n^{n-1} \colon N_n^{n-2}\right|$ is linked to the number of unrefinable partitions of n with a condition on their minimal excludant (mex), i.e., the minimum positive integer number that does not appear in the partition.

The purpose of this thesis is to study unrefinable partitions into distinct parts, which have not yet been investigated. Clearly, the condition of being unrefinable imposes on the partition a non-trivial limitation on the possible distributions of the parts. First of all, we studied some arithmetic properties of unrefinable partitions that allowed us to construct two algorithms, one capable of recognising whether or not a sequence is an unrefinable partition (see Algorithm 2), and the other capable of enumerating all unrefinable partitions of a given weight (Algorithm 4), [ACCL23]. Analysing the data of the enumeration algorithm, we realised that there exists a non-trivial limitation on the size of the largest part of the unrefinable partitions. We proved a $O\left(n^{1/2}\right)$-upper bound for the largest part in an unrefinable partition of n (see Proposition 4.1), and we call *maximal* (Definition 4.2) those partitions that reach the bound [ACCL22]. We showed a complete classification of maximal unrefinable partitions for triangular numbers [ACCL22] and then we completed the general classification of the maximal unrefinable partitions [ACC22]. These classifications allowed us to show explicit bijections between unrefinable partitions and suitable subsets of partitions into distinct parts, in particular if we consider the maximal unrefinable partitions of n-th triangular number T_n we obtain (see Theorem 4.3, Theorem 4.17)

$$\#\widetilde{\mathcal{U}}_{T_n} = \begin{cases} \#\mathcal{D}_k & \text{if } n = 2k-1, \\ 1 & \text{if } n \quad \text{is even.} \end{cases}$$

where $\mathcal{D}_k$ is the set of partitions into distinct parts of weight k, while if we consider the non-triangular numbers the result is similar but more complicated and depends by the distance from the next triangular number. As proved in Theorem 5.28 and Theorem 5.39, if $T_{n,d} = T_n - d$ and n is an odd number greater than 11, then

$$\#\widetilde{\mathcal{U}}_{T_{n,d}} = \begin{cases} 1 + \#\mathcal{D}_{(n-d+1)/2} & \text{if } d > 3 \text{ is even,} \\ \#\mathcal{D}^{\mathcal{O}}_{n-d+2} & \text{if } d > 3 \text{ is odd,} \\ 1 & \text{if } d \in \{1,2,3\}. \end{cases}$$

if n is even

$$\#\widetilde{\mathcal{U}}_{T_{n,d}} = \begin{cases} 1 + \#\mathcal{D}_{(n-d+1)/2} & \text{if } d > 2 \text{ is odd,} \\ \#\mathcal{D}^{\mathcal{O}}_{n-d+2} & \text{if } d > 2 \text{ is even,} \\ 1 & \text{if } d \in \{1,2\}. \end{cases}$$

where $\mathcal{D}^{\mathcal{O}}_k$ is the set of partitions into distinct parts of weight k and whose parts are odd numbers.

Moreover, it can be established an important connection between unrefinable partitions and numerical semigroups, additive submonoids of the non-negative integers which include 0 and such that the complementary sets have finitely many elements. This relationship allowed us to find other methods to recognise when a partition is unrefinable or not only looking the hooksets of the Young tableau associated to the numerical semigroup (see Lemma 6.4). Moreover, we can describe other kinds of subsets of unrefinable partitions fixing the largest part and maximising the number of missing parts. In this case we find a relation with the set of symmetric numerical semigroups (SNS) when the largest part λ_t is a prime number, i.e.,

$$\#\bar{\mathcal{U}}(\lambda_t) = \#\left\{S \in SNS \mid F(S) = \lambda_t\right\},$$

where λ_t is the largest part and $F(S)$ is the maximal element of the complementary of S (Theorem 6.15).

A future research could be devoted to understand such connection more deeply. Another interesting aspect of future research is to find some combinatorial properties of unrefinable partitions in order to define their generating functions or in order to understand their density.

The thesis is organised as follows.
In the first chapter we analyse integer partitions by dwelling on generating functions and their representation by Ferrer and Young's tableaux, then showing some combinatorial relations between subsets of integer functions ([And84],[HWHBS08]). In the second part we analyse some arithmetic properties of partitions ([Ram19],[AHAW00]) by showing the Rogers-Ramanujan identity. In the last part we introduce Dyck paths by explaining a way to count

them through partitions ([ALW16], [Biz54], [BM23]). The topics discussed in this chapter will not be taken up in the rest of the thesis, but have been included to give a sense of completeness to the partitions, analysing them from the point of view of combinatorics and historical evolution.

In Chapter 2 we introduce the numerical semigroups. In the first section we describe a covariety of numerical semigroups and represent it as a finite tree. After a characterisation of the children of an arbitrary vertex in this tree, we present an algorithm to describe the covariety [MFR23]. In the second part we show some relations between numerical semigroups and integer partitions [BNST23], looking in particular at the hookset. Then we introduce the symmetryic and pseudo-symmetyric numerical semigroups showing four operations [SY21] that give us some relations to obtain other numerical semigroups.

In the third chapter we finally introduce unrefinable partitions and we show some basic properties. In section 2 we present an algorithm that checks if a strictly increasing sequence of integer numbers is an unrefinable partition or not, showing its correctness. In the last part we use the verification algorithm to create a new algorithm capable of enumerating all the unrefinable partitions of a given weight.

In Chapter 4 we analyse the unrefinable partitions of a triangular number finding a bound for their maximal part. Then we present a constructive method that allow us to classify and count the maximal unrefinable partitions showing a bijective correspondence with the set of partitions into distinct parts.

In the fifth chapter we generalise the classification of maximal unrefinable partitions to the non-triangular case, exploiting the method used in the previous chapter. In this case we show that we can express maximal unrefinable partitions in terms of suitable partitions into distinct parts depending by the distance of the weight from the next triangular number.

In the last chapter we present some relations between unrefinable partitions and numerical semigroups that can be useful to find new properties about unrefinable partitions. We conclude the chapter by formulating two conjectures about the density of unrefinable partitions trying to count the unrefinable partitions fixing the largest part (see Conjecture 1) and the number of missing parts (Conjecture 2).

Contents

Chapter 1

Integer Partitions

In this chapter we introduce the integer partitions. We analyse Euler's theorem and generating functions of subsets of integer partitions in the first section, introducing Ferrer's diagrams and Young's tableaux. In the second section we show arithmetic properties of partitions. We conclude the chapter by showing relations between partitions and Dyck paths.

1.1 Generating functions and Euler Theorem

Definition 1.1. *A* ***partition*** $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{t-1}, \lambda_t)$ *of an integer* n *is a finite non-increasing sequence of positive integers such that* $\sum_{i=1}^{t} \lambda_i = n$. *If* λ *is a partition of* n *we write* $\lambda \vdash n$
Each λ_i *is called* ***part*** *of partition and the element* λ_1 *is called* ***maximal part****.*
We call ***length*** *of a partition the number of parts of partition and we write* $len(\lambda) = t$.
For all $1 \leq i \leq t$, $m_i(\lambda)$ *is the number of parts of* λ *equal to* i.
We can also see a partition of an integer as an increasing sequences of positive integer with their multiplicity, so we can write
$\lambda = \left(1^{m_1(\lambda)}, 2^{m_2(\lambda)}, 3^{m_3(\lambda)}, \cdots, j^{m_j(\lambda)}\right) \vdash n$, *where* $\sum_{1 \leq h \leq j} m_h(\lambda) h = n$.
We define a useful number for partitions

$$z_\lambda = \prod_{i=1}^{j} i^{m_i(\lambda)} m_i(\lambda)!$$

One of the first problems that arise about integer partitions is counting them. Euler found the solution using the generating function

$$F(x) = \sum_{n \geq 0} p(n) x^n$$

where $p(n)$ is the number of partitions of n.

If we take the infinite product

$$\prod_{n\geq 1}\left(1+x^n+x^{2n}+x^{3n}+\cdots\right) \tag{1.1}$$

we obtain a polynomial $1+a_1x+a_2x^2+a_3x^3+\cdots$ where every coefficient a_i coincides with $p(i)$, because every partition of i contributes just 1 to the coefficient of x^i. For example, $x^5=\left(x^1\right)^5=\left(x^1\right)^3\left(x^2\right)^1=\left(x^1\right)^2\left(x^3\right)^1=\left(x^1\right)^1\left(x^2\right)^2=\left(x^1\right)^1\left(x^4\right)^1=\left(x^2\right)^1\left(x^3\right)^1=\left(x^5\right)^1$, indeed the partitions of 5 are 7: $\left(1^5\right),\left(1^3,2\right),\left(1^2,3\right),\left(1,2^2\right),(1,4),(2,3),(5)$.

Theorem 1.2 (Euler)**.** *When $|x|<1$*

$$F(x)=\prod_{n\geq 1}\frac{1}{1-x^n} \tag{1.2}$$

Starting from Euler formula, it is possible to count particular types of integer partitions. First we have to give the following definition.

Definition 1.3. *Let $\mathcal{O}$, $\mathcal{E}$, $\mathcal{D}$, $\mathcal{D}^{\mathcal{O}}$, $\mathcal{D}^{\mathcal{E}}$ be respectively the sets of integer partitions into parts which are odd, even, distinct, both distinct and odd and both distinct and even.*
We denote by $\mathcal{P}_{\leq m}$ the set of partitions whose each part is lower or equal to m.

For obtaining the generating function of partitions whose each part is lower than m, it is enough to consider (1.1) up to m

$$F\left(x,\mathcal{P}_{\leq m}\right)=\prod_{n=1}^{m}\frac{1}{1-x^n}.$$

In the same way, for describing the partitions whose parts belonging to a subset $H\subset\mathbb{N}$ we have to replace the infinite product in (1.1) with $\prod_{i\in H}$ and so we have

$$F\left(x,\mathcal{O}\right)=\prod_{i=1}^{\infty}\frac{1}{1-x^{2i-1}}$$
$$F\left(x,\mathcal{E}\right)=\prod_{i=1}^{\infty}\frac{1}{1-x^{2i}}$$

while for the set $\mathcal{D}$ we obtain

$$F\left(x,\mathcal{D}\right)=(1+x)\left(1+x^2\right)\left(1+x^3\right)\cdots \tag{1.3}$$

The following results give the relationship between $F\left(x,\mathcal{D}\right)$ and $F\left(x,\mathcal{O}\right)$.

Proposition 1.4. *The number of partitions of n into distinct parts is equal to the number of partitions in odd parts*

Proof. It is enough to notice the following equalities

$$\begin{aligned} F(x,\mathcal{D}) = (1+x)\left(1+x^2\right)\left(1+x^3\right)\cdots &= \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\cdots \\ &= \frac{1}{(1-x)(1-x^3)\cdots} \\ &= F(x,\mathcal{O}) \end{aligned}$$

□

Notice that it is possible to prove the previous proposition without considering the generating functions. Let us consider for every positive integer the corresponding binary form $\sum_{i\in I} 2^i$ for some I finite subset of $\mathbb{N}$.
Let $\lambda = (1^{m_1}, 3^{m_2}, \cdots) \in \mathcal{O}$, where every $m_i = 2^{a_i} + 2^{b_i} + \cdots$. Then we can take the sequence $\left(2^{a_1}, 2^{b_1}, \cdots, 2^{a_2}3, 2^{b_2}3, \cdots\right)$ and, after reordering in decreasing order we obtain a partition into distinct parts.

Let

$$K(a) = K(a,x) = (1+ax)\left(1+ax^3\right)\left(1+ax^5\right)\cdots \tag{1.4}$$

so we can write $K(a) = 1 + \sum_{i\geq 1} c_i a^i$, where $c_i = c_i(x)$. Clearly $K(a) = (1+ax)\,K\left(ax^2\right)$, and so

$$1 + c_1 a + c_2 a^2 + \cdots = (1+ax)\left(1 + c_1 a x^2 + c_2 a^2 x^4 + \cdots\right)$$

and collecting the coefficients we obtain $c_1 = x + c_1 x^2$, $c_2 = c_1 x^3 + c_2 x^4$, $\ldots$, $c_m = c_{m-1}x^{2m-1} + c_m x^{2m}$, so:

$$\begin{aligned} c_m &= \frac{x^{2m-1}}{1-x^{2m}} c_{m-1} \\ &= \frac{x^{2m-1}}{1-x^{2m}} \frac{x^{2m-3}}{1-x^{2m-2}} c_{m-2} \\ &\vdots \\ &= \frac{x^{m^2}}{(1-x^2)\cdots(1-x^{2m-2})(1-x^{2m})} \end{aligned}$$

Finally we can rewrite (1.4)

$$K(a) = 1 + \frac{x}{1-x^2} a + \frac{x^4}{(1-x^2)(1-x^4)} a^2 + \cdots$$

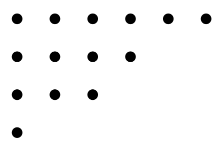
Proposition 1.5.

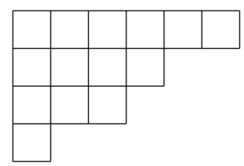
$$\begin{aligned} F\left(x,\mathcal{D}^{\mathcal{O}}\right) &= 1 + \frac{x}{(1-x^2)} + \frac{x^4}{(1-x^2)(1-x^4)} + \cdots \\ F\left(x,\mathcal{D}^{\mathcal{E}}\right) &= 1 + \frac{x^2}{(1-x^2)} + \frac{x^6}{(1-x^2)(1-x^4)} + \cdots \end{aligned}$$

Proof. In Equation (1.4) it is enough to substitute $a = 1$ in the case of odd and distinct parts, and $a = x$ in the case of even and distinct parts. □

It is useful represent the partitions with the Ferrers graphs or the Young diagrams. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t)$ be a partition, for representing λ with a Ferrers graph we draw λ_1 point in a first line, λ_2 in the second one and so on; for representing λ with a Young diagram, we draw in the same way squares instead of points.

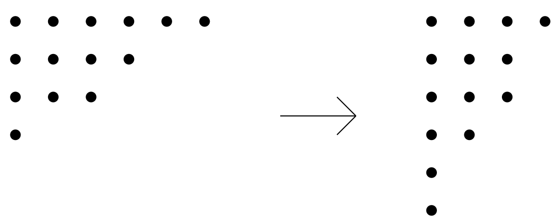
Example 1.6. *For $\lambda = (6, 4, 3, 1)$ the representations of λ by Ferrers graph and Young diagram are respectively*





Definition 1.7. *Let $\lambda = (\lambda_1, \cdots, \lambda_t)$ be a partition, we can define a new partition $\lambda' = (\lambda'_1, \lambda'_2, \cdots, \lambda'_k)$ choosing λ'_i as the number of parts of λ that are greater or equal to i. We call λ' the* ***conjugate*** *of λ.*
If we obtain $\lambda = \lambda'$ we call λ a ***self-conjugate*** *partition. We denote by $\mathcal{S}$ the set of self-conjugate partitions.*

Example 1.8. *The definition of conjugate partition is more clear in the graphical representation. The partition λ' is obtain by counting the dots in the columns; in other words, the conjugate is obtained by reflecting the main diagonal.*



From the partition $\lambda = (6, 4, 3, 1)$ we obtain $\lambda' = (4, 3, 3, 2, 1, 1)$

Proposition 1.9. *The number of partitions of n with at most m parts equals the number of partitions of n in which no part exceeds m.*
In particular the number of partitions of n into m parts is equal to the number of partitions of n into parts the largest of which is m.

Proof. The conjugate map is a one-to-one correspondence. □

Definition 1.10. *For every point in the Ferrers representation of a partition or cell in Young diagram we define the* ***arm*** *of a point as the number of points (or cells) in the same row on the right and the* ***leg*** *as the number of points (or cells) down in the same column.*
We call ***hook*** *of a point the sum of the arm and the leg of a point plus one, hence the* ***hookset*** *the set of all hooks.*

Example 1.11. *If we consider the same partition of the previous examples* $\lambda = (6,4,3,1)$*, we can fill the cells of its Young diagram with the value of their hook, and we obtain*

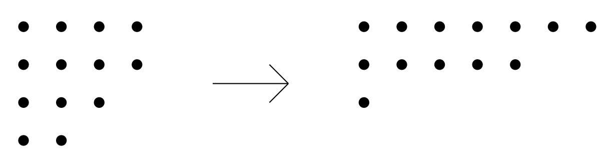
9	7	6	4	2	1
6	4	3	1		
4	2	1			
1					

The hookset of λ *is* $\{1,2,3,4,6,7,9\}$

Proposition 1.12. *The number of partitions* λ *of* n *such that* $\lambda \in \mathcal{D}^{\mathcal{O}}$ *is equal to the number of partitions of* n *that are self-conjugate.*

Proof. Let λ be a self-conjugate partition. If we take a point on the main diagonal it is possible to see that its arm and its leg are equal by the symmetry, so the hook for a point in the main diagonal is a odd number.
We can also see that all the hooks in the main diagonal are different and strictly decreasing starting from the top left.
We can construct a new partition λ' whose parts are the hooks of the points in the main diagonal of λ, and this is a partition into both odd and distinct parts. □

Example 1.13. *If we take the partition* $\lambda = (4,4,3,2)$*:*



we obtain $\lambda' = (7,5,1)$.

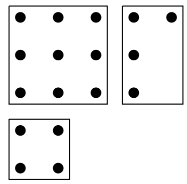
It is possible to prove the previous proposition using the respective generating functions.
A self-conjugate partition of n has a central square of m^2 points identified by the main diagonal and two conjugate tails that represent a partition of $\frac{1}{2}\left(n-m^2\right)$ in at most m parts.
Conversely if $m^2 \leq n$, then there is a set of self-conjugate partitions of n based on a square of m^2 points.
We know that the number of partitions of $\frac{1}{2}\left(n-m^2\right)$ whose parts do not exceed m corresponds to

$$\frac{x^{m^2}}{(1-x^2)\cdots(1-x^{2m})},$$

which corresponds, by Proposition 1.9, to the number of partitions of $\frac{1}{2}\left(n-m^2\right)$ in at most m parts, so we have

$$F\left(x,\mathcal{S}\right) = 1 + \sum_{m\geq 1} \frac{x^{m^2}}{(1-x^2)\cdots(1-x^{2m})} = F\left(x,\mathcal{D}^{\mathcal{O}}\right).$$

We observe that not only the self-conjugate partitions contain a square in their Ferrers representation. Let $\lambda \vdash n$, the graphical representation contains a square which its diagonal coincides with the main diagonal of Ferrers graph (it is called **Durfee** square), and two tails.



Similarly it is possible to reformulate the Euler's theorem (Theorem 1.2).

Proposition 1.14. *Let* $|x| < 1$

$$\prod_{i\geq 1} \frac{1}{1-x^i} = 1 + \sum_{j\geq 1} \frac{x^{j^2}}{(1-x)^2 \cdots (1-x^j)^2}$$

Proof. The left part of the equation is the generating function of any partition, so we have to show that the right side is a method to count the partitions too. Let $\lambda \vdash n$, we can assume that the Dufree square contains j^2 points and the two tails represent a partition of a positive integer m into no more j parts and a partition of a positive integer l into parts don't exceed j; obviously n, m, j and l satisfy $j^2 + m + l = n$.
We know that the number of partitions of l into parts lower than j corresponds to the coefficient of x^l of

$$\frac{1}{(1-x)\cdots(1-x^j)}$$

and the number of partitions of m into no more j parts is the coefficient of x^m in the same expression. So the number of possible pairs of tails in a partition of n whose the Dufree square is j^2 is the coefficient of x^{n-j^2} of

$$\left(\frac{1}{(1-x)\cdots(1-x^j)}\right)^2$$

or the coefficient of x^n in

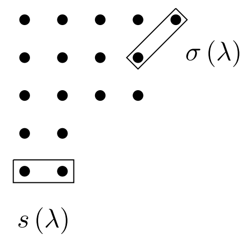
$$\frac{x^{j^2}}{(1-x)^2 \cdots (1-x^j)^2}$$

□

We conclude this section proving the Euler's theorem on pentagonal numbers.

Definition 1.15. *Let* $\lambda = (\lambda_1, \cdots, \lambda_t)$ *be a partition. The term* λ_t *is the* ***smallest*** *part of partition, we denote it* $s(\lambda) = \lambda_t$ *and call it* ***base*** *of the*

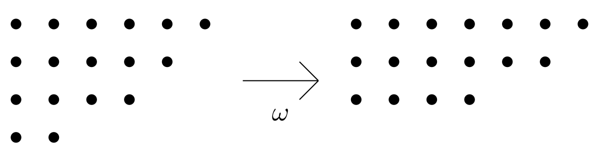
Ferrers graph.
We define the ***slope*** *of* λ *the number of consecutive parts of* λ*, starting from* λ_1*, whose difference is equal to 1, or in other words the maximal* j *such that* $\lambda_j = \lambda_1 - j + 1$*. We denote by* $\sigma(\lambda)$ *the slope*



Theorem 1.16 (Euler's Pentagonal Number Theorem)**.** *Let* $p_e(\mathcal{D}, n)$ *and* $p_o(\mathcal{D}, n)$ *the number of partitions of* n *into an even or odd number of parts. Then:*

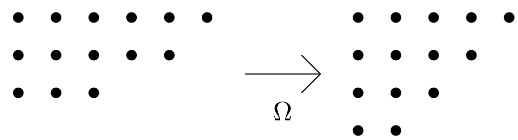
$$p_e(\mathcal{D}, n) - p_o(\mathcal{D}, n) = \begin{cases} (-1)^m & \text{if } n = \frac{1}{2}m(3m \pm 1) \\ 0 & \text{otherwise} \end{cases}$$

Proof. We introduce two new graphical transformations of a partition. A operation ω that removes the base of the partition and glues it in parallel and outside position of the slope, so the base become the new slope, and the inverse operation Ω that removes the slope and glues it as the new base of the partition. We have three cases.
Case 1. $s(\lambda) < \sigma(\lambda)$
We can only apply operation ω, because if we do Ω there is a violation in the decreasing order of partition of partition.



Notice that the new partition still be a partition into distinct parts, but it changes the parity of the number of parts.
Case 2 $s(\lambda) = \sigma(\lambda)$
We cannot apply Ω because otherwise we obtain a new partition where the two smallest parts are not distinct.
It is possible to apply only ω, but we have a limitation when $s(\lambda)$ and $\sigma(\lambda)$ have intersection, in particular when λ is a partition in m parts and $s(\lambda) = m$. In this case it is easy to calculate the weight of partition because also the slope is equal to m and then we obtain $n = m + (m+1) + \cdot + (2m-1) = \frac{1}{2}m(3m-1)$.
Case 3 $s(\lambda) > \sigma(\lambda)$

In the last case we cannot apply ω but it is possible to apply Ω.



Also Ω sends a partition into distinct parts in another partition into distinct parts and changes the parity of parts. Also in this case we have a limit case. It is not possible apply Ω when the slope intersects the base and $s(\lambda) = \sigma(\lambda) + 1$, because after the transformation we obtain the two smallest parts equal. This is the case when λ has m parts and $s(\lambda) = m + 1$, so we have again $n = (m+1) + (m+2) + \cdots + (2m) = \frac{1}{2}m(3m+1)$.
When $n \neq \frac{1}{2}m(3m \pm 1)$, all the foregoing procedures change the parity of the number of parts of the partitions and exactly one case is applicable to any partition of n. So the operations establish a one-to-one correspondence and then we obtain $p_e(\mathcal{D}, n) = p_o(\mathcal{D}, n)$. When $n = \frac{1}{2}m(3m \pm 1)$ we have an excess of one partition into an even or an odd number of parts respectively when m is even or odd, so $p_e(\mathcal{D}, n) - p_o(\mathcal{D}, n) = (-1)^m$. □

Theorem 1.17 (Euler pentagonal numbers)**.** *Let* $|x| < 1$

$$\prod_{n \geq 1} (1 - x^n) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(3n+1)} \tag{1.5}$$

Proof. If we look at the infinite product

$$(1-x)\left(1-x^2\right)\left(1-x^3\right)\cdots$$

it is possible see that every coefficient of x^i has contribution 1 if we multiply a even number of x^{j_h}, in other words if we consider a partition of i into an even number of different parts; and it has contribution -1 if we have a partition of i into an odd number of different parts. So:

$$\prod_{n \geq 1} (1 - x^n) = 1 + \sum_{n \geq 1} \left(p_e(\mathcal{D}, n) - p_o(\mathcal{D}, n)\right) x^n$$

By Theorem 1.16 we can rewrite the right part of the equation:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(3n+1)} &= 1 + \sum_{n=1}^{\infty} (-1)^n x^{\frac{1}{2}n(3n+1)} + \sum_{n=-1}^{-\infty} (-1)^n x^{\frac{1}{2}n(3n+1)} \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n x^{\frac{1}{2}n(3n+1)} + \sum_{n=1}^{\infty} (-1)^n x^{\frac{1}{2}n(3n-1)} \end{aligned}$$

and we obtain the equivalence of the two equation. □

Corollary 1.18. *Let $n > 0$:*

$$p(n) - p(n-1) - p(n-2) + \cdots$$
$$\cdots + (-1)^m p\left(n - \frac{1}{2}m(3m-1)\right) + (-1)^m p\left(n - \frac{1}{2}m(3m+1)\right) + \cdots = 0$$

where $p(M) = 0$ if $M < 0$

Proof. Let a_n the left side of the equation. So

$$\begin{aligned}\sum_{n\geq 0} a_n x^n &= \sum_{n\geq 0} p(n) x^n \left(1 + \sum_{m\geq 0} (-1)^m \left(x^{\frac{1}{2}m(3m-1)} + x^{\frac{1}{2}m(3m+1)}\right)\right) \\ &= \left(\prod_{n\geq 1} (1-x^n)^{-1}\right)\left(\prod_{n\geq 1}(1-x^n)\right) \\ &= 1\end{aligned}$$

then $a_n = 0$ when $n > 0$. □

1.2 Arithmetic properties of $p(n)$

It is possible to demonstrate Theorem 1.17 as a special case of Jacobi's theorem.

Theorem 1.19 (Jacobi)**.** *If $|x| < 1$, then*

$$\begin{aligned}\prod_{n=1}^{\infty}\left(1-x^{2n}\right)\left(1+x^{2n-1}z\right)\left(1+x^{2n-1}z^{-1}\right) &= 1 + \sum_{n=1}^{\infty} x^{n^2}\left(z^n + z^{-n}\right) \\ &= \sum_{n=-\infty}^{\infty} x^{n^2} z^n\end{aligned}$$

Proof. See [HWHBS08] section 19.8 □

If we write x^k instead of x, $-x^l$ instead of z and n instead of $n+1$ on the left hand we obtain

$$\prod_{n=0}^{\infty}\left(1-x^{2kn+2k}\right)\left(1-x^{2kn+k+l}\right)\left(1-x^{2kn+2k-l}\right) = \sum_{n=-\infty}^{\infty}(-1)^n x^{kn^2+ln}, \tag{1.6}$$

and if we set $k = \frac{3}{2}$ and $l = \frac{1}{2}$ we obtain

$$\prod_{n=0}^{\infty}\left(1-x^{3n+3}\right)\left(1-x^{3n+2}\right)\left(1-x^{3n+1}\right) = \sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{1}{2}n(3n+1)},$$

that is the Euler's Pentagonal Number Theorem.
Now we see two other special cases of Jacobi's theorem that are useful to show some arithmetic properties of $p(n)$.

Corollary 1.20. *If $k = \frac{5}{2}$ and $l = \frac{3}{2}$ in equation* (1.6) *we obtain*

$$\prod_{n=0}^{\infty} \left(1 - x^{5n+1}\right)\left(1 - x^{5n+4}\right)\left(1 - x^{5n+5}\right) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(5n+3)}; \tag{1.7}$$

and if $k = \frac{5}{2}$ and $l = \frac{1}{2}$ give

$$\prod_{n=0}^{\infty} \left(1 - x^{5n+2}\right)\left(1 - x^{5n+3}\right)\left(1 - x^{5n+5}\right) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(5n+1)}; \tag{1.8}$$

Another application of Jacobi's theorem gives us the following equation.

Theorem 1.21.

$$\prod_{n=1}^{\infty} (1 - x^n)^3 = \sum_{m=0}^{\infty} (-1)^m (2m+1) x^{\frac{1}{2}m(m+1)} \tag{1.9}$$

Examining Macmahon's table of $p(n)$ [Mac16], Ramanujan conjectured and proved three arithmetic properties associated with the moduli of some prime numbers.

Theorem 1.22 (Ramanujan)**.** *Let $m \geq 0$, then*

$$p(5m+4) \equiv 0 \pmod 5$$

Proof. We consider

$$x \left(\prod_{n=1}^{\infty} 1 - x^n\right)^4$$

using (1.5) and (1.9) we can write

$$\begin{aligned} x \left(\prod_{n=1}^{\infty} 1 - x^n\right)^4 &= x \left(\sum_{r=-\infty}^{\infty} (-1)^r x^{\frac{1}{2}r(3r+1)}\right)\left(\sum_{s=0}^{\infty} (2s+1) x^{\frac{1}{2}s(s+1)}\right) \\ &= \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} (-1)^{r+s} x^k, \end{aligned}$$

where $k = 1 + \frac{1}{2}r(3r+1) + \frac{1}{2}s(s+1)$.
If we suppose $k \equiv 0 \pmod 5$ then

$$\begin{aligned} 8k &= 8 + 12r^2 + 4r + 4s^2 + 4s \\ &= 2(r+1)^2 + (2s+1)^2 + 10r + 5 \equiv 0 \pmod 5 \end{aligned}$$

so we obtain $2(r+1)^2 + (2s+1)^2 \equiv 0 \pmod 5$. It is possible to see that $2(r+1)^2 \equiv 0, 2, 3 \pmod 5$ and $(2s+1)^2 \equiv 0, 1, 4 \pmod 5$.
So we have $2(r+1)^2 \equiv 0 \pmod 5$ and $(2s+1)^2 \equiv 0 \pmod 5$, in particular

$(2s+1) \equiv 0 \pmod 5$, and thus the coefficient of x^{5m+5} in $x\left(\left(1-x\right)\left(1-x^2\right)\cdots\right)^4$ is divisible by 5.
Now we consider $\left(1-x\right)^{-5}$ and its binomial expansion

$$\frac{1}{\left(1-x\right)^5} = \sum_{i=0}^{\infty} \binom{i+4}{i} x^i$$

It is easy to see that all the coefficients are divisible by 5, except those of x^{5m} which are equal 1 modulo 5. We can express this by writing

$$\frac{1}{\left(1-x\right)^5} \equiv \frac{1}{1-x^5} \pmod 5$$

where this notation implies that the coefficients of every power of x are congruent modulo 5.
It follows that

$$\frac{\left(1-x^5\right)\left(1-x^{10}\right)\cdots}{\left(1-x\right)^5\left(1-x^2\right)^5\cdots} \equiv 1 \pmod 5$$

Hence the coefficient of x^{5m+5} in

$$x\left(\left(1-x\right)\left(1-x^2\right)\cdots\right)^4 \frac{\left(1-x^5\right)\left(1-x^{10}\right)\cdots}{\left(\left(1-x\right)\left(1-x^2\right)\cdots\right)^5} = x\frac{\left(1-x^5\right)\left(1-x^{10}\right)\cdots}{\left(1-x\right)\left(1-x^2\right)\cdots}$$

is a multiple of 5.
Then

$$x\frac{\left(1-x^5\right)\left(1-x^{10}\right)\cdots}{\left(1-x\right)\left(1-x^2\right)\cdots}\left(1+x^5+x^{10}\cdots\right)\left(1+x^{10}+x^{20}\cdots\right) = \frac{x}{\left(1-x\right)\left(1-x^2\right)\cdots} = x + \sum_{n=2}^{\infty} p\left(n-1\right)x^n$$

so the coefficient of x^{5m} is a multiple of 5. □

Other two properties have been proved using the same method.

Theorem 1.23 (Ramanujan)**.** *Let $m \geq 0$, then*

$$p\left(7m+5\right) \equiv 0 \pmod 7$$
$$p\left(11m+6\right) \equiv 0 \pmod{11}$$

He also made a general conjecture [Ram19] that

$$p\left(n\right) \equiv 0 \pmod \delta$$

if $\delta = 5^a, 7^b, 11^c$ and $24n \equiv 1 \pmod \delta$. Ramanujan proved it for $5^2, 7^2, 11^2$ [AHAW00]. Watson generalised the proof for 5^a [Wat38], and Atkin for 11^c [Atk67]. Then, after Gupta found a counterexample for $p\left(243\right)$, [Gup80], Watson modified the conjecture, stating and proving it for only special exponents of 7 [Wat38].

Now we state two identities which resemble Proposition 1.5

Theorem 1.24 (Rogers-Ramanujan identities)**.** *Let* $|x| < 1$

$$1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)\cdots(1-x^m)} = \prod_{m=0}^{\infty} \frac{1}{(1-x^{5m+1})(1-x^{5m+4})} \tag{1.10}$$

$$1 + \sum_{m=1}^{\infty} \frac{x^{m(m+1)}}{(1-x)\cdots(1-x^m)} = \prod_{m=0}^{\infty} \frac{1}{(1-x^{5m+2})(1-x^{5m+3})} \tag{1.11}$$

Proof. We start to prove equation (1.10).
Let us denote

$$P_0 = 1 \qquad P_r = \prod_{s=1}^{r} \frac{1}{1-x^s}$$

$$\lambda(r) = \frac{1}{2} r(5r+1) \qquad Q_r = Q_r(a) = \prod_{s=r}^{\infty} \frac{1}{1-ax^s}$$

and define the operator η by

$$\eta f(a) = f(ax)$$

Let $H_m(a)$ be the auxiliary function

$$H_m(a) = \sum_{r=0}^{\infty} (-1)^r a^{2r} x^{\lambda(r)-mr} \left(1 - a^m x^{2mr}\right) P_r Q_r \tag{1.12}$$

where $m = 0, 1$ or 2.
Let us consider

$$\begin{aligned} H_m - H_{m-1} &= \sum_{r=0}^{\infty} (-1)^r a^{2r} x^{\lambda(r)} P_r Q_r \left(x^{-mr} - a^m x^{mr} - x^{r(1-m)} + a^{m-1} x^{mr-r}\right) \\ &= \sum_{r=0}^{\infty} (-1)^r a^{2r} x^{\lambda(r)} P_r Q_r \left(x^{-mr}(1-x^r) + a^{m-1} x^{r(m-1)}(1-ax^r)\right) \end{aligned}$$

Notice that $(1-x^r) P_r = P_{r-1}$ and $(1-ax^r) Q_r = Q_{r+1}$ and so

$$\begin{aligned} H_m - H_{m-1} = &\sum_{r=0}^{\infty} (-1)^r a^{2r+m-1} x^{\lambda(r)+r(m-1)} P_r Q_{r+1} \\ &+ \sum_{r=1}^{\infty} (-1)^r a^{2r} x^{\lambda(r)+mr} P_{r-1} Q_r \end{aligned}$$

We change r into $r+1$ in the second sum and we note that $\lambda(r+1) - \lambda(r) =$

$5r+3$, so we have

$$\begin{aligned}
H_m - H_{m-1} &= \sum_{r=0}^{\infty} (-1)^r P_r Q_{r+1} \left(a^{2r+m-1} x^{\lambda(r)+r(m-1)} - a^{2(r+1)} x^{\lambda(r+1)-m(r+1)} \right) \\
&= \sum_{r=0}^{\infty} (-1)^r P_r Q_{r+1} \left(a^{2r+m-1} x^{\lambda(r)+r(m-1)} \left(1 - a^{3-m} x^{(2r+1)(3-m)}\right)\right) \\
&= \sum_{r=0}^{\infty} (-1)^r P_r Q_{r+1} a^{m-1} (ax)^{2r} x^{\lambda(r)+r(m-3)} \left(1 - (ax)^{3-m} x^{2r(3-m)}\right) \\
&= \sum_{r=0}^{\infty} (-1)^r P_r Q_{r+1} a^{m-1} \eta \left(a^{2r} x^{\lambda(r)-r(3-m)} \left(1 - a^{3-m} x^{2r(3-m)}\right)\right)
\end{aligned}$$

Since $\eta(Q_r) = Q_{r+1}$, we have

$$\begin{aligned}
H_m - H_{m-1} &= a^{m-1} \eta \left(\sum_{r=0}^{\infty} (-1)^r a^{2r} x^{\lambda(r)-r(3-m)} \left(1 - a^{3-m} x^{2r(3-m)}\right)\right) \\
&= a^{m-1} \eta (H_{3-m})
\end{aligned}$$

If we set $m=1$ and $m=2$, recalling that $H_0 = 0$, we have

$$\begin{aligned}
H_1 &= \eta H_2 \\
H_2 - H_1 &= a \eta H_1
\end{aligned}$$

and summing the two equations we obtain

$$H_2 = \eta H_2 + a \eta^2 H_2 \tag{1.13}$$

Now we can expand H_2 in powers of a

$$H_2 = \sum_{s=0}^{\infty} c_s a^s$$

where c_s are independent of a. Using (1.13) we obtain

$$\sum c_s a^s = \sum c_s x^s a^s + \sum c_s x^{2s} a^s$$

and we can explicit the value of c_s equaling the coefficients of a^s, recalling that $c_0 = 1$. So we have

$$c_s = \frac{x^{2s-2}}{1-x^s} c_{s-1} = \frac{x^{2+4+\cdots+2(s-1)}}{(1-x)\cdots(1-x^s)} = x^{s(s-1)} P_s$$

and finally

$$H_2(a) = \sum_{s=0}^{\infty} a^s x^{s(s-1)} P_s$$

If we set $a = x$ we obtain the left side of (1.10).
Note that

$$P_r Q_r(x) = \prod_{s=1}^{r} \frac{1}{1-x^s} \prod_{s=r}^{\infty} \frac{1}{1-x^{s+1}} = \prod_{s=1}^{\infty} \frac{1}{1-x^s} = P_\infty$$

and by Equation (1.8), we obtain

$$\begin{aligned}
H_2(x) &= P_\infty \sum_{r=0}^{\infty} (-1)^r x^{\lambda(r)} \left(1 - x^{2(2r+1)}\right) \\
&= P_\infty \left(\sum_{r=0}^{\infty} (-1)^r x^{\lambda(r)} - \sum_{r=0}^{\infty} (-1)^r x^{\lambda(r)+2(2r+1)} \right) \\
&= P_\infty \left(\sum_{r=0}^{\infty} (-1)^r x^{\lambda(r)} + \sum_{r=1}^{\infty} (-1)^r x^{\lambda(r-1)+2(2r-1)} \right) \\
&= P_\infty \left(1 + \sum_{r=1}^{\infty} (-1)^r \left(x^{\frac{1}{2}r(5r+1)} + x^{\frac{1}{2}r(5r-1)} \right) \right) \\
&= P_\infty \sum_{r=-\infty}^{\infty} (-1)^r x^{\frac{1}{2}r(5r+1)} \\
&= P_\infty \prod_{n=0}^{\infty} \left(\left(1 - x^{5n+2}\right) \left(1 - x^{5n+3}\right) \left(1 - x^{5n+5}\right) \right) \\
&= \prod_{n=0}^{\infty} \frac{1}{\left(1 - x^{5n+1}\right) \left(1 - x^{5n+4}\right)},
\end{aligned}$$

that is the right side of (1.10).
To prove Equation (1.11) we can proceed in the same way using $h_1 = \eta H_2(a) = \sum a^s x^{s^2} P_s$ and Equation (1.7). □

We conclude this section describing a combinatorial interpretation of Theorem 1.24.
The sum in the left side of equation (1.10) is the generating function for partitions of $n - m^2$ whose parts are lower or equal to m or, by Proposition 1.9, for partitions of $n - m^2$ into at most m parts.
Note that $m^2 = 1 + 3 + \cdots + (2m-1)$, so if we represent it as Ferrers graph, we obtain a graph with m rows. Adding a partition of $n - m^2$ into at most m parts to the graph, we obtain a partition of n in distinct parts whose minimal difference is 2.
For example, if $n = 9$ and $m = 2$ we can take $\lambda = (3, 2)$ as partition of 5

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \circ & \circ & \circ \\ \bullet & \circ & \circ & & & \end{array}$$

and we obtain $\lambda' = (6, 3) \vdash 9$.
The right side of equation (1.10) enumerates partitions into parts of the form $5m + 1$ and $5m + 4$.

Proposition 1.25. *The number of partitions of n with minimal difference 2 is equal to the number of partitions into parts of the form $5m+1$ and $5m+4$.*

Example 1.26. *Let $n = 11$, so $m = 1, 4$ or 9. When $m = 1$ we obtain (11), $m = 4$ gives $(10,1)$, $(9,2)$, $(8,3)$, $(7,4)$ and for $m = 9$ we have $(7,3,1)$ and $(6,4,1)$. So in total we have 7 partitions.*
If we consider the partitions into parts of the forms $5m+1$ and $5m+4$ we have (11), $\left(9,1^2\right)$, $(6,4,1)$, $\left(6,1^5\right)$, $\left(4^2,1^3\right)$, $\left(4,1^7\right)$, $\left(1^{11}\right)$, again 7 partitions.

The combinatorial interpretation of (1.11) is obtained in the same way noting that $m(m+1) = 2+4+\cdots+2m$.

Proposition 1.27. *The number of partitions of n into parts not less than 2 and with minimal difference 2 is equal to the number of partitions of n into parts of the form $5m+2$ and $5m+3$.*

1.3 Relation with Dyck paths and Parking functions

In this section we will see a combinatorical aspect of partitions and their relation with the Dyck paths and symmetric functions and parking functions, analysing the standard case [Whi70].
We start introducing an important kind of numbers, a sequence of natural numbers that occur in more than 100 counting problems [Sta99].

