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A MODULAR IDEALIZER CHAIN AND UNREFINABILITY OF PARTITIONS WITH REPEATED PARTS*

BY

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ABSTRACT

Recently Aragona et al. have introduced a chain of normalizers in a Sylow 2-subgroup of $\mathrm{Sym}(2^n)$, starting from an elementary abelian regular subgroup. They have shown that the indices of consecutive groups in the chain depend on the number of partitions into distinct parts and have given a description, by means of rigid commutators, of the first $n-2$ terms in the chain. Moreover, they proved that the $(n-1)$-th term of the chain is described by means of rigid commutators corresponding to unrefinable partitions into distinct parts. Although the mentioned chain can be defined in a Sylow p-subgroup of $\mathrm{Sym}(p^n)$, for $p>2$ computing the chain of normalizers becomes a challenging task, in the absence of a suitable notion of rigid commutators. This problem is addressed here from an alternative point of view. We propose a more general framework for the normalizer chain, defining a chain of idealizers in a Lie ring over $\mathbb{Z}_m$ whose elements are represented by integer partitions. We show how the corresponding idealizers are generated by subsets of partitions into at most $m-1$ parts and we conjecture that the idealizer chain grows as the normalizer chain in the symmetric group. As evidence of this, we establish a correspondence between the two constructions in the case $m=2$.

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1. Introduction

Let $n \geq 3$ be an integer and $\Sigma \leq \mathrm{Sym}(2^n)$ be a Sylow 2-subgroup containing an elementary abelian regular subgroup T. Let us define $N_0 = N_\Sigma(T)$ and recursively let N_i be the normalizer in Σ of the previous term, i.e.,

$$N_i = N_\Sigma(N_{i-1}). \tag{1}$$

Aragona et al. [ACGS21b] have recently shown that, for $1 \leq i \leq n-2$, a transversal of N_{i-1} in N_i can be put in one-to-one correspondence with a set of partitions into distinct parts in such a way that, denoting by $\{q_{2,i}\}_{i\geq 1}$ the partial sum of the sequence $\{p_{2,i}\}_{i\geq 1}$ of partitions into distinct parts, the following equality is satisfied:

$$\log_2|N_i : N_{i-1}| = q_{2,i+2}. \tag{2}$$

The first numbers of the mentioned sequences and the relative OEIS references are displayed in Table 1.

Table 1. First values of the sequences $\{p_{2,i}\}$ and $\{q_{2,i}\}$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	OEIS
$p_{2,i}$	0	0	1	1	2	3	4	5	7	9	11	14	17	21	26	31	A111133
$q_{2,i}$	0	0	1	2	4	7	11	16	23	32	43	57	74	95	121	152	A317910

In a subsequent work [ACGS22], the authors introduced the concept of unrefinable partitions and proved that a transversal of N_{n-2} in N_{n-1} is in one-to-one correspondence with a set of unrefinable partitions whose minimal excludant satisfies an additional requirement. The study of the chain on normalizers $(N_i)_{i\geq 0}$ has been carried out up to the $(n-1)$-th term by means of rigid commutators [ACGS21b], a set of generators of Σ, which is closed under commutation and which was intentionally designed for the purpose. However, the technique of rigid commutators could not be easily generalized to the odd case of the normalizer chain, i.e., the one defined in a Sylow p-subgroup of $\mathrm{Sym}(p^n)$, with p odd. Understanding the behavior of the chain in the odd case was indeed left as an open problem by the authors.

1.1. Overview of the new contributions. In an attempt to achieve results in this direction, we introduce the graded Lie ring associated to the lower central series of Σ, which is the iterated wreath product of Lie rings of rank one, and reflects the construction of the Sylow p-subgroup of $\mathrm{Sym}(p^n)$ (cf. also Sushchansky and Netreba [SN05]), for any prime $p \geq 2$.

More generally, given any integer $m \geq 2$, we endow the set of partitions, where each part can be repeated no more than $m-1$ times, with the Lie ring structure mentioned above. We call it the Lie ring of partitions (cf. Section 2). In this ring we recursively define the analog of the chain of normalizers, i.e., the idealizer chain, starting from an abelian subring that plays the role of the elementary abelian regular subgroup T. Notice that, when $m = 2$, no part can be repeated, i.e., that we have the same combinatorial setting as in Aragona et al. [ACGS21b]. Not surprisingly, we could notice that the behavior of the first $n-2$ terms of the chain of idealizers is in complete accordance with that of the chain of normalizers, i.e., Equation (2) has an analogous version for the terms of the idealizer chain, summarized in Theorem 2.14. Interestingly, this result can be made even more general in the setting of the Lie ring of partitions. Indeed the mentioned theorem holds in the case when m is any integer greater than two, provided that partitions with at most $m-1$ repeated parts are considered in place of partitions into distinct parts. In Theorem 2.15 we prove that the growth of the idealizer chain is related to the partial sums of the sequence of the number of partitions with at most $m-1$ repeated parts. This result involves the first $n-1$ terms of the idealizer chain, one more than the case $m = 2$. We conjecture that Theorem 2.15 is the p-analog of the chain of normalizers in $\mathrm{Sym}(p^n)$, where $m = p$ is odd.

Section 3 is totally devoted to the case $m = 2$, where we show that the terms of the normalizer chain can be actually computed via the Lie ring structure described in this paper (see Theorem 3.4). Precisely, we define a bijection (cf. Definition 3.2) from the basis elements of the Lie ring of partitions to the set of rigid commutators which preserves commutators.

In Section 4 we address the problem of first idealizer not following the rules of Theorems 2.14 and 2.15, i.e., the $(n - \delta_{m,2})$-th. If $m = 2$, it has been proved by Aragona et al. [ACGS22] that $\log_2|N_{n-1} : N_{n-2}|$ depends on the number of a suitable subset of unrefinable partitions satisfying some additional constraints. We introduce here a natural generalization of the concept of unrefinability for partitions with at most $m-1$ repeated parts. We prove, in the Lie ring context, that the $(n - \delta_{m,2})$-th idealizer is determined by unrefinable partitions with at most $m - 1$ repeated parts satisfying the same additional constraints as in Aragona et al. [ACGS22] (see Theorem 4.5). We conclude the section by giving a characterization of the n-th idealizer (cf. Theorem 4.7), which, by virtue of Theorem 3.4, also allows to give a precise characterization of the n-th normalizer N_n, improving already known results [ACGS21b, ACGS22].

Section 5 concludes the paper with some comments on open problems.

1.2. Related works in the combinatorics on integer partitions. The original notion of unrefinability for partitions into distinct parts is at least as old as the OEIS entry A179009 [OEI] (due to David S. Newman in 2011) and has been formally introduced by Aragona et al. [ACGS22]. In that paper, unrefinable partitions satisfying a special condition on the minimum excludant appear in a natural way in connection to the chain of normalizers [ACGS21b]. The notion of minimum excludant has been studied in the context of integer partitions by other authors [AN19, BM20, HSS22, DT23], although it also appears in combinatorial game theory [Gur12, FP15]. Partial combinatorial equalities regarding unrefinable partitions have been recently shown in [ACCL22, ACC22], and the study of the algorithmic complexity of generating all the unrefinable partitions of a given integer has been addressed [ACCL23].

2. A polynomial representation of partitions of integers

Let $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ be a sequence of non-negative integers with finite support, i.e., such that

$$\mathrm{wt}(\Lambda) = \sum_{i=1}^{\infty} i\lambda_i < \infty.$$

The sequence Λ defines a **partition** of $N = \mathrm{wt}(\Lambda)$. Each non-zero i is a **part** of the partition, the integer λ_i is the multiplicity of the part i in Λ and the support of Λ is denoted by $\mathrm{supp}(\Lambda) = \{i \mid \lambda_i \neq 0\}$. The maximal part of Λ is the maximum i such that $\lambda_i \neq 0$, i.e., $\max\mathrm{supp}(\Lambda)$. The set of the partitions whose maximal part is at most j is denoted by $\mathcal{P}(j)$ and we define for each $m > 0$

$$\mathcal{P}_m(j) = \{\Lambda \in \mathcal{P}(j) \mid \lambda_i \leq m - 1 \text{ for all } i\}$$

as the set of partitions with maximal part at most j and where each part has multiplicity at most $m - 1$. We set also

$$\mathcal{P}_m = \bigcup_{j \geq 1} \mathcal{P}_m(j).$$

2.1. Power monomials. In the polynomial ring $\mathbb{Z}[x_k]_{k=1}^{\infty}$ we consider the monomials x_k^i where i is a non-negative integer. The **power monomial** x^Λ, where Λ is a partition, is defined as

$$x^\Lambda = \prod_i x_i^{\lambda_i}.$$

These monomials clearly form a basis for $\mathbb{Z}[x_k]_{k=1}^{\infty}$ as a free $\mathbb{Z}$-module. The set of power monomials in at most n variables is denoted by

$$\mathrm{Mon}_n = \{x^\Lambda \mid \Lambda \in \mathcal{P}(n)\}.$$

The degree of the power monomials x^Λ is defined as $\deg(x^\Lambda) = \sum_{i\geq 1} \lambda_i$.

Note that

$$x^\Lambda x^\Theta = \prod_i x_i^{\lambda_i+\theta_i} = x^{\Lambda+\Theta}.$$

In particular the $\mathbb{Z}$-module $\mathbb{Z}[x_1, \dots, x_n]$, with basis Mon_n, has a natural structure of $\mathbb{Z}$-algebra and is the ring of polynomials in n variables with coefficients in $\mathbb{Z}$.

The k-partial derivative is defined by

$$\partial_k(x^\Lambda) = \begin{cases} 0 & \text{if } \lambda_k = 0, \\ \lambda_k x^{D_k(\Lambda)} & \text{otherwise,} \end{cases}$$

where $D_k(\Lambda) = \{\lambda_i - \delta_{ik}\}_{i=1}^{\infty}$. In particular ∂_k can be extended by linearity to a derivation over $\mathbb{Z}[x_1, \dots, x_n]$.

Let m be a positive integer and consider the ideal $I = (x_1^m, \dots, x_n^m)$ of $\mathbb{Z}[x_1, \dots, x_n]$. Clearly $\partial_k(I) \subseteq m\,\mathbb{Z}[x_1, \dots, x_n]$ and so the k-th partial derivative can be seen also as a derivation defined on the **ring of power monomials modulo** m in n variables (see also Strade [Str17])

$$\mathcal{O}_m(n) = \mathbb{Z}_m[x_1, \dots, x_n]/(x_1^m, \dots, x_n^m).$$

Starting from a modular Lie ring $\mathfrak{g}$ over $\mathbb{Z}$, let us define $\mathfrak{g}^\uparrow = \mathcal{O}_m(1) \otimes_{\mathbb{Z}} \mathfrak{g}$. We also define the **inflated Lie algebra** as

$$\mathrm{Inf}(\mathfrak{g}) = \langle \partial \otimes 1 \rangle \ltimes \mathfrak{g}^\uparrow,$$

where ∂ is the standard derivative.

2.2. Lie rings of partitions. The **Lie ring $\mathfrak{L}(n)$ over $\mathbb{Z}_m$ of partitions with maximal part at most** $n-1$ is obtained starting from the trivial Lie ring $\mathfrak{L}(1)=\mathbb{Z}_m$ and defining iteratively $\mathfrak{L}(i)=\operatorname{Inf}(\mathfrak{L}(i-1))$. For the sake of brevity, we shall write $\mathfrak{L}$ in place of $\mathfrak{L}(n)$.

In order to have a description which is more suitable for computations, $\mathfrak{L}$ can be seen as the free $\mathbb{Z}_m$-module with basis $\mathcal{B}=\bigcup_{i=1}^n \mathcal{B}_i$, where

$$\mathcal{B}_i=\{x^\Lambda\partial_i \mid x^\Lambda\in\mathcal{O}_m(n) \text{ with } \Lambda\in\mathcal{P}_m(i-1)\}.$$

The Lie bracket is defined on this basis by

$$\begin{aligned}[x^\Lambda\partial_k, x^\Theta\partial_j] &= \partial_j(x^\Lambda)x^\Theta\partial_k - x^\Lambda\partial_k(x^\Theta)\partial_j\\ &= \begin{cases}\partial_j(x^\Lambda)x^\Theta\partial_k & \text{if } j<k,\\ -x^\Lambda\partial_k(x^\Theta)\partial_j & \text{if } j>k,\\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{3}$$

and is extended to $\mathfrak{L}$ by bilinearity. If $\mathfrak{L}_i$ is the $\mathbb{Z}_m$-linear span of $\mathcal{B}_i$ then $\mathfrak{L}_i$ is an abelian subring of $\mathfrak{L}$ and $[\mathfrak{L}_i,\mathfrak{L}_j]\subseteq\mathfrak{L}_{\max(i,j)}$ and, as a $\mathbb{Z}_m$-module,

$$\mathfrak{L}(n)=\bigoplus_{i=1}^n\mathfrak{L}_i=\mathfrak{L}(n-1)\oplus\mathfrak{L}_n\,.$$

Moreover $\mathfrak{L}_n$ is an ideal and so $\mathfrak{L}(n)=\mathfrak{L}(n-1)\ltimes\mathfrak{L}_n$, as a Lie ring.

For a subset $\mathcal{H}$ of $\mathfrak{L}$ we set

$$\mathbb{Z}_m\,\mathcal{H}=\{ax^\lambda\partial_k \mid a\in\mathbb{Z}_m \text{ and } x^\lambda\partial_k\in\mathcal{H}\}.$$

Let $\varphi_{\Theta,j}\colon\mathcal{B}\to\mathbb{Z}_m\,\mathcal{B}$ be the right adjoint map defined by

$$\varphi_{\Theta,j}(x^\Lambda\partial_k)=[x^\Lambda\partial_k,x^\Theta\partial_j].$$

Lemma 2.1: *Let $x^\Theta\partial_j\in\mathcal{B}$ and $\mathcal{E}=\{x^\Lambda\partial_k\in\mathcal{B}\mid\varphi_{\Theta,j}(x^\Lambda\partial_k)\neq 0\}$. Then the restriction of $\varphi_{\Theta,j}$ to $\mathcal{E}$ is injective.*

Proof. Assume that $x^\Lambda\partial_k, x^\Xi\partial_l\in\mathcal{E}$ are such that

$$\varphi_{\Theta,j}(x^\Lambda\partial_k)=\varphi_{\Theta,j}(x^\Xi\partial_l). \tag{4}$$

Since both $\varphi_{\Theta,j}(x^\Lambda\partial_k)$ and $\varphi_{\Theta,j}(x^\Xi\partial_l)$ are non-trivial, we either have $j>\max(k,l)$ or $j<\min(k,l)$. In the first case, assuming without loss of generality that $k\le l<j$, from Equation (4) we obtain $\partial_k(x^\Theta)x^\Lambda\partial_j=\partial_l(x^\Theta)x^\Xi\partial_j$, i.e.,

$$x^\Lambda\partial_k(x^\Theta)=x^\Xi\partial_l(x^\Theta). \tag{5}$$

If we assume by contradiction that $k < l$, since $\lambda_l = \xi_l = 0$, we have the exponent of x_l is left unchanged by the derivative ∂_k in the left term of Equation (5) while it is decreased by one in the right term of Equation (5), a contradiction. Hence we have $k = l$, from which we obtain $x^\Lambda \partial_k(x^\Theta) = x^\Xi \partial_k(x^\Theta)$, and therefore $x^\Lambda = x^\Xi$, the claim.