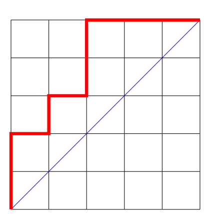
Definition 1.28. *We call* ***Catalan numbers*** *a sequence of natural numbers that satisfies the recurrence relations*

$$C_0 = 1 \qquad\qquad C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i}$$

The n-th Catalan number can be expressed directly in terms of the binomial coefficient

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Definition 1.29. *A* ***classical Dyck path*** *of order n is a minimal lattice path in $\mathbb{Z}^2$ from the point $(0,0)$ to the point (n,n) consisting of east and north steps which stay above the diagonal $y = x$. We write DP_n to indicate the set of all classical Dyck path of order n.*



*Given positive integer $a, b \in \mathbb{N}$, we can define an (a, b)-**Dyck path** as a minimal lattice path from the point $(0,0)$ to the point (b, a) consisting of east and north steps which stay above the diagonal $y = \frac{a}{b}x$. If a and b are coprime we call these **rational Dyck paths**. We denote this set by $DP_{a,b}$.*
*We can define a **Dyck word** from a Dyck path labelling every step with letter N or E.*

For counting the number of Dyck paths of order n, we need the following equality.

Lemma 1.30. *Let $n \geq 1$, then*

$$\sum_{i=0}^{n} \frac{1}{i+1}\binom{2i}{i}\binom{2(n-i)}{n-i} = \binom{2n+1}{n}$$

Proof. By the binomial Theorem

$$(1-4x)^{-\frac{1}{2}} = 1 + \binom{2}{1}x + \binom{4}{2}x^2 + \cdots + \binom{2i}{i}x^i + \cdots$$

$$(1-4x)^{\frac{1}{2}} = 1 - 2x - 2\binom{2}{1}\frac{x^2}{2} + \cdots - 2\binom{2i}{i}\frac{x^{i+1}}{i+1} + \cdots$$

Since the product of $(1-4x)^{-\frac{1}{2}}$ and $(1-4x)^{\frac{1}{2}}$ is 1, then all the coefficients of x^n are equal to 0 for each $n \geq 1$. If we consider the coefficient of $x^n + 1$, we have

$$\binom{2(n+1)}{n+1} - 2\left(\binom{2n}{n} + \cdots + \binom{2i}{i}\frac{1}{n-i+1}\binom{2(n-i)}{n-i} + \cdots + \frac{1}{n+1}\binom{2n}{n}\right) = 0,$$

so

$$\sum_{i=0}^{n-1}\binom{2i}{i}\frac{1}{n+1-i}\binom{2(n-i)}{n-i} = \frac{1}{2}\binom{2(n+1)}{n+1} = \binom{2n+1}{n}$$

□

Proposition 1.31. *The number of classical Dyck paths of order n is equal to C_n*

Proof. We consider a lattice in $\mathbb{Z}^2$ from $(0,0)$ to the point (n, n). The number of minimal paths is $\binom{2n}{n}$.
Notice that every path that is not a Dick path passes the right of the diagonal at least one time, so we can classify them according to the point at which they cross the diagonal for the first time.
We consider a path that passes the diagonal for the first time in the point (k, k) and we cut it in two parts. The first one is a Dyck path from $(0,0)$ to (k, k), and the second one is a path from $(k+1, k)$ to (n, n). So the total number of this kind of paths is

$$\#DP_k\binom{2(n-k)-1}{n-k}$$

We can observe that this kind of paths is equal to paths that are Dyck paths until the point (k, k) and then go north, so

$$\#DP_k \binom{2(n-k)-1}{n-k} = \frac{1}{2} \#DP_k \binom{2(n-k)}{n-k}.$$

If we sum k from 0 to $n-1$ we obtain all the complementary of Dyck paths set

$$\sum_{k=0}^{n-1} \frac{1}{2} \#DP_k \binom{2(n-k)}{n-k} = \binom{2n}{n} - \#DP_n$$

Notice that $\#DP_0 = 1 = \binom{0}{0}$ and $\#DP_1 = 1 = \frac{1}{2}\binom{2}{1}$, so we suppose that $\#DP_i = \frac{1}{i+1}\binom{2i}{i}$ for $i < n$ and we obtain

$$2\binom{2n}{n} - 2\#DP_n = \sum_{k=0}^{n-1} \frac{1}{1+k} \binom{2k}{k} \binom{2(n-k)}{n-k}$$

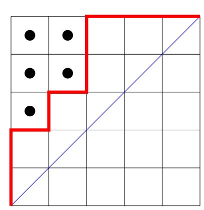
and using Lemma 1.30 we have

$$2\binom{2n}{n} - 2\#DP_n = \binom{2n+1}{n} - \frac{1}{n+1}\binom{2n}{n}$$

so

$$\#DP_n = \frac{1}{2}\left(2 + \frac{1}{n+1} - \frac{2n+1}{n+1}\right)\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n} = C_n$$

□



Remark 1.32. *We can observe that every Dyck path defines in a unique way a Ferrers graph in the cells above it.*
We denote by δ_n the Ferrers graph associated to the Dyck path of length $2n$ that touches the diagonal in every point, hence $\delta_n = (n-1, n-2, \ldots, 1)$.
This is the Ferrers graph with the larger area and all the other partitions are contained in it. We indicate this fact by $\lambda \subset \delta_n$.

Notice that by Proposition 1.31

$$\#\{\lambda | \lambda \subseteq \delta_n\} = C_n$$

Definition 1.33. *A* ***semistandard*** *Young tableau is obtained filling the boxes of a Young diagram with elements of a totally ordered set such that the entries weakly increase along each row and strictly increase down each column.*
A semistandard Young tableau of n cells is called ***standard*** *if its entries are in bijection with $[n]$, in other words the entries strictly increase along each row.*

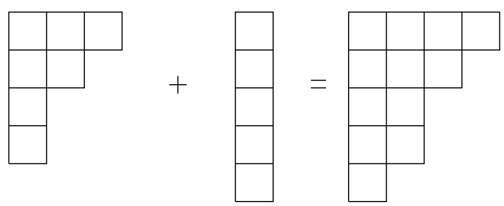
Example 1.34. *Let us consider the Young tableau associated to the partition $\lambda = (6, 4, 3, 1)$. If we fill the boxes as follows*

1	2	2	3	6	9
3	3	5	6		
4	6	7			
5					

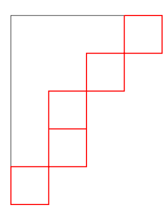
1	2	6	8	9	14
3	7	10	12		
4	11	13			
5					

we obtain a semistandard Young tableau on the left and a standard Young tableau on the right.

We define now two operations with the Young diagrams. Let λ be a partition represented by a Young diagram and let 1^n the partition of n into n parts we define the sum $\lambda + 1^n$ graphically by



and we obtain the partition $\lambda^* = (\lambda_1 + 1, \lambda_2 + 1, \cdots, \lambda_t + 1)$.
Let λ be a partition and $\mu \subset \lambda$ a subpartition, we define the **skew** diagram the set-theoretic difference of the Young diagrams λ and μ. We denote this by λ/μ. For example if $\lambda = (4, 3, 2, 2, 1)$ and $\mu = (3, 2, 1, 1)$ we obtain the skew shape



A skew diagram represents a skew standard tableau if all the entries increase down each column.

We introduce now a new set of functions.

Definition 1.35. *A* ***parking function*** *of size n is a sequence $(a_1, a_2, \ldots, a_n)$ of positive integers such that its increasing rearrangement $(b_1, b_2, \ldots, b_n)$ satisfies $b_i \leq i$. We denote the set of parking functions of size n by PF_n.*

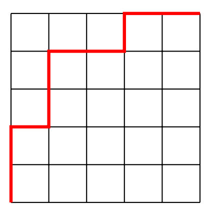
The parking functions were introduced by Konheim and Weiss [KW66] to study the widely used storage device of hashing. They show that there are $(n+1)^{n-1}$ parking functions of size n. There are other combinatorial demonstrations as in the enumerative theory of trees and forests by Riordan [Rio69].

Example 1.36. *The parking functions of length* 3 *are:*

$$\begin{array}{cccc} (1,1,1) & (1,1,2) & (1,2,1) & (2,1,1) \\ (1,1,3) & (1,3,1) & (3,1,1) & (1,2,2) \\ (2,1,2) & (2,2,1) & (1,2,3) & (1,3,2) \\ (2,1,3) & (2,3,1) & (3,2,1) & (3,1,2) \end{array}$$

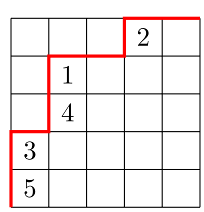
We can observe that there are $4^2 = 16$ *parking functions of size* 3*, as expected.*

Notice that there exists a bijection between increasing rearrangement of parking functions and Dych paths. Let $B = (b_1, b_2, \cdots, b_n)$ an increasing parking function, we call m_i the number of times that i occurs in B, so we can define the Path word $N^{m_1}EN^{m_2}E\cdots EN^{m_n}E$. For example if $B = (1,1,2,2,4)$ and so N^2EN^2EENEE, that correspond to



Definition 1.37. *Given a parking function* $P = (a_1, \cdots, a_n)$ *we draw the Dyck path corresponding to the increasing rearrangement. The* i*-th vertical run of the path has length* m_i*. If* $m_i = k$ *we have* $i = a_{j_1} = \cdots = a_{j_k}$ *and then we label the* i*-th vertical run by the set of indices* $\{j_1, \cdots, j_k\}$*. We do this by filling the boxes to the right of* i*-th vertical run with the labels* $j_1 < \cdots < j_k$ *increasing down the column. This is a* ***labeled Dyck path****.*

For example if $P = (2,4,1,2,1)$, we obtain



Notice that it is possible to represent every labelled Dyck path associated to a partition λ as a skew standard Young tableau of shape $(\lambda + 1^n)/\lambda$.
We can observe that S_n acts on parking functions by permuting subscripts. So for any $\sigma \in S_n$ and $P \in PF_n$, we obtain that P and $\sigma \cdot P$ have the same Dyck path, hence the orbits of this action are in bijection with the Dyck paths.

Translating the action of S_n on labelled Dyck paths, we can observe that S_n acts on the labelled path by permuting the labels and then reordering the labels in each column, so if $P \in PF_n$, $\sigma \cdot P$ is a skew standard tableau of the same shape of P. Notice that if a permutation $\tau \in S_n$ permutes only the labels in a single column, then $\tau \cdot P$ is the same skew standard Young tableau of P.
Suppose λ be a partition of n and a given Dyck path has $m_i = m_i(\lambda)$ vertical runs of length i, for $1 \leq i \leq n$. For example if $\lambda = (2, 2, 1)$ we can obtain the previous Dyck path. The orbit corresponding to this Dyck path has stabiliser isomorphic to the Young subgroup

$$S^\lambda \cong S_1^{m_1} \times S_2^{m_2} \times \cdots \times S_n^{m_n}. \tag{1.14}$$

For example if $\lambda = (2, 2, 1)$ and we choose $P = (2, 4, 1, 2, 1)$ the stabiliser is generated by the permutation $(1, 4)$ and $(3, 5)$.
Now we count the number of possible Dyck paths with a vertical structure defined by λ. To do this we count the rational Dyck paths with a fixed vertical structure.

Proposition 1.38. *Let a and b be coprime positive integer. Let $m_0, m_1, \ldots, m_a$ be non-negative integers such that $\sum_{i\geq 0} im_i = a$ and $\sum_{i\geq 0} m_i = b$.
Then the number of (a, b)-Dyck paths with m_i vertical runs of length i is*

$$\frac{(b-1)!}{m_0! m_1! \cdots m_a!}$$

Proof. Let Y be the set of minimal lattice paths π from the point $(0, 0)$ to the point (b, a) such that π has m_i vertical runs of length i, for all $1 \leq i \leq a$.
Let X be the set of words on the alphabet $\{v_0, \ldots, v_a\}$ containing m_i copies of v_i for each i.
We can define a function $f\colon X \to \{N, E\}^*$ by replacing each letter v_i by a word $N^i E$, so $f(v_0)$ is equal to E. Notice that, if $w \in X$, $f(w)$ has a letter equal to N and b letter equal to E and the last letter of $f(w)$ is a E, so $f(w) \in Y$. Notice that f is a bijection, hence

$$\#Y = \#X = \binom{b}{m_0, m_1, \cdots, m_a} = \frac{b!}{m_0! \cdots m_a!}$$

Let $\pi \in Y$, we define the level l_i to the i-th lattice point of π by recurrence. Let us set $l_0 = 0$ and for each $i \geq 1$

$$l_i = \begin{cases} l_{i-1} + b, & \text{if the } \quad i\text{-th step of} \quad \pi \quad \text{is north} \\ l_{i-1} - a, & \text{if the } \quad i\text{-th step of} \quad \pi \quad \text{is east} \end{cases}$$

Notice that the level of points (x, y) is equal to $by - ax$. Indeed $l_0 = 0$ and if we suppose the point (x, y) has level equal to $by - ax$, the next point in the path might be $(x + 1, y)$ and its level is $by - a(x + 1)$ or it might be $(x, y + 1)$ and its level is equal to $b(y + 1) - ax$.
Let (x_i, y_i) and (x_j, y_j) be the i-th and j-th points of π, with $i < j$, so we have

$0 \leq x_i \leq x_j \leq b$ and $0 \leq y_i \leq y_j \leq a$
Suppose $l_i = l_j$, hence $by_i - ax_i = by_j - ax_j$ that is equal to $b(y_j - y_i) = a(x_j - x_i)$. Since a and b are coprime and $y_j - y_i$ and $x_j - x_i$ are both positive, we obtain that a divides $y_j - y_i$ and b divides $x_j - x_i$. The only possible solution is $x_i = y_i = 0$, $x_j = b$ and $y_j = a$, and we obtain the starting and the finishing points of the path. So the levels $l_1, l_2, \cdots, l_{a+b}$ are all different.
Notice that a path in Y is a (a,b)-Dyck path if and only if every level are non-negative.
Now we define a equivalence relation $\sim$ in Y: $\pi_1 \sim \pi_2$ if and only if there exist $w_1, w_2 \in X$ such that $f(w_1) = \pi_1$ and $f(w_2) = \pi_2$ and w_2 is a cyclic shift of the letters of w_1.
If we fix $w \in X$, then $f(w) = \pi \in Y$. By cyclically shifting w we obtain b paths $\pi = \pi_0, \pi_1, \ldots, \pi_{b-1}$ which are equivalent to π.
Suppose π has east steps at position $i_1 < i_2 < \cdots < i_b = a+b$ and let $i_0 = 0$. Cyclically shifting w by k steps has the effect of cyclically shifting π by i_k steps, where $0 \leq k \leq b-1$. So the sequence of the levels $(l_0, l_1, \cdots, l_{a+b-1})$ for π became in π_k

$$(l_{i_k} - l_{i_k}, l_{i_k+1} - l_{i_k}, \ldots, l_{a+b} - l_{i_k}, l_1 - l_{i_k}, \ldots, l_{i_k-1} - l_{i_k})$$

Let m_0 be the minimum level in π, hence $m_0 - l_{i_k}$ is the minimum level in π_k for all k. Since all the levels in π are distinct then the b paths $\pi_0, \ldots, \pi_{b-1}$ all have distinct minimum level and hence these paths are distinct. So every equivalence class in Y has b distinct elements.
Notice that the minimum level m_0 in π occurs at the end of an east step, so $m_0 = l_{i_j}$, for some j. For every k the minimum level is $l_{i_j} - l_{i_k}$, which is non-negative if and only if $l_{i_j} \geq l_{i_k}$. Since l_{i_j} is the minimum level in π and all levels are distinct, the minimum level in π_k is non-negative if and only if $k = j$. So we have exactly one (a,b)-Dyck path in the equivalence class.
So we showed that Y is decomposed into a disjoint union of subsets with b elements and exactly one (a,b)-Dyck path, so

$$\#DP_{a,b} = \frac{b!}{m_0! \cdots m_a!} \frac{1}{b} = \frac{(b-1)!}{m_0! \cdots m_a!}$$

□

Proposition 1.39. *Let λ be a partition of n. The number of Dyck paths with $m_i = m_i(\lambda)$ vertical runs of length i is*

$$\frac{n!}{m_0! \cdots m_n!}$$

where $m_0 = n + 1 - len(\lambda)$.

Proof. Observe that the all $(n, n+1)$-Dyck paths finish with an east step, so

$$\#DP_n = \#DP_{n,n+1}$$

By Proposition 1.38, fixing $a = n$ and $b = n+1$ we have

$$\#DP_n = \frac{(n+1)!}{m_0! \cdots m_n!}$$

where $\sum i m_i = n$ is the weight of λ and $\sum m_i = n+1$. Since $m_1 + \cdots + m_n = len(\lambda)$, we obtain $m_0 = n + 1 - len(\lambda)$. $\square$

Definition 1.40. *Let $\boldsymbol{x} = \{x_1, x_2, \ldots\}$ be a infinite set of variables and let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a partition of n. We denote by $\boldsymbol{x}^\lambda$ the monomial $= x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \ldots x_{i_t}^{\lambda_t}$ of degree n. The function $f(\boldsymbol{x}) \in \mathbb{C}[[\boldsymbol{x}]]$ in the formal power series ring is* ***homogeneous*** *of degree n if every monomial in $f(\boldsymbol{x})$ has degree n.*
For every $n \in \mathbb{N}$, there is a natural action of $\sigma = (\sigma_1, \ldots, \sigma_n) \in S_n$ on $f(\boldsymbol{x})$ that permutes the variables

$$\sigma \cdot f(x_1, x_2, \cdots) = f(x_{\sigma_1}, x_{\sigma_2}, \cdots).$$

We say that f is a ***symmetric function*** *if*

$$\sigma \cdot f(\boldsymbol{x}) = f(\boldsymbol{x})$$

The simplest functions that are fixed by this action are the ***monomial symmetric function*** *m_λ corresponding to λ, that is*

$$m_\lambda = \sum_{\alpha \in \tau(\lambda)} \boldsymbol{x}^\lambda$$

where $\tau(\lambda)$ is the set of rearrangements of vector $(\lambda_1, \ldots, \lambda_t, 0, 0, \ldots)$.

For example if $\lambda = (2,1)$, m_λ results

$$x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + \cdots$$

Definition 1.41. *The ring of symmetric functions is*

$$\Lambda = \Lambda(\boldsymbol{x}) = \mathbb{C}[m_\lambda \mid \lambda \in \mathcal{P}]$$

i.e., the vector space spanned by all the m_λ.

Notice that Λ is a graded vector space $\Lambda = \bigoplus_{n \geq 0} \Lambda^n$, where Λ^n is the set of homogeneous symmetric functions of degree n, thus a vector space of dimension $p(n)$ whose basis is

$$\{m_\lambda \mid \lambda \vdash n\}$$

It is possible to define other kinds of symmetric functions.

Definition 1.42. *The* ***power sum symmetric function*** *of degree n is*

$$p_n = m_{(n)} = \sum_{i \geq 1} x_i^n$$

The ***elementary symmetric function*** *of degree n is*

$$e_n = m_{(1^n)} = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}$$

The ***complete homogeneous symmetric functions*** *of degree n is*

$$h_n = \sum_{\lambda \vdash n} m_\lambda$$

If $\mu = (\mu_1, \mu_2, \ldots, \mu_t)$ is a partition it is possible define the complete homogeneous (power sum, elementary) symmetric function $h_\mu = h_{\mu_1} h_{\mu_2} \cdots h_{\mu_t}$, ($p_\mu$, e_μ, respectively).

Example 1.43. *Let $\lambda = (2, 1)$, we have*

$$\begin{aligned} p_{2,1} = p_2 p_1 &= \left(x_1^2 + x_2^2 + \cdots\right)(x_1 + x_2 + \cdots) \\ &= \left(x_1^3 + x_2^3 + \cdots\right) + \left(x_1 x_2^2 + x_1^2 x_2 + \cdots\right) = m_3 + m_{2,1} \end{aligned}$$

$$\begin{aligned} e_{2,1} = e_2 e_1 &= (x_1 x_2 + x_1 x_3 + x_2 x_3 + \cdots)(x_1 + x_2 + \cdots) \\ &= \left(x_1^2 x_2 + x_1 x_2^2 + \cdots\right) + 3\,(x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots) = m_{2,1} + 3 m_{1,1,1} \end{aligned}$$

$$\begin{aligned} h_{2,1} = h_2 h_1 &= \left(x_1^2 + x_2^2 + \cdots + x_1 x_2 + x_1 x_3 + \cdots\right)(x_1 + x_2 + \cdots) \\ &= \left(x_1^3 + x_2^3 + \cdots\right) + 2\left(x_1^2 x_2 + x_1 x_2^2 + \cdots\right) + 3\,(x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots) \\ &= m_3 + 2 m_{2,1} + 3 m_{1,1,1} \end{aligned}$$

Notice that the elementary and the complete homogeneous symmetric functions can be expressed by generating functions very similar to the generating functions of partitions. In the elementary e_n we have all the exponents of variables equal to one, so coincides with the partition in distinct parts and using Equation (1.3)

$$E(t) := \sum_{n \geq 0} e_n(\mathbf{x})\, t^n = \prod_{i \geq 1} (1 + x_i t),$$

while in the complete homogeneous h_n the sum of the exponents of the variables of the monomials has all the degree n, so using Equation (1.2)

$$H(t) := \sum_{n \geq 0} h_n(\mathbf{x})\, t^n = \prod_{i \geq 1} \frac{1}{1 - x_i t}$$

We can define a map from the the space of class functions on S_n to the ring of symmetric functions

Definition 1.44. *Let $R^n = R(S_n)$ be the space of class functions on S_n, the* ***Frobenius characteristic map*** *is $ch^n \colon R^n \to \Lambda^n$ defined by*

$$ch^n(\chi) = \sum_{\lambda \vdash n} \chi_\lambda \frac{p_\lambda}{z_\lambda}$$

where χ_λ is the value of χ on the class λ

Theorem 1.45. *The number of parking functions of size n is equal to*

$$\# PF_n = (n+1)^{n-1}$$

Proof. Let us denote by $Frob_{PF_n}$ the image of character of PF_n under the Frobenius characteristic map. By [Mac95, p. 113,114] each orbit of PF_n with stabiliser isomorphic to S^λ (Equation (1.14)) contributes a term $h_\lambda(\mathbf{x})$, so

$$\begin{aligned} Frob_{PF_n} &= \sum_{\lambda \vdash n} \frac{1}{n+1} \binom{n+1}{n+1-len(\lambda), m_1(\lambda), \dots, m_n(\lambda)} h_\lambda(\mathbf{x}) \\ &= \frac{1}{n+1} \sum_{\lambda \vdash n} \binom{n+1}{n+1-len(\lambda), m_1(\lambda), \dots, m_n(\lambda)} h_1(\mathbf{x})^{m_1(\lambda)} \cdots h_n(\mathbf{x})^{m_n(\lambda)} \end{aligned}$$

Notice that $h_0(\mathbf{x}) = 1$, so using the multinomial theorem

$$(y_1 + \dots + y_m)^n = \sum_{k_1 + \dots + k_m = n; k_1, \dots, k_m \geq 0} \binom{n}{k_1, \dots, k_m} \prod_{t=1}^{m} y_t^{k_t}$$

we obtain

$$\begin{aligned} Frob_{PF_n} &= \frac{1}{n+1} (h_0(\mathbf{x}) + \dots + h_n(\mathbf{x}))^{n+1} \\ &= \frac{1}{n+1} [H(t)]^{n+1}|_{t^n} \\ &= \frac{1}{n+1} \left(\prod_{i \geq 1} \frac{1}{1 - x_i t} \right)^{n+1} \Bigg|_{t^n} \\ &= \frac{1}{n+1} \prod_{i \geq 1} \prod_{j \geq 1} \frac{1}{1 - x_i y_j} \Bigg|_{t^n} \end{aligned}$$

where we set $n+1$ of the y variables equal to t and the rest equal to 0.
The Cauchy product $\prod \prod \frac{1}{1 - x_i y_j}$ is equal to $\sum \frac{p_\lambda(\mathbf{x}) p_\lambda(\mathbf{y})}{z_\lambda}$ [Mac95, I.4.6], so,

setting the y variables as in precedent case, we have

$$\begin{aligned}
Frob_{PF_n} &= \frac{1}{n+1}\sum_{\lambda\in\mathcal{P}}\left(t^{\lambda_1}+\cdots+t^{\lambda_1}\right)\cdots\left(t^{\lambda_n}+\cdots+t^{\lambda_n}\right)\frac{p_\lambda(\mathbf{x})}{z_\lambda}\Bigg|_{t^n}\\
&= \frac{1}{n+1}\sum_{\lambda\in\mathcal{P}}(n+1)^{len(\lambda)}\,t^{|\lambda|}\frac{p_\lambda(\mathbf{x})}{z_\lambda}\Bigg|_{t^n}\\
&= \sum_{\lambda\in\mathcal{P}}(n+1)^{len(\lambda)-1}\,t^{|\lambda|}\frac{p_\lambda(\mathbf{x})}{z_\lambda}\Bigg|_{t^n}\\
&= \sum_{\lambda\in\mathcal{P}}(n+1)^{len(\lambda)-1}\,\frac{p_\lambda(\mathbf{x})}{z_\lambda}
\end{aligned}$$

Applying the inverse of the Frobenius map, this formula tell us the character of the S_n-module PF_n. Let $w\in S_n$, so $\chi(w)$ is the number of parking functions in PF_n fixed by the action of w. If we consider w the identity permutation in S_n that fixes all parking functions in PF_n and has a cycle type $\lambda=(1^n)$ we obtain $\#PF_n=(n+1)^{n-1}$. □

We have showed that fixing n we have $C_n=\frac{1}{n+1}\binom{2n}{n}$ (Definition 1.28) partitions λ contained in the maximal Dyck path δ_n (Remark 1.32) and $(n+1)^{n-1}$ skew standard Young tableaux of shape $(\lambda+1^n)/\lambda$. Furthermore given a $\lambda\vdash n$ we can construct $\frac{n!}{(n+1-len(\lambda))!m_1(\lambda)!\dots m_n(\lambda)!}$ Dyck paths with m_i vertical runs of length i and we can associate to them $(n+1)^{n-1}$ skew standard Young tableaux.

Chapter 2

Numerical Semigroups

In this chapter we introduce the numerical semigroups. In the first part we describe a covariety of numerical semigroups and represent it as a finite tree. After a characterisation of the children of an arbitrary vertex in this tree, we present an algorithm to describe the covariety [MFR23].
Then we show some relation between numerical semigroups and integer partitions [BNST23], looking in particular at the hookset. Then we introduce four operations [SY21] that give us some relation with symmetric numerical semigroups.

2.1 Covariety of numerical semigroups

Definition 2.1. *A* ***numerical set*** S *is a subset of the non-negative integers* $\mathbb{N}_0$ *such that* S *includes* 0 *and the complementary set* $\mathbb{N}_0 \setminus S = S^c$ *has finitely many elements. A numerical set* S *is a* ***numerical semigroup*** *if it is an additive submonoid of* $\mathbb{N}_0$*, in other words if the sum of two elements of* S *is an element of* S*. We denote by* NS *the set of numerical semigroups.*
The set S^c *is called set of* ***gaps*** *and its cardinality* $G(S)$ *is called* ***genus*** *of* S*. The maximal element* $F(S)$ *of* S^c *is called* ***Frobenius number****. Finally, let us define the* ***multiplicity*** $M(S)$ *of* S *as the lowest non-zero element of* S*. These three positive integers* $G(S)$*,* $F(S)$ *and* $M(S)$ *are invariant of* S*.*

Definition 2.2. *If* A *is a nonempty subset of* $\mathbb{N}_0$*, we denote by* $\langle A \rangle$ *the submonoid generated by* A*, i.e.,*

$$\langle A \rangle = \{i_1 a_1 + \cdots + i_n a_n \mid n \in \mathbb{N}_0 \setminus \{0\}, \{a_1, \ldots, a_n\} \subseteq A, \{i_1, \ldots, i_n\} \in \mathbb{N}_0\}$$

If S *is a numerical semigroup such that* $S = \langle A \rangle$*, then we say that* A *is a* ***system of generator*** *of* S*. If* $S \neq \langle B \rangle$ *for all* $B \subset A$ *then we say that* A *is a* ***minimal system of generators*** *of* S*, we denote the minimal system of generators of* S *by* $msg(S)$ *and its cardinality by* $e(S)$*, called* ***embedding dimension*** *of* S*.*

Example 2.3. *Let* $S = \{0, 3, 6, 8, 9, 11, 12, 14, \rightarrow\}$*, where* $\rightarrow$ *means that each integer number greater than* 14 *is in* S*. Then* S *is a numerical semigroup whose set of gaps is* $S^c = \{1, 2, 4, 5, 7, 10, 13\}$*, its genus is* $G(S) = 7$*, its Frobenius number is* $F(S) = 13$ *and its multiplicity is* $M(S) = 3$*. Moreover* $msg(S) = \{3, 8, 16\}$ *and then* $e(S) = 3$*.*

Lemma 2.4. *Let* A *be a nonempty subset of* $\mathbb{N}_0$*. Then* $\langle A \rangle$ *is a numerical semigroup if and only if* $gcd(A) = 1$

Proof. See [RGS09, Lemma 2.1]. □

Definition 2.5. *Let* S *be a numerical semigroup and* $n \in S \setminus \{0\}$*. For* $1 \leq i \leq n - 1$*, let* $w(i)$ *be the smallest integer in* S *congruent to* i *modulo* n*. The set* $Ap(S, n) = \{0, w(1), \ldots, w(n-1)\}$ *is called the* ***Apéry set*** *of* S *respect* n*.*

Let $S = \{0, 3, 6, 8, 9, 11, 12, 14, \rightarrow\}$ as before, if $n = 3$ we obtain $Ap(S, 3) = \{0, 16, 8\}$, while if $n = 5$ we have $Ap(S, 5) = \{0, 6, 12, 8, 9\}$.

Definition 2.6. *A* ***covariety*** $\mathcal{C}$ *is a nonempty family of numerical semigroups such that*

- $\mathcal{C}$ *has a minimum element with respect to set inclusion, denoted by* $\Delta(\mathcal{C})$
- *If* $S, T \in \mathcal{C}$*, then* $S \cap T \in \mathcal{C}$
- *If* $S \in \mathcal{C}$ *and* $S \neq \Delta(\mathcal{C})$*, then* $S \setminus \{M(S)\} \in \mathcal{C}$*.*

Let $F \in \mathbb{N}_0$, we consider the set $\mathcal{A}(F) = \{S \mid S \in NS, F(S) = F\}$, now we will see some properties about the covarieties that lead us to construct an algorithm to compute $\mathcal{A}(F)$

Proposition 2.7. *Every covariety has finite cardinality.*

Proof. Let $\mathcal{C}$ be a covariety, then $\mathcal{C} \subseteq \{S \mid S \in NS, \Delta(S) \subseteq S\}$.
If T is a numerical semigroup, the set

$$\{S \mid S \in NS, T \subseteq S\}$$

is a finite set because T^c is finite and then we can obtain at most $2^{\#T^c}$ numerical semigroups. So $\{S \mid S \in NS, \Delta(S) \subseteq S\}$ is a finite set and then $\mathcal{C}$ has finite cardinality. □

Proposition 2.8. *If* $F \in \mathbb{N}_0$*, then* $\mathcal{A}(F)$ *is a covariety.*

Proof. The set $\{0, F+1, \rightarrow\}$ is a numerical semigroup such that its Frobenius number is F and it is the numerical semigroup with Frobenius number F with fewer elements, so

$$\Delta(\mathcal{A}(F)) = \{0, F+1, \rightarrow\}$$

Let $S, T \in \mathcal{A}(F)$, then $S \cap T$ is a numerical semigroup such that $F(S \cap T) = max\{F(S), F(T)\}$, so $S \cap T \in \mathcal{A}(F)$.
Let now $S \in \mathcal{A}(F)$ be different from $\Delta(\mathcal{A}(F))$, then $M(S) < F$ and more precisely $M(S) = \min(msg(S))$, so we obtain that $S \setminus M(S) \in \mathcal{A}(F)$. □

Lemma 2.9. *Let $\mathcal{C}$ be a covariety and $S \in \mathcal{C}$.*
Let $\{S_n\}_{n\in\mathbb{N}}$ be the sequence defined by

$$S_0 = S$$
$$S_{n+1} = \begin{cases} S_n \setminus \{M(S_n)\} & \textit{if} \quad S_n \neq \Delta(\mathcal{C}) \\ \Delta(\mathcal{C}) & \textit{otherwise} \end{cases}$$

then there exists $k \in \mathbb{N}_0$ such that $\Delta(\mathcal{C}) = S_k \subset S_{k-1} \subset \ldots \subset S_0 = S$. Moreover the cardinality of $S_i \setminus S_{i+1}$ is equal to 1, for $0 \leq i \leq k-1$.

If $\mathcal{C}$ is a covariety, we can construct a directed graph $\mathcal{G}(\mathcal{C})$, whose set of vertices coincides with the elements of the covariety $\mathcal{C}$ and $(S,T) \in \mathcal{C} \times \mathcal{C}$ is an edge if and only if $T = S \setminus \{M(S)\}$, see Example 2.15. So we can deduce easily the lemma.

Lemma 2.10. *If $\mathcal{C}$ is a covariety then $\mathcal{G}(\mathcal{C})$ is a tree with root $\Delta(\mathcal{C})$.*

Proof. Direct consequence of Lemma 2.9. □

Fixing $F \in \mathbb{N}$, it is possible to construct recursively a tree containing all the numerical semigroups with F as Frobenius number. Moreover the root is $\{0, F+1, \rightarrow\}$. Now we will introduce two new objects that permit to construct the tree.

Definition 2.11. *An integer x is a* ***pseudo-Frobenius*** *number of a numerical semigroup S if $x \notin S$ and $x + s \in S$ for every $s \in S$. We denote by $PF(S)$ the set of pseudo-Frobenius number of S.*
We call set of ***special gaps*** *the set $SG(S) = \{x \in PF(S) \mid 2x \notin PF(S)\}$.*

Lemma 2.12. *Let S be a numerical semigroup and $x \in \mathbb{N}\setminus S$. Then $x \in SG(S)$ if and only if $S \cup \{x\}$ is a numerical semigroup.*

Proof. See [RGS09, Proposition 4.33].

□

Proposition 2.13. *If $\mathcal{C}$ is a covariety and $s \in \mathcal{C}$, then the set formed by the children of S in the associated tree of the covariety, is*

$$\{S \cup \{x\} \mid x \in SG(S), x < M(S), S \cup \{x\} \in \mathcal{C}\}$$

Proof. Let T be a child of S, so $T \in \mathcal{C}$ and $T \setminus \{M(T)\} = S$, hence $S \cup \{M(T)\} = T \in \mathcal{C}$, and, by Lemma 2.12, $M(T) \in SG(S)$ and $M(T) < M(S)$.
If $x < M(S)$, then $M(S \cup \{x\}) = x$ and so $S \cup \{x\} \in \mathcal{C}$. Therefore result $(S \cup \{x\}) \setminus M(S \cup \{x\}) = S$, so $S \cup \{x\}$ is a child of S. □

Similarly we can specialize the previous proposition in terms of the Frobenius number.

Proposition 2.14. *Let $F \in \mathbb{N}$ and $S \in \mathcal{A}(F)$. Then the set formed by the children of S in the associated tree is $\{S \cup \{x\} \mid x \in SG(S), x < M(S), x \neq F\}$.*

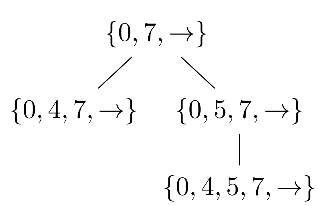
We can observe that if S is a numerical semigroup, $x \in SG(S)$ and $n \in S \setminus \{0\}$, then the element $x + n$ is clearly an element of $Ap(S, n)$, because $x \in PF(S)$ and hence $x + n \in S$. If we consider the new numerical semigroup $S \cup \{x\}$, then the Apéry set changes with respect to $Ap(S, n)$. Indeed we added the element $x \equiv x + n \pmod{n}$ to the numerical semigroup, obtaining

$$Ap(S \cup \{x\}) = (Ap(S, x) \setminus \{x + n\}) \cup \{x\} .$$

Now we can construct recursively the tree associated to $\mathcal{A}(F)$ following the pseudocode presented in [MFR23].

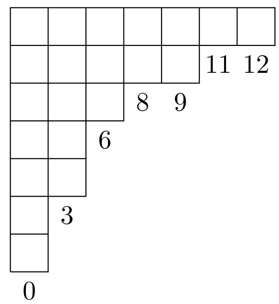
1. Let F be a positive integer.
2. Set $\mathcal{A}(F) = \{\Delta(\mathcal{A}(F))\}$ and $B = \{\Delta(\mathcal{A}(F))\}$.
3. For every $S \in B$ compute
$$\theta(S) = \{x \in SG(S) \mid x < M(S), x \neq F\} .$$
4. If $\bigcup_{S \in B} \theta(S) = \emptyset$ returns $\mathcal{A}(F)$.
5. Set $C = \bigcup_{S \in B} \{S \cup \{x\} \mid x \in \theta(S)\}$.
6. Update the sets $\mathcal{A}(F) = \mathcal{A}(F) \cup C$ and $B = C$.

Example 2.15. *Let $F = 6$. We set $\mathcal{A}(F) = \{0, 7, \rightarrow\} = B$. Hence $Ap(B, 7) = \{0, 8, 9, 10, 11, 12, 13\}$ and it is easy deduce the sets $PF(B) = \{1, 2, 3, 4, 5, 6\}$ and $SG(B) = \{4, 5, 6\}$, so $\theta(B) = \{4, 5\}$. Then we obtain two new numerical semigroups $S = \{0, 4, 7 \rightarrow\}$ and $T = \{0, 5, 7, \rightarrow\}$.*
In the first case we have $Ap(S, 7) = \{0, 4, 8, 9, 10, 12, 13\}$, $PF(S) = \{3, 5, 6\}$, $SG(S) = \{5, 6\}$ and $\theta(S) = \emptyset$.
In the second case we have $Ap(T, 7) = \{0, 5, 8, 9, 10, 11, 13\}$, $PF(T) = \{2, 4, 6\}$, $SG(T) = \{4, 6\}$ and $\theta(S) = \{4\}$ and we obtain $T' = \{0, 4, 5, 7, \rightarrow\}$.
If we continue in this way we can observe that $\theta(T') = \emptyset$ and so we obtain



2.2 Numerical semigroups and integer partitions

Let S be a numerical set, Keith and Nath showed in [KN11] that every numerical set uniquely defines a integer partition. Given a numerical set $S = \{s_0, s_1, \ldots, s_n, \rightarrow\}$ it is possible to construct a Young diagram by drawing a contiguous polygonal path that starts from the origin in $\mathbb{Z}^2$. Starting with $s = 0$ we draw a east step if $s \in S$ or a north step otherwise, then we continue with $s+1$ and we stop when we reach $F(S)$. For example if $S = \{0, 3, 6, 8, 9, 11, 12, 14, \rightarrow\}$ we obtain



the partition $\lambda_S = (7, 5, 3, 2, 2, 1, 1)$. The relation is clearly a bijection and we can apply this method in either direction, so the number of numerical sets is equal to the number of integer partitions.
Notice that the Young diagram Y_S has number of rows equal to $G(S)$ and a number of columns equal to n, where $S = \{s_0, s_1, \ldots, s_n\}$.
If we consider the hook of every cell we can observe some properties [TKG19].

Proposition 2.16. *Let $S = \{0, s_1, \ldots, s_n, \rightarrow\}$ be a numerical set with corresponding Young diagram Y_S. Then:*

1. *The hook length of the box in the first column and ith row is the ith gap of S;*

2. *For each $0 \leq i \leq n-1$ the hook length of the top box of the ith column of Y_S is equal to $F(S) - s_i$;*

3. *The set S is a numerical semigroup if and only if every length of the hook of the boxes of Y_S is contained in the first column.*

Proof. Let $1 \leq j \leq n-1$ be the lowest integer such that $s_j \neq j$, so we have $s_1^c = j$, where $S^c = \{s_1^c, s_2^c, \ldots, F(S)\}$. This means that we have j east steps in the first row of Y_S and the length of the first hook coincides with $j = s_1^c$.
Proceeding by induction, we suppose that every length of hook in the first column from the first row to the ith row coincides with the gaps of S, hence the hook length of the cell in the first column and ith row is s_i^c. Let k such that $s_{k-1} < s_i^c < s_k$. If $s_k \neq s_i^c + 1$, then $s_{i+1}^c = s_i^c + 1$ and the $(i+1)$th row of Y_S has the same number of columns of the ith row, so the hook length is $s_i^c + 1$. Otherwise, if $s_k = s_i^c + 1$, we consider the lowest integer $j > k$ such that $s_j \neq s_i^c + j + 1 - k$, so there are $j - k$ integer between s_i^c and $s_{i+1}^c = s_i^c + j + 1 - k$, or in other words the $(i+1)$th row of Y_S has $j - k$ columns more than the ith row. So we have proved Part (1).
Now, the upper left corner hook coincides with $F(S) = F(S) - s_0$. We suppose that the hook in the ith column is $F(S) - s_i$. Let $k = s_{i+1} - s_i$, so after the ith column we have $k - 1$ north steps in Y_S, then the hook of the $(i+1)$th column

is equal to $F(S) - s_i - (k-1) - 1 = F(S) - k = F(S) - s_{i+1}$ and we obtain Part (2).
Part (3) remains to check. Let $S = \{s_0, \dots, s_n, \rightarrow\}$ be a numerical semigroup. Notice that every $\widehat{S}$ such that $\widehat{S}^c = \{s_1^c, \dots, s_i^c\}$, where $s_i^c < F(S)$, is a numerical semigroup. Furthermore, by Point (2), the hook of the ith row and jth column in Y_S is equal to $s_i^c - s_j$. We suppose that exists a cell with hook length $s_i^c - s_j$ that doesn't appear in the first column, so this number is not in S^c and we have $s_h = s_i^c - s_j \in S$, but $s_h + s_j = s_i^c \in S$ and this is a contradiction.
Viceversa if all the boxes in the i-th row are signed by numbers appearing in the first column,then by Point (1) they are element of S^c. So do not exist two elements of S such that the sum is equal to s_i^c and then we can conclude that S is a numerical semigroup. □

Example 2.17. *If we consider* $S = \{0, 3, 6, 8, 9, 11, 12, 14, \rightarrow\}$ *and write the hook length of every cell, we obtain*

13	10	7	5	4	2	1
10	7	4	2	1		
7	4	1				
5	2					
4	1					
2						
1						

We can establish some relation between numerical semigroups and integer partitions in order to count them.

Lemma 2.18. *Let S be a numerical set. If $F(S) < 2 \cdot M(S)$, then S is a numerical semigroup.*

Proof. Let $s_1, s_2 \in S \setminus \{0\}$, by definition of multiplicity we have $s_1 + s_2 > 2 \cdot M(S) > F(S)$, so this implies that S is closed under addition. □

Proposition 2.19 ([BNST23])**.** *The number of partitions of n into g parts corresponding to a numerical semigroup with genus g is equal to the number of partitions of $n - g$, for any positive integer n and $g \geq \frac{2}{3}n$.*

Proof. Notice that every S numerical semigroup such that $S^c \neq \emptyset$ contains 1. Let $\lambda = \left(\lambda_1, \dots, \lambda_k, 1^{g-k}\right)$ be a partition of n into g parts corresponding to a numerical semigroup S_λ such that $\lambda_k \geq 2$. We define

$$\phi(\lambda) : = (\lambda_1 - 1, \dots, \lambda_k - 1)$$

that associates to λ a partition of weight $n - g$. Clearly this function is injective. Let $\mu = (\mu_1, \dots, \mu_k)$ be a partition of $n - g$ and let

$$\lambda = \phi^{-1}(\mu) = \left(\mu_1 + 1, \dots, \mu_k + 1, 1^{g-k}\right).$$

Notice that λ is a partition of n into g parts and $M(S_\lambda) = g - k + 1$. Among all the possible Frobenius numbers of λ, we get the maximum when $\lambda = \left(\lambda_1, 2^{k-1}, 1^{g-k}\right)$, so we have

$$F(S_\lambda) = n - k + 2 - g \leq n - k + 1$$

Since $n \leq \frac{3}{2}g$ and $k \leq n - g \leq \frac{1}{2}g$, we obtain

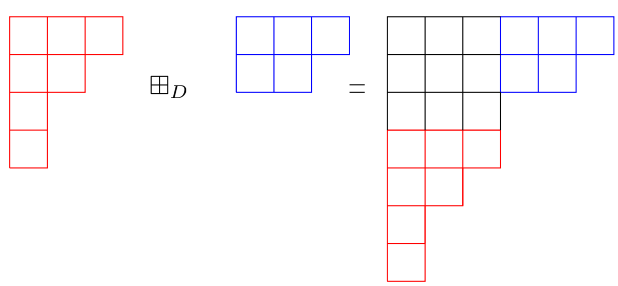
$$F(S_\lambda) \leq \frac{3}{2}g - k + 1 \leq \frac{3}{2}g - k + 1 + \left(\frac{1}{2} - k\right) < 2\,(g - k + 1) = 2M\,(S_\lambda)$$

By Lemma 2.18 we have that S_λ is a numerical semigroup, so ϕ is surjective too. □

Now we define four operation on Young diagrams introduced in [SY21] .

Definition 2.20. *Let Y be a Young diagram with n columns and k rows, Z be the Young diagram with m columns and l rows. Gluing Z above Y as putting a row of boxes of length n above Y and then uniting the top right corner of this row and the bottom left corner of the first column of Z is called the* ***discrete sum*** *of Y and Z, denoted by $Y \boxplus_D Z$, which is a Young diagram with $n + m$ columns and $k + l + 1$ rows.*
We can see the discrete sum in terms of partitions. Let $\lambda_Y = (n, \lambda_{Y_1}, \ldots, \lambda_{Y_k})$ and $\lambda_Z = (m, \lambda_{Z_2}, \ldots, \lambda_{Z_l})$ be the partitions associated with Y and Z respectively, then $\lambda_Y \boxplus_D \lambda_z = (n + m, n + \lambda_{Z_2}, \ldots, n + \lambda_{Z_l}, n, n, \lambda_{Y_2}, \ldots, \lambda_{Y_k})$.

For example, if we take $\lambda = (3, 2, 1, 1)$ and $\mu = (3, 2)$, the discrete sum of Y_λ and Z_μ results



and we obtain the partition $\lambda \boxplus_D \mu = (6, 5, 3, 3, 2, 1, 1)$.
Notice that, due to bijection between the set of Young diagrams and the set of numerical semigroup, we can define the sum for numerical semigroup. Let $S_Y = \{0, s_1, \ldots, s_n, \rightarrow\}$ be the numerical set associated to the Young diagram Y and $T_Z = \{0, t_1, \ldots, t_m, \rightarrow\}$ be the numerical set associated to the Young diagram Z. The discrete sum preserves the shape of Y until the $F(S_Y)$-th step, makes a north step at s_n-th step and then continues with the shape of Z, so we obtain

$$S_Y \boxplus_D T_Z = \{0, s_1, \ldots, s_{n-1}, s_n + 1, s_n + t_1, \ldots, s_n + t_m, \rightarrow\}.$$

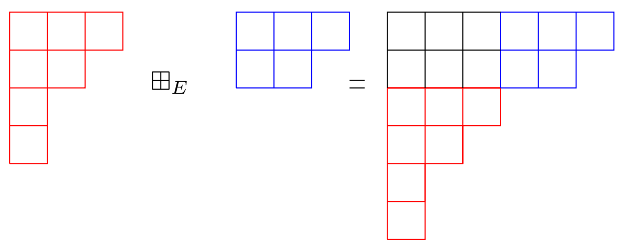
If we observe the set of gaps we obtain

$$(S_Y \boxplus_D T_Z)^c = \{s_1^c, \dots, s_k^c, s_k^c + 1, s_k^c + t_1^c + 2, \dots, s_k^c + t_l^c + 2\},$$

where $S_Y^c = \{s_1^c, \dots, s_k^c\}$ and $T_Z = \{t_1^c, \dots, t_l^c\}$. Moreover $F(S_Y \boxplus_D T_z) = s_k^c + t_l^c + 2$.

Definition 2.21. *Let Y be a Young diagram with n columns and k rows, Z be the Young diagram with m columns and l rows. Gluing Z above Y as uniting the top right corner of the first row of Y and the bottom left corner of the first column of Z is called the* ***end-to-end sum*** *of Y and Z, denoted by $Y \boxplus_E Z$, which is a Young diagram with $n+m$ columns and $k+l$ rows.*
Let $\lambda_Y = (n, \lambda_{Y_1}, \dots, \lambda_{Y_k})$ and $\lambda_Z = (m, \lambda_{Z_2}, \dots, \lambda_{Z_l})$ be the partitions associated with Y and Z respectively, then
$\lambda_Y \boxplus_E \lambda_z = (n+m, n+\lambda_{Z_2}, \dots, n+\lambda_{Z_l}, n, \lambda_{Y_2}, \dots, \lambda_{Y_k})$.

Taking λ and μ as before



and so $\lambda \boxplus_E \mu = (6,5,3,2,1,1)$. In this case the end-to-end sum overlaps the s_n-th step of S_Y and the $t_0 = 0$ step of T_Z, then we have

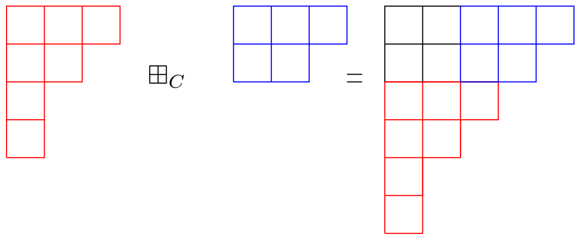
$$S_Y \boxplus_E T_Z = \{0, s_1, \dots, s_n, s_n + t_1, \dots, s_n + t_m, \rightarrow\}.$$

Furthermore $F(S_Y \boxplus_E T_Z) = s_k^c + t_l^c + 1$ and

$$(S_Y \boxplus_E T_Z) = \{s_1^c, \dots, s_k^c, s_k^c + t_1^c + 1, \dots, s_k^c + t_l^c + 1\}.$$

Definition 2.22. *Let Y be a Young diagram with n columns and k rows, Z be the Young diagram with m columns and l rows. Gluing Z above Y as putting the first column of Z on top of the last column of Y is called the* ***conjoint sum*** *of Y and Z, denoted by $Y \boxplus_C Z$, which is a Young diagram with $n+m-1$ columns and $k+l$ rows.*
Let $\lambda_Y = (n, \lambda_{Y_1}, \dots, \lambda_{Y_k})$ and $\lambda_Z = (m, \lambda_{Z_2}, \dots, \lambda_{Z_l})$ be the partitions associated with Y and Z respectively, then
$\lambda_Y \boxplus_C \lambda_z = (n+m-1, n+\lambda_{Z_2}-1, \dots, n+\lambda_{Z_l}-1, n, \lambda_{Y_2}, \dots, \lambda_{Y_k})$.

In this case we obtain



$\lambda \boxplus_C \mu = (5, 4, 3, 2, 1, 1)$. Again we can define the sum for the numerical sets and we obtain

$$S_Y \boxplus_C T_Z = \{0, s_1, \ldots, s_{n-1}, s_n + t_1 - 1, \ldots, s_n + t_m - 1, \rightarrow\}$$

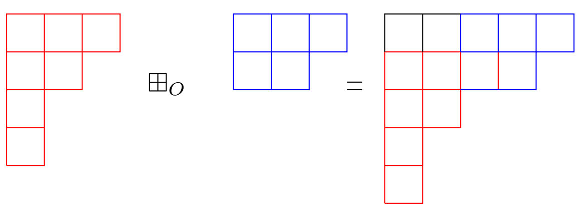
and the set of gaps too

$$(S_Y \boxplus_C T_Z) = \{s_1^c, \ldots, s_k^c, s_k^c + t_1^c, \ldots, s_k^c + t_l^c\}.$$

hence $F(S_Y \boxplus_C T_Z) = s_k^c + t_l^c$

Definition 2.23. *Let Y be a Young diagram with n columns and k rows, Z be the Young diagram with m columns and l rows. Gluing Z above Y as overlapping the last box of first column of Z and the last box of the first row of Y is called the* ***overlap sum*** *of Y and Z, denoted by $Y \boxplus_O Z$, which is a Young diagram with $n + m - 1$ columns and $k + l - 1$ rows.*
Let $\lambda_Y = (n, \lambda_{Y_1}, \ldots, \lambda_{Y_k})$ and $\lambda_Z = (m, \lambda_{Z_2}, \ldots, \lambda_{Z_l})$ be the partitions associated with Y and Z respectively, then
$\lambda_Y \boxplus_O \lambda_z = (n + m - 1, n + \lambda_{Z_2} - 1, \ldots, n + \lambda_{Z_l} - 1, \lambda_{Y_2}, \ldots, \lambda_{Y_k})$.

The last one operation give us



$\lambda \boxplus_O \mu = (5, 4, 2, 1, 1)$. In this case if we look at numerical sets we obtain

$$S_Y \boxplus_O T_Z = \{0, s_1, \ldots, s_{n-1}, s_n + t_1 - 2, \ldots, s_n + t_m - 2, \rightarrow\}$$

The set of gaps results

$$(S_Y \boxplus_C T_Z) = \left\{s_1^c, \ldots, s_{k-1}^c, s_k^c + t_1^c - 1, \ldots, s_k^c + t_l^c - 1\right\}.$$

and the Frobenius number is $S_k^c + t_l^c - 1$.

Definition 2.24. *A numerical semigroup S is called* ***symmetric*** *if $F(S)$ is odd and if $x \in \mathbb{N}_0 \setminus S$ implies that $F(S) - x \in S$. We denote by SNS the set of symmetric numerical semigroups.*
Also a numerical semigroup is called ***pseudo-symmetric*** *if $F(S)$ is even and $x \in \mathbb{N}_0 \setminus S$ implies $F(S) - x \in S$ or $x = \frac{F(S)}{2}$.*

Lemma 2.25. *Let S be a numerical semigroup. Then S is symmetric if and only if $G(S) = \frac{F(S)+1}{2}$. Furthermore S is pseudo-symmetric if and only if $G(S) = \frac{F(S)+2}{2}$.*

Proof. See [RGS09] Corollary 4.5. □

Definition 2.26. *Let S be a numerical semigroup with $S^c = \{s_1^c, \ldots, s_k^c\}$ and Frobenius number $F(S) = s_k^c$. We define the dual of S as*

$$S^* = \left\{0, F(S) - s_{k-1}^c, \ldots, F(S) - s_1^c, s_n, \rightarrow\right\}$$

Notice that if S is symmetric we obtain the following equivalence $S = S^*$.

Theorem 2.27. *For every symmetric numerical semigroup S, there exist a unique numerical semigroup T such that $S = T \boxplus_E T^*$ or $S = T \boxplus_O T^*$.*

Proof. Let $S = \{0, s_1, \ldots, s_n, \rightarrow\}$ be a symmetric numerical semigroup. We recall that $s_n = F(S) + 1$, $S^c = \{s_1^c, \ldots, s_n^c\}$ and

$$S = S^* = \left\{0, F(S) - s_{n-1}^c, \ldots, F(S) - s_1^c, s_n, \rightarrow\right\},$$

where $F(S) = S_n^c$.
Let k be such that $s_k \leq \frac{F(S)+1}{2} < s_{k+1}$. If $s_k = \frac{F(S)+1}{2}$ then we define $T = (0, s_1, \ldots, s_k, \rightarrow)$. Clearly T is a numerical semigroup such that $F(T) = s_k - 1$ and $T^c = \left\{s_1^c, \ldots, s_{n-k}^c\right\}$.
Then it results $T^* = \left\{0, F(T) - s_{n-k-1}^c, \ldots, F(T) - s_1^c, s_k, \rightarrow\right\}$. By definition of end-to-end sum on numerical semigroups we obtain

$$\begin{aligned} T \boxplus_E T^* &= \left\{0, s_1, \ldots, s_k, s_k + F(T) - s_{n-k-1}^c, \ldots, s_k + F(T) - s_1^c, s_k + s_k, \rightarrow\right\} \\ &= \left\{0, s_1, \ldots, s_k, 2s_k - 1 - s_{n-k-1}^c, \ldots, 2s_k - 1 - s_1^c, 2s_k, \rightarrow\right\} \\ &= \left\{0, s_1, \ldots, s_k, F(S) - s_{n-k-1}^c, \ldots, F(S) - s_1^c, F(S) + 1, \rightarrow\right\} \\ &= \{0, s_1, \ldots, s_k, s_{k+1}, \ldots, s_{n-1,s_n,\rightarrow}\} \\ &= S \end{aligned}$$

Otherwise if $s_k \neq \frac{F(S)+1}{2}$, we have $F(S) - \frac{F(S)+1}{2} = \frac{F(S)+1}{2} - 1 \in S$, and hence $s_k = \frac{F(S)+1}{2} - 1$. Let $T = \left\{0, s_1, \ldots, s_k, \frac{F(S)+1}{2} + 1, \rightarrow\right\}$. Obviously T is a numerical semigroup, furthermore we have $T^c = \left\{s_1^c, \ldots, s_{n-k-1}^c, \frac{F(S)+1}{2}\right\}$ and $T^* = \left\{0, F(T) - s_{n-k-1}^c, \ldots, F(T) - s_1^c, \frac{F(S)+1}{2} + 1\right\}$, where $F(T) = \frac{F(S)+1}{2}$.

Applying the overlap sum to numerical sets we obtain

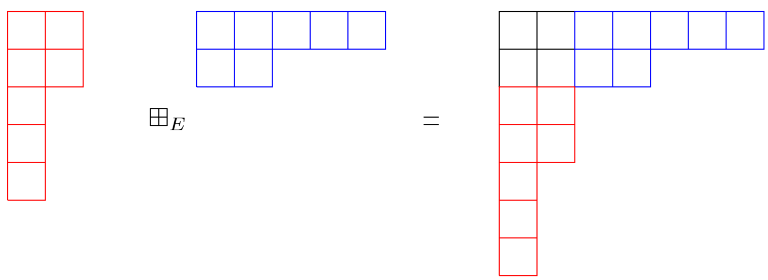
$$\begin{aligned} T \boxplus_O T^* &= \left\{0, \ldots, s_k, 2\left(\frac{F(S)+1}{2}\right) - s^c_{n-k-1} - 1, \right. \\ &\left. \ldots, 2\left(\frac{F(S)+1}{2}\right) - s^c_1 - 1, 2\left(\frac{F(S)+1}{2} + 1\right) - 2, \rightarrow \right\} \\ &= \{0, \ldots, s_k, F(S) - s^c_{n-k-1}, \ldots, F(S) - s^c_1, F(S) + 1, \rightarrow\} \\ &= \{0, s_1, \ldots, s_k, s_{k+1}, \ldots, s_{n-1}, s_n, \rightarrow\} \\ &= S \end{aligned}$$

$\square$

Example 2.28. *Let* $S = (0, 4, 7, 8, 10, 11, 12, 14, \rightarrow)$. *$S$ is a symmetrical semigroup such that* $G(S) = (1, 2, 3, 5, 6, 9, 13)$, $F(S) = 13$ *and* $\frac{F(S)+1}{2} = 7 \in S$.
By Theorem 2.27, we have $T = (0, 4, 7, \rightarrow)$ *and* $G(T) = (1, 2, 3, 5, 6)$, *so we can obtain*

$$T^* = (0, 1, 3, 4, 5, 7 \rightarrow)$$

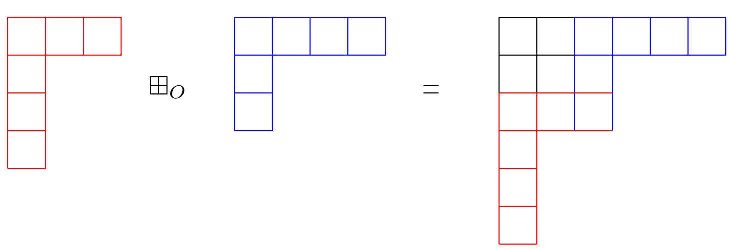
where $T \boxplus_E T^* = S$. *We obtain*



Example 2.29. *Let* $S = \{0, 4, 5, 8, 9, 10, 12, \rightarrow\}$. *$S$ is a symmetrical semigroup such that* $G(S) = (1, 2, 3, 6, 7, 11)$, $F(S) = 11$ *and* $\frac{F(S)+1}{2} = 6 \notin S$.
By Theorem 2.27, we have $T = \{0, 4, 5, 7, \rightarrow\}$ *and* $G(T) = (1, 2, 3, 6)$, *so we can obtain*

$$T^* = \{0, 3, 4, 5, 7 \rightarrow\}$$

where $T \boxplus_O T^* = S$. *Graphically*



Chapter 3

Verification and generation of unrefinable partitions

In this chapter we introduce the main object of this thesis, the unrefinable partitions, showing some basic properties. Then we present two algorithms: a *Verification* Algorithm which verifies if a sequence of distinct positive integers is unrefinable or not and a *Enumerating* Algorithm that lists all the unrefinable partitions of a given weight.

3.1 Unrefinable partitions into distinct parts

From now on, we will consider only partitions into distinct parts and so, in particular, we will be taking strictly increasing sequences of positive integers.

Definition 3.1. *Let $\lambda = (\lambda_1, \dots, \lambda_t)$ be a partition into distinct parts, i.e., $\lambda_1 < \lambda_2 < \cdots < \lambda_t$, such that $t \geq 2$. We call the **set of missing parts** of λ the set $\mathcal{M}_\lambda = \{\mu_1, \dots, \mu_m\}$ composed by the integers that do not belong to λ lower than λ_t*

$$\mathcal{M}_\lambda \colon = \{1, 2, \dots, \lambda_t\} \setminus \{\lambda_1, \dots, \lambda_t\} .$$

*The least integer which is not part of λ, i.e., μ_1, is the **minimal excludant** of λ. We denote this by writing $\mu_1 = mex\,(\lambda)$, assuming $mex\,(\lambda) = 0$ when $\mathcal{M}_\lambda = \emptyset$.*

Definition 3.2. *Let $N \in \mathbb{N}$. Let $\lambda = (\lambda_1, \dots, \lambda_t)$ be a partition of N into distinct parts and let $\mu_1 < \mu_2 < \cdots < \mu_m$ be its missing parts. The partition λ is **refinable** if there exist $1 \leq \ell \leq t$ and $1 \leq i_1 < \ldots < i_k \leq m$, where $2 \leq k$, such that $\mu_{i_1} + \cdots + \mu_{i_k} = \lambda_\ell$, and **unrefinable** otherwise. The set of unrefinable partitions is denoted by $\mathcal{U}$, and by $\mathcal{U}_N$ we denote those whose sum of the parts is N.*

For example, the partition $\lambda = (1, 2, 3, 5, 6, 8, 9, 11, 13)$ is refinable because we can write 11 as $4 + 7$, while $\lambda = (1, 2, 3, 5, 6, 8, 9, 13)$ is unrefinable. Clearly, the condition of being unrefinable imposes on partitions a non-trivial limitation on the size of the largest part and on the possible distributions of the parts. Now we show some basic properties of such partitions.

Lemma 3.3. *Let $\lambda = (\lambda_1, \dots, \lambda_t)$ be a integer partition into distinct parts. If $\#\mathcal{M}_\lambda = \{0, 1\}$, then $\lambda \in \mathcal{U}$.*

Proof. This is a direct consequence of the definition of unrefinable partition, because we can not substitute any parts of partition with two or more missing parts. □

Observe that Lemma 3.3 is not an if and only if statement, hence if we take the partition $\lambda = (1, 2, 3, 5, 6, 8, 9, 13)$, as in the previous example, we have $\lambda \in \mathcal{U}$ and $\#\mathcal{M}_\lambda = 5$.

Definition 3.4. *Let $n \in \mathbb{N}$. We denote by T_n the **n-th triangular number**, i.e.*

$$T_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

*The **complete partition** $\pi_n = (1, 2, \dots, n)$ is the partition of T_n with no missing parts.*
Let $n \geq 3$ and $1 \leq d \leq n-1$. We denote by $T_{n,d}$ the integer number $T_n - d$ and define the partition $\pi_{n,d} = (1, \dots, d-1, d+1, \dots, n) \vdash T_{n,d}$.

Notice that for every $n \geq 3$ and $1 \leq d \leq n-1$ we have $T_{n-1} < T_{n,d} < T_n$

Corollary 3.5. *Let $N \in \mathbb{N}$ be such that $N > 2$, then $\#\mathcal{U}_N \geq 1$.*

Proof. If $N = T_n$ by Lemma 3.3 and Definition 3.4 we have $\pi_n \in \mathcal{U}_N$, otherwise if $N \neq T_n$ there exists $1 \leq d \leq n-1$ such that $N = T_{n,d}$ and then we obtain $\pi_{n,d} \in \mathcal{U}_N$ by Lemma 3.3. □

Lemma 3.6. *Let $\lambda = (\lambda_1, \dots, \lambda_t)$ be an unrefinable partition and let $\mu_1 < \dots < \mu_m$ be the missing parts. Then the number of missing parts m is bounded by*

$$m \leq \left\lfloor \frac{\lambda_t}{2} \right\rfloor. \tag{3.1}$$

Proof. Let us start by observing that $\lambda_t - \mu_i \in \lambda$ for $1 \leq i \leq m$, otherwise from $\lambda_t - \mu_i$, $\mu_i \in \mathcal{M}_\lambda$, we obtain $(\lambda_t - \mu_i) + \mu_i = \lambda_t \in \lambda$ and thus λ is refinable. We prove the claim considering the complete partition π_{λ_t} and removing from this the maximum number of parts different from λ_t. From the previous observation, each candidate part μ_i to be removed has a counterpart $\lambda_t - \mu_i$ in the partition. The bound of Equation (3.1) depends on the fact that this process can be repeated no more than $\lfloor \lambda_t/2 \rfloor$ times. □

3.2 Algorithm to check refinability

Proposition 3.7. *If a partition $\lambda = (\lambda_1, \dots, \lambda_t)$ has some refinement, then its smallest refinable part λ_r has a refinement of the form $\lambda_r = a + b$, where $a, b \in \mathcal{M}_\lambda$.*

Proof. Let λ_r be the smallest refinable part for which there exists some refinement $\lambda_r = \mu_{i_1} + \cdots + \mu_{i_t}$. If $t = 2$ there is nothing to prove. Otherwise, let us fix $a = \mu_{i_1}$ and $b = \mu_{i_2} + \cdots + \mu_{i_t}$. If $b \in \lambda$, then $b = \mu_{i_2} + \cdots + \mu_{i_t}$ would be a refinement itself, but $b < \lambda_r$ and this would violate the minimality of λ_r, hence b is not a part of λ.
This shows that $\lambda_r = a + b$ is indeed a refinement of λ. □

Let λ be an unrefinable partition, notice that once we know $mex\,(\lambda) = \mu_1$ and we find another missing part μ_i, then, by definition, the element $\mu_i + \mu_1$ can not be part of λ, so it is the same for the element $\mu_i + 2\mu_1 = (\mu_i + \mu_1) + \mu_1$ and, recursively, every element of the form $\mu_i + k\mu_1$, such that $k \in \mathbb{N}_0$, is not a part of λ. So we can observe that every element $x \equiv \mu_i \pmod{\mu_1}$ such that $x \geq \mu_i$ is not in λ.
It is possible to see $\lambda = (v_1, \dots, v_l)$ as a sequence of parts, where $v_i = \{0, i\}$. The sequence corresponds to a partition when $v_l = l$, in other words a sequence of integer parts is a partition when the last term is not 0.

Definition 3.8. *Let λ be a sequence of strictly increasing integers. We can define a vector $\vec{p}_\lambda$, whose length is $mex\,(\lambda) = \mu_1$, and, for every $1 \leq i \leq \mu_1$, the element p_i in position i is the lowest missing part greater than μ_1 congruent to $i - 1$ modulo μ_1. If there does not exist a missing part $\mu_1 < a < \lambda_t$ such that $a \equiv i \pmod{\mu_1}$ we set the element $p_{i+1} = \infty$.*

Notice that if we consider the partition $\pi_{n,d}$, the vector $\vec{p}_{\pi_{n,d}}$ is a vector of length d made up of ∞ in every positions.
Let λ be an unrefinable partition, $\mu_1 = mex\,(\lambda)$ and let us suppose that exists $\mu_2 \in \mathcal{M}_\lambda$, we can observe that

- if μ_1 and μ_2 are coprime, so we have $\mu_1 + \mu_2 \in \mathcal{M}_\lambda$ and $\mu_1 + \mu_2 \equiv \mu_2 \equiv j \pmod{\mu_1}$, where $1 \leq j \leq \mu_1 - 1$. Then also $\mu_1 + 2\mu_2 \in \mathcal{M}_\lambda$ and therefore $\mu_1 + 2\mu_2 \equiv 2j \neq 0 \pmod{\mu_1}$. So if $\mu_1 + 2\mu_2$ is lower than λ_t we obtain a new element of $\vec{p}_\lambda$. By the coprimality of μ_1 and μ_2 we have that $\mu_1 + k\mu_2$, where $2 \leq k \leq \mu_1$, is an element of $\vec{p}_\lambda$;

- if $(\mu_1, \mu_2) = d \neq \{1, \mu_1\}$, as before, we obtain $\frac{\mu_1}{d} - 1$ elements of $\vec{p}_\lambda$ which are $\mu_1 + k\mu_2$, where $2 \leq k \leq \frac{\mu_1}{d}$;

- if $\mu_2 = k\mu_1$, we obtain only the element p_0 because $\mu_1 + 2\mu_2 \equiv 0 \pmod{\mu_1}$.

Example 3.9. *Let λ be a partition such that $mex\,(\lambda) = 6$. If $\mu_2 = 7$, we obtain*

$$\vec{p}_\lambda = \boxed{48 \mid 7 \mid 20 \mid 27 \mid 34 \mid 41}$$

otherwise if $\mu_2 = 10$ *we have*

$$\vec{p}_\lambda = \boxed{36}\boxed{\infty}\boxed{26}\boxed{\infty}\boxed{10}\boxed{\infty}$$

and in the last case if $\mu_2 = 18$

$$\vec{p}_\lambda = \boxed{18}\boxed{\infty}\boxed{\infty}\boxed{\infty}\boxed{\infty}\boxed{\infty}$$

Definition 3.10. *Let* $\lambda = (v_1, \ldots, v_l)$ *be an unrefinable sequence of parts with* $mex(\lambda) = \mu_1$ *and let* $\vec{p}_\lambda$ *be its vector of lowest missing parts modulo* μ_1*. We say that* λ *is* ***saturated*** *when*

$$|\{p_j \leq \ell \mid 1 \leq j \leq \mu_1\}| = \mu_1.$$

Lemma 3.11. *Let* λ *be an unrefinable sequence of parts such that* $\mathcal{M}_\lambda \geq 2$*. If there exists* μ_i *such that* $(\mu_1, \mu_i) = 1$*, then* λ *has a finite length, otherwise* λ *has infinite length.*

Example 3.12. *Let* λ *be the sequence of all odd numbers, i.e.,* $\lambda = (1, 3, 5, \ldots)$*. Then* λ *is an infinite unrefinable sequence, because all the missing parts are even and the sum of even numbers is even.*

Let λ be a sequence of parts, we want to see how the vector $\vec{p}_\lambda$ changes every time we find a new missing part. Let $\mu_i \in \mathcal{M}_\lambda$ such that $\mu_i \equiv j \pmod{\mu_1}$, if exists $\mu_k \equiv j \pmod{\mu_1}$ such that $\mu_k < \mu_i$, then the element p_{j+1} remains μ_k and we only have to check if the sum $\mu_k + \mu_i \equiv 2j \equiv r \pmod{\mu_1}$ is lower than element p_{r+1}, in this case we update $\vec{p}_\lambda$. Otherwise, if $\mu_i < p_{j+1}$ we have to check all the sums $\mu_i + p_k$, where $1 \leq k \leq \mu_1$ and $k \neq j$, and to update $\vec{p}_\lambda$ when the sum is lower than the elements in the respective modulo class. So we can define the following algorithm.

Given a strictly increasing sequence of positive integers λ, we can determinate if λ is an unrefinable partition or not only seeing the integers $\mu_1 + 1 \leq r \leq \lambda_t$

- if $r \notin \lambda$ we have to update the vector $\vec{p}_\lambda$ to define which numbers would contradict unrefinability;
- if $r \in \lambda$ we need to check if it has a refinement, i.e., if $r \equiv j \pmod{\mu_1}$ and $p_{j+1} < r$.

Then we construct an algorithm that checks refinability.

In the next sections of this Chapter we use the same notation for the integer partitions and for the sequences of positive numbers. Furthermore if a number does not appear in a partition we put a $\star$ in the corresponding position.

Example 3.13. *Let* $\lambda = (1, 2, 3, 4, 5, \star, 7, 8, \star, \star, 11, 12, 13, \star, \ldots)$ *and consider all calls to* Update *when Algorithm 2 reaches* 14 *in* λ*. We have* $\mu_1 = 6$*. The first call sets* $p_4 = 9$*. The second call sets* $p_5 = 10$*, and since* $19 = 9 + 10$*, we*

Algorithm 1: UPDATE (improves p_js after a new missing part r is discovered)

Input : $\vec{p} = (p_1, \ldots, p_{\mu_1})$, r a newly discovered missing part
Returns: $\vec{p} = (p_1, \ldots, p_{\mu_1})$, updated

$j \leftarrow r \pmod{\mu_1}$
if $r > p_{j+1}$ **then**
  $t \leftarrow r + p_{j+1} \pmod{\mu_1}$
  $p_{t+1} \leftarrow \min(p_{t+1}, r + p_{j+1})$
else
  $p_{j+1} \leftarrow r$
  for j' *in* $\{1, \ldots, \mu_1 - 1\} \setminus \{j\}$ **do**
    $t \leftarrow j + j' \pmod{\mu_1}$
    $p_{t+1} \leftarrow \min(p_{t+1}, p_{j+1} + p_{j'+1})$
  end
end
return $(p_1, \ldots, p_{\mu_1})$

Algorithm 2: VERIFY (an algorithm to check refinability)

Input : $\lambda = (\lambda_1, \ldots, \lambda_t)$
Returns: REFINABLE or UNREFINABLE

$\mu_1 \leftarrow mex(\lambda)$
if $\mu_1 = 0$ **then return** UNREFINABLE
$\vec{p_\lambda} = (p_1, \ldots, p_\mu) \leftarrow (\infty, \infty, \ldots, \infty)$
for r *in* $(\mu + 1), \ldots, \lambda_t$ **do**
  $j \leftarrow r \pmod{\mu_1}$
  if $r \in \lambda$ *and* $r \geq p_{j+1}$ **then return** REFINABLE
  if $r \notin \lambda$ **then** $\vec{p_\lambda} \leftarrow$UPDATE$(\vec{p_\lambda}, r)$
end
return UNREFINABLE

need 19 *to be forbidden as well. This happens in the* ***for*** *loop that sets* $p_2 = 19$*. At this stage we have* $(p_1, p_2, p_3, p_4, p_5, p_6) = (\infty, 19, \infty, 9, 10, \infty)$*. The third call happens when the scan reaches* 14*. Here the algorithm sets* $p_3 = 14$*, and afterwards the* ***for*** *loop in line 6 computes the forbidden values*

$$19 + 14 = 33, \quad 9 + 14 = 23, \quad 10 + 14 = 24.$$

The information that 33 *is forbidden is included in* p_4 *(previously set to* 9*), while the information* $p_6 = 23$ *and* $p_1 = 24$ *is newly determined. When* 14 *is reached and processed, the information on forbidden numbers is represented by*

$$(p_1, p_2, p_3, p_4, p_5, p_6) = (24, 19, 14, 9, 10, 23).$$

The partition λ *may continue either with* 15 *or with* $\star$*. Notice that* $15 = 3 \pmod 6$ *and* $15 > p_4 = 9$*, therefore* $15 \in \lambda$ *would prove refinability (indeed* $15 = 6 + 9$*). Therefore* λ *can only continue with* $\star$*.*

Example 3.14. *Let* $\lambda = (1, 2, 3, \star, 5, 6, \star, 8, 9, \star, \star, 12, 13, \star, \dots)$ *and consider the call to* Update *when Algorithm 2 reaches* 14 *in* λ*. We have* $\mu = 4$*. By the time the algorithm scans position* 14 *we know that the sequence misses parts* 10 *and* 14*, therefore* 24 *must be forbidden as well. Indeed in this call we have* $r = 14$ *and* $p_3 = 10$*, and line 4 runs and sets* p_1 *to* 24 *as desired.*

Now we discuss the correctness Algorithm 2. First we prove the correctness of the algorithm on unrefinable and refinable sequences of parts separately.

Lemma 3.15. *Algorithm 2 outputs* Unrefinable *on every unrefinable* λ*.*

Proof. Consider an unrefinable sequences of parts λ. We start by proving that when the algorithm assigns a value w to some p_j, it means that $w \notin \lambda$. We prove this by induction on the iterations of the main loop at line 4. The base of the induction trivially holds because before the loop all p_j are set to ∞.
For the inductive step we discuss all the ways these assignments occur in the Update function described in Algorithm 1. If we set p_j to w at line 6, then $w \notin \lambda$ because Update would have been called when $v_w = \star$. If we update p_t to value w either at line 4 or at line 9, we already know that $v_r = \star$ and, by induction, that p_j and $p_{j'}$ are not in λ. Since λ is unrefinable, the new value of p_t (namely w) cannot be in λ either.
We just proved that, at any moment in the algorithm, every finite valued p_j is not in λ. We improve this by showing that the same holds for $p_j + t\mu_1$ for $t \geq 0$, by induction on t. The case $t = 0$ is what we have proved so far. Assuming $p_j + t\mu_1 \notin \lambda$, then by unrefinability the same holds for $p_j + (t+1)\mu_1$.
To conclude, observe that the only possible way for Algorithm 2 to be incorrect is to return at line 6. This happens when there is some $v_r = r$ which is greater than both μ_1 and p_{j+1}, and that it is equal to j modulo μ_1. Hence $r = p_{j+1} + t\mu_1$ for some $t > 0$. But we just showed that these values are not in λ, therefore the algorithm cannot return at line 6. □

To prove the correctness of Algorithm 2 on refinable partition we use the following two propositions.

Proposition 3.16. *Consider the iteration r of the main loop of Algorithm 2, where $r \notin \lambda$ and $r = j \pmod{\mu_1}$. After that iteration, $p_{j+1} \leq r$.*

Proof. UPDATE is called, and when it reaches line 2 either the test $r > p_{j+1}$ passes, or p_{j+1} is set to r. Hence at the end of iteration r we have that $p_{j+1} \leq r$. Successive iterations can only decrease the value of p_{j+1}. □

Proposition 3.17. *Assume that Algorithm 2 reaches iteration r, and let j be the residue class of r modulo μ_1. The assignment at line 6 of* UPDATE *is executed if and only if r is the smallest number strictly greater than μ_1 in residue class j with $r \notin \lambda$.*

Proof. If there is a smaller $r' \notin \lambda$ in the same residue class j, then by Proposition 3.16 we have $p_{j+1} \leq r' < r$. In that case, line 6 is not reached.
In the other direction, let r be the smallest number in the residue class j for which $r \notin \lambda$. If $r \leq p_{j+1}$ at the time the main loop reaches iteration r, then line 6 is executed. Otherwise, p_{j+1} must have been assigned to the current value at lines 4 or 9 in some iteration r' earlier than r. In both cases the assigned value is strictly larger than r'. Hence we have $r' < p_{j+1} < r$ and therefore $p_{j+1} \in \lambda$ by hypothesis. Algorithm 2 returns REFINABLE at iteration p_j or earlier, and therefore never reaches iteration r as assumed. □

Now we can prove the correctness in the refinable case.

Lemma 3.18. *Algorithm 2 outputs* REFINABLE *on every refinable λ.*

Proof. By Proposition 3.7 we know that the smallest refinable part r is refinable as $a + b$ with $a, b \notin \lambda$. Let us denote $j_a = a \pmod{\mu_1}$, $j_b = b \pmod{\mu_1}$, and $j_r = r \pmod{\mu_1}$. Clearly $j_r = j_a + j_b \pmod{\mu}$.
If the algorithm does not reach iteration r, it must be because it returned REFINABLE earlier and so there is nothing to prove. Otherwise let us show that it must return REFINABLE at iteration r.
The case of $j_a = 0$ is simple: we have that $j_r = j_b$ and $p_{j_r+1} \leq b$ by Proposition 3.16. Therefore we get $r > b \geq p_{j_r+1}$ and the algorithm returns at line 6. The case of $j_b = 0$ is symmetric.
For the remaining case of $j_a \neq 0$ and $j_b \neq 0$ we split into two further subcases: when $j_a = j_b$ and when $j_a \neq j_b$.
When $j_a \neq 0$, $j_b \neq 0$ and $j_a = j_b$, we may assume without loss of generality that $a < b$. By the time the algorithm reaches iteration b, we have that $p_{j_a+1} \leq a$ because of Proposition 3.16. The test at line 2 at call UPDATE$(\vec{p}, b)$ can be rewritten as $b > p_{j_a+1}$, hence the value p_{j_r} is assigned to a number smaller or equal than $r = a + b$ in the residue class of j_r, in line 4. At the time the main loop reaches the iteration r, the algorithm reaches line 6 and returns REFINABLE.
When $j_a \neq 0$ and $j_b \neq 0$ and $j_a \neq j_b$, we need to consider the smallest missing elements a' and b' that are equal to j_a and j_b, respectively, modulo μ_1. We assume without loss of generality that $a' < b'$. When the algorithm reaches

iteration b' we have that $p_{j_a+1} \leq a'$ because of Proposition 3.16, and that assignment $p_{j_b+1} \leftarrow b'$ in line 6 is executed because of Proposition 3.17. In the for loop right after line 6, we know that $j_a \in \{1, \ldots, \mu_1 - 1\} \setminus j_b$, therefore we get that p_{j_r+1} is set to some value smaller or equal to $a' + b'$ and in particular to a value smaller or equal than r. In the successive iteration the value never increases, and at iteration r we know that line 6 gets executed. □

Putting together Lemmas 3.15 and Lemma 3.18 we obtain the main theorem of this section.