In the second case, $j < \min(k, l)$ means

$$\partial_j(x^\Lambda)x^\Theta\partial_k = \partial_j(x^\Xi)x^\Theta\partial_l, \tag{6}$$

from which immediately $k = l$. Then Equation (6) implies $\partial_j(x^\Lambda) = \partial_j(x^\Xi)$, therefore $x^\Lambda = x^\Xi$. ∎

Definition 2.2: A Lie subring $\mathfrak{H}$ of $\mathfrak{L}$ is said to be **homogeneous** if it is the $\mathbb{Z}_m$-linear span of a subset $\mathcal{H}$ of $\mathcal{B}$.

Example 2.3: The $\mathbb{Z}_m$-submodule $\mathfrak{T}$ of $\mathfrak{L}$ spanned by $\mathcal{T} = \{\partial_1, \ldots, \partial_n\}$ is a homogeneous (abelian) Lie subring. Notice that ∂_i is the generator of the center of $\mathfrak{L}(i)$. When m is prime, this shows that $\mathfrak{T}$ is the natural counterpart for the elementary abelian regular subgroup of the Sylow p-subgroup of $\mathrm{Sym}(p^n)$.

Definition 2.4: If $\mathcal{H}$ is a subset of $\mathcal{B}$, then its **idealizer** is defined as

$$N_{\mathcal{B}}(\mathcal{H}) = \{b \in \mathcal{B} \mid [b, h] \in \mathbb{Z}_m \mathcal{H} \text{ for all } h \in \mathcal{H}\}.$$

The following theorem shows that the idealizers of homogeneous subrings $\mathfrak{H}$ can be efficiently computed directly from the intersection $\mathfrak{H} \cap \mathcal{B}$.

THEOREM 2.5: *Let $\mathfrak{H}$ be a homogeneous subring of $\mathfrak{L}$ having basis $\mathcal{H} \subseteq \mathcal{B}$. The idealizer of $\mathfrak{H}$ in $\mathfrak{L}$ is the homogeneous subring of $\mathfrak{L}$ spanned by $N_{\mathcal{B}}(\mathcal{H})$ as a free Z_m-module.*

Proof. Let $\mathfrak{N} = N_{\mathfrak{L}}(\mathfrak{H})$ be the idealizer of $\mathfrak{H}$ in $\mathfrak{L}$ and let

$$z = \sum_{x^\Lambda\partial_k \in \mathcal{B}} l_{\Lambda,k} x^\Lambda \partial_k \in \mathfrak{N}.$$

We need to show that $l_{\Lambda,k} x^\Lambda \partial_k \in \mathfrak{N}$ for all Λ and k. Since $\mathfrak{H}$ is a homogeneous subring then for all $x^\Theta\partial_j \in \mathcal{H}$ it suffices to show that if $[l_{\Lambda,k} x^\Lambda \partial_k, x^\Theta\partial_j] \neq 0$ then $[l_{\Lambda,k} x^\Lambda \partial_k, x^\Theta\partial_j] = l_{\Lambda,k}\varphi_{\Theta,j}(x^\Lambda\partial_k) \in \mathbb{Z}_m \mathcal{H}$. Indeed, if $x^\Theta\partial_j \in \mathcal{H}$, then

$$\mathfrak{H} \ni [z, x^\Theta\partial_j] = \sum_{x^\Lambda\partial_k \in \mathcal{B}} l_{\Lambda,k}\varphi_{\Theta,j}(x^\Lambda\partial_k).$$

Since $\varphi_{\Theta,j}(x^\Lambda \partial_k) \in \mathbb{Z}_m \mathcal{B}$, since the set $\mathcal{B}$ is a basis for $\mathfrak{L}$ and since the subset $\mathcal{H} \subseteq \mathcal{B}$ is a basis for $\mathfrak{H}$, by Lemma 2.1 we have $l_{\Lambda,k}\varphi_{\Theta,j}(x^\Lambda \partial_k) \in \mathbb{Z}_m \mathcal{H}$ as required. ∎

2.3. The idealizer chain. Let us now define the bases for the chain of idealizers, starting from the subring $\mathcal{T}$ defined in Example 2.3.

Definition 2.6: For $-1 \leq i \leq n-1-\delta_{m,2}$, set

$$\mathcal{U} = \mathcal{T} \cup \{x_j \partial_k \mid 1 \leq j < k \leq n\},$$

$$\mathcal{N}_i = \begin{cases} \mathcal{T} & \text{if } i = -1 \\ \mathcal{U} & \text{if } i = 0 \\ \mathcal{N}_{i-1} \uplus \mathcal{W}_i & \text{otherwise} \end{cases} \tag{7}$$

where

$$\mathcal{W}_i = \{x^\Lambda \partial_k \in \mathcal{B} \mid n-i+1 \leq k \leq n \text{ and } \operatorname{wt}(\Lambda) = k+i-n+1+\delta_{m,2}\}. \tag{8}$$

Remark 1: The need for the symbol $\delta_{m,2}$, as will be clearer later, depends on the fact that the case $m = 2$ is different from the other cases since there is no partition of 2 into at least two distinct parts.

Remark 2: Note that from (7) it follows that

$$\mathcal{N}_{n-1-\delta_{m,2}} = \{x^\Lambda \partial_k \in \mathcal{B} \mid \operatorname{wt}(\Lambda) \leq k\},$$
$$\mathcal{N}_{n-2-\delta_{m,2}} = \{x^\Lambda \partial_k \in \mathcal{B} \mid \operatorname{wt}(\Lambda) \leq k-1\},$$

and in general for $3 + \delta_{m,2} \leq i \leq n$

$$\mathcal{N}_{n-i} = \{x^\Lambda \partial_k \in \mathcal{B} \mid \operatorname{wt}(\Lambda) \leq k-i+1+\delta_{m,2}\} \cup \mathcal{U}.$$

Definition 2.7: The **idealizer chain** starting from the $\mathbb{Z}_m$-submodule $\mathfrak{T}$ of $\mathfrak{L}$ (cf. Example 2.3) is defined as follows:

$$\mathfrak{N}_i = \begin{cases} N_{\mathfrak{L}}(\mathfrak{T}) & i = 0, \\ N_{\mathfrak{L}}(\mathfrak{N}_{i-1}) & i \geq 1. \end{cases} \tag{9}$$

We will prove that for $0 \leq i \leq n-1$ the Lie subring $\mathfrak{N}_i$ is the $\mathbb{Z}_m$-linear span of $\mathcal{N}_i$. To do so, we need the next results.