Theorem 3.19. *Algorithm 2, executed on the sequence of parts $\lambda = (v_1, \ldots, v_\ell)$, returns* UNREFINABLE *if and only if the partition λ is unrefinable.*

Lemma 3.20 (Running time)**.** *Algorithm 2, executed on the sequence of parts $\lambda = (v_1, \ldots, v_\ell)$ with $\mu = mex(\lambda)$, runs in time $O(\ell + \mu^2)$.*

Proof. The initialization of the p_js and the computation of $\mu = mex(\lambda)$ takes $O(\ell)$ steps. The main loop in Algorithm 2 is executed at most ℓ times. The inner loop in Algorithm 1 is executed in at most μ of them. The total running time is therefore $O(\ell + \mu^2)$. □

3.3 Algorithm to enumerate unrefinable partitions

The verification via Algorithm 2 of a sequence of parts $\lambda = (v_1, \ldots, v_l)$ with $\mu_1 = mex(\lambda)$ starts by scanning the interval $\mu_1 + 1, \ldots, l$. Up to the point when some index r is under scrutiny, the algorithm uses no information about the elements of λ of successive indexes. More concretely, the values $\vec{p}_\lambda$ computed at iteration r are completely determined by the same old values computed at iteration $r-1$ and by the fact that r is either in λ or not. Therefore we can design the enumeration process as the visit of the tree (see Figure 3.1) of all possible sequences of parts, so that the Verification Algorithm is run on the sequence corresponding to any branch of the tree. A branch is pruned as soon as the the corresponding sequence has no possible extensions that are unrefinable and of sum at most N. When the sum of a sequence corresponding to a surviving branch equals the goal value N, the sequence is returned as output.
It is convenient to enumerate separately all unrefinable partitions of N that have the same minimal excludant. Given N, we set n as the largest positive integer such that

$$\sum_{i=1}^{n} i \leq N$$

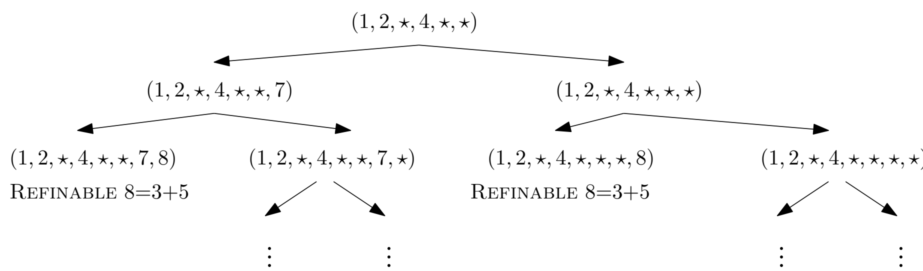


Figure 3.1: The branching from the sequence $(1, 2, \star, 4, \star, \star)$. For any two sequences in the tree, the running of Algorithm 2 proceeds identically up to the point that the corresponding branches diverge.

and then we partition the search space of sequence of parts according to prefixes:

$$\begin{aligned}
\lambda^{\dagger} &= (1, 2, 3, 4, \dots, n-2, n-1, n), \\
\lambda^{n} &= (1, 2, 3, 4, \dots, n-2, n-1, \star), \\
\lambda^{n-1} &= (1, 2, 3, 4, \dots, n-2, \star), \\
&\vdots \\
\lambda^{4} &= (1, 2, 3, \star), \\
\lambda^{3} &= (1, 2, \star), \\
\lambda^{2} &= (1, \star), \\
\lambda^{1} &= (\star),
\end{aligned} \tag{3.2}$$

where we use $\star$ to indicate that $v_i = 0$.
If N is triangular then the sequence $\lambda^{\dagger}$ itself is the unique unrefinable partition of N with no minimal excludant, and it must be in the output of the enumeration. If N is not triangular, i.e., if $n(n+1)/2 < N$, there is no unrefinable partition with prefix $\lambda^{\dagger}$: any additional part would make the sum exceed N.

Any other unrefinable partition of N must have minimal excludant $1 \leq \mu_1 \leq n$, and for a given value of μ_1 there is a one-to-one correspondence between these partitions and the sequence of parts λ that

- are unrefinable,
- have $\lambda \vdash N$,
- have prefix $\lambda^{\mu_1} = (1, 2, 3, 4, \dots, \mu_1 - 1, \star)$,
- have $v_l = l$ (i.e., not ending with $\star$).

In order to enumerate them, we describe the recursive algorithm Enumerate.

Enumerate starts with a sequence of parts λ with $mex(\lambda) = \mu_1$ and extends it in all possible ways in a binary tree-like fashion (cf. Figure 3.1). When visiting the node of the tree corresponding to sequence λ, the algorithm decides

Algorithm 3: Enumerate

Input : N
$\lambda = (1, 2, 3, \ldots, \mu_1 - 1, \star)$, unrefinable
$\vec{p}_\lambda = (p_1, \ldots, p_{\mu_1})$

Output : All unrefinable partitions of N with prefix λ, not ending with $\star$.

$r \leftarrow |\lambda| + 1$
$j \leftarrow r \pmod{\mu_1}$
`// Cases when we extend with` r`, if possible`
if $r < p_{j+1}$ *and* $sum(\lambda) + r = N$ **then output** $\lambda \cup \{r\}$
if $r < p_{j+1}$ *and* $sum(\lambda) + r < N$ **then** Enumerate(N, $\lambda \cup \{r\}$, $\vec{p}_\lambda$)
`// Case when we extend with` $\star$
$\vec{p}_\lambda \leftarrow$Update($\vec{p}_\lambda, r$)
if $\lambda \cup \{\star\}$ *is not saturated* **then** Enumerate(N, $\lambda \cup \{\star\}$, $\vec{p}_\lambda$)

whether to branch on $\lambda \cup \{r\}$, and successively whether to branch on $\lambda \cup \{\star\}$. Therefore, the tree is visited in lexicographic order. A branch is pruned either when a partition of N is reached, when an extension goes over the goal value N, when it introduces a refinable part, or when the sequence of parts is saturated according to Definition 3.10, and therefore no non-trivial extension would ever be unrefinable.

Walking along the tree, we update the values $\vec{p}_\lambda$ using the same Update function that we used in Algorithm 2. The idea is that the computation done by the recursive process on the sequence corresponding to some path is the same as the one done by Algorithm 2 on the same sequence. Formally we consider

P_1 the set of pairs $(\lambda, \vec{p}_\lambda)$ such that λ is unrefinable and not saturated, $mex(\lambda) = \mu_1$, $sum(\lambda) < N$, and such that running Algorithm 2 on λ computes the values $\vec{p}_\lambda$;

P_2 the set of pairs $(\lambda, \vec{p}_\lambda)$ such that the execution of Enumerate $(N, \lambda^{\mu_1}, (\infty, \ldots, \infty))$ produces a recursive call Enumerate$(N, \lambda, \vec{p}\ )$.

Lemma 3.21. *The two sets P_1 and P_2 are equal.*

Proof. We prove this statement by induction on the length of the sequence. For the base case, the sequence of parts λ^{μ_1}, paired with all p_js set to ∞, is both in P_1 and P_2 because $sum(\lambda^{\mu_1}) < N$.

For the induction step, consider the pair $(\lambda, \vec{p}_\lambda)$ for which we know that λ is unrefinable, is not saturated, that $mex(\lambda) = \mu_1$ and $sum(\lambda) < N$, and that a recursive call Enumerate$(N, \lambda, \vec{p}_\lambda)$ occurs.

For the extension $\lambda \cup \{r\}$ the values of $\vec{p}_\lambda$ do not change in both algorithms, therefore if λ is not saturated, neither is $\lambda \cup \{r\}$. The pair $(\lambda \cup \{r\}, \vec{p}_\lambda)$ is in P_1 if and only if $r < p_j$ for $r = j \pmod{\mu_1}$ and $sum(\lambda) + r < N$. But these are exactly the same condition for the recursive call Enumerate$(N, \lambda \cup \{r\}, \vec{p}_\lambda\ )$.

Considering the extension $\lambda \cup \{\star\}$, this is of course as unrefinable as λ and the sum does not change either. Let $\vec{q} \leftarrow$Update($\vec{p}_\lambda, r$). The pair $(\lambda \cup \{\star\}, \vec{q}\ )$ is in P_1 if and only if $\lambda \cup \{\star\}$ it is not saturated , and that is the exact same condition for the recursive call Enumerate($N, \lambda \cup \{\star\}, \vec{q}\ $) to happen. □

We are ready to show that, provided the appropriate input, Enumerate correctly produces all the unrefinable partitions of N with a given minimal excludant μ_1.

Lemma 3.22. *The recursive algorithm* Enumerate($N, \lambda^{\mu_1}, \vec{p}_\lambda$)*, where* $\vec{p}_\lambda = (p_0, \ldots, p_{\mu_1 - 1})$ *are all set to* ∞*, outputs the unrefinable sequence of parts whose sum is* N *with minimum excludant* μ_1*, and without* $\star$ *in the last position.*

Proof. By definition, the output of the enumeration only includes sequence of parts of N, not ending with $\star$. We need to prove that the output includes all unrefinable ones and no refinable ones.

Any unrefinable sequence of parts of N with minimal excludant μ_1, not ending with $\star$, can be written as $\lambda \cup \{r\}$ where $mex(\lambda) = \mu_1$ and $sum(\lambda) = N - r < N$. By Lemma 3.21, there is a recursive call Enumerate($N, \lambda, \vec{p}_\lambda$) where $\vec{p}_\lambda$ are the values computed by Algorithm 2 on λ. By the correctness of Algorithm 2 it must be $r < p_{j+1}$ for $j = r \pmod{\mu_1}$ since $\lambda \cup \{r\}$ is unrefinable. Hence the call Enumerate($N, \lambda, \vec{p}_\lambda$) outputs $\lambda \cup \{r\}$.

Now we want to show that no refinable sequence of parts of N is in the output. Consider the shortest prefix $\lambda \cup \{r\}$ of any such sequence where λ is unrefinable and $\lambda \cup \{r\}$ is refinable. It still holds that $mex(\lambda) = \mu_1$ and that $sum(\lambda) < N$, therefore, by Lemma 3.21, there is a recursive call Enumerate($N, \lambda, \vec{p}_\lambda$) where $\vec{p}_\lambda$ are the values computed by Algorithm 2 on λ. By the correctness of Algorithm 2, it must be $r \geq p_{j+1}$ for $j = r \pmod{\mu_1}$ since $\lambda \cup \{r\}$ is refinable. Hence Enumerate skips $\lambda \cup \{r\}$ and all its extensions. □

We are ready to describe the algorithm that enumerates all unrefinable partitions of N.

Algorithm 4: UnrefinablePartitions (enumerate all unrefinable partitions of N)

Input : N
Output : All unrefinable partitions of N.

$n \leftarrow$ largest n such that $\sum_{i=1}^{n} \leq N$
if $\sum_{i=1}^{n} = N$ **then output** $(1, 2, 3, \ldots, n)$
for μ_1 *in* $\{n, n-1, \ldots, 2, 1\}$ **do**
  $\vec{p}_\lambda = (p_1, \ldots, p_{\mu_1}) \leftarrow (\infty, \infty, \ldots, \infty)$
  $\lambda^{\mu_1} \leftarrow (1, 2, 3, \ldots, \mu_1 - 1, \star)$
  Enumerate(N,λ^{μ_1}, $\vec{p}_\lambda$)
end

Chapter 4

Classification of maximal unrefinable partition of triangular numbers

In this chapter we first prove a matching upper bound for the maximal part of unrefinable partitions and then we define *maximal unrefinable partitions* as those which reach the bound. As a main contribution, we provide a complete classification of maximal unrefinable partitions for triangular numbers. We constructively prove, denoting by T_n the n-th triangular number, that for even n there exists exactly one maximal unrefinable partition of T_n. For odd n, we obtain a lower bound for the minimal excludant for the maximal unrefinable partitions of T_n. The knowledge of a bound on the minimal excludant, among other considerations, allows us to show an explicit bijection between the set of the maximal unrefinable partitions of T_n and the set of partitions of $\lceil n/2 \rceil$ into distinct parts in the classical sense. From now on the set $\mathcal{D}$ is the set of partitions into distinct parts that have at least two parts.
In Section 4.1 we prove two upper bounds for the maximal part in an unrefinable partition of n, distinguishing the case when n is a triangular number and when it is not. The classification theorem, i.e., Theorem 4.3, is proved in Section 4.2, which also contains the result on triangular numbers of an even number. The odd case is developed in Section 4.3, which concludes the chapter. In particular, we show in Theorem 4.17 a bijective proof that the number of maximal unrefinable partitions of T_n equals the number of partitions of $\lceil n/2 \rceil$ into distinct parts.

4.1 Upper bound

Let $n \leq 5$, it is easy to check that the complete partitions π_n is the only unrefinable partitions for the triangular number T_n. In the general case of T_n

for $n \geq 6$, this is not true. For example, the partition $(1,2,3,7,8) \vdash 21 = T_6$ is unrefinable. As a more complex example, in the case of T_9 we can calculate that

$$\begin{array}{ll} (1,2,3,4,5,6,7,8,9) & (1,2,3,5,6,7,10,11) \\ (1,2,3,4,6,8,10,11) & (1,2,3,4,5,9,10,11) \\ (1,2,3,4,6,7,10,12) & (1,2,3,4,5,8,10,12) \\ (1,2,3,4,5,7,11,12) & (1,2,3,4,5,7,10,13) \\ (1,2,3,4,5,6,11,13) & (1,2,3,4,5,6,10,14) \\ (1,2,4,5,8,11,14) & \end{array}$$

are all the unrefinable partitions of $45 = T_9$.

It is clear that the property of being unrefinable imposes on the one hand an upper limitation on the size of the largest part which is admissible in the partition, and on the other a lower limitation on the minimal excludant.

Proposition 4.1. *Let $n \in \mathbb{N}$ and $N = T_n$. For every $\lambda \in \mathcal{U}$ such that $\lambda = (\lambda_1, \ldots, \lambda_t) \vdash N$ we have*

$$n \leq \lambda_t \leq 2n - 4. \tag{4.1}$$

Equivalently,

$$\frac{\sqrt{1+8N}-1}{2} \leq \lambda_t \leq \sqrt{1+8N} - 5.$$

Proof. Let us start by considering the complete partition $\pi_n \vdash N$. Other unrefinable partitions of N are obtained from π_n by removing some parts smaller than or equal to n and replacing them with parts larger than n. Hence, the lower bound for the maximal part in any partition of N is n, obtained when no part is removed. Since $N = n(n+1)/2$, n is the positive solution of $n^2 + n - 2N = 0$ and so we have

$$\lambda_t \geq \frac{\sqrt{1+8N}-1}{2}.$$

Let $h, j \in \mathbb{N}$ and let us denote by $1 \leq a_1 < a_2 < \cdots < a_h \leq n$ the candidate parts to be removed from π_n to obtain a new unrefinable partition of N, and by $n+1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_j$ the corresponding replacements. Since $\sum a_i = \sum \alpha_i$ we have $h > j$. Moreover $j > 1$, otherwise from $\sum a_i = \alpha_1$ the obtained partition is refinable. Hence we obtain

$$h \geq 3, \; j \geq 2, \text{ and } h > j \,.$$

There are h missing parts in the interval $\{1, 2, \ldots, n\}$ and exactly j parts appear in the interval $\{n+1, n+2, \ldots, \alpha_j\}$. Therefore the number of missing parts of λ is

$$m = h + \alpha_j - n - j.$$

To prove $\alpha_j \leq 2n - 4$ we consider the cases where α_j is either equal to $2n-3$, equal to $2n-2$, or strictly larger than $2n-2$. We derive a contradiction in each case. Let us observe that

$$\sum_{i=1}^{h} a_i \leq n + (n-1) + \cdots + (n-(h-1)) = hn - \frac{(h-1)h}{2}.$$

In the case $\alpha_j = 2n-3$ we obtain $m = h+n-3-j$. By Lemma 3.6, we have $m \leq \lfloor \alpha_j/2 \rfloor = n-2$, hence $h \leq j+1$, and so $h = j+1$. Notice that

$$\alpha_1 + \cdots + \alpha_j > (j-1)n + 2n - 3 = (j+1)n - 3 = hn - 3.$$

Therefore, since $\sum a_i = \sum \alpha_i$, we have

$$3 > \frac{(h-1)h}{2},$$

which is satisfied if $h < 3$, a contradiction.

In the case $\alpha_j = 2n-2$ we obtain $m = h+n-2-j$. By Lemma 3.6, we have $m \leq \lfloor \alpha_j/2 \rfloor = n-1$, hence $h \leq j+1$, and so again $h = j+1$. Notice that

$$\alpha_1 + \cdots + \alpha_j > (j-1)n + 2n - 2 = (j+1)n - 2 = hn - 2.$$

Therefore, since $\sum \alpha_i = \sum a_i$, we have

$$2 > \frac{(h-1)h}{2},$$

which is satisfied if $h < (1+\sqrt{17})/2 < 3$, a contradiction.

To conclude we consider the last case $\alpha_j > 2n-2$. We have $n-1 < \alpha_j/2$ and so

$$\alpha_j - (n-1) > \frac{\alpha_j}{2} \geq \left\lfloor \frac{\alpha_j}{2} \right\rfloor.$$

Hence, since $h \geq j+1$, we have

$$m = h + \alpha_j - n - j > \left\lfloor \frac{\alpha_j}{2} \right\rfloor + h - j - 1 \geq \left\lfloor \frac{\alpha_j}{2} \right\rfloor,$$

which contradicts Lemma 3.6. □

Notice that the upper bound of Equation 4.1 is tight. Indeed, let us define the following partition:

$$\widetilde{\pi}_n = (1, 2, \ldots, n-3, n+1, 2n-4). \tag{4.2}$$

It is easy to notice that $\widetilde{\pi}_n \vdash N$ and that $\widetilde{\pi}_n$ is unrefinable, since its least missing parts are $n-2$ and $n-1$, and $2n-4 < (n-2)+(n-1)$. In the notation of the proof of Proposition 4.1, $\widetilde{\pi}_n$ is obtained in the case $h = 3$ and $j = h-1 = 2$. We now introduce maximal unrefinable partitions as those partitions $\lambda = (\lambda_1, \ldots, \lambda_t) \in \widetilde{\mathcal{U}}_N$ whose λ_t is maximal.

Definition 4.2. *Let $N \in \mathbb{N}$. An unrefinable partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ of N is called **maximal** if*

$$\lambda_t = \max_{(\eta_1, \eta_2, \ldots, \eta_t) \in \mathcal{U}_N} \eta_t.$$

We denote by $\widetilde{\mathcal{U}}_N$ the set of the maximal unrefinable partitions of N.

In the case of triangular numbers $N = T_n$ for some $n \geq 6$, by virtue of Proposition 4.1, we have that $\lambda = (\lambda_1, \dots, \lambda_t) \in \widetilde{\mathcal{U}}_N$ is maximal if and only if $\lambda_t = 2n - 4$.
As already observed in the proof of Proposition 4.1, for each $\lambda \in \mathcal{U}_N$ there exist $1 < j < h$, $1 \leq a_1 < a_2 < \cdots < a_h \leq n$ and $\alpha_1, \alpha_2, \dots, \alpha_j \geq n + 1$ such that λ is obtained from π_n by removing the parts a_is which are replaced by the parts α_is. Consequently, $\#\widetilde{\mathcal{U}}_N$ coincides with the number of choices which lead to partitions meeting the mentioned conditions and, in addition, the condition $\lambda_t = 2n - 4$. In the remainder of the work, when $\lambda \in \mathcal{U}_N$ we will refer to a_is, α_is, j and h as intended here.

4.2 Classification of maximal unrefinable partitions

We are now ready to prove our first main contribution. Using arguments similar to those of the proofs in the previous section, we classify maximal unrefinable partitions for triangular numbers.

Theorem 4.3. *Let $n \in \mathbb{N}$, $n \geq 6$, and $N = T_n$. Then*

1. *if n is even, then $\widetilde{\mathcal{U}}_N = \{\widetilde{\pi}_n\}$;*
2. *if n is odd, then $\widetilde{\pi}_n \in \widetilde{\mathcal{U}}_N$ and the other partitions $\lambda \in \widetilde{\mathcal{U}}_N$, $\lambda \neq \widetilde{\pi}_n$, are such that $j = h - 2$ and the following conditions hold:*
   (i) the removed parts $a_1, \dots, a_{h-3}$ are replaced by $2n - 4 - a_1, 2n - 4 - a_2, \dots, 2n - 4 - a_{h-3}$, and
   (ii) the triple (a_{h-2}, a_{h-1}, a_h) is one of the following

$$(n-4, n-3, n-2), (n-4, n-2, n-1), (n-3, n-2, n), (n-2, n-1, n).$$

Proof. Let $\lambda \in \widetilde{\mathcal{U}}_N$ and let $a_1, a_2, \dots, a_h$ and $\alpha_1, \alpha_2, \dots, \alpha_j = 2n - 4$ as before. We already know that $h \geq 3$. From the hypotheses on λ we have that

$$m = h + \alpha_j - n - j = h + n - 4 - j.$$

By Lemma 3.6 we have $h - j \leq 2$, and, since $h > j$, we obtain $j \in \{h-1, h-2\}$. Notice that if $a \in \{a_1, \dots, a_h\}$ is such that $a < n - 4$, then $\alpha = 2n - 4 - a$ must belong to $\{\alpha_1, \dots, \alpha_{j-1}\}$, otherwise $\alpha + a = 2n - 4 = \alpha_j \in \lambda$, and so λ is refinable. Then each removed part a_i such that $a_i < n - 4$ is in one-to-one correspondence with its *replacement* which, for the sake of simplicity, we will denote from now on by α_i. On the other side, for the same symmetry argument, no part in the interval $\{n - 4, \dots, n\}$ has a replacement. In such an interval we may choose at most 5 parts. However, we are not allowed to remove, at the same time, parts from one of the pairs $(n - 4, n)$ and $(n - 3, n - 1)$ without contradicting the unrefinability of λ. Analogously, we are not allowed to remove more than three parts. Moreover, we cannot choose to pick only one part to be removed in that interval, otherwise we would obtain $h-1$ replacements but at most $j-1$ are allowed, and $h>j$.

We are left to consider the cases of two or three parts to be removed in the interval $\{n-4,\dots,n\}$, both with the assumptions $j=h-1$ or $j=h-2$. In particular, we will show that in both settings of j, there exists no maximal partition with two removed parts in the selected interval. Moreover, in the case $j=h-1$ and three removed parts, we show that the only admissible partition is $\widetilde{\pi}_n$. Finally, partitions in the case $j=h-2$ and three removed parts are only possible for odd n as claimed in (2). Let us address each of the four cases separately.

Let us suppose $j=h-1$ and $1\le a_1<a_2<\cdots<a_{h-2}\le n-5$ and $n-4\le a_{h-1}<a_h\le n$. For each $1\le i\le j-1=h-2$ we have $\alpha_i=\alpha_j-a_i$. We will now show that this configuration leads to a contradiction. To do this, we estimate $\sum a_i$ and $\sum \alpha_i$ from above and from below, respectively. This is clearly accomplished by noticing that $\alpha_{h-2}\ge n+1,\alpha_{h-3}\ge n+2,\dots,\alpha_1\ge n+h-2$ and $a_h\le n, a_{h-1}\le n-1$, obtaining $a_{h-2}\le \alpha_j-\alpha_{h-2}\le n-5, a_{h-3}\le n-6,\dots,a_1\le n-h-2$. Hence

$$\sum_{i=1}^{h} a_i \le hn-\sum_{i=1}^{h+2} i+2+3+4 = hn-\frac{h^2+5h+6}{2}+9, \text{ and}$$

$$\sum_{i=1}^{j} \alpha_i \ge hn+\sum_{i=1}^{h-2} i-4 = hn+\frac{h^2-3h+2}{2}-4.$$

For $\sum a_i=\sum \alpha_i$ we obtain an inequality which is satisfied for $h<3$, which is a contradiction.

In the second case, i.e., $j=h-1$, and $1\le a_1<a_2<\cdots<a_{h-3}\le n-5$, $n-4\le a_{h-2}<a_{h-1}<a_h\le n$, we have $\alpha_i=\alpha_j-a_i$ for every $1\le i\le h-3=j-2$. Notice that, in this case, the part $\alpha_{j-1}=\alpha_{h-2}$ is not determined by one of the a_is. Proceeding as before, since $\alpha_{h-3}\ge n+1,\alpha_{h-4}\ge n+2,\dots,\alpha_1\ge n+h-3$, $\alpha_{h-2}\ge n+h-2$ and $a_h\le n, a_{h-1}\le n-1, a_{h-2}\le n-2$, we determine $a_{h-3}\le \alpha_j-\alpha_{h-3}\le n-5, a_{h-4}\le n-6,\dots,a_1\le n-h-1$ and we obtain the bounds

$$\sum a_i \le hn-\sum_{i=1}^{h+1} i+3+4 = hn-\frac{h^2+3h+2}{2}+7, \text{ and}$$

$$\sum \alpha_i \ge hn+\sum_{i=1}^{h-2} i-4 = hn+\frac{h^2-3h+2}{2}-4.$$

From $\sum a_i=\sum \alpha_i$ we obtain an inequality which is satisfied only for $h=3$, which corresponds to the partition (cf. also Equation (4.2))

$$(1,2,\dots,n-3,n+1,2n-4)=\widetilde{\pi}_n\in\widetilde{\mathcal{U}}_N.$$

The third case $j = h-2$ with two removed parts is immediately contradictory, since the parts $a_1, a_2, \ldots, a_{h-2}$ determine $h-2 = j$ replacements but at most $j-1$ are possible.

The last case to be considered is the one where $j = h-2$ and the three largest parts a_is are chosen in the interval $\{n-4, \ldots, n\}$. As already observed, since λ is unrefinable, the only possible choices are

$$(a_{h-2}, a_{h-1}, a_h) \in \{(n-4, n-3, n-2), (n-4, n-2, n-1), \\ (n-3, n-2, n), (n-2, n-1, n)\},$$

which means $a_{h-2}+a_{h-1}+a_h \in \{3n-9, 3n-7, 3n-5, 3n-3\}$. From $\sum a_i = \sum \alpha_i$ we obtain

$$a_1 + a_2 + \cdots + a_h = (\alpha_{h-2} - a_1) + (\alpha_{h-2} - a_2) + \cdots + (\alpha_{h-2} - \alpha_{h-3}) + \alpha_{h-2}$$

and so, since $\alpha_{h-2} = \alpha_j = 2n-4$,

$$2(a_1 + a_2 + \cdots + a_{h-3}) + (a_{h-2} + a_{h-1} + a_h) = (h-2)(2n-4). \tag{4.3}$$

Since the right side of Equation (4.3) is even and $a_{h-2} + a_{h-1} + a_h$ is even only if n is odd, Equation (4.3) can be satisfied only in the case when n is odd. This proves (2) when n is odd and that the partition $\widetilde{\pi}_n$ of Equation (4.2) is the only maximal unrefinable partition of T_n when n is even, i.e., (1). □

From Theorem 4.3 we obtain that the description of maximal unrefinable partitions for the triangular number of an even integer is completed. The odd case is addressed in the following section.

Corollary 4.4. *Let $k \in \mathbb{N}$ and $N = T_{2k}$. Then $\#\widetilde{\mathcal{U}}_N = 1$.*

4.3 Odd triangular numbers

Let N will denote the triangular number of an odd integer. More precisely, let $n = 2k-1 \in \mathbb{N}$ be such that $N = T_n$.

From Theorem 4.3 we have that the set of maximal unrefinable partitions of triangular numbers of odd integers can be partitioned in the following way

$$\{\widetilde{\pi}_n \mid n \text{ odd}\} \,\dot\cup\, \widetilde{\mathcal{A}} \,\dot\cup\, \widetilde{\mathcal{B}} \,\dot\cup\, \widetilde{\mathcal{C}} \,\dot\cup\, \widetilde{\mathcal{D}},$$

where

$$\widetilde{\mathcal{A}} = \bigcup_{h\geq 4} \widetilde{\mathcal{A}}_h, \quad \widetilde{\mathcal{B}} = \bigcup_{h\geq 4} \widetilde{\mathcal{B}}_h, \quad \widetilde{\mathcal{C}} = \bigcup_{h\geq 4} \widetilde{\mathcal{C}}_h, \quad \widetilde{\mathcal{D}} = \bigcup_{h\geq 4} \widetilde{\mathcal{D}}_h$$

and

$$
\begin{aligned}
\widetilde{\mathcal{A}}_h &= \bigcup_{n \text{ odd}} \{\lambda \mid \lambda \in \widetilde{\mathcal{U}}_{T_n}, (a_{h-2}, a_{h-1}, a_h) = (n-4, n-3, n-2)\}, \\
\widetilde{\mathcal{B}}_h &= \bigcup_{n \text{ odd}} \{\lambda \mid \lambda \in \widetilde{\mathcal{U}}_{T_n}, (a_{h-2}, a_{h-1}, a_h) = (n-4, n-2, n-1)\}, \\
\widetilde{\mathcal{C}}_h &= \bigcup_{n \text{ odd}} \{\lambda \mid \lambda \in \widetilde{\mathcal{U}}_{T_n}, (a_{h-2}, a_{h-1}, a_h) = (n-3, n-2, n)\}, \\
\widetilde{\mathcal{D}}_h &= \bigcup_{n \text{ odd}} \{\lambda \mid \lambda \in \widetilde{\mathcal{U}}_{T_n}, (a_{h-2}, a_{h-1}, a_h) = (n-2, n-1, n)\}.
\end{aligned}
$$

Each set $\widetilde{\mathcal{A}}_h, \widetilde{\mathcal{B}}_h, \widetilde{\mathcal{C}}_h, \widetilde{\mathcal{D}}_h$ is called a *class* of maximal unrefinable partitions. If $\lambda \in \widetilde{\mathcal{A}}_h$ (resp. $\widetilde{\mathcal{B}}_h, \widetilde{\mathcal{C}}_h$ or $\widetilde{\mathcal{D}}_h$) for some h we say that λ is a *partition of class* $\widetilde{\mathcal{A}}_h$ (resp. $\widetilde{\mathcal{B}}_h, \widetilde{\mathcal{C}}_h$ or $\widetilde{\mathcal{D}}_h$).

The following consideration is a trivial but important consequence of Theorem 4.3.

Corollary 4.5. *Let* $n \in \mathbb{N}$ *and* $N = T_n$. *If* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) \in \widetilde{\mathcal{U}}_N$, *then* $\lambda_i \neq n-2$ *for every* $1 \leq i \leq t$.

Remark 4.6 (Anti-symmetric property)**.** *From Theorem 4.3(2) and Corollary 4.5 we derive that every partition* $\lambda \in \widetilde{\mathcal{U}}_{T_n}, \lambda \neq \widetilde{\pi}_n$, *is* anti-symmetric *with respect to* $n-2$, *i.e., for* $1 \leq a < 2n-4$ *we have*

$$
a \notin \lambda \iff 2n-4-a \in \lambda,
$$

provided that $a \neq n-2$.

Example 4.7. *Let us fix* $n = 13$. *In Table 4.1 we have displayed the three different partitions of* $\widetilde{\mathcal{U}}_{T_{13}} \setminus \{\widetilde{\pi_{13}}\}$, *where a black dot means that the corresponding integer is a part and the white dot means otherwise. Disregarding the last part which is fixed to be* $2n-4$ *due to the maximality constraint, the anti-symmetric property with respect to* $n-2$ *can be appreciated. Notice also that* $\min_{\lambda \in \widetilde{\mathcal{U}}_{T_{13}}} mex(\lambda) = 5 = (n-3)/2$ *and that* $(n-2)-5+1 = 7 = \lceil n/2 \rceil$.

										$n-2$		n									λ_t
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
●	●	●	●	●	●	○	●	○	○	○	●	●	○	●	○	○	○	○	○	○	●
●	●	●	●	●	○	●	●	○	●	○	○	●	○	○	●	○	○	○	○	○	●
●	●	●	●	○	●	●	●	●	○	○	●	○	○	○	○	●	○	○	○	○	●

Table 4.1: The anti-symmetric property shown on the partitions $\lambda \in \widetilde{\mathcal{U}}_{T_{13}}$, $\lambda \neq \widetilde{\pi_{13}}$.

Example 4.8. *As another significative example, we show in Table 4.2 all the partitions in* $\widetilde{\mathcal{U}}_{T_{27}} = \widetilde{\mathcal{U}}_{378}$, *classified according to the description of Theorem 4.3.*

$1 \le \lambda_i \le 22$	$23 \le \lambda_i \le 27$	$28 \le \lambda_i \le 50$	class
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22	23 24	28 50	$\widetilde{\pi_{27}}$
1 2 3 4 5 6 7 8 9 10 11 12 13 15 16 17 18 19 20 21 22	26 27	36 50	$\widetilde{\mathcal{A}}_4$
1 2 3 4 5 6 7 8 9 10 11 12 14 15 16 17 18 19 20 21 22	24 27	37 50	$\widetilde{\mathcal{B}}_4$
1 2 3 4 5 6 7 8 9 10 11 13 14 15 16 17 18 19 20 21 22	23 26	38 50	$\widetilde{\mathcal{C}}_4$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 21 22	26 27	30 31 50	$\widetilde{\mathcal{A}}_5$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 19 20 22	26 27	29 32 50	$\widetilde{\mathcal{A}}_5$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 18 19 20 21	26 27	28 33 50	$\widetilde{\mathcal{A}}_5$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 19 21 22	24 27	30 32 50	$\widetilde{\mathcal{B}}_5$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 18 19 20 22	24 27	29 33 50	$\widetilde{\mathcal{B}}_5$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 17 18 19 20 21	24 27	28 34 50	$\widetilde{\mathcal{B}}_5$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 20 21 22	23 26	31 32 50	$\widetilde{\mathcal{C}}_5$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 18 19 21 22	23 26	30 33 50	$\widetilde{\mathcal{C}}_5$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 17 18 19 20 22	23 26	29 34 50	$\widetilde{\mathcal{C}}_5$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 16 17 18 19 20 21	23 26	28 35 50	$\widetilde{\mathcal{C}}_5$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 18 20 21 22	23 24	31 33 50	$\widetilde{\mathcal{D}}_5$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 17 18 19 21 22	23 24	30 34 50	$\widetilde{\mathcal{D}}_5$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 16 17 18 19 20 22	23 24	29 35 50	$\widetilde{\mathcal{D}}_5$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19	24 27	28 29 30 50	$\widetilde{\mathcal{B}}_6$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 20	23 26	28 29 31 50	$\widetilde{\mathcal{C}}_6$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 21	23 24	28 30 31 50	$\widetilde{\mathcal{D}}_6$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 19 20	23 24	28 29 32 50	$\widetilde{\mathcal{D}}_6$

Table 4.2: Maximal unrefinable partitions of $378 = T_{27}$ and the corresponding classes.

It is important to notice that, when $h \ge 5$, partitions in the same class may appear with different multiplicities. Here all the parts λ_is of the partitions are listed, divided in three areas $1 \le \lambda_i \le n-5$, $n-4 \le \lambda_i \le n$ *and* $n+1 \le \lambda_i \le 2n-4$ *naturally induced by Theorem 4.3. Notice again that we have* $\min_{\lambda \in \widetilde{\mathcal{U}}_{T_{27}}} mex(\lambda) = 12 = (n-3)/2$.

It is natural to wonder, recalling that in general $h \ge 4$, what is an upper bound for h in a maximal unrefinable partition $\lambda \in \widetilde{\mathcal{U}}_{T_{2k-1}}$. The answer to this question is provided in Proposition 4.11, from which we also derive the result on the lower bound for the minimal excludant in maximal unrefinable partitions (cf. Corollary 4.15). Let us address before the two extremal cases $h = 4$ and $h = 5$.

Proposition 4.9. *Let $n \ge 7$ be odd. We have:*

1. $\widetilde{\mathcal{U}}_{T_n} \cap \widetilde{\mathcal{D}}_4 = \emptyset$,
2. $\widetilde{\mathcal{U}}_{T_7} \cap \widetilde{\mathcal{C}}_4 = \emptyset$ *and if* $n \ge 9$, *then* $\#(\widetilde{\mathcal{U}}_{T_n} \cap \widetilde{\mathcal{C}}_4) = 1$,
3. $\widetilde{\mathcal{U}}_{T_7} \cap \widetilde{\mathcal{B}}_4 = \widetilde{\mathcal{U}}_{T_9} \cap \widetilde{\mathcal{B}}_4 = \emptyset$ *and if* $n \ge 11$, *then* $\#(\widetilde{\mathcal{U}}_{T_n} \cap \widetilde{\mathcal{B}}_4) = 1$,
4. $\widetilde{\mathcal{U}}_{T_7} \cap \widetilde{\mathcal{A}}_4 = \widetilde{\mathcal{U}}_{T_9} \cap \widetilde{\mathcal{A}}_4 = \emptyset$ *and if* $n \ge 11$, *then* $\#(\widetilde{\mathcal{U}}_{T_n} \cap \widetilde{\mathcal{A}}_4) = 1$.

Proof. Let $\lambda \in \widetilde{\mathcal{U}}_{T_n}$ be obtained by removing the integers a_1, a_2, a_3, a_4 and adding the replacements α_1 and $\alpha_2 = 2n-4$, which need to satisfy the following conditions:

(i) $1 \leq a_1 \leq n-5$,

(ii) $n-4 \leq a_2 < a_3 < a_4 \leq n$,

(iii) $n+1 \leq \alpha_1 = 2n-4-a_1$,

(iv) $a_1 + a_2 + a_3 + a_4 = \alpha_1 + \alpha_2$.

For a contradiction, let us assume that $\lambda \in \widetilde{\mathcal{D}}_4$, and so $a_2 = n-2, a_3 = n-1$ and $a_4 = n$. From Equation (iv) we obtain

$$a_1 = \frac{n-5}{2} \quad \text{and} \quad \alpha_1 = \frac{3n-3}{2}.$$

Notice that $a_1 + (n+1) = \alpha_1$ and, by hypothesis, $\alpha_1 \geq n+1$. If $\alpha_1 = n+1$, we obtain $n = 5$, a contradiction. Otherwise, since $(n+1) \notin \lambda$, we obtain that λ is refinable. The claim (1) is then proved.

Let us now address the case (2). Similarly as before, we now have $a_2 = n-3, a_3 = n-2$ and $a_4 = n$ and so we determine

$$a_1 = \frac{n-3}{2} \quad \text{and} \quad \alpha_1 = \frac{3n-5}{2}.$$

Notice that from $\alpha_1 \geq n+1$ we obtain $n \geq 7$. However, assuming $n = 7$ leads to $a_1 + a_2 = 6 = n-1 \in \lambda$, a contradiction since λ is unrefinable. Let us now prove that the obtained partition

$$\lambda = \left(\ldots, \widehat{\frac{n-3}{2}}, \ldots, n-4, n-1, \frac{3n-5}{2}, 2n-4\right)$$

is unrefinable by showing that each possible sum $a_i + a_j$, with $1 \leq i < j \leq 4$, is different from α_1. Recall that by the classification of Theorem 4.3 we have already ruled out those partitions which contradict the unrefinability in $2n-4$. Since $n \geq 9$, we have that $a_1 + a_2 = (3n-9)/2 > n-1$. Consequently, every sum of missing parts is larger than $n-1 \in \lambda$. Moreover, $a_1 + a_2 \neq \alpha_1$, $a_1 + a_3 = (3n-7)/2 \neq \alpha_1$, $a_1 + a_4 = (3n-3)/2 > \alpha_1$, $a_2 + a_3 = 2n-5 > \alpha_1$ and therefore $a_2 + a_4, a_3 + a_4 > \alpha_1$. Therefore $\lambda \in \widetilde{\mathcal{C}}_4$ and it is unique by construction, which proves the claim (2).

In the case of $\widetilde{\mathcal{B}}_4$, we find

$$a_1 = \frac{n-1}{2} \quad \text{and} \quad \alpha_1 = \frac{3n-7}{2}.$$

From $\alpha_1 \geq n+1$ we have $n \geq 9$, and assuming $n = 9$ contradicts again the unrefinability; therefore $n \geq 11$. With arguments similar to those of the previous case the partition

$$\left(\ldots, \widehat{\frac{n-1}{2}}, \ldots, n-5, n-3, n, \frac{3n-7}{2}, 2n-4\right)$$

is proved unrefinable and unique by construction, hence (3) is obtained.

Finally, considering the case of $\widetilde{\mathcal{A}}_4$, we obtain

$$a_1 = \frac{n+1}{2} \quad \text{and} \quad \alpha_1 = \frac{3n-9}{2}.$$

Now, $\alpha_1 \geq n+1$ implies $n \geq 11$ and $a_1 + a_2 = (3n-7)/2 > \alpha_1$. This proves that

$$\left(\ldots, \widehat{\frac{n+1}{2}}, \ldots, n-5, n-1, n, \frac{3n-9}{2}, 2n-4\right) \in \widetilde{\mathcal{A}}_4,$$

i.e., the claim (4). □

Proposition 4.10. *Let $n \geq 7$ be odd and $k \geq 0$. We have:*

1. *if $n < 15$, then $\widetilde{\mathcal{U}}_{T_n} \cap \widetilde{\mathcal{C}}_5 = \emptyset$ and if $n \geq 15$, then $\#(\widetilde{\mathcal{U}}_{T_{15+2k}} \cap \widetilde{\mathcal{C}}_5) = \lfloor k/2 \rfloor + 1$,*
2. *if $n < 17$, then $\widetilde{\mathcal{U}}_{T_n} \cap \widetilde{\mathcal{B}}_5 = \emptyset$ and if $n \geq 17$, then $\#(\widetilde{\mathcal{U}}_{T_{17+2k}} \cap \widetilde{\mathcal{B}}_5) = \lfloor k/2 \rfloor + 1$,*
3. *if $n < 17$, then $\widetilde{\mathcal{U}}_{T_n} \cap \widetilde{\mathcal{D}}_5 = \emptyset$ and if $n \geq 17$, then $\#(\widetilde{\mathcal{U}}_{T_{17+2k}} \cap \widetilde{\mathcal{D}}_5) = \lfloor k/2 \rfloor + 1$,*
4. *if $n < 19$, then $\widetilde{\mathcal{U}}_{T_n} \cap \widetilde{\mathcal{A}}_5 = \emptyset$ and if $n \geq 19$, then $\#(\widetilde{\mathcal{U}}_{T_{19+2k}} \cap \widetilde{\mathcal{A}}_5) = \lfloor k/2 \rfloor + 1$.*

Proof. Let us proceed as in the proof of Proposition 4.9. Let $\lambda \in \widetilde{\mathcal{U}}_{T_n}$ be obtained by removing the integers $a_1, a_2, \ldots, a_5$ and adding the replacements α_1, α_2 and $\alpha_3 = 2n-4$, which need to satisfy the following conditions:

(i) $1 \leq a_1 < a_2 \leq n-5$,

(ii) $n-4 \leq a_3 < a_4 < a_5 \leq n$,

(iii) $n+1 \leq \alpha_2 = 2n-4-a_2 < \alpha_1 = 2n-4-a_1$,

(iv) $a_1 + a_2 + a_3 + a_4 + a_5 = \alpha_1 + \alpha_2 + \alpha_3$.

First, let us address the case (1). We have $a_3 = n-3, a_4 = n-2$ and $a_5 = n$ and, from Equation (iii) and Equation (iv), a_1 and a_2 satisfy the condition

$$a_1 + a_2 = \frac{3n-7}{2}. \tag{4.4}$$

We first consider the case when a_2 is maximal, i.e., $a_2 = n-5$, in which we have $a_1 = (n+3)/2$ and consequently $\alpha_1 = (3n-11)/2$ and $\alpha_2 = n+1$. Notice then that the condition of Equation (4.4) can be met in $\lfloor (a_2 - a_1 - 1)/2 \rfloor$ other ways by taking the first two parts to be removed as $a_1 + i$ and $a_2 - i$, for $1 \leq i \leq \lfloor (a_2 - a_1 - 1)/2 \rfloor$. Now, from $a_1 < a_2$ we obtain $n \geq 15$. If $n = 15$, the partition

$$\left(\ldots, \widehat{\frac{n+3}{2}}, \ldots, \widehat{n-5}, n-4, n-1, n+1, \frac{3n-11}{2}, 2n-4\right)$$

is unique by construction and is unrefinable since $a_1 + a_2 > \alpha_1$. In the other cases, which are

$$\left\lfloor \frac{a_2 - a_1 - 1}{2} \right\rfloor = \left\lfloor \frac{n-15}{4} \right\rfloor, \tag{4.5}$$

we obtain an unrefinable partition since, letting $\alpha_1' = 2n - 4 - (a_1 + i)$ and $\alpha_2' = 2n - 4 - (a_2 - i)$, we have

$$(a_1 + i) + (a_2 - i) = a_1 + a_2 > \alpha_1 > \alpha_1' > \alpha_2'.$$

The claim (1) is then obtained writing $n = 15 + 2k$ in Equation (4.5).

The proofs for (2) and (4) are obtained in the same way. When $n = 17$, the partition

$$\left(\dots, \widehat{\frac{n+5}{2}}, \dots, \widehat{n-5}, n-3, n, n+1, \frac{3n-13}{2}, 2n-4 \right) \in \widetilde{\mathcal{B}}_5$$

and is unique by construction, and when $n > 17$ it can be modified in $\lfloor (a_2 - a_1 - 1)/2 \rfloor = \lfloor (n-17)/4 \rfloor$ ways as in the proof of (1). Analogously, when $n = 19$ the partition

$$\left(\dots, \widehat{\frac{n+7}{2}}, \dots, \widehat{n-5}, n-1, n, n+1, \frac{3n-15}{2}, 2n-4 \right) \in \widetilde{\mathcal{A}}_5$$

and is unique by construction, and when $n > 19$ it can be modified in $\lfloor (n-19)/4 \rfloor$ ways.

It remains to prove the slightly different case (3). Here, we have $a_1 + a_2 = (3n-9)/2$ and, proceeding as above, from $a_2 = n - 5$ we obtain $a_1 = (n+1)/2$ and $\alpha_1 = (3n-9)/2$. This leads to the contradiction $a_1 + a_2 = \alpha_1$. The argument of (1) is here replicated starting from $a_2 = n - 6$. It is now easy to see that, when $n = 17$, the partition

$$\left(\dots, \widehat{\frac{n+3}{2}}, \dots, \widehat{n-6}, \dots, n-3, n+2, \frac{3n-11}{2}, 2n-4 \right),$$

unique by construction, is unrefinable. When $n > 17$, it can be modified in $\lfloor (n-17)/4 \rfloor$ ways, which proves (3). □

Proposition 4.11. *Let $n \geq 7$ be odd and $h \geq 6$. We have*

1. *$\widetilde{\mathcal{U}}_{T_n} \cap \widetilde{\mathcal{D}}_h \neq \emptyset$ if and only if $n \geq h^2 - h - 7$,*
2. *$\widetilde{\mathcal{U}}_{T_n} \cap \widetilde{\mathcal{C}}_h \neq \emptyset$ if and only if $n \geq h^2 - h - 5$,*
3. *$\widetilde{\mathcal{U}}_{T_n} \cap \widetilde{\mathcal{B}}_h \neq \emptyset$ if and only if $n \geq h^2 - h - 3$,*
4. *$\widetilde{\mathcal{U}}_{T_n} \cap \widetilde{\mathcal{A}}_h \neq \emptyset$ if and only if $n \geq h^2 - h - 1$.*

Proof. We proceed as in Proposition 4.9 and Proposition 4.10, assuming the conditions

(i) $1 \le a_1 < a_2 < \cdots < a_{h-3} \le n-5$,

(ii) $n-4 \le a_{h-2} < a_{h-1} < a_h \le n$,

(iii) $n+1 \le \alpha_{h-3} = 2n-4-a_{h-3} < \alpha_{h-4} = 2n-4-a_{h-4} < \cdots < \alpha_1 = 2n-4-a_1$,

(iv) $\sum a_i = \sum \alpha_i$.

If $\lambda \in \widetilde{\mathcal{U}}_{T_n} \cap \widetilde{\mathcal{D}}_h$, then $a_{h-2}+a_{h-1}+a_h = 3n-3$ and therefore, from Equation (iii) and Equation (iv),

$$a_1 + a_2 + \cdots + a_{h-3} = \frac{(h-2)(2n-4)-(3n-3)}{2} = \frac{(2h-7)n+11-4h}{2}.$$

Let us now assume that $a_{h-3} = n-5, a_{h-4} = n-6, \ldots, a_2 = n-h$, i.e., let us maximize the sum $a_2 + \cdots + a_{h-3}$. We obtain

$$a_2 + \cdots + a_{h-3} = (h-4)n - \sum_{i=5}^{h} i = (h-4)n - \frac{h(h+1)}{2} + 10,$$

from which we can calculate

$$a_1 = \frac{n+h^2-3h-9}{2}.$$

Imposing $a_1 < a_2$ we obtain $n > h^2 - h - 9$. In this setting, we have

$$\alpha_1 = \frac{3n-h^2+3h+1}{2} \quad \text{and} \quad a_1 + a_2 = \frac{3n+h^2-5h-9}{2}.$$

Notice that $a_1 + a_2 > \alpha_1$ is satisfied for $h \ge 6$, hence the provided construction leads to a partition λ which belongs to $\widetilde{\mathcal{D}}_h$ if and only if $n \ge h^2-h-7$, i.e., (1).

In the cases (2), (3) and (4) we proceed analogously, maximising $a_2 + a_3 + \ldots a_{h-3}$, provided that a_{h-2}, a_{h-1}, a_h are modified accordingly. In particular, when considering $\widetilde{\mathcal{C}}_h$ we have

$$a_1 = \frac{n+h^2-3h-7}{2} \quad \text{and} \quad \alpha_1 = \frac{3n-h^2+3h-1}{2}.$$

From $a_1 < a_2$ we have $n \ge h^2-h-5$ and from

$$\alpha_1 < a_1 + a_2 = \frac{3n+h^2-5h-7}{2}$$

we obtain $\lambda \in \widetilde{\mathcal{C}}_h$, i.e., the claim (2) follows.

In the case of $\widetilde{\mathcal{B}}_h$ we have

$$a_1 = \frac{n + h^2 - 3h - 5}{2} \quad \text{and} \quad \alpha_1 = \frac{3n - h^2 + 3h - 3}{2}.$$

From $a_1 < a_2$ we have $n \geq h^2 - h - 3$ and from

$$\alpha_1 < a_1 + a_2 = \frac{3n + h^2 - 5h - 5}{2}$$

we obtain $\lambda \in \widetilde{\mathcal{B}}_h$, i.e., the claim (3) is proved.

Finally, assuming the conditions of $\widetilde{\mathcal{A}}_h$ we have

$$a_1 = \frac{n + h^2 - 3h - 3}{2} \quad \text{and} \quad \alpha_1 = \frac{3n - h^2 + 3h - 5}{2}.$$

From $a_1 < a_2$ we have $n \geq h^2 - h - 1$ and from

$$\alpha_1 < a_1 + a_2 = \frac{3n + h^2 - 5h - 3}{2}$$

we obtain $\lambda \in \widetilde{\mathcal{A}}_h$, from which the desired result (4) follows. □

By interchanging the role of n and h in the statements of Proposition 4.11, we obtain the following description of the set of maximal unrefinable partitions of triangular numbers of an odd number, where we can read the upper bound for h in each different class.

Corollary 4.12. *Let $n \geq 7$ be odd. Then*

$$\widetilde{\mathcal{U}}_{T_n} = \{\widetilde{\pi}_n\} \,\dot\cup \left(\bigcup_{h=5}^{\lfloor \frac{1+\sqrt{29+4n}}{2} \rfloor} \widetilde{\mathcal{D}}_h \,\dot\cup \bigcup_{h=4}^{\lfloor \frac{1+\sqrt{21+4n}}{2} \rfloor} \widetilde{\mathcal{C}}_h \,\dot\cup \bigcup_{h=4}^{\lfloor \frac{1+\sqrt{13+4n}}{2} \rfloor} \widetilde{\mathcal{B}}_h \,\dot\cup \bigcup_{h=4}^{\lfloor \frac{1+\sqrt{5+4n}}{2} \rfloor} \widetilde{\mathcal{A}}_h \right).$$

Remark 4.13. *In the proof of Proposition 4.11 we have exhibited an example of unrefinable partition for each class, constructed by maximising $a_2 + a_3 + \ldots a_{h-3}$ and consequently by determining a_1. The unrefinability of the obtained partition is then granted from the fact that $a_1 + a_2 > \alpha_1$. Notice that, each other partition λ' of the same class is determined by the removed parts $a'_1, a'_2, \ldots, a'_{h-3}$ such that $a'_1 = a_1 + i$ and $a'_s = a_s - i_{s-1}$ for $s > 1$ and $i_s \geq 0$, where $i = \sum_{s=1}^{h-4} i_s$, provided that $a'_i < a'_s$ for $i < s$. The unrefinability of λ' is then easily proved, since*

$$a'_1 + a'_2 = a_1 + i + a_2 - i_i \geq a_1 + a_2 \geq \alpha_1 \geq \alpha'_1.$$

Example 4.14. *Let $n = 49$. For the bound in the previous corollary, when considering partitions of class $\widetilde{\mathcal{D}}$ we have $5 \leq h \leq (1 + \sqrt{29 + 4n})/2 = 8$. Let us fix $h = 7$ and construct all the partitions in $\widetilde{\mathcal{U}}_{T_{49}} \cap \widetilde{\mathcal{D}}_7$. We recall that, for Theorem 4.3, a partition of class $\widetilde{\mathcal{D}}_7$ is given when $a_1, a_2 \ldots, a_{h-3} = a_4$ are specified. Therefore, for the sake of simplicity, we denote the partitions just by* *listing the removed parts* (a_1, a_2, a_3, a_4). *Let us start, as in Proposition 4.11, from the partition*

$$\left(\frac{n+h^2-3h-9}{2}, n-7, n-6, n-5\right) = (34, 42, 43, 44).$$

All the remaining partitions in $\widetilde{\mathcal{D}}_7$*, obtained as in Remark 4.13, are:*

$$\begin{array}{cc} (35,41,43,44) & (36,40,43,44) \\ (37,39,43,44) & (36,41,42,44) \\ (37,40,42,44) & (38,39,42,44) \\ (38,40,41,44) & (37,41,42,43) \\ (38,40,42,43) & (39,40,41,43) \end{array}$$

The partitions in other classes are obtained analogously.

We have already highlighted in Example 4.7 and in Example 4.8 what $\min_{\lambda \in \widetilde{\mathcal{U}}_{T_n}} mex(\lambda)$ looks like. The intuition can now be easily proved as a consequence of the previous propositions.

Corollary 4.15. *Let* $n \geq 7$ *be odd. For each* $\lambda \in \widetilde{\mathcal{U}}_{T_n}$ *we have*

$$mex(\lambda) = \mu_1 \geq \frac{(n-3)}{2}.$$

Proof. Notice that $\mu_1 = a_1$. The claim is trivial if $\lambda = \widetilde{\pi}_n$. Otherwise it follows from Propositions 4.9, 4.10 and 4.11, recalling that a_1 was calculated in order to be minimal, since $a_2 + a_3 + \cdots + a_{h-3}$ was maximised. The results are summarised in Table 4.3, where it is not hard to check that $(n-3)/2$ is the smaller value that a_1 can assume.

□

4.4 The bijection

Notice that, by the anti-symmetric property (Remark 4.6) and by the bound on the minimal excludant (Corollary 4.15), a partition in $\widetilde{\mathcal{U}}_{T_n}$ is determined by at most

$$(n-2) - \frac{n-3}{2} = \frac{n+1}{2} - 1 = \left\lceil \frac{n}{2} \right\rceil - 1$$

parts. The following theorem is used to establish a bijection between $\widetilde{\mathcal{U}}_{T_{2k-1}}$ and $\mathcal{D}_k$.

Theorem 4.16. *Let* $a_1, a_2, \ldots, a_u$ *be the missing parts smaller than or equal to* $n-3$ *of an unrefinable partition* $\lambda \neq \widetilde{\pi}_n$ *of* T_n*, for some odd integer* $n \geq 7$*. Then* n*,* λ *and its class are uniquely determined.*

class	a_1
$\widetilde{\mathcal{C}}_4$	$(n-3)/2$
$\widetilde{\mathcal{B}}_4$	$(n-1)/2$
$\widetilde{\mathcal{A}}_4$	$(n+1)/2$
$\widetilde{\mathcal{D}}_5$	$(n+3)/2$
$\widetilde{\mathcal{C}}_5$	$(n+3)/2$
$\widetilde{\mathcal{B}}_5$	$(n+5)/2$
$\widetilde{\mathcal{A}}_5$	$(n+7)/2$
$\widetilde{\mathcal{D}}_h$	$(n+h^2-3h-9)/2$
$\widetilde{\mathcal{C}}_h$	$(n+h^2-3h-7)/2$
$\widetilde{\mathcal{B}}_h$	$(n+h^2-3h-5)/2$
$\widetilde{\mathcal{A}}_h$	$(n+h^2-3h-3)/2$

Table 4.3: Values of a_1 in the construction of Propositions 4.9, 4.10 and 4.11, for $h = 4$, $h = 5$ and $h \geq 6$.

Proof. Let us start by proving that n can be obtained from knowing $a_1, a_2, \dots, a_u$. In particular, let us prove that

$$n = \frac{2\sum_{i=1}^{u} a_i + 1 + 4u}{2u - 1} \tag{4.6}$$

by distinguishing the four possible classes. Let us first assume $\lambda \in \widetilde{\mathcal{D}}_h$ for some $h \geq 5$. Recalling that $a_u \leq n-3$ and, by the definition of $\widetilde{\mathcal{D}}_h$ and by Remark 4.6, since $a_{h-1} = n-1$ and $a_h = n$, we have $a_u \notin \{n-3, n-4\}$. Therefore $a_u \leq n-5$, and so $u = h-3$. Recalling that the following conditions hold

(i) $1 \leq a_1 < a_2 < \cdots < a_{h-3} \leq n-5$,

(ii) $n-4 \leq a_{h-2} < a_{h-1} < a_h \leq n$,

(iii) $n < \alpha_{h-3} = 2n-4-a_{h-3} < \alpha_{h-4} = 2n-4-a_{h-4} < \cdots < \alpha_1 = 2n-4-a_1$,

(iv) $\sum a_i = \sum \alpha_i$,

we obtain

$$\sum_{i=1}^{u} a_i = \frac{(u+1)(2n-4) - (3n-3)}{2},$$

from which we determine n as claimed.

Let us consider the class $\widetilde{\mathcal{C}}_h$. In this case, reasoning as above, we have $a_u = n-3$ and $a_{u-1} \leq n-5$, which means $u-1 = h-3$. Therefore

$$\sum_{i=1}^{u} a_i - (n-3) = \sum_{i=1}^{u-1} a_i = \frac{u(2n-4) - (3n-5)}{2},$$

from which we obtain again Equation (4.6).

When $\lambda \in \widetilde{\mathcal{B}}_h$, we have $a_u = n-4$, which means $h = u+2$, so

$$\sum_{i=1}^{u} a_i - (n-4) = \sum_{i=1}^{u-1} a_i = \frac{u(2n-4)-(3n-7)}{2},$$

from which the same n is determined.

In conclusion, if $\lambda \in \widetilde{\mathcal{A}}_h$, we have $a_u = n-3$ and $a_{u-1} = n-4$, so $h = u+1$ and Equation (4.6) is satisfied since

$$\sum_{i=1}^{u} a_i - (n-4) - (n-3) = \sum_{i=1}^{u-2} a_i = \frac{(u-1)(2n-4)-(3n-9)}{2}.$$

Now that n is determined from $a_1, a_2, \dots, a_u$, the class of the partition can be recognised by looking at a_{u-1} and a_u. In particular

- $a_u < n-4 \iff \lambda \in \widetilde{\mathcal{D}}_{u+3}$,
- $a_u = n-3$ and $a_{u-1} < n-4 \iff \lambda \in \widetilde{\mathcal{C}}_{u+2}$,
- $a_u = n-4 \iff \lambda \in \widetilde{\mathcal{B}}_{u+2}$,
- $a_u = n-3$ and $a_{u-1} = n-4 \iff \lambda \in \widetilde{\mathcal{A}}_{u+1}$.

To conclude, we determine the partition by using the anti-symmetric property (cf. Remark 4.6). □

We are now ready to prove our last main result of the chapter, in which we show that the number of maximal unrefinable partitions of T_{2k-1} coincides with the number of partitions of k into distinct parts $\mathcal{D}_k$.
Denoting by $\mathcal{D}$ the set of all the partitions into distinct parts, let us define the following subsets of $\mathcal{D}$:

$$\begin{aligned}
\widetilde{\mathcal{A}}^*_t &= \{\lambda = (\lambda_1, \dots, \lambda_t) \mid \lambda \in \mathcal{D},\ \lambda_1 = 1, \lambda_2 = 2, t \geq 3\},\\
\widetilde{\mathcal{B}}^*_t &= \{\lambda = (\lambda_1, \dots, \lambda_t) \mid \lambda \in \mathcal{D},\ \lambda_1 = 2, t \geq 2\},\\
\widetilde{\mathcal{C}}^*_t &= \{\lambda = (\lambda_1, \dots, \lambda_t) \mid \lambda \in \mathcal{D},\ \lambda_1 = 1, \lambda_2 > 2, t \geq 2\},\\
\widetilde{\mathcal{D}}^*_t &= \{\lambda = (\lambda_1, \dots, \lambda_t) \mid \lambda \in \mathcal{D},\ \lambda_1 > 3, t \geq 2\}.
\end{aligned}$$

It is not hard to notice that

$$\mathcal{D} = \widetilde{\mathcal{A}}^* \,\dot\cup\, \widetilde{\mathcal{B}}^* \,\dot\cup\, \widetilde{\mathcal{C}}^* \,\dot\cup\, \widetilde{\mathcal{D}}^*,$$

where

$$\widetilde{\mathcal{A}}^* = \bigcup_{t\geq 3} \widetilde{\mathcal{A}}^*_t, \quad \widetilde{\mathcal{B}}^* = \bigcup_{t\geq 2} \widetilde{\mathcal{B}}^*_t, \quad \widetilde{\mathcal{C}}^* = \bigcup_{t\geq 2} \widetilde{\mathcal{C}}^*_t, \quad \widetilde{\mathcal{D}}^* = \bigcup_{t\geq 2} \widetilde{\mathcal{D}}^*_t.$$

Theorem 4.17. *Let $k \in \mathbb{N}$, $k \geq 7$, $n = 2k-1$ and $N = T_n$. Let $\tau \colon \widetilde{\mathcal{U}}_{T_n} \to \mathcal{D}_k$ be such that*

$$\lambda \mapsto \begin{cases} (3, k-3) & \lambda = \widetilde{\pi}_n \\ (n-2-a_u, \dots, n-2-a_2, n-2-a_1) & \lambda \neq \widetilde{\pi}_n \end{cases},$$

where, if $\lambda \neq \widetilde{\pi}_n$, then $(a_1, a_2, \dots, a_u)$ are the missing parts of $\mathcal{M}_\lambda \cap \{1, 2, \dots, n-3\}$ as in Theorem 4.16. Then τ is bijective, therefore $\#\widetilde{\mathcal{U}}_{T_{2k-1}} = \#\mathcal{D}_k$.

Proof. For the sake of brevity and by virtue of Theorem 4.16, we will denote each $\lambda \in \widetilde{\mathcal{U}}_{T_n} \setminus \{\widetilde{\pi}_n\}$ by listing its missing parts $a_1, a_2, \dots, a_u$ in $\mathcal{M}_\lambda \cap \{1, 2, \dots, n-3\}$. We prove that τ is bijective by proving explicitly that partitions of $\widetilde{\mathcal{A}} \cap \widetilde{\mathcal{U}}_{T_{2k-1}}$ are in one-to-one correspondence with those of $\widetilde{\mathcal{A}}^* \cap \mathcal{D}_k$ and that the same holds respectively for $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}^*$, $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{C}}^*$, and $\widetilde{\mathcal{D}} \cup \{\widetilde{\pi}_n\}$ and $\widetilde{\mathcal{D}}^*$.

Let us start by proving that τ is well defined, i.e., for each $\lambda \in \widetilde{\mathcal{U}}_{T_n}$ we have that $\tau(\lambda)$ is a partition of k into distinct parts. If $\lambda = \widetilde{\pi}_n$ there is nothing to prove, otherwise, since the missing parts of λ are distinct, so are the parts $n-2-a_u < \cdots < n-2-a_2 < n-2-a_1$ of $\tau(\lambda)$. We now prove that the sum of the parts of $\tau(\lambda)$ is k in each possible case, making extensive use of Proposition 4.9, Proposition 4.10 and Proposition 4.11 without further mention. If $\lambda \in \widetilde{\mathcal{A}}_4$, then $\lambda = ((n+1)/2, n-4, n-3)$ and so

$$\begin{aligned}\tau(\lambda) = \left(1, 2, n-2-\frac{n+1}{2}\right) = \left(1, 2, \frac{n-5}{2}\right) &= \left(1, 2, \frac{n+1}{2} - 3\right) \\ &= (1, 2, k-3) \in \mathcal{D}_k.\end{aligned}$$

Notice that, in particular, $\tau(\lambda) \in \mathcal{D}_k \cap \widetilde{\mathcal{A}}_3^*$. Similarly, if $\lambda \in \widetilde{\mathcal{B}}_4$, then $\lambda = ((n-1)/2, n-4)$ and $\tau(\lambda) = (2, (n+1)/2-2) = (2, k-2) \in \mathcal{D}_k \cap \widetilde{\mathcal{B}}_2^*$. If $\lambda \in \widetilde{\mathcal{C}}_4$ we have $\lambda = ((n-3)/2, n-3)$, so $\tau(\lambda) = (1, (n+1)/2-1) = (1, k-1) \in \mathcal{D}_k \cap \widetilde{\mathcal{C}}_2^*$.
If $\lambda \in \widetilde{\mathcal{A}}_5$, then $\lambda = ((n+7)/2+i, n-5-i, n-4, n-3)$, for $0 \leq i \leq \lfloor (n-19)/4 \rfloor$, and

$$\begin{aligned}\tau(\lambda) = \left(1, 2, 3+i, \frac{n-11}{2} - i\right) &= \left(1, 2, 3+i, \frac{n+1}{2} - 6 - i\right) \\ &= (1, 2, 3+i, k-6-i) \in \mathcal{D}_k,\end{aligned}$$

for $0 \leq i \leq \lfloor (n-19)/4 \rfloor$. In particular, $\tau(\lambda) \in \mathcal{D}_k \cap \widetilde{\mathcal{A}}_4^*$. Similarly, if $\lambda \in \widetilde{\mathcal{B}}_5$, then $\lambda = ((n+5)/2+i, n-5-i, n-4)$ and

$$\tau(\lambda) = \left(2, 3+i, \frac{n+1}{2} - 5 - i\right) = (2, 3+i, k-5-i) \in \mathcal{D}_k \cap \widetilde{\mathcal{B}}_3^*,$$

for $0 \leq i \leq \lfloor (n-17)/4 \rfloor$. If $\lambda \in \widetilde{\mathcal{C}}_5$, we have $\lambda = ((n+3)/2+i, n-5-i, n-3)$ and so

$$\tau(\lambda) = \left(1, 3+i, \frac{n+1}{2} - 4 - i\right) = (1, 3+i, k-4-i) \in \mathcal{D}_k \cap \widetilde{\mathcal{C}}_3^*,$$

for $0 \leq i \leq \lfloor (n-15)/4 \rfloor$. In the case when $\lambda = ((n+3)/2+i, n-6-i) \in \widetilde{\mathcal{D}}_5$, we have

$$\tau(\lambda) = \left(4+i, \frac{n+1}{2} - 4 - i\right) = (4+i, k-4-i) \in \mathcal{D}_k \cap \widetilde{\mathcal{D}}_2^*,$$

for $0 \leq i \leq \lfloor (n-17)/4 \rfloor$.
Let us now consider $\lambda \in \widetilde{\mathcal{D}}_h$ for $h \geq 6$. In this case

$$\lambda = \left((n+h^2-3h-9)/2+i, n-h-i_1, \ldots, n-5-i_{h-4}\right),$$

where $i = \sum_{s=1}^{h-4} i_s$, and so

$$\begin{aligned}\tau(\lambda) &= \left(3+i_{h-4}, \ldots, h-2+i_1, \frac{n-h^2+3h+5}{2} - i\right) \\ &= \left(3+i_{h-4}, \ldots, h-2+i_1, k+2-\frac{h^2-3h}{2} - i\right).\end{aligned}$$

Notice that

$$\begin{aligned}(3+i_{h-4}) + (4+i_{h-5}) + \cdots + (h-2+i_1) + \\ + \left(k+2-\frac{h^2-3h}{2} - i\right) = \\ \sum_{j=3}^{h-2} j + \sum_{s=1}^{h-4} i_s + k + 2 - \frac{h^2-3h}{2} - i = \\ \frac{(h-2)(h-1)}{2} - 3 + k + 2 - \frac{h^2-3h}{2} = \\ \frac{h^2-3h}{2} - 2 + k + 2 - \frac{h^2-3h}{2} = k,\end{aligned}$$

and so $\tau(\lambda) \in \mathcal{D}_k \cap \widetilde{\mathcal{D}}_{h-3}^*$. Similarly, if $\lambda \in \widetilde{\mathcal{C}}_h$, then

$$\lambda = \left(\frac{n+h^2-3h-7}{2} + i, n-h-i_1, \ldots, n-5-i_{h-4}, n-3\right)$$

and

$$\tau(\lambda) = \left(1, 3+i_{h-4}, \ldots, h-2+i_1, k+1-\frac{h^2-3h}{2} - i\right) \in \mathcal{D}_k \cap \widetilde{\mathcal{C}}_{h-2}^*$$

since the sum of the first $h-3$ terms is $(h^2-3h)/2 - 1 + \sum_{s=1}^{h-4} i_s$. If $\lambda \in \widetilde{\mathcal{B}}_h$, we have

$$\lambda = \left(\frac{n+h^2-3h-5}{2} + i, n-h-i_1, \ldots, n-5-i_{h-4}, n-4\right)$$

an so

$$\tau(\lambda) = \left(2, 3 + i_{h-4}, \ldots, h - 2 + i_1, k - \frac{h^2 - 3h}{2} - i\right) \in \mathcal{D}_k \cap \widetilde{\mathcal{B}}^*_{h-2},$$

since the sum of the first $h-3$ terms is $(h^2-3h)/2 + \sum_{s=1}^{h-4} i_s$. Finally, in the case when

$$\lambda = \left(\frac{n + h^2 - 3h - 3}{2} + i, n - h - i_1, \ldots, n - 5 - i_{h-4}, n - 4, n - 3\right) \in \widetilde{\mathcal{A}}_h, \tag{4.7}$$

we obtain

$$\tau(\lambda) = \left(1, 2, 3 + i_{h-4}, \ldots, h - 2 + i_1, k - 1 - \frac{h^2 - 3h}{2} - i\right) \in \mathcal{D}_k \cap \widetilde{\mathcal{A}}^*_{h-1}, \tag{4.8}$$

noticing that the sum of the first $h-2$ terms is $(h^2-3h)/2 + 1 + \sum_{s=1}^{h-4} i_s$.

We proved that τ is well defined. Notice also that τ is trivially injective. Therefore it remains to prove that τ is surjective. In particular, it suffices to check that for each partition $\lambda^* \in (\widetilde{\mathcal{A}}^* \dot\cup \widetilde{\mathcal{B}}^* \dot\cup \widetilde{\mathcal{C}}^* \dot\cup \widetilde{\mathcal{D}}^*) \cap \mathcal{D}_k$, $\lambda^* \neq (3, k-3)$, there exists $\lambda \in (\widetilde{\mathcal{A}} \dot\cup \widetilde{\mathcal{B}} \dot\cup \widetilde{\mathcal{C}} \dot\cup \widetilde{\mathcal{D}}) \cap \widetilde{\mathcal{U}}_{T_{2k-1}}$ such that $\tau(\lambda) = \lambda^*$, since $\tau(\widetilde{\pi}_n) = (3, k-3)$ by definition. Given $\lambda^* = (\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_t) \in \mathcal{D}_k$, by the definition of τ we have that the partition λ denoted by its missing parts $(n-2-\lambda^*_t, \ldots, n-2-\lambda^*_2, n-2-\lambda^*_1)$ is such that $\tau(\lambda) = \lambda^*$. It remains to prove that such λ is a maximal unrefinable partition of n. The full details of the proof are here omitted since they can be obtained by arguments very similar to those used for proving that τ is well defined. As an example, let us consider the case when $\lambda^* \in \widetilde{\mathcal{A}}^*_t \cap \mathcal{D}_k$, for $t \geq 5$, and let us prove that λ^* is the image of an unrefinable partition λ of class $\widetilde{\mathcal{A}}$. Since λ^* is a partition of k into t distinct parts and contains 1 and 2 by definition we can write

$$\lambda^* = \left(1, 2, 3 + i_1, \ldots, t - 1 + i_{t-3}, k - \sum_{s=1}^{t-1} \lambda^*_s\right), \tag{4.9}$$

where

$$\sum_{s=1}^{t-1} \lambda^*_s = \frac{(t-1)t}{2} + \sum_{s=1}^{t-3} i_s$$

for some $i_1, i_2, \ldots, i_{t-3} \geq 0$ (cf. also Equation (4.8)). We can now substitute $t-1$ to $h-2$ and $(n+1)/2$ to k in Equation (4.9). Applying the correspondence $\lambda_i \leftrightarrow n - 2 - \lambda^*_{t-i+1}$ and denoting the obtained partition λ by listing its missing parts, we obtain

$$\lambda = \left(\frac{n + h^2 - 3h - 3}{2} + i, n - h - i_{h-4}, \ldots, n - 5 - i_1, n - 4, n - 3\right)$$

as in Equation (4.7). This proves that $\lambda \in \widetilde{\mathcal{U}}_{T_{2k-1}} \cap \widetilde{\mathcal{A}}_{t+1}$ is unrefinable (cf. Proposition 4.11 and Remark 4.13) and such that $\tau(\lambda) = \lambda^*$. The remaining cases are similar. □

Remark 4.18. *The bijection τ is not well defined when $k < 7$. However, it can be easily shown that the result of Theorem 4.17 is still valid when $k = 4$ and $k = 5$, where we have $\#\widetilde{\mathcal{U}}_{T_7} = \#\mathcal{D}_4 = 1$ and $\#\widetilde{\mathcal{U}}_{T_9} = \#\mathcal{D}_5 = 2$, respectively. The claim is false instead in the case $k = 6$, where we have $\#\widetilde{\mathcal{U}}_{T_{11}} = 4$ and $\#\mathcal{D}_6 = 3$.*

Corollary 4.19. *Let $\tilde{u}(i) = \#\widetilde{\mathcal{U}}_{T_i}$ and let $q(i)$ be the i-th coefficient of the polynomial $\prod_{j\geq 1}(1 + x^j)$. We obtain*

$$\tilde{u}(2k-1) = q(2k-1) - 1$$

Proof. It is a direct consequence of Theorem 4.17 and Equation 1.3. $\square$

In the proof of Theorem 4.17 we showed that τ is a bijection from $\widetilde{\mathcal{U}}_{T_{2k-1}}$ to $\mathcal{D}_k$. Moreover, we also proved that τ is bijective when it is restricted to each class.

Corollary 4.20. *The function τ of Theorem 4.17 maps in a bijective way*

(i) $\widetilde{\mathcal{A}}_h \cap \widetilde{\mathcal{U}}_{T_{2k-1}}$ *to* $\widetilde{\mathcal{A}}^*_{h-1} \cap \mathcal{D}_k$,

(ii) $\widetilde{\mathcal{B}}_h \cap \widetilde{\mathcal{U}}_{T_{2k-1}}$ *to* $\widetilde{\mathcal{B}}^*_{h-2} \cap \mathcal{D}_k$,

(iii) $\widetilde{\mathcal{C}}_h \cap \widetilde{\mathcal{U}}_{T_{2k-1}}$ *to* $\widetilde{\mathcal{C}}^*_{h-2} \cap \mathcal{D}_k$,

(iv) $\widetilde{\mathcal{D}}_h \cap \widetilde{\mathcal{U}}_{T_{2k-1}}$ *to* $\left(\widetilde{\mathcal{D}}^*_{h-3} \setminus \{(3, k-3)\}\right) \cap \mathcal{D}_k$.

Example 4.21. *Coming back to the case of Example 4.7, we represent in Table 4.4 the bijection τ between maximal unrefinable partitions of 13 obtained in the case $h = j - 2$ (hence those different from $\widetilde{\pi_{13}}$), represented by black dots, and the partitions of 7 into distinct parts, represented by $\star$. Notice that the partition $(3, 4)$ is not displayed since it corresponds to $\widetilde{\pi_{13}}$. Here x corresponds to*

$$x = \min_{\lambda\in\widetilde{\mathcal{U}}_{T_{13}}} mex\,(\lambda)\,.$$

Equivalently, by the anti-symmetric property, partitions of 7 into distinct parts can be read looking at the black dots on the right side of the table.

				x						$n-2$		n									λ_t
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
				6	5	4	3	2	1												
•	•	•	•	•	•	◦	•	◦	◦	◦	•	•	◦	•	◦	◦	◦	◦	◦	◦	•
•	•	•	•	•	◦	•	•	◦	•	◦	◦	•	◦	◦	•	◦	◦	◦	◦	◦	•
•	•	•	•	◦	•	•	•	•	◦	◦	•	◦	◦	◦	◦	•	◦	◦	◦	◦	•

Table 4.4: The bijection τ shown on the partitions of $\lambda \in \widetilde{\mathcal{U}}_{T_{13}}$, $\lambda \neq \widetilde{\pi_{13}}$. Like in Table 4.1 we indicate with $\bullet$ the integers that are in the partition and with $\circ$ the missing parts. The function τ acts on the missing parts signed by $\circ$
.

class	$f(n,h)$	$g(n,h)$
$\widetilde{\mathcal{A}}_h$	$\frac{-h^3+6h^2+(n-8)h-4n+2}{2}$	$\frac{n-h^2+3h-5}{2}$
$\widetilde{\mathcal{B}}_h$	$\frac{-h^3+6h^2+(n-6)h-4n-6}{2}$	$\frac{n-h^2+3h-3}{2}$
$\widetilde{\mathcal{C}}_h$	$\frac{-h^3+6h^2+(n-4)h-4n-14}{2}$	$\frac{n-h^2+3h-1}{2}$
$\widetilde{\mathcal{D}}_h$	$\frac{-h^3+6h^2+(n-2)h-4n-22}{2}$	$\frac{n-h^2+3h+1}{2}$

Table 4.5: The values of $f(n,h)$ and $g(n,h)$ for each class.

Remark 4.22. *Another combinatorial equality can be derived from the provided construction for $\widetilde{\mathcal{U}}_N$. Indeed, assuming $n = 2k-1$ for $k \geq 7$, $h \geq 6$, and reasoning as in Example4.14, it can be easily shown that $\#\left(\widetilde{\mathcal{U}}_{T_n} \cap \widetilde{\mathcal{D}}_h\right)$ equals the number of partitions in $h-3$ parts of $f(n,h)$ in which each part is smaller than or equal to $g(n,h)$, where*

$$f(n,h) = \frac{-h^3 + 6h^2 + (n-2)h - 4n - 22}{2} \quad \text{and} \quad g(n,h) = \frac{n - h^2 + 3h + 1}{2}.$$

The proof is obtained from Proposition 4.11, considering the bijection

$$a_i \leftrightarrow a_i - a_1 + 1. \tag{4.10}$$

In Table 4.5 the result is summarised for each class. Notice that, using the bijection of Equation (4.10) on the partitions shown in Example 4.14, one can recover the eleven partitions of 31 in 4 parts, where each part is not larger than 11.

Chapter 5

Classification of maximal unrefinable partitions in general case

In this chapter we complete the classification of maximal unrefinable partitions, extending the previous result to the case of non-triangular numbers. If $N \in \mathbb{N}$ is non-triangular, then it is uniquely determined a pair (n, d) where $n \in \mathbb{N}$ and $1 \leq d \leq n-1$ such that $N = T_n - d$, and we denote such integer N by $T_{n,d}$.
We proceed in the same way of Chapter 4. First of all we show the bound that λ_t reach in Section 5.1. The bounds are obtained constructively, i.e., we show actual partitions which reach the bounds. Such constructions are then extended in Section 5.2 in a complete classification of maximal unrefinable partitions reaching the corresponding bounds. With similar arguments but slightly different computations, we address the cases $\lambda_t \leq 2n-4$ and $\lambda_t \leq 2n-5$ in two separate subsections, i.e., respectively in Subsection 5.2.1 and in Subsection 5.2.2. In particular, the already mentioned counting result proved by a bijective argument can be read in Theorem 5.28 and in Theorem 5.39. As in the case of triangular numbers, maximal unrefinable partitions can be expressed in terms of suitable partitions into distinct parts, for sufficiently large n. We will show the following result.

5.1 Upper bounds

Proposition 5.1. *Let $N \in \mathbb{N}$ be such that $T_{n-1} < N < T_n$ for some $n \in \mathbb{N}$. For every unrefinable partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ of N we have*

$$n \leq \lambda_t \leq 2n-2. \tag{5.1}$$

Equivalently,

$$\frac{\sqrt{1+8(N+d)}-1}{2} \le \lambda_t \le \sqrt{1+8(N+d)}-3,$$

where $d = T_n - N$.

Proof. Let us start by considering d and the partition $\pi_{n,d} \vdash N$. Other partitions of N are obtained from $\pi_{n,d}$ by removing some parts smaller than or equal to n which are replaced by d and other parts larger than n or only by other parts larger than n. Proceeding as in the proof of Proposition 4.1, let $h, j \in \mathbb{N}$ and let us denote by $1 \le a_1 < a_2 < \cdots < a_h \le n$ the candidate parts to be removed from $\pi_{n,d}$ to obtain a new partition of N, and by $\alpha_1 < \alpha_2 < \cdots < \alpha_j$ the corresponding replacements. Since $\sum a_i = \sum \alpha_i$ we have $h \ge j > 1$, and we may obtain $h = j$ only if $\alpha_1 = d$. For this reason, we need to consider the two cases separately.

Let us assume $\alpha_i > n$, for every $1 \le i \le j$. Reasoning as in the proof of Proposition 4.1 we can count $m = (h+1) + \alpha_j - n - j$. On the other hand, if $\alpha_1 = d$ and $\alpha_i > n$ for every $2 \le i \le j$, then we obtain just h missing parts in the interval $\{1, 2, \ldots, n\}$ and exactly $j-1$ parts appear in the interval $\{n+1, n+2, \ldots, \alpha_j\}$, therefore we obtain the same formula for the number of missing parts $m = h + \alpha_j - n - (j-1)$. By Lemma 3.6 we obtain

$$h + \left\lceil \frac{\alpha_j}{2} \right\rceil - n - j + 1 \le 0. \tag{5.2}$$

If $\alpha_j > 2n-2$, then $\lceil \alpha_j/2 \rceil \ge n$ and from Equation (5.2) we obtain $h \le j-1$, a contradiction. □

From now on, let us assume $n \ge 11$. In this section we show that the bound for the largest part in an unrefinable partition for a non-triangular number depends on the parity of the distance from the index of the successive triangular number. To do this, we start from $\pi_{n,d}$, a privileged partition.
In order to construct other partitions of $T_{n,d}$, we will proceed as follows: starting from $\pi_{n,d} \in T_{n,d}$, we create a new partition λ by removing from $\pi_{n,d}$ some of its parts, namely $a_1, a_2, \ldots, a_h \in \{1, 2, \ldots, d-1, d+1, \ldots, n\}$, and, at the same time, by adding to λ new parts $\alpha_1, \alpha_2, \ldots, \alpha_j \in \{d\} \cup \{s \mid s \ge n+1\}$, for some positive integers h and j. This leads to the creation of a partition λ that, provided $j \le h$ and

$$\sum_{i=1}^{h} a_i = \sum_{i=1}^{j} \alpha_i,$$

is such that $\lambda \vdash T_{n,d}$, and which, in general, may not be unrefinable. This notation will be used in the reminder of the paper and this strategy, in addition to further unrefinability checks, will lead to the classification of $\widetilde{\mathcal{U}}_{T_{n,d}}$.

Notice that when λ is an unrefinable partition of $\widetilde{\mathcal{U}}_{T_{n,d}}$, the missing part d in $\pi_{n,d}$ can be either one of the *replacements* α_is or not. We will show in Proposition 5.2 that, depending on this, we will obtain two different bounds.