Lemma 2.8: *$\mathcal{N}_0 = N_{\mathcal{B}}(\mathcal{T})$.*

Proof. We clearly have $\mathcal{T} \subseteq N_{\mathcal{B}}(\mathcal{T})$. Now, if $x_i\partial_j \in \mathcal{U}$ with $i < j$, then

$$\mathbb{Z}_m\,\mathcal{T} \ni [x_i\partial_j, \partial_k] = \begin{cases} \partial_j & k = i, \\ 0 & k \neq i, \end{cases}$$

therefore $\mathcal{U} \subseteq N_{\mathcal{B}}(\mathcal{T})$.

Conversely, let $x^\Lambda\partial_j \in N_{\mathcal{B}}(\mathcal{T})$. For $1 \leq k \leq n$ we have

$$[x^\Lambda\partial_j, \partial_k] = \partial_k(x^\Lambda)\partial_j \in \mathbb{Z}_m\,\mathcal{T}.$$

This is possible when $\Lambda = 0$ or if $x^\Lambda = x_k$ for some $1 \leq k \leq n$, i.e., $x^\Lambda\partial_j \in \mathcal{N}_0$. ∎

The following result, which will be useful later on, is straightforward.

Lemma 2.9: *If $[x^\Lambda\partial_j, x^\Theta\partial_k] = cx^\Gamma\partial_u$, where $0 \neq c \in \mathbb{Z}_m$, then $u = \max(j,k)$ and $\mathrm{wt}(\Gamma) = \mathrm{wt}(\Lambda) + \mathrm{wt}(\Theta) - \min(j,k)$.*

Lemma 2.10: *If $1 \leq i \leq n - 1 - \delta_{m,2}$, then $[\mathcal{U}, \mathcal{W}_i] \subseteq \mathbb{Z}_m\,\mathcal{N}_{i-1}$.*

Proof. Let $x^\Lambda\partial_j \in \mathcal{W}_i$ and $x_h^{e_h}\partial_k \in \mathcal{U}$, where $0 \leq e_h \leq 1$ and let

$$cx^\Gamma\partial_u = [x^\Lambda\partial_j, x_h^{e_h}\partial_k],$$

where $0 \neq c \in \mathbb{Z}_m$. If $x^\Gamma\partial_u \in \mathcal{U}$ there is nothing to prove, so assume $x^\Gamma\partial_u \notin \mathcal{U}$. If $k \leq j$, then either $c = 0$ or, since $\mathrm{wt}(\Gamma) < \mathrm{wt}(\Lambda)$, $x^\Gamma\partial_u \in \mathcal{N}_{i-1}$. Otherwise, if $k > j$, then $cx^\Gamma\partial_u = \partial_j(x_h^{e_h})x^\Lambda\partial_k \neq 0$ if and only if $h = j$ and $e_h = 1$. Moreover, since we are assuming $x^\Gamma\partial_u \notin \mathcal{U}$, then it satisfies Equation (8), and we have

$$\mathrm{wt}(\Lambda) = j + 1 - (n-1) + \delta_{m,2}.$$

Now, $cx^\Gamma\partial_u = x^\Lambda\partial_k$ and

$$\begin{aligned} \mathrm{wt}(\Gamma) = \mathrm{wt}(\Lambda) &= j + i - (n-1) + \delta_{m,2} \\ &\leq k + i - 1 - (n-1) + \delta_{m,2}, \end{aligned}$$

therefore $x^\Gamma\partial_u \in \mathcal{N}_{i-1}$. ∎

Lemma 2.11: *If $1 \leq i < h \leq n - 1 - \delta_{m,2}$ then $[\mathcal{W}_i, \mathcal{W}_h] \subseteq \mathbb{Z}_m\,\mathcal{N}_{h-1}$.*

Proof. Let $x^\Lambda\partial_j \in \mathcal{W}_i$ and $x^\Theta\partial_k \in \mathcal{W}_h$. Let us denote $[x^\Lambda\partial_j, x^\Theta\partial_k] = cx^\Gamma\partial_u$ with $c \neq 0$ and let us assume that $x^\Gamma\partial_u \notin \mathcal{U}$ otherwise, as before, there is nothing to prove. By $x^\Lambda\partial_j \in \mathcal{W}_i$ we obtain $\mathrm{wt}(\Lambda) = j + i - (n-1) + \delta_{m,2}$ and by $x^\Theta\partial_k \in \mathcal{W}_h$ we obtain $\mathrm{wt}(\Theta) = k+h-(n-1)+\delta_{m,2}$. Now, by Lemma 2.9 we have

$$\begin{aligned}\mathrm{wt}(\Gamma) &= \mathrm{wt}(\Lambda)+\mathrm{wt}(\Theta)-\min(j,k)\\ &= j+i-(n-1)+\delta_{m,2}+k+h-(n-1)+\delta_{m,2}-\min(j,k)\\ &= \max(j,k)+i-(n-1)+\delta_{m,2}+h-(n-1)+\delta_{m,2}\\ &= u+h-(n-1)+\delta_{m,2}+i-n+1+\delta_{m,2}\\ &\le u+(h-1)-(n-1)+\delta_{m,2},\end{aligned}$$

which implies $x^\Gamma\partial_u \in \mathcal{N}_{h-1}$. ∎

PROPOSITION 2.12: *If $1 \le i \le n-1-\delta_{m,2}$, then $\mathcal{N}_i = N_{\mathcal{B}}(\mathcal{N}_{i-1})$.*

Proof. The inclusion $\mathcal{N}_i \subseteq N_{\mathcal{B}}(\mathcal{N}_{i-1})$ follows from the previous lemmas. It remains to prove that $N_{\mathcal{B}}(\mathcal{N}_{i-1}) \subseteq \mathcal{N}_i$. Let $x^\Lambda\partial_j \in N_{\mathcal{B}}(\mathcal{N}_{i-1})$. Then for each $1 \le l \le n-1-\delta_{m,2}$ and for each $x^\Theta\partial_k \in \mathcal{W}_l$ we have $[x^\Lambda\partial_j, x^\Theta\partial_k] \in \mathcal{N}_{i-1} \setminus \mathcal{N}_0$. Let $k < j$ be minimum such that $\lambda_k \neq 0$, and let $x^\Theta\partial_k = x_{k-1}\partial_k$. Then, since $\lambda_k \neq 0$, we have $[x^\Lambda\partial_j, x_{k-1}\partial_k] \neq 0$ and, by hypothesis, $[x^\Lambda\partial_j, x_{k-1}\partial_k]$ is a scalar multilple of an element $x^\Gamma\partial_j \in \mathcal{N}_{i-1}$ such that

$$\begin{aligned}\mathrm{wt}(\Gamma) = \mathrm{wt}(\Lambda)-1 &\le j+i-1-(n-1)+\delta_{m,2}\\ &< j+i-(n-1)+\delta_{m,2}.\end{aligned}$$

Therefore $\mathrm{wt}(\Lambda) \le j+i-(n-1)+\delta_{m,2}$, i.e., $x^\Lambda\partial_j \in \mathcal{N}_i$. ∎

Based on the previous Lemma for all $i \in \mathbb{N}$ we may define

$$\mathcal{N}_i = N_{\mathcal{B}}(\mathcal{N}_{i-1}). \tag{10}$$

THEOREM 2.13: *The Lie subring $\mathfrak{N}_i$ is homogeneous and the Lie subring $\mathfrak{N}_i$ is the $\mathbb{Z}_m$-linear span of $\mathcal{N}_i$.*

Proof. The statement is a straightforward consequence of Theorem 2.5 and of Lemma 2.8 and Proposition 2.12. ∎

2.4. CONNECTIONS WITH INTEGER PARTITIONS. Let $p_{m,i}$ be the number of partitions of i into at least two parts, where each part can be repeated at most $m-1$ times, and let $q_{m,i}$ be the partial sum

$$q_{m,i} = \sum_{j=1}^{i} p_{m,j}.$$

The first values of the sequences are shown in Table 2. Notice that the last three OEIS entries of the table include the partition of i with a single part that we do not consider.