Proposition 5.2. *Let $N = T_{n,d}$ with $n \in \mathbb{N}$ and $1 \leq d \leq n-1$, and let $\lambda = (\lambda_1 \dots, \lambda_t) \in \widetilde{\mathcal{U}}_N$. Then*

$$\lambda_t \leq \begin{cases} 2n-2 & d \in \lambda, \\ 2n-4 & d \notin \lambda. \end{cases}$$

Proof. Let us first assume that $d \notin \lambda$, then the number of missing parts of λ is $h+1+(\lambda_t - n - j)$. From Lemma 3.6, we have $h+1+\lambda_t - n - j \leq \lfloor \lambda_t/2 \rfloor$ and so

$$\begin{aligned} h &\leq \lfloor \lambda_t/2 \rfloor - \lambda_t + n + j - 1 \\ &= n + j - 1 - \lceil \lambda_t/2 \rceil. \end{aligned} \tag{5.3}$$

If $d \notin \lambda$, then we have $j < h$ and so $n - 1 - \lceil \lambda_t/2 \rceil \geq 1$, i.e., $\lambda_t \leq 2n-4$. If we assume $d \in \lambda$, from Equation (5.3) and from $j \leq h$ we have $n - 1 - \lceil \lambda_t/2 \rceil \geq 0$, i.e., $\lambda_t \leq 2n-2$. □

Remark 5.3. *First notice that $1 \leq a_1 < a_2 < \cdots < a_h \leq n$, $d < n$ and $n+1 \leq \alpha_i \leq \lambda_t$ for each $1 \leq i \leq j$ such that $\alpha_i \neq d$, hence, since $\sum a_i = \sum \alpha_i$, if $d \in \lambda$ then $j \leq h$, otherwise $j < h$. In particular if $\lambda_t \in \{2n-3, 2n-2\}$, which is by Proposition 5.2 only possible when $d \in \lambda$, we have $j \leq h$. Moreover, from Equation (5.3) we have $h \leq j + n - 1 - \lceil \lambda_t/2 \rceil = j$. In other words, if $\lambda_t \in \{2n-3, 2n-2\}$, then $h = j$. In the case when $\lambda_t = 2n-4$, if $d \in \lambda$ we have $j \leq h$ and, from Equation (5.3), $h \leq j+1$, and so $h \in \{j, j+1\}$; instead if $d \notin \lambda$, since $j < h$, we obtain $h = j+1$.*

Remark 5.4 (Anti-symmetry)**.** *Let $\lambda = (\lambda_1, \dots, \lambda_t) \vdash N$ be unrefinable. Notice that if an integer x in $\{1, \dots, \lambda_t - n - 1\}$ is such that $x \notin \lambda$, then it corresponds to an element $x' = \lambda_t - x \in \{n+1, \dots, \lambda_t - 1\}$ such that $x' \in \lambda$, otherwise $x + x' = \lambda_t$ and λ is refinable. Therefore, the parts of λ can belong to three consecutive* areas *of $\{1, 2, \dots, \lambda_t - 1\}$, as shown in Figure 5.1. We call*

- *the first area the set $\{s \in \mathbb{N} \mid 1 \leq s \leq \lambda_t - n - 1\}$,*
- *the free area the set $\{s \in \mathbb{N} \mid \lambda_t - n \leq s \leq n\}$,*
- *the last area the set $\{s \in \mathbb{N} \mid n+1 \leq s < \lambda_t\}$.*

Choosing elements in the first area implies fixing parts in the last one. For this reason, if we consider $\pi_{n,d}$ and if we obtain a new unrefinable partition $\lambda \vdash T_{n,d}$ from $\pi_{n,d}$ removing $a_1, a_2, \dots, a_h \leq n$ and replacing them with $\alpha_1, \alpha_2, \dots, \alpha_j$, then each a_i in the first area determines $\lambda_t - a_i \in \{\alpha_i\}_{i=1}^{j-1}$. Accordingly, we denote the element $\lambda_t - a_i$ by α_i. In particular $\lambda_t = \alpha_j$ and, when $d \in \lambda$, we denote d by α_{j-1}.

By Proposition 5.2 we know that if $\lambda \in \widetilde{\mathcal{U}}_{T_{n,d}}$, then $\lambda_t \leq 2n-2$. In the following sections, we will distinguish all the possible cases for λ_t and we will provide the corresponding constructions.

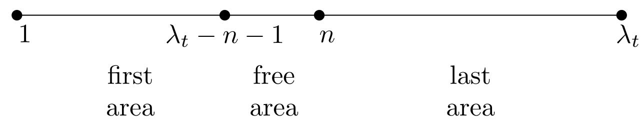


Figure 5.1: The three areas of the parts in an unrefinable partition

5.1.1 The case $\lambda_t = 2n - 2$

By virtue of Proposition 5.2 we know that if $\lambda \in \widetilde{\mathcal{U}}_{T_{n,d}}$, then $d \in \lambda$ implies $\lambda_t \leq 2n-2$. Let us now show that the bound is tight only for a single choice of d.

Proposition 5.5. *Let $\lambda = (\lambda_1, \ldots, \lambda_t) \in \widetilde{\mathcal{U}}_{T_{n,d}}$. If $\lambda_t = 2n-2$, then $d = 1$ and such partition is unique.*

Proof. From Proposition 5.2 we have that $\lambda_t = 2n-2$ implies $d \in \lambda$ and by Remark 5.3 we also know that $h = j \geq 2$. From the hypothesis $\lambda_t = 2n-2$ we obtain that the free area corresponds to the set $\{n-2, n-1, n\}$. By Remark 5.4 we have that $\alpha_{h-1} = d$ and $\alpha_h = 2n-2$ are fixed. Therefore, since $h = j$, the free area can contain two or three parts, but we must rule out the second option since it would violate unrefinability. We are then only left with the case of two parts chosen among $\{(n-2, n-1), (n-1, n)\}$. The case $(n-2, n)$ is not considered since $n-2+n = 2n-2$ and λ is unrefinable. Let us distinguish all the possible cases for h.

Let $h = 2$. Since α_1 and α_2 are already fixed, we have that a_1 and a_2 are free elements. From $\sum a_i = \sum \alpha_i$ we have that either

$$(n-2) + (n-1) = d + 2n - 2 \quad \text{or}$$
$$(n-1) + n = d + 2n - 2.$$

From the first equation we obtain $d = -1$, a contradiction. From the second one we obtain $d = 1$, as claimed. Indeed the obtained partition $\lambda = (1, 2, \ldots, n-2, 2n-2)$ is unrefinable since the sum of the first two missing parts $n-1$ and n is larger than $\lambda_t = 2n-2$. Let us now prove that the remaining cases lead to contradictions.

Let $h = 3$. From the hypothesis and from Remark 5.4, we have $\alpha_3 = 2n-2$, $\alpha_2 = d$, $\alpha_1 = 2n-2-a_1$ is determined by the choice of a_1 and $(a_2, a_3) \in \{(n-2, n-1), (n-1, n)\}$. Let us assume that $(a_2, a_3) = (n-1, n)$. Then, by $\sum a_i = \sum \alpha_i$, we obtain

$$a_1 + n - 1 + n = 2n - 2 - a_1 + d + 2n - 2,$$

from which

$$a_1 = \frac{2n-3+d}{2}.$$

By checking if $a_1 \leq n-3$ as it should be, we determine a contradiction on d. The other option for (a_2, a_3) corresponds to a larger value for a_1, even more so a contradiction.

Let $h \geq 4$. We are assuming $\alpha_h = 2n-2$, $\alpha_{h-1} = d$, $(a_{h-1}, a_h) \in \{(n-2, n-1), (n-1, n)\}$ and $1 \leq a_1 < a_2 \cdots < a_{h-2} \leq n-3$, which determine $\alpha_1, \alpha_2, \ldots, \alpha_{h-2}$ as $\alpha_i = 2n-2-a_i$. From $\sum a_i = \sum \alpha_i$ we obtain

$$a_1 + a_2 + \cdots + a_{h-2} = \frac{(h-1)(2n-2) + d - (a_{h-1} + a_h)}{2}.$$

Proceeding as in the previous case, we can choose to maximise $a_2 + \cdots + a_h$ by setting $a_{h-2} = n-3, a_{h-3} = n-4, \ldots, a_2 = n-h+1$ and $a_{h-1} = n-1$, $a_h = n$. From this we obtain

$$a_1 = \frac{2n + h^2 - 3h - 3 + d}{2}.$$

Checking $a_1 < a_2 = n-h+1$, we obtain $h^2 - h - 5 + d < 0$, which is impossible in the current setting where $d > 0$ and $h \geq 4$. Notice that the choice of maximising $a_2 + \cdots + a_h$ leads to the minimum value for a_1. Any other choice of $a_2, \ldots, a_h$ would lead to a contradiction even more so. □

Example 5.6. *Let* $\lambda = (1, 2, 3, 4, 5, 6, 7, 16)$*. We can observe that* $\lambda \in \mathcal{U}$*, due to the sum of the two smallest missing parts* $8+9 > 16$*. Furthermore* $\lambda \vdash 44 = T_{9,1}$ *and the largest part is* $16 = 2 \cdot 9 - 2$*, hence* $\lambda \in \widetilde{\mathcal{U}}_{T_{9,1}}$*.*
We can notice that we obtain the partition λ *from the partition* $\pi_{9,1} = (2, 3, 4, 5, 6, 7, 8, 9)$ *removing the* 8 *and* 9 *and adding* 1 *and* 16*.*

Notice that in the previous proof only one construction was successful. Therefore, the following consequence is trivially obtained.

Corollary 5.7. $\#\widetilde{\mathcal{U}}_{T_{n,1}} = 1$.

In the following sections we will investigate the remaining possibilities for λ_t. Notice that we will mimic the arguments of Proposition 5.5. As before, given the value of λ_t, we will determine the free area and the number of elements that can be chosen in the free area. Then we will attempt to construct partitions for each possible value of h. In the general case we will derive the conclusion starting from the choice which maximises the sum of the values assigned to $a_2, a_3, \ldots, a_h$, and minimises a_1. We will use this strategy also in the following proofs, without further mention.

5.1.2 The case $\lambda_t = 2n - 3$

Proposition 5.8. *Let* $\lambda = (\lambda_1, \ldots, \lambda_t) \in \widetilde{\mathcal{U}}_{T_{n,d}}$*. If* $\lambda_t = 2n-3$*, then* $d = 2$ *and such partition is unique.*

Proof. From the hypothesis and from Remark 5.3 we obtain that $d \in \lambda$ and $h = j$. The free parts are those belonging to $\{n-3, \ldots, n\}$. We have already fixed two of the α_is and it is not possible to choose more that two parts in the free area without reaching a contradiction on the unrefinability of λ. Therefore, we are left with the case of two free parts and $h-2$ parts in the first area to be determined. Only four conditions on (a_{h-1}, a_h) do not contradict the unrefinability on $2n-3$, namely

$$(a_{h-1}, a_h) \in \{(n-3, n-2), (n-3, n-1), (n-2, n), (n-1, n)\}.$$

Let us distinguish the possible cases for h.

Let $h = 2$. From $\sum a_i = \sum \alpha_i$ we obtain four equations

$$\begin{aligned}(n-3) + (n-2) &= 2n-5 = d + 2n - 3,\\ (n-3) + (n-1) &= 2n-4 = d + 2n - 3,\\ (n-2) + n &= 2n-2 = d + 2n - 3,\\ (n-1) + n &= 2n-1 = d + 2n - 3.\end{aligned}$$

From the first two equations we obtain the contradiction of d being a negative integer. From the third equation we obtain $d = 1$, which means that the partition is not maximal (cf. Proposition 5.5). From the last one we obtain $d = 2$, as claimed. Notice that the obtained partition $\lambda = (1, 2, \ldots, n-2, 2n-3)$ is unrefinable since the sum of the least missing parts $n-1$ and n is larger than $2n-3$. In the remainder of the proof we will show that the remaining cases lead to contradictions.

Let $h = 3$. In the current setting we have $\alpha_2 = d$, $\alpha_3 = 2n-3$, $(a_2, a_3) \in \{(n-3, n-2), (n-3, n-1), (n-2, n), (n-1, n)\}$, $1 \le a_1 \le n-4$ and $\alpha_1 = 2n-3-a_1$. Proceeding as usual, let us consider the case where $a_2 + \cdots + a_h$ is maximal, which corresponds to the choice $a_2 = n-1$ and $a_3 = n$. From $\sum a_i = \sum \alpha_i$ we obtain

$$a_1 = \frac{2n-5+d}{2},$$

and checking if $a_1 \le n-4$ we obtain a contradiction on d.

Let $h \ge 4$. Maximising $a_2 + \cdots + a_h$, i.e., setting $a_{h-2} = n-4, a_{h-3} = n-5, \ldots, a_2 = n-h$, $a_{h-1} = n-1$ and $a_h = n$, from $\sum a_i = \sum \alpha_i$ we have

$$a_1 = \frac{2n + h^2 - 2h - 8 + d}{2}.$$

Imposing $a_1 < a_2 = n-h$ leads to a contradiction. □

Example 5.9. *Let $\lambda = (1, 2, 3, 4, 5, 6, 7, 8, 17)$. We can observe that $\lambda \in \mathcal{U}$, due to the sum of the two smallest missing parts 9 and 10 is bigger than 17. Furthermore $\lambda \vdash 53 = T_{10,2}$ and the largest part is $17 = 2 \cdot 10 - 3$, hence $\lambda \in \widetilde{\mathcal{U}}_{T_{10,2}}$.*
We can notice that we obtain the partition λ from the partition $\pi_{10,2} = (1, 3, 4, 5, 6, 7, 8, 9, 10)$ removing the 9 and 10 and adding 2 and 17.

Corollary 5.10. $\#\widetilde{\mathcal{U}}_{T_{n,2}} = 1$.

5.1.3 The case $\lambda_t = 2n-4$

In this case, by Proposition 5.2, we have to consider both cases $d \notin \lambda$ and $d \in \lambda$. Let us start by showing that the first assumption gives only one contribution.

Proposition 5.11. *Let* $\lambda = (\lambda_1, \ldots, \lambda_t) \in \widetilde{\mathcal{U}}_{T_{n,d}}$ *be such that* $d \notin \lambda$*. If* $\lambda_t = 2n-4$*, then* $d = n-5$ *and such partition is unique.*

Proof. We derive the claim by proving the following two statements:

1. if $d \leq n-5$, then $d = n-5$ and there exists only one partition;
2. no partition exists if $n-4 \leq d < n$.

Let us now prove each claim separately.

1. If $\lambda_t = 2n-4$ the free area is $\{n-4, \ldots, n\}$ and we have, by Remark 5.3, that $h = j+1$. Moreover, from the fact that $d \notin \lambda$ and $d \leq n-5$, or in other words d is outside the free area, we must have $\lambda_t - d \in \lambda$ since λ is unrefinable. Hence we are left with $j-2$ parts in the last area to be determined. Now, choosing four parts in the free area would contradict the unrefinability of λ. We also obtain a contradiction choosing less than two parts in the free area, i.e., more than $h-2 = j-1$ parts in the first area. We conclude we can only choose three parts in the free area. In particular we have only four possible cases, i.e., $(a_{h-2}, a_{h-1}, a_h) \in \{(n-4, n-3, n-2), (n-4, n-2, n-1), (n-3, n-2, n), (n-2, n-1, n)\}$.

   Let $h = 3$. From $\sum a_i = \sum \alpha_i$ we obtain four equations by the four possible options in the free area:

$$\begin{aligned} 3n-9 &= 4n-8-d, \\ 3n-7 &= 4n-8-d, \\ 3n-5 &= 4n-8-d, \\ 3n-3 &= 4n-8-d. \end{aligned}$$

   The first three equations lead to a contradiction on d while from the last one we obtain $d = n-5$, corresponding to the partition

$$\lambda = (1, 2, \ldots, n-6, n-4, n-3, n+1, 2n-4)$$

   which is unrefinable since, by hypothesis, we have $n \geq 11$.

   Let $h = 4$. As usual, maximising $a_2 + a_3 + a_4$, from $\sum a_i = \sum \alpha_i$ we determine

$$a_1 = \frac{3n-9-d}{2}.$$

   Imposing $a_1 < n-4$ we obtain $d > n-1$, a contradiction.

   Let $h \geq 5$. Maximising $a_2 + a_2 + \cdots + a_h$, from $\sum a_i = \sum \alpha_i$ we obtain

$$a_1 = \frac{3n + h^2 - 3h - 13 - d}{2},$$

being meaningful when $a_1 < a_2 = n - h$, from which we obtain $n - d + (h^2 - h - 13) < 0$, a contradiction if $h \geq 5$.

2. Notice that, since $d \notin \lambda$ and $n - 4 \leq d < n$, we can only choose a_{h-1} and a_h in the free area, being the third spot occupied already by d. From $\sum a_i = \sum \alpha_i$, in this case we have

$$a_1 + a_2 + \cdots + a_{h-2} = \frac{(h-1)(2n-4) - (a_{h-1} + a_h)}{2}. \tag{5.4}$$

Let us now examine each possible choice of d. If $d = n-4$ or $d = n-2$, then $a_{h-1} + a_h$ is odd, therefore Equation (5.4) cannot be satisfied. Let us now assume that $d = n-3$. In this case $(a_{h-1}, a_h) \in \{(n-4, n-2), (n-2, n)\}$.

Let $h = 3$. Maximising $a_2 + a_3$, we can calculate

$$a_1 = \frac{4n - 8 - (a_2 + a_3)}{2},$$

and so $a_1 > n - 5$, a contradiction.

Let $h \geq 4$. Maximising $a_2 + a_3 + \cdots + a_h$, from $\sum a_i = \sum \alpha_i$ we obtain

$$a_1 = \frac{2n + h^2 - h - 12}{2}.$$

Checking if $a_1 < a_2 = n - h - 1$, we derive that $h^2 + h - 10 < 0$, a contradiction. The same contradiction is obtained when $d = n - 1$. □

Let us address the remaining case $d \in \lambda$. Recall that, in this case, by Remark 5.3 we have $h \in \{j, j+1\}$.

Proposition 5.12. *Let $\lambda = (\lambda_1, \ldots, \lambda_t) \in \widetilde{\mathcal{U}}_{T_{n,d}}$ be such that $\lambda_t = 2n - 4$ and $d \in \lambda$. If $h = j$, then $d = 3$ and such partition is unique. If $h = j + 1$, then for each $1 \leq k \leq \lfloor (n-2)/2 \rfloor$ there exists $\lambda \in \widetilde{\mathcal{U}}_{T_{n,d}}$ with $d = n - (2k-1)$ and there not exists $\lambda \in \widetilde{\mathcal{U}}_{T_{n,d}}$ with $d = n - 2k$.*

Proof. Let us assume that $h = j$. Since, by Remark 5.4, $\alpha_{h-1} = d$ and $\alpha_h = 2n - 4$ are already fixed, then a_{h-1} and a_h are parts of the free area. Notice that the free area cannot contain more than three parts.

Let us first assume that it only contains two parts, i.e., $(a_{h-1}, a_h) \in \{(n-4, n-3), (n-4, n-2), (n-4, n-1), (n-3, n-2), (n-3, n), (n-2, n-1), (n-2, n), (n-1, n)\}$.

Let $h = 2$. We must have $a_1 + a_2 = 2n - 4 + d$. In the case when $(a_1, a_2) = (n-1, n)$ we obtain $d = 3$. which is the claim, since the corresponding partition $\lambda = (1, 2, \ldots, n-2, 2n-4)$ is unrefinable. If $(a_1, a_2) = (n-2, n-1)$ or $(n-2, n)$, we respectively obtain $d = 1$ and $d = 2$, which, by Proposition 5.5 and Proposition 5.8, contradicts the maximality of λ. In the remaining cases, we obtain $d \leq 0$ which is a contradiction.

Let $h = 3$. From $\sum a_i = \sum \alpha_i$, we have

$$a_1 = \frac{4n - 8 - (a_1 + a_2) + d}{2}.$$

Considering the maximal choice $(a_2, a_3) = (n-1, n)$ we obtain $a_1 = (2n - 7 + d)/2 < n - 4$ when $d < -1$, a contradiction.

Let $h \geq 4$. From $\sum a_i = \sum \alpha_i$ we obtain

$$a_1 + a_2 + \cdots + a_{h-2} = \frac{(h-1)(2n-4) - (a_{h-1} + a_h) + d}{2}.$$

Maximising $a_2 + \ldots + a_h$, we have

$$a_1 = \frac{2n + h^2 - h - 13 + d}{2},$$

which satisfies $a_1 < a_2$ when $h^2 + h - 11 + d < 0$, a contradiction when $h \geq 4$.

Under the assumption that $h = j$, it remains to consider the case of three parts in the free area, i.e., $(a_{h-2}, a_{h-1}, a_h) \in \{(n-4, n-3, n-2), (n-4, n-2, n-1), (n-3, n-2, n), (n-2, n-1, n)\}$. In this case $a_1, a_2, \ldots, a_{h-3}$ determine $\alpha_1, \alpha_2, \ldots, \alpha_{h-3=j-3}$, while the part $n+1 \leq \alpha_{j-2} \leq 2n-5$ is not determined by one of the α_is.

Let $h = 3$. We have $a_1 + a_2 + a_3 = 2n - 4 + d + \alpha_1$, with $n + 1 \leq \alpha_1 \leq 2n - 5$ and each possible choice of the parts in the free area implies that $\alpha_1 < n + 1$.

Let $h = 4$. From $\sum a_i = \sum \alpha_i$ we have

$$a_1 = \frac{2(2n-4) - (a_2 + a_3 + a_4) + d + \alpha_2}{2},$$

and maximising $a_2 + a_3 + a_4$ we obtain $a_1 = (n - 5 + d + \alpha_2)/2$. Since $a_1 < n - 4$, then $\alpha_2 < n - 3 - d < n - 3$, a contradiction.

Let $h \geq 5$. From $\sum a_i = \sum \alpha_i$ we have

$$a_1 + a_2 + \cdots + a_{h-3} = \frac{(h-2)(2n-4) - (a_{h-2} + a_{h-1} + a_h) + d + \alpha_{h-2}}{2}.$$

From the maximal choice of $a_2 + a_3 + \cdots + a_h$, we obtain

$$a_1 = \frac{n + h^2 - 3h - 9 + d + \alpha_{h-2}}{2},$$

and checking if $a_1 < a_2 = n - h$ leads to $(\alpha_{h-2} - n) + (h^2 - h - 9) + d < 0$ which is not compatible with $h \geq 5$.

This concludes the case $h = j$.

Let us now address the remaining case $h = j + 1$. In this setting we have only three parts in the free area and, as before, the possible choices are the following triple of elements

$$(a_{h-2}, a_{h-1}, a_h) \in \{(n-4, n-3, n-2), (n-4, n-2, n-1), (n-3, n-2, n), (n-2, n-1, n)\}. \tag{5.5}$$

Let $h = 3$. We have $a_1 + a_2 + a_3 = 2n - 4 + d$. In the case $(a_1, a_2, a_3) = (n-2, n-1, n)$ we obtain $d = n+1$, a contradiction. In the other three cases we obtain d equals $n-1$, $n-3$ and $n-5$, or, in other words, $d = n - (2k-1)$ for $1 \leq k \leq 3$ as claimed. The remaining cases $3 \leq d \leq n-7$ are considered by showing partitions obtained in the case $h = 4$.

Let $h = 4$. We have

$$a_1 = \frac{2(2n-4) - (a_2 + a_3 + a_4) + d}{2}.$$

Notice that, for each choice of (a_2, a_3, a_4), we have $a_2 + a_3 + a_4 = 3n - (2t+1)$, for some $t \geq 0$. Therefore, since a_1 is an integer, n is even if and only if d is odd. Precisely, $d = n - (2k-1)$ for some $1 \leq k \leq \lfloor (n-2)/2 \rfloor$ (recall that, by Proposition 5.5 and Proposition 5.8, the cases $d = 1, 2$ are not maximal when $\lambda_t = 2n-4$). To prove that for all $3 \leq d \leq n-7$ there exists $\lambda \in \widetilde{\mathcal{U}}_{T_{n,d}}$, consider, for example, the assignment $(a_2, a_3, a_4) = (n-3, n-2, n)$. In this case, from $\sum a_i = \sum \alpha_i$, we obtain $a_1 = (n-3+d)/2$ which satisfies $a_1 < n-4$ if and only if $d < n-5$ and the corresponding partition is unrefinable. Indeed, we can violate the refinability only if either $a_1 + a_2 = \alpha_1$ or $a_1 + a_3 = \alpha_1$ or $a_1 + a_4 = \alpha_1$, and this is only possible if $d \in \{-1, 1, 2\}$, a contradiction.

If $h \geq 5$, from $\sum a_i = \sum \alpha_i$ we have

$$a_1 + a_2 + \cdots + a_{h-3} = \frac{(h-2)(2n-4) - (a_{h-2} + a_{h-1} + a_h) + d}{2},$$

from which we obtain again that n is even if and only if d is odd, i.e., $d = n - (2k-1)$ for some positive integer k. $\square$

Corollary 5.13. *Let n be odd. Then $\#\widetilde{\mathcal{U}}_{T_{n,3}} = 1$.*

Proof. By Proposition 5.11, if $d \notin \lambda$ then there not exists any maximal unrefinable partition with $d = 3$. By Proposition 5.12, if $d \in \lambda$ then we have that, for $h = j+1$, n odd implies d even and, for $h = j$, there exists only one maximal unrefinable partition with $d = 3$. $\square$

5.1.4 The case $\lambda_t = 2n - 5$

Also in this case, by Proposition 5.2, we have to consider both cases $d \notin \lambda$ and $d \in \lambda$. Let us address the two cases separately.

Proposition 5.14. *Let $\lambda = (\lambda_1, \ldots, \lambda_t) \in \widetilde{\mathcal{U}}_{T_{n,d}}$ such that $d \notin \lambda$. If $\lambda_t = 2n-5$, then $d = n-6$ and such partition is unique.*

Proof. We derive the claim from proving the following two statements:

1. if $d \leq n-6$, then $d = n-6$ and there exists only one partition;

2. no partition exists if $n-5 \leq d < n$.

Let us now prove each claim separately.

1. If $\lambda_t = 2n-5$ the free area is $\{n-5,\dots,n\}$ and we have, by Remark 5.3, that $h = j+1$. Moreover, from the fact that $d \notin \lambda$ and $d \le n-6$, or in other words d is outside the free area, we must have $\lambda_t - d \in \lambda$ since λ is unrefinable. Hence we are left with $j-2 = h-3$ parts in the last area to be determined. Now, as already concluded in the case $\lambda_t = 2n-4$, we can only choose three parts in the free area. In particular we have only eight possible cases, i.e.,

$$(a_{h-2}, a_{h-1}, a_h) = \left\{ \begin{array}{l} (n-5, n-4, n-3), (n-5, n-4, n-2), \\ (n-5, n-3, n-1), (n-5, n-2, n-1), \\ (n-4, n-3, n), (n-4, n-2, n), \\ (n-3, n-1, n), (n-2, n-1, n) \end{array} \right\}.$$

Let $h = 3$. From $\sum a_i = \sum \alpha_i$ we obtain eight equations by the eight possible options in the free area

$$\begin{aligned} 4n-10-d &= 3n-3, \\ 4n-10-d &= 3n-4, \\ 4n-10-d &= 3n-6, \\ 4n-10-d &= 3n-7, \\ 4n-10-d &= 3n-8, \\ 4n-10-d &= 3n-9, \\ 4n-10-d &= 3n-11, \\ 4n-10-d &= 3n-12. \end{aligned}$$

In the first case we obtain $d = n-7$ which is a contradiction since if $\lambda \in \widetilde{\mathcal{U}}_{T_{n,n-7}}$ then $\lambda_t = 2n-4$. In the last six cases we have $d > n-5$ and so we obtain a contradiction. From the second one we obtain $d = n-6$, corresponding to the partition

$$\lambda = (1, \dots, n-7, n-5, n-4, n-2, n+1, 2n-5),$$

which is unrefinable for $n \ge 11$.

Let $h = 4$. As usual, maximising $a_2 + a_3 + a_4$, from $\sum a_i = \sum \alpha_i$ we determine

$$a_1 = \frac{3n-12-d}{2}.$$

Imposing $a_1 < n-5$ we obtain $n-2-d < 0$, a contradiction.

Let $h \ge 5$. Maximising $a_2 + a_2 + \dots + a_h$, from $\sum a_i = \sum \alpha_i$ we obtain

$$a_1 = \frac{3n + h^2 - 2h - 20 - d}{2},$$

being meaningful when $a_1 < a_2 = n-h-1$, from which we obtain

$$n - d + (h^2 - 18) < 0,$$

a contradiction if $h \ge 5$.

2. Notice that, since $d \notin \lambda$ and $n-5 \le d < n$, we can only choose a_{h-1} and a_h in the free area, being the third spot occupied already by d. From $\sum a_i = \sum \alpha_i$, in this case we have

$$a_1 + a_2 + \cdots + a_{h-2} = \frac{(h-1)(2n-5) - (a_{h-1} + a_h)}{2}. \tag{5.6}$$

We already know that if $d = n-(2k-1)$ and $\lambda \in \widetilde{\mathcal{U}}_{T_{n,d}}$, then $\lambda_t = 2n-4$, so we can suppose that $d \in \{n-4, n-2\}$.

First suppose that $d = n-4$. In this case

$$(a_{h-1}, a_h) \in \{(n-5, n-3), (n-5, n-2), (n-3, n), (n-2, n)\}.$$

Let $h = 3$. Maximising $a_2 + a_3$, we can calculate

$$a_1 = \frac{4n - 10 - (a_2 + a_3)}{2}, \tag{5.7}$$

so $a_2 + a_3$ must be an even number. Now if $a_2 + a_3 = 2n-2$, we obtain $a_1 = n-4$, a contradiction, and if the sum is $a_2 + a_3 = 2n-8$, we obtain $a_1 = n-1$, again a contradiction.

Let $h \ge 4$. Maximising $a_2 + a_3 + \cdots + a_h$, from $\sum a_i = \sum \alpha_i$ we obtain

$$a_1 = \frac{2n + h^2 - 17}{2}.$$

Checking if $a_1 < a_2 = n-h-2$, we derive that $h^2 + h - 13 < 0$, a contradiction if $h \ge 4$.

Now suppose that $d = n-2$. In this case

$$(a_{h-1}, a_h) \in \{(n-5, n-4), (n-5, n-1), (n-4, n), (n-1, n)\}.$$

Let $h = 3$. Maximising $a_2 + a_3$, from Equation (5.7) $a_2 + a_3$ must be an even number. Now if $a_2 + a_3 = 2n-4$, we obtain $a_1 = n-3$, a contradiction, and if the sum is $a_2 + a_3 = 2n-6$, we obtain $a_1 = n-2$, again a contradiction.

Let $h \ge 4$. Maximising $a_2 + a_3 + \cdots + a_h$, from $\sum a_i = \sum \alpha_i$ we obtain

$$a_1 = \frac{2n + h^2 - 18}{2}.$$

Checking if $a_1 < a_2 = n-h-2$, we derive that $h^2 + h - 14 < 0$, a contradiction if $h \ge 4$. □

Let us address the case $d \in \lambda$. Recall that, in this case, by Remark 5.3 we have $h \in \{j, j+1\}$.

Proposition 5.15. *Let $\lambda = (\lambda_1, \ldots, \lambda_t) \in \widetilde{\mathcal{U}}_{T_{n,d}}$ be such that $\lambda_t = 2n-5$ and $d \in \lambda$. If $h = j$, then $d = 4$, n is even and such partition is unique. If $h = j+1$, then for each $1 \leq k \leq \lfloor (n-4)/2 \rfloor$ there exists $\lambda \in \widetilde{\mathcal{U}}_{T_{n,d}}$ with $d = n - 2k$.*

Proof. Let us assume that $h = j$. Since, by Remark 5.4, $\alpha_{h-1} = d$ and $\alpha_h = 2n-5$ are already fixed, then a_{h-1} and a_h are parts of the free area. Notice that the free area cannot contain more than three parts.

Let us first assume that it only contains two parts, i.e.,

$$(a_{h-1}, a_h) \in \left\{ \begin{array}{l} (n-5, n-4), (n-5, n-3), \\ (n-5, n-2), (n-5, n-1), \\ (n-4, n-3), (n-4, n-2), \\ (n-4, n), (n-3, n-1), \\ (n-3, n), (n-2, n-1), \\ (n-2, n), (n-1, n) \end{array} \right\}.$$

Let $h = 2$. We must have $a_1 + a_2 = 2n - 5 + d$. In the case when $(a_1, a_2) = (n-1, n)$ we obtain $d = 4$. which is the claim, since the corresponding partition $\lambda = (1, 2, \ldots, n-2, 2n-5)$ is unrefinable. In all the other cases we obtain either $0 < d < 4$ which, by Proposition 5.5, Proposition 5.8 and Proposition 5.12, contradicts the maximality of λ, or $d \leq 0$ which is also a contradiction.

Let $h = 3$. From $\sum a_i = \sum \alpha_i$, we have

$$a_1 = \frac{4n - 10 - (a_1 + a_2) + d}{2}.$$

Considering the maximal choice $(a_2, a_3) = (n-1, n)$ we obtain $a_1 = (2n - 9 + d)/2 < n - 5$ when $d < -1$, a contradiction.

Let $h \leq 4$. From $\sum a_i = \sum \alpha_i$ we obtain

$$a_1 + a_2 + \cdots + a_{h-2} = \frac{(h-1)(2n-5) - (a_{h-1} + a_h) + d}{2}.$$

Maximising $a_2 + \ldots + a_h$, we have

$$a_1 = \frac{2n + h^2 - h - 18 + d}{2},$$

which satisfies $a_1 < a_2 = n - h - 2$ when $h^2 + h - 14 + d < 0$, a contradiction when $h \geq 4$.

Under the assumption that $h = j$, it remains to consider the case of three parts in the free area, i.e.,

$$(a_{h-2}, a_{h-1}, a_h) \in \left\{ \begin{array}{l} (n-5, n-4, n-3), (n-5, n-4, n-2), \\ (n-5, n-3, n-1), (n-5, n-2, n-1), \\ (n-4, n-3, n), (n-4, n-2, n), \\ (n-3, n-1, n), (n-2, n-1, n) \end{array} \right\}. \tag{5.8}$$

In this case $a_1, a_2, \ldots, a_{h-3}$ determine $\alpha_1, \alpha_2, \ldots, \alpha_{h-3=j-3}$, while the element $n+1 \leq \alpha_{j-2} \leq 2n-6$ is not determined by one of the α_is.

Let $h = 3$. We have $a_1+a_2+a_3 = 2n-5+d+\alpha_1$, with $n+1 \leq \alpha_1 \leq 2n-5$. If $a_1 = n-2$, we obtain $d+\alpha_1 = n+2$ and so we have the only possibility of $d = 1$ and $\alpha_1 = n+1$, which, by Proposition 5.5, contradicts the maximality of λ. In all the other cases we obtain $d < 0$, a contradiction.

Let $h = 4$. From $\sum a_i = \sum \alpha_i$ we have

$$a_1 = \frac{4n-10-(a_2+a_3+a_4)+d+\alpha_2}{2},$$

and maximising $a_2+a_3+a_4$ we obtain $a_1 = (n-7+d+\alpha_2)/2$. Since $a_1 < n-5$, then $\alpha_2 < n-3-d < n-3$, a contradiction.

Let $h \geq 5$. From $\sum a_i = \sum \alpha_i$ we have

$$a_1 + a_2 + \cdots + a_{h-3} = \frac{(h-2)(2n-5)-(a_{h-2}+a_{h-1}+a_h)+d+\alpha_{h-2}}{2}.$$

From the maximal choice of $a_2 + a_3 + \cdots + a_h$, we obtain

$$a_1 = \frac{n+h^2-2h-15+d+\alpha_{h-2}}{2},$$

and checking if $a_1 < a_2 = n-h-1$ leads to $(\alpha_{h-2}-n)+(h^2-13)+d < 0$ which is not compatible with $h \geq 5$.

This concludes the case $h = j$.

Let us now address the remaining case $h = j+1$. In this setting we have only three parts in the free area and the possible choices are those in the set presented in Equation (5.8).

Let $h = 3$. We have $a_1+a_2+a_3 = 2n-5+d$. In the case $(a_1, a_2, a_3) \in \{(n-3, n-1, n), (n-2, n-1, n)\}$ we obtain $d > n$, a contradiction. If $(a_1, a_2, a_3) = (n-4, n-2, n)$ or $(n-5, n-2, n-1)$ or $(n-5, n-4, n-3)$ then we obtain respectively $d = n-1$, $n-3$ and $n-7$ which, by Proposition 5.12, contradict the maximality of λ. In the other three cases we obtain d equals $n-2$, $n-4$ and $n-6$, or, in other words, $d = n-2k$ for $1 \leq k \leq 3$ as claimed. The remaining cases $4 \leq d \leq n-8$ are considered by showing partitions obtained in the case $h = 4$.

Let $h = 4$. We have

$$a_1 = \frac{2(2n-5)-(a_2+a_3+a_4)+d}{2}.$$

Notice that, for each choice of (a_2, a_3, a_4) in $\{(n-2, n-1, n), (n-4, n-3, n), (n-5, n-3, n-1), (n-5, n-4, n-2)\}$, we have that n is even if and only if d is odd, which, by Proposition 5.12, contradicts the maximality of λ. In the other four cases, since a_1 is an integer, we obtain that $d = n-2k$, for some integer k. To prove that for all $4 \leq d \leq n-8$ there exists $\lambda \in \widetilde{\mathcal{U}}_{T_{n,d}}$, consider for example, the assignment $(a_2, a_3, a_4) = (n-4, n-2, n)$. In this case, from

$\sum a_i = \sum \alpha_i$, we obtain $a_1 = (n-4+d)/2$ which satisfies $a_1 < n-5$ if and only if $d < n-6$ and the corresponding partition is unrefinable. Indeed, we can violate the refinability only if either $a_1+a_2 = \alpha_1$ or $a_1+a_3 = \alpha_1$ or $a_1+a_4 = \alpha_1$, and this is only possible if $d \in \{-1, 1, 3\}$, a contradiction by Proposition 5.5, Proposition 5.12 and since $d > 0$.

If $h \geq 5$, from $\sum a_i = \sum \alpha_i$ we have

$$a_1 + a_2 + \cdots + a_{h-3} = \frac{(h-2)(2n-5) - (a_{h-2} + a_{h-1} + a_h) + d}{2}.$$

Since $2n-5$ is odd, we have to consider the parity of $h-2$, and so of h. If h is even, then we obtain a contradiction for (a_{h-2}, a_{h-1}, a_h) in $\{(n-2, n-1, n), (n-4, n-3, n), (n-5, n-3, n-1), (n-5, n-4, n-2)\}$ and $d = n-2k$ for some positive integer k in the other cases. Instead if h is odd we obtain a contradiction for (a_{h-2}, a_{h-1}, a_h) in $\{(n-3, n-1, n), (n-4, n-2, n), (n-5, n-2, n-1), (n-5, n-4, n-3)\}$ and again $d = n-2k$ for some positive integer k in the other cases. □

Example 5.16. *Let $\lambda = (1, 2, 3, 4, 5, 6, 11)$ be a partition. We can observe that λ is unrefinable and $\lambda \vdash 32 = 36 - 4$, hence $\lambda \in \mathcal{U}_{T_{8,4}}$. Furthermore $\lambda_t = 11 = 2 \cdot 8 - 5$, then $\lambda \in \widetilde{\mathcal{U}}_{T_{n,d}}$. We can note that we have removed 7 and 8 from $\pi_{n,d}$ and we add 4 and 11, so $h = j = 2$.*
If we consider a partition $\eta = (1, 2, 3, 4, 5, 6, 7, 13)$, similar to the previous one, we have that η is unrefinable and is weigth is $T_{9,4} = 41$, but the partition is not maximal because $n = 9$ is odd. For example we have $\gamma = (1, 2, 3, 4, 8, 9, 14)$ that is unrefinable and has the same weigth of η, but $\eta_8 < \gamma_7$.

Example 5.17.

5.2 Counting maximal unrefinable partitions

In the previous section we proved the existence of maximal unrefinable partitions with specific parameters. We use those results in the current section to specify all the possible configurations meeting the requirements and therefore counting the corresponding number of partitions. The two cases to be considered are addressed in this section using the same strategy. Therefore, despite the problems having a slightly different combinatorial structure, we try to use a similar notation and terminology in Section 5.2.1 and in Section 5.2.2.

5.2.1 The case $\lambda_t = 2n - 4$

We have already proved that if $\lambda \in \widetilde{\mathcal{U}}_{T_{n,d}}$, then $d = 3$ or $d = n - (2k - 1)$, for $1 \leq k \leq \lfloor (n-2)/2 \rfloor$. We denote by $(\alpha_1, \ldots, \alpha_j) \setminus (a_1, \ldots, a_h)$ the partition λ obtained from $\pi_{n,d}$ by removing the elements a_is and replacing them with the elements α_is. We have already shown in the previous section that, when $h \leq 3$, only the following partitions belong to $\widetilde{\mathcal{U}}_{T_{n,d}}$ (cf. the proof of Proposition 5.12, computing α_i from the corresponding a_i):

- $(3, 2n-4) \setminus (n-1, n)$ for $d = 3$,
- $(n-1, 2n-4) \setminus (n-3, n-2, n)$ for $d = n-1$,
- $(n-3, 2n-4) \setminus (n-4, n-2, n-1)$ for $d = n-3$,
- $(n-5, 2n-4) \setminus (n-4, n-3, n-2)$ for $d = n-5$,
- $(n+1, 2n-4) \setminus (n-2, n-1, n)$ for $d = n-5$ (cf. Proposition 5.11).

Recall that for $h \geq 4$ we have the following choices for the free area, $(a_{h-2}, a_{h-1}, a_h) \in \{(n-4, n-3, n-2), (n-4, n-2, n-1), (n-3, n-2, n), (n-2, n-1, n)\}$. From now on, according to Proposition 5.11, we must only consider the case $d \in \lambda$. Indeed, the only maximal unrefinable partition with $d \notin \lambda$ and $\lambda_t = 2n-4$ is the fifth partition in the previous list.