Table 2. First values of the sequences $(p_{m,i})$ and $(q_{m,i})$ for $2 \leq m \leq 5$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	OEIS
$p_{2,i}$	0	0	1	1	2	3	4	5	7	9	11	14	17	21	26	31	A111133
$q_{2,i}$	0	0	1	2	4	7	11	16	23	32	43	57	74	95	121	152	A317910
$p_{3,i}$	0	1	1	3	4	6	8	12	15	21	26	35	43	56	69	88	A000726
$q_{3,i}$	0	1	2	5	9	15	23	35	50	71	97	132	175	231	300	388	
$p_{4,i}$	0	1	2	3	5	8	11	15	21	28	37	49	63	81	104	131	A001935
$q_{4,i}$	0	1	3	6	11	19	30	45	66	94	131	180	243	324	428	559	
$p_{5,i}$	0	1	2	4	5	9	12	18	24	33	43	59	75	99	126	163	A035959
$q_{5,i}$	0	1	3	7	12	21	33	51	75	108	151	210	285	384	510	673	

From Theorem 2.13 we derive the following corollaries, here stated in the case $m = 2$ and $m > 2$ separately.

THEOREM 2.14: *Let $m = 2$ and $1 \leq i \leq n-2$. Then, for $n-i+1 \leq k \leq n$ we have $|\mathcal{W}_i \cap \mathcal{B}_k| = p_{2,k+2+i-n}$ and therefore the free $\mathbb{Z}_2$-module $\mathfrak{N}_i/\mathfrak{N}_{i-1}$ has rank $q_{2,i+2}$.*

Notice that the result of Theorem 2.14 is in complete accordance with the analogous result found in the case of the chain of normalizers in the Sylow 2-subgroup of $\mathrm{Sym}(2^n)$ starting from an elementary abelian regular subgroup ([ACGS21b, Corollary 5]). This is not surprising: we will indeed prove in Section 3 that there exists a correspondence between the two constructions. More importantly, the use of the Lie ring of partitions introduced here allows to easily generalize the result to the case $m > 2$.

THEOREM 2.15: *Let $m > 2$ and $1 \leq i \leq n-1$. Then, for $n-i+1 \leq k \leq n$ we have $|\mathcal{W}_i \cap \mathcal{B}_k| = p_{m,k+1+i-n}$ and therefore the free $\mathbb{Z}_m$-module $\mathfrak{N}_i/\mathfrak{N}_{i-1}$ has rank $q_{m,i+1}$.*

3. An explicit correspondence in the case $m = 2$

In this section we will assume $m = 2$ without further reference. As already anticipated above, we now prove that for any $i \geq 1$ the ranks of the quotients $\mathfrak{N}_i/\mathfrak{N}_{i-1}$ are equal to the logarithms $\log_2 |N_i : N_{i-1}|$ of the factors of the normalizer chain in the Sylow 2-subgroup of $\mathrm{Sym}(2^n)$ starting from an elementary abelian regular subgroup. This is constructively accomplished by showing a bijection which maps rigid commutators into basis elements of the Lie ring of partitions and which preserves commutators.

3.1. Correspondence with Sylow 2-subgroups of $\mathrm{Sym}(2^n)$. We recall here some fundamental facts about rigid commutators, although we advise the reader to refer to Aragona et al. [ACGS21b] for notation and results. We use the punctured notation as in the mentioned paper. More precisely, if $\{s_1, s_2, \dots, s_n\}$ is the considered set of generators of the Sylow 2-subgroup of $\mathrm{Sym}(2^n)$ and $X = \{x_1 > x_2 > \dots > x_\ell\}$ is a subset of $\{1, \dots, n\}$, we denote by $[X]$ the left normed commutator $[s_{x_1}, s_{x_2}, \dots, s_{x_\ell}]$. The **rigid commutator based** at b and **punctured** at I is

$$\vee[b; I] = [\{1, \dots, b\} \setminus I] \in \mathcal{R}^*,$$

where $1 \leq b \leq n$ and $I \subseteq \{1, \dots, b-1\}$ and the symbol $\mathcal{R}^*$ denotes the set of non-trivial rigid commutators. We also denote $\mathcal{R} = \mathcal{R}^* \cup \{[\emptyset]\}$. We will use the commutator formula

$$(11) \quad [\vee[a; I], \vee[b; J]] = \begin{cases} \vee[\max(a,b); (I \cup J) \setminus \{\min(a,b)\}] & \text{if } \min(a,b) \in I \cup J \\ 1 & \text{otherwise} \end{cases}$$

proved in Proposition 4 of the referenced paper. We also recall that the elementary abelian regular group T is obtained in terms of rigid commutators as $T = \langle t_1, \dots, t_n \rangle$, where $t_i = \vee[i; \emptyset]$ for $1 \leq i \leq n$.

The mentioned bijection that will be soon defined relies crucially on the representation of rigid commutators provided by the following result.

Lemma 3.1: *Let $\mathcal{S} \subseteq \mathcal{R}$ be normalized by $\{t_1, \dots, t_n\}$. If $\vee[a; X]$ is any rigid commutator normalizing $\mathcal{S}$ and $\vee[b; Y] \in \mathcal{S}$, then there exists a rigid commutator $\vee[b; Z] \in \mathcal{S}$ such that $Z \cap X = \emptyset$ and*

$$[\vee[a; X], \vee[b; Y]] = [\vee[a; X], \vee[b; Z]].$$

Proof. Let $i \in X \cap Y$. Note that

$$[\vee[a;X], \vee[b;Y \setminus \{i\}]] = [\vee[a;X], [\vee[b;Y], t_i]] = [\vee[a;X], \vee[b;Y]].$$

In this way we can remove one by one from Y all the elements in $X \cap Y$ obtaining Z and preserving the commutator. ∎

Let us now define the bijection f between basis elements of the Lie ring and the set of rigid commutators. We will show later that f preserves commutators.

Definition 3.2: Let $f \colon \mathcal{B} \cup \{0\} \to \mathcal{R}$ be defined by letting

$$f(0) = [\emptyset] \quad \text{and} \quad f(x^{\Lambda}\partial_k) = \vee[k; \operatorname{supp}(\Lambda)].$$

Remark 3: By Equation (11) we have that if either $[x^{\Lambda}\partial_k, x^{\Gamma}\partial_h] \neq 0$ or $\Lambda \cap \Gamma = \emptyset$, then

$$f([x^{\Lambda}\partial_k, x^{\Gamma}\partial_h]) = [f(x^{\Lambda}\partial_k), f(x^{\Gamma}\partial_h)].$$

We note indeed that if $[x^{\Lambda}\partial_k, x^{\Gamma}\partial_h] = 0$ and $\Lambda \cap \Gamma = \emptyset$, then $k \notin \Gamma$ and $h \notin \Lambda$ and hence both members of the previous equation are the identity element.

LEMMA 3.3: *If $S \subseteq \mathcal{B} \cup \{0\}$ is normalized by $\mathcal{T} = \{\partial_1, \ldots, \partial_n\}$ and is closed under commutation, then $x^{\Theta}\partial_u$ normalizes S if and only if $f(x^{\Theta}\partial_u)$ normalizes $\mathcal{S} = f(S)$.*

Proof. We show first that $\mathcal{S}$ is closed under commutation. Notice that, since $[\mathcal{T}, S] \subseteq S$, by Remark 3 we have that $f(\mathcal{T}) = \{t_1, \ldots, t_n\}$ normalizes $\mathcal{S}$. Let $f(x^{\Lambda}\partial_k)$ and $f(x^{\Gamma}\partial_h)$ be two elements in $\mathcal{S}$. By Lemma 3.1, and by Remark 3 we have

$$[f(x^{\Lambda}\partial_k), f(x^{\Gamma}\partial_h)] = [f(x^{\Lambda}\partial_k), f(x^{\Gamma'}\partial_h)] = f([x^{\Lambda}\partial_k, x^{\Gamma'}\partial_h]) \in \mathcal{S}$$

for some Γ' such that $\operatorname{supp}(\Lambda) \cap \operatorname{supp}(\Gamma') = \emptyset$.

Let $x^{\Lambda}\partial_k \in S$ and $x^{\Theta}\partial_u$ be a basis element in the Lie ring normalizing S. The commutator $[x^{\Theta}\partial_u, x^{\Lambda}\partial_k] \in S$, hence, by Lemma 3.1, there exists Λ' such that $\operatorname{supp}(\Lambda') \cap \operatorname{supp}(\Theta) = \emptyset$ and

$$[f(x^{\Theta}\partial_u), f(x^{\Lambda}\partial_k)] = [f(x^{\Theta}\partial_u), f(x^{\Lambda'}\partial_k)] = f([x^{\Theta}\partial_u, x^{\Lambda'}\partial_k]) \in \mathcal{S}.$$

Therefore $f(x^{\Theta}\partial_u)$ normalizes $\mathcal{S}$. Conversely, if $f(x^{\Theta}\partial_u)$ normalizes $\mathcal{S}$ and $f(x^{\Lambda'}\partial_k) \in \mathcal{S}$, then $[f(x^{\Theta}\partial_u), f(x^{\Lambda}\partial_k)] \in \mathcal{S}$. Thus either

$$[x^{\Theta}\partial_u, x^{\Lambda}\partial_k] = 0 \in S$$

or

$$[f(x^\Theta\partial_u), f(x^\Lambda\partial_k)] = f([x^\Theta\partial_u, x^\Lambda\partial_k]) \in \mathcal{S} = f(S).$$

Hence $[x^\Theta\partial_u, x^\Lambda\partial_k] \in S$, as f is a bijection, and so $x^\Theta\partial_u$ normalizes S. ∎

We are finally ready to prove the claimed result.

THEOREM 3.4: *For all non-negative integers i the term N_i of the normalizer chain is the saturated subgroup generated by the saturated set of rigid commutators $f(\mathcal{N}_i)$. In particular, the following equality holds for each $i \geq 1$:*

$$\operatorname{rk}(\mathfrak{N}_i/\mathfrak{N}_{i-1}) = \log_2|N_i : N_{i-1}|.$$

Proof. This is a straightforward consequence of the previous lemma applying Theorem 2.13 and [ACGS21b, Corollary 2 and Proposition 5]. ∎

4. Unrefinable partitions with repeated parts and the $(n-1)$-th idealizer

The definition of unrefinability of a partition into distinct parts has been given in Aragona et al. [ACGS22] in connection with the $(n-1)$-th term in the chain of normalizers in $\operatorname{Sym}(2^n)$. We introduce here a natural generalization to partitions whose parts can be repeated at most $m-1$ times and we show the connection (cf. Theorem 4.5) with the first idealizer not following the rules of Theorems 2.14 and 2.15, i.e., the $(n-\delta_{m,2})$-th.

Definition 4.1: Let $\Lambda \in \mathcal{P}_m$ be a partition where each part has multiplicitiy at most $m-1$ and such that there exist indices $j_1 < \cdots < j_\ell < j$ satisfying the conditions

- $j = \sum_{i=1}^{\ell} a_i j_i$, with $a_i \leq m-1-\lambda_{j_i}$,
- $\lambda_j \geq 1$.

The partition Θ obtained from Λ removing the part j and inserting the parts $j_1, \ldots, j_\ell$, each taken a_i times, is said to be an a-refinement of Λ where $a = \sum a_i$. We shall write $\Theta \prec \Lambda$ to mean that Θ is a 2-refinement of Λ. A partition admitting a refinement is said to be **refinable** in $\mathcal{P}_m$, otherwise it is said to be **unrefinable** in $\mathcal{P}_m$.

Remark 4: Notice that, although the part j can appear with multiplicity up to $m-1$, the operation of refinement as in Definition 4.1 is performed on a single part.

PROPOSITION 4.2: *Every a-refinement of a partition Λ is obtained applying exactly $a-1$ subsequent 2-refinements.*

Proof. Let j be the part of Λ replaced by a_1 repetitions of $j_1, \ldots,$ and a_ℓ repetitions of j_ℓ. We split the proof in two cases, depending on $\lambda_{j_1+j_2} \geq 1$ or $\lambda_{j_1+j_2} = 0$, and we argue by induction, the statement being trivial when $a = 2$. Let $\lambda_{j_1+j_2} \geq 1$. First we apply the 2-refinement that inserts j_1 and j_2 in place of $j_1 + j_2$. Subsequently we apply the induction argument on the refinement replacing j by inserting $j_1 + j_2, j_3, \ldots, j_\ell$ via $a-2$ subsequent 2-refinements. Suppose now $\lambda_{j_1+j_2} = 0$. We first apply the $(a-2)$-refinement replacing j by inserting $j_1 + j_2, j_3, \ldots, j_\ell$ and subsequently we apply the 2-refinement that inserts j_1 and j_2 in place of $j_1 + j_2$. In both cases by induction a number $a-1$ of 2-refinement are applied. Since every 2-refinement increases by one the total number of the parts, $a-1$ is the minimum possible number of 2-refinements that we can subsequently perform to obtain the final a-refinement. ∎

Definition 4.3: Let $\Lambda \in \mathcal{P}_m$ and $t > 0$ be an integer. We say that Λ is **0-step refinable** if it is unrefinable in $\mathcal{P}_m$. We say that Λ is **t-step refinable** if t is maximal such that there exists a sequence made of t subsequent proper 2-refinements $\Lambda_t \prec \Lambda_{t-1} \prec \cdots \prec \Lambda_0 = \Lambda$ such that Λ_t is unrefinable. In other words, t is the maximum number of 2-refinements to be subsequently applied starting from Λ in order to obtain some partition that is unrefinable in $\mathcal{P}_m$.

Remark 5: A straightforward consequence of Proposition 4.2 is that a partition Λ in $\mathcal{P}_m$ is t-step refinable if and only if t is maximal among the a such that Λ admits an a-refinement.