Let $h = 4$. We have

$$a_1 = \frac{2(2n-4) - (a_2 + a_3 + a_4) + d}{2},$$

and, assigning all the possible values to a_2, a_3, a_4, we obtain the partitions:

- $(d, (3n-3-d)/2, 2n-4) \setminus ((n-5+d)/2, n-2, n-1, n)$, with $d < n-5$, otherwise $a_1 \geq a_2$, and $d \neq 3$, otherwise $a_1 + a_2 = \alpha_2$;
- $(d, (3n-5-d)/2, 2n-4) \setminus ((n-3+d)/2, n-3, n-2, n)$, with $d < n-5$;
- $(d, (3n-7-d)/2, 2n-4) \setminus ((n-1+d)/2, n-4, n-2, n-1)$, with $d < n-7$;
- $(d, (3n-9-d)/2, 2n-4) \setminus ((n+1+d)/2, n-4, n-3, n-2)$, with $d < n-9$.

Let $h \geq 5$. We have

$$a_1 = \frac{n + (h^2 - 3h - 9) + d}{2}$$

obtained from the maximal choice for $a_2 + a_3 + \cdots + a_{h-3}$ and from $a_{h-2} = n-2$, $a_{h-1} = n-1$ and $a_h = n$, which is also the maximal choice in the free area, and we obtain the partition

$$\left(d, n+1, \ldots, n+h-4, \frac{3n - (h^2 - 3h - 1) - d}{2}, 2n-4\right) \setminus$$
$$\left(\frac{n + (h^2 - 3h - 9) + d}{2}, n-h, \ldots, n-5, n-2, n-1, n\right),$$

with $d < n - (h^2 - h - 9)$. Notice that $a_1 + a_2 > \alpha_1$, therefore the obtained partition is unrefinable.

All the others, obtained for the remaining possibilities for $a_2 + a_3 + \cdots + a_{h-3}$, are obtained replacing $(a_1, a_2, \ldots, a_{h-3})$ with $(a_1 + i, a_2 - i_1, \ldots, a_{h-3} - i_{h-4})$, where $i = \sum_{r=1}^{h-4} i_r$ and such that $a_1 + i < a_2 - i_1 < \cdots < a_{h-3} - i_{h-4}$.

We proceed similarly for the other three choices in the free area. All the results are summarised in Table 5.1 (displayed at the end of the section). The first row of the table is Corollary 5.13 and the next four rows are summarised in the following three results.

Corollary 5.18. $\#\widetilde{\mathcal{U}}_{T_{n,n-1}} = 1$.

Corollary 5.19. $\#\widetilde{\mathcal{U}}_{T_{n,n-3}} = 1$.

Corollary 5.20. $\#\widetilde{\mathcal{U}}_{T_{n,n-5}} = 2$.

We are now ready to address the remaining cases, i.e., to compute explicitly the number of partitions $\#\widetilde{\mathcal{U}}_{T_{n,d}}$ when $3 \le d \le n-7$ and $d = n-(2k-1)$. Notice that, by Proposition 5.11 and Proposition 5.12, we know that $d \in \lambda$ and that the partition is uniquely determined when we are given n, d and the elements $a_1, a_2, \dots, a_h$ to be removed. Moreover, from Equation (5.5) we have four possible choices for the three elements in the free area which are symmetric with respect to $n-2$, therefore the partitions are determined by the list of the a_is which are smaller than or equal to $n-3$. Only one partition is exceptional with respect to this representation, i.e., the partition

$$\pi = (3, 2n-4) \setminus (n-1, n) = (1, 2, \dots, n-2, 2n-4).$$

Definition 5.21. *Let $d = n-(2k-1)$ with $3 \le d \le n-7$. Let us define the set of missing parts, for each $\lambda \in \widetilde{\mathcal{U}}_{T_{n,d}}$, which are smaller than or equal to $n-3$:*

$$\widetilde{\mathcal{U}}^*_{T_{n,d}} = \{\eta = (\eta_1, \eta_2, \dots, \eta_s) \mid s \ge 0, \eta_i \in \mathcal{M}_\lambda, \lambda \in \widetilde{\mathcal{U}}_{T_{n,d}}, \eta_i \le n-3\}.$$

Notice that π corresponds to the empty partition $() \in \widetilde{\mathcal{U}}^*_{T_{n,3}}$ obtained for $s = 0$.

From the previous argument, $\widetilde{\mathcal{U}}_{T_{n,d}}$ is in one-to-one correspondence with $\widetilde{\mathcal{U}}^*_{T_{n,d}}$. In order to prove the claimed bijection, let us introduce a partition of the set $\widetilde{\mathcal{U}}^*_{T_{n,d}}$ which is convenient for our purposes.

Definition 5.22. *Let n, d and h be positive integers. Let us define*

$$\begin{aligned}
\widetilde{\mathcal{A}}_{n,d,h} &= \left\{\eta \in \widetilde{\mathcal{U}}^*_{T_{n,d}} \mid |\eta| = h-3, \eta_{h-3} \le n-5\right\},\\
\widetilde{\mathcal{B}}_{n,d,h} &= \left\{\eta \in \widetilde{\mathcal{U}}^*_{T_{n,d}} \mid |\eta| = h-2, n_{h-3} \le n-5, \eta_{h-2} = n-3\right\},\\
\widetilde{\mathcal{C}}_{n,d,h} &= \left\{\eta \in \widetilde{\mathcal{U}}^*_{T_{n,d}} \mid |\eta| = h-2, \eta_{h-3} \le n-5, \eta_{h-2} = n-4\right\},\\
\widetilde{\mathcal{D}}_{n,d,h} &= \left\{\eta \in \widetilde{\mathcal{U}}^*_{T_{n,d}} \mid |\eta| = h-1, \eta_{h-2} = n-4, \eta_{h-1} = n-3\right\},
\end{aligned}$$

and

$$\widetilde{\mathcal{A}}_{n,d} = \bigcup_{h\ge 4} \widetilde{\mathcal{A}}_{n,d,h}, \qquad \widetilde{\mathcal{B}}_{n,d} = \bigcup_{h\ge 4} \widetilde{\mathcal{B}}_{n,d,h},$$
$$\widetilde{\mathcal{C}}_{n,d} = \bigcup_{h\ge 4} \widetilde{\mathcal{C}}_{n,d,h}, \qquad \widetilde{\mathcal{D}}_{n,d} = \bigcup_{h\ge 4} \widetilde{\mathcal{D}}_{n,d,h}.$$

Reading Table 5.1, we can note that

$$\widetilde{\mathcal{U}}^*_{T_{n,d}} = \begin{cases} \widetilde{\mathcal{A}}_{n,d} \,\dot\cup\, \widetilde{\mathcal{B}}_{n,d} \,\dot\cup\, \widetilde{\mathcal{C}}_{n,d} \,\dot\cup\, \widetilde{\mathcal{D}}_{n,d} & 4 \le d \le n-7,\\ \{()\} \,\dot\cup\, \bigcup_{h\ge 5} \widetilde{\mathcal{A}}_{n,d,h} \,\dot\cup\, \widetilde{\mathcal{B}}_{n,d} \,\dot\cup\, \widetilde{\mathcal{C}}_{n,d} \,\dot\cup\, \widetilde{\mathcal{D}}_{n,d} & d = 3. \end{cases} \tag{5.9}$$

Analogously, let us now introduce a convenient partition of $\mathcal{D}_r$, that we will prove to be related with that of Definition 5.22.

Definition 5.23. *Let r and s be positive integers. Let us define*

$$\begin{aligned}\widetilde{\mathcal{A}}^*_{r,s} &= \{\lambda \in \mathcal{D}_{r,s} \mid \lambda_1 \geq 3\}\,,\\ \widetilde{\mathcal{B}}^*_{r,s} &= \{\lambda \in \mathcal{D}_{r,s} \mid \lambda_1 = 1, \lambda_2 \geq 3\}\,,\\ \widetilde{\mathcal{C}}^*_{r,s} &= \{\lambda \in \mathcal{D}_{r,s} \mid \lambda_1 = 2\}\,,\\ \widetilde{\mathcal{D}}^*_{r,s} &= \{\lambda \in \mathcal{D}_{r,s} \mid \lambda_1 = 1, \lambda_2 = 2\}\,.\end{aligned}$$

It is clear that

$$\mathcal{D}_r = \bigcup_{s\geq 2} \widetilde{\mathcal{A}}^*_{r,s} \dot\cup \bigcup_{s\geq 2} \widetilde{\mathcal{B}}^*_{r,s} \dot\cup \bigcup_{s\geq 2} \widetilde{\mathcal{C}}^*_{r,s} \dot\cup \bigcup_{s\geq 3} \widetilde{\mathcal{D}}^*_{r,s}. \tag{5.10}$$

Finally, let us define the following correspondence from $\widetilde{\mathcal{U}}^*_{T_{n,d}} \setminus \left\{\widetilde{\mathcal{A}}_{n,d,4} \cup \{()\}\right\}$ to $\mathcal{D}$. We will discuss later how to extend the values of the function on the partitions of $\left\{\widetilde{\mathcal{A}}_{n,d,4} \cup \{()\}\right\}$.

Definition 5.24. *Let us denote*

$$\begin{array}{rccl}\phi: & \widetilde{\mathcal{U}}^*_{T_{n,d}} \setminus \left\{\widetilde{\mathcal{A}}_{n,d,4} \cup \{()\}\right\} & \longrightarrow & \mathcal{D}\\ & (\eta_1, \eta_2, \ldots, \eta_t) & \longmapsto & (n-2-\eta_t, \ldots, n-2-\eta_2, n-2-\eta_1).\end{array}$$

The two following result will be used in Theorem 5.28 to prove the part of our main result related to the case $\lambda_t = 2n-4$.

Proposition 5.25. *Let $d = n-(2k-1)$ such that $3 \leq d \leq n-7$ and $h \geq 5$. Then ϕ sends bijectively $\widetilde{\mathcal{A}}_{n,d,h}$ into $\widetilde{\mathcal{A}}^*_{k,h-3}$.*

Proof. Let us start by proving that the correspondence is well defined, i.e., if $\eta \in \widetilde{\mathcal{A}}_{n,n-(2k-1),h}$, then $\phi(\eta) \in \widetilde{\mathcal{A}}^*_{k,h-3}$. Let $\eta \in \widetilde{\mathcal{A}}_{n,n-(2k-1),h}$. Then, by Table 5.1,

$$\eta = \left(\frac{n + \left(h^2 - 3h - 9\right) + d}{2} + i, n-h-i_1, \ldots, n-5-i_{h-4}\right),$$

for some positive integers $i, i_1 \geq i_2 \geq \ldots \geq i_{h-4}$ such that $i = \sum_{j=1}^{h-4} i_j$. By definition of ϕ we have

$$\phi(\eta) = \left(3 + i_{h-4}, 4 + i_{h-5}, \ldots, h-2+i_1, k + \frac{-h^2+3h+4}{2} - i\right).$$

Notice that $|\phi(\eta)| = h-3$, $\phi(\eta)_1 \geq 3$ and that $\phi(\eta) \vdash k$. Therefore $\phi(\eta) \in \widetilde{\mathcal{A}}^*_{k,h-3}$.

Notice also that ϕ is trivially injective and so, in order to conclude the proof, it remains to prove that ϕ is surjective from $\widetilde{\mathcal{A}}_{n,n-(2k-1),h}$ to $\widetilde{\mathcal{A}}^*_{k,h-3}$. For this purpose, let $\rho = (\rho_1, \rho_2, \ldots, \rho_{h-3}) \in \widetilde{\mathcal{A}}^*_{k,h-3}$. Then the general expression for such ρ is

$$\rho = \left(3 + i_1, 4 + i_2, \ldots, h - 2 + i_{h-4}, k + \frac{-h^2 + 3h + 4}{2} - i\right),$$

for some positive integers $i, i_1 \leq i_2 \leq \ldots \leq i_{h-4}$ such that $i = \sum_{j=1}^{h-4} i_j$. It is easy to see that

$$\eta = (n - 2 - \rho_{h-3}, \ldots, n - 2 - \rho_2, n - 2 - \rho_1)$$

is such that $\phi(\eta) = \rho$. We need to prove that $\eta \in \widetilde{\mathcal{A}}_{n,n-(2k-1),h}$. We have

$$\eta = \left(n - 2 - \left(k + \frac{-h^2 + 3h + 4}{2} - i\right), n - 2 - (h - 2 + i_{h-4}), \ldots, n - 2 - (3 + i_1)\right)$$
$$= \left(\frac{n + d + h^2 - 3h - 9}{2} + 1, n - h - i_{h-4}, \ldots, n - 5 - i_1\right),$$

which, from Table 5.1, is exactly the generic form of a partition in $\widetilde{\mathcal{A}}_{n,n-(2k-1),h}$. □

Similar computations lead to the corresponding results for $\widetilde{\mathcal{B}}^*$, $\widetilde{\mathcal{C}}^*$ and $\widetilde{\mathcal{D}}^*$. Precisely:

Proposition 5.26. *Let $d = n - (2k - 1)$ such that $3 \leq d \leq n - 7$ and $h \geq 5$. Then ϕ sends bijectively*

1. $\widetilde{\mathcal{B}}_{n,n-(2k-1),h}$ *into* $\widetilde{\mathcal{B}}^*_{k,h-2}$,
2. $\widetilde{\mathcal{C}}_{n,n-(2k-1),h}$ *into* $\widetilde{\mathcal{C}}^*_{k,h-2}$,
3. $\widetilde{\mathcal{D}}_{n,n-(2k-1),h}$ *into* $\widetilde{\mathcal{D}}^*_{k,h-1}$.

Moreover, in the case $h = 4$ we have

1. $\widetilde{\mathcal{B}}_{n,n-(2k-1),4} \overset{\phi}{\longleftrightarrow} \{(1, k - 1)\} = \widetilde{\mathcal{B}}^*_{k,2}$,
2. $\widetilde{\mathcal{C}}_{n,n-(2k-1),4} \overset{\phi}{\longleftrightarrow} \{(2, k - 2)\} = \widetilde{\mathcal{C}}^*_{k,2}$,
3. $\widetilde{\mathcal{D}}_{n,n-(2k-1),4} \overset{\phi}{\longleftrightarrow} \{(1, 2, k - 3)\} = \widetilde{\mathcal{D}}^*_{k,3}$.

Remark 5.27. *Notice that $\widetilde{\mathcal{A}}_{n,d,4}$ contains only the (trivial) partition $\tilde{\eta} = (\frac{n-5+d}{2})$, and so, extending the function ϕ on $\widetilde{\mathcal{A}}_{n,d,4}$ implies to consider the (trivial) partition of k in a single part, i.e., $\phi(\tilde{\eta}) = (k)$. Similarly, with an abuse of notation we can assume $() \overset{\phi}{\mapsto} ()$. This extends the definition of ϕ also on $\left\{\widetilde{\mathcal{A}}_{n,d,4} \cup \{()\}\right\}$, making the function defined on the whole set $\widetilde{\mathcal{U}}^*_{T_{n,d}}$.*

We can finally summarise the above results. From Proposition 5.25, Proposition 5.26, Remark 5.27 and from Equation (5.10) we obtain:

Theorem 5.28. *Let* $d = n - (2k-1)$ *be such that* $3 \leq d \leq n-7$. *Then*

$$\#\widetilde{\mathcal{U}}_{T_{n,n-(2k-1)}} = 1 + \#\mathcal{D}_k.$$

Proof. Let us assume first that $4 \leq d \leq n-7$. Then, since the empty partition () appears only in the case $d = 3$ which is not considered, we have (cf. Equation (5.9))

$$\begin{aligned}
\widetilde{\mathcal{U}}_{T_{n,d}} \leftrightarrow \widetilde{\mathcal{U}}^*_{T_{n,d}} &= \left(\bigcup_{h\geq 4} \widetilde{\mathcal{A}}_{n,d,h}\right) \cup \left(\bigcup_{h\geq 4} \widetilde{\mathcal{B}}_{n,d,h}\right) \cup \left(\bigcup_{h\geq 4} \widetilde{\mathcal{C}}_{n,d,h}\right) \cup \left(\bigcup_{h\geq 4} \widetilde{\mathcal{D}}_{n,d,h}\right) \\
&\overset{\phi}{\leftrightarrow} \{(k)\} \cup \left(\bigcup_{s\geq 2} \widetilde{\mathcal{A}}^*_{k,s}\right) \cup \left(\bigcup_{s\geq 2} \widetilde{\mathcal{B}}^*_{k,s}\right) \cup \left(\bigcup_{s\geq 2} \widetilde{\mathcal{C}}^*_{k,s}\right) \cup \left(\bigcup_{s\geq 3} \widetilde{\mathcal{D}}^*_{k,s}\right) \\
&= \{(k)\} \cup \mathcal{D}_k,
\end{aligned}$$

from which we obtained the desired claim. In the remaining case $d = 3$, we proceed in the same way and, using the corresponding description of Equation (5.9), we obtain

$$\widetilde{\mathcal{U}}_{T_{n,d}} \leftrightarrow \widetilde{\mathcal{U}}^*_{T_{n,d}} \overset{\phi}{\leftrightarrow} \{()\} \cup \mathcal{D}_k.$$

□

d	$(a_1, \dots, a_h)$	$(\alpha_1, \dots, \alpha_j)$
3	$(n-1, n)$	$(d, 2n-4)$
$n-1$	$(n-3, n-2, n)$	$(d, 2n-4)$
$n-3$	$(n-4, n-2, n-1)$	$(d, 2n-4)$
$n-5$	$(n-4, n-3, n-2)$	$(d, 2n-4)$
$n-5$	$(n-2, n-1, n)$	$(n+1, 2n-4)$
$3 < n-(2k-1) \leq n-7$	$\left(\frac{n-5+d}{2}, n-2, n-1, n\right)$	$\left(d, \frac{3n-3-d}{2}, 2n-4\right)$
$3 \leq n-(2k-1) \leq n-7$	$\left(\frac{n-3+d}{2}, n-3, n-2, n\right)$	$\left(d, \frac{3n-5-d}{2}, 2n-4\right)$
$3 \leq n-(2k-1) \leq n-9$	$\left(\frac{n-1+d}{2}, n-4, n-2, n-1\right)$	$\left(d, \frac{3n-7-d}{2}, 2n-4\right)$
$3 \leq n-(2k-1) \leq n-11$	$\left(\frac{n+1+d}{2}, n-4, n-3, n-2\right)$	$\left(d, \frac{3n-9-d}{2}, 2n-4\right)$
$3 \leq n-(2k-1) \leq n-\left(h^2-h-7\right)$ for $h \geq 5$	$\Big(\frac{n+\left(h^2-3h-9\right)+d}{2} + i, n-h-i_1, \dots$ $n-5-i_{h-4}, n-2, n-1, n)$	$(d, n+1+i_{h-4}, \dots, n-4+h+i_1,$ $\frac{3n-\left(h^2-3h-1\right)-d}{2} - i, 2n-4\Big)$
$3 \leq n-(2k-1) \leq n-\left(h^2-h-5\right)$ for $h \geq 5$	$\Big(\frac{n+\left(h^2-3h-7\right)+d}{2} + i, n-h-i_1, \dots$ $\dots, n-5-i_{h-4}, n-3, n-2, n)$	$(d, n+1+i_{h-4}, \dots, n-4+h+i_1,$ $\frac{3n-\left(h^2-3h+1\right)-d}{2}, 2n-4\Big)$
$3 \leq n-(2k-1) \leq n-\left(h^2-h-3\right)$ for $h \geq 5$	$\Big(\frac{n+\left(h^2-3h-5\right)+d}{2} + i, n-h-i_1,$ $\dots, n-5-i_{h-4}, n-4, n-2, n-1)$	$(d, n+1+i_{h-4}, \dots, n-4+h+i_1,$ $\frac{3n-\left(h^2-3h+3\right)-d}{2}, 2n-4\Big)$
$3 \leq n-(2k-1) \leq n-\left(h^2-h-1\right)$ for $h \geq 5$	$\Big(\frac{n+\left(h^2-3h-3\right)+d}{2} + i, n-h-i_1,$ $\dots, n-5-i_{h-4}, n-4, n-3, n-2)$	$(d, n+1+i_{h-4}, \dots, n-4+h+i_1,$ $\frac{3n-\left(h^2-3h+5\right)-d}{2}, 2n-4\Big)$

Table 5.1: List of all the possible maximal constructions when $\lambda_t = 2n-4$

5.2.2 The case $\lambda_t = 2n-5$

We can now count the number of maximal unrefinable partitions in the case of $\lambda_t = 2n-5$. We use the same notation of Section 5.2.1 and using similar argument, although the combinatorial nature of the problem is more complex. We have already proved that if $\lambda \in \widetilde{\mathcal{U}}_{T_{n,d}}$, then $d = n-2k$, for $1 \leq k \leq \lfloor (n-4)/2 \rfloor$. Moreover, we have already proved in Section 5.1.4 that, when $h \leq 3$, only the following partitions belong to $\widetilde{\mathcal{U}}_{T_{n,d}}$ (cf. the proof of Proposition 5.15):

- $(4, 2n-5) \setminus (n-1, n)$ for $d = 4$,
- $(n-2, 2n-5) \setminus (n-4, n-3, n)$ for $d = n-2$,
- $(n-4, 2n-5) \setminus (n-5, n-3, n-1)$ for $d = n-4$,
- $(n-6, 2n-5) \setminus (n-5, n-4, n-2)$ for $d = n-6$,
- $(n+1, 2n-5) \setminus (n-3, n-1, n)$ for $d = n-6$ (cf. Proposition 5.14).

For $h \geq 4$ we have the following eight choices for the free area, i.e.,

$$(a_{h-2}, a_{h-1}, a_h) \in \left\{ \begin{array}{l} (n-5, n-4, n-3), (n-5, n-4, n-2), \\ (n-5, n-3, n-1), (n-5, n-2, n-1), \\ (n-4, n-3, n), (n-4, n-2, n), \\ (n-3, n-1, n), (n-2, n-1, n) \end{array} \right\}.$$

Also in this case, we only consider the case $d \in \lambda$ (cf. Proposition 5.14), indeed the only maximal unrefinable partition with $\lambda_t = 2n-5$, obtained assuming $d \notin \lambda$ is the last of the previous list.

Let $h=4$. Since n is even if and only if d is even, and since, from $\sum a_i = \sum \alpha_i$, we can calculate

$$a_1 = \frac{2\,(2n-5)+d-(a_2+a_3+a_4)}{2},$$

then $a_2+a_3+a_4$ is even if and only if n is even. Therefore, the only possible choices compliant with the previous requirement are

$$(a_2,a_3,a_4) = \{(n-5,n-4,n-3),(n-5,n-2,n-1),(n-4,n-2,n),(n-3,n-1,n)\}.$$

We obtain the partitions

- $(d,(3n-4-d)/2,2n-5)\setminus((n-6+d)/2,n-3,n-1,n)$, with $d<n-6$, otherwise $a_1 \geq a_2$, and $d \neq 4$, otherwise $a_1+a_2=\alpha_2$;
- $(d,(3n-6-d)/2,2n-5)\setminus((n-4+d)/2,n-4,n-2,n)$, with $d<n-6$;
- $(d,(3n-8-d)/2,2n-5)\setminus((n-2+d)/2,n-5,n-2,n-1)$, with $d<n-8$;
- $(d,(3n-12-d)/2,2n-5)\setminus((n+2+d)/2,n-5,n-4,n-3)$, with $d<n-12$.

If $h \geq 5$, we need to distinguish the two cases h odd and h even, as already observed at the end of Proposition 5.15. The only difference between the two cases is in the triple (a_{h-2},a_{h-1},a_h) to be chosen in the free area. Let $h \geq 5$, h odd. We have

$$a_1 = \frac{n+\left(h^2-2h-15\right)+d}{2}$$

obtained from the maximal choice for $a_2+a_3+\cdots+a_{h-3}$ and from $a_{h-2}=n-2$, $a_{h-1}=n-1$ and $a_h=n$, which is also the maximal choice in the free area, and we obtain the partition

$$\left(d,n+1,\ldots,n+h-4,\frac{3n-\left(h^2-2h-5\right)-d}{2}\right)\setminus$$
$$\left(\frac{n+\left(h^2-2h-15\right)+d}{2},n-h-1,\ldots,n-6,n-2,n-1,n\right)$$

with $d \leq n-\left(h^2-11\right)$. Notice that $a_1+a_2>\alpha_1$, therefore the obtained partition is unrefinable. The remaining cases for h are treated analogously.

All the other partitions, obtained for the remaining possibilities for $a_2+a_3+\cdots+a_{h-3}$, are obtained replacing $(a_1,a_2,\ldots,a_{h-3})$ by $(a_1+i,a_2-i_1,\ldots,a_{h-3}-i_{h-4})$, where $i=\sum_{r=1}^{h-4} i_r$ and such that $a_1+i<a_2-i_1<\cdots<a_{h-3}-i_{h-4}$.

We proceed similarly for the other seven choices in the free area. All the results are summarised in Table 5.3 (displayed at the end of the section), and the following consequences are easily noted.

Corollary 5.29. $\#\widetilde{\mathcal{U}}_{T_{n,n-2}} = 1$.

Corollary 5.30. $\#\widetilde{\mathcal{U}}_{T_{n,n-4}} = 1$.

Corollary 5.31. $\#\widetilde{\mathcal{U}}_{T_{n,n-6}} = 2$.

As in the previous section, it remains to compute $\#\widetilde{\mathcal{U}}_{T_{n,d}}$ when $4 \le d \le n-8$ and $d = n-2k$. Notice that the partition is uniquely determined when we are given n, d and the list of the a_is which are smaller than or equal to $n-3$. Only one partition is exceptional with respect to this representation, i.e., the partition

$$\tau = (4, 2n-5) \setminus (n-1, n) = (1, 2, \ldots, n-2, 2n-5).$$

The following definition is the counterpart of Definition 5.21 for the case under consideration here. The defined set will be again in one-to-one correspondence with $\widetilde{\mathcal{U}}_{T_{n,d}}$.

Definition 5.32. *Let $d = n-2k$ with $4 \le d \le n-8$. Let us define the set of missing parts, for each $\lambda \in \widetilde{\mathcal{U}}_{T_{n,d}}$, which are smaller than or equal to $n-3$:*

$$\widetilde{\mathcal{U}}^*_{T_{n,d}} = \{(\eta_1, \eta_2, \ldots, \eta_s) \mid s \ge 0, \eta_i \in \mathcal{M}_\lambda, \lambda \in \widetilde{\mathcal{U}}_{T_{n,d}}, \eta_i \le n-3\}.$$

Notice that τ corresponds to the empty partition $() \in \widetilde{\mathcal{U}}^*_{T_{n,4}}$ obtained for $s = 0$.

Notice that when $h \ge 5$ we have (cf. Table 5.3)

$$a_1 = \frac{n + d + \left(h^2 - 2h + t\right)}{2}$$

for some $t \in \mathbb{Z}$. Since the numerator must be even, we have that h is even if and only if t is even. From this, we obtain a convenient partition of the set $\widetilde{\mathcal{U}}^*_{T_{n,d}}$, similar to that introduced in Section 5.2.1, but which takes into account also the parity of h.

Definition 5.33. *Let n, d and h be positive integers. If h is odd, let us define*

$$\begin{aligned}
\widetilde{\mathcal{E}}^1_{n,d,h} &= \left\{\eta \in \widetilde{\mathcal{U}}^*_{T_{n,d}} \mid |\eta| = h-3, \eta_{h-3} \le n-6\right\}, \\
\widetilde{\mathcal{E}}^2_{n,d,h} &= \left\{\eta \in \widetilde{\mathcal{U}}^*_{T_{n,d}} \mid |\eta| = h-1, n_{h-3} \le n-6, \eta_{h-2} = n-4, \eta_{h-1} = n-3\right\}, \\
\widetilde{\mathcal{E}}^3_{n,d,h} &= \left\{\eta \in \widetilde{\mathcal{U}}^*_{T_{n,d}} \mid |\eta| = h-1, \eta_{h-2} = n-5, \eta_{h-1} = n-3\right\}, \\
\widetilde{\mathcal{E}}^4_{n,d,h} &= \left\{\eta \in \widetilde{\mathcal{U}}^*_{T_{n,d}} \mid |\eta| = h-1, \eta_{h-2} = n-5, \eta_{h-1} = n-4\right\}.
\end{aligned}$$

Moreover

$$\widetilde{\mathcal{E}}^1_{n,d} = \bigcup_{h \ge 5} \widetilde{\mathcal{E}}^1_{n,d,h}, \quad \widetilde{\mathcal{E}}^2_{n,d} = \bigcup_{h \ge 5} \widetilde{\mathcal{E}}^2_{n,d,h},$$
$$\widetilde{\mathcal{E}}^3_{n,d} = \bigcup_{h \ge 5} \widetilde{\mathcal{E}}^3_{n,d,h}, \quad \widetilde{\mathcal{E}}^4_{n,d} = \bigcup_{h \ge 5} \widetilde{\mathcal{E}}^4_{n,d,h}.$$

If h is even, let us define

$$\widetilde{\mathcal{F}}^1_{n,d,h} = \left\{\eta \in \widetilde{\mathcal{U}}^*_{T_{n,d}} \mid |\eta| = h-2, \eta_{h-3} \leq n-6, \eta_{h-2} = n-3\right\},$$
$$\widetilde{\mathcal{F}}^2_{n,d,h} = \left\{\eta \in \widetilde{\mathcal{U}}^*_{T_{n,d}} \mid |\eta| = h-2, n_{h-3} \leq n-6, \eta_{h-2} = n-4\right\},$$
$$\widetilde{\mathcal{F}}^3_{n,d,h} = \left\{\eta \in \widetilde{\mathcal{U}}^*_{T_{n,d}} \mid |\eta| = h-3, \eta_{h-3} \leq n-6, \eta_{h-2} = n-5\right\},$$
$$\widetilde{\mathcal{F}}^4_{n,d,h} = \left\{\eta \in \widetilde{\mathcal{U}}^*_{T_{n,d}} \mid |\eta| = h, \eta_{h-2} = n-5, \eta_{h-1} = n-4, \eta_h = n-3\right\}.$$

Moreover

$$\widetilde{\mathcal{F}}^1_{n,d} = \bigcup_{h\geq 4} \widetilde{\mathcal{F}}^1_{n,d,h}, \quad \widetilde{\mathcal{F}}^2_{n,d} = \bigcup_{h\geq 4} \widetilde{\mathcal{F}}^2_{n,d,h},$$
$$\widetilde{\mathcal{F}}^3_{n,d} = \bigcup_{h\geq 4} \widetilde{\mathcal{F}}^3_{n,d,h}, \quad \widetilde{\mathcal{F}}^4_{n,d} = \bigcup_{h\geq 4} \widetilde{\mathcal{F}}^4_{n,d,h}.$$

Finally, let us denote

$$\widetilde{\mathcal{E}}_{n,d} = \widetilde{\mathcal{E}}^1_{n,d} \cup \widetilde{\mathcal{E}}^2_{n,d} \cup \widetilde{\mathcal{E}}^3_{n,d} \cup \widetilde{\mathcal{E}}^4_{n,d},$$
$$\widetilde{\mathcal{F}}_{n,d} = \widetilde{\mathcal{F}}^1_{n,d} \cup \widetilde{\mathcal{F}}^2_{n,d} \cup \widetilde{\mathcal{F}}^3_{n,d} \cup \widetilde{\mathcal{F}}^4_{n,d}.$$

Reading Table 5.3, we can note that

$$\widetilde{\mathcal{U}}^*_{T_{n,d}} = \begin{cases} \widetilde{\mathcal{E}}_{n,d} \,\dot\cup\, \widetilde{\mathcal{F}}_{n,d} & 5 \leq d \leq n-8, \\ \widetilde{\mathcal{E}}_{n,d} \,\dot\cup\, \{()\} \,\dot\cup\, \left(\bigcup_{h\geq 6} \widetilde{\mathcal{F}}^1_{n,d,h}\right) \dot\cup\, \widetilde{\mathcal{F}}^2_{n,d} \,\dot\cup\, \widetilde{\mathcal{F}}^3_{n,d} \,\dot\cup\, \widetilde{\mathcal{F}}^4_{n,d} & d = 4. \end{cases} \tag{5.11}$$

The sets defined next play in this section the same role of those defined in Definition 5.23.

Definition 5.34. *Let n, d and h be positive integers. If h is odd, let us define*

$$\widetilde{\mathcal{E}}^{1^*}_{n,d,h} = \left\{\rho \in \mathcal{D}_{k+(h-1)/2} \mid |\rho| = h-3, \rho_1 \geq 4\right\},$$
$$\widetilde{\mathcal{E}}^{2^*}_{n,d,h} = \left\{\rho \in \mathcal{D}_{k+(h+1)/2} \mid |\rho| = h-1, \rho_1 = 1, \rho_2 = 2, \rho_3 \geq 4\right\},$$
$$\widetilde{\mathcal{E}}^{3^*}_{n,d,h} = \left\{\rho \in \mathcal{D}_{k+(h+1)/2} \mid |\rho| = h-1, \rho_1 = 1, \rho_2 = 3\right\},$$
$$\widetilde{\mathcal{E}}^{4^*}_{n,d,h} = \left\{\rho \in \mathcal{D}_{k+(h+1)/2} \mid |\rho| = h-1, \rho_1 = 2, \rho_2 = 3\right\}.$$

If h is even, let us define

$$\widetilde{\mathcal{F}}^{1^*}_{n,d,h} = \left\{\rho \in \mathcal{D}_{k+h/2} \mid |\rho| = h-2, \rho_1 = 1, \rho_2 \geq 4\right\},$$
$$\widetilde{\mathcal{F}}^{2^*}_{n,d,h} = \left\{\rho \in \mathcal{D}_{k+h/2} \mid |\rho| = h-2, \rho_1 = 2, \rho_2 \geq 4\right\},$$
$$\widetilde{\mathcal{F}}^{3^*}_{n,d,h} = \left\{\rho \in \mathcal{D}_{k+h/2} \mid |\rho| = h-2, \rho_1 = 3, \rho_2 \geq 4\right\},$$
$$\widetilde{\mathcal{F}}^{4^*}_{n,d,h} = \left\{\rho \in \mathcal{D}_{k+1+h/2} \mid |\rho| = h, \rho_1 = 1, \rho_2 = 2, \rho_3 = 3\right\}.$$

In the following definition we adapt the description of ϕ (cf. Definition 5.24) to the current representation of $\widetilde{\mathcal{U}}^*_{T_{n,d}}$.

Definition 5.35. *Let us define the following correspondence from $\widetilde{\mathcal{U}}^*_{T_{n,d}} \setminus \{()\}$ to $\mathcal{D}$. We will discuss later how to extend the values of the function on the empty partition $()$. We denote*

$$\begin{array}{rccl} \phi: & \widetilde{\mathcal{U}}^*_{T_{n,d}} \setminus \{()\} & \longrightarrow & \mathcal{D} \\ & (\eta_1, \eta_2, \ldots, \eta_t) & \longmapsto & (n-2-\eta_t, \ldots, n-2-\eta_2, n-2-\eta_1). \end{array}$$

Proposition 5.36. *Let $d = n-2k$ such that $4 \leq d \leq n-8$ and $h \geq 4$. Then, for $1 \leq i \leq 4$, ϕ sends bijectively*

1. *$\widetilde{\mathcal{E}}^i_{n,d,h}$ into $\widetilde{\mathcal{E}}^{i^*}_{n,d,h}$,*
2. *$\widetilde{\mathcal{F}}^i_{n,d,h}$ into $\widetilde{\mathcal{F}}^{i^*}_{n,d,h}$,*

Proof. Let us prove that $\widetilde{\mathcal{E}}^1_{n,d,h} \overset{\phi}{\leftrightarrow} \widetilde{\mathcal{E}}^{1^*}_{n,d,h}$. The other claims can be proved in the same way. Let us start by proving that the correspondence is well defined, i.e., if $\eta \in \widetilde{\mathcal{E}}^1_{n,d,h}$, then $\phi(\eta) \in \widetilde{\mathcal{E}}^{1^*}_{n,d,h}$. Let $\eta \in \widetilde{\mathcal{E}}^1_{n,d,h}$. Then, by Table 5.3,

$$\eta = \left(\frac{n + \left(h^2 - 2h - 15\right) + d}{2} + i, n-h-1-i_1, \ldots, n-6-i_{h-4} \right),$$

for some positive integers $i, i_1 \geq i_2 \geq \ldots \geq i_{h-4}$ such that $i = \sum_{j=1}^{h-4} i_j$. By definition of ϕ we have

$$\phi(\eta) = \left(4 + i_{h-4}, \ldots, h-1+i_1, \frac{n - h^2 + 2h + 11 - d}{2} - i \right).$$

Notice that $|\phi(\eta)| = h-3$, $\phi(\eta)_1 \geq 4$ and that $\phi(\eta) \vdash k + (h-1)/2$. Therefore $\phi(\eta) \in \widetilde{\mathcal{E}}^{1^*}_{n,d,h}$.

Notice also that ϕ is trivially injective and so, in order to conclude the proof, it remains to prove that ϕ is surjective from $\widetilde{\mathcal{E}}^1_{n,d,h}$ to $\widetilde{\mathcal{E}}^{1^*}_{n,d,h}$. For this purpose, let $\rho = (\rho_1, \rho_2, \ldots, \rho_{h-3}) \in \widetilde{\mathcal{E}}^{1^*}_{n,d,h}$. Then the general expression for such ρ is

$$\rho = \left(4 + i_1, 5 + i_2, \ldots, h-1+i_{h-4}, k + \frac{-h^2 + 2h + 11}{2} - i \right)$$

for some positive integers $i, i_1 \leq i_2 \leq \ldots \leq i_{h-4}$ such that $i = \sum_{j=1}^{h-4} i_j$. It is easy to see that

$$\eta = (n-2-\rho_{h-3}, \ldots, n-2-\rho_2, n-2-\rho_1)$$

is such that $\phi(\eta) = \rho$. We need to prove that $\eta \in \widetilde{\mathcal{E}}^1_{n,d,h}$. We have

$$\begin{aligned} \eta &= \left(n - 2 - \left(k + \frac{-h^2 + 2h + 11}{2} - i \right), n-2-(h-1+i_{h-4}), \ldots, n-2-(4+i_1) \right) \\ &= \left(\frac{2n - 2k + h^2 - 2h - 15}{2} + i, n-h-1-i_{h-4}, \ldots, n-6-i_1 \right). \end{aligned}$$

which, from Table 5.3, is exactly the generic form of a partition in $\widetilde{\mathcal{E}}^1_{n,d,h}$. □

Notice that, as in Proposition 5.26, each of the sets $\widetilde{\mathcal{F}}^{i^*}_{n,d,h}$, with $h = 4$, contains only one partition.

Remark 5.37. *Let $h = 4$ and $d = n - 2k$ be such that $4 \le d \le n - 8$. We have*

1. $\widetilde{\mathcal{F}}^{1}_{n,d,h} \overset{\phi}{\leftrightarrow} \widetilde{\mathcal{F}}^{1^*}_{n,d,h} = \left(1, k + \frac{h}{2} - 1\right)$, *for* $d \neq 4$,
2. $\widetilde{\mathcal{F}}^{2}_{n,d,h} \overset{\phi}{\leftrightarrow} \widetilde{\mathcal{F}}^{2^*}_{n,d,h} = \left(2, k + \frac{h}{2} - 2\right)$,
3. $\widetilde{\mathcal{F}}^{3}_{n,d,h} \overset{\phi}{\leftrightarrow} \widetilde{\mathcal{F}}^{3^*}_{n,d,h} = \left(3, k + \frac{h}{2} - 3\right)$,
4. $\widetilde{\mathcal{F}}^{4}_{n,d,h} \overset{\phi}{\leftrightarrow} \widetilde{\mathcal{F}}^{4^*}_{n,d,h} = \left(1, 2, 3, k + \frac{h}{2} - 5\right)$.

We now show how the partitions of $\widetilde{\mathcal{E}}^{i^*}_{n,d,\_}$ and $\widetilde{\mathcal{F}}^{i^*}_{n,d,\_}$ represent a convenient partition of the set $\mathcal{D}_{k+(h-1)/2}$, which will be used to prove the claimed bijection.

Proposition 5.38. *Let $d = n - 2k$ such that $4 \le d \le n - 8$ and $h \ge 5$ be odd. Then we have*

$$\begin{aligned}\mathcal{D}_{k+(h-1)/2,\,h-3} = &\widetilde{\mathcal{E}}^{1^*}_{n,d,h} \cup \widetilde{\mathcal{E}}^{2^*}_{n,d,h-2} \cup \widetilde{\mathcal{E}}^{3^*}_{n,d,h-2} \cup \widetilde{\mathcal{E}}^{4^*}_{n,d,h-2} \cup \\ &\cup \widetilde{\mathcal{F}}^{1^*}_{n,d,h-1} \cup \widetilde{\mathcal{F}}^{2^*}_{n,d,h-1} \cup \widetilde{\mathcal{F}}^{3^*}_{n,d,h-1} \cup \widetilde{\mathcal{F}}^{4^*}_{n,d,h-3},\end{aligned} \tag{5.12}$$

where the set $\mathcal{D}_{k+(h-1)/2,\,h-3}$ indicates the set of partitions of $k+(h-1)/2$ into $h-3$ distinct parts.