Definition 4.4: Let $\Lambda \in \mathcal{P}_m(n-1)$. Consider the monomial $f = \prod_{i=1}^{n-1} x_i^{m-1}$ and let

$$x_{e_1}^{\mu_1} \cdots x_{e_s}^{\mu_s} = f / x^\Lambda,$$

where $e_1 < \cdots < e_s$ and $\mu_i \geq 1$. The index e_i is said to be the **i-th excludant** of Λ and μ_i is its multiplicity. The first excludant of Λ is also called its **minimum excludant**. We say that $x^\Lambda \partial_k$ satisfies the **i-th excludant condition** if i is the minimum index such that $n < k + e_i$. Moreover, we say that $x^\Lambda \partial_k$ satisfies the **weak i-th excludant condition** if i is the minimum index such that $n < k + e_1 + \cdots + e_i$.

Note that if a partition satisfies the i-th excludant condition then it also satisfies the weak j-th excludant condition for some $j \leq i$.

We define the **filler element** as

$$\mathrm{fil}_{i,j} = x_i x_j \partial_{i+j} \in \mathfrak{N}_{n-1-\delta_{m,2}} \setminus \mathfrak{N}_{n-2-\delta_{m,2}}. \tag{12}$$

Let $\Lambda \in \mathcal{P}_m(n-1)$ be a partition with excludants $e_1 < \cdots < e_s$ and suppose that $x^\Lambda \partial_k \in \mathfrak{N}_j$ for some $j \geq n - \delta_{m,2}$. If $k + e_i \leq n$, then the commutator operation

$$[x^\Lambda \partial_k, \mathrm{fil}_{e_i,k}] = x_{e_i} x^\Lambda \partial_{k+e_i} \in \mathfrak{N}_{j-1}$$

has the effect of **filling** the i-th excludant of Λ.

We now deal with the main result of the section. The condition for a partition $\Lambda \in \mathcal{P}_m(k-1)$ to be refinable is equivalent to the fact that there exists a partition $\Theta \in \mathcal{P}_m(h-1)$ with $h = \mathrm{wt}(\Theta) < k$, such that $[x^\Lambda \partial_k, x^\Theta \partial_h] \neq 0$.

Theorem 4.5: *The elements of the set $\mathcal{N}_{n-\delta_{m,2}} \setminus \mathcal{N}_{n-1-\delta_{m,2}}$ are of the form $x^\Lambda \partial_k \in \mathcal{B}$, where $\Lambda \in \mathcal{P}_m(n-1)$ is an unrefinable partition of $k+1$ satisfying the first excludant condition.*

Proof. We prove the claim assuming $m > 2$. The proof of the case $m = 2$ is nearly identical, and also unnecessary, by virtue of the correspondence shown in Section 3.

Let $x^\Lambda \partial_k \in \mathcal{N}_n \setminus \mathcal{N}_{n-1}$ and let e be the minimal excludant of Λ. By Remark 2, since $x^\Lambda \partial_k \notin \mathcal{N}_{n-1}$, we have $\mathrm{wt}(\Lambda) \geq k+1$. Let $h = \min\{j \mid \lambda_j \neq 0\}$ and let

$$\mathcal{N}_0 \ni x^\Gamma \partial_h = \begin{cases} \partial_1 & \text{if } h = 1, \\ x_{h-1}\partial_h & \text{if } h > 1. \end{cases}$$

Since $\mathcal{N}_{n-1} \ni [x^\Gamma \partial_h, x^\Lambda \partial_k] = x^\Theta \partial_k \neq 0$ it follows that $\mathrm{wt}(\Theta) = \mathrm{wt}(\Lambda) - 1 \leq k$. Hence $\mathrm{wt}(\Lambda) = k+1$.

Let Ξ be any partition of weight $k+1$. By Lemma 2.9 $[x^\Xi \partial_k, \mathcal{N}_{n-2}] \subseteq \mathcal{N}_{n-1}$, again by Lemma 2.9 and Remark 2 it follows that $x^\Xi \partial_k \in \mathcal{N}_n \setminus \mathcal{N}_{n-1}$ if and only if $[x^\Xi \partial_k, \mathcal{W}_{n-1}] = 0$. Let then $x^\Sigma \partial_h \in \mathcal{W}_{n-1}$, so that $\mathrm{wt}(\Sigma) = h$. The condition $[x^\Sigma \partial_h, x^\Lambda \partial_k] = 0$ for all $x^\Sigma \partial_h \in \mathcal{W}_{n-1} \setminus \mathcal{U}$ with $h \leq k$ is equivalent to $x^\Lambda \partial_k$ being unrefinable. So we assume $h > k$ and $x^\Theta \partial_h = [x^\Sigma \partial_h, x^\Lambda \partial_k] \neq 0$. In particular $\mathrm{wt}(\Sigma) \geq e + k$, since $\sigma_k \geq 1$ and since Σ can have non-zero components σ_i only if $i \neq k$ is an excludant of Λ. Hence

$$n \geq h = 1 + \mathrm{wt}(\Sigma) \geq e + k$$

yielding $k \leq n - e$. Conversely if $0 < k \leq n - e$ then $\mathrm{fil}_{e,k} \in \mathcal{N}_{n-1}$ and

$$[x^\Lambda \partial_k, \mathrm{fil}_{e,k}] = x_e x^\Lambda \partial_{e+k} \neq 0.$$

Hence if $\mathrm{wt}(\Lambda) = k + 1$ then $[x^\Lambda \partial_k, \mathcal{W}_{n-1} \setminus \mathcal{U}] = 0$ if and only if $n - e < k \leq n$ and $x^\Lambda \partial_k$ is unrefinable in $\mathcal{P}_m$. ∎

4.1. One more idealizer. In this last section we set again $m = 2$ and we aim at the characterization of the n-th term of the idealizer chain defined in Equation (9). By virtue of the results of Section 3.1, the characterization automatically extends to the n-th normalizer in $\mathrm{Sym}(2^n)$ of Equation (1). The next contributions are rather technical and will really show the cost, in terms of combinatorial complexity, of trying to go beyond the 'natural' limit of the $(n-1)$-th idealizer/normalizer.

Let $x^\Lambda \partial_k \in \mathfrak{N}_n \setminus \mathfrak{N}_{n-1}$ and let $e_1 < \cdots < e_s$ be the excludants of Λ. We start by giving some necessary conditions that $x^\Lambda \partial_k$ has to satisfy since it belongs to $\mathfrak{N}_n \setminus \mathfrak{N}_{n-1}$.

By Theorem 2.13, we have $\mathrm{wt}(\Lambda) \geq k + 1$. Suppose first that $\mathrm{wt}(\Lambda) = k + 1$. By Theorem 4.5 either Λ is refinable or Λ is unrefinable and $k \leq n - e_1$. If Λ is refinable, then there exists a partition Γ with $h = \mathrm{wt}(\Gamma) < k$, such that $\mathfrak{N}_{n-1} \ni x^\Theta \partial_k = [x^\Lambda \partial_k, x^\Gamma \partial_h] \neq 0$; the partition Θ is then unrefinable and the minimal excludant e of Θ is such that $k > n - e$. Suppose that Λ satisfies the j-th excludant condition with $j \geq 1$. Since there exists an unrefinable 2-refinement Θ of Λ obtained replacing a part λ_u with two excludants e_s and e_t, then we have $j \leq 3$. Moreover, if $j \geq 2$, then the commutator element

$$x^\Xi \partial_{e_1+k} = [x^\Lambda \partial_k, \mathrm{fil}_{e_1,k}] = x_{e_1} x^\Lambda \partial_{e_1+k} \in \mathfrak{N}_{n-1},$$

therefore $x^\Lambda \partial_k$ satisfies the weak second excludant condition and the partition Ξ obtained by Λ by filling its minimum excludant is an unrefinable partition. This implies that any refinement of Λ has 1 in the e_1-th component. In the more specific case $j = 3$, the same argument applies replacing e_1 with e_2. Thus every refinement Θ of Λ has each of the e_1-th and e_2-th component set to 1. From Proposition 4.2, we have that if $j = 3$, then $\lambda_{e_1+e_2} = 1$ and Θ is obtained from Λ inserting 0 in the $(e_1 + e_2)$-th component and 1 in the e_1-th and e_2-th component of Λ. A similar argument shows that if Λ is unrefinable, then it has to satisfy the second weak excludant condition. Let us summarize the previous conditions as follows:

Definition 4.6: The element $x^\Lambda\partial_k$ satisfies the **1-step excludant condition** if $wt(\Lambda) = k+1$ and Λ satisfies one of the following:

(a) Λ is 1-step refinable and it satisfies the first excludant condition,

(b) Λ is 1-step refinable and it satisfies the second excludant condition and every refinement Θ is such that $\theta_{e_1} = 1$,

(c) Λ is 1-step refinable and satisfies both the third excludant condition and the second weak excludant condition, $\lambda_{e_1+e_2} = 1$, and the only refinement Θ of Λ is such that $x^\Theta = x_{e_1}x_{e_2}x^\Lambda/x_{e_1+e_2}$,

(d) Λ is unrefinable and it has to satisfy the second weak excludant condition.

We are now left with the case $\mathrm{wt}(\Lambda) \geq k+2$. If $\lambda_1 = 1$, then

$$x^\Theta\partial_k = [x^\Lambda\partial_k, \partial_1] \in \mathfrak{N}_{n-1},$$

and so $k+1 \leq \sum_{i\geq 2} i\lambda_i \leq k+1$ implies

$$\mathrm{wt}(\Lambda) = k+2.$$

The minimal excludant of Θ is 1, which implies $k > n-1$, i.e., $k = n$. Moreover, Θ has to be unrefinable and so if $\lambda_i = \theta_i = 0$ for some $i \geq 2$, then $\lambda_{i+1} = \theta_{i+1} = 0$ as well. This implies that there exists an index t such that $\lambda_i = 1$ for $1 \leq i \leq t$ and $\lambda_i = 0$ for $i > t$. Thus Λ is a triangular partition. Suppose now that $\lambda_1 = 0$. Let h be an index such that $\lambda_{h-1} = 0$ and $\lambda_h = 1$. We want to show that $h = 2$. If $h > 2$, then the commutator element $x^\Theta\partial_k = [x^\Lambda\partial_k, \mathrm{fil}_{1,h-1}] \in \mathfrak{N}_{n-1}$ where $\mathrm{wt}(\Theta) = k+2$, a contradiction. This implies that there exists an index $t > 2$ such that $\lambda_i = 0$ for $i > t$. We will then say that Λ is a **weak-triangular partition**. In particular, $\lambda_2 = 1$ and so the commutator element

$$[x^\Lambda\partial_k, x_1\partial_2] = x^\Theta\partial_k \in \mathfrak{N}_{n-1},$$

where the minimum excludant of Θ is 2. Thus $n-2 < k \leq n$, i.e., k is either n or $n-1$. Note that the case $k = n-1$ cannot occur since then $[x^\Lambda\partial_k, \mathrm{fil}_{1,k}] = x^\Theta\partial_n \neq 0 \in \mathfrak{N}_{n-1}$ which yields the contradiction $\mathrm{wt}(\Theta) = k+3 = n+2 > n+1$.

We conclude summarizing below what was previously discussed and showing that the mentioned conditions are also sufficient, with some sporadic exceptions in the case $n = 8$.

THEOREM 4.7: *With the sole exclusion of the cases $n = 8$ and*

$$x^\Lambda \partial_k = x_2 x_7 \partial_8,$$
$$x^\Lambda \partial_k = x_4 x_5 \partial_8,$$
$$x^\Lambda \partial_k = x_2 x_4 \partial_5,$$

the element $x^\Lambda \partial_k$ belongs to $\mathfrak{N}_n \setminus \mathfrak{N}_{n-1}$ if and only if one of the following conditions is satisfied:

(1) $\mathrm{wt}(\Lambda) = k + 1$ *and $x^\Lambda \partial_k$ satisfies the* 1-step excludant condition,
(2) $\mathrm{wt}(\Lambda) = k + 2$, $k = n$ *and one of the following holds,*
    (i) $n + 2$ *is the t-th triangular number and $x^\Lambda = x_1 \cdots x_t$, i.e., Λ is the t-th triangular partition,*
    (ii) $n + 3$ *is the t-th triangular number and $x^\Lambda = x_2 \cdots x_t$, i.e., Λ is the t-th weak-triangular partition.*

Proof. Due to the intricate combinatorial nature of the problem, the long proof of the result is rather tedious as it is articulated in several cases and sub-cases and it is therefore omitted. It can be made immediately available by the authors to the interested reader upon request. ∎

5. Conclusions and open problems

Computing the chain of normalizers of Equation (1) is a computationally challenging task which soon clashes with the exponential growth of the order of the considered groups. In fact, as already pointed out in [ACGS21a], computing the chain of normalizers up to the $(n-2)$-th term and more would not have been possible without introducing rigid commutators [ACGS21b]. Unfortunately, it appears that there is no natural way to generalize the notion of rigid commutators when p is odd in such a way that these turn out to be closed under commutation. An odd version of the rigid commutator machinery, as described in the cited paper for $p = 2$, would be indeed the key ingredient that could prove helpful in computing the chain of normalizers in $\mathrm{Sym}(p^n)$. This task is otherwise computationally unfeasible when $p \geq 3$, even when minimal values of n are considered.

With this goal in mind, in this work we have introduced a new framework which moves the setting from the symmetric group to a Lie ring with a basis of elements represented by partitions of integers which parts can be repeated no more than $m-1$ times. In this framework, the construction of the Lie ring reflects the construction of the Sylow p-subgroup of $\mathrm{Sym}(p^n)$ when $m=p$ is prime, and still provides meaningful results when m is composite. We defined the corresponding idealizer chain in the Lie ring and proved, as expected, that the growth of the idealizer chain goes as in the case of $\mathrm{Sym}(2^n)$ when $m=2$, and proceeds according to its natural generalization when $m>2$ (cf. Theorem 2.14 and Theorem 2.15). In particular, when $m=2$ an explicit bijection between generators which preserves commutators is provided (cf. Definition 3.2 and Theorem 3.4).

The possible obvious extensions of the notion of rigid commutators in the case p odd, to which will correspond a bijection similar to that given in Definition 3.2, do not produce a set of commutators that turns out to be closed under commutation, a property that is crucial in the proof of Theorem 3.4. If a commutation-closed extension were found, it would not be hard to believe that a natural correspondence preserving commutators between the new rigid commutators and the basis elements of the Lie ring, as the one of Definition 3.2, may exist. This would imply that Theorem 2.15 is the p-analog of the chain of normalizers in $\mathrm{Sym}(p^n)$, where $m=p$ is odd, which at the time of writing remains a very plausible conjecture for which this paper, in the absence of any computational evidence, represents a source of support.

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