Proof. It follows from Definition 5.34 that each partition in one of the sets in the right side of Equation (5.12) is a partition of $k+(h-1)/2$ into $h-3$ distinct parts, therefore we have

$$\begin{aligned}\mathcal{D}_{k+(h-1)/2,\,h-3} \supseteq &\widetilde{\mathcal{E}}^{1^*}_{n,d,h} \cup \widetilde{\mathcal{E}}^{2^*}_{n,d,h-2} \cup \widetilde{\mathcal{E}}^{3^*}_{n,d,h-2} \cup \widetilde{\mathcal{E}}^{4^*}_{n,d,h-2} \cup \\ &\cup \widetilde{\mathcal{F}}^{1^*}_{n,d,h-1} \cup \widetilde{\mathcal{F}}^{2^*}_{n,d,h-1} \cup \widetilde{\mathcal{F}}^{3^*}_{n,d,h-1} \cup \widetilde{\mathcal{F}}^{4^*}_{n,d,h-3}.\end{aligned}$$

To prove the converse, it is enough to notice that the claimed sets form a partition of the set $\mathcal{D}_{k+(h-1)/2,\,h-3}$, indeed we can write

$$\begin{aligned}\widetilde{\mathcal{E}}^{1^*}_{n,d,h} &= \left\{\lambda \in \mathcal{D}_{k+(h-1)/2,\,h-3} \mid \lambda_1 \ge 4\right\}, \\ \widetilde{\mathcal{F}}^{1^*}_{n,d,h-1} &= \left\{\lambda \in \mathcal{D}_{k+(h-1)/2,\,h-3} \mid \lambda_1 = 1, \lambda_2 \ge 4\right\}, \\ \widetilde{\mathcal{F}}^{2^*}_{n,d,h-1} &= \left\{\lambda \in \mathcal{D}_{k+(h-1)/2,\,h-3} \mid \lambda_1 = 2, \lambda_2 \ge 4\right\}, \\ \widetilde{\mathcal{F}}^{3^*}_{n,d,h-1} &= \left\{\lambda \in \mathcal{D}_{k+(h-1)/2,\,h-3} \mid \lambda_1 = 3, \lambda_2 \ge 4\right\}, \\ \widetilde{\mathcal{E}}^{2^*}_{n,d,h-2} &= \left\{\lambda \in \mathcal{D}_{k+(h-1)/2,\,h-3} \mid \lambda_1 = 1, \lambda_2 = 2, \lambda_3 \ge 4\right\}, \\ \widetilde{\mathcal{E}}^{3^*}_{n,d,h-2} &= \left\{\lambda \in \mathcal{D}_{k+(h-1)/2,\,h-3} \mid \lambda_1 = 1, \lambda_2 = 3, \lambda_3 \ge 4\right\}, \\ \widetilde{\mathcal{E}}^{4^*}_{n,d,h-2} &= \left\{\lambda \in \mathcal{D}_{k+(h-1)/2,\,h-3} \mid \lambda_1 = 2, \lambda_2 = 3, \lambda_3 \ge 4\right\}, \\ \widetilde{\mathcal{F}}^{4^*}_{n,d,h-3} &= \left\{\lambda \in \mathcal{D}_{k+(h-1)/2,\,h-3} \mid \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3\right\}.\end{aligned}$$

□

We now use Proposition 5.38 to show the claimed bijection related to the case $\lambda_t = 2n-5$ of the main result. Recall that $\mathcal{D}_k^{\mathcal{O}}$ indicates the set of partitions in distinct parts of k such that every part is an odd number.

Theorem 5.39. *Let $d = n - 2k$ such that $4 \leq d \leq n-8$. Then*

$$\widetilde{\mathcal{U}}_{T_{n,d}} \leftrightarrow \mathcal{D}^{\mathcal{O}}_{2(k+1)}.$$

Proof. Let us start assuming $d > 4$. We obtain the claim by showing first that $\widetilde{\mathcal{U}}_{T_{n,d}} \leftrightarrow \bigcup_{i\geq 0} \mathcal{D}_{k+2+i,2+2i}$ and successively that $\bigcup_{i\geq 0} \mathcal{D}_{k+2+i,2+2i} \leftrightarrow \mathcal{D}^{\mathcal{O}}_{2(k+1)}$. The first claim follows directly from Proposition 5.38, indeed

$$\begin{aligned}
\widetilde{\mathcal{U}}_{T_{n,d}} \leftrightarrow & \widetilde{\mathcal{U}}^*_{T_{n,d}} \\
= & \widetilde{\mathcal{E}}_{n,d} \cup \widetilde{\mathcal{F}}_{n,d} \\
= & \bigcup_{h\geq 5} \widetilde{\mathcal{E}}^1_{n,d,h} \cup \bigcup_{h\geq 5} \widetilde{\mathcal{E}}^2_{n,d,h} \cup \bigcup_{h\geq 5} \widetilde{\mathcal{E}}^3_{n,d,h} \cup \bigcup_{h\geq 5} \widetilde{\mathcal{E}}^4_{n,d,h} \cup \\
& \cup \bigcup_{h\geq 4} \widetilde{\mathcal{F}}^1_{n,d,h} \cup \bigcup_{h\geq 4} \widetilde{\mathcal{F}}^2_{n,d,h} \cup \bigcup_{h\geq 4} \widetilde{\mathcal{F}}^3_{n,d,h} \cup \bigcup_{h\geq 4} \widetilde{\mathcal{F}}^4_{n,d,h} \\
\leftrightarrow & \bigcup_{h\geq 5} \widetilde{\mathcal{E}}^{1^*}_{n,d,h} \cup \bigcup_{h\geq 5} \widetilde{\mathcal{E}}^{2^*}_{n,d,h} \cup \bigcup_{h\geq 5} \widetilde{\mathcal{E}}^{3^*}_{n,d,h} \cup \bigcup_{h\geq 5} \widetilde{\mathcal{E}}^{4^*}_{n,d,h} \cup \\
& \cup \bigcup_{h\geq 4} \widetilde{\mathcal{F}}^{1^*}_{n,d,h} \cup \bigcup_{h\geq 4} \widetilde{\mathcal{F}}^{2^*}_{n,d,h} \cup \bigcup_{h\geq 4} \widetilde{\mathcal{F}}^{3^*}_{n,d,h} \cup \bigcup_{h\geq 4} \widetilde{\mathcal{F}}^{4^*}_{n,d,h} \\
= & \bigcup_{\substack{h\geq 5\\ h \text{ odd}}} \left(\widetilde{\mathcal{E}}^{1^*}_{n,d,h} \cup \widetilde{\mathcal{E}}^{2^*}_{n,d,h-2} \cup \widetilde{\mathcal{E}}^{3^*}_{n,d,h-2} \cup \widetilde{\mathcal{E}}^{4^*}_{n,d,h-2}\right) \cup \\
& \bigcup_{\substack{h\geq 5\\ h \text{ odd}}} \left(\widetilde{\mathcal{F}}^{1^*}_{n,d,h-1} \cup \widetilde{\mathcal{F}}^{2^*}_{n,d,h-1} \cup \widetilde{\mathcal{F}}^{3^*}_{n,d,h-1} \cup \widetilde{\mathcal{F}}^{4^*}_{n,d,h-3}\right) \\
= & \bigcup_{\substack{h\geq 5\\ h \text{ odd}}} \mathcal{D}_{k+(h-1)/2,\, h-3} \\
= & \bigcup_{i\geq 0} \mathcal{D}_{k+2+i,\, 2+2i}.
\end{aligned}$$

Notice that the union in the last equation does not provide any contribution when i is sufficiently large, therefore it represents a finite union of sets. Notice that the largest number of parts that can appear in a partition of $T_{n,d}$ is approximatively the square root of n, while there is a linear dependence in i between $k+2+i$ and $2+2i$.

Let us now prove that $\bigcup_{i\geq 0} \mathcal{D}_{k+2+i,2+2i} \leftrightarrow \mathcal{D}^{\mathcal{O}}_{2(k+1)}$. First notice that, if $\lambda \in \mathcal{D}^{\mathcal{O}}_{2(k+1)}$, then $|\lambda|$ is even, therefore the following equation trivially holds

$$\mathcal{D}^{\mathcal{O}}_{2(k+1)} = \bigcup_{i\geq 0} \mathcal{D}^{\mathcal{O}}_{2(k+1),\, 2+2i},$$

where the last union is again only formally infinite. Let us define

$$\begin{array}{rccc} \psi\colon & \mathcal{D}_{k+2+i,\,2+2i} & \to & \mathcal{D}^{\mathcal{O}}_{2(k+1),\,2+2i} \\ & (\lambda_1,\dots,\lambda_{2+2i}) & & (2\lambda_1-1,\dots,2\lambda_{2+2i}-1) \end{array}$$

and let us prove that ψ is bijective. Clearly ψ is well defined, indeed if $\lambda \in \mathcal{D}_{k+2+i,\,2+2i}$, then

$$\begin{aligned} \psi(\lambda) \vdash 2\lambda_1 - 1 + \cdots + 2\lambda_{2+2i} - 1 &= 2(\lambda_1 + \cdots + \lambda_{2+2i}) - 2 - 2i \\ &= 2(k+2+i) - 2 - 2i \\ &= 2k+2. \end{aligned}$$

Let us now prove that ψ is surjective. Let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{2+2i}) \in \mathcal{D}^{\mathcal{O}}_{2(k+1),\,2+2i}$. It is easy to verify that

$$\rho = \left(\frac{\sigma_1+1}{2}, \frac{\sigma_2+1}{2}, \dots, \frac{\sigma_{2+2i}+1}{2}\right)$$

is such that $\psi(\rho) = \sigma$. Since ψ is trivially injective, the claim is proved for $d > 4$. In the case $d = 4$, from Equation (5.11) we have

$$\begin{aligned} \widetilde{\mathcal{U}}_{T_{n,4}} \leftrightarrow & \widetilde{\mathcal{U}}^*_{T_{n,4}} \\ &= \widetilde{\mathcal{E}}_{n,4} \mathbin{\dot\cup} \{()\} \mathbin{\dot\cup} \left(\bigcup_{h\geq 6} \widetilde{\mathcal{F}}^1_{n,4,h}\right) \mathbin{\dot\cup} \widetilde{\mathcal{F}}^2_{n,4} \mathbin{\dot\cup} \widetilde{\mathcal{F}}^3_{n,4} \mathbin{\dot\cup} \widetilde{\mathcal{F}}^4_{n,4}. \end{aligned}$$

The claim is obtained as before, only noticing the empty partition () replaces the partition of $\widetilde{\mathcal{F}}^1_{n,d,4}$, which is not defined when $d = 4$ (cf. Remark 5.37).

□

Now we completed the classification of maximal unrefinable partitions. We have that, if N is the triangular number T_n, then the number of maximal unrefinable partitions of T_n is one if n is even and coincides with the number of partitions of $(n+1)/2$ into distinct parts if n is odd. If N is non-triangular, i.e., if $N = T_{n,d}$ for some $n \geq 11$ and $1 \leq d \leq n-1$, from Theorem 5.28 and Theorem 5.39 we obtain:

Corollary 5.40. *Let $n \geq 11$ and $1 \leq d \leq n-1$ be integers. If n is odd, then*

$$\#\widetilde{\mathcal{U}}_{T_{n,d}} = \begin{cases} 1 + \#\mathcal{D}_{(n-d+1)/2} & \text{if } d > 3 \text{ is even,} \\ \#\mathcal{D}^{\mathcal{O}}_{n-d+2} & \text{if } d > 3 \text{ is odd,} \\ 1 & \text{if } d \in \{1,2,3\}. \end{cases}$$

Otherwise

$$\#\widetilde{\mathcal{U}}_{T_{n,d}} = \begin{cases} 1 + \#\mathcal{D}_{(n-d+1)/2} & \text{if } d > 2 \text{ is odd,} \\ \#\mathcal{D}^{\mathcal{O}}_{n-d+2} & \text{if } d > 2 \text{ is even,} \\ 1 & \text{if } d \in \{1,2\}. \end{cases}$$

N	$\lambda_{t\max}$	$\#\widetilde{\mathcal{U}}_N$	N	$\lambda_{t\max}$	$\#\widetilde{\mathcal{U}}_N$
T_{n-1}	$2n-4$	$\mathcal{D}_{n/2}$	T_n	$2n-4$	1
$T_{n,n-1}$	$2n-4$	1	$T_{n+1,n}$	$2n-4$	1
$T_{n,n-2}$	$2n-5$	1	$T_{n+1,n-1}$	$2n-5$	1
$T_{n,n-3}$	$2n-4$	1	$T_{n+1,n-2}$	$2n-4$	1
$T_{n,n-4}$	$2n-5$	1	$T_{n+1,n-3}$	$2n-5$	1
$T_{n,n-5}$	$2n-4$	2	$T_{n+1,n-4}$	$2n-4$	2
$T_{n,n-6}$	$2n-5$	2	$T_{n+1,n-5}$	$2n-5$	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$T_{n,n-(2k-1)}$	$2n-4$	$\mathcal{D}_{k+1}$	$T_{n+1,n+1-(2k-1)}$	$2n-4$	$\mathcal{D}_{k+1}$
$T_{n,n-(2k)}$	$2n-5$	$\mathcal{D}^{\mathcal{O}}_{2k+2}$	$T_{n+1,n+1-(2k)}$	$2n-5$	$\mathcal{D}^{\mathcal{O}}_{2k+2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$T_{n,4}$	$2n-5$	$\mathcal{D}^{\mathcal{O}}_{n-2}$	$T_{n+1,5}$	$2n-5$	$\mathcal{D}^{\mathcal{O}}_{n-2}$
$T_{n,3}$	$2n-4$	$\mathcal{D}_{(n-2)/2}$	$T_{n+1,4}$	$2n-4$	$\mathcal{D}_{(n-2)/2}$
$T_{n,2}$	$2n-3$	1	$T_{n+1,3}$	$2n-4$	1
$T_{n,1}$	$2n-2$	1	$T_{n+1,2}$	$2n-3$	1
T_n	$2n-4$	1	$T_{n+1,1}$	$2n-2$	1
$\star$	$\star$	$\star$	T_{n+1}	$2n-4$	$\mathcal{D}_{1+n/2}$

Table 5.2: The number of maximal unrefinable partitions between two consecutive triangular numbers. Here n is an even number.

The two results can be appreciated in Table 5.2, where we list the number of maximal unrefinable partitions for integers included between two consecutive triangular numbers. Precisely, we start from an even integer n and list the number $\#\widetilde{\mathcal{U}}_{T_{n,d}}$ and the corresponding maximum λ_t, for each integer in $\{s \in \mathbb{N} \mid T_{n-1} \leq s \leq T_{n+1}\}$. The same combinatorial structure replicates in other intervals between two consecutive triangular numbers, according to the rules of Theorem 5.28 and Theorem 5.39.

d	$(a_1,\ldots,a_h)$	$(\alpha_1,\ldots,\alpha_j)$
4	$(n-1,n)$	$(d,2n-5)$
$n-2$	$(n-4,n-3,n)$	$(d,2n-5)$
$n-4$	$(n-5,n-3,n-1)$	$(d,2n-5)$
$n-6$	$(n-5,n-4,n-2)$	$(d,2n-5)$
$n-6$	$(n-3,n-1,n)$	$(n+1,2n-5)$
$4<n-2k\leq n-8$	$\left(\frac{n-6+d}{2},n-3,n-1,n\right)$	$\left(d,\frac{3n-4-d}{2},2n-5\right)$
$4\leq n-2k\leq n-8$	$\left(\frac{n-4+d}{2},n-4,n-2,n\right)$	$\left(d,\frac{3n-6-d}{2},2n-5\right)$
$4\leq n-2k\leq n-10$	$\left(\frac{n-2+d}{2},n-5,n-2,n-1\right)$	$\left(d,\frac{3n-8-d}{2},2n-5\right)$
$4\leq n-2k\leq n-14$	$\left(\frac{n+2+d}{2},n-5,n-4,n-3\right)$	$\left(d,\frac{3n-12-d}{2},2n-5\right)$
$4\leq n-2k\leq n-\left(h^2-11\right)$ for $h\geq 5, h$ odd	$\left(\frac{n+\left(h^2-2h-15\right)+d}{2}+i,n-h-1-i_1,\ldots\right.$ $\left.n-6-i_{h-4},n-2,n-1,n\right)$	$(d,n+1+i_{h-4},\ldots,n-4+h+i_1,$ $\left.\frac{3n-\left(h^2-2h-5\right)-d}{2}-i,2n-5\right)$
$4\leq n-2k\leq n-\left(h^2-10\right)$ for $h\geq 5, h$ even	$\left(\frac{n+\left(h^2-2h-14\right)+d}{2}+i,n-h-1-i_1,\ldots\right.$ $\left.n-6-i_{h-4},n-3,n-1,n\right)$	$(d,n+1+i_{h-4},\ldots,n-4+h+i_1,$ $\left.\frac{3n-\left(h^2-2h-4\right)-d}{2}-i,2n-5\right)$
$4\leq n-2k\leq n-\left(h^2-8\right)$ for $h\geq 5, h$ even	$\left(\frac{n+\left(h^2-2h-12\right)+d}{2}+i,n-h-1-i_1,\ldots\right.$ $\left.n-6-i_{h-4},n-4,n-2,n\right)$	$(d,n+1+i_{h-4},\ldots,n-4+h+i_1,$ $\left.\frac{3n-\left(h^2-2h-2\right)-d}{2}-i,2n-5\right)$
$4\leq n-2k\leq n-\left(h^2-7\right)$ for $h\geq 5, h$ odd	$\left(\frac{n+\left(h^2-2h-11\right)+d}{2}+i,n-h-1-i_1,\ldots\right.$ $\left.n-6-i_{h-4},n-4,n-3,n\right)$	$(d,n+1+i_{h-4},\ldots,n-4+h+i_1,$ $\left.\frac{3n-\left(h^2-2h-1\right)-d}{2}-i,2n-5\right)$
$4\leq n-2k\leq n-\left(h^2-6\right)$ for $h\geq 5, h$ even	$\left(\frac{n+\left(h^2-2h-10\right)+d}{2}+i,n-h-1-i_1,\ldots\right.$ $\left.n-6-i_{h-4},n-5,n-2,n-1\right)$	$(d,n+1+i_{h-4},\ldots,n-4+h+i_1,$ $\left.\frac{3n-\left(h^2-2h\right)-d}{2}-i,2n-5\right)$
$4\leq n-2k\leq n-\left(h^2-5\right)$ for $h\geq 5, h$ odd	$\left(\frac{n+\left(h^2-2h-9\right)+d}{2}+i,n-h-1-i_1,\ldots\right.$ $\left.n-6-i_{h-4},n-5,n-3,n-1\right)$	$(d,n+1+i_{h-4},\ldots,n-4+h+i_1,$ $\left.\frac{3n-\left(h^2-2h+1\right)-d}{2}-i,2n-5\right)$
$4\leq n-2k\leq n-\left(h^2-3\right)$ for $h\geq 5, h$ odd	$\left(\frac{n+\left(h^2-2h-7\right)+d}{2}+i,n-h-1-i_1,\ldots\right.$ $\left.n-6-i_{h-4},n-5,n-4,n-2\right)$	$(d,n+1+i_{h-4},\ldots,n-4+h+i_1,$ $\left.\frac{3n-\left(h^2-2h+3\right)-d}{2}-i,2n-5\right)$
$4\leq n-2k\leq n-\left(h^2-2\right)$ for $h\geq 5, h$ even	$\left(\frac{n+\left(h^2-2h-6\right)+d}{2}+i,n-h-1-i_1,\ldots\right.$ $\left.n-6-i_{h-4},n-5,n-4,n-3\right)$	$(d,n+1+i_{h-4},\ldots,n-4+h+i_1,$ $\left.\frac{3n-\left(h^2-2h+4\right)-d}{2}-i,2n-5\right)$

Table 5.3: List of all the possible maximal constructions when $\lambda_t = 2n-5$

Chapter 6

Conclusion and future reasearch

In this last chapter we show some relations between unrefinable partitions and numerical semigroups. We also present some possible future directions to study.

6.1.1 Unrefinable partitions and numerical semigroups

Let $S = \{0, s_1, \ldots, s_n, \rightarrow\}$ be a numerical semigroup and $S^c = \{s_1^c, \ldots, s_t^c\}$ be the set of gaps of S. We can observe, by definition of numerical semigroups, that the sum of $s_i, s_j \in S$ is such that $s_i + s_j \notin S^c$, in other words every element of S^c cannot be obtained as a sum of two or more elements that are not in S^c. We obtain that every set of gaps of a numerical semigroup S^c coincides with an unrefinable partition λ. More precisely we have the following correspondences

$$\begin{array}{lll} S^c = \{s_1^c, \ldots, s_t^c\} & \rightarrow & \lambda = (\lambda_1, \ldots, \lambda_t) \\ S = \{0, s_1, \ldots, s_t + 1, \rightarrow\} & \rightarrow & \{0\} \cup \mathcal{M}_\lambda \cup \{\lambda_t + 1, \rightarrow\} \\ G(S) & \rightarrow & len(\lambda) \\ F(S) & \rightarrow & \lambda_t \\ M(S) & \rightarrow & mex(\lambda) = \mu_1 \\ Ap(S, s_1) & \rightarrow & \vec{p}_\lambda \end{array}$$

We can observe another relation between numerical semigroups and unrefinable partitions. The Apery set respect s_1 coincides with the vector of forbidden parts of the unrefinable partition, except for the first position where in the case of the Apery set appears 0, while in the vector $2s_1$.

Example 6.1. *Let $S = \{0, 4, 7, 8, 10, 11, 12, 14, \rightarrow\}$ be a numerical semigroup, then the comlementary $S^c = \{1, 2, 3, 5, 6, 9, 13\}$ defines the unrefinable partition $\lambda = (1, 2, 3, 5, 6, 9, 13)$. We can observe that $\mathcal{M}_\lambda = \{4, 7, 8, 10, 11, 12\}$ coincides with $S^c \setminus (\{0\} \cup \{14, \rightarrow\})$. Moreover, the set of gaps represents the length of the* *partition* $G(S) = 7 = len(\lambda)$*, the Frobenius number coincides with the maximal part of* λ $F(S) = 13 = \lambda_7$ *and* $M(S) = 4$ *is the* $mex(\lambda)$*. The Apery set of* S *respect to* 4 *is* $\{0, 17, 10, 7\}$ *while the vector* $\vec{p}_\lambda = \{8, 17, 10, 7\}$*.*

As a direct consequence of these correspondences we have the following result.

Proposition 6.2. *Let* NS *be the set of numerical semigroups and* $\mathcal{U}$ *be the set of unrefinable partitions into distinct parts. We have*

$$NS \subset \mathcal{U}$$

Notice that the other inclusion is false. If $\lambda \in \mathcal{U}$ then, by definition every $\lambda_i \in \lambda$ cannot be substituted as the sum of two equal missing parts, so if $\mu_1 \in \mathcal{M}_\lambda$ then $2\mu_1$ might be a part of λ, contrary to the case of numerical semigroups for which if a positive integer is an element of S then its double must also be in S. So every unrefinable partition λ, in general every partition into distinct parts, defines a numerical set S_λ such that $S_\lambda^c = \lambda$.

Example 6.3. *The partition* $\lambda = (1, 2, 5, 6, 8)$ *is an unrefinable partition, while the set* $S_\lambda = \{0, 3, 4, 7, 9, \rightarrow\}$ *is not a numerical semigroup because* $6 = 3 + 3$ *and* $8 = 4 + 4$ *are not in* S_λ*.*

By the characterisation of hooksets of numerical semigroups (Proposition 2.16), we can prove a similar result in the case of unrefinable partition.

Lemma 6.4. *Let* $\lambda = (\lambda_1, \ldots, \lambda_t, \rightarrow)$ *be an unrefinable partition corresponding Young tableau* Y_{S_λ}*, where* S_λ *is the numerical set associates to* λ*. Then:*

1. *The hook length of the box in the first column and ith row is* λ_i*;*
2. *For each* $2 \leq i \leq \#\mathcal{M}_\lambda$ *the hook length of the top box of the ith column of* Y_{S_λ} *is equal to* $\lambda_t - \mu_{i-1}$*;*
3. λ *is a unrefinable partition if and only every length of the hook of the boxes of* Y_{S_λ}
   - *(a) is contained in the first column* Y_{S_λ}*;*
   - *(b) does not appear in the first column of* Y_{S_λ}*, then the length of the hook of the cell in the first column and the same row is its double.*

Proof. The proof of Part (1) and Part (2) is the same that we used in Proposition 2.16, and it is true in general for all partitions into distinct parts.
Let λ be an unrefinable partitions. If λ corresponds to a numerical semigroup S_λ then, by Proposition 2.16 Part (3), all the cells are marked with numbers of the first column, so the statement (a) is proved. If $S_\lambda \notin NS$, then there exists $\mu_j \in \mathcal{M}_\lambda$ such that $k\mu_j \in \lambda$. In particular we can set $k = 2$, otherwise the partition is refinable. By the fact that $2\mu_j \in \lambda$ we know that there exists one row whose its first box is marked by $2\mu_j$ (Part (1)) and all the other boxes in

the row are such that $2\mu_j - s_i$, with $s_i \in S_\lambda$ and $s_i < \mu_j$, so we have a box marked by $2\mu_j - \mu_j = \mu_j$. If neither the condition (a) and the condition (b) are satisfied, then there exists a box marked by x such that it does not appear in the first column, so, by Part (1), $x \notin \lambda$. Moreover, let z be the first element in the same row of x, we have $x = z - s_i$, with $s_i \in S_\lambda$. By the fact that $z \neq 2x$ and $z = x + s_i$ we obtain a contradiction because λ is refinable.
Conversely, if only condition (a) is satisfied then we obtain a numerical semigroup S_λ and then λ is unrefinable. If conditions (a) and (b) are verified there is a box not in the first column signed by μ_j and the first element of its row is $2\mu_j$ then $2\mu_j \in \lambda$ by Part (1), which cannot be replaced by the sum $\mu_j + \mu_j$, so the partition is unrefinable. □

Notice that the previous result gives another method for recognising if a partition into distinct parts is unrefinable or not

Example 6.5. *If we consider the unrefinable partition $\lambda = (1, 2, 5, 6, 8)$, as in the Example 6.3, and write the hook length of every cell, we obtain*

8	5	4	1
6	3	2	
5	2	1	
2			
1			

We can observe that the condition (a) and (b) of Lemma 6.4 Part (3) are satisfied, then λ is unrefinable.

Let us introduce another subset of the set containing all the unrefinable partitions useful for obtaining relations with numerical semigroups.

Definition 6.6. *Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be an unrefinable partition. We denote by $\mathcal{U}(\lambda_t)$ the set of the unrefinable partitions with the maximal part is equal to λ_t. Let us define the subset of $\mathcal{U}(\lambda_t)$ composed by all the partitions with maximal number of missing parts*

$$\bar{\mathcal{U}}(\lambda_t) = \left\{ \lambda \in \mathcal{U} \mid \#\mathcal{M}_\lambda = \left\lfloor \frac{\lambda_t}{2} \right\rfloor \right\}$$

Lemma 6.7. *Let $NS(k)$ be the set of numerical semigroups such that $F(S) = k$. Then we obtain*

$$\#NS(k) < \#\mathcal{U}(k).$$

Proof. Is a direct consequence of Proposition 6.2. □

We can observe some properties.

Lemma 6.8. *If $\lambda \in \bar{\mathcal{U}}(\lambda_t)$ and let $x \neq \frac{\lambda_t}{2}$, then $x \in \lambda$ if and only if $\lambda_t - x \in \mathcal{M}_\lambda$.*

Proof. By the anti-symmetry property if $\lambda_t - x \notin \lambda$ then x must be a part of λ. Conversely, if we suppose that $x \in \lambda$ and $\lambda_t - x \in \lambda$ then, since $\#\mathcal{M}_\lambda = \left\lfloor \frac{\lambda_t}{2} \right\rfloor$, there exists $y \notin \lambda$ such that $\lambda_t - y \notin \lambda$ which is a contradiction. □

Lemma 6.9. *If $\lambda = (\lambda_1, \dots, \lambda_t) \in \bar{\mathcal{U}}(\lambda_t)$, then $\frac{\lambda_t}{2} \notin \lambda$.*

Proof. If λ_t is an odd number then the element $\frac{\lambda_t}{2}$ is not an integer number, so it cannot appear in λ.
If λ_t is an even number, we have $\frac{\lambda_t}{2}$ missing parts in the integer interval $[1, \lambda_t - 1]$. Let $x \in \left[1, \frac{\lambda_t}{2} - 1\right]$, by Lemma6.8 we have that if $x \notin \lambda$ then $\lambda_t - x \in \lambda$, otherwise if $x \in \lambda$ the element $\lambda_t - x \notin \lambda$. So we set $\frac{\lambda_t}{2} - 1$ missing parts in the interval $\left[1, \frac{\lambda_t}{2} - 1\right] \cup \left[\frac{\lambda_t}{2} + 1, \lambda_t - 1\right]$, then $\frac{\lambda_t}{2} \notin \lambda$. □

Proposition 6.10. *Let $\widehat{\mathcal{U}}$ be the subset of $\widetilde{\mathcal{U}}$ defined as follows*

$$\begin{aligned}\widehat{\mathcal{U}} = &\{(1, \dots, 2k-3, 4k-6) \mid k \geq 4\} \\ &\bigcup \{(1, \dots, 2k-2, 4k-5) \mid k \geq 4\} \bigcup \{\widetilde{\pi}_n \mid n \geq 6\}.\end{aligned}$$

The subset of maximal unrefinable partitions $\widetilde{\mathcal{U}} \setminus \widehat{\mathcal{U}}$ is contained in the set of unrefinable partitions with maximal number of missing parts $\bar{\mathcal{U}}$.

$$\left(\widetilde{\mathcal{U}} \setminus \widehat{\mathcal{U}}\right) \subseteq \bar{\mathcal{U}}$$

Proof. In Chapter 4 and in Chapter 5 we construct maximal unrefinable partitions. In particular in Theorem 4.3 we define the maximal unrefinable partition $\widetilde{\pi}_n = (1, \dots, n-3, n+1, 2n-4)$ obtained from π_n removing three parts and adding two new elements, hence we have $\#\mathcal{M}_{\widetilde{\pi}_n} = n - 3 < n - 2 = \left\lfloor \frac{2n-4}{2} \right\rfloor$. So, for every $n \geq 6$ the partition $\widetilde{\pi}_n \notin \bar{\mathcal{U}}$.
In the first part of Proposition 5.12 we obtain a maximal unrefinable partition $\lambda = (1, \dots, n-2, 2n-4)$ of $T_{n,3}$ when n is an odd number. In this case $\#\mathcal{M}_\lambda = n - 3 < n - 2$, hence $\lambda \notin \bar{\mathcal{U}}$.
Similarly, in Proposition 5.15 we obtain a maximal unrefinable partition for $T_{n,4}$ when n is an even number. In this case the partition $\eta = (1, n-2, 2n-5)$ has $n - 4$ missing parts, while $\left\lfloor \frac{2n-5}{2} \right\rfloor = n - 3$, so $\eta \notin \bar{\mathcal{U}}$.
All the others maximal unrefinable partitions obtained in Theorem 4.3, Proposition 5.5, Proposition 5.8, Proposition 5.11, Proposition 5.12, Proposition 5.14, Proposition 5.15 have the maximal number of missing parts. □

Lemma 6.11. *Let $\lambda \in \bar{\mathcal{U}}(\lambda_t)$ and let λ_t be an odd number such that $\lambda_t \neq 3\mu_i$ for all $\mu_i \in \mathcal{M}_\lambda$. Then the numerical set S_λ associated to λ is a numerical semigroup.*

Proof. We suppose that there exists a numerical set S_λ that is not a numerical semigroup, in other words exists $\mu_i \in \mathcal{M}_\lambda$ such that $k\mu_i \in \lambda$. We can set $k = 2$, otherwise if $2\mu_i \notin \lambda$ then all the multiples of μ_i are not in λ. So we have $2\mu_i \in \lambda$ and then $\lambda_t - 2\mu_i \notin \lambda$ by Lemma 6.8. If $\lambda_t - 2\mu_i \neq \mu_i$ we have $\lambda_t - 2\mu_i + \mu_i = \lambda_t + \mu_i \notin \lambda$, but this is a contradiction because $\lambda_t - \mu_i \in \lambda$ by Lemma6.8. If $\lambda_t - 2\mu_i = \mu_i$, then $\lambda_t = 3\mu_i$ and this is another contradiction. □

Example 6.12. *Let* $\lambda = (1,2,3,4,7,9,10,15)$ *and* $\mu = (1,2,3,5,7,9,11,15)$. *We can observe that it holds* $\#\mathcal{M}_\lambda = \#\mathcal{M}_\mu = 7 = \left\lfloor \frac{15}{2} \right\rfloor$, *and hence* $\lambda, \mu \in \bar{\mathcal{U}}(15)$. *However, while* $S_\lambda = \{0,5,6,8,11,12,13,14,16,\rightarrow\}$ *is not a numerical semigroup, indeed* $10, 15 \notin S_\lambda$, S_μ *is.*

Corollary 6.13. *Let* $\lambda \in \bar{\mathcal{U}}(\lambda_t)$, *such that* λ_t *is an odd number coprime with* 3, *then the numerical set associated* S_λ *is a numerical semigroup.*

This result follows directly from Lemma 6.11, but it is possible to state even a stronger version.

Corollary 6.14. *Let* $\lambda \in \bar{\mathcal{U}}(\lambda_t)$, *such that* λ_t *is a prime number, then the numerical set associated* S_λ *is a numerical semigroup.*

Theorem 6.15. *Let* λ_t *be a prime number. Then we have*

$$\#\bar{\mathcal{U}}(\lambda_t) = \# \{S \in SNS \mid F(S) = \lambda_t\}$$

Proof. The claim follows from Corollary 6.14 and Definition 2.24. □

Now we introduce two sets those induce a useful decomposition of $\mathcal{U}(\lambda_t)$. Let us define

$$\mathcal{U}(\lambda_t, \mu_1) = \{\lambda \in \mathcal{U}(\lambda_t) \mid \mu_1 = mex(\lambda)\},$$
$$\bar{\mathcal{U}}(\lambda_t, \mu_1) = \left\{\lambda \in \bar{\mathcal{U}}(\lambda_t) \mid \mu_1 = mex(\lambda)\right\}$$

We can obtain the first set starting from the second one. Let $\lambda \in \bar{\mathcal{U}}(\lambda_t)$ such that λ_t is prime and let $\vec{p}_\lambda$ be the vector of forbidden integers (Definition 3.8). By Theorem 6.15 S_λ is a numerical semigroup then we have $(\vec{p}_\lambda)_1 = 2\mu_1$, the other $(\vec{p}_\lambda)_i$, with $2 \le i \le \mu_1$, are equal to the lower integers modulo μ_1 such that they are not a parts of λ. If there exists $1 \le i \le \mu_1$ such that $(\vec{p}_\lambda)_i \neq (\vec{p}_\lambda)_j + (\vec{p}_\lambda)_k$ for $1 \le j < k \le \mu_1$ such that $j, k \neq i$, we can obtain a new unrefinable partition $\lambda^* \in \mathcal{U}(\lambda_t, \mu_1)$ such that $\lambda^* = \lambda \cup \{(\vec{p}_\lambda)_i\}$. The vector $\vec{p}_{\lambda^*}$ changes from $\vec{p}_\lambda$ only in the position i, i.e., $(\vec{p}_{\lambda^*})_i = (\vec{p}_\lambda)_i + \mu_1$.

Example 6.16. *Let* $\lambda_t = 13$ *and* $\mu_1 = 3$, *we have* $\bar{\mathcal{U}}(\lambda_t, \mu_1) = \{\lambda\}$, *where* $\lambda = (1,2,4,5,7,10,13)$ *and* $\vec{p}_\lambda = (6,16,8)$. *Since* $13 \equiv 1 \pmod 3$, *all the lower integers in the same modulo class are in* λ. *We can add to* λ *the integers of the others modulo classes* μ_1, *respecting the unrefinability. So we can obtain new unrefinable partitions adding these integers to* λ *(see fig. 6.1).*

Here we have proved just some relationship between unrefinable partitions and numerical semigroups. A future research could be devoted to understand such connection more deeply and finding formulas for $\bar{\mathcal{U}}(\lambda_t)$.

6.1.2 Unrefinable partitions and Polytopes

Definition 6.17. *A* ***polytope*** P *is a non-empty and bounded intersection of finitely many closed halfspaces in* $\mathbb{R}^n$. *Any intersection of some of the halfspaces defining the polytope* P *and* P *itself is called* ***face***. *The face of dimension* 0 *are the vertices of* P. *If the vertex coordinates are rational, then* P *is a* ***rational polytope****[Zie95].*

$$\begin{array}{ccccccc}
\{8,11\} & \longrightarrow & \{6,8,11\} & \longrightarrow & \{6,8,9,11\} & \longrightarrow & \{6,8,9,11,12\} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\{8\} & \longrightarrow & \{6,8\} & \longrightarrow & \{6,8,9\} & \longrightarrow & \{6,8,9,12\} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\{0\} & \longrightarrow & \{6\} & \longrightarrow & \{6,9\} & \longrightarrow & \{6,9,12\}
\end{array}$$

Figure 6.1: Given $\lambda = \{1, 2, 4, 5, 7, 10, 13\}$, any directed path from $\{0\}$ to $\{6, 8, 9, 11, 12\}$ gives a sequence of integers that, added to λ, keep it an unrefinable partition. These are all the unrefinable partitions such that $\lambda_t = 13$ and $\mu_1 = 3$.

Definition 6.18. *A non-empty set $C \subseteq \mathbb{R}^n$ is a **cone** if $\alpha x + \beta y \in C$ for each $x, y \in C$ and $\alpha, \beta \in \mathbb{R}_{\geq 0}$. A cone is **polyhedral** if it is finitely generated*

Let S be a numerical semigroup and $M(S) = m$.

Proposition 6.19 ([Kap17])**.** *Let S be a numerical semigroup with Apery set $\{m, k_1 m + 1, \ldots, k_{m-1} m + m - 1\}$ with respect m, where $k_i \geq 1$. Then*

$$\sum_{i=1}^{m-1} k_i = G(S)$$

Notice that the different choices of k_i are a composition of $G(S)$. Recall that a composition of an integer n is a way of writing n as a sum of positive integers where the order of these integers does matter.

Proposition 6.20 ([Kap17])**.** *The set $\{m, k_1 m + 1, \ldots, k_{m-1} m + m - 1\}$ is the Apery set of the numerical semigroup $S = \langle m, k_1 m + 1, \ldots, k_{m-1} m + m - 1\rangle$ if and only if for all $1 \leq i \leq m - 1$, $(k_i - 1)\, m + i \notin S$*

There are some necessary conditions for $\{m, k_1 m + 1, \ldots, k_{m-1} m + m - 1\}$ to be an Apery set.The results of [RGSGGB02] and [Kun87] are that these conditions completely determine which compositions of $G(S)$ lead to a valid Apery set. Consider the following set of inequalities

$$\begin{array}{ll}
x_i \geq 1 & \forall i \in \{1, \ldots, m-1\} \\
x_i + x_j \geq x_{i+j} & \forall 1 \leq i \leq j \leq m-1, i + j \leq m - 1 \\
x_i + x_j + 1 \geq x_{i+j-m} & \forall 1 \leq i \leq j \leq m-1, i + j > m - 1 \\
x_i \in \mathbb{Z} & \forall i \in \{1, \ldots, m-1\}
\end{array}$$

Proposition 6.21 ([Kun87],[RGSGGB02]). *There is a one to one correspondence between solutions $\{k_1, \ldots, k_{m-1}\}$ to the above inequalities and the Apery sets of numerical semigroups with multiplicity m. If we add the condition that $\sum_{i=1}^{m-1} k_i = G(S)$, there is a one to one correspondence between solution $\{k_1, \ldots, k_{m-1}\}$ to the above inequalities and the Apery sets of numerical semigroups with multiplicity m and genus $G(S)$.*

Each of above inequalities defines a half space in $\mathbb{R}^{m-1}$ and their intersection defines a rational polyhedral cone. If we fix $\sum_{i=1}^{m-1} k_i = G(S)$ then each $k_i \leq G(S)$ and we obtain a rational polytope. It is possible to use the theory of integer points in rational polytopes to count the numerical semigroups with genus $G(S)$ and multiplicity $M(S)$, and we might extend such method for counting unrefinable partitions with fixed minimal excludant and fixed number of parts.

6.1.3 Density and generating function of unrefinable partitions

Definition 6.22. *Let f and g two functions, we denote $f(\lambda_t) \sim g(\lambda_t)$, if for $\lambda_t \to \infty$ we have*

$$\frac{f(\lambda_t)}{g(\lambda_t)} = 1$$

Fixing λ_t as the maximal part of unrefinable partitions λ, we try to count the elements of $\mathcal{U}(\lambda_t)$ depending on the cardinality of $\mathcal{M}_\lambda$.
Let $0 \leq h \leq \left\lfloor \frac{\lambda_t}{2} \right\rfloor$ be the cardinality of $\mathcal{M}_\lambda$. It is easy to verify that $\#\mathcal{U}(\lambda_t, h = 0) = 1$ and $\#\mathcal{U}(\lambda_t, h = 1) = \lambda_t - 1$. If $h = 2$ we can count these partition depending on $2 \leq \mu_1 \leq \lambda_t - 2$. We can observe that if $2 \leq \mu_1 \leq \left\lceil \frac{\lambda_t}{2} \right\rceil - 1$ then μ_2 is such that $\lambda_t - \mu_1 < \mu_2 < \lambda_t$, while if $\left\lceil \frac{\lambda_t}{2} \right\rceil \leq \mu_1 \leq \lambda_t - 2$, then $\mu_1 < \mu_2 < \lambda_t$. We obtain

$$\#\mathcal{U}(\lambda_t, 2) \sim \frac{(\lambda_t - 1)^2}{4},$$

Also if $h = 3$ we obtain something similar

$$\#\mathcal{U}(\lambda_t, 3) \sim \frac{\lambda_t^3}{24}$$

When h becomes bigger it is more difficult to understand in which way the missing parts can be disposed.

Conjecture 1. *Let $0 \leq h \leq \left\lfloor \frac{\lambda_t}{2} \right\rfloor$*

$$\#\mathcal{U}(\lambda_t, h = k) \sim \frac{\lambda_t^k}{2^{k-1} k!}$$

We also can observe that the number of partitions into distinct parts with maximal part λ_t and h missing parts is equal to $\binom{\lambda_t - 1}{h} \sim \frac{\lambda_t^h}{h!}$, so we conjecture the following

Conjecture 2.

$$\lim_{\lambda_t \to \infty} \frac{\#\mathcal{U}(\lambda_t)}{\#\mathcal{D}_{\lambda_t}} = 0$$

Another interesting aspect of further research is to find some combinatorial properties of unrefinable partitions into distinct parts in order to define their generating functions.

